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PURE-PERIODIC MODULES AND A STRUCTURE OF
PURE-PROJECTIVE RESOLUTIONS

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PURE-PERIODIC MODULES AND A STRUCTURE OF PURE-PROJECTIVE RESOLUTIONS

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Dedicated to Stanisław Balcerzyk on the occasion of his seventieth birthday

We investigate the structure of pure-syzygy modules in a pure-projective resolution of any right R -module over an associative ring R with an identity element. We show that a right R -module M is pure-projective if and only if there exists an integer $n \geq 0$ and a pure-exact sequence $0 \rightarrow M \rightarrow P_n \rightarrow \dots \rightarrow P_0 \rightarrow M \rightarrow 0$ with pure-projective modules P_n, \dots, P_0 . As a consequence we get the following version of a result in Benson and Goodearl, 2000: A flat module M is projective if M admits an exact sequence $0 \rightarrow M \rightarrow F_n \rightarrow \dots \rightarrow F_0 \rightarrow M \rightarrow 0$ with projective modules F_n, \dots, F_0 .

1. Introduction.

Throughout this paper R is an associative ring with an identity element. We denote by $\text{Mod}(R)$ the category of all right R -modules. We recall (see [12]) that an exact sequence $\dots \rightarrow X_{n-1} \rightarrow X_n \rightarrow X_{n+1} \rightarrow \dots$ in $\text{Mod}(R)$ is said to be **pure** (in the sense of Cohn [6]) if the induced sequence $\dots \rightarrow X_{n-1} \otimes_R L \rightarrow X_n \otimes_R L \rightarrow X_{n+1} \otimes_R L \rightarrow \dots$ of abelian groups is exact for any left R -module L . An epimorphism $f : Y \rightarrow Z$ in $\text{Mod}(R)$ is said to be **pure** if the exact sequence $0 \rightarrow \text{Ker } f \rightarrow Y \xrightarrow{f} Z \rightarrow 0$ is pure. A submodule X of a right R -module Y is said to be **pure** if the exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Y/X \rightarrow 0$ is pure. A module P in $\text{Mod}(R)$ is said to be **pure-projective** if for any pure epimorphism $f : Y \rightarrow Z$ in $\text{Mod}(R)$ the induced group homomorphism $\text{Hom}_R(P, f) : \text{Hom}_R(P, Y) \rightarrow \text{Hom}_R(P, Z)$ is surjective. The following facts are well-known (see [14], [15], [29], [31]):

(i) *A module P in $\text{Mod}(R)$ is pure-projective if and only if P is a direct summand of a direct sum of finitely presented modules.*

(ii) *Every module M in $\text{Mod}(R)$ admits a pure-projective pure resolution \mathbf{P}_* in $\text{Mod}(R)$, that is, there is a pure-exact sequence*

$$(1.1) \quad \dots \rightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$$

where the modules P_0, \dots, P_n, \dots are pure-projective.

The main results of the paper are the following two theorems:

Theorem 1.2. *Let M be a right R -module and (1.1) a pure-exact sequence in $\text{Mod}(R)$ such that the modules P_0, \dots, P_n, \dots are pure-projective. Then, for each $n \geq 0$, the n -th pure-syzygy module $\text{Ker } d_n$ of M is an \aleph_n -directed union of \aleph_n -generated pure-projective R -modules, which are pure submodules of P_n and of $\text{Ker } d_n$.*

Theorem 1.3. *Let M be a right R -module. If there exists a pure-exact sequence*

$$0 \rightarrow M \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

in $\text{Mod}(R)$ such that the modules P_0, \dots, P_n are pure-projective, then M is pure-projective.

In other words, every pure-periodic R -module M is pure-projective. As a consequence of Theorem 1.3 we get the following version of a recent result by Benson and Goodearl in [5].

Corollary 1.4. *Let M be a right flat R -module. If there exists an exact sequence*

$$0 \rightarrow M \rightarrow F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

in $\text{Mod}(R)$ such that the modules F_0, \dots, F_n are projective, then M is projective.

Our Theorem 1.2 extends the main projective resolution structure theorem [23, Theorem 1.5] (see also [17, Theorem 3.3] for $n = 0$) from flat modules to arbitrary modules. The Corollary 1.4 with $n = 0$ coincides with [5, Theorem 2.5].

The main results of the paper are proved in Sections 3 and 4. In Section 2 we collect preliminary facts and notation we need throughout the paper. In Section 5 we show that Theorems 1.2 and 1.3 remain valid in any locally finitely presented Grothendieck category \mathcal{A} .

Throughout this paper we use freely the module theory terminology and notation introduced in [1] and [12]. The reader is referred to [8], [12], [14], [16], [26], [29], [31] and to the expository papers [9] and [28] for a basic background and historical comments on purity and pure homological dimensions.

2. Preliminaries on the pure-projective dimension.

We start this section by collecting basic definitions, notation and elementary facts we need throughout this paper.

Given right R -modules M and N the n -th **pure extension group** $\text{Pext}_R^n(M, N)$ is defined to be the n -th cohomology group of the complex $\text{Hom}_R(\mathbf{P}_*, N)$, where \mathbf{P}_* is a pure-projective resolution of M in $\text{Mod}(R)$.

The **pure-projective dimension** $\text{P.pd } M$ of M is defined to be the minimal integer $m \geq 0$ (or infinity) such that $\text{Pext}_R^m(M, -) = 0$. The **right pure global dimension** $\text{r.P.gl.dim } R$ of R is defined to be the minimal integer $n \geq 0$ (or infinity) such that $\text{Pext}_R^n = 0$. Following [27] we call the ring R **right pure semisimple** if $\text{r.P.gl.dim } R = 0$.

The left pure global dimension $\text{l.P.gl.dim } R$ of a ring R was introduced in 1967 by R. Kiełpiński [14] and in 1970 by P. Griffith [7]. It was shown in [26, Theorem 2.12] that the right pure global dimension $\text{r.P.gl.dim } R$ of the ring R is the supremum of $\text{P.pd } M$, where M runs through all right R -modules M such that $\text{P.pd } M$ is finite. This means that the right finitistic pure global dimension of R and the right pure global dimension of R coincide.

Throughout this paper we denote by \aleph an infinite cardinal number and by \aleph_0 the cardinality of a countable set. A right R -module M is said to be \aleph -generated if it is generated by a set of cardinality \aleph , and M is \aleph -presented if M is \aleph -generated and for any epimorphism $f : L \rightarrow M$ with \aleph -generated module L the kernel $\text{Ker } f$ is \aleph -generated, or equivalently, M is a limit of a direct system $\{M_j, h_{ij}\}$ of cardinality \aleph consisting of finitely presented modules M_j (see [18], [20], [22], [26]). We say that M is an **\aleph -directed union** of submodules M_j , $j \in J$, if for each subset J_0 of J of cardinality \aleph there exist $j_0 \in J$ such that $M_t \subseteq M_{j_0}$ for all $t \in J_0$.

A union $\bigcup_{\xi < \gamma} M_\xi$ of submodules M_ξ of M is **well-ordered and continuous** if γ is an ordinal number, $M_0 = (0)$, $M_\xi \subseteq M_\eta$ for $\xi < \eta < \gamma$, and $M_\tau = \bigcup_{\xi < \tau} M_\xi$ for any limit ordinal number $\tau \leq \gamma$ (see [16]).

The following pure version of the well-known Auslander result [2, Proposition 3] is of importance.

Proposition 2.1. *Assume that the right R -module M is a continuous well-ordered union of submodules M_ξ , with $\xi < \gamma$, where γ is an ordinal number. If $\text{P.pd } M_{\xi+1}/M_\xi \leq m$ for all $\xi < \gamma$, then $\text{P.pd } M \leq m$.*

Proof. The arguments of Auslander in the proof of [2, Proposition 3] generalise to our situation (see [16, Proposition 1.2]). \square

We also need the following pure version of the well-known Osofsky result [19] (see also [3]) proved in [16, Corollary 1.4], [8], and in [26, Theorem 2.12] in a general context of Grothendieck categories.

Proposition 2.2. *Assume that $m \geq 0$ is an integer and M is an arbitrary \aleph_m -presented right R -module. Then $\text{P.pd } M \leq m + 1$.*

By applying the definition of a pure submodule one proves the following useful criterion:

Lemma 2.3. *Assume that P is a pure-projective right R -module and let K be a submodule of P . The following conditions are equivalent:*

- (a) K is a pure submodule of P .
- (b) For any finitely generated submodule X of K there exists an R -homomorphism $\varphi : P \rightarrow K$ such that $\text{Im } \varphi$ is contained in a finitely generated R -submodule of K and $\varphi|_X = \text{id}_X$.
- (c) For any finitely generated submodule X of K there exists an R -homomorphism $\varphi : P \rightarrow K$ such that $\varphi|_X = \text{id}_X$.

Proof. Since the module P is pure projective, there exists a module P' such that $P \oplus P'$ is a direct sum of finitely presented modules. Assume that K is a submodule of P and let $u : K \rightarrow P$ be the embedding.

(a) \Rightarrow (b) Assume that $u : K \rightarrow P$ is a pure monomorphism and X is a finitely generated submodule of K . Then the monomorphism $(u, 0) : K \rightarrow P \oplus P'$ is pure and there exists a finitely presented direct summand L of $P \oplus P'$ such that $(u, 0)(X) \subseteq L$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & K & \xrightarrow{(u,0)} & P \oplus P' & \xrightarrow{\pi} & \text{Coker}(u, 0) & \longrightarrow & 0 \\
 & & \uparrow h' & & \uparrow h & & \uparrow h'' & & \\
 0 & \longrightarrow & X & \xrightarrow{u'} & L & \xrightarrow{p} & \bar{L} & \longrightarrow & 0
 \end{array}$$

with exact rows, where h' is the embedding of X into K , h is a direct summand embedding, π is a pure epimorphism and the module \bar{L} is finitely presented. It follows that there exists $v'' \in \text{Hom}_R(\bar{L}, P \oplus P')$ such that $\pi v'' = h''$, and consequently there exists $v' \in \text{Hom}_R(L, K)$ such that $v'u' = h'$. Let $\varphi' : P \oplus P' \rightarrow K$ be an extension of v' to $P \oplus P'$ such that $v' = \varphi'h$ and $\text{Im } \varphi'$ is finitely generated. Let $\varphi : P \rightarrow K$ be the restriction of φ' to P . It follows that $\text{Im } \varphi$ is contained in the finitely generated R -submodule $\text{Im } \varphi'$ of K and, for any $x \in X$, we have $x = h'(x) = v'u'(x) = \varphi'hu'(x) = \varphi'(h'(x), 0) = \varphi'(x, 0) = \varphi(x)$. This shows that $\varphi|_X = \text{id}_X$ and (b) follows.

The implication (b) \Rightarrow (c) is obvious.

(c) \Rightarrow (a) Assume that, for any finitely generated submodule X of K , there exists an R -homomorphism $\varphi : P \rightarrow K$ such that $\varphi|_X = \text{id}_X$. We shall prove that K is a pure submodule of P by showing that the canonical epimorphism $\pi : P \rightarrow P/K$ is pure. Let $f : L \rightarrow P/K$ be a homomorphism from a finitely presented module L to P/K . Then $L \cong F/N$, where F is a finitely generated free module and N is a finitely generated submodule of F . It is clear that there exists a commutative diagram

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & K & \xrightarrow{u} & P & \xrightarrow{\pi} & P/K & \longrightarrow & 0 \\
 & & \uparrow f'' & & \uparrow f' & & \uparrow f & & \\
 0 & \longrightarrow & N & \xrightarrow{u'} & F & \xrightarrow{p} & L & \longrightarrow & 0
 \end{array}$$

with exact rows, where p is the canonical epimorphism and u' is the canonical embedding. Then $X = f''(N)$ is a finitely generated submodule of K and,

according to our assumption, there exists an R -homomorphism $\varphi : P \rightarrow K$ such that $\varphi|_X = \text{id}_X$. Note that the homomorphism $v' = \varphi f' : F \rightarrow K$ satisfies the equality $f'' = v'u'$. It follows that there exists $v'' \in \text{Hom}_R(L, P)$ such that $\pi v'' = f$. This shows that π is a pure epimorphism and finishes the proof of the lemma. \square

Let P be a pure-projective right R -module and let K be a pure submodule of P . Following [23, Proposition 1.4] and [24] we define a **pure-closure** L^\diamond of any R -submodule L of K as follows. Set $L_0 = L$ and fix a set L' of generators of L . By Lemma 2.3, for any finite subset λ of L' we find an R -homomorphism $\varphi_\lambda : P \rightarrow K$ such that $\text{Im } \varphi_\lambda$ is contained in a finitely generated R -submodule K_λ of K , and $\varphi_\lambda|_\lambda = \text{id}_\lambda$. Let L_1 be the R -submodule of L generated by the set $L'' = \bigcup_{\lambda \subseteq L'} K_\lambda$, over all finite subsets of L' . It is clear that $L = L_0 \subseteq L_1$ and, for any finitely generated submodule X of $L_0 = L$, there exists an R -homomorphism $\varphi : P \rightarrow L_1$ such that $\text{Im } \varphi$ is contained in a finitely generated R -submodule of L_1 , and $\varphi|_X = \text{id}_X$. By choosing a set L'_1 of generators of L_1 and applying the procedure above with L' and L'_1 interchanged, we construct a submodule L_2 containing L_1 such that for any finitely generated submodule X of L_1 there exists an R -homomorphism $\varphi : P \rightarrow L_2$ such that $\text{Im } \varphi$ is contained in a finitely generated R -submodule of L_2 , and $\varphi|_X = \text{id}_X$. Continuing this way we define an ascending sequence

$$L = L_0 \subseteq L_1 \subseteq L_2 \subseteq \dots \subseteq L_m \subseteq L_{m+1} \subseteq \dots$$

of R -submodules of K , and sets $L'_0, L'_1, L'_2, \dots, L'_m, L'_{m+1}, \dots$ of their generators in such a way that, for each $m \geq 0$ and for any finitely generated submodule X of L_m , there exists an R -homomorphism $\varphi : P \rightarrow L_{m+1}$ such that $\text{Im } \varphi$ is contained in a finitely generated R -submodule of L_{m+1} , and $\varphi|_X = \text{id}_X$. By Lemma 2.3, the submodule

$$(2.4) \quad L^\diamond = \bigcup_{m=0}^\infty L_m$$

of K is a pure submodule of P (and of K), and we call it a **pure-closure** of the R -submodule L of K . It is clear that L^\diamond is not determined uniquely by L and depends on the choice of the modules K_λ , the sets $L'_0, L'_1, L'_2, \dots, L'_m, L'_{m+1}, \dots$ and the R -homomorphisms $\varphi_\lambda : P \rightarrow K$. However, if \aleph is an infinite cardinal number and the module L is \aleph -generated then the sets $L'_0, L'_1, L'_2, \dots, L'_m, L'_{m+1}, \dots$ can be chosen of cardinality \aleph and we get the following result.

Lemma 2.5. *Assume that P is a pure-projective right R -module, K a pure submodule of P and L an \aleph -generated submodule of K , where \aleph is an infinite*

cardinal number. Then there exists an \aleph -generated submodule L^\diamond of K such that $L \subseteq L^\diamond$ and L^\diamond is a pure submodule of P (and of K).

We also need the following technical result.

Lemma 2.6. *Assume that \aleph is an infinite cardinal number, $h : P \rightarrow K$ is a pure epimorphism in $\text{Mod}(R)$, P is an \aleph -generated pure-projective module and K is a pure submodule of a pure-projective module.*

- (a) *The module K has a directed union form $K = \bigcup_{\lambda \in \Omega} K_\lambda$, where Ω is a set of cardinality $\leq \aleph$ and K_λ is a countably generated pure-projective pure submodule of K , for each $\lambda \in \Omega$.*
- (b) *The module $\text{Ker } h$ is \aleph -generated.*

Proof. Let $h : P \rightarrow K$ be a pure-epimorphism. We set $L = \text{Ker } h$ and assume that the module P is \aleph -generated. Then there exist a set Ω of cardinality $\leq \aleph$ and a family of finitely generated submodules P_λ of P , with $\lambda \in \Omega$, such that $P = \bigcup_{\lambda \in \Omega} P_\lambda$ is a directed union. By our assumption, K is a pure submodule of a pure-projective module P_0 . Let P'_0 be a right R -module such that $P_0 \oplus P'_0$ is a direct sum of finitely presented modules.

For each $\lambda \in \Omega$, we consider the commutative diagram

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & L \cap P_\lambda & \xrightarrow{u_\lambda} & P_\lambda & \xrightarrow{g_\lambda} & \overline{P}_\lambda & \longrightarrow & 0 \\
 & & \downarrow u'_\lambda & & \downarrow u''_\lambda & & \downarrow r_\lambda & & \\
 (2.7) & & 0 & \longrightarrow & L & \xrightarrow{v} & P & \xrightarrow{h} & K & \longrightarrow & 0
 \end{array}$$

with exact rows, where $\overline{P}_\lambda = P_\lambda / L \cap P_\lambda$, $u_\lambda, u'_\lambda, u''_\lambda, v$ are the embeddings and r_λ is the natural R -module homomorphism induced by u''_λ . Since $V_\lambda = \text{Im } r_\lambda = h(P_\lambda)$ is a finitely generated submodule of K and K is a pure submodule of $P_0 \oplus P'_0$ then V_λ is \aleph_0 -generated and, according to Lemma 2.5, there exists an \aleph_0 -generated pure submodule V_λ^\diamond of $P_0 \oplus P'_0$ contained in K and containing V_λ . It follows that V_λ^\diamond is a pure submodule of an \aleph_0 -generated direct summand P' of $P_0 \oplus P'_0$. Then the module P'/V_λ^\diamond is \aleph_0 -presented and Proposition 2.2 yields (see also [10])

$$\text{P.pd } P'/V_\lambda^\diamond \leq 1.$$

It follows that the submodule V_λ^\diamond of K is pure-projective. If we set $K_\lambda = V_\lambda^\diamond$, then obviously $K = \bigcup_{\lambda \in \Omega} K_\lambda$ is a directed union and K_λ is a countably generated pure-projective pure submodule of K for each $\lambda \in \Omega$. This proves Statement (a).

Since the epimorphism $h : P \rightarrow K$ is pure, the embedding $w_\lambda : V_\lambda^\diamond \hookrightarrow K$ extends to an R -module homomorphism $f_\lambda : V_\lambda^\diamond \rightarrow P$ such that $hf_\lambda =$

w_λ . Then the composed R -module homomorphism $\psi_\lambda = f_\lambda r_\lambda : \overline{P}_\lambda \longrightarrow P$ satisfies $h\psi_\lambda = r_\lambda$ and, by the commutativity of the diagram (2.7), there exists an R -module homomorphism $\varphi_\lambda : P_\lambda \rightarrow L$ such that $\varphi_\lambda u_\lambda = u'_\lambda$. Hence we easily conclude that $L = \sum_{\lambda \in \Omega} \text{Im } \varphi_\lambda$ and therefore L is \aleph -generated, because $|\Omega| \leq \aleph$ and $\text{Im } \varphi_\lambda$ is finitely generated for any $\lambda \in \Omega$. This finishes the proof. \square

3. A pure-projective structure of pure-syzygy modules.

The aim of this section is to prove Theorem 1.2 on the pure-projective structure of the n -th pure-syzygy module of any right R -module M , that is, the pure submodule $\text{Ker } d_n$ of P_n in a pure-projective resolution (1.1) of M .

We start with the following key proposition.

Proposition 3.1. *Assume that R is a ring, \aleph is an infinite cardinal number, M is a right R -module, $n \geq 0$ an integer and*

$$(*) \quad 0 \rightarrow K_n \rightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$$

is a pure-exact sequence, where $K_n = \text{Ker } d_n$ and the modules P_0, \dots, P_n are pure-projective.

- (a) *For any \aleph -generated submodule N of K_n and any \aleph -generated submodule L of $K_0 = \text{Ker } d_0$ there exist an \aleph -generated pure submodule $N^{\blacklozenge n}$ of P_n , and \aleph -generated pure submodule $L^{\blacklozenge 0}$ of P_0 , an \aleph -generated direct summands P'_1, \dots, P'_n of P_1, \dots, P_n , respectively, such that $d_j(P'_j) \subseteq P'_{j-1}$ for $j = 1, \dots, n$, $N \subseteq N^{\blacklozenge n} \subseteq K_n = \text{Ker } d_n$, $L \subseteq L^{\blacklozenge 0} \subseteq K_0 = \text{Ker } d_0$, and, for each $n \geq 1$, the sequence*

$$(**) \quad 0 \rightarrow N^{\blacklozenge n} \rightarrow P'_n \xrightarrow{d'_n} P'_{n-1} \xrightarrow{d'_{n-1}} \dots \xrightarrow{d'_2} P'_1 \xrightarrow{d'_1} L^{\blacklozenge 0} \rightarrow 0$$

is pure-exact, where d'_j is the restriction of d_j to P'_j . In case $n = 0$ we have $N^{\blacklozenge 0} = L^{\blacklozenge 0}$.

- (b) *If $n \geq 1$ and there is an R -module isomorphism $K_n \cong K_0$, then there exists a pure-exact sequence $(**)$ satisfying the conditions listed in (a) and such that $N^{\blacklozenge n} \cong L^{\blacklozenge 0}$.*

Proof. (a) Since any pure-projective module is a direct summand of a direct sum of finitely presented modules then, according to the well-known Kaplansky theorem [13], there are pairwise disjoint sets I_0, I_1, \dots, I_n and countably generated pure-projective modules Q_t , with $t \in I_0 \cup I_1 \cup \dots \cup I_n$, such that, for each for $j \in \{0, 1, \dots, n\}$, the pure-projective module P_j in $(*)$ has the form

$$P(I_j) = \bigoplus_{t \in I_j} Q_t,$$

up to isomorphism. Without loss of generality we can suppose that $P_j = P(I_j)$ for $j = 0, 1, \dots, n$.

Instead of the statement (a) we prove inductively on $n \geq 0$, like [23, Proposition 1.4], the following stronger form of (a).

Assume that, for each $j \in \{0, 1, \dots, n\}$, the pure-projective module P_j in (*) has the form $P(I_j)$ as above. Then the following two statements hold.

- (a1) For any \aleph -generated submodule N of K_n and any \aleph -generated submodule L of $K_0 = \text{Ker } d_0$ there exist an \aleph -generated pure submodule $N^{\blacklozenge n}$ of $P_n = P(I_n)$, \aleph -generated pure submodule $L^{\blacklozenge 0}$ of $P_0 = P(I_0)$, and subsets I'_0, \dots, I'_n of I_0, \dots, I_n , respectively, of cardinality $\leq \aleph$ such that $d_j(P(I'_j)) \subseteq P(I'_{j-1})$ for $j = 1, \dots, n$, $N \subseteq N^{\blacklozenge n} \subseteq K_n = \text{Ker } d_n$, $L \subseteq L^{\blacklozenge 0} \subseteq K_0 = \text{Ker } d_0$, and, for each $n \geq 1$, the sequence (**) is pure-exact, where $P'_j = P(I'_j)$ and d'_j is the restriction of d_j to P'_j . In case $n = 0$ we have $N^{\blacklozenge 0} = L^{\blacklozenge 0}$.
- (a2) Assume that $N, L, N^{\blacklozenge n}, L^{\blacklozenge 0}$ and I'_0, \dots, I'_n are such that the statement (a1) holds, and let N' and L' be \aleph -generated submodules of K_n and K_0 containing N and L , respectively. Then there exist an \aleph -generated pure submodule $N'^{\blacklozenge n}$ of $P_n = P(I_n)$, \aleph -generated pure submodule $L'^{\blacklozenge 0}$ of $P_0 = P(I_0)$ and subsets I''_0, \dots, I''_n of I_0, \dots, I_n , respectively, of cardinality $\leq \aleph$ such that $d_j(P(I''_j)) \subseteq P(I''_{j-1})$ for $j = 1, \dots, n$, $N' \subseteq N'^{\blacklozenge n} \subseteq K_n$, $L' \subseteq L'^{\blacklozenge 0} \subseteq K_0$, $N^{\blacklozenge n} \subseteq N'^{\blacklozenge n}$, $L^{\blacklozenge 0} \subseteq L'^{\blacklozenge 0}$, the diagram

$$\begin{array}{ccccccc}
 0 \rightarrow N^{\blacklozenge n} & \rightarrow & P(I'_n) & \xrightarrow{d'_n} & P(I'_{n-1}) & \rightarrow & \dots \xrightarrow{d'_2} P(I'_1) \xrightarrow{d'_1} L^{\blacklozenge 0} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 (***) & & & & & & & & & & \\
 0 \rightarrow N'^{\blacklozenge n} & \rightarrow & P(I''_n) & \xrightarrow{d''_n} & P(I''_{n-1}) & \rightarrow & \dots \xrightarrow{d''_2} P(I''_1) \xrightarrow{d''_1} L'^{\blacklozenge 0} \rightarrow 0
 \end{array}$$

is commutative and has pure-exact rows, where the vertical arrows are natural embeddings induced by the inclusions $I'_0 \subseteq I''_0, \dots, I'_n \subseteq I''_n$ and d''_j is the restriction of d_j to $P(I''_j)$ for $j = 1, \dots, n$. In case $n = 0$ we have $N'^{\blacklozenge 0} = L'^{\blacklozenge 0}$.

Assume that $n = 0$. Since the submodules N and L of K_0 are \aleph -generated then applying Lemma 2.5 to the \aleph -generated submodule $N + L$ of K_0 we get an \aleph -generated pure submodule $(N + L)^{\blacklozenge}$ of $P(I_0)$ and of K_0 . It follows that there is a subset I'_0 of I_0 of cardinality $\leq \aleph$ such that $(N + L)^{\blacklozenge}$ is a pure submodule of $P(I'_0) \subseteq P(I_0)$. If we set $N^{\blacklozenge 0} = L^{\blacklozenge 0} = (N + L)^{\blacklozenge}$ we get (a1). Statement (a2) follows in a similar way.

Assume that $n \geq 1$. First we prove the following:

Claim. For any \aleph -generated submodule Y of K_{n-1} and any subset \mathcal{Y} of I_n of cardinality $\leq \aleph$ there exists an \aleph -generated pure submodule Y' of K_{n-1}

containing Y and a subset \mathcal{Y}' of I_n of cardinality $\leq \aleph$ containing \mathcal{Y} such that:

- (c1) $d_n(P(\mathcal{Y}')) = Y'$,
- (c2) the restriction $d'_n : P(\mathcal{Y}') \rightarrow Y'$ of d_n to $P(\mathcal{Y}')$ is a pure epimorphism, and
- (c3) the submodule $\text{Ker } d'_n$ of $P(\mathcal{Y}')$ is \aleph -generated.

Let Y be an \aleph -generated submodule of K_{n-1} and \mathcal{Y} a subset of I_n of cardinality $\leq \aleph$. We construct the subset \mathcal{Y}' of I_n as the union $\mathcal{Y}' = \bigcup_{j=1}^{\infty} \mathcal{Y}_j$ of subsets

$$(+) \quad \mathcal{Y} \subseteq \mathcal{Y}_1 \subseteq \mathcal{Y}_2 \subseteq \dots \subseteq \mathcal{Y}_j \subseteq \mathcal{Y}_{j+1} \subseteq \dots$$

of I_n of cardinality $\leq \aleph$, and the module Y' as the union $Y' = \bigcup_{j=1}^{\infty} Y^{(j)}$ of \aleph -generated pure submodules

$$(++) \quad Y \subseteq Y^{(1)} \subseteq Y^{(2)} \subseteq \dots \subseteq Y^{(j)} \subseteq Y^{(j+1)} \subseteq \dots$$

of K_{n-1} such that the image of the restriction $d^{(j)} : P(\mathcal{Y}^{(j)}) \rightarrow K_{n-1}$ of d_n to $P(\mathcal{Y}^{(j)})$ contains $Y^{(j)}$ and it is contained in $Y^{(j+1)}$ for $j \geq 1$, and for any finitely generated R module Z and any R -homomorphism $f : Z \rightarrow Y^{(j)}$ there exists an R -homomorphism $f' : Z \rightarrow P(\mathcal{Y}^{(j)})$ such that $f = d^{(j)} f'$.

It is clear that the above properties imply Conditions (c1) and (c2) of claim. In view of Lemma 2.6, Condition (c3) is a consequence of (c2), because Y' is an \aleph -generated pure submodule of K_{n-1} .

We construct the sequences (+) and (++) inductively as follows. By applying Lemma 2.5 to the pure submodule $K = K_{n-1}$ of the pure-projective module $P = P(I_{n-1})$ and $L = Y$ we get an \aleph -generated pure submodule Y^\diamond of K_{n-1} containing Y . We set $Y^{(1)} = Y^\diamond$. By Lemma 2.6, the module $Y^{(1)}$ has a directed union form $Y^{(1)} = \bigcup_{\lambda \in \Omega_1} Y_\lambda^{(1)}$, where Ω_1 is a set of cardinal-

ity $\leq \aleph$ and $Y_\lambda^{(1)}$ is a countably generated pure-projective pure submodule of K_{n-1} for each $\lambda \in \Omega_1$. Since the epimorphism $d_n : P(I_n) \rightarrow K_{n-1}$ is pure and $Y_\lambda^{(1)}$ is pure-projective, then for each $\lambda \in \Omega_1$ the embedding $v_\lambda : Y_\lambda^{(1)} \rightarrow Y^{(1)}$ has a factorisation $v_\lambda = d_n f_\lambda$, where $f_\lambda \in \text{Hom}_R(Y_\lambda^{(1)}, P(I_n))$. Since $f_\lambda(Y_\lambda^{(1)})$ is a countably generated submodule of $P(I_n)$, $|\Omega_1| \leq \aleph$ and $\aleph \geq \aleph_0$, then there exists a subset $\mathcal{Y}^{(1)}$ of I_n of cardinality $\leq \aleph$ containing \mathcal{Y} such that $\sum_{\lambda \in \Omega_1} f_\lambda(Y_\lambda^{(1)}) \subseteq P(\mathcal{Y}^{(1)})$. It follows that the image of the

restriction $d^{(1)} : P(\mathcal{Y}^{(1)}) \rightarrow K_{n-1}$ of d_n to $P(\mathcal{Y}^{(1)})$ contains $Y^{(1)} \supset Y$. Moreover, for any finitely generated R module Z and any R -homomorphism $f : Z \rightarrow Y^{(1)}$ there exists an R -homomorphism $f' : Z \rightarrow P(\mathcal{Y}^{(1)})$ such

that $f = d^{(1)}f'$. Indeed, $\text{Im } f$ is a finitely generated submodule of $Y^{(1)}$ and therefore there exists $\lambda \in \Omega_1$ such that $\text{Im } f \subseteq Y_\lambda^{(1)}$. If we set $f' = f_\lambda f$, we get the required equality $f = d^{(1)}f'$. Hence we conclude $Y^{(1)} \subseteq \text{Im } d^{(1)}$.

Since $|\mathcal{Y}^{(1)}| \leq \aleph$, the submodule $\text{Im } d^{(1)}$ of K_{n-1} is \aleph -generated, and according to Lemma 2.5 there exists an \aleph -generated pure submodule $(\text{Im } d^{(1)})^\diamond$ of K_{n-1} containing $\text{Im } d^{(1)}$. We set $\mathcal{Y}^{(2)} = (\text{Im } d^{(1)})^\diamond$.

If $j \geq 1$ and $Y^{(j)}, \mathcal{Y}^{(j)}$ are constructed, we construct $Y^{(j+1)}$ and $\mathcal{Y}^{(j+1)}$ by applying the above construction of $Y^{(1)}, \mathcal{Y}^{(1)}$ and $Y^{(2)}$ to $Y^{(j)}$ and the set $\mathcal{Y}^{(j)}$. The details are left to the reader. This finishes the proof of claim.

Now we prove the inductive step. Assume that $n \geq 1$ and that Statements (a1) and (a2) hold for $n - 1$. In order to prove (a1) and (a2) for n , we assume that N is an \aleph -generated submodule of K_n and L is an \aleph -generated submodule of K_0 . We set $L_0 = L$. By Lemma 2.5, there exists an \aleph -generated pure submodule $N_0^{\diamond n}$ of $P(I_n)$ such that $N \subseteq N_0^{\diamond n} \subseteq K_n$.

Let $J'_{n,0}$ be a subset of I_n of cardinality $\leq \aleph$ such that $N_0^{\diamond n} \subseteq P(J'_{n,0}) \subseteq P(I_n)$. Then the submodule $T_0 = d_n(P(J'_{n,0}))$ of $K_{n-1} = \text{Ker } d_{n-1} \subseteq P(I_{n-1})$ is \aleph -generated. By applying the induction hypothesis to $T_0 \subseteq K_{n-1}$ and $L_0 = L \subseteq K_0$ one gets subsets $J_{n-1,0} \subseteq I_{n-1}, \dots, J_{0,0} \subseteq I_0$ of cardinality $\leq \aleph$, an \aleph -generated pure submodule $T_0^{\diamond n-1} \subseteq K_{n-1}$ of $P(J_{n-1,0})$ containing T_0 , an \aleph -generated pure submodule $L_0^{\diamond 0} \subseteq K_0$ of $P(J_{0,0})$ containing L_0 such that the sequence

$$0 \rightarrow T_0^{\diamond n-1} \rightarrow P(J_{n-1,0}) \xrightarrow{d_{n-1,0}} P(J_{n-2,0}) \rightarrow \dots \rightarrow P(J_{1,0}) \xrightarrow{d_{1,0}} L_0^{\diamond 0} \rightarrow 0$$

is pure-exact, where $d_{j,0}$ is the restriction of d_j to $P(J_{j,0})$ for $j = 1, \dots, n-1$.

By our claim applied to $Y = T_0^{\diamond n-1}$ and $\mathcal{Y} = J'_{n,0}$, there exist a subset $J_{n,0}$ of I_n of cardinality $\leq \aleph$ containing $J'_{n,0}$ and an \aleph -generated pure submodule $T_1 = (T_0^{\diamond n-1})'$ of K_{n-1} containing $T_0^{\diamond n-1}$ such that $J'_{n,0} \subseteq J_{n,0}$, the restriction of d_n to $P(J_{n,0})$ yields a pure epimorphism

$$d_{n,0} : P(J_{n,0}) \rightarrow T_1$$

and the pure submodule $\text{Ker } d_{n,0}$ of $P(J_{n,0})$ is \aleph -generated. It is clear that $N \subseteq N_0^{\diamond n} \subseteq \text{Ker } d_{n,0}$.

By applying the induction hypothesis to $T_1 \subseteq K_{n-1}$ and $L_1 = L_0^{\diamond 0} \subseteq K_0$, one gets subsets $J_{n-1,1} \subseteq I_{n-1}, \dots, J_{0,1} \subseteq I_0$ of cardinality $\leq \aleph$, an \aleph -generated pure submodule $T_1^{\diamond n-1} \subseteq K_{n-1}$ of $P(J_{n-1,1})$ containing T_1 , an \aleph -generated pure submodule $L_1^{\diamond 0} \subseteq K_0$ of $P(J_{0,1})$ containing L_1 such that

the sequence

$$0 \rightarrow T_1^{\diamond n-1} \rightarrow P(J_{n-1,1}) \xrightarrow{d_{n-1,1}} P(J_{n-2,1}) \rightarrow \dots$$

$$\rightarrow P(J_{1,1}) \xrightarrow{d_{1,1}} L_1^{\diamond 0} \rightarrow 0$$

is pure-exact, where $d_{j,1}$ is the restriction of d_j to $P(J_{j,1})$ and $J_{j,0} \subseteq J_{j,1} \subseteq I_j$ for $j = 1, \dots, n - 1$.

By our claim applied to $Y = T_1^{\diamond n-1}$ and $\mathcal{Y} = J_{n,0}$, there exist a subset $J_{n,1}$ of I_n of cardinality $\leq \aleph$ containing $J_{n,0}$ and an \aleph -generated pure submodule $T_2 = (T_1^{\diamond n-1})'$ of K_{n-1} containing $T_1^{\diamond n-1}$ such that the restriction of d_n to $P(J_{n,1})$ yields a pure epimorphism

$$d_{n,1} : P(J_{n,1}) \rightarrow T_2,$$

the submodule $\text{Ker } d_{n,1}$ of $P(J_{n,1})$ is \aleph -generated and $N \subseteq N_0^{\diamond n} \subseteq \text{Ker } d_{n,0} \subseteq \text{Ker } d_{n,1}$.

Continuing this way, we construct two sequences

- $T_0 \subseteq T_0^{\diamond n-1} \subseteq T_1 \subseteq T_1^{\diamond n-1} \subseteq \dots \subseteq T_s \subseteq T_s^{\diamond n-1} \subseteq \dots,$
- $L = L_0 \subseteq L_1 = L_0^{\diamond 0} \subseteq L_2 = L_1^{\diamond 0} \subseteq \dots \subseteq L_s = L_{s-1}^{\diamond 0} \subseteq \dots$

of \aleph -generated submodules of $K_{n-1} \subseteq P(I_{n-1})$ and $K_0 \subseteq P(I_0)$, respectively, and, for each $j \in \{1, \dots, n\}$, a chain

$$J_{j,0} \subseteq J_{j,1} \subseteq J_{j,2} \subseteq \dots \subseteq J_{j,s} \subseteq J_{j,s+1} \subseteq \dots$$

of subsets $J_{j,s}$ of I_j such that $|J_{j,s}| \leq \aleph$, $T_s^{\diamond n-1} \subseteq P(J_{n-1,s})$ and $L_s \subseteq P(J_{0,s-1})$ are pure embeddings and the restriction of d_n to $P(J_{n,s})$ yields a pure epimorphism

$$d_{n,s} : P(J_{n,s}) \rightarrow T_{s+1}.$$

It follows that, for each $j \in \{1, \dots, n\}$, there is a chain

$$P(J_{j,0}) \subseteq P(J_{j,1}) \subseteq P(J_{j,2}) \subseteq \dots \subseteq P(J_{j,s}) \subseteq P(J_{j,s+1}) \subseteq \dots$$

of submodules $P(J_{j,s})$ of $P(I_j)$, and we get an infinite commutative diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & T_0^{\diamond n-1} & \longrightarrow & P(J_{n-1,0}) & \xrightarrow{d_{n-1,0}} & \dots \longrightarrow P(J_{1,0}) \xrightarrow{d_{1,0}} L_1 \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 P(J_{n,0}) & \xrightarrow{d_{n,0}} & T_1 & \rightarrow & 0 & & & & \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & T_1^{\diamond n-1} & \longrightarrow & P(J_{n-1,1}) & \xrightarrow{d_{n-1,1}} & \dots \longrightarrow P(J_{1,1}) \xrightarrow{d_{1,1}} L_2 \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 P(J_{n,1}) & \xrightarrow{d_{n,1}} & T_2 & \rightarrow & 0 & & & & \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & T_2^{\diamond n-1} & \longrightarrow & P(J_{n-1,2}) & \xrightarrow{d_{n-1,2}} & \dots \longrightarrow P(J_{1,2}) \xrightarrow{d_{1,2}} L_3 \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 P(J_{n,2}) & \xrightarrow{d_{n,2}} & T_3 & \rightarrow & 0 & & & & \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots
 \end{array}$$

with pure-exact rows, where the vertical homomorphisms are the R -module embeddings constructed above. Let

$$0 \rightarrow N^{\blacklozenge n} \rightarrow P(I'_n) \xrightarrow{d'_n} P(I'_{n-1}) \xrightarrow{d'_{n-1}} \dots \xrightarrow{d'_2} P(I'_1) \xrightarrow{d'_1} L^{\blacklozenge 0} \rightarrow 0$$

be the direct limit of the above system of pure-exact sequences, where

$$N^{\blacklozenge n} = \bigcup_{s=1}^{\infty} \text{Ker } d_{n,s}, \quad L^{\blacklozenge 0} = \bigcup_{s=1}^{\infty} L_s \quad \text{and} \quad I'_j = \bigcup_{s=0}^{\infty} J_{j,s}$$

for $j = 1, \dots, n$. It follows that the limit sequence is pure-exact, consists of \aleph -generated modules, $N^{\blacklozenge n} = \text{Ker } d'_n$ is a pure submodule of $P(I'_n)$ (and of K_n) containing N , the module

$$\text{Im } d'_n = \bigcup_{s=1}^{\infty} T_s = \bigcup_{s=1}^{\infty} T_s^{\diamond n-1} = \text{Ker } d'_{n-1}$$

is a pure submodule of K_{n-1} and $L^{\blacklozenge 0} = \bigcup_{s=1}^{\infty} L_s$ is a pure submodule of $P(I'_0)$ as well as of K_0 . By Lemma 2.6, the module $N^{\blacklozenge n} = \text{Ker } d'_n$ is \aleph -generated.

This finishes the proof of (a1). The method used in the inductive step of (a1) above also proves the inductive step of (a2). We leave it to the reader. This finishes the proof of (a).

(b) Assume that $n \geq 1$ and $K_n \cong K_0$. Let N be an \aleph -generated submodule of K_n and L an \aleph -generated submodule of K_0 . Fix an R -module isomorphism $f : K_n \rightarrow K_0$. Keeping the notation above and by applying (a), we construct inductively an infinite commutative diagram

$$\begin{array}{ccccccccc}
 0 \rightarrow N_1^{\blacklozenge n} & \longrightarrow & P(I'_{n,1}) & \xrightarrow{d_{n,1}} & P(I'_{n-1,1}) & \xrightarrow{d_{n-1,1}} & \dots & \longrightarrow & P(I'_{1,1}) & \xrightarrow{d_{1,1}} & L_1^{\blacklozenge 0} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 0 \rightarrow N_2^{\blacklozenge n} & \longrightarrow & P(I'_{n,2}) & \xrightarrow{d_{n,2}} & P(I'_{n-1,2}) & \xrightarrow{d_{n-1,2}} & \dots & \longrightarrow & P(I'_{1,2}) & \xrightarrow{d_{1,2}} & L_2^{\blacklozenge 0} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 0 \rightarrow N_3^{\blacklozenge n} & \longrightarrow & P(I'_{n,3}) & \xrightarrow{d_{n,3}} & P(I'_{n-1,3}) & \xrightarrow{d_{n-1,3}} & \dots & \longrightarrow & P(I'_{1,3}) & \xrightarrow{d_{1,3}} & L_3^{\blacklozenge 0} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & & & \vdots & & \vdots
 \end{array}$$

with pure-exact rows, where the vertical homomorphisms are R -module embeddings induced by the inclusions $I'_{j,1} \subseteq I'_{j,2} \subseteq I'_{j,3} \subseteq \dots$, for $j = 1, \dots, n$. We set $N_1 = N + f^{-1}(L)$ and $L_1 = f(N) + L$. If the modules N_j, L_j and $N_j^{\blacklozenge n}, L_j^{\blacklozenge 0}$ are defined we set

$$N_{j+1} = N_j^{\blacklozenge n} + f^{-1}(L_j^{\blacklozenge 0}) \quad \text{and} \quad L_{j+1} = f(N_j^{\blacklozenge n}) + L_j^{\blacklozenge 0}.$$

It is clear that $N_j \subseteq N_j^{\blacklozenge n} \subseteq N_{j+1}, L_j \subseteq L_j^{\blacklozenge 0} \subseteq L_{j+1}, f(N_1) = L_1$ and, for each $j \geq 1$, we get $f(N_{j+1}) = L_{j+1}$. Let

$$0 \rightarrow N^{\blacklozenge} \longrightarrow P(I'_n) \xrightarrow{d'_n} P(I'_{n-1}) \xrightarrow{d'_{n-1}} \dots \xrightarrow{d'_2} P(I'_1) \xrightarrow{d'_1} L^{\blacklozenge} \rightarrow 0$$

be the direct limit of the above system of pure-exact sequences, where

$$N^{\blacklozenge} = \bigcup_{s=1}^{\infty} N_s^{\blacklozenge n} = \bigcup_{s=1}^{\infty} N_s, \quad L^{\blacklozenge} = \bigcup_{s=1}^{\infty} L_s^{\blacklozenge 0} = \bigcup_{s=1}^{\infty} L_s \quad \text{and} \quad I'_j = \bigcup_{s=0}^{\infty} I'_{j,s}$$

for $j = 1, \dots, n$. It is easy to see that $f(N^{\blacklozenge}) = L^{\blacklozenge}$. Thus the modules $N^{\blacklozenge}, L^{\blacklozenge}$ are isomorphic and the statement (b) follows. This finishes the proof. □

The claim proved in the above proof yields the following useful result.

Corollary 3.2. *Assume that \aleph is an infinite cardinal number, $h : P \rightarrow K$ is a pure epimorphism in $\text{Mod}(R)$, P is an \aleph -generated pure-projective module and K is a pure submodule of a pure-projective module. For any \aleph -generated submodule Y of K and any subset \mathcal{X} of P of cardinality $\leq \aleph$ there exist an \aleph -generated direct summand P' of P containing \mathcal{X} and an \aleph -generated pure submodule Y' of K containing Y such that $h(P') = Y'$, the restriction $h' : P' \rightarrow Y'$ of h to P' is a pure epimorphism and the module $\text{Ker } h'$ is \aleph -generated.*

Proof. Let $h : P \rightarrow K$ be a pure-epimorphism. By our assumption, K is a pure submodule of a pure-projective module P_0 . We apply the claim in the proof above to $n = 1$, $P_1 = P$, $d_1 = h$, $M = P_0/K$ and $d_0 : P_0 \rightarrow M$ the canonical epimorphism. By Kaplansky theorem [13], the module P has the form $P_1 = P(I_1) = \bigoplus_{t \in I_1} Q_t$, where Q_t is a countably generated pure-projective module for $t \in I_1$, in the notation introduced above. Since \mathcal{X} is a subset of P of cardinality $\leq \aleph$ there exist a subset \mathcal{Y} of I_1 of cardinality $\leq \aleph$ containing \mathcal{X} . Then the corollary is an immediate consequence of Statements (c1), (c2) and (c3) of the claim. \square

One of the main results of this paper is the following theorem describing a pure-projective structure of pure-syzygy modules of any R -module (compare with [23, Theorem 1.5]).

Theorem 3.3. *Let R be a ring, M a right R -module and*

$$\dots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$$

a pure-exact sequence in $\text{Mod}(R)$ such that the modules P_0, \dots, P_n, \dots are pure-projective. Then, for each $n \geq 0$, the n th pure-syzygy module $\text{Ker } d_n$ of M is an \aleph_n -directed union of \aleph_n -generated pure-projective pure R -submodules of $\text{Ker } d_n$, which are pure submodules of P_n .

Proof. Fix $n \geq 0$ and consider the n th pure-syzygy submodule $K_n = \text{Ker } d_n$ of P_n . Set $K_0 = \text{Ker } d_0$. It is sufficient to show that any \aleph_n -generated submodule N of K_n is a submodule of an \aleph_n -generated pure-projective submodule of K_n , which is pure in K_n . Let N be an \aleph_n -generated submodule of K_n . By applying Proposition 3.1 to $N \subseteq K_n$ and to the submodule $L = (0)$ of K_0 , one gets a pure-exact sequence

$$(3.4) \quad 0 \rightarrow N^{\blacklozenge n} \longrightarrow P'_n \xrightarrow{d'_n} P'_{n-1} \xrightarrow{d'_{n-1}} \dots \xrightarrow{d'_2} P'_1 \xrightarrow{d'_1} L^{\blacklozenge 0} \rightarrow 0$$

consisting of \aleph_n -generated modules, where P'_n, \dots, P'_1 are pure-projective modules, $N^{\blacklozenge n}$ is a pure submodule of K_n , $L^{\blacklozenge 0}$ is a pure submodule of P_0 and $N \subseteq N^{\blacklozenge n}$. In case $n = 0$ we get just $N^{\blacklozenge 0} = L^{\blacklozenge 0}$.

Let P''_0 be a right R -module such that the module $P' = P_0 \oplus P''_0$ is a direct sum of finitely presented modules. Since $L^{\blacklozenge 0}$ is a pure submodule of P_0 , it is a pure submodule of P' and therefore $L^{\blacklozenge 0}$ is a pure submodule of an \aleph_n -generated pure-projective direct summand P of P_0 . Then the module $P/L^{\blacklozenge 0}$ is \aleph_n -presented and Proposition 2.2 yields

$$\text{P.pd } P/L^{\blacklozenge 0} \leq n + 1.$$

It follows that the submodule N_n^{\blacklozenge} of P'_n in the pure-exact sequence (3.4) is pure-projective, because P'_n, \dots, P'_1 are pure-projective modules. This finishes the proof. \square

As a consequence of Theorem 3.3 we get the following structure theorem on syzygy modules of flat modules proved by the author in [23, Theorem 1.5].

Corollary 3.5. *Let R be a ring, M a right flat R -module and $n \geq 0$ an integer. If*

$$(3.6) \quad 0 \rightarrow K_n \rightarrow F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

is an exact sequence in $\text{Mod}(R)$ such that the modules F_0, \dots, F_n are projective, then the n -th syzygy module K_n of M is an \aleph_n -directed union of \aleph_n -generated projective pure R -submodules of K_n (which are pure submodules of F_n).

Proof. Since M is flat then the sequence (3.6) is pure-exact and the projective modules F_0, \dots, F_n are obviously pure-projective. It follows from Theorem 3.3 that K_n is an \aleph_n -directed union of \aleph_n -generated pure-projective R -submodules U_μ of K_n , with $\mu \in \Omega_n$, which are pure submodules of the projective module F_n . Hence each of the modules U_μ is flat and therefore any epimorphism $h_\mu : P_\mu \rightarrow U_\mu$ from a projective module P_μ to U_μ is a pure epimorphism. Since the module U_μ is pure-projective, then the epimorphism h_μ splits and, consequently, the module U_μ is projective for any $\mu \in \Omega_n$. This completes the proof. \square

4. Pure-periodic modules.

Definition 4.1. Let R be a ring. A right R -module T is defined to be **pure-periodic** of period $p \geq 1$ if there exists a pure-exact sequence

$$0 \rightarrow T \rightarrow P_p \rightarrow P_{p-1} \rightarrow \dots \rightarrow P_1 \rightarrow T \rightarrow 0$$

in $\text{Mod}(R)$ such that the modules P_1, \dots, P_p are pure-projective.

We start with two useful lemmata on pure-periodic modules.

Lemma 4.2. *Assume that T is a pure-periodic right R -module of period $p \geq 1$ and \aleph is an infinite cardinal number. Then, for any \aleph -generated*

submodule U of T , there exists an \aleph -generated pure submodule U^\blacklozenge of T such that $U \subseteq U^\blacklozenge$ and U^\blacklozenge is pure-periodic of the same period p .

Proof. Since T is pure-periodic of period $p \geq 1$, then T is a pure submodule of a pure-projective module P_0 and there exists a pure-exact sequence

$$0 \rightarrow K_p \longrightarrow P_p \xrightarrow{d_p} P_{p-1} \xrightarrow{d_{p-1}} \dots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$$

where $M = P_0/T$, d_0 is the canonical projection, $\text{Ker } d_0 = T \cong K_p = \text{Ker } d_p$ and the modules P_1, \dots, P_p are pure-projective. Let $f : T \rightarrow K_p$ be an R -module isomorphism and let $N = f(U)$. Then Proposition 3.1 (b) applies to the above sequence with $K_0 = T$ and $N \cong L = U$. Consequently, there exists an \aleph -generated pure submodule U^\blacklozenge of T satisfying the required conditions. □

Lemma 4.3. *Assume that $\aleph \geq \aleph_1$ is an infinite cardinal number and T an \aleph -generated pure submodule of a pure-projective right R -module.*

- (a) *The module T is a continuous well-ordered union of pure submodules T_ξ , with $\xi < \gamma$, such that, for each ξ , the module T_ξ is generated by a set of cardinality $< \aleph$.*
- (b) *If, in addition, T is pure-periodic of period $p \geq 1$ then, for each $\xi < \gamma$, the pure submodule T_ξ of T can be chosen pure-periodic of the same period p .*

Proof. Let $\aleph' < \aleph$ be an infinite cardinal. It follows from Lemma 2.5 that any \aleph' -generated submodule X of T can be embedded in an \aleph' -generated pure submodule X^\blacklozenge of T . If, in addition, T is pure-periodic of period $p \geq 1$ then, according to Lemma 4.2, X can be embedded in an \aleph' -generated pure-periodic pure submodule X^\blacklozenge of T of the same period p .

It is well-known that T can be represented as a continuous well-ordered union of submodules U_ξ , with $\xi < \gamma$, such that, for each ξ , the module U_ξ is generated by a set of cardinality $< \aleph$ (see [11, Lemme 1.4] and [19, Lemma 2.2]). Let us define a transfinite increasing chain of pure-submodules T_ξ of T , with $\xi < \gamma$, having $< \aleph$ generators as follows (compare with [19, Theorem 1.5] and [16, Lemma 1.7]). We set $T_0 = (0)$ and $T_1 = U_1^\blacklozenge$. If T_ξ is defined and $\xi + 1 < \gamma$, we set $T_{\xi+1} = (T_\xi + U_{\xi+1})^\blacklozenge$. Finally, we set $T_\tau = \bigcup_{\xi < \tau} T_\xi$ if τ

is a limit ordinal number. This proves Statement (a).

In order to prove (b), we assume that T is pure-periodic of period $p \geq 1$. It follows that T is a pure submodule of a pure-projective module P_0 and there exists a pure-exact sequence

$$0 \rightarrow K_p \longrightarrow P_p \xrightarrow{d_p} P_{p-1} \xrightarrow{d_{p-1}} \dots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0,$$

where $M = P_0/T$, d_0 is the canonical projection, $\text{Ker } d_0 = T \cong K_p = \text{Ker } d_p$ and the modules P_1, \dots, P_p are pure-projective. Let $f : T \rightarrow K_p$ be an R -module isomorphism and let $N = f(U)$.

there exists a pure-exact sequence

$$0 \rightarrow \text{Ker } d_p \rightarrow P_p \xrightarrow{d_p} P_{p-1} \rightarrow \dots \rightarrow P_1 \xrightarrow{d_1} P_0 \rightarrow M \rightarrow 0$$

such that the modules P_0, \dots, P_p are pure-projective, $M = P_0/T$ and $T = \text{Im } d_1$. Without loss of generality we can suppose that $p \geq n$. In that case T is also \aleph_p -generated, and therefore we can assume $p = n$.

By Theorem 3.3, the p th pure-syzygy module $\text{Ker } d_p \cong T$ of P_p is an \aleph_p -directed union of \aleph_p -generated pure-projective pure submodules. It follows that T is pure-projective, because it is \aleph_p -generated.

Next, we assume that T is an arbitrary \aleph -generated pure-periodic R -module, $\aleph > \aleph_n$ for all integers $n \geq 1$, and the theorem holds for all cardinals smaller than \aleph . By Lemma 4.3, T is a continuous well-ordered union of pure-periodic pure submodules T_ξ , $\xi < \gamma$, such that for each $\xi < \gamma$ the module T_ξ is generated by a set of cardinality $< \aleph$. By the inductive hypothesis, each of the modules T_ξ is pure-projective and therefore $\text{P.pd } T_{\xi+1}/T_\xi \leq 1$. It follows from Proposition 2.1 that $\text{P.pd } T \leq 1$. Hence the submodule $\text{Ker } d_2$ of P_2 in the pure-exact sequence above is pure-projective and consequently the monomorphism $T \cong \text{Ker } d_p \hookrightarrow P_p$ splits. This shows that T is pure-projective and finishes the proof. \square

Corollary 4.5. *Let M be a right R -module and*

$$(*) \quad \dots \rightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$$

a pure-exact sequence, where the modules P_0, \dots, P_n, \dots are pure-projective. Assume that there exist two integers $n \geq 0$ and $p \geq 1$ such that $\text{Ker } d_{n+p} \cong \text{Ker } d_n$. Then $\text{P.pd } M \leq n + 1$.

Proof. By our assumption, the n th pure syzygy module $\text{Ker } d_n$ is pure-periodic of period $p \geq 1$. It then follows from Theorem 4.4 that the module $\text{Ker } d_n$ is pure-projective and, consequently, $\text{P.pd } M \leq n + 1$. \square

As a consequence of Theorem 4.4 we get the following form of [5, Theorem 2.5]:

Corollary 4.6. *Let R be a ring, M a right flat R -module and $n \geq 1$ an integer. If there exists an exact sequence*

$$(4.7) \quad 0 \rightarrow M \rightarrow F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

in $\text{Mod}(R)$ such that the modules F_0, \dots, F_n are projective, then M is projective.

Proof. Since M is flat, the sequence (4.7) is pure-exact and the projective modules F_0, \dots, F_n are obviously pure-projective. It follows that M is pure-periodic and, according to Theorem 4.4, the sequence (4.7) splits. Consequently, M is projective. \square

We finish this section with the following interesting question suggested by referee and related with the problems studied in [4] and [5].

Problem 4.8. Assume that R is an associative ring with an identity element and H is a subgroup of finite index in a group G . Let M be an arbitrary right module over the group ring RG . Is the RG -module M pure-projective, if M is pure-projective, when viewed as an RH -module?

5. A structure of pure-projective resolutions in Grothendieck categories.

We show in this section that the main results of Sections 3 and 4 on pure-syzygies and pure-periodic modules generalize from the module category $\text{Mod}(R)$ to an arbitrary locally finitely presented Grothendieck category \mathcal{A} (see [21]).

Throughout we denote by \mathcal{A} a locally finitely presented Grothendieck category. We recall that an object L of \mathcal{A} is said to be finitely presented if the additive functor $\text{Hom}_{\mathcal{A}}(L, -) : \mathcal{A} \rightarrow \mathcal{A}b$ commutes with filtered direct limits (see [21], [26], [30]). A long exact sequence $\dots \rightarrow X_{n-1} \rightarrow X_n \rightarrow X_{n+1} \rightarrow \dots$ in \mathcal{A} is said to be **pure** if the induced sequence $\dots \rightarrow \text{Hom}_{\mathcal{A}}(L, X_{n-1}) \rightarrow \text{Hom}_{\mathcal{A}}(L, X_n) \rightarrow \text{Hom}_{\mathcal{A}}(L, X_{n+1}) \rightarrow \dots$ of abelian groups is exact for any finitely presented object L . An epimorphism $f : Y \rightarrow Z$ in \mathcal{A} is said to be **pure** if the exact sequence $0 \rightarrow \text{Ker} f \rightarrow Y \xrightarrow{f} Z \rightarrow 0$ is pure. A subobject X of Y in \mathcal{A} is said to be **pure** if the exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Y/X \rightarrow 0$ is pure. An object P in \mathcal{A} is said to be **pure-projective** if for any pure-epimorphism $f : Y \rightarrow Z$ in \mathcal{A} the induced group homomorphism $\text{Hom}_{\mathcal{A}}(P, f) : \text{Hom}_{\mathcal{A}}(P, Y) \rightarrow \text{Hom}_{\mathcal{A}}(P, Z)$ is surjective.

It is well-known that:

- An object P in locally finitely presented Grothendieck category \mathcal{A} is pure-projective if and only if P is a direct summand of a coproduct of finitely presented objects.
- Every object M in \mathcal{A} admits a pure-projective pure resolution in \mathcal{A} , that is, there is a pure-exact sequence

$$(5.1) \quad \dots \rightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0,$$

where P_0, \dots, P_n, \dots are pure-projective objects (see [26], [30]).

Then, for each $n \geq 0$, the functor $\text{Pext}_{\mathcal{A}}^n : \mathcal{A}^{op} \times \mathcal{A} \rightarrow \mathcal{A}b$ is naturally defined and, for any object M in \mathcal{A} , the pure-projective dimension $\text{P.pd } M$ of M is defined in a natural way (see [30] and [26]).

The main results of Sections 3 and 4 extend to the context of locally finitely presented Grothendieck categories as follows:

Theorem 5.2. *Let M be an arbitrary object of a locally finitely presented Grothendieck category \mathcal{A} . If (5.1) is a pure-exact sequence in \mathcal{A} such that*

the objects P_0, \dots, P_n, \dots are pure-projective, then for each $n \geq 0$ the n -th pure-syzygy object $\text{Ker } d_n$ of M is an \aleph_n -directed union of \aleph_n -generated pure-projective objects, which are pure subobjects of P_n .

Theorem 5.3. *Let M be an object of a locally finitely presented Grothendieck category \mathcal{A} . If there exists a pure-exact sequence*

$$0 \rightarrow M \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

in \mathcal{A} such that the objects P_0, \dots, P_n are pure-projective, then M is pure-projective.

In other words, every pure-periodic object M of \mathcal{A} is pure-projective.

Outline of the proof.

First we note that Proposition 2.1 (a pure version of a theorem of Auslander [2]) extends to an arbitrary Grothendieck category \mathcal{A} (see [25] and [26, Proposition 2.6]). Further, by [26, Theorem 2.12], if M is an \aleph_n -presented object of a locally finitely presented Grothendieck category \mathcal{A} and $n \geq 0$, then $\text{P.pd } M \leq n + 1$. Finally, we note that the proof of Lemma 2.5 uses only categorical arguments and therefore extends to our situation. It then follows that also our Lemma 2.6 remains valid with $\text{Mod}(R)$ and \mathcal{A} interchanged. Since the Kaplansky theorem [13] also remains valid for objects of a Grothendieck category \mathcal{A} (see [21]) then the proof of Proposition 3.1 works with $\text{Mod}(R)$ and \mathcal{A} interchanged. Thus, applying the arguments in the proof of Theorems 3.3 and 4.4, we easily get Theorems 5.2 and 5.3. The details are left to the reader. \square

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