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Let $\overline{M}$ be a smoothly bounded orientable pseudoconvex CR manifold of finite type with $\dim_{\mathbb{R}} M \geq 7$ and assume that the Levi-form of M has at most one degenerate eigenvalue. Then we extend the given CR structure on M to an integrable almost complex structure on S_g which is a complex manifold containing M as a (locally) real hypersurface.

1. Introduction.

Suppose that $\overline{M}$ is an abstract smoothly bounded orientable CR manifold of dimension $2n-1$ with a given integrable CR structure $\mathcal{S}$ of dimension $n-1$. Since $\overline{M}$ is orientable, there are a smooth real nonvanishing 1-form η and a smooth real vector field X_0 on $\overline{M}$ so that $\eta(X) = 0$ for all $X \in \mathcal{S}$ and $\eta(X_0) = 1$. We define the Levi form of $\mathcal{S}$ on $\overline{M}$ by $i\eta([X', \overline{X}''])$, $X', X'' \in \mathcal{S}$. We may assume that $\overline{M} \subset \widetilde{M}$, in C^∞ sense, where $\widetilde{M}$ is a smooth manifold.

Ever since the discovery of non-realizable CR-structures on M with $\dim_{\mathbb{R}} M = 3$ [**15**], the question of local embeddability of M as a real hypersurface in $\mathbb{C}^n$ has been one of the main interests in CR-geometry. In [**12**], Jacobowitz and Treves also showed that there is $(M, \mathcal{S})$, of nondegenerate Levi-form with only one positive eigenvalue, which can not be realizable. However, Kuranishi [**13**] showed that $(M, \mathcal{S})$ can be realizable provided M is strongly pseudoconvex and $\dim_{\mathbb{R}} M \geq 9$. Later, Akahori [**1**], Webster [**17**] proved the same result when M is strongly pseudoconvex and $\dim_{\mathbb{R}} M \geq 7$. Recently, under certain conditions on the Levi-form, Catlin [**5**] extended the given CR structure on M to an integrable almost complex structure on a $2n$-dimensional manifold Ω with boundary so that the extension is smooth up to the boundary and so M lies in $b\Omega$. This leads to a solution of the local embedding problem provided M is pseudoconvex and the Levi-form of M has at least three positive eigenvalues (so $\dim_{\mathbb{R}} M \geq 7$).

In this paper, we consider a local embedding problem of a given CR structure on M when M is a pseudoconvex CR manifold of finite type with one degenerate eigenvalue and $\dim_{\mathbb{R}} M \geq 7$. For given positive continuous functions g_1, g_2 on M, where $g_1 = g_2 = 0$ on bM, we define

$$S_{g_1}^- = \{(x,t) \in M \times (-\infty, 0] : \ -g_1(x) \le t \le 0\},$$
$$S_{g_2}^+ = \{(x,t) \in M \times [0,\infty) : \ 0 \le t \le g_2(x)\}.$$

Theorem 1.1. *Let $(\overline{M}, \mathcal{S})$ be a smoothly bounded pseudoconvex* CR *manifold of finite type and the Levi-form has corank one and* $\dim_{\mathbb{R}} M \ge 7$. *Then there exists a positive continuous function g_1 on M and a smooth integrable almost complex structure $\mathcal{L}$ on $S_{g_1}^-$ such that for all $x \in M$, $\mathcal{L}_{(x,0)} \cap \mathbb{C}TM = \mathcal{S}_x$. Furthermore, if $\mathcal{J}_{\mathcal{L}} : TS_{g_1}^- \to TS_{g_1}^-$ is the map associated with the complex structure $\mathcal{L}$, then $dt(\mathcal{J}_{\mathcal{L}}(X_0)) < 0$ at all points of $M_0 = \{(x,0); x \in M\}$.*

In [**7**, **8**], the author showed that the given CR structure on M can be extended smoothly to an integrable almost complex structure on $S_{g_2}^+$ (i.e., the concave side of M), for some g_2, when M is a pseudoconvex CR manifold of finite type and the Levi-form has co-rank one and $\dim_{\mathbb{R}} M \ge 3$. Since the extension of CR structures is essentially unique [**5**, Theorem 4.2], we can patch the integral structures on $S_{g_2}^+$ and $S_{g_1}^-$ smoothly to get an integrable almost complex structure $S_g = S_{g_1}^- \cup S_{g_2}^+$ which contains M as a real hypersurface. By virtue of Newlander-Nirenberg theorem [**14**], S_g is then a complex manifold and we have the following local embedding theorem.

Theorem 1.2. *Let $(\overline{M}, \mathcal{S})$ be as in Theorem 1.1. Then $(\overline{M}, \mathcal{S})$ can be locally realized as a real hypersurface in $\mathbb{C}^n$.*

Remark 1.3. When $\dim_{\mathbb{R}} M = 7$, all the previous results [**1**, **5**, **17**] were for strongly pseudoconvex CR manifolds while Theorem 1.2 includes local embedding theorem for some pseudoconvex CR manifolds of $\dim_{\mathbb{R}} M = 7$. We leave the local embedding theorem of general pseudoconvex CR manifold of finite type as open.

In [**5**], Catlin has introduced certain nonlinear equations which come from deformation theory of an almost complex structure (Section 2). The linearized forms of these equations are simply the $\overline{\partial}$-operator from $\Lambda^{0,1} \otimes T^{1,0}$ to $\Lambda^{0,2} \otimes T^{1,0}$ (Section 2). The solutions of these equations represent sucessive corrections that must be made in the iterative process of solving the nonlinear equation.

In order to solve this $\overline{\partial}$-type equation, we take, for each $x_0 \in M$, a special smooth complex valued coordinates $\zeta = (\zeta_1, \dots, \zeta_n)$ defined near x_0. Then a careful analysis of the local geometry of M near x_0 will give us a family of plurisubharmonic functions with maximal Hessian near x_0. Thanks to these functions, we construct a smooth positive function $g(x,t) = g(x)$, $x \in M$, $g(x) = 0$ on bM, so that the hypersurface $M_g := \{(x,t) \in M \times (-1,0]; t = -g(x)\}$ is pseudoconvex (a bumping theorem), and it patches smoothly with

bM. Using this function g we can define an almost complex manifold S_g^- defined as above. In solving $\overline{\partial}$-type equation on S_g^-, we assign the Neumann condition on M_g and a Dirichlet condition on M to preserve the existing CR structure on M.

To overcome difficulties in subelliptic estimates for $\overline{\partial}$ near bM, we choose a Hermitian metric on S_g^- so that S_g^- takes on the form $S_\varepsilon = M \times [-\varepsilon, 0]$ (Section 4). To this end, we choose, for each $x_0 \in M$, a non-isotropic ball of size $\delta = g(x_0)$ in the transverse holomorphic direction and of size $\delta^{1/2}$ in strongly pseudoconvex tangential holomorphic directions, and of size $\tau(x_0, \delta)$ in the weakly pseudoconvex tangential holomorphic direction.

Section 5 is devoted to showing subelliptic estimates for the $\overline{\partial}$-type equation on each non-isotropic ball. Since the Levi-form has at least $(n-2)$-positive eigenvalues, the usual $1/2$ subelliptic estimates hold for the components which contain the normal component, or for the components which do not contain the normal component but contain the weakly pseudoconvex component. To get a $(1/m)$-subelliptic estimates for the remaining components we use an important feature of M_g, that is, M_g is strongly pseudoconvex with the estimates $\eta([L, \overline{L}])(x,t) \gtrsim g(x)|L|^2$ on M_g. This estimate is also a key one in controlling the boundary integral terms on M_g occuring from integration by parts.

Then we get uniform subelliptic estimates for $\overline{\partial}$ on each non-euclidean ball, and then we get the so called “tame estimates” which are required in the simplified version of Nash-Moser theorem [**16**] for the approximate solution to the linearized equation.

2. Deformation of almost complex structures.

Let $(M, \mathcal{S})$ be a CR manifold as in Section 1 and set $\Omega = M \times (-1, 1)$. In this section we extend the given CR structure $\mathcal{S}$ on M to an almost complex manifold $(\Omega, \mathcal{L})$, and consider a deformation problem of the almost complex structure $\mathcal{L}$ on Ω so that the new (deformed) amost complex structure is integrable (or close to be integrable).

Assume that $\mathcal{L}$ is an almost complex structure on Ω. Let A be a smooth section of $\Gamma^1(\mathcal{L}) = \Lambda^{0,1}(\mathcal{L}) \otimes \mathcal{L}$, where $\Lambda^{0,1}(\mathcal{L})$ denotes the set of $(0,1)$ forms with respect to $\mathcal{L}$. Observe that if A is sufficiently small, then the bundle $\mathcal{L}^A = \{L + \overline{A}(L);\ L \in \mathcal{L}\}$ defines a new almost complex structure. If ω is a section of $\Lambda^{1,0}(\mathcal{L})$, then $\omega - A^*\omega$ is a section of $\Lambda^{1,0}(\mathcal{L}^A)$ where the adjoint A^* maps from $\Lambda^{1,0}(\mathcal{L})$ to $\Lambda^{0,1}(\mathcal{L})$ and is defined by $(A^*\omega)(\overline{L}) = \omega(A(\overline{L}))$, for all $\overline{L} \in \overline{\mathcal{L}}$ and $\omega \in \Lambda^{1,0}$. We want to choose A so that

$$(\omega - A^*\omega)([L' + A(L'), L'' + A(L'')]) = 0. \tag{2.1}$$

Let $L = L' + L''$ denote the decomposition of a vector $L \in \mathbb{C}T_z$ where $L' \in \mathcal{L}_z$ and $L'' \in \overline{\mathcal{L}}_z$. For sections $\overline{L}_1$, $\overline{L}_2$ of $\overline{\mathcal{L}}$, we define

$$(D_2A)(\overline{L}_1, \overline{L}_2) = [\overline{L}_1, A(\overline{L}_2)]' - [\overline{L}_2, A(\overline{L}_1)]' - A([\overline{L}_1, \overline{L}_2]''), \quad \text{and} \tag{2.2}$$

$$F(\overline{L}_1, \overline{L}_2) = [\overline{L}_1, \overline{L}_2]'. \tag{2.3}$$

Note that these definitions are linear in $\overline{L}_1$ and $\overline{L}_2$ and hence D_2A and F are section of $\Gamma^2 = \Lambda^{0,2}(\mathcal{L}) \otimes \mathcal{L}$, and F measures the extend to which $\mathcal{L}$ fails to be integrable. By (2.2) and (2.3) we can write the linearized equation (in A) of (2.1) as

$$D_2A = -F. \tag{2.4}$$

Also if we define $D_3 : \Gamma^2 \longrightarrow \Gamma^3 = \Lambda^{0,3}(\mathcal{L}) \otimes \mathcal{L}$ by

(2.5)
$$\begin{aligned} D_3B(\overline{L}_1, \overline{L}_2, \overline{L}_3) = {} & [\overline{L}_1, B(\overline{L}_2, \overline{L}_3)]' - [\overline{L}_2, B(\overline{L}_1, \overline{L}_3)]' + [\overline{L}_3, B(\overline{L}_1, \overline{L}_2)]' \\ & - B([\overline{L}_1, \overline{L}_2]'', \overline{L}_3) + B([\overline{L}_1, \overline{L}_3]'', \overline{L}_2) - B([\overline{L}_2, \overline{L}_3]'', \overline{L}_1), \end{aligned}$$

for $B \in \Gamma^2$, then it follows that $D_3F = 0$ [**5**, Lemma 3.2].

If $\mathcal{L}$ defines a CR structure on $M \subset b\Omega$ and if we want $\mathcal{L}^A$ to define the same CR structure on M, then this means that A must satisfy $A(\overline{L}') = 0$ on M whenever $\overline{L}'$ is a section of $\overline{\mathcal{L}}$ that is tangent to M. This is a Dirichlet condition on some of the components of the solution of (2.4). Then by solving the extension problem formally, we get the following proposition [**5**, Theorem 4.1].

Theorem 2.1. *Suppose that M is an orientable* CR *manifold of dimension $2n-1$ such that the* CR *dimension equals $n-1$. Then there exists an almost complex structure $\mathcal{L}^*$ on $\Omega = M \times (-1, 1)$ such that $\mathcal{L}^*$ is an extension of the* CR *structure on M, and such that it is integrable to infinite order at M in the sense that if ω is a section of $\Lambda^{1,0}(\mathcal{L}^*)$ and $\overline{L}_1$, $\overline{L}_2$ are sections of $\overline{\mathcal{L}}^*$, then $\omega([\overline{L}_1, \overline{L}_2])$ vanishes to infinite order along M.*

By Theorem 2.1, we have an almost complex structure $\mathcal{L}^*$ on Ω, that is integrable to infinite order along $M_0 = \{(x, 0); x \in M\}$. Then we have the following theorem which is a formal solution of local embedding problem. One can refer a proof from [**3**, Proposition 3].

Theorem 2.2. *Let $x_0 \in M$. Then there are a small neighborhood U of x_0 and a constant $c > 0$ so that for each $x \in M \cap U$, there are* (*almost*) *holomorphic functions $f_1, \ldots, f_n$ defined on $\overline{U}$ so that if $F_x = (f_1, \ldots, f_n)$, then $F_x(x) = 0$, and*

(a) *$|dF_x| \geq c$ on $\overline{U}$, and*
(b) *$\overline{L}f_j$ vanishes to infinite order at x_0, for each $L \in \mathcal{L}$.*

Remark 2.3. Suppose $L \in S_{(x,0)}$. Then $(F_{x_0})_*L$ differs from a section of $T^{1,0}(F_{x_0}(M))$ by a vector field which vanishes to infinite order at 0. Therefore the image $F_{x_0}(M)$ is a smooth real hypersurface in $\mathbb{C}^n$ with defining function given by $r(w) = t \circ F_{x_0}^{-1}(w)$.

In order to define the type of $x_0 \in M$, we use the (almost) holomorphic function F_x constructed in Theorem 2.2:

Definition 2.4. Let (x_0, U, F_{x_0}) be as in Theorem 2.2. Then we let the type of x_0 be equal to the type of $F_{x_0}(x_0) = 0 \in \mathbb{C}^n$, on the hypersurface $F_{x_0}(M)$, in the sense of D'Angelo [**9**].

Set $T(x_0) =$ the type of $x_0 \in M$, and set

$$T(\overline{M}) = \max\{T(x_0) : x_0 \in \overline{M}\} = m. \tag{2.6}$$

Assuming that the Levi-form of M has $(n-2)$-positive eigenvalues, we may assume that m is an even integer.

Let us take $(\Omega, \mathcal{L}^*)$ constructed in Theorem 2.1. Choose a smooth real vector field X_0 on Ω that satisfies $X_0 t \equiv 0$ and $\eta(X_0) \equiv 1$ in Ω. Set $Y_0 = -\mathcal{J}_{\mathcal{L}^*}(X_0)$ so that $X_0 + iY_0$ is a section of $\mathcal{L}^*$ that is transverse to the level set of t. Let $G : \Omega \longrightarrow \Omega$ be a diffeomorphism such that G fixes M_0 and

$$G_* Y_{0|_{(x,0)}} = \frac{\partial}{\partial t|_{(x,0)}}, \quad x \in M.$$

Since M is orientable, we may assume that $dt(\mathcal{J}_{\mathcal{L}^*}(X_0)) < 0$. Thus $dt(Y_0) > 0$ along M_0, which shows that G preserves the sides of M_0; i.e., G maps $\Omega^- = \{(x,t); -1 < t \leq 0\}$ into itself. If we set $\mathcal{L}^0 = G_*\mathcal{L}^*$, then clearly $\widetilde{Z} = -iG_*(X_0 + iY_0)$ is a section of $\mathcal{L}^0$ such that along M_0,

$$\widetilde{Z} = -iX_0 + \frac{\partial}{\partial t}.$$

We write $\widetilde{Z} = \widetilde{X} + g(x,t)\frac{\partial}{\partial t}$ where $\widetilde{X}t \equiv 0$, and set $L_n = g^{-1}\widetilde{Z}$. Then $L_n = \frac{\partial}{\partial t} + X$ where $Xt \equiv 0$. We fix a smooth metric $\langle\ ,\ \rangle_0$ that is Hermitian with respect to the structure $\mathcal{L}^0$ on Ω.

3. A bumping family of pseudoconvex CR manifolds.

Let M, Ω, X_0 and $\mathcal{L}^0$ be as in Section 2. In this section, we will construct one parameter family of pseudoconvex CR manifolds $\{M_s\}_{s>0}$ which connect smoothly with bM. For this purpose we use the analysis of local geometry near $x_0 \in M$ [**8**, Section 3].

Assume that $\widetilde{x}_0 \in M$. Then there are coordinate functions $x_1, \ldots x_{2n}$ defined on a neighborhood $\overline{U}$ of $\widetilde{x}_0$ with the property that $x_{2n} = t$ and that $x_k(x', t) = x_k(x', 0)$, $k < 2n$, for $(x', t) \in \overline{U}$, and that $\frac{\partial}{\partial x_{2n-1}} = -X_0$ at all points of $\overline{U} \cap M$. We take an orthonormal frame $\{\widetilde{L}_1, \ldots, \widetilde{L}_n\}$ of $\mathcal{L}^0$

defined on $\overline{U}$. Let $x_0 \in U$ be fixed for a moment. If $\widetilde{L}_j$ is replaced by $L_j = \sum_{k=1}^{n-1} U_{jk}\widetilde{L}_k$, where $U = (U_{jk})$ is a suitably chosen unitary matrix, then we have:

$$\frac{i}{2}\eta([L_j, \overline{L}_k])(x_0) = \delta_{jk} d_j(x_0) := d_{j,k}(x_0), \quad 1 \le j, k \le n-1, \tag{3.1}$$

where $d_2 \le \cdots \le d_{n-1}$, and $d_j(x_0)$ is a smooth function defined on $\overline{U}$ satisfying $d_2(x) \ge d_0 > 0$ on $\overline{U}$ for a uniform constant $d_0 > 0$.

In the sequal we let ∂_l, $1 \le l \le n$, denote the holomorphic partial derivatives in l-th variable of the local complex valued coordinates. We also let $\widetilde{\partial}_\beta$ denotes ∂_β or $\overline{\partial}_\beta$. We recall the following special coordinates near $x_0 \in M$ [**8**, Proposition 3.1].

Proposition 3.1. *For each $x_0 \in U$ and positive integer m, there are smooth complex valued coordinates $\zeta = (\zeta_1, \ldots, \zeta_n)$, $\zeta_n = t + ix_{2n-1}$, defined near x_0 so that in ζ-coordinates, the vector fields $L_1, \ldots, L_{n-1}$, can be written as*

$$\begin{aligned} L_1 &= \frac{\partial}{\partial \zeta_1} + \sum_{l=1}^{n-1} a_l(\zeta)\frac{\partial}{\partial \zeta_l} + \sum_{l=1}^{n-1} b_l(\zeta)\frac{\partial}{\partial \overline{\zeta}_l} + \big(e(\zeta) + id(\zeta)\big)\frac{\partial}{\partial x_{2n-1}}, \\ L_\alpha &= \frac{\partial}{\partial \zeta_\alpha} + \sum_{l=1}^{n-1} a_l^\alpha(\zeta)\frac{\partial}{\partial \zeta_l} + \sum_{l=1}^{n-1} b_l^\alpha(\zeta)\frac{\partial}{\partial \overline{\zeta}_l} + \big(e_\alpha(\zeta) + id_\alpha(\zeta)\big)\frac{\partial}{\partial x_{2n-1}}, \end{aligned} \tag{3.2}$$

where $2 \le \alpha \le n-1$. Also the coefficient functions satisfy

$$\begin{aligned} &\partial_1^j \overline{\partial}_1^k b_l(0) = \partial_1^j \overline{\partial}_1^k a_l^\alpha(0) = \partial_1^j \overline{\partial}_1^k b_l^\alpha(0) = 0,\ j+k \le m,\ 2 \le l \le n-1, \\ &\widetilde{\partial}_\beta^i \partial_1^j \overline{\partial}_1^k e(0) = 0,\ i = 0, 1,\ i+j+k \le m,\ 2 \le \beta \le n-1,\ \textit{and} \\ &(\partial_1 - \overline{\partial}_1)^s d(0) = (\partial_1 - \overline{\partial}_1)^s e_\alpha(0) = (\partial_1 - \overline{\partial}_1)^s d_\alpha(0) = 0,\ s \le m. \end{aligned} \tag{3.3}$$

Now assume that $x_0 \in M \cap U$ and let us take the smooth complex valued coordinates $\zeta = (\zeta_1, \ldots, \zeta_n)$ defined near x_0 as in Proposition 3.1, and write the vector fields $L_1, \ldots, L_{n-1}$ in this special coordinates. Let $b(\zeta) = e(\zeta) + id(\zeta)$ be the coefficient function of $\partial/\partial x_{2n-1}$ in L_1, and let $b_{m-1}(\zeta)$ be the $(m-1)$-th order Taylor polynomial, in ζ_1 and $\overline{\zeta}_1$, of $b(\zeta)$. Let $a(\zeta)$ be a real valued function defined by

$$a(\zeta) := Im\left[\frac{\partial}{\partial \zeta_1}\overline{b}_{m-1}\right] := \sum_{0 \le j+k \le m-2} a_{j,k}\zeta_1^j \overline{\zeta}_1^k,$$

and set

$$A_l(x_0) = \max\{|a_{j,k}| : j+k = l\}, \quad l = 0, 1, \ldots, m-2.$$

For each $\delta > 0$, we define

$$\tau(x_0, \delta) = \min_{0 \le l \le m-2}\{(\delta/A_l(x_0))^{1/l+2}\}. \tag{3.4}$$

Then the author showed in [8] that

$$(3.5)\quad |\partial_1^j\overline{\partial}_1^k a_l(0)|,\ |\widetilde{\partial}_\beta\partial_1^j\overline{\partial}_1^k b(0)| \lesssim \delta^{1/2}\tau^{-(j+k+1)+\gamma},\quad j+k\le m/2-1,\ \text{and}$$

$$(3.6)\quad |\partial_1^j\overline{\partial}_1^k\eta([L_1,\overline{L}_\alpha])(0)| \lesssim \delta^{1/2}\tau^{-(j+k+1)+\gamma},\ j+k\le m/2-1,$$

for $2\le\alpha,\beta,l\le n-1$, and for $\gamma=(10\times(m/2)!)^{-1}$.

By virtue of the definition of $\tau(x_0,\delta)$ in (3.4) it follows that $\delta^{1/2}\lesssim\tau\lesssim\delta^{1/m}$ and if $\delta'<\delta''$, then

$$(3.7)\qquad (\delta'/\delta'')^{1/2}\tau(x_0,\delta'')\lesssim\tau(x_0,\delta')\lesssim(\delta'/\delta'')^{1/m}\tau(x_0,\delta'').$$

For each $\delta>0$ and $x_0\in M$, we define the type of x_0 with respect to $\delta>0$ as

$$(3.8)\qquad T(x_0,\delta)=\min\{l+2;(\delta/A_l(x_0))^{1/l+2}=\tau(x_0,\delta)\}.$$

Now let us cover $\overline{M}$ by a finite number of neighborhoods U_ν, $\nu=1,\dots,N$, in Ω so that in each U_ν, Proposition 3.1 holds. Let $\{\chi_\nu\}$ be a partion of unity subordinated to the coordinate neighborhoods $\{U_\nu\}$ of Ω, and let m be a given positive integer.

For any $j,k\ge 0$, $j\ge 1$, we define

$$\mathcal{L}^\nu_{j,k}\eta(x)=\frac{i}{2}L_1^{j-1}\overline{L}_1^{k-1}\eta([L_1,\overline{L}_1])(x),\quad x\in U_\nu,$$

and set

$$C_l^\nu(x)=\sum_{j+k=l}|\mathcal{L}^\nu_{j+k}\eta(x)|^2,\quad 1\le l\le m-1,$$

and then set

$$C_l(x)=\sum_{\nu=1}^N\chi_\nu C_l^\nu(x).$$

Set $M=(m+1)!$ and for each $\delta>0$, we define

$$(3.9)\qquad \mu(x,\delta)=\left(\sum_{l=1}^m C_l^{M/l+1}(x)\delta^{-2M/l+1}\right)^{-1/2M}.$$

Note that $\sum_{l=1}^m C_l(x)>0$ if the type at x is less than or equal to m. Therefore $\mu(x,\delta)$ is defined intrinsically and it is a smooth function of $\delta>0$ and x, for x satisfying $\sum_{l=1}^m C_l(x)>0$.

Let us fix $x_0\in M\cap U$ and take the smooth complex valued coordinates $\zeta=(\zeta_1,\dots,\zeta_n)$ defined on $M\cap U$ as in Proposition 3.1. For each $\delta>0$, set $\tau_1=\tau=\tau(x_0,\delta)$ and $\tau_k=\delta^{1/2}$, $2\le k\le n-1$, and then set

$$(3.10)\qquad P_\delta(x_0)=\{x\in\mathbb{R}^{2n}:|\zeta_i|\le\tau_i,\ 1\le i\le n-1,\ |\zeta_n|\le\delta\}.$$

Thanks to the estimates in (3.5), (3.6) and the definition of $T(x_0,\delta)$ in (3.8), and the Taylor's theorem argument, we have the following proposition.

Proposition 3.2. *If $x \in P_\delta(x_0)$, then*

$$\tau(x_0,\delta) \approx \mu(x,\delta). \tag{3.11}$$

Corollary 3.3. *Suppose $x \in P_\delta(x_0)$. Then*

$$\tau(x_0,\delta) \approx \tau(x,\delta). \tag{3.12}$$

Proof. If we set $x = x_0$ in (3.11), we see that $\mu(x,\delta) \approx \tau(x_0,\delta)$. Since this holds for $x_0 = x$, it follows that $\mu(x,\delta) \approx \tau(x,\delta)$. Hence (3.12) follows. □

Note that $\mu(x,\delta)$ is defined intrinsically. That is, it does not depend on the choice of a specific coordinates. Proposition 3.2 and Corollary 3.3 show that the quantity $\tau(x,\delta)$ is also defined invariantly, up to a universal constant, with respect to the coordinate functions.

For each $\varepsilon > 0$, we set $\Omega_\varepsilon = M \times (-1,\varepsilon)$ and set $S(\varepsilon) = M \times (-\varepsilon,\varepsilon)$. Then one can construct bounded plurisubharmonic weight functions so that the Hessian of these functions satisfy certain essentially maximal bounds in a thin strip $S(\varepsilon)$ of M_0. The heart of these construction is the so called "doubling property" of $P_\delta(x_0)$, which comes from the relation in (3.12). For a detailed proof of the following theorem, one can refer Section 3 of [**8**]. For each small $\delta > 0$, we set $\tau_1(x) = \tau(x,\delta), \tau_2(x) = \dots \tau_{n-1}(x) = \delta^{1/2}$, and $\tau_n(x) = \delta$ as before.

Theorem 3.4. *For all small $\delta > 0$, there is a plurisubharmonic function $h_\delta \in C^\infty(\Omega_\delta)$ with the following properties:*

(i) $|h_\delta(x)| \le 1$, $x \in U \cap \Omega_\delta$.

(ii) *For all $L = \sum_{j=1}^n b_j L_j$ at $x \in U \cap S(\delta)$,*

$$\partial\bar\partial h_\delta(x)(L,\bar L) \approx \sum_{j=1}^n |b_j(x)|^2 \tau_j^{-2}(x), \quad \textit{and} \tag{3.13}$$

(iii) $|D^\alpha h_\delta(x)| \lesssim C_\alpha \prod_{k=1}^n \tau_k^{-\alpha_k}(x)$, *where* $D^\alpha = \partial_1^{\beta_1}\bar\partial_1^{\gamma_1}\dots\partial_n^{\beta_n}\bar\partial_n^{\gamma_n}$ *and* $\alpha_i = \beta_i + \gamma_i$.

In D_2-equation, we will assign a Dirichlet condition on one side of bS_g^-, and the Neumann condition on the other side of bS_g^-. This fact leads us another difficulty which was not occurred in 1/2-subelliptic estimates of Catlin in [**5**]. To overcome this difficulty, we need the following lemma.

Lemma 3.5. *Let $x_0 \in M \cap U$ and set $c_{1k}^n = \partial\bar\partial t(L_1,\bar L_k)$, $1 \le k \le n$. Then for each small $\delta > 0$, we have*

$$|c_{11}^n(x)| \lesssim \delta\tau(x_0,\delta)^{-2}, \quad x \in P_\delta(x_0), \quad \textit{and} \tag{3.14}$$

$$|c_{1k}^n(x)| \lesssim \delta^{1/2}\tau(x_0,\delta)^{-1+\gamma}, \quad x \in P_\delta(x_0),\ 2 \le k \le n, \tag{3.15}$$

where $\gamma = (10 \times (m/2)!)^{-1}$.

Proof. See [8, Lemma 3.7]. □

Now we want to construct one parameter family of pseudoconvex CR manifolds $\{M_s\}_{s>0}$, which connect up smoothly with bM, the boundary of M. In the sequel, we let m be the type of $\overline{M}$ defined in (2.6) and let $\sigma_1(x,k)=\sigma_1(x,k^{-3m})=\tau_1(x,k^{-3m})$ denote the quantity defined in (3.4) for $\delta=k^{-3m}$ and set $\sigma_j(x,k)=k^{-3m/2}$, $j=2,\dots,n-1$, and $\sigma_n(x,k)=k^{-3m}$. We also set $x'=(x',0)\in M$. Let $d(x)=d(x')\in C^\infty(\overline{M})$ be a smooth defining function of bM, independent of t, such that $d(x')>0$ for all $x'\in M$.

Theorem 3.6. *There exist a smooth one parameter family of pseudoconvex* CR *manifolds M_s, $0\le s<s_0$, $M_0=M$, each defined by $M_s=\{(x',t)\in\Omega; r_s(x',t)=t+sG(x')=0\}$, and a large integer K_0 such that:*

(i) *$G(x')$ is a smooth plurisubharmonic function in $x'\in M$, $G(x')>0$ on M and $G(x')=0$ on bM,*

(ii) *if $L=\sum_{j=1}^n b_jL_j$ satisfies $Lr_s(x',t)=0$, and if $d(x')\approx\frac{1}{\log k}$, for $k\ge K_0$, then $G(x')\approx 2^{-k}\cdot(\log k)^{-1}$, and*

$$\partial\overline{\partial}r_s(L,\overline{L})(x',t)\gtrsim s\cdot G(x')\sum_{j=1}^{n}\sigma_j(x,k)^{-2}|b_j|^2,\quad \textit{and} \tag{3.16}$$

(iii) $s|L_1G(x')|\lesssim\sigma_1(x,k)\cdot\partial\overline{\partial}r_s(L_1,\overline{L}_1)$.

Proof. We cover bM by a finite number of neighborhoods $B(x_\nu,a)$, $x_\nu\in bM$, $a>0$, $\nu=1,2,\dots,N_0$. Since $\operatorname{dist}(x',bM)\approx d(x')$ near bM, there is a small constant $b_0>0$ such that

$$M_{b_0}:=\{x\in M; d(x)>b_0\}\supset\overline{M}-\cup_{\nu=1}^{N_0}B(x_\nu,2a).$$

We cover $\overline{M}-\cup_{\nu=1}^{N_0}B(x_\nu,2a)$ by finite number of neighborhoods $B(x_\nu,a')$, where $B(x_\nu,2a')\subset M_{b_0}$, $\nu=N_0+1,\dots,N$. Set $V_\nu=B(x_\nu,3a)$, $\nu=1,\dots,N_0$, and $V_\nu=B(x_\nu,2a')$, $\nu=N_0+1,\dots,N$. In each V_ν, we may assume that there exist coordinates $x=(x_1,\dots,x_{2n})$ with the property that $x_{2n}=t$ and that $x_k(x',t)=x_k(x',0)$, $k<2n$, for $(x',t)\in V_\nu$, and $\partial/\partial x_{2n-1}=-X_0$ at all points of $\overline{M}\cap V_\nu$, and that Proposition 3.1 holds. Let $\phi_\nu(x',t)=\phi_\nu(x',0)$ be C^∞ functions supported in V_ν, $\nu=1,\dots,N$, and $\phi_\nu=1$ on $B(x_\nu,2a)$, $\nu=1,\dots,N_0$, and $\phi_\nu=1$ on $B(x_\nu,a')$, $\nu=N_0+1,\dots,N$.

For each V_ν, $\nu=1,\dots,N_0$, we consider a function g_ν defined by $g_\nu(x')=d(x')\phi_\nu(x')$. Then $\operatorname{supp}g_\nu\subset V_\nu$, and $g_\nu(x',t)=g_\nu(x')=d(x')\approx\operatorname{dist}(x',bM)$ on $B(x_\nu,2a)$. We take the complex valued coordinate functions $\zeta=(\zeta_1,\dots,\zeta_n)$ defined in V_ν as in Proposition 3.1 and let $x=(x_1,\dots,x_{2n-1},t)$ be the real coordinates of ζ. Set $D_k=\partial/\partial x_k$, $1\le k\le 2n$. We may assume that these coordinates satisfy $\{\zeta\in\overline{M}:\sum_{i=1}^{2n-1}|x_i(\zeta)|^2\le 1\}\subset V_\nu\cap\overline{M}$.

Let K_0 be a large integer still to be chosen. For all $k > K_0$ let $\psi_{k,\nu}(x',t) = \psi_{k,\nu}(x')$ denote a function defined on V_ν, $\nu = 1,\dots,N_0$, and satisfies

$$\begin{aligned}\psi_{k,\nu}(x') &= 1 \quad \text{if } \frac{1}{\log(k+1)} < d(x') < \frac{1}{\log(k+2)},\\ \psi_{k,\nu}(x') &= 0 \quad \text{if } d(x') > \frac{1}{\log(k+3)}, \quad \text{or } d(x') < \frac{1}{\log k},\end{aligned} \tag{3.17}$$

and

$$|D^\alpha \psi_{k,\nu}(x)| \le C_\alpha k^{2|\alpha|}. \tag{3.18}$$

Similarly we define $\psi_{K_0,\nu}$ as

$$\begin{aligned}\psi_{K_0,\nu}(x') &= 1 \quad \text{if } d(x') > \frac{1}{\log K_0}\\ \psi_{K_0,\nu}(x') &= 0 \quad \text{if } d(x') < \frac{1}{\log(K_0+2)},\end{aligned}$$

and so that the direct analog of (3.18) holds for $k = K_0$.

For all $k \ge K_0$, define $\Lambda_{k,\nu}(x) = h_{k^{-3m}}(x)$, where h_δ is the function constructed in Theorem 3.4, for $\delta = k^{-3m}$. Let $L = \sum_{j=1}^n b_j L_j$ and $x = (x',t) \in \operatorname{supp}\psi_{k,\nu}$. Then (3.13) shows that if $|t| < k^{-3m}$,

$$\partial\overline{\partial}\Lambda_{k,\nu}(L,\overline{L})(x) \approx \sum_{l=1}^n |b_l|^2 \sigma_l(x,k)^{-2}. \tag{3.19}$$

Set

$$G_1(x',t) = G_1(x') = \sum_{\nu=1}^{N_0}\sum_{k=K_0}^{\infty} 2^{-k} g_\nu(x)\psi_{k,\nu}(x)(\Lambda_{k,\nu}(x',0)+2).$$

Then for those x with $d(x) \approx \frac{1}{\log k}$, we can show, as in the proof of Theorem 2.3 in [6], that

$$\partial\overline{\partial}G_1(x)(L,\overline{L}) \approx 2^{-k}(\log k)^{-1}\sum_{j=1}^n \sigma_j(x,k)^{-2}|b_j|^2. \tag{3.20}$$

By virtue of (3.18), (3.20) and from the property (iii) of Theorem 3.4, we have

(3.21)

$$|L_1G_1(x)| \lesssim 2^{-k}\sigma_1(x,k^{-3m})^{-1}\cdot(\log k)^{-1} \lesssim \sigma_1(x,k^{-3m})\partial\overline{\partial}G_1(x)(L_1,\overline{L}_1),$$

and by virtue of (3.17) and (3.18), we easily obtain that

$$|D^\alpha G_1(x)| \le C_\alpha 2^{-k}k^{3m|\alpha|}, \quad x \in \operatorname{supp}\psi_{k,\nu}.$$

Thus as $d(x) \longrightarrow 0$, G_1 and all of its derivatives vanish to infinite order.

Note that we may assume that $d(x) > b_0 \geq 1/\log K_0$ on M_{b_0} provided K_0 is sufficiently large. Set $\lambda_0 = h_{K_0^{-3m}}$, where $h_{K_0^{-3m}}$ is defined as in Theorem 3.4. Set

$$G_0(x', t) = G_0(x') = 2^{-K_0} \sum_{\nu=N_0+1}^{N} \phi_\nu(x)(\lambda_0(x', 0) + 2),$$

where $\operatorname{supp} \phi_\nu \subset B(x_\nu, 2a')$, $\phi_\nu \equiv 1$ on $B(x_\nu, a')$, $\nu = N_0 + 1, \ldots, N$. Thus if $x = (x', t)$ satisfies $|t| < K_0^{-3m}$, and $d(x) \geq b_0$, then as in the proof of Theorem 2.3 in [**6**], we also have that

$$\partial\overline{\partial} G_0(L, \overline{L})(x) \approx 2^{-K_0} \sum_{j=1}^{n} \sigma_j(x, K_0)^{-2} |b_j|^2. \tag{3.22}$$

Note that g_ν, $\psi_{k,\nu}$, ϕ_ν and d are independent of t and hence so do G_0 and G_1. Set

$$G(x) = G(x') = G_0(x) + G_1(x). \tag{3.23}$$

Then (3.20) and (3.22) show that $G(x)$ is a smooth plurisubharmonic function which vanishes on bM, independent of t. For all $0 \leq s < 1$, set

$$r_s(x', t) = t(x') + sG(x'). \tag{3.24}$$

Since $\Lambda_{k,\nu} + 2 \approx 1$, it follows that

$$G(x) \approx 2^{-k} \cdot (\log k)^{-1},$$

for those $x = (x', t)$ with $d(x') \approx 1/\log k$, $k \geq K_0$, provided K_0 is sufficiently large. Since $|sG(x)| \lesssim s 2^{-K_0} << 1$, (3.20) and (3.22) imply that

$$\partial\overline{\partial} G(x)(L, \overline{L}) \approx 2^{-k} \cdot (\log k)^{-1} \sum_{j=1}^{n} |b_j|^2 \sigma_j(x, k)^{-2}, \ |t| < k^{-3m}, \tag{3.25}$$

because $\sigma_1(x, k^{-3m}) \approx \sigma_1(x', k^{-3m})$ by (3.12).

For all small $s > 0$, the set $M_s = \{x; r_s(x) = 0\}$ then defines a smooth manifold of dimension $(2n - 1)$ and can be joined smoothly with bM. Let $L = \sum_{j=1}^{n} b_j L_j$ satisfies $Lr_s = 0$. Then as in the proof of Theorem 2.3 in [**6**] we can show that

$$\partial\overline{\partial} r_s(x', t)(L, \overline{L}) \gtrsim s 2^{-k} \cdot (\log k)^{-1} \sum_{j=1}^{n} \sigma_j(x', k)^{-2} |b_j|^2,$$

for those $(x', t) \in M_s$ such that $d(x') \approx \frac{1}{\log k}$, $k \geq K_0$. This proves (3.16). Property (iii) follows from (3.21) and (3.22). □

To obtain a subelliptic estimates near M_s, we want to construct a family of plurisubharmonic functions with large Hessian defined near M_s (instead of M_0 in Theorem 3.4). The estimates in (3.16), at $t = -sG(x')$, is an important ingredient to the construction of these functions.

Assume $x_0 \in M$ and let $x = (x_1, \ldots, x_{2n-1}, t) = (x', t)$ be coordinate functions defined on a neighborhood $\overline{U}$ of x_0 such that $\partial/\partial x_{2n-1} = -X_0$ along $M \cap \overline{U}$. We may assume that $d(x_0) \approx (\log k)^{-1}$ for some $k \geq K_0$ and hence that $G(x_0) \approx 2^{-k}$. In terms of these coordinate functions, let $x_s = (x'_s, -sG(x'_s)) \in M_s \cap \overline{U}$ be fixed for a moment. We also assume that $G(x'_s) \approx 2^{-k} \cdot (\log k)^{-1}$. Let $\{\widetilde{L}_1, \ldots, \widetilde{L}_n\}$ be an orthonormal frame of $\mathcal{L}^0$ defined on $\overline{U}$. Set

$$\begin{aligned} &L_j = \widetilde{L}_j - (\widetilde{L}_j r_s)(\widetilde{L}_n r_s)^{-1}\widetilde{L}_n, \quad 1 \leq j \leq n-1, \quad \text{and} \\ &L_n = \widetilde{L}_n. \end{aligned} \tag{3.26}$$

Then $L_j r_s \equiv 0$, $j = 1, 2, \ldots, n-1$.

Define new coordinates $\widetilde{x} = (\widetilde{x}_1, \ldots, \widetilde{x}_{2n})$ on $\overline{U}$ by:

$$\begin{aligned} \widetilde{x}_i &= x_i, \quad i = 1, 2, \ldots, 2n-1, \quad \text{and} \\ \widetilde{x}_{2n} &= t + sG(x') = r_s(x', t). \end{aligned}$$

In new coordinates, $M_0 \cap \overline{U}$ and $M_s \cap \overline{U}$ corresponds to the points on $\overline{U}$ where $\widetilde{x}_{2n} = sG(x')$ and 0, respectively. Therefore

$$X_0 = -\frac{\partial}{\partial \widetilde{x}_{2n-1}} - sG_{2n-1}(x')\frac{\partial}{\partial \widetilde{x}_{2n}},$$

along $M \cap \overline{U}$, where $G_{2n-1}(x') = \partial/\partial x_{2n-1} G(x')$. By taking affine transformation $C_{x_s} : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$, $C_{x_s}(\widetilde{x}) = (u_1, \ldots, u_{2n})$, L_k can be written as:

$$L_k|_{x_s} = \frac{\partial}{\partial u_{2k-1}} - i\frac{\partial}{\partial u_{2k}}, \quad 1 \leq k \leq n-1. \tag{3.27}$$

By virtue of Proposition 3.1, for the point x_s (instead of x_0), we then have local complex valued smooth coordinate functions $\zeta = (\zeta_1, \ldots, \zeta_n)$ defined near $x_s \in M_s$ so that the vector fields $L_1, \ldots, L_{n-1}$ can be written as in (3.2) satsfying (3.3) in ζ coordinates.

Let $x = (x_1, \ldots, x_{2n})$ be the real coordinates of ζ, where $x_{2n} = t + sG(x') = r_s(x', t)$, and set $D_k = \partial/\partial x_k$, $1 \leq k \leq 2n$. For $l \geq 0$ we set

$$|G^{(l)}(x')| = \sum_{|\alpha'| \leq l} |D^{\alpha'} G(x')|,$$

where $\alpha' = (\alpha_1, \ldots, \alpha_{2n-1})$'s are multi-indices. Let ω^j be dual of L_j, $1 \leq j \leq n$. Let $0 \leq \varepsilon \leq s$ and set $x_\varepsilon = (x', \varepsilon G(x'))$. So $x_\varepsilon \in M_s$ if and only if $\varepsilon = 0$, and $x_\varepsilon \in M_0$ if and if $\varepsilon = s$.

Assuming that $G(x'_s) \approx G(x') \approx 2^{-k} \cdot (\log k)^{-1}$, for some $k \geq K_0$, we obtain from (3.16) that

$$\partial\overline{\partial} r_s(L_1, \overline{L}_1)(x_\varepsilon) \gtrsim (s - \varepsilon) G(x') \sigma_1(x, k)^{-2}. \tag{3.28}$$

Since $L_n = \partial/\partial x_{2n} - iX_0$ along $M \cap U$, it follows that $\omega^n = 1/2(dx_{2n} + i\eta)$ along $M \cap U$. Therefore we can write, at x_ε, as

$$\omega^n = \frac{1}{2}(dx_{2n} + i\eta) + (s-\varepsilon)\mathcal{O}\left[\sum_{l=1}^{2n}\left(D_l G(x') + G(x')\right)dx_l\right].$$

Because $L_n = (1 + \mathcal{O}(s|G^{(1)}(x')|))\partial/\partial x_{2n} + X$ where $Xx_{2n} \equiv 0$ on $\overline{U}$ it follows that $\partial r_s = (1 + \mathcal{O}(s|G^{(1)}(x')|))\omega^n$ on $\overline{U}$. Since $\mathcal{L}^0$ is integrable to infinite order along M_0 one obtains, for $1 \leq i, j \leq n-1$, that

(3.29)
$$\begin{aligned}\partial\overline{\partial} r_s(L_i, \overline{L}_j) &= \partial r_s([L_i, \overline{L}_j]) + (s-\varepsilon)^3\mathcal{O}(G(x')^3)\\ &= \omega^n([L_i, \overline{L}_j])(1 + s\mathcal{O}(|G^{(1)}(x')|)) + (s-\varepsilon)^3\mathcal{O}(G(x')^3),\end{aligned}$$

at the point x_ε. Combining (3.28) and (3.29) one then obtains that

$$\partial\overline{\partial} r_s(L_1, \overline{L}_1)(x_\varepsilon) \approx \omega^n([L_1, \overline{L}_1])(x_\varepsilon), \tag{3.30}$$

independent of $\varepsilon > 0$ for all $0 \leq \varepsilon \leq s$.

For each small $\rho > 0$, set $\delta = \rho G(x'_s)$. Following the notations in (3.2)-(3.4), we let $\tau(x_s, \delta)$ be the quantity defined in (3.4) for the point x_s. Set $\tau_1 = \tau(x_s, \delta)$, $\tau_j = \delta^{1/2}$, $2 \leq j \leq n-1$, $\tau_n = \delta$, and set

$$P_\rho(x_s) = \{\zeta : |\zeta_j| \leq \tau_j, \quad 1 \leq j \leq n\}.$$

Since $\tau_1 \lesssim \delta^{1/m} \ll (\log k)^{-1}$, it follows that $d(x') \approx (\log k)^{-1}$ on $P_\rho(x_s)$, and hence that

$$G(x') \approx 2^{-k} \cdot (\log k)^{-1}, \quad \text{for } (x', x_{2n}) \in P_\rho(x_s). \tag{3.31}$$

Also the estimate in (3.18) implies that

$$|G^{(m+1)}(x')| \lesssim 2^{-k} k^{3m(m+1)} \lesssim G(x'_s)^{1-\gamma/m+1}, \tag{3.32}$$

for $(x', x_{2n}) \in P_\rho(x_s)$, provided K_0 is sufficiently large. Here $\gamma = (10 \times (m/2)!)^{-1}$. Note that for those x' with $(x', x_s) \in P_\rho(x_s)$, $x_\varepsilon \in P_\rho(x_s)$ if and only if $0 \leq \varepsilon \leq \rho$. In the sequal, we assume that $2\rho \leq s$.

By virtue of the definition of $\tau(x_s, \rho G(x'_s))$, and by (3.12), (3.31), we obtain that

$$|\omega^n([L_1, \overline{L}_1])(x)| \lesssim \rho G(x')\tau(x, \rho G(x'))^{-2}, \quad x \in P_\rho(x_s). \tag{3.33}$$

Also the estimates leading to (3.6) hold similarly except the error terms $s^3|G^{(m+1)}(x')|^3 \lesssim s^2 G(x')^2$ (by 3.32), and hence one obtains that

(3.34)
$$|\omega^n([L_1, \overline{L}_\alpha])(x)| \lesssim \delta^{1/2}\tau^{-1+\gamma} + s^2 G(x')^2, \quad x \in P_\rho(x_s),\ 2 \leq \alpha \leq n-1.$$

Assuming that $2\rho \leq s$, we obtain, from (3.28) and (3.29), that

$$\omega^n([L_1, \overline{L}_1])(x) \gtrsim sG(x')\sigma_1(x, k)^{-1} \gg s^2 G(x')^2, \quad x \in P_\rho(x_s), \tag{3.35}$$

and hence the error term $s^2G(x')^2$ in (3.34) can be absorbed into the estimates.

For each $\rho > 0$, set

$$S_\rho = \{(x', t) : (-\rho - s)G(x') \le t \le (\rho - s)G(x')\} \cap U,$$

in (x', t) coordinates. (Note that $x_{2n} = t + sG(x')$.) Set $\tau_1^\rho(x) = \tau(x, \rho G(x'))$, $\tau_j^\rho(x) = (\rho G(x'))^{1/2}$, $2 \le j \le n-1$, and $\tau_n^\rho(x) = \rho G(x')$.

We now consider the two cases, $\omega^n([L_1, \overline{L}_1]) \approx \delta\tau^{-2}$ and $\omega^n([L_1, \overline{L}_1]) \ll \delta\tau^{-2}$ on $P_\rho(x_s)$. The first case corresponds to the case that $T(x_s, \delta) = 2$ and the later case corresponds to the case that $T(x_s, \delta) \ge 3$, where $T(x_s, \delta)$ is defined as in (3.8). Then as in Theorem 2.1 in [**6**], one can construct a family of local plurisubharmonic functions $g_{x_s,\rho}$ satisfying all the properties in the theorem. Then by adding up these functions $g_{x_s,\rho}$, we can prove the following theorem as in Theorem 3.4, in each thin neighborhood S_ρ of M_s (instead of M_0). This is a crucial one to get subelliptic estimates for the forms supported near M_s in Section 5.

Theorem 3.7. *For all small $\rho > 0$, there is a plurisubharmonic function $h_\rho \in C^\infty(\Omega)$ with the following properties:*

(i) $|h_\rho(x)| \le 1$ *on* Ω,
(ii) *if* $L = \sum_{j=1}^n b_j L_j$, *then*

$$\partial\overline{\partial} h_\rho(x)(L, \overline{L}) \approx \sum_{j=1}^n |b_j|^2 \tau_j(x)^{-2}, \quad x \in S_\rho \cap U, \tag{3.36}$$

(iii) $|D^\alpha h_\rho(x)| \le C_\alpha \prod_{j=1}^n \tau_j^{-\alpha_j}(x)$.

Set $c_{1l}^n(x) = \omega^n([L_1, \overline{L}_l])(x)$, $1 \le l \le n-1$. Using (3.33), (3.34) and the property (iii) of Theorem 3.7, we can prove the following proposition.

Proposition 3.8. *For all small $\rho > 0$, and for each $\alpha = (\alpha_1, \dots \alpha_n)$, we have*

$$|c_{11}^n(x) D^\alpha h_\rho(x)| \le C_\alpha \tau_n^\rho(x) \tau_1^\rho(x)^{-2} \prod_{k=1}^n (\tau_k^\rho(x))^{-\alpha_k}, \tag{3.37}$$

and, for $2 \le l \le n-1$,

(3.38)
$$|c_{1l}^n(x) D^\alpha h_\rho(x)| \le C_\alpha \left(\tau_n^\rho(x)^{\frac{1}{2}} \tau_1^\rho(x)^{-1+\gamma} + s^2 G(x')^2 \right) \prod_{k=1}^n (\tau_k^\rho(x))^{-\alpha_k},$$

for all $x \in S(\rho) \cap U$.

4. Special frames for almost complex structures.

In this section we want to construct a special dilated coordinates defined near $x_0 \in M$. Let us take the smooth function $G(x')$, $(x', t) \in \Omega$, and one parameter family of pseudoconvex CR manifolds M_s with defining function $r_s = t + sG(x')$ as in the previous section. Assume that $T(\overline{M}) = m < \infty$, where $T(\overline{M})$ is defined as in (2.6).

For any ε, σ, $0 < \varepsilon \leq \sigma \leq 1$, we set $s = \varepsilon\sigma^{2m}$ and set $r_s(x', t) = t + \varepsilon\sigma^{2m}G(x')$ and define

$$S_{\varepsilon,\sigma} = \{(x', t) \in \Omega; G(x') > 0 \text{ and } -\varepsilon\sigma^{2m}G(x') \leq t \leq 0\}. \tag{4.1}$$

Remark 4.1. The quantities ε and σ will be fixed later. If we set $g(x') = \varepsilon \cdot \sigma^{2m} \cdot G(x')$, then g is the required positive function in the definition of S_g^- in Section 1 and $S_{\varepsilon,\sigma}$ equals S_g^-.

We define a subbundle of $\mathcal{L}^0$ on $S_{\varepsilon,\sigma}$ by letting $\mathcal{R}_{(x,t)} = \{L \in \mathcal{L}^0_{(x,t)}; Lr_s = 0\}$. Clearly the map H defined by $H(L) = L - (Lr_s)(L_n r_s)^{-1}L_n$ defines an isomorphism of $\mathcal{S}$ onto $\mathcal{R}$ (at all points of $S_{\varepsilon,\sigma}$). Set $\mu_1(x) = \mu(x, \varepsilon G(x'))$, $\mu_2(x) = \ldots, \mu_{n-1}(x) = \varepsilon^{1/2}G(x')^{1/2}$, and $\mu_n(x) = \varepsilon G(x')$. We define a weighted metric $\langle\ ,\ \rangle$ on $\mathcal{L}^0$ by the relations

$$\begin{aligned}
\langle H(L_j), H(L_k)\rangle &= \mu_j(x)^{-1}\mu_k(x)^{-1}\langle L_j, L_k\rangle_0, \quad 1 \leq j, k \leq n-1,\\
\langle L_n, L_n\rangle &= \varepsilon^{-2}\varphi(x)^{-4m}, \quad \text{and}\\
\langle L_n, H(L_l)\rangle &= 0, \quad 1 \leq l \leq n-1,
\end{aligned}$$

where $L_l \in \mathcal{S}$, $1 \leq l \leq n-1$. Since $\mu(x, \delta)$ is a smooth function of x and δ, it follows that $<\ ,\ >$ is a smooth Hermitian metric on $\mathcal{L}^0$. Now using the special coordinates defined in Proposition 3.1, we will cover $S_{\varepsilon,\sigma}$ by special dialated coordinate neighborhoods such that on each such neighborhood, there is a frame $\mathcal{L}$ that satisfies required good estimates.

Proposition 4.2. *There exist constants ε_0 and σ_0 such that if $0 < \varepsilon < \varepsilon_0$ and $0 < \sigma < \sigma_0$, then on $S_{\varepsilon,\sigma}$ there exist for all $x_0 = (x_0', 0) \in M$ with $G(x_0') > 0$ a neighborhood $W(x_0) \subset S_{\varepsilon,\sigma}$ with the following properties:*

(i) *On $W(x_0)$ there are smooth coordinates $y = (y', y_{2n})$ where $y' = (y_1, y_2, \ldots, y_{2n-1})$ is independent of t and where the function y_{2n} is defined by $y_{2n} = \varepsilon^{-1}G(x')^{-1}r_s(x', t) - \sigma^{2m}$, so that $W(x_0) = \{y; |y'| < \sigma, -\sigma^{2m} \leq y_{2n} \leq 0\}$. Thus, $M_0 \cap W(x_0)$ and $M_s \cap W(x_0)$ correspond to the points in $W(x_0)$ where $y_{2n} = 0$ and $-\sigma^{2m}$, respectively. Moreover, the point $(x_0', 0) \in \Omega$ corresponds to the origin.*

(ii) *The above coordinate charts are uniformly smoothly related in the sense that if $W(\widetilde{p}_0)$ and $W(x_0)$ intersect, and if $\widetilde{y}$ and y_0 are the associated coordinates, then*

$$|D^\alpha(\widetilde{y} \circ (y_0)^{-1})| \leq C_{|\alpha|}$$

holds on that portion of $\mathbb{R}^{2n}$ *where* $\widetilde{y}\circ(y_0)^{-1}$ *is defined. The constant* $C_{|\alpha|}$ *is independent of* ε, σ, *and* x_0.

(iii) *On* $W(x_0)$, *there exists a smooth frame* $L_1,\dots,L_n$ *for* $\mathcal{L}$ *such that if* $\omega^1,\dots,\omega^n$ *is the dual frame, and if* L_k *and* ω^k *are written as* $\sum_{j=1}^{2n} b_{k_j}\frac{\partial}{\partial y_j}$ *and* $\sum_{j=1}^{2n} d_{kj}dy_j$, *then*

$$\sup_{y\in W(x_0)}\left\{|D_y^\alpha b_{kj}(y)|+|D_y^\alpha d_{kj}(y)|\right\}\le C_{|\alpha|},$$

where $C_{|\alpha|}$ *is independent of* x_0, j,k, ε *and* σ.

(iv) *With the frames as in* (iii), *set* $c_{1l}^n=\omega^n([L_1,\overline{L}_l])$, $l=2,\dots n$. *Then there is an independent constant* $C>0$ *such that*

$$\sup_{y\in W(x_0)}|c_{1l}^n(y)|\le C\sigma^{2m}. \tag{4.2}$$

(v) *There are independent constants* $c>0$ *and* $C>0$ *such that if* $B_b(x)$ *denotes the ball of radius* b *about* $x\in S_{\varepsilon,\sigma}$ *with respect to the metric* $\langle\ ,\ \rangle$, *then*

$$B_{c\sigma}(x_0)\subset W(x_0)\subset B_{C\sigma}(x_0), \tag{4.3}$$

and if $\operatorname{Vol}B_b(x_0)$ *denotes the volume of* $B_b(x_0)$ *with respect to* $\langle\ ,\ \rangle$, *then*

$$cb^{2n-1}\sigma^{2m}\le \operatorname{Vol}B_b(x_0)\le Cb^{2n-1}\sigma^{2m}. \tag{4.4}$$

Proof. We will sketch the proof. For a detailed proof, one can refer [**5**, Proposition 5.1]. We first cover $\overline{M}$ by a finite number of neighborhoods V_ν, $\nu=1,\dots,N$, in Ω such that in each V_ν there exist coordinates $(u_1,\dots,u_{2n})$ with the property that $u_{2n}=t$ and that $u_k(u',t)=u_k(u',0)$, $k<2n$, for $(u',t)\in V_\nu$, and that $\partial/\partial u_{2n-1}=-X_0$, at all points of $M\cap V_\nu$. Also we can arrange the neighborhoods V_ν so that Proposition 3.1 holds on each V_ν.

For any point $x_0\in M\cap V_\nu$, we take coordinate functions $\zeta^\nu=(\zeta_1^\nu,\dots,\zeta_n^\nu)$ constructed as in Proposition 3.1. Let us denote by $\widetilde{L}_k^\nu$, the vector fields $\widetilde{L}_k$, $1\le k\le n$, written in ζ^ν-coordinates as in (3.2), satisfying (3.3). Set $\theta_0=\varepsilon G(x_0')$. Let $\tau(x_0,\theta_0)$ and $\mu(x,\theta_0)$ be the quantities defined in (3.4) and (3.9) respectively with respect to the point $x_0\in M_0$. By Proposition 3.2 it follows that $\mu(x_0,\theta_0)\approx\tau(x_0,\theta_0)$.

Set $\mu_0=\mu(x_0,\theta_0)$ for a convenience. Let $x=(x',t)$ be the real coordinates of ζ^ν. We define new coordinates $y=D_{\varepsilon,x_0}(x)=(y_1,\dots,y_{2n})$ by means of dilation map $D_{\varepsilon,x_0}:\mathbb{R}^{2n}\longrightarrow\mathbb{R}^{2n}$ given by:

$$y(x',t)=(\mu_0^{-1}x_1,\mu_0^{-1}x_2,\theta_0^{-1/2}x_3,\dots,\theta_0^{-1/2}x_{2n-2},\theta_0^{-1}x_{2n-1},\varepsilon^{-1}G(x')^{-1}t). \tag{4.5}$$

In terms of the y-coordinates, we define an open set $W_b(x_0)$ by

$$W_b(x_0) = \{x \in V_\nu \cap S_{\varepsilon,\sigma};\ |y_k(x)| < b,\qquad k = 1, 2, \dots 2n-1,\ -\sigma^{2m} \le y_{2n}(x) \le 0\}. \tag{4.6}$$

Note that in $W_b(x_0)$, $y_{2n} = 0$ and $y_{2n} = -\sigma^{2m}$ coincide with $r_s(x', t) = \varepsilon\sigma^{2m}G(x')$ and $r_s(x', t) = 0$, respectively, the boundaries of $S_{\varepsilon,\sigma}$. As in (3.26) we set $L^\nu_j = \widetilde{L}^\nu_j - e_j\widetilde{L}^\nu_n$, $j \le n-1$, and $L^\nu_n = \widetilde{L}^\nu_n$, where $e_j = (\widetilde{L}^\nu_k r_s)(\widetilde{L}^\nu_n r_s)^{-1}$, and then define a frame $L_1, \dots, L_n$ in $W_b(x_0)$ by setting

$$\begin{aligned} &L_1 = \mu(x, \varepsilon G(x'))L^\nu_1, \quad L_k = \varepsilon^{1/2}G(x')^{\frac{1}{2}}L^\nu_k, \quad 2 \le k \le n-1,\\ &L_n = \varepsilon G(x')L^\nu_n. \end{aligned} \tag{4.7}$$

Then it follow from (3.32) that

$$|e_k| \lesssim \varepsilon\sigma^{2m}|G^{(1)}(x'_0)| << \sigma^{2m}\theta_0\mu(x_0, \theta_0)^{-1/m}, \quad 1 \le k \le n-1.$$

We set $W(x_0) := W_\sigma(x_0)$ for a sufficiently small $\sigma > 0$. Then (4.2) follows by using the estimates in (3.37) and (3.38). The other properties follow as in the proof of Proposition 5.1 in [**5**]. □

Let us take the dilated coordinates $y = (y_1, \dots, y_{2n})$ defined in (4.5). In the sequel we let L^ν_j and ω^j_ν, $1 \le j \le n$, be the vector field and its dual form written in x-coordinates as in (3.26) defined on $V_\nu \ni x_0$. Also let L_j be the vector fields defined in (4.7) and let ω^j be its dual frame, $1 \le j \le n$.

We may assume that $d(x_0) :=$ dist $(x_0, bM) \approx (\log k)^{-1} \approx d(x')$, $(x', t) \in W(x_0)$, for some $k \ge K_0$ as in Section 3. Also it follows that $G(x') \approx 2^{-k} \cdot (\log k)^{-1}$ for $(x', t) \in W(x_0)$. By (3.28)-(3.30) we then have

$$\begin{aligned} \omega^n([L_1, \overline{L}_1]) &= \varepsilon^{-1}G(x')^{-1}\mu_1(x, \varepsilon G(x'))^2\omega^n_\nu([L^\nu_1, \overline{L}^\nu_1])\\ &\gtrsim \varepsilon^{-1}G(x')^{-1}\mu(x, \varepsilon G(x'))^2 \cdot sG(x')\sigma_1(x, k)^{-2}\\ &\gtrsim \sigma^{2m}\mu(x, \varepsilon G(x'))^2 \cdot \sigma_1(x, k)^{-2}, \quad \text{on } M_\sigma. \end{aligned} \tag{4.8}$$

We also want to get estimates for $\omega^n([L_1, \overline{L}_k])$, $2 \le k \le n-1$, on M_σ. Let us set $\delta = (\varepsilon\sigma^{2m}G(x'_0))^{m/\gamma}$, where $\gamma = (10 \times (m/2)!)^{-1}$. For each $x_{s_\nu} \in M_\sigma$, we consider the coordinate functions $\zeta = (\zeta_1, \dots, \zeta_n)$ constructed in Proposition 3.1 about the point $x_{s_\nu} \in M_\sigma$, and set $\tau^\nu_1 = \tau(x_{s_\nu}, \delta)$, $\tau^\nu_j = \delta^{1/2}$, $2 \le j \le n-1$, $\tau^\nu_n = \delta$, and set

$$P_\delta(x_{s_\nu}) = \{\zeta; |\zeta_j| \le \tau^\nu_j, \quad 1 \le j \le n\}.$$

We cover $M_\sigma \cap W(x_0)$ by polydiscs, $P_\delta(x_{s_\nu})$, $x_{s_\nu} \in M_\sigma \cap \overline{W(x_0)}$ so that any $(N+1)$ intersection of $P_\delta(x_{s_\nu})$'s are empty. Let $x_s \in M_\sigma$. Then $x_s \in P_\delta(x_{s_\nu})$ for some $x^\nu_s \in M_\sigma$. If we express the vector fields $L_1, \dots, L_n$ in terms of the special coordinates about the point x_{s_ν}, it follows from (3.34) that

$$
\begin{aligned}
(4.9)\qquad & |\omega^n([L_1,\overline{L}_\alpha])(x_s)|^2 \\
&= \varepsilon^{-1}G(x_s')^{-1}\mu(x_s,\varepsilon G(x_s'))^2|\omega_\nu^n([L_1^\nu,\overline{L}_\alpha^\nu])(x_s)|^2 \\
&\lesssim \varepsilon^{-1}G(x_s')^{-1}\mu(x_s,\varepsilon G(x_s'))^2\left(\delta\tau(x_{s_\nu},\delta)^{-2+2\gamma}+s^4G(x_s')^4\right) \\
&\lesssim \varepsilon^{-1}G(x_s')^{-1}\mu(x_s,\varepsilon G(x_s'))^2(\varepsilon^2\sigma^{4m}G(x')^2) \\
&\lesssim \varepsilon\sigma^{2m}G(x_s')\mu(x_s,\varepsilon G(x_s'))^2,\quad 2\le\alpha\le n-1,
\end{aligned}
$$

where we have used the fact that $\delta\tau^{-2+2\gamma}\lesssim\tau^{2\gamma}\lesssim\varepsilon^2\sigma^{4m}G(x_0)^2$.

From (4.8) and (4.9) one obtains that

$$
(4.10)\qquad |\omega^n([L_1,\overline{L}_\alpha])(x)|^2\lesssim\varepsilon G(x_0)\omega^n([L_1,\overline{L}_1]),\quad 2\le\alpha\le n-1.
$$

Using the special coordinates $y_1,\dots,y_{2n}$ and the special frames $L_1,\dots L_n$ defined in (4.7), we want to define L^2-operators with mixed boundary conditions. We set

$$
|f|_{k,W_b(x_0)} = \sup\{|D_y^\alpha f(y)|; y\in W_b(x_0),\ |\alpha|\le k\},
$$

and we extend this norm to vector fields and 1-forms.

In the process of subelliptic estimates for D_2-operator, we will see that a certain boundary integral terms on M_0 occurs. To handle these boundary integral terms, we need the following lemma.

Lemma 4.3. *There are a frame $X_1,\dots,X_n$ for $\mathcal{L}$ and its dual frame $\eta^1,\dots,\eta^n$ so that if we set $c_{kn}^n=\eta^n([X_k,\overline{X}_n])$, $1\le k\le n-1$, then*

$$
\begin{aligned}
&c_{kn}^n = 0 \quad \text{on}\quad W(x_0),\quad k=2,\dots,n-1,\quad\text{and}\\
&|c_{1n}^n|_{s,W(x_0)}\le C_s\sigma^{2m}.
\end{aligned}
$$

Proof. See [8, Lemma 4.3]. □

Recall that a deformation of $\mathcal{L}^0$ is a section A of the bundle $\Gamma^1(S_{\varepsilon,\sigma})$. In terms of the special frames in $W(x_0)$, we write $A=\sum_{j,l=1}^n A_{jl}\overline{\omega}^l\otimes L_j$, and then define

$$
\begin{aligned}
|A(y)|_s &= \sum_{|\alpha|\le s}\sum_{j,l=1}^n|D_y^\alpha A_{jl}(y)|,\quad\text{and}\\
|A|_{s,W(x_0)} &= \sup\{|A(y)|_s; y\in W(x_0)\},
\end{aligned}
$$

and suppose that A satisfies

$$
(4.11)\qquad |A|_{m+2n+3,W(x_0)}\le\varepsilon_0,
$$

for a sufficiently small $\varepsilon_0>0$.

We define $\mathcal{A}(S_{\varepsilon,\sigma})$ to be the space of sections $A \in \Gamma^{0,1}(S_{\varepsilon,\sigma};0)$ such that along M_0, $A(\overline{L}) = 0$ whenever $\overline{L} \in T^{0,1} \cap \mathbb{C}TM_0$. From now on, we assume that $A \in \mathcal{A}(S_{\varepsilon,\sigma})$. Then we can define a deformation $\mathcal{L}^A$ of $\mathcal{L}^0$ by

$$\overline{\mathcal{L}}^A = \{\overline{L} + A(\overline{L}); \overline{L} \in \overline{\mathcal{L}^0_z}, \quad z \in S_{\varepsilon,\sigma}\}.$$

In terms of the frame $X_1, \dots, X_n$, and its dual frame $\eta^1, \dots, \eta^n$ in $W(x_0)$ constructed in Lemma 4.3, we define

$$X_j^A = X_j + \overline{A}(X_j), \quad j = 1, \dots, n,$$

and let η_A^j be the dual frame. Set

(4.12)
$$L_j^A = \sigma^{\frac{1}{4}}\left(X_j^A - (X_j^A r)(X_n^A r)^{-1} X_n^A\right), \quad 1 \le j \le n-1, \quad L_n^A = X_n^A,$$

and

(4.13)
$$\omega_A^j = \sigma^{-\frac{1}{4}}\eta_A^j, \quad 1 \le j \le n-1, \quad \omega_A^n = \left(\eta_A^n + \sum_{j=1}^{n-1}(X_j^A r)(X_n^A r)^{-1}\eta_A^j\right).$$

Obviously, the frame ω_A^j, for $j = 1, \dots, n$, is dual to L_j^A, and $L_j^A r \equiv 0$ for $1 \le j \le n-1$.

Assuming that A satisfies (4.11) for sufficiently small $\varepsilon_0 \le \sigma^{2m^2 \cdot \gamma^{-1}}$, it follows from Lemma 4.3 that

$$\sup_{y \in W(x_0)} |\omega_A^n([L_k^A, \overline{L}_n^A])(y)| \lesssim \sigma^{1/4}|A|_1, \quad 2 \le k \le n-1, \tag{4.14}$$

$$\sup_{y \in W(x_0)} |\omega_A^n([L_1^A, \overline{L}_n^A])(y)| \le C\sigma^{2m+\frac{1}{4}} + \sigma^{1/4}|A|_1, \tag{4.15}$$

where the constant $C > 0$ is independent of x_0, σ and ε.

In order to measure how L_j^A, $j = 1, 2, \dots, n$, depend on A, we define

$$P_k(y; A) = \sum_{\substack{k_1, \dots, k_N \\ |k_1| + \dots + |k_N| \le k}} \prod_{\nu=1}^{N} |A(y)|_{k_\nu}.$$

Lemma 4.4. *If A satisfies (4.11) for sufficiently small ε_0, then the following pointwise estimates hold for $y \in W(x_0)$:*

$$|L_k^A - \sigma^{\frac{1}{4}} L_k|_s \le C_s \sigma^{1/4} P_s(y; A), \quad |L_n^A - L_n|_s \le C_s P_s(y; A), \tag{4.16}$$

$$|\omega_A^k - \sigma^{-\frac{1}{4}}\omega^k|_s \le C_s \sigma^{-\frac{1}{4}} P_s(y; A), \quad |\omega_A^n - \omega^n|_s \le C_s P_s(y; A). \tag{4.17}$$

Proof. From the expression of L_k^A and ω_A^k in (4.12) and (4.13), the error terms are the finite product of derivatives as in (4.16) and (4.17). $\square$

Lemma 4.5. *Assume A satisfies (4.11). Then*

$$|\omega_A^n([L_1^A,\overline{L}_k^A])|\lesssim \varepsilon G(x')\omega_A^n([L_1^A,\overline{L}_1^A]),\quad 2\le k\le n-1. \tag{4.18}$$

Proof. By virtue of Lemma 4.4, we can write

$$\omega_A^n([L_1^A,\overline{L}_k^A])=\sigma^{1/2}\omega^n([L_1,\overline{L}_k])+\mathcal{O}(|A|_2).$$

Since A vanishes to infinite order along M_0 (in x-coordinates), we have $|D_x^\alpha A(x)|\lesssim \varepsilon^4\sigma^{8m}G(x')^4$, $|\alpha|\le 2$. In terms of y-coordinates, we then have

$$|A|_2\lesssim \varepsilon^2\sigma^{4m}G(x')^2. \tag{4.19}$$

Hence (4.18) follows by combining (4.8), (4.10) and (4.19). □

For the subelliptic estimates on the forms supported on M_σ, we still have to construct a family of plurisubharmonic functions with large Hessian in dilated coordinates y. By virtue of Theorem 3.7, there is a family of plurisubharmonic functions $\{h_{\varepsilon\rho}(x)\}_{\rho>0}$ defined on $S_\rho^\varepsilon=\{(x',t):(-\varepsilon\rho-\varepsilon\sigma^{2m})G(x')\le t\le(\varepsilon\rho-\varepsilon\sigma^{2m})G(x')\}$. In y-coordinates, we can write

$$S_\rho:=S_\rho^\varepsilon=\{(y',y_{2n});|y_{2n}+\sigma^{2m}|\le\rho\}.$$

Set $\widetilde{W}(x_0)=W_{C\sigma}(x_0)$, for some $C>1$, and set

$$\mu_1(x)=\mu(x,\varepsilon G(x')),\quad \mu_k(x)=\varepsilon^{1/2}G(x')^{1/2},\quad 2\le k\le n-1,$$
$$\mu_n(x)=\varepsilon G(x'),$$

and for any $\rho>0$, we set

$$\mu_1^\rho(x)=\mu(x,\rho\varepsilon G(x')),\quad \mu_k^\rho(x)=(\rho\varepsilon G(x'))^{1/2},\quad 2\le k\le n-1,$$
$$\mu_n^\rho(x)=\rho\varepsilon G(x').$$

Theorem 4.6. *Assume that A satisfies (4.11) for a sufficiently small $\varepsilon_0>0$. Then for each small $\rho>0$, there exists a C^∞ plurisubharmonic function λ_ρ defined on $\widetilde{W}(x_0)\Supset W(x_0)$ such that*

(i) *$|\lambda_\rho|\le 1$ in $\widetilde{W}(x_0)$,*
(ii) *for all $y\in S_\rho\cap\widetilde{W}(x_0)$, and $L^A=\sum_{j=1}^n b_jL_j^A$, we have*

$$\partial\overline{\partial}\lambda_\rho(y)(L^A,\overline{L}^A)\approx\sum_{j=1}^n|b_j|^2\mu_j(x)^2\mu_j^\rho(x)^{-2}, \tag{4.20}$$

(iii) *$|L^A\lambda_\rho|^2\lesssim\partial\overline{\partial}\lambda_\rho(y)(L^A,\overline{L}^A)$,*
(iv) *$|D^\alpha\lambda_\rho|\le C_\alpha\prod_{k=1}^n\mu_k^{\alpha_i}(x)(\mu_k^\rho(x))^{-\alpha_i}$.*

Proof.

Let $\{h_{\varepsilon\rho}\}_{\varepsilon\rho>0}$ be the family of plurisubharmonic functions constructed in Theorem 3.7. Set $\lambda_\rho(y)=h_{\varepsilon\rho}\circ D_{x_0}^{-1}(y)$, where D_{x_0} is the dilation function defined in (4.5). It is clear that λ_ρ is plurisubharmonic and satisfies (i). Let

$\{\widetilde{L}_1^\nu, \dots, \widetilde{L}_n^\nu\}$ be an orthonormal frame defined on V_ν in x-coordinates, and let $L_j^\nu = \widetilde{L}_j^\nu - e_j \widetilde{L}_n^\nu$, $1 \le j \le n-1$, $L_n^\nu = \widetilde{L}_n^\nu$, be the vector fields defined in (3.26) where $e_j = (\widetilde{L}_j^\nu r_s)(\widetilde{L}_n^\nu r_s)^{-1}$. Note that the frame $\{L_1, \dots L_n\}$ defined in (4.7) can be written as $L_j = \mu_j(x) L_j^\nu$, $1 \le j \le n$. If we set $L = \sum_{j=1}^n b_j L_j$, then it follows from (3.36) and by functoriality that

$$\begin{aligned} \partial\overline{\partial}\lambda_\rho(y)(L, \overline{L}) &= \partial\overline{\partial} h_{\varepsilon\rho}(x)(dD_{x_0}^{-1} L, dD_{x_0}^{-1}\overline{L}) \\ &= \partial\overline{\partial} h_{\varepsilon\rho}(x) \left(\sum_{j=1}^n b_j \mu_j L_j^\nu, \sum_{j=1}^n \bar{b}_j \mu_j \overline{L}_j^\nu \right) \\ &\approx \sum_{j=1}^n |b_j(x)|^2 \mu_j(x)^2 \mu_j^\rho(x)^{-2}. \end{aligned} \tag{4.21}$$

Note that the vector fields L_j^A's and its dual forms ω_A^j's are written as in (4.12) and (4.13). Since A vanishes to infinite order along M_0, it follows that $|A|_2 \lesssim \varepsilon^2 \sigma^{4m} G(x')^2$. Therefore one obtains, from (4.21), that

$$\partial\overline{\partial}\lambda_\rho(y)(L^A, \overline{L}^A) \approx \partial\overline{\partial}\lambda_\rho(y)(L, \overline{L}) + \mathcal{O}(|A|_2)|b|^2 \approx \sum_{j=1}^n |b_j|^2 \mu_j(x)^2 \mu_j^\rho(x)^{-2}.$$

This completes the proof. $\square$

Next we show that there exists a smooth Hermitian metric on $S_{\varepsilon,\sigma}$ such that for all $x_0 \in M$ the frame $L_1^A, \dots, L_n^A$ given by (4.12) is orthonormal.

For $L \in \mathcal{L}^0$ and $A \in \mathcal{A}(S_{\varepsilon,\sigma})$ satisfying (4.11), define a bundle isomorphism $P_A : \mathcal{L}^0 \to \mathcal{L}^A$ by $P_A(L) = L + A(L)$. Define a homomorphism $H_A : \mathcal{L}^A \to \mathcal{R}^A$, where $\mathcal{R}^A = \{L \in \mathcal{L}^A; Lr = 0\}$, by

$$H_A(L) = L - \frac{Lr}{X_n^A r} X_n^A = L - \frac{L y_{2n}}{L_n^A y_{2n}} L_n^A.$$

Then $H_A \circ P_A$ is an isomorphism of $\mathcal{R}$ onto $\mathcal{R}^A$. We define a metric $\langle\ ,\ \rangle_A$ on $\mathcal{L}^A$ by

$$\begin{aligned} &\langle (H_A \circ P_A)\overline{L}_1, (H_A \circ P_A)\overline{L}_2 \rangle_A = \langle \overline{L}_1, \overline{L}_2 \rangle, \quad \overline{L}_1, \overline{L}_2 \in \mathcal{R}, \\ &\langle L_n^A, L_n^A \rangle_A = 1, \quad \text{and} \\ &\langle (H_A \circ P_A)\overline{L}_1, L_n^A \rangle_A = 0, \quad \overline{L}_1 \in \mathcal{R}. \end{aligned}$$

Note that L_n^A is actually globally defined, so that the above conditions determine a metric on $\mathcal{L}^A$. Since L_j, $j = 1, 2, \dots, n-1$, defined in (4.7), are an orthonormal basis of $\mathcal{L}$, it follows that $L_j^A = (H_A \circ P_A) L_j$, $j = 1, 2, \dots, n-1$, are an orthonormal basis of $\mathcal{L}^A$ with respect to $\langle\ ,\ \rangle_A$.

Let dV denote the volume form associated with the Riemannian metric $\langle\ ,\ \rangle$. In the coordinates $(y_1, \dots, y_{2n})$ in $W(x_0)$, we can write $dV = V(y)dy$, where $dy = dy_1 \dots dy_{2n}$, and where V satisfies

$$|V|_{s,W(x_0)} \le C_s, \quad \text{and} \quad \underset{y\in W(x_0)}{\to} \inf V(y) > c > 0,$$

where c is independent of σ, ε, and x_0. Let $S_{\varepsilon,\sigma}$ be defined as in (4.1). We will define the inner product for two functions g, $h \in C^\infty(S_{\varepsilon,\sigma})$ by

$$(g,h) = \int_{S_{\varepsilon,\sigma}} g\overline{h}\, dV. \tag{4.22}$$

Let $\Lambda^{0,q}(S_{\varepsilon,\sigma}; A)$ denote the space of $(0,q)$-forms with respect to $\mathcal{L}^A$ on $S_{\varepsilon,\sigma}$, and set

$$\Gamma^{0,q}(S_{\varepsilon,\sigma}; A) = \Lambda^{0,q}(S_{\varepsilon,\sigma}; A) \otimes \mathcal{L}^A.$$

We can extend the inner product of (4.22) to the forms $\Gamma^{0,q}(S_{\varepsilon,\sigma}; A)$ using the component functions. We recall the D_q operators defined in (2.4) and (2.5). Then by integration by parts it follows that, if $U = \sum_\nu U_\nu L_\nu^A \in \Gamma^{0,k}(S_\varepsilon; A)$ is supported in $W(x_0)$, D'_q, the adjoint operator of D_q, is defined by

$$D'_k U = \sum_\nu \left(\overline{\partial}^* U_\nu - \sum_\mu \sum_j \partial\overline{\omega_A^\mu}(L_j^A, \overline{L}_\nu^A)(\overline{L}_j^A \rfloor U_\mu) \right) L_\nu^A, \tag{4.23}$$

where

$$\begin{aligned} \overline{\partial}^* U_\nu = &- \sum_{|J|=k-1} \sum_{j=1}^n (L_j^A U_\nu^{jJ} + e_j U_\nu^{jJ})\overline{\omega}_A^J \\ &- \sum_{|K|=k-2} \sum_{\ell=1}^n \sum_{i<j} \omega_A^\ell([L_i^A, L_j^A]) U_\nu^{ijK} \overline{\omega}_A^{\ell K}. \end{aligned} \tag{4.24}$$

Now we extend the definition of the operators D_q and D'_q to the L^2–spaces with mixed boundary conditions as in [**5**, Section 6], and we denote their extensions by T_q and T_q^*. Let $\mathcal{B}^q(S_{\varepsilon,\sigma}; A) \subset \Gamma^{0,k}(S_\varepsilon; A)$ denote the set of forms $U = \sum_l \sum_{|J|=q} U_l^J L_l^A$ such that U_l^J vanishes on M_g when $n \in J$, and vanishes on M_0 when $n \notin J$. Then we can approximate $U \in \mathrm{Dom}(T_{q+1}) \cap \mathrm{Dom}(T_q^*)$ by $U_\mu \in \mathcal{B}^q(S_{\varepsilon,\sigma}; A)$ in the graph norm of T_{q+1} and T_q^* [**5**, Lemma 6.4]:

Lemma 4.7. *Let $U \in \mathrm{Dom}(S) \cap \mathrm{Dom}(T^*)$. Then there exists $U_\mu \in \mathcal{B}^q(S_{\varepsilon,\sigma}; A)$ such that*

$$\lim_{\mu\to\infty} (\|U_\mu - U\| + \|SU_\mu - SU\| + \|T^*U_\mu - T^*U\|) = 0.$$

Finally suppose that we have proved the estimate

$$\|U\|^2 \le C(\|T^*U\|^2 + \|SU\|^2) \tag{4.25}$$

for all $U \in \mathcal{B}^q(S_{\varepsilon,\sigma}; A)$. Then Lemma 4.7 shows that (4.25) holds for all $U \in \operatorname{Dom} T^* \cap \operatorname{Dom} S$. Then from the usual $\overline{\partial}$-Neumann theory it follows that for all $G \in L^2_q(S_{\varepsilon,\sigma}; T^{1,0}_A)$, there exists an element $NG \in \operatorname{Dom}(T^*) \cap \operatorname{Dom}(S)$ such that

$$\|NG\| \leq C^2 \|G\|,$$

and

$$(G, V) = (T^*(NG), T^*V) + (SNG, SV), \quad V \in \operatorname{Dom}(T^*) \cap \operatorname{Dom}(S).$$

We will call N the Neumann operator associated with D_q.

5. The subelliptic estimate for D_q.

In this section we prove a subelliptic estimate for the D_q-Neumann problem with almost-complex structure $\mathcal{L}^A$. We set $q = 2$ in this section.

We first define tangential norms that will be used in the estimates. For any $s \in \mathbb{R}$, set

$$|||f|||_s^2 = \int_0^{\sigma^{2m}} \int_{\mathbb{R}^{2n-1}} |\hat{f}(\xi, y_{2n})|^2 (1 + |\xi|^2)^s d\xi \, dy_{2n},$$

where $\hat{f}(\xi, y_{2n}) = \int_{\mathbb{R}^{2n-1}} e^{-iy' \cdot \xi} f(y', y_{2n}) dy'$. For any integer $k \geq 0$ and any $s \in \mathbb{R}$, set

$$\|f\|_{s,k}^2 = \sum_{j=0}^{k} \left|\left|\left| \frac{\partial^j f}{\partial y_{2n}^j} \right|\right|\right|_{s-j}^2.$$

Finally for any integer $m \geq 0$ and $f \in C^\infty(W(x_0))$, set

$$\|f\|_m^2 = \sum_{|\alpha| \leq m} \|D_y^\alpha f\|^2.$$

By using the coefficients of U, we can easily define all of the above norms for any section U of $\Gamma^{0,q}$. We recall that $\mathcal{A}(S_{\varepsilon,\sigma})$ is the space of sections $A \in \Gamma^{0,1}(S_{\varepsilon,\sigma}; 0)$ such that along M_0, $A(\overline{L}) = 0$ whenever $\overline{L} \in T^{0,1} \cap \mathbb{C}TM_0$. Then the goal of this section is to prove the following subelliptic estimate:

Theorem 5.1. *Suppose $T(\overline{M}) = m < \infty$ and that A is a section of $\mathcal{A}(S_{\varepsilon,\sigma})$ that satisfies (4.11) for some small $\varepsilon_0 > 0$. Then there exist small positive constants σ_1 and ε_1 so that if $\varepsilon < \varepsilon_1$, if $\sigma < \sigma_1$, and if $|A|_{m+2n+3, W(x_0)} \leq \varepsilon$, then the D_q-Neumann problem on $S_{\varepsilon,\sigma}$ with coefficient σ for the almost–complex structure $\mathcal{L}^A$ satisfies the following estimate for all forms $U \in \mathcal{B}^q(S_{\varepsilon,\sigma}; A)$ that are compactly supported in $W(x_0)$:*

$$\sigma^{-3} \|U\|^2 + L^A(U) + \sigma^{\frac{1}{2}} \|U\|^2_{\frac{1}{m},1} \leq C(\|SU\|^2 + \|T^*U\|^2), \tag{5.1}$$

where $L^A(U) = L^A(U') + L^A(U'')$, *and* $L^A(U'') = L^A(\widetilde{U}) + L^A(\widetilde{\widetilde{U}})$, *and where*

$$L^A(U') = \sum_{\ell=1}^{n}\sum_{k=1}^{n-1}\sum_{\substack{|J|=q\\ n\in J}} \left(\|L_k^A U_\ell^J\|^2 + \|\overline{L}_k^A U_l^J\|^2\right) + \sum_{\ell=1}^{n}\sum_{\substack{|J|=q\\ n\in J}} \|L_n^A U_l^J\|^2,$$

$$L^A(\widetilde{U}) = \sum_{\ell=1}^{n}\sum_{k=1}^{n-1}\sum_{\substack{|J|=q\\ 1\in J\\ n\notin J}} \left(\|L_k^A U_\ell^J\|^2 + \|\overline{L}_k^A U_l^J\|^2\right) + \sum_{\ell=1}^{n}\sum_{\substack{|J|=q\\ 1\in J\\ n\notin J}} \|\overline{L}_n^A U_\ell^J\|^2, \quad \textit{and}$$

$$L^A(\widetilde{\widetilde{U}}) = \sum_{\ell=1}^{n}\sum_{k=1}^{n-1}\sum_{\substack{|J|=q\\ 1,n\notin J}} \|L_k^A U_\ell^J\|^2 + \sum_{\ell=1}^{n}\sum_{\substack{|J|=q\\ 1,n\notin J}} \|\overline{L}_1^A U_l^J\|^2 + \sum_{\ell=1}^{n}\sum_{\substack{|J|=q\\ 1,n\notin J}} \|\overline{L}_n^A U_l^J\|^2.$$

We let $C > 1$ and $0 < c \le 1$ be the independent constants which may vary in various estimates. For convenience, in all that follows, we omit the notation A from the frames $L_1^A, \dots, L_n^A$, and $\omega_A^1, \dots, \omega_A^n$, and $L^A(U)$. We first state some necessary lemmas for the proof of Theorem 5.1.

If $n \ge 3$, then the $(n-2)$ positive eigenvalue condition on the Levi-form of M guarantees the existence of at least one positive eigenvalue. Then the classical theorem of Hörmander shows that:

Lemma 5.2. *Assume that* $n \ge 3$ *and* A *satisfies* (4.11)*. Then for all* $f \in C_0^\infty(W(x_0))$,

$$\sigma^{\frac{1}{2}} |||f|||_{\frac{1}{2}}^2 \le C \sum_{k=1}^{n-1} \left(\|L_k^A f\|^2 + \|\overline{L_k^A} f\|^2\right) + C\|f\|^2. \tag{5.2}$$

Note that in $W(x_0)$, we have technically chosen so that $y_{2n} = 0$ and $y_{2n} = \sigma^{2m}$ coincide with the boundaries of $S_{\varepsilon,\sigma}$. Then the following lemma can be proved by modifying the proof of Lemma 7.7 in [**5**].

Lemma 5.3. *Suppose that* $f \in C_0^\infty(W(x_0))$ *and that* f *vanishes either on* M_0 *or on* M_σ*. If* σ *is sufficiently small, say* $\sigma < \sigma_1$*, then there exists a constant* C *independent of* ε, σ, *and* x_0 *so that*

$$\sigma^{-4}\|f\|^2 \le C(\|f\|_{\frac{1}{m}}^2 + \|\widetilde{L}_n f\|^2), \tag{5.3}$$

where $\widetilde{L}_n = L_n$ *or* $\overline{L}_n$.

To handle the commutator terms, we need the following lemma:

Lemma 5.4. *Assume that* $n \ge 3$*. Let* $U \in \mathcal{B}_q(S_{\varepsilon,\sigma}; A)$ *be compactly supported in* $W(x_0)$*, and suppose that* A *satisfies* (4.11)*. Assume that* $|K| =$

$q-1$ with $n \notin K$ and that $1 \leq k \leq n-1$. Set $c^n_{kn} = w^n([L_k, \overline{L}_n])$ and $d^n_{nk} = \overline{\omega}^n([L_n, \overline{L}_k])$. Then

$$|(c^n_{kn} L_n U^{kK}_\ell, U^{nK}_\ell)| \leq C(\sigma L^A(U) + \sigma^{-1}\|U\|^2), \tag{5.4}$$

$$|(d^n_{kn} \overline{L}_n U^{nK}_\ell, U^{kK}_\ell)| \leq C(\sigma L^A(U) + \sigma^{-1}\|U\|^2). \tag{5.5}$$

Proof. (4.14) and (4.15) are the key estimates for the proof of (5.4) and (5.5). One can refer a proof from [**8**]. □

For each small $\rho > 0$, we set

$$S_\rho = \{(y', y_n); |y_{2n} + \sigma^{2m}| < \rho\} \cap \widetilde{W}(x_0),$$

and let $\lambda_\rho = h_{\rho\delta_o} \circ D^{-1}_{x_0}(y)$ be the plurisubharmonic weight functions constructed in Theorem 4.6 where $\delta_0 = \varepsilon G(x_0)$.

Lemma 5.5. *For each k, $1 \leq k \leq n-1$, set $c^n_{1k} = \omega^n([L_1, \overline{L}_k])$. Then*

$$\sup_{y\in S_\rho} |c^n_{11}(L_n\lambda_\rho)(y)| \lesssim \sigma^{\frac{1}{2}} \tau(x,\delta_0)^2 \tau(x,\rho\delta_0)^{-2}, \tag{5.6}$$

$$\sup_{y\in S_\rho} |c^n_{1k}(L_n\lambda_\rho)(y)| \lesssim \sigma\rho^{-\frac{1}{2}} \tau(x,\delta_0) \tau(x,\rho\delta_0)^{-1}, \quad 2 \leq k \leq n-1. \tag{5.7}$$

Proof. Note that $c^n_{1k} = \omega^n_A([L^A_1, \overline{L}^A_k])$, where ω^n_A and L^A_k's are defined in (4.12) and (4.13). Therefore it follows that

$$c^n_{1k}(y) = \sigma^{\frac{1}{2}} \omega^n([L_1, \overline{L}_k])(y) + \mathcal{O}(|A|_1), \quad 1 \leq k \leq n-1. \tag{5.8}$$

Since $|A|_1 \lesssim \varepsilon\rho^2$ in S_ρ, (5.6) and (5.7) follow from the estimates in (3.37), (3.38), (5.8) and from the fact that $\varepsilon\rho^2 \lesssim \rho\tau(x,\delta_0)\tau(x,\rho\delta_0)^{-1}$. □

We now want to prove Theorem 5.1. We follow the standard $\overline{\partial}$-type estimates, while we need a special attention to the components U^J_l, with $1, n \notin J$, on M_σ. Assume $U = \sum_{l=1}^n \sum_{|J|=q} U^J_l \overline{\omega}^J \cdot L_l \in \mathcal{B}^q(S_{\varepsilon,\sigma}, A)$ with $\operatorname{supp} U \subset W(x_0)$. Then from (4.23) and (4.24) it follows that

$$T^*U = D'_q U = BU + C|U|,$$

where

$$BU = -\sum_{l=1}^n \sum_{|K|=q-1} \sum_{j=1}^n (L_j U^{jK}_l) \overline{\omega}^K \cdot L_l. \tag{5.9}$$

Also (2.5) shows that

$$SU = D_{q+1}U = AU + C|U|, \tag{5.10}$$

where

$$AU = \sum_{l=1}^n \sum_{|J|=q} \sum_{j=1}^n (\overline{L}_j U^J_l) \overline{\omega}^{jJ} \cdot L_l. \tag{5.11}$$

Combining (5.9)-(5.11) one obtains that

$$\|AU\|^2 + \|BU\|^2 \le 2\|SU\|^2 + 2\|T^*U\|^2 + C\|U\|^2. \tag{5.12}$$

Let us write $U = U' + U''$, where

$$U' = \sum_{l=1}^{n} \sum_{\substack{|J|=q \\ n\in J}} U_l^J \overline{\omega}^J \cdot L_l, \quad \text{and,} \quad U'' = \sum_{l=1}^{n} \sum_{\substack{|J|=q \\ n\notin J}} U_l^J \overline{\omega}^J \cdot L_l,$$

and let $L(U) = L(U') + L(U'')$ be defined as in the statement of Theorem 5.1. Then we can write:

(5.13)

$$\|AU\|^2 + \|BU\|^2 = \|AU''\|^2 + \|BU''\|^2 + \|AU'\|^2 + \|BU'\|^2 + E(U', U''),$$

where $E(U', U'')$ denotes the (sum of) inner products (AU', AU'') and (BU', BU'').

Note that the Levi-form of M_σ has at least $(n-2)$-positive eigenvalues and $U' = 0$ along M_σ. Since $n \ge 4$, there are at least two positive eigenvalues along M_σ. Therefore we may proceed in the standard way as in [**10**, **11**] for U' and we get

$$\|AU'\|^2 + \|BU'\|^2 \ge c\Big(L(U') + \int_{M_0} |U'|^2 dS\Big). \tag{5.14}$$

Set $E(\sigma, U) = \sigma^{1/4} L(U) + \sigma^{-5/2}\|U\|^2$. Note that $U_l^{kK} = 0$ on M_0 and $U_l^{nK} = 0$ on M_σ. Then by typical integration by parts method and by Lemma 5.4 one can show that

$$|E(U', U'')| \le CE(\sigma, U). \tag{5.15}$$

Combining (5.13)-(5.15) we conclude that

$$\begin{aligned}\|AU\|^2 + \|BU\|^2 \ge c\left(L(U') + \int_{M_0} |U'|^2 ds\right) \\ + \|AU''\|^2 + \|BU''\|^2 - CE(\sigma, U).\end{aligned} \tag{5.16}$$

Let us write $U'' = \widetilde{U} + \widetilde{\widetilde{U}}$, where

$$\widetilde{U} = \sum_{l=1}^{n} \sum_{\substack{|J|=2 \\ 1\in J \\ n\notin J}} U_l^J \omega^J \cdot L_l, \quad \text{and} \quad \widetilde{\widetilde{U}} = \sum_{l=1}^{n} \sum_{\substack{|J|=2 \\ 1,n\notin J}} U_l^J \overline{\omega}^J \cdot L_l.$$

Then we can write

(5.17)

$$\|AU''\|^2 + \|BU''\|^2 = \|A\widetilde{U}\|^2 + \|B\widetilde{U}\|^2 + \|A\widetilde{\widetilde{U}}\|^2 + \|B\widetilde{\widetilde{U}}\|^2 + E(\widetilde{U}, \widetilde{\widetilde{U}}),$$

where $E(\widetilde{U}, \widetilde{\widetilde{U}})$ denotes the (sum of) inner products $(A\widetilde{U}, A\widetilde{\widetilde{U}})$ and $(B\widetilde{U}, B\widetilde{\widetilde{U}})$. A typical term of $E(\widetilde{U}, \widetilde{\widetilde{U}})$ looks like:

$$(L_1 U_l^{1k}, L_j U_l^{jk}) - (\overline{L}_j U_l^{1k}, \overline{L}_1 U_l^{jk}),\ 2 \le j, k \le n-1.$$

If we perform integration by parts again, all the terms, except the term $(c_{1j}^n L_n U_l^{1k}, U_l^{jk})$, are bounded by $E(\sigma, U'')$. Observe that

$$\begin{aligned}(c_{1j}^n L_n U_l^{1k}, U_l^{jk}) = -(c_{1j}^n U_l^{1k}, \overline{L}_n U_l^{jk}) - ((L_n c_{1j}^n) U_l^{1k}, U_l^{jk}) \\ - \int_{M_\sigma} c_{1j}^n U_l^{1k} \overline{U}_l^{jk} ds.\end{aligned}$$

Therefore one obtains that

(5.18)
$$|E(\widetilde{U}, \widetilde{\widetilde{U}})| \le CE(\sigma, U'') + C \left(\sum_{j,k=2}^{n-1} \int_{M_\sigma} (\sigma^{\frac{1}{2}} |U_l^{1k}|^2 + \sigma^{-\frac{1}{2}} |c_{1j}^n|^2 |U_l^{jk}|^2) ds \right).$$

Let us estimate $\|A\widetilde{U}\|^2 + \|B\widetilde{U}\|^2$. Note that $n \ge 4$, and the Levi-form of M_σ has at least $(n-2)$-positive eigenvalues (and hence at least two positive eigenvalues). Hence for each fixed J with $1 \in J$, $n \notin J$, there is at least one k, $2 \le k \le n-1$, such that $k \notin J$. This is the property for the estimates for $\widetilde{U}$ in [**10**], and we get

$$\|A\widetilde{U}\|^2 + \|B\widetilde{U}\|^2 \ge c \left(L(\widetilde{U}) + \int_{M_\sigma} |\widetilde{U}|^2 ds \right). \tag{5.19}$$

Combining (5.16)-(5.19) we conclude that

$$\begin{aligned}\|AU\|^2 + \|BU\|^2 \ge c \left(L(U') + L(\widetilde{U}) + \int_{M_0} |U'|^2 ds + \int_{M_\sigma} |\widetilde{U}|^2 ds \right) \\ - C\sigma^{-\frac{1}{2}} \sum_{j,k=2}^{n-1} \int_{M_\sigma} |c_{1j}^n|^2 |U_l^{jk}|^2 ds \\ + \|A\widetilde{\widetilde{U}}\|^2 + \|B\widetilde{\widetilde{U}}\|^2 - CE(\sigma, U'').\end{aligned} \tag{5.20}$$

Finally let us estimate $\|A\widetilde{\widetilde{U}}\|^2 + \|B\widetilde{\widetilde{U}}\|^2$. In this case we can not get the boundary integral term $\int_{M_\sigma} |\widetilde{\widetilde{U}}|^2 ds$ which is necessary for the 1/2-subelliptic estimates for the D_2 equation. Therefore we need to use the family of plurisubharmonic functions with large Hessian near M_σ constructed in Theorem 4.6.

If $n \ge 5$ then Catlin's theorem [**5**, Theorem 7.1] shows that 1/2 subelliptic estimates hold because there are at least three positive eigenvalues.

Therefore let us assume that $n=4$. Then we can write

$$\widetilde{\widetilde{U}} = \sum_{l=1}^{n} U_l^{23}\overline{\omega}^{23}\cdot L_l,$$

and hence
(5.21)

$$\|A\widetilde{\widetilde{U}}\|^2 + \|B\widetilde{\widetilde{U}}\|^2 = \sum_{l=1}^{n}\left[\|\overline{L}_1 U_l^{23}\|^2 + \|\overline{L}_n U_l^{23}\|^2 + \|L_2 U_l^{23}\|^2 + \|L_3 U_l^{23}\|^2\right].$$

By integration by parts, we get

$$\frac{1}{3}\|\overline{L}_1 U_l^{23}\|^2 = \frac{1}{3}\|L_1 U_l^{23}\|^2 + \frac{1}{3}\int_{M_\sigma} c_{11}^n |U_l^{23}|^2 ds + \mathcal{O}(E(\sigma,\widetilde{\widetilde{U}})). \tag{5.22}$$

Combining (5.20)-(5.22) one obtains

$$\begin{aligned}\|AU\|^2 + \|BU\|^2 \geq c&\left(L(U) + \int_{M_0}|U'|^2 ds + \int_{M_\sigma}|\widetilde{U}|^2 ds\right)\\ &+ \frac{1}{3}\|L_1\widetilde{\widetilde{U}}\|^2 + \frac{1}{3}\int_{M_\sigma} c_{11}^n|\widetilde{\widetilde{U}}|^2 ds\\ &- C\sigma^{-\frac{1}{2}}\sum_{j=2}^{3}\int_{M_\sigma}|c_{1j}^n|^2|\widetilde{\widetilde{U}}|^2 ds - C\sigma^{-\frac{5}{2}}\|U\|^2,\end{aligned} \tag{5.23}$$

provided σ is sufficiently small.

Let $\lambda \in C^\infty(\widetilde{W}(x_0))$ with $|\lambda|\leq 1$ and for $f\in C^\infty(W(x_0))$ we define

$$\|f\|_\lambda^2 = \int_{W(x_0)} |f|^2 e^{-\lambda} dV.$$

Then $\frac{1}{3}\|L_1 U_l^{23}\|^2 \geq \frac{1}{9}\|L_1 U_l^{23}\|_\lambda^2$ because $e^{-\lambda}\geq 1/3$. Let us estimate $\|L_1 U_l^{23}\|_\lambda^2$.

With the notation $\delta_1 = e^\lambda L_1 e^{-\lambda}$, we can write
(5.24)

$$(\delta_1 U_l^{23}, \delta_1 U_l^{23})_\lambda = \|L_1 U_l^{23}\|_\lambda^2 + \mathcal{O}(\sigma^{-1/4}\|(L_1\lambda)U_l^{23}\|^2) + \mathcal{O}(E(\sigma,\widetilde{\widetilde{U}})).$$

Since we have the following commutation relation

$$[\delta_1, \overline{L}_1] = \overline{L}_1 L_1\lambda + [L_1,\overline{L}_1] - (L_1\lambda)\overline{L}_1,$$

we obtain from (5.24) that

$$\begin{aligned}\|L_1 U_l^{23}\|^2 = &\|\overline{L}_1 U_l^{23}\|_\lambda^2 + ((\overline{L}_1 L_1\lambda)U_l^{23}, U_l^{23})_\lambda\\ &+ \left(\sum_{i=1}^{n} c_{11}^i L_i U_l^{23}, U_l^{23}\right)_\lambda + \left(\sum_{i=1}^{n} d_{11}^i \overline{L}_i U_l^{23}, U_l^{23}\right)_\lambda\\ &+ \mathcal{O}(\sigma^{-1/4}\|(L_1\lambda)U_l^{23}\|_\lambda^2) + \mathcal{O}(E(\sigma,\widetilde{\widetilde{U}})).\end{aligned} \tag{5.25}$$

Note that the terms $(d_{11}^n \overline{L}_n U_l^{23}, U_l^{23})_\lambda$ and $(\sum_{i=1}^{n-1} c_{11}^i L_i U_l^{23}, U_l^{23})_\lambda$ are dominated by the error term $CE(\sigma, \widetilde{\widetilde{U}})$. By integration by parts one obtains

$$(5.26)\qquad \begin{aligned}(c_{11}^n L_n U_l^{23}, U_l^{23})_\lambda &= (c_{11}^n (L_n\lambda) U_l^{23}, U_l^{23})_\lambda \\ &\quad - \int_{M_\sigma} c_{11}^n |U_l^{23}|^2 e^{-\lambda} ds + \mathcal{O}(E(\sigma, \widetilde{\widetilde{U}})),\end{aligned}$$

and for $1 \le i \le n-1$,

$$(5.26')\quad \left(\sum_{i=1}^{n-1} d_{11}^i \overline{L}_i U_l^{23}, U_l^{23}\right)_\lambda = \left(\sum_{i=1}^{n-1} d_{11}^i (\overline{L}_i\lambda) U_l^{23}, U_l^{23}\right)_\lambda + \mathcal{O}(E(\sigma, \widetilde{\widetilde{U}})).$$

Combining (5.23)-(5.26) we conclude that

(5.27)

$$\begin{aligned}\|AU\|^2 + \|BU\|^2 &\ge c\left(L(U) + \int_{M_0} |U'|^2 ds + \int_{M_\sigma} |\widetilde{U}|^2 ds\right) \\ &\quad + \frac{1}{9}((\overline{L}_1 L_1 \lambda)\widetilde{\widetilde{U}}, \widetilde{\widetilde{U}})_\lambda + \frac{1}{9}\left(\sum_{i=1}^{n-1} d_{11}^i (L_i\lambda)\widetilde{\widetilde{U}}, \widetilde{\widetilde{U}}\right) \\ &\quad + \int_{M_\sigma}\left(\frac{2}{9}c_{11}^n - C\sigma^{-\frac{1}{2}}\sum_{j=2}^{3}|c_{1j}^n|^2\right)\left|\widetilde{\widetilde{U}}\right|^2 ds - C\sigma^{-\frac{5}{2}}\|U\|^2 \\ &\quad + \left(c_{11}^n(\overline{L}_n\lambda)\widetilde{\widetilde{U}}, \widetilde{\widetilde{U}}\right)_\lambda - C_1\sigma^{-\frac{1}{4}}\|(L_1\lambda)\widetilde{\widetilde{U}}\|_\lambda^2.\end{aligned}$$

By virtue of the estimates in (4.18) and from the fact that $\varepsilon G(x') \ll \sigma$, the boundary integral terms in (5.27) satisfy

$$(5.28)\quad \frac{2}{9}\int_{M_\sigma} c_{11}^n |\widetilde{\widetilde{U}}|^2 ds - C\sigma^{-\frac{1}{2}}\sum_{j=2}^{3}\int_{M_\sigma}|c_{1j}^n|^2|\widetilde{\widetilde{U}}|^2 ds \ge \frac{1}{9}\int_{M_\sigma} c_{11}^n |\widetilde{\widetilde{U}}|^2 ds.$$

With the notation

$$\partial\overline{\partial}\lambda = \sum_{j,k=1}^{n} \lambda_{jk}\omega^j \wedge \overline{\omega}^k,$$

the second line of (5.27) is equal to $(\lambda_{11} U_l^{23}, U_l^{23})_\lambda$. Now suppose that $|\lambda| \le 1$ on $\widetilde{W}(x_0)$. Let C_1 be the constant appeared in the last line of (5.27) and let $\chi(t)$ denotes the function $(\sigma^{1/4}/3C_1)e^t$ and set $\phi = \chi(\lambda)$. Then

$\chi''(t) \geq C_1\sigma^{-\frac{1}{4}}\chi'(t)^2$ and we get

$$\phi_{11}U_l^{23}t_1\bar{t}_1 = \chi'(t)\lambda_{11}t_1\bar{t}_1 + \chi''(t)|(L_1\lambda)t_1|^2 \tag{5.29}$$
$$\geq \frac{\sigma^{\frac{1}{4}}}{9C_1}\lambda_{11}t_1\bar{t}_1 + C_1\sigma^{-\frac{1}{4}}\chi'(t)^2|(L_1\lambda)t_1|^2.$$

Thus if we replace λ by ϕ in (5.27), we obtain, from (5.28) and (5.29) that

(5.30)

$$\|AU\|^2 + \|BU\|^2 \geq c\left(L(U) + \int_{M_0}|U'|^2 ds + \int_{M_\sigma}|\widetilde{U}|^2 ds\right)$$
$$+ c\sigma^{\frac{1}{4}}(\lambda_{11}\widetilde{\widetilde{U}}, \widetilde{\widetilde{U}}) + c(c_{11}^n(\overline{L}_n\lambda)\widetilde{\widetilde{U}}, \widetilde{\widetilde{U}})_\lambda - C\sigma^{-5/2}\|\widetilde{\widetilde{U}}\|^2.$$

Now we take the family $\{\lambda_\rho\}_{\rho>0}$ of plurisubharmonic functions with large Hessian constructed in Theorem 4.6 and replace λ in (5.27) by these functions. By Lemma 5.5, and by the fact that $\tau(x,\delta) \approx \mu(x,\delta)$, we have the following pointwise estimates at the point $y = D_{x_0}(x) \in S_\rho$:

$$|c_{11}^n(y)(\overline{L}_n\lambda_\rho)(y)|\widetilde{\widetilde{U}}(y)|^2| \lesssim \sigma^{\frac{1}{2}}\mu(x,\delta_0)^2\mu(x,\rho\delta_0)^{-2}|\widetilde{\widetilde{U}}(y)|^2. \tag{5.31}$$

Also it follows from (4.20) that

$$\lambda_{11}(y) \gtrsim \tau(x,\delta_0)^2\tau(x,\rho\delta_0)^{-2}. \tag{5.32}$$

Combining (5.30)-(5.32) and the fact that $\tau(x,\delta_0)^2\tau(x,\rho\delta_0)^{-2} \gtrsim \rho^{-2/m}$ (by (3.7)) it follows that, for each $0 < \rho \leq \sigma^{2m}/2$, there is $\lambda = \lambda_\rho$ so that

$$\|A\widetilde{\widetilde{U}}\|^2 + \|B\widetilde{\widetilde{U}}\|^2 \gtrsim \sigma^{\frac{1}{4}}(\lambda_{11}\widetilde{\widetilde{U}}, \widetilde{\widetilde{U}}) + L(\widetilde{\widetilde{U}}) \gtrsim \sigma^{1/4}\rho^{-\frac{2}{m}}\|\widetilde{\widetilde{U}}\|^2 + L(\widetilde{\widetilde{U}}).$$

Then by the theorem of Catlin [**2**] the subelliptic estimates of order $1/m$ hold for the components $\widetilde{\widetilde{U}}$, and hence we get

$$\sigma^{\frac{1}{4}}|||\widetilde{\widetilde{U}}|||^2_{\frac{1}{m}} + L(\widetilde{\widetilde{U}}) \leq C(\|A\widetilde{\widetilde{U}}\|^2 + \|B\widetilde{\widetilde{U}}\|^2). \tag{5.33}$$

Recall that the estimates in (5.14) and (5.19) give us 1/2-subelliptic estimates for the components U' and $\widetilde{U}$. Hence we have, from (5.12), (5.30) and (5.33), that

$$\sigma^{\frac{1}{4}}|||U|||^2_{\frac{1}{m}} + L(U) \leq C(\|SU\|^2 + \|T^*U\|^2) + C\sigma^{-5/2}\|U\|^2.$$

By (5.3) we also have that $\sigma^{-4}\|U\|^2 \leq C(\|U\|^2_{\frac{1}{m}} + \|\widetilde{L}U\|^2)$, and hence we conclude that

$$\sigma^{-3}\|U\|^2 + L(U) + \sigma^{\frac{1}{4}}|||U|||^2_{\frac{1}{m}} \leq C(\|SU\|^2 + \|T^*U\|^2), \tag{5.34}$$

for all $U \in \mathcal{B}^q(S_{\varepsilon,\sigma}; A)$ provided σ is sufficiently small.

For the estimates of the non–tangential derivatives of U, we note that $L_n^A = \frac{\partial}{\partial y_{2n}} + X$, where $X = \sum_{j=1}^{2n-1} b_j(y)\frac{\partial}{\partial y_j}$. Therefore a standard argument yields the inequality

(5.35)
$$\left|\left|\left|\frac{\partial f}{\partial y_{2n}}\right|\right|\right|^2_{-1+\frac{1}{m}} \le C\left(1 + \sum_{j=1}^{2n-1} |b_j|^2_{\widetilde{W}(x_0),2n+3}\right)(|||f|||^2_{\frac{1}{m}} + \|\overline{L}_n f\|^2 + \|f\|^2),$$

for all $f \in C_0^\infty(\widetilde{W}(x_0))$. This inequality can be applied with $f = U_l^J$ and one obtains (5.1) from (5.35). This completes the proof of Theorem 5.1.

We now define Sobolev spaces for sections of $\Gamma^{0,q}(S_{\varepsilon,\sigma}; A)$. Recall that the open sets $B_b(x_0)$ satisfy (4.3) and (4.4) for each $x_0 \in M$. Choose a set $T_\sigma = \{x_i^\sigma \in M, i \in I\}$ such that the sets $B_{c\sigma/2}(x_i^\sigma)$, $i \in I$, cover $S_{\varepsilon,\sigma}$, and such that no two points x_i^σ and x_j^σ satisfy $|x_i^\sigma - x_j^\sigma| \le c\sigma/4$ where $|\quad|$ is the distance function on $S_{\varepsilon,\sigma}$. It follows that the sets $W(x_i^\sigma)$, $i \in I$, cover $S_{\varepsilon,\sigma}$ and that there exists an integer $\widetilde{N}$ such that no point of $S_{\varepsilon,\sigma}$ lies in more than $\widetilde{N}$ of the open sets $W(x_i^\sigma)$. Furthermore, there exist functions ζ_i, ζ_i' (that are independent of y_{2n}) $\in C_0^\infty(W(x_i^\sigma))$ such that $\sum\limits_{i\in I} \zeta_i^2 \equiv 1$, such that if $x \in \operatorname{supp}\zeta_i$, then $\zeta_i' \equiv 1$ in $B_{c'\sigma}(x)$, and such that both ζ_i and ζ_i' satisfy

$$|\zeta_i|_{k,W(x_i^\sigma)} + |\zeta_i'|_{k,W(x_i^\sigma)} \le C_k\sigma^{-k}. \tag{5.36}$$

Now let F be any section of $\Gamma^{0,q}(S_{\varepsilon,\sigma}; A)$. We define

$$\|F\|^2_{k,A} = \sum_{i\in I} \|\zeta_i F\|^2_{k,A,W(x_i^\sigma)},$$

where

$$\|\zeta_i F\|^2_{k,A,W(x_i^\sigma)} = \sum_{l=1}^{n}\sum_{|J|=q} \|\zeta_i F_l^J\|^2_{k,W(x_i^\sigma)},$$

and where $F = \sum\limits_{l=1}^{n}\sum\limits_{|J|=q} F_l^J \overline{\omega}_A^J \cdot L_j^A$ is the decomposition of F in terms of the frame of $W(x_i^\sigma)$. Moreover, the Sobolev norm $\|\quad\|_{k,W(x_i^\sigma)}$ is taken with respect to the y–coordinates of $W(x_i^\sigma)$. We define $H_k^{0,q}(S_{\varepsilon,\sigma}; T_A^{1,0})$ to be the set of all sections F of $\Gamma^{0,q}(S_{\varepsilon,\sigma}; A)$ for which $\|F\|_{k,A} < \infty$.

We want to get an estimate in global form. Define $Q(U,U) = \|T^*U\|^2 + \|SU\|^2$. From the estimates in Theorem 5.1 and by using the partition of unity as defined above, we obtain:

Corollary 5.6. *Suppose that A satisfies (4.11) for all $x_0 \in \overline{M}$. Then there exist a fixed small σ and a constant $\varepsilon_1 > 0$ such that for all $\varepsilon, 0 < \varepsilon < \varepsilon_1$, and all $U \in \operatorname{Dom}(T^*) \cap \operatorname{Dom}(S)$,*

$$\|U\|^2 \le CQ(U,U).$$

Now let us fix $\sigma > 0$ satisfying Corollary 5.6 and set $W(x_0) = W_\sigma(x_0)$. Using Theorem 5.1 and the standard "bootstrap" method, we can get regularity estimates for the linearized equation. The proof is similar to that in Section 9 of [**5**]. Here we use $1/m$ subelliptic estimates instead of $1/2$ subellitic estimates. Set $\Box = D_q D_q^* + D_{q+1}^* D_{q+1}$. Assume that A satisfies

$$\|A\|_{2n+3} \leq \varepsilon_0. \tag{5.37}$$

Theorem 5.7. *Suppose that* (5.37) *holds and that* U *is the solution of* $\Box U = G$ *where* $G \in H_k^{0,q}(S_\varepsilon; T_A^{1,0})$ *for all* $k > 0$. *Then*

$$\|D_q^* U\|_k + \|D_{q+1} U\|_k \lesssim \|G\|_k + (1 + \|A\|_{k+2})\|G\|_{n+2}.$$

Now set $E = D_{q+1}^* D_{q+1} U$. Then we have the following estimates for the error term E [**5**, Theorem 10.3].

Theorem 5.8. *Suppose that* A *satisfies* (5.37) *and that* $\Box U = G$, *where* $D_{q+1} G = 0$ *and* $G \in H_k^{0,q}(S_\varepsilon; T_A^{1,0})$ *for all* k. *Then* $E = D_{q+1}^* D_{q+1} U$ *satisfies*

$$\|E\|_{k-1} \lesssim \|G\|_k \|F^A\|_{n+1} + \|G\|_{n+1} \|F^A\|_k + (1 + \|A\|_{k+2})\|G\|_{n+1}\|F^A\|_{n+1}.$$

Note that F^A is D_3-closed. Since $q = 2$, we immediately obtain:

Corollary 5.9. *If* A *satisfies* (5.37) *and if* U *is the solution with respect to* $\mathcal{L}^A$ *of* $\Box U = F^A$, *then* $V' = D_2^* U$ *satisfies for all* $k = n+1, n+2, \ldots$

$$\|D_2 V' - F^A\|_k \lesssim \|F^A\|_{k+1} \|F^A\|_{n+1} + \|A\|_{k+3} \|F^A\|_{n+1}^2.$$

6. Embedding of CR structures.

In this section we will prove Theorem 1.1 using the estimates in Section 5. First, we describe the nonlinear extension operator. For the details, one can refer Section 11 of [**5**].

If $A \in \mathcal{A}(S_{\varepsilon,\sigma})$ is sufficiently small and if we set $P_A(\overline{L}) = \overline{L} + A(\overline{L})$, then $\overline{\mathcal{L}}_A = \{P_A(\overline{L}); \overline{L} \in \overline{\mathcal{L}}\}$. If we set $Q_A(\omega) = \omega - A^*\omega$, then $\Lambda_A^{1,0} = \{Q_A(\omega); \omega \in \Lambda^{1,0}(\mathcal{L})\}$. We define a nonlinear operator $\Phi : \mathcal{A}(S_{\varepsilon,\sigma}) \to \Gamma^{0,2}(S_{\varepsilon,\sigma})$ by:

$$\Phi(A)(\overline{L}', \overline{L}'', \omega) = Q_A(\omega)([P_A(\overline{L}'), P_A(\overline{L}'')]). \tag{6.1}$$

Obviously, if $\Phi(A) = 0$, then $\mathcal{L}_A$ is an integrable almost complex structure on $S_{\varepsilon,\sigma}$. Note that there is a natural map $\mathcal{P}_A : \Gamma_A^{0,2} \to \Gamma^{0,2}$, defined by:

$$(\mathcal{P}_A B)(\overline{L}_1, \overline{L}_2, \omega) = B(P_A(\overline{L}_1), P_A(\overline{L}_2), Q_A(\omega)), \quad B \in \Gamma_A^{0,2}.$$

Therefore it follows from the definition of F^A in (2.4) that $\Phi(A) = \mathcal{P}_A(F^A)$.

If we use the error estimates in Theorem 5.8 and Corollary 5.9, we then obtain the following good estimates for the approximate solution of $\Phi(A) + \Phi'(A)(d) = 0$.

Theorem 6.1. *Suppose that $A \in H_k(S_{\varepsilon,\sigma}, \mathcal{A})$ for all k and that A satisfies* (5.37)*. Then there exists $d_A \in H_k(S_{\varepsilon,\sigma}, \mathcal{A})$ for all k so that if $k \geq n+2$,*

$$\|d_A\|_k \lesssim \|\Phi(A)\|_k + \|A\|_{k+2}\|\Phi(A)\|_{n+2}, \quad \textit{and} \tag{6.2}$$

$$\|\Phi(A) + \Phi'(A)(d_A)\|_{k-1} \lesssim \|\Phi(A)\|_k\|\Phi(A)\|_{n+2} + \|A\|_{k+2}\|\Phi(A)\|^2_{n+2}. \tag{6.3}$$

Note that the properties (6.2) and (6.3) of the nonlinear operator Φ are the crucial ingrediants in the application of simplified Nash-Moser iteration process [**16**]. We now can prove the main theorem of this paper:

Proof of Theorem 1.1*.* We will show that $\|\Phi(0)\|_D < b$ for the small $b > 0$ and the integer D which are appeared in the variant of Nash-Moser theorem [**16**]. The rest properties for the $\Phi(A)$ in the hypothesis of Nash-Moser theorem can be proved using the relations in (6.2) and (6.3), and the estimates for $\Box$ operator in Section 5.

We recall that F^A vanishes in infinite order along M_0 (in x-coordinates!). This can be stated in y-coordinates, and hence it follows that for each $i \in I$,

$$\|\zeta_i F^0\|^2_{k,0} \leq C_{k,N}\varepsilon^N \varphi(x_i^\sigma)^N,$$

where ζ_i's are defined before (5.36). After summing up over x_i^σ, we get

$$\|F^0\|^2_{k,0,\Phi} \leq C_{k,N} \sum_{i\in I} \varphi(x_i^\sigma)^N \varepsilon^N. \tag{6.4}$$

Since the choice of the points that was made before (5.36) shows that the balls $B_{\frac{c\sigma}{8}}(x_i^\sigma)$, $i \in I$, are all disjoint, we can obtain an upper bound on $N(\ell)$, which is defined to be the number of $i \in I$ such that $2^{-\ell-1} \leq \varphi(x_i^\sigma) < 2^{-\ell}$. In fact, in terms of the $\langle\ ,\ \rangle_0$–metric introduced at the end of Section 2, the volume of $B_{\frac{c\sigma}{8}}(x_i^\sigma)$ is roughly bounded below by $\varepsilon^{n+1}\sigma^{2n-1+2m}\varphi(x_i^\sigma)^{2m(n+2)} \sim \varepsilon^{n+1}\sigma^{2n-1+2m} \cdot 2^{-2\ell m(n+2)}$, and the $\langle\ ,\ \rangle_0$–volume of the region in $S_{\varepsilon,\sigma}$ with $2^{-\ell-1} \leq \varphi(x) \leq 2^{-\ell}$ is roughly bounded above by $\varepsilon\sigma^{2m} \cdot 2^{-2m\ell}$. Thus, we conclude that

$$N(\ell) \lesssim \varepsilon^{-n}\sigma^{-(2n-1)}2^{2m\ell(n+1)}. \tag{6.5}$$

Thus (6.4) and (6.5) imply that if $N = 2m\ell(n+1)+1$, then

$$\|\Phi(A_0)\|_k = \|F^0\|_{k,0} \lesssim C_k \cdot \varepsilon$$

for sufficiently small ε. In particular, if we set $k = D$, and choose ε to be sufficiently small, then it follow that $\|\Phi(A)\|_D < b$. $\Box$

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