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DISCRETE LOGARITHMIC ENERGY ON THE SPHERE

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In this article we consider the problem posed by Whyte, about the distribution of N point charges on the unit sphere, whose mutual distances have maximal geometric mean. Some properties of the extremal points are discussed. In the case when $N = 5$ the optimal configuration is established rigorously, which solves an open problem communicated by Rakhmanov.

1. Introduction.

In 1952 L.L. Whyte [25] posed the question of distributing N points on the sphere so that the product of their mutual distances is as large as possible. Such points are sometimes referred to as *logarithmic points* because they minimize the *discrete logarithmic energy*

$$E_0(\omega_N) := \sum_{i<j} \log \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{1.1}$$

among all point configurations $\omega_N := \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ on the unit sphere $S^2 := \{\mathbf{x} \in \mathbf{R}^3 : |\mathbf{x}| = 1\}$. They are also known as *elliptic Fekete points* (see [19]).

The Whyte's problem stems from a more general minimal energy problem. Given $\alpha > 0$, the α-*energy* of $\omega_N \subset S^2$ is defined by

$$E_\alpha(\omega_N) := \sum_{i<j} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|^\alpha}. \tag{1.2}$$

The *minimal α-energy problem* is to find the global minimum of (1.2) and configuration(s) ω_N^* which realize it. Such configurations are called α-*extremal*. For the special case of logarithmic points (or $\alpha = 0$) we shall use in this paper the term *optimal N-configurations*.

When $\alpha = 1$ the problem is called *J.J. Thomson's problem* and dates back to the beginning of twentieth century, when Föppl [8], under the suggestion of Hilbert, investigated the so-called Thomson arrangements. He observed that the point charges tend to distribute in sequences of rings. However, Föppl considered only configurations that shared high symmetry properties. In [5] the authors claim that the Föppl's arrangements are optimal for the special values of $N = 1 - 6, 12$, which they derive from Leech's paper [14].

The latter paper though only implies that the configurations are local extrema or saddle points. We couldn't find a reference which rigorously solves the Thomson problem for $N = 5$. For $N = 6$ the problem has been resolved in [**26**] (the solution is the vertices of regular octahedron). In [**1**] the author obtains that the vertices of the regular icosahedron solve the Whyte's problem ($\alpha = 0$), and claims that the methods he uses can be applied to solve the Thompson problem for $N = 12$. For other values of N, the problem has been constantly attacked with numerical methods (see [**5**], [**9**], [**15**], [**16**], [**17**]), with better configurations appearing in the literature occasionally.

Another distinguished problem is the *Tammes problem*, which occurs when $\alpha \to \infty$. Since now the main contribution in (1.2) comes from the smallest mutual distance, the problem reduces to maximizing the smallest distance in the configuration. This problem is also referred to as *the best packing problem* on the sphere, namely how to pack N congruent spherical caps on the sphere, so that their radius is as large as possible. The problem has been solved for $N = 1 - 12$ and $N = 24$ (see [**10**], [**7**], [**18**]).

The other limiting case, when $\alpha \to 0$, leads to the Whyte's problem, stated at the beginning. It is easy to solve the problem for $N = 2 - 4$. We cited earlier the result of Andreev that the regular icosahedron is a solution of the Whyte's problem for $N = 12$. For $N = 5 - 7$ the problem has been communicated by Rakhmanov in [**4**] as an open problem. In [**11**] the authors obtained the solution of the Whyte's problem for $N = 6$, using heavily the fact that the vertices of the octahedron form a spherical design. They also noted that the solution is unknown for $N = 5$. It this paper we derive the following

Theorem 1. *The five-point configuration that maximizes the geometric mean of the mutual distances is unique up to rotations, and has two antipodal points, say in the North and South Pole, and three that form an equilateral triangle on the Equator.*

Using Föppl notation, we can denote the optimal configuration as $\{1, 3, 1\}$, where the 1's are at the poles and the 3 stands for regular 3-gon. Two other configurations will be used in the paper, $\{5\}$ for the regular pentagon on a great circle, and $\{1, 4\}$ for the configuration with one point at the North Pole and a square inscribed in $\{z = -1/4\} \cap S^2$.

We should also point out that similar problems are considered when $\alpha < 0$. Then one asks for the maximum in (1.2). When $\alpha = -1$ this is, except for some small values of N, a long standing open problem in discrete geometry. We refer the reader to [**6**], [**20**], [**21**], [**11**] for more on this problem. For generalizations of the α-energy problems to higher dimension see [**13**], [**3**].

Another important trend of investigation is asymptotical results concerning the minimal energy and the optimal configurations as $N \to \infty$. We refer the reader for details to [**16**], [**17**], [**12**], [**13**], [**22**], [**23**], [**24**].

In Section 2 we consider some properties of the optimal configurations and the Proof of Theorem 1 is presented in Section 3.

2. Some properties of optimal configurations.

In the next proposition we list some properties of the optimal configurations ($\alpha = 0$) in the d-dimensional case. These properties are essentially found in [**2**] for $d = 3$. Although the generalization is not difficult, we choose to include a proof to make things self-contained.

Proposition 2. *Let $X_1, X_2, \ldots, X_N$ be an optimal configuration on the unit sphere S^{d-1} in $\mathbf{R}^d$. Then the center of mass of the configuration coincides with the center of the sphere O and the following force conditions hold:*

$$\sum_{j\neq i} \frac{\overrightarrow{X_j X_i}}{|X_j - X_i|^2} = f_i \overrightarrow{OX_i} \quad i = 1, \ldots, N, \tag{2.1}$$

where $f_1 = \cdots = f_N = (N-1)/2$. Moreover,

$$\sum_{j\neq i} |X_j - X_i|^2 = 2N \quad i = 1, \ldots, N. \tag{2.2}$$

Proof. First, we shall prove (2.1). Without loss of generality we can assume that $i = 1$ and $X_1 = (0, \ldots, 0, 1)$. Let $X_j = (x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(d)})$, $j = 2, \ldots, N$ be fixed, and let $X = (x^{(1)}, x^{(2)}, \ldots, x^{(d)})$ vary on S^{d-1}. The function

$$f(X) = \sum_{j=2}^{N} \frac{1}{2} \log \frac{1}{|X - X_j|^2} + \sum_{2\leq l<j\leq N} \log \frac{1}{|X_l - X_j|}$$

has a minimum at $X_1 = (0, \ldots, 0, 1)$, therefore its partial derivatives $f_{x^{(k)}}$, $k = 1, \ldots, d-1$, vanish at this point. In a neighborhood of $(0, \ldots, 0, 1)$ we can write $x^{(d)} = x^{(d)}(x^{(1)}, \ldots, x^{(d-1)})$ and from $|X|^2 = 1$ we have $\partial x^{(d)}/\partial x^{(k)} = -x^{(k)}/x^{(d)}$. Thus,

$$\begin{aligned} f_{x^{(k)}}(0, \ldots, 0, 1) &= -\sum_{j=2}^{N} \frac{(x^{(k)} - x_j^{(k)}) + (x^{(d)} - x_j^{(d)})\partial x^{(d)}/\partial x^{(k)}}{|X - X_j|^2}\Bigg|_{(0,\ldots,0,1)} \\ &= -\sum_{j=2}^{N} \frac{-x_j^{(k)}}{|X_1 - X_j|^2} = 0 \quad k = 1, \ldots, d-1. \end{aligned} \tag{2.3}$$

This implies (2.1) with some f_1. To show that $f_1 = (N-1)/2$ we multiply (2.1) for $i = 1$ with $\overrightarrow{OX_1}$ and use the fact that

$$\begin{aligned}\overrightarrow{X_jX_1} \cdot \overrightarrow{OX_1} &= (\overrightarrow{OX_1} - \overrightarrow{OX_j}) \cdot \overrightarrow{OX_1} \\ &= 1 - \overrightarrow{OX_j} \cdot \overrightarrow{OX_1} = (1/2)|\overrightarrow{OX_j} - \overrightarrow{OX_1}|^2.\end{aligned}$$

Since the choice $i = 1$ was arbitrary, we conclude that (2.1) is true for any i.

Next, we derive that the center of mass is O. To do so we sum Equations (2.1) for $i = 1, \ldots, N$. The left-hand side is obviously zero, and since all f_i's are equal, the right-hand side will be $((N-1)/2)\sum \overrightarrow{OX_i}$, which implies that O is the center of mass of the system of points.

Finally, we show that (2.2) holds true whenever O is the center of mass of the point configuration $\{X_i\}$ on the unit sphere. Indeed, for fixed i we get

$$\begin{aligned}\sum_{j\neq i}(\overrightarrow{OX_j} - \overrightarrow{OX_i})^2 &= \sum_{j\neq i}(2 - 2\overrightarrow{OX_j} \cdot \overrightarrow{OX_i}) \\ &= 2(N-1) - 2\overrightarrow{OX_i} \cdot \sum_{j\neq i} \overrightarrow{OX_j} = 2N.\end{aligned}$$

□

Remark. It is certainly a remarkable fact that in the case of logarithmic interaction, if the charges are in equilibrium, then their center of mass coincides with the center of the sphere. Another important consequence is formula (2.2), which explains the phenomenon that when in equilibrium, the charges tend to arrange so that there are fewer different distances. For example, this formula enables us to derive an elementary proof of the fact that the regular simplex inscribed in the unit sphere S^{d-1} is the only α-extremal configuration with $d+1$ points for any $\alpha \geq 0$ (compare with [**11**]).

Corollary 3. *The only α-extremal configuration with $d+1$ points on S^{d-1} for $\alpha \geq 0$ is the regular $d+1$-simplex inscribed in S^{d-1}.*

Proof. First, we consider the case $\alpha = 0$. Suppose that $X_1, X_2, \ldots, X_{d+1}$ is an optimal configuration on S^{d-1}. Then from the arithmetic-geometric mean and (2.2) we get

$$\begin{aligned}\prod_{i=1}^{d+1}\left(\prod_{j\neq i}|X_i - X_j|^2\right) &\leq \prod_{i=1}^{d+1}\left(\frac{1}{d}\sum_{j\neq i}|X_i - X_j|^2\right)^d \\ &= \left(\frac{2(d+1)}{d}\right)^{d(d+1)}.\end{aligned} \tag{2.4}$$

Since equality holds only when all mutual distances $|X_i - X_j|$ are equal, the corollary easily follows in this case.

In the general case we again use the arithmetic-geometric mean, as well as (2.4), to derive

$$\sum_{1\le i<j\le d+1} \frac{1}{|X_i-X_j|^\alpha} \ge \frac{d(d+1)/2}{\left(\prod_{1\le i<j\le d+1}|X_i-X_j|\right)^\alpha} \ge \frac{d(d+1)/2}{(2(d+1)/d)^{\alpha d(d+1)/4}}, \tag{2.5}$$

with equality holding only for the regular simplex. □

3. Optimal configuration for five points.

In this section we prove Theorem 1. First, we consider only configurations that have antipodal points and show that among these $\{1,3,1\}$ has maximal geometrical mean (Lemma 4 below). Next, in Lemma 5 we show that in an optimal configuration there must be at least two adjacent edges with equal length. Using the symmetrical properties arising from this, we shall prove the theorem at the end of the section.

Lemma 4. *Among all configurations with a pair of antipodal points, the one that maximizes the geometrical mean is the* $\{1,3,1\}$ *configuration.*

Proof. Let D and E be two antipodal points on the sphere. Without loss of generality we may assume that $D=(0,0,1)$ and $E=(0,0,-1)$. Since $\triangle ADE$ is a right triangle inscribed in a circle of radius one, $AD\,.\,AE\le 2$. Similarly, $BD\,.\,BE\le 2$ and $CD\,.\,CE\le 2$. Observe, that equality holds in all three inequalities if and only if $A,B,C\in xy$-plane.

On the other hand, we have that $AB\,.\,AC\,.\,BC = 4\,.\,R\,.\,S_{\triangle ABC}$, where R is the radius of the circumscribed circle and $S_{\triangle ABC}$ is the area of $\triangle ABC$. It is clear that the larger R is, the larger the product is, and for fixed R the maximum of $S_{\triangle ABC}$ occurs only when the triangle is equilateral. Therefore, $AB\,.\,AC\,.\,BC\le 3\,.\sqrt{3}$. Thus, we get that for any configuration with D and E fixed at the poles, the maximum of the product of the ten mutual distances occurs only at the $\{1,3,1\}$ configuration. □

Next, we show that an optimal configuration has at least two equal adjacent edges.

Lemma 5. *If* $\omega=\{A,B,C,D,E\}$ *is an optimal* 5*-configuration which maximizes the geometrical mean of the mutual distances, then there are at least two adjacent edges that have equal length.*

Proof. We first introduce some notation. We shall use lower letter index for the corresponding coordinates of the points, for example $A=(x_A,y_A,z_A)$. Set E to be the North Pole and let A', B', C', and D' be images in the xy-plane of the corresponding points on the sphere under a stereographical projection. Denote with A^*, B^*, C^*, and D^* the projections onto the z-axis of A, B, C, and D, respectively. The length of a vector, say $\overrightarrow{AB}$, we will denote with AB only.

We now focus on how the force conditions (2.1) affect the the vectors $\overrightarrow{OA'}$, $\overrightarrow{OB'}$, $\overrightarrow{OC'}$, and $\overrightarrow{OD'}$. The force condition on E is:

$$\frac{\overrightarrow{EA}}{EA^2} + \frac{\overrightarrow{EB}}{EB^2} + \frac{\overrightarrow{EC}}{EC^2} + \frac{\overrightarrow{ED}}{ED^2} = 2\overrightarrow{EO}. \tag{3.1}$$

We have the representation

$$\begin{aligned} \overrightarrow{EA} = \overrightarrow{EA^*} + \overrightarrow{A^*A} &= (1 - z_A)\overrightarrow{EO} + (1 - z_A)\overrightarrow{OA'} \\ &= EA^2(\overrightarrow{EO} + \overrightarrow{OA'})/2. \end{aligned} \tag{3.2}$$

Similar equations hold for $\overrightarrow{EB}$, $\overrightarrow{EC}$, and $\overrightarrow{ED}$. Substituting these in (3.1) we get

$$\overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'} + \overrightarrow{OD'} = 0. \tag{3.3}$$

Equation (3.3) is an interesting observation, which is true for any N, namely if ω_N is an optimal configuration and we apply a stereographical projection with one of the points being the pole, the center of mass of the images of the other $N-1$ points is at the origin (recall that O is also a center of mass of ω_N).

Next, we consider the center of mass condition. It can be written as

$$\overrightarrow{EA} + \overrightarrow{EB} + \overrightarrow{EC} + \overrightarrow{ED} = 5\overrightarrow{EO}.$$

Using similar manipulation and the fact that $EA^2 + EB^2 + EC^2 + ED^2 = 10$ (see (2.2)) we derive

$$EA^2\overrightarrow{OA'} + EB^2\overrightarrow{OB'} + EC^2\overrightarrow{OC'} + ED^2\overrightarrow{OD'} = 0. \tag{3.4}$$

We now consider the condition at A (the conditions at B, C, and D being similar),

$$\frac{\overrightarrow{AB}}{AB^2} + \frac{\overrightarrow{AC}}{AC^2} + \frac{\overrightarrow{AD}}{AD^2} + \frac{\overrightarrow{AE}}{AE^2} = 2\overrightarrow{AO}.$$

We introduce E in the vectors above to get

$$\frac{\overrightarrow{EB} - \overrightarrow{EA}}{AB^2} + \frac{\overrightarrow{EC} - \overrightarrow{EA}}{AC^2} + \frac{\overrightarrow{ED} - \overrightarrow{EA}}{AD^2} + \frac{-\overrightarrow{EA}}{AE^2} = 2(\overrightarrow{EO} - \overrightarrow{EA}),$$

which after regrouping becomes

$$\overrightarrow{EA}\left(2 - \frac{1}{AB^2} - \frac{1}{AC^2} - \frac{1}{AD^2} - \frac{1}{AE^2}\right) + \frac{\overrightarrow{EB}}{AB^2} + \frac{\overrightarrow{EC}}{AC^2} + \frac{\overrightarrow{ED}}{AD^2} = 2\overrightarrow{EO}.$$

Applying (3.2) and its companions to this equation, and using the orthogonality of $\overrightarrow{EO}$ to $\overrightarrow{OA'}$, $\overrightarrow{OB'}$, $\overrightarrow{OC'}$, $\overrightarrow{OD'}$ we get

$$EA^2\left(2-\frac{1}{AB^2}-\frac{1}{AC^2}-\frac{1}{AD^2}-\frac{1}{AE^2}\right)\overrightarrow{EO} = \left(4-\frac{EB^2}{AB^2}-\frac{EC^2}{AC^2}-\frac{ED^2}{AD^2}\right)\overrightarrow{EO} \tag{3.5}$$

and

$$EA^2\left(2-\frac{1}{AB^2}-\frac{1}{AC^2}-\frac{1}{AD^2}-\frac{1}{AE^2}\right)\overrightarrow{OA'} + \frac{EB^2}{AB^2}\overrightarrow{OB'}+\frac{EC^2}{AC^2}\overrightarrow{OC'}+\frac{ED^2}{AD^2}\overrightarrow{OD'}=0,$$

which in the end provides

$$\left(4-\frac{EB^2}{AB^2}-\frac{EC^2}{AC^2}-\frac{ED^2}{AD^2}\right)\overrightarrow{OA'}+\frac{EB^2}{AB^2}\overrightarrow{OB'}+\frac{EC^2}{AC^2}\overrightarrow{OC'}+\frac{ED^2}{AD^2}\overrightarrow{OD'}=0. \tag{3.6}$$

Interchanging $A \longleftrightarrow B$, $A \longleftrightarrow C$, $A \longleftrightarrow D$ we get

$$\frac{EA^2}{BA^2}\overrightarrow{OA'}+\left(4-\frac{EA^2}{BA^2}-\frac{EC^2}{BC^2}-\frac{ED^2}{BD^2}\right)\overrightarrow{OB'}+\frac{EC^2}{BC^2}\overrightarrow{OC'}+\frac{ED^2}{BD^2}\overrightarrow{OD'}=0, \tag{3.7}$$

$$\frac{EA^2}{CA^2}\overrightarrow{OA'}+\frac{EB^2}{CB^2}\overrightarrow{OB'}+\left(4-\frac{EA^2}{CA^2}-\frac{EB^2}{CB^2}-\frac{ED^2}{CD^2}\right)\overrightarrow{OC'}+\frac{ED^2}{CD^2}\overrightarrow{OD'}=0, \tag{3.8}$$

$$\frac{EA^2}{DA^2}\overrightarrow{OA'}+\frac{EB^2}{DB^2}\overrightarrow{OB'}+\frac{EC^2}{DC^2}\overrightarrow{OC'}+\left(4-\frac{EA^2}{DA^2}-\frac{EB^2}{DB^2}-\frac{EC^2}{DC^2}\right)\overrightarrow{OD'}=0. \tag{3.9}$$

Observe that (3.4) is obtained as a sum of (3.3), (3.6), (3.7), (3.8), and (3.9) (using of course the equality for the coefficients in (3.5) and its companions), so in general we could use only five of these six equations. In addition we have Equations (2.2).

Now suppose that there is an optimal configuration A, B, C, D, and E with no two adjacent edges of equal length. Because of Lemma 4 we need to consider only the case when there is no point at the South Pole, meaning that the vectors $\overrightarrow{OA'}$, $\overrightarrow{OB'}$, $\overrightarrow{OC'}$, and $\overrightarrow{OD'}$ are all nonzero. Then no two of these vectors are colinear. Indeed, suppose for example that $\overrightarrow{OB'} = k\overrightarrow{OA'}$. From (3.3) and (3.4) we get that

$$(EA^2-ED^2)\overrightarrow{OA'}+(EB^2-ED^2)\overrightarrow{OB'}+(EC^2-ED^2)\overrightarrow{OC'}=0, \tag{3.10}$$

which shows that $\overrightarrow{OC'} = l\overrightarrow{OA'}$ (recall that $EC^2 \neq ED^2$ according to our assumption). Then, from (3.3) $\overrightarrow{OD'} = -(1+l+k)\overrightarrow{OA'}$, which implies that all five points A, B, C, D, and E lie on a great circle. The point configuration that maximizes the geometric mean in this case will be the regular pentagon (denoted $\{5\}$ in Föppl notation). Comparing the product of distances for the two configurations $\{5\}$ and $\{1,3,1\}$ we see that the former is not an optimal configuration

$$\prod\nolimits_{\{5\}} = 2^{10}\sin^5(\pi/5)\sin^5(2\pi/5) = 55.901... < 83.138... = 48\sqrt{3} = \prod\nolimits_{\{1,3,1\}}.$$

Consider now the set of equations (3.3), (3.4), (3.6), (3.7), (3.8), and (3.9). Since all the vectors $\overrightarrow{OA'}$, $\overrightarrow{OB'}$, $\overrightarrow{OC'}$, and $\overrightarrow{OD'}$ are mutually independent, the rank of the matrix formed by their coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) respectively, is two, which shows that (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) are linearly independent. Then the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ EA^2 & EB^2 & EC^2 & ED^2 \\ 4 - \frac{EB^2}{AB^2} - \frac{EC^2}{AC^2} - \frac{ED^2}{AD^2} & \frac{EB^2}{AB^2} & \frac{EC^2}{AC^2} & \frac{ED^2}{AD^2} \\ \frac{EA^2}{BA^2} & 4 - \frac{EA^2}{BA^2} - \frac{EC^2}{BC^2} - \frac{ED^2}{BD^2} & \frac{EC^2}{BC^2} & \frac{ED^2}{BD^2} \\ \frac{EA^2}{CA^2} & \frac{EB^2}{CB^2} & 4 - \frac{EA^2}{CA^2} - \frac{EB^2}{CB^2} - \frac{ED^2}{CD^2} & \frac{ED^2}{CD^2} \\ \frac{EA^2}{DA^2} & \frac{EB^2}{DB^2} & \frac{EC^2}{DC^2} & 4 - \frac{EA^2}{DA^2} - \frac{EB^2}{DB^2} - \frac{EC^2}{DC^2} \end{pmatrix}$$

has rank two because its kernel contains (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4), and because

$$\det\begin{pmatrix} 1 & 1 \\ EA^2 & EB^2 \end{pmatrix} \neq 0.$$

Now consider the submatrix formed by the first three rows

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ EA^2 & EB^2 & EC^2 & ED^2 \\ 4 - \frac{EB^2}{AB^2} - \frac{EC^2}{AC^2} - \frac{ED^2}{AD^2} & \frac{EB^2}{AB^2} & \frac{EC^2}{AC^2} & \frac{ED^2}{AD^2} \end{pmatrix}. \tag{3.11}$$

We know that its rank is two. Applying elementary operations, we get (recall (2.2))

$$\operatorname{rank}\begin{pmatrix} 4 & 1 & 1 & 1 \\ 10 & EB^2 & EC^2 & ED^2 \\ 4 & \frac{EB^2}{AB^2} & \frac{EC^2}{AC^2} & \frac{ED^2}{AD^2} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 2 & \frac{1}{EB^2} & \frac{1}{EC^2} & \frac{1}{ED^2} \\ 2 & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \\ 2 & \frac{1}{AB^2} & \frac{1}{AC^2} & \frac{1}{AD^2} \end{pmatrix} = 2, \tag{3.12}$$

from which we derive the sequence of equations (similar equations hold when we interchange $A \longleftrightarrow B$, $A \longleftrightarrow C$, $A \longleftrightarrow D$)

$$\frac{\frac{2}{5} - \frac{1}{EB^2}}{\frac{2}{5} - \frac{1}{AB^2}} = \frac{\frac{2}{5} - \frac{1}{EC^2}}{\frac{2}{5} - \frac{1}{AC^2}} = \frac{\frac{2}{5} - \frac{1}{ED^2}}{\frac{2}{5} - \frac{1}{AD^2}}. \tag{3.13}$$

Observe that no vertex can have an edge of length $\sqrt{5/2}$, because this immediately leads to an adjacent pair of equal length (indeed, from (3.12) if $1/EB^2 = 2/5$, then $1/AB^2 = 2/5$ or $1/EC^2 = 2/5$). Since the sum of the squares of the distances coming out of the same vertex equals 10 (see (2.2)), we get that from every vertex there will be an edge $< \sqrt{5/2}$ and an edge $> \sqrt{5/2}$.

Suppose now that there is a vertex with three of the edges stemming out greater than $\sqrt{5/2}$. Without loss of generality we may assume this vertex to be E, and we can choose the smallest edge to be EA. Let us order the rest of the edges, say

$$EA^2 < 5/2 < EB^2 < EC^2 < ED^2.$$

Clearly, then the denominators of all three fractions of (3.13) must have the same sign. Since $EA^2 < 5/2$ and $EA^2 + AB^2 + AC^2 + AD^2 = 10$, this is only possible if they are all positive. If $EB^2 > AB^2$ ($EB^2 < AB^2$), then the first fraction in (3.13) is greater (less) than 1, which implies that $EC^2 > AC^2$ and $ED^2 > AD^2$ ($EC^2 < AC^2$ and $ED^2 < AD^2$), which is a contradiction with $EA^2 + AB^2 + AC^2 + AD^2 = 10 = EA^2 + EB^2 + EC^2 + ED^2$.

Similarly, we derive a contradiction in the case when there are three edges less than $\sqrt{5/2}$. Then we choose

$$EB^2 < EC^2 < ED^2 < 5/2 < EA^2,$$

and the rest of the argument is similar.

Now we can assume that all vertices have two edges that are $> \sqrt{5/2}$ and two edges that are $< \sqrt{5/2}$. Without loss of generality we may assume that

$$EA^2 < EB^2 < 5/2 < EC^2 < ED^2.$$

Then the denominators of the second and the third fraction in (3.13) have the same sign, and because $AE^2 < 5/2$, this sign is positive, i.e., $AC^2 > 5/2$ and $AD^2 > 5/2$. A similar argument can be applied to the companion of (3.13) when $A \longleftrightarrow B$

$$\frac{\frac{2}{5} - \frac{1}{EA^2}}{\frac{2}{5} - \frac{1}{BA^2}} = \frac{\frac{2}{5} - \frac{1}{EC^2}}{\frac{2}{5} - \frac{1}{BC^2}} = \frac{\frac{2}{5} - \frac{1}{ED^2}}{\frac{2}{5} - \frac{1}{BD^2}}. \tag{3.14}$$

Here $EB^2 < 5/2$ forces $BC^2 > 5/2$ and $BD^2 > 5/2$. But this leads to a contradiction, because we obtained three edges CA, CB, and CE coming out of C and $> \sqrt{5/2}$.

The latter contradiction proves the lemma. $\square$

We now are ready to prove our main result.

Proof of Theorem 1. Let A, B, C, D, and E be an optimal configuration. By Lemmas 4 and 5 we may assume that no two points are antipodal and that there is a pair of adjacent edges with equal length, say $EC = ED$. Multiplying (3.3) by EC^2 and subtracting it from (3.4) we obtain

$$(EA^2 - EC^2)\overrightarrow{OA'} + (EB^2 - EC^2)\overrightarrow{OB'} = 0.$$

If $EB = EC$, then $EA = EC = \sqrt{5/2}$ (see (2.2)) and the configuration must be $\{1,4\}$ in Föppl notation (i.e., $z_A = z_B = z_C = z_D = -1/4$ and $ABCD$ is a square). In this case we can verify that

$$\prod_{\{1,4\}} = \frac{5^5 3^3}{2^{10}} = 82.397... < 83.138... = 48\sqrt{3} = \prod_{\{1,3,1\}}.$$

Thus we may assume that $EB \neq EC$. Then $\overrightarrow{OB'} = k\overrightarrow{OA'}$. Since $EC = ED$ implies that $OC' = OD'$, we obtain from (3.3) that $\overrightarrow{OM} = -((1+k)/2)\,\overrightarrow{OA'}$, where M is the midpoint of $C'D'$. This means that E, A, and B lie on a great circle and the plane that they form bisects CD. Then $AC = AD$ and $BC = BD$.

Now consider $\triangle EAB$. We claim that it has at least two equal sides. Suppose not.

Equations (3.3), (3.4), and (3.6) become

$$\begin{aligned} \overrightarrow{OA'} \quad & +\overrightarrow{OB'} \quad & +2\overrightarrow{OM} \quad &= 0, \\ EA^2\overrightarrow{OA'} \quad & +EB^2\overrightarrow{OB'} \quad & +2EC^2\overrightarrow{OM} \quad &= 0, \\ \left(4 - \frac{EB^2}{AB^2} - 2\frac{EC^2}{AC^2}\right)\overrightarrow{OA'} \quad & +\frac{EB^2}{AB^2}\overrightarrow{OB'} \quad & +2\frac{EC^2}{AC^2}\overrightarrow{OM} \quad &= 0. \end{aligned} \tag{3.15}$$

As in Lemma 5 we can derive

$$\frac{\frac{2}{5}-\frac{1}{EB^2}}{\frac{2}{5}-\frac{1}{AB^2}}=\frac{\frac{2}{5}-\frac{1}{EC^2}}{\frac{2}{5}-\frac{1}{AC^2}}. \tag{3.16}$$

Interchanging $A \longleftrightarrow B$ we obtain also that

$$\frac{\frac{2}{5}-\frac{1}{EA^2}}{\frac{2}{5}-\frac{1}{AB^2}}=\frac{\frac{2}{5}-\frac{1}{EC^2}}{\frac{2}{5}-\frac{1}{BC^2}}. \tag{3.17}$$

Utilizing Equations (2.2) for the vertices E, A, and B we get

$$EA^2+EB^2+2EC^2=10, \qquad AB^2+2AC^2+EA^2=10,$$
$$AB^2+2BC^2+EB^2=10.$$

Subtracting the first and the second equation we get

$$EB^2-AB^2=2(AC^2-EC^2), \tag{3.18}$$

and subtracting the first and the third equation

$$EA^2-AB^2=2(BC^2-EC^2). \tag{3.19}$$

After simplification (3.16) can be written as

$$2EC^2AC^2(EB^2-AB^2)+2AB^2EB^2(AC^2-EC^2)$$
$$+5EC^2AB^2-5EB^2AC^2=0,$$

which using (3.18) can be factored as

$$(EB^2-AB^2)\left[2EC^2AC^2+AB^2EB^2-\frac{5}{2}AB^2-5AC^2\right]=0.$$

Since $EB \neq AB$ we get

$$2EC^2AC^2+AB^2EB^2-\frac{5}{2}AB^2-5AC^2=0. \tag{3.20}$$

Equations (3.17) and (3.19) are obtained from (3.16) and (3.18) by interchanging $A \longleftrightarrow B$, therefore we derive similarly

$$2EC^2BC^2+AB^2EA^2-\frac{5}{2}AB^2-5BC^2=0. \tag{3.21}$$

Subtracting (3.20) and (3.21) and using the fact that $EB^2-EA^2=2(AC^2-BC^2)$ (which we get from (3.18) and (3.19)), we finally obtain that

$$(AC^2-BC^2)(2EC^2+2AB^2-5)=0. \tag{3.22}$$

If $AC=BC$, then $AC=AD=BC=BD$ (recall that $AC=AD$ and $BC=BD$), which means that C and D lie on the perpendicular bisector of AB. Since O is the center of mass of A, B, C, D, and E, we get that $EA=EB$, which contradicts our assumption. This means that $2EC^2+2AB^2=5$.

By symmetry $2AC^2 + 2EB^2 = 5$ and $2BC^2 + 2EA^2 = 5$. But then $EA^2 < 5/2$, $EB^2 < 5/2$, and $EC^2 = ED^2 < 5/2$. This contradicts (2.2). Therefore, $\triangle EAB$ has at least two equal sides.

Without loss of generality we may assume that $EA = EB$. Then it is easy to see that $AC = BC$ (recall that we already have $EC = ED$, $AC = AD$, $BC = BD$, and $CD \perp EAB$). Let $E = (0,0,1)$, and let $EAB \in xz-$plane, $ECD \in yz-$plane. Denote $z_A = z_B = h$ and $z_C = z_D = g$, where $h + g = -1/2$ (because O is the center of mass of the configuration). The coordinates of the points are:

$$A = (\sqrt{1-h^2}, 0, h), \qquad B = (-\sqrt{1-h^2}, 0, h),$$
$$C = (0, \sqrt{1-g^2}, g), \qquad D = (0, -\sqrt{1-g^2}, g).$$

We now have that $\overrightarrow{OA'} = -\overrightarrow{OB'}$ and $\overrightarrow{OC'} = -\overrightarrow{OD'}$. Then from (3.6) and (3.8) we get

$$\frac{EA^2}{AB^2} + \frac{EC^2}{AC^2} = 2 \tag{3.23}$$

$$\frac{EA^2}{AC^2} + \frac{EC^2}{CD^2} = 2. \tag{3.24}$$

We have $EA^2 = 2(1-h)$, $EC^2 = 2(1-g)$, $AB^2 = 4(1-h^2)$, $CD^2 = 4(1-g^2)$ and $AC^2 = 2(1-gh)$. After substitution in (3.23) and (3.24) we get

$$\frac{1}{2(1+h)} + \frac{1-g}{1-gh} = 2 \tag{3.25}$$

$$\frac{1}{2(1+g)} + \frac{1-h}{1-gh} = 2. \tag{3.26}$$

Subtracting the two equations we obtain

$$\frac{(g-h)(-1-2(h+g)-3hg)}{2(1+g)(1+h)(1-gh)} = 0$$

and since $h + g = -1/2$, we get that $hg(g-h) = 0$. If $g - h = 0$ then $g = h = -1/4$ and the configuration is $\{1, 4\}$, which is not an optimal. If $g = 0$, then $h = -1/2$, which implies that AB is a diameter and $\triangle ECD$ is an equilateral triangle on a great circle perpendicular to AB. Similarly when $h = 0$ and $g = -1/2$. Thus, if a configuration is optimal, then up to rotation it must be $\{1, 3, 1\}$.

On the other hand, by the compactness of the sphere the global maximum of the product of distances exists. This proves the theorem. $\square$

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