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Following Parreau's work in 1951-52, we give a unified definition of parabolic Riemann surfaces, with or without boundary. A surface is parabolic under the unified definition implies that it is either relative parabolic or parabolic under the classical definitions.

Then we study the conformal structures of noncompact, proper, branched minimal surfaces in $\mathbb{R}^3$ and prove several criteria of such surfaces (with or without boundary) being parabolic. Using these criteria we then prove two graph theorems, they are noncompact versions of the classical graph theorem of Radó, generalized in various directions.

1. Introduction.

In this paper we will prove that certain complete branched minimal surfaces are actually minimal graphs. For the convenience of readers, we first describe surfaces.

A **parametrized surface** in $\mathbb{R}^3$ is a C^1 mapping $X : M \to \mathbb{R}^3$, where M is a 2-dimensional manifold and $X = (X_1, X_2, X_3)$. The topology of the surface is the topology of M. Let (u, v) be a local coordinate of M, and

$$X_u = \frac{\partial X}{\partial u} = \left(\frac{\partial X_1}{\partial u}, \frac{\partial X_2}{\partial u}, \frac{\partial X_3}{\partial u}\right),$$

etc., then $p \in M$ is a **regular point** of the surface if and only if

$$(X_u \wedge X_v)(p) \neq (0, 0, 0), \tag{1}$$

where $\wedge$ is the external product in $\mathbb{R}^3$. Otherwise p is a **singular point** of the surface. If every point of M is a regular point, $X : M \to \mathbb{R}^3$ is called a **regular surface**, or an **immersion**. If X is an immersion and also a one-to-one map, then X is called an **embedding** and $X(M)$ is an embedded surface. The inverse mapping theorem guarantees that any surface is locally an embedding at a regular point.

Following [**12**], we say that a surface $X : M \to \mathbb{R}^3$ is **complete** if and only if any path $\alpha \subset M$ that leaves every compact subset of M, then $X(\alpha) \subset \mathbb{R}^3$ has infinite length. Note that it is possible that $\partial M \neq \emptyset$.

The **spherical map (unit normal map)** at a regular point of a surface $X : M \to \mathbb{R}^3$ is given by

$$N(p) = \frac{X_u \wedge X_v}{|X_u \wedge X_v|}(p) \in S^2. \tag{2}$$

A surface $X : M \to \mathbb{R}^3$ is **minimal** if and only if X is a **conformal harmonic** mapping. Let $\triangle_M$ be the Laplace operator on M and $z = u + iv$ be a local complex coordinate of the **Riemann surface** M, i.e., (u, v) is an **isothermal coordinate**, then X is harmonic if and only if for $j = 1, 2, 3$,

$$\triangle_M X_j \equiv 0, \quad \text{or equivalently} \quad \frac{\partial^2 X_j}{\partial u^2} + \frac{\partial^2 X_j}{\partial v^2} \equiv 0. \tag{3}$$

If we write in vector form, then X is harmonic if and only if

$$\triangle_M X \equiv (0,0,0), \quad \text{or equivalently} \quad X_{uu} + X_{vv} \equiv (0,0,0). \tag{4}$$

Using $U \bullet V$ and $|U| = \sqrt{U \bullet U}$ to denote the inner product and the norm in $\mathbb{R}^3$, X is conformal if and only if

$$|X_u|^2 \equiv |X_v|^2, \quad X_u \bullet X_v \equiv 0. \tag{5}$$

Thus $p \in M$ is a singular point if and only if $|X_u|^2(p) = |X_v|^2(p) = 0$.

It is well-known that a minimal surface may have at most isolated singular points, also called **branch points** in minimal surface theory. A nice property of minimal surface is that the spherical map N of a minimal surface can be analytically extended to branch points. A minimal surface with singular points is called a **branched minimal surface**. For more details, please see [**18**].

Our Riemann surface M may have boundary, i.e., $\partial M \neq \emptyset$. Denote Int$M$ the interior of M, we will alway assume that $M = \text{Int}M \cup \partial M$.

Let $X : M \to \mathbb{R}^3$ be a branched minimal surface. At any regular point $p \in \text{Int}M$, $X(M)$ can be locally expressed as a **minimal graph** generated by a function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ satisfying the **minimal surface equation**

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0. \quad (x, y) \in \Omega. \tag{6}$$

But globally a complete minimal surface being a minimal graph is a very restricted hypothesis. For example, if u is defined in the whole $\mathbb{R}^2$, then u must be an affine function and the surface is a plane, this is the famous Bernstein theorem [**2**].

For compact minimal surfaces, a famous graph theorem is Radó's theorem, [**20**]. It states that if $X : \overline{\mathbb{D}} \to \mathbb{R}^3$ is a compact minimal disk and $X(\partial\mathbb{D})$ can be one-to-one orthogonally projected onto a planar convex Jordan curve δ, then $X(\mathbb{D})$ is a minimal graph over the convex planar domain bounded by δ.

In [**13**], Meeks generalized Radó's theorem to any compact minimal surface, regardless the topology, and replaced the assumption of one-to-one orthogonal projection by monotone central or orthogonal projection. Moreover, the surface may have branch points.

In this paper we prove two noncompact versions of Radó's graph theorem.

To state our results, we need another general terminology. Let $f : M \to N$ be continuous and M and N be Hausdorff. We say that f is **proper** if and only if for any compact set $C \subset N$, $f^{-1}(C)$ is also compact. Thus $X : M \to \mathbb{R}^3$ is proper implies that as a surface, $X : M \to \mathbb{R}^3$ is complete.

Theorem 1.1. *Let δ be a C^2 noncompact, non-flat, complete convex curve in $\mathbb{R}^2$ and Ω be the convex domain bounded by δ. Let M be a simply connected Riemann surface with $C^{1,\alpha}$ boundary and $X : M \to \mathbb{R}^3$ be a branched proper minimal surface such that $X(M) \subset \overline{\Omega} \times \mathbb{R}$ and X on ∂M is a diffeomorphsim onto a $C^{1,\alpha}$ graph over $\delta = \partial\Omega$, then $X(M)$ is a minimal graph over $\overline{\Omega}$.*

Remark 1.1. Nitsche has proved that the Dirichlet problem with continuous boundary value $f : \delta = \partial\Omega \to \mathbb{R}$ is solvable, see [**16**]. Collin [**3**] has shown that there may exist more than one solution with the same boundary value. Thus Theorem 1.1 can be stated as that all simply connected solutions to the Plateau problem with boundary $\Gamma = X(\partial M)$ and contained in $\overline{\Omega} \times \mathbb{R}$ are graphs. An interesting problem is to determine how many solutions are there.

Theorem 1.1 is a generalization of Radó's theorem in that the projection is no longer compact. Another possible generalization of Radó's theorem is that although the surface is noncompact, the projection of the boundary is compact. The most famous example of such unbounded minimal surfaces is the Jenkins-Serrin minimal graph, it is a minimal graph defined on a domain Ω bounded by a convex $2n$-gon (maybe reduced, that is, Q is a k-gon, $3 \le k \le 2n$), with alternative boundary segments A_j, B_j, $1 \le j \le n$, such that $u(x, y) \to \infty$ when (x, y) approachs A_j and $u(x, y) \to -\infty$ when (x, y) approachs B_j. When $n = 2$ and Ω is a square, the Jenkins-Serrin graph is Scherk's surface, see pages 124, 151, and 156 of Volume 1 of [**5**] for the pictures of Scherk's surface. It is Finn who first studied the Dirichlet problem with infinite boundary value in a line segment of the boundary, see [**7**].

Jenkins and Serrin proved in [**10**] that such a minimal graph exists if and only if

$$\sum_{j=1}^{n} |A_j| = \sum_{j=1}^{n} |B_j|, \tag{7}$$

where $|A_j|$ is the length of A_j, etc.

Looking at the surface, we see that a Jenkins-Serrin graph is an embedded simply connected minimal surface bounded by $2n$ straight lines. We will prove that the Jenkins-Serrin minimal graphs are essentially the only such minimal surfaces, even under much relaxed hypotheses.

We need more notations. Let $\mathcal{P} : \mathbb{R}^3 \to \mathbb{R}^2$ be the perpendicular projection. Let $Q \subset \mathbb{R}^2$ be a convex k-gon, $k \geq 3$. Let q_j, $1 \leq j \leq n$, $n \geq k$, be boundary points of Q including all the vertices, labeled by counterclockwise orientation. Let $l_{j,j+1}$ be the consecutive boundary line segments of Q, connecting q_j and q_{j+1}, $1 \leq j \leq n$, here we denote $q_{n+1} = q_1$. Let L_j be the straight lines passing through q_j and be perpendicular to $\mathbb{R}^2$, $1 \leq j \leq n$.

Theorem 1.2. *Let $X : M \to \mathbb{R}^3$ be a proper, branched, simply connected minimal surface such that $\Omega := \mathcal{P} \circ X(\mathrm{Int}M) \subset \mathbb{R}^2$ is a domain bounded by a convex k-gon Q, $k \geq 3$.*

If X maps ∂M one-to-one onto the parallel lines $\cup_{j=1}^n L_j$, then $X(\mathrm{Int}M)$ is a Jenkins-Serrin minimal graph over Ω. Hence such a proper branched minimal surface exists if and only if $n = 2m \geq 4$ and

$$\sum_{j=1}^{m} |l_{2j-1,2j}| = \sum_{j=1}^{m} |l_{2j,2j+1}|. \tag{8}$$

In particular, X is a minimal embedding. Furthermore, let $\mathbb{D} \subset \mathbb{C}$ be the unit disk, then conformally

$$M = \overline{\mathbb{D}} - \cup_{j=1}^n \{e^{i\theta_j}\}, \quad 0 = \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi.$$

Remark 1.2. In [**12**], Theorem II.2, it is proved that the Scherk's surface is the only complete minimal surface whose interior is a graph over a parallelogram and is bounded by four parallel straight lines over the veritices of the parallelogram. The conditions in Theorem 1.2 are much weaker than the conditions in [**12**].

The proofs of the above graph theorems are based on the concept of **parabolic** Riemann surface. The concept is very important in the study of minimal surfaces.

Let S be a connected noncompact Riemann surface. If $\partial S = \emptyset$, then S is parabolic if and only if on S there are no nonconstant negative subharmonic functions, see page 204 of [**1**] and page 164 of [**6**]. Otherwise, S is hyperbolic. Parreau gave an equivalent definition of parabolic surfaces, see page 117 of [**19**].

If $\partial S \neq \emptyset$, S is **relative parabolic** if and only if there are no nonnegative bounded harmonic functions vanishing on ∂S, see page 212 of [**1**]. Note that if ∂S is regular (i.e., there exists a local barrier at each point of ∂S) for solving Dirichlet problem by Perron's method, see for example, page 139 of [**1**] or page 25 of [**8**], then we can replace harmonic by subharmonic in the definition of relative parabolic. Parreau took this definition, see page 118 of [**19**].

Regardless $\partial M = \emptyset$ or $\partial M \neq \emptyset$, in this paper we will follow Parreau to use a unified definition of a noncompact Riemann surface M being parabolic.

Definition 1.1. A noncompact Riemann surface M is parabolic if and only if there are no nonconstant nonnegative bounded subharmonic functions vanishing on ∂M. Otherwise M is hyperbolic.

Remark 1.3. Note that when $\partial M = \emptyset$, the last hypothesis is void.

This unified definition is equivalent to the classical definitions in cases of $\partial M = \emptyset$ or M has regular boundary as mentioned above. In general, if $\partial M \neq \emptyset$, M is parabolic under our definition must be relative parabolic under the classical definition; while M is relative parabolic under the classical definition may not be parabolic under our definition.

On a relative parabolic or parabolic Riemann surface under the classical definitions, bounded subharmonic functions satisfy the maximum principle, see for example, pages 204 and 214 of [**1**] and pages 164-5 of [**6**]. Under the unified definition of parabolic surfaces, the above maximum principle still holds.

Since a minimal surface has harmonic coordinate functions and holomorphic Gauss map, if we know that the surface is parabolic, then we can get some inferences from the boundary behaviour of the surface.

For example, if u satisfies the minimal surface equation over $\mathbb{R}^2$, $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = u(x_1, x_2)\}$, then both $\tan^{-1} u_{x_1}$ and $\tan^{-1} u_{x_2}$ are bounded harmonic functions on M. If we know that M is parabolic, as proved by Nitsche in [**14**] and [**15**], also see §5 of [**18**], then u_{x_1} and u_{x_2} must be constant functions, therefore, M must be a plane. This is an alternative proof of the Bernstein Theorem.

Many authors have studied criteria of a minimal surface to be parabolic, and then use the parabolicity to obtain related results. For example, Collin and Krust in [**4**] used parabolicity to prove that if u satisfies the minimal surface equation in a strip, and has affine boundary value, then the graph of u must be a piece of a helicoid. Rodríguez and Rosenberg also established criteria of parabolic surfaces in [**21**], [**22**], and [**23**].

We will establish some criteria of proper minimal surfaces to be parabolic in the next section. These criteria are simple and interesting in their own right. We will also give a simple proof of the Collin-Krust theorem. In the end of Section 2, we make a conjecture. Sections 3 and 4 are contributed to the proofs of Theorems 1.1 and 1.2. In Section 5 we discuss briefly the higher dimensional generalizations of parabolicity and criteria proved in Section 2.

2. Criteria of parabolicity.

Lemma 2.1. *Let $\vec{n} \in S^2$ and $X : M \to \mathbb{R}^3$ be an noncompact branched minimal surface. If $X \bullet \vec{n}$ is a proper function, then M is parabolic.*

Proof. Select a coordinate system such that $\vec{n} = (0, 0, 1)$ and $X = (X_1, X_2, X_3)$. Then X_3 is proper.

Suppose that w is a nonnegative bounded subharmonic function vanishing on ∂M. If w is not a constant function, then we may assume that $\sup_M w = 1$.

Let $C_0 = X_3^{-1}(0)$. Since X_3 is proper, C_0 is compact. Take $m = \max_{p\in C_0} w(p)$ if $C_0 \neq \emptyset$, or $m = 0$. Since C_0 is compact, by the maximum principle for subharmonic functions on compact manifold we have $0 \leq m < 1$.

For any $a \in \mathbb{R}$, let $M^a = X_3^{-1}((-\infty, a])$, $M_a = X_3^{-1}([a, \infty))$, and $M_c^d = M_c \cap M^d$, $-\infty < c < d < \infty$. Since X_3 is proper, M_c^d is compact.

For any $a > 1$, consider the superharmonic function

$$u_a(p) = a^{-1}X_3(p) - w(p) + m, \quad p \in M.$$

We claim that $u_a \geq 0$ on M_0. In fact, $u_a(p) \geq 0$ for $p \in M_a$. On the compact set M_0^a, since $u_a \geq 0$ on ∂M_0^a, by the minimum principle for superharmonic functions on compact manifold, $u_a \geq 0$ on M_0^a. Therefore, $u_a \geq 0$ on M_0.

Since $a > 1$ was arbitrary, letting $a \to \infty$, we see that $w \leq m$ on M_0. Similarly, $w \leq m$ on M^0. Therefore, $w \leq m < 1$ on M, a contradiction to the fact that $\sup_M w = 1$. This contradiction proves that w must be a constant. □

Let $P_t = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = t\}$, $t \in \mathbb{R}$. For $t > 0$, let $C(t) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq t^2\}$.

Lemma 2.1 has two immediate corollaries.

Corollary 2.1. *Let $X : M \to \mathbb{R}^3$ be an noncompact, proper, branched minimal surface such that $X(M) \subset \Omega \times \mathbb{R}$, where Ω is a bounded planar domain. Then M is parabolic.*

Proof. Let $\vec{n}$ be the normal direction of Ω, we may assume that $\vec{n} = (0, 0, 1)$ and $\Omega \subset P_0$. We need only prove that X_3 is proper. Let $C \subset P_0$ be compact and $\Omega \subset C$. Let $I \subset \mathbb{R}$ be compact then $X_3^{-1}(I) \subset X^{-1}(C \times I)$ is compact. □

Corollary 2.2. *Let $X : M \to \mathbb{R}^3$ be an noncompact, proper, branched minimal surface. If for every $t \in \mathbb{R}$, $X(M) \cap P_t$ is compact, then M is parabolic.*

Proof. We only need prove that X_3 is proper. In fact, let $d(t)$ be the diameter of $M \cap P_t$, then $d(t)$ is a continuous function of t. Let $I \subset \mathbb{R}$ be a compact interval, then for some $D > 0$, $d(t) \leq D$ for $t \in I$. Thus $X_3^{-1}(I) \subset X^{-1}(C(2D) \times I)$. Since X is proper, $X^{-1}(C(2D) \times I)$ is compact. Therefore, $X_3^{-1}(I)$ is compact. □

Remark 2.1. The proper condition is necessary, since there are non-proper minimal surfaces contained in a slab $S = \{(x_1, x_2, x_3) : 0 < x_3 < 1\}$ such that $X(M) \cap P_t$ is compact. See [**11**] and [**24**].

The next corollary is obvious.

Corollary 2.3. *Let $X : M \to \mathbb{R}^3$ be an noncompact, proper branched minimal annulus. If for every $t \in \mathbb{R}$, $X(M) \cap P_t$ is a Jordan curve, then conformally $M = \mathbb{C} - \{0\}$.*

Theorem 2.1. *Let $X : M \to \mathbb{R}^3$ be a connected, noncompact, proper branched minimal surface, if there are two different planes P and Q, $P \cap Q \neq \emptyset$, such that $(P \cup Q) \cap X(M)$ is compact, then M is parabolic.*

Proof. Select coordinates (x_1, x_2, x_3) of $\mathbb{R}^3$, such that $P = \{x_1 = 0\}$, $Q = \{x_2 = 0\}$. Such a coordinate system is not necessarily orthonormal. Under this coordinate system, $X = (X_1, X_2, X_3)$ and each component is harmonic.

Let w be a nonnegative bounded subharmonic function vanishing on ∂M. If w is not constant, we may assume that $\sup_M w = 1$. Since X is proper and $X(M) \cap (P \cup Q)$ is compact, $X^{-1}(P \cup Q) = X^{-1}[X(M) \cap (P \cup Q)]$ is compact. If $(P \cup Q) \cap X(M) = \emptyset$, let $m = 0$. If $(P \cup Q) \cap X(M) \neq \emptyset$, let $m = \max_{X^{-1}(P \cup Q)} w$. By the maximum principle for subharmonic functions on compact manifold, in both cases we have $0 \leq m < 1$.

For any $a > 1$, $b > 1$, define a superharmonic function

$$u_{a,b}(p) = a^{-1} X_1(p) + b^{-1} X_2(p) - w(p) + m, \quad p \in M.$$

We claim that $u_{a,b} \geq 0$ on $X^{-1}([0, \infty) \times [0, \infty) \times \mathbb{R})$. In fact, on $X^{-1}(([a, \infty) \times [0, \infty) \times \mathbb{R}) \cup ([0, \infty) \times [b, \infty) \times \mathbb{R}))$, $u_{a,b} \geq 0$. On $D_{a,b} = X^{-1}([0, a] \times [0, b] \times \mathbb{R})$, $u_{a,b}$ is a bounded superharmonic function and $u_{a,b} \geq 0$ on $\partial D_{a,b}$. By Corollary 2.1, $D_{a,b}$ is either compact or parabolic. Thus by the minimum principle for bounded superharmonic functions on compact or parabolic surfaces, $u_{a,b} \geq 0$ on $D_{a,b}$.

Letting $a, b \to \infty$, we see that $w \leq m$ on $X^{-1}([0, \infty) \times [0, \infty) \times \mathbb{R})$. Similarly, we can prove that $w \leq m$ on $X^{-1}((-\infty, 0] \times [0, \infty) \times \mathbb{R})$, $X^{-1}((-\infty, 0] \times (-\infty, 0] \times \mathbb{R})$, and $X^{-1}([0, \infty) \times (-\infty, 0] \times \mathbb{R})$.

Therefore, $w \leq m < 1$ on M, a contradiction to $\sup_M w = 1$. So w must be a constant. □

Collin and Krust in [**4**] used the parabolic property of M to prove that if $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1\}$ and $u : \Omega \to \mathbb{R}$ has affine boundary values and satisfies the minimal surface equation, then the graph of u is a piece of a helicoid. Their proof of parabolicity is quite involved. We now give a simple proof of the Collin-Krust theorem, a different proof based on PDE method can be found in [**9**]. First an easy corollary of Theorem 2.1.

Corollary 2.4. *Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1\}$ be a strip and $X : M \to \mathbb{R}^3$ be a minimal surface that is a graph over $\overline{\Omega}$, then M is parabolic.*

Proof. Since X is a homeomorphism of M to a closed subset of $\mathbb{R}^3$, clearly X is proper. Take P as any plane parallel to the x_1x_3-plane and $Q = \{x_1 = 2\}$, then $(P \cup Q) \cap X(M)$ is compact. □

Theorem 2.2 (Collin-Krust [**4**]). *Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1\}$ be a strip and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies the minimal surface equation in Ω and u has affine boundary values, then the minimal graph generated by u is a piece of a helicoid.*

Proof. Let $M := \{(x_1, x_2, u(x_1, x_2));\ (x_1, x_2) \in \overline{\Omega}\} \subset \mathbb{R}^3$ be the simply connected minimal graph generated by u. Corollary 2.4 shows that M is a parabolic Riemann surface.

By assumption, u_{x_2} is constant on each boundary component of M, that is there are constants C_1 and C_2 such that $\tan^{-1} u_{x_2}(0, x_2) \equiv C_1$ and $\tan^{-1} u_{x_2}(1, x_2) \equiv C_2$.

Let $\triangle_M$ be the Laplace operator on M, in the (x_1, x_2) coordinate $\triangle_M \phi = 0$ if and only if

$$(1 + u_{x_2}^2)\phi_{x_1x_1} - 2u_{x_1}u_{x_2}\phi_{x_1x_2} + (1 + u_{x_1}^2)\phi_{x_2x_2} = 0.$$

It is not hard to calculate that $\triangle_M \tan^{-1} u_{x_2} = 0$, Bernstein first observed this in [**2**]. Note that $\triangle_M x_1 = 0$. Since $\tan^{-1} u_{x_2}$ and $(C_2 - C_1)x_1 + C_1$ are both bounded and have the same boundary value, by the maximum principle for bounded harmonic functions on parabolic surfaces, we have

$$\tan^{-1} u_{x_2}(x_1, x_2) = (C_2 - C_1)x_1 + C_1, \quad (x_1, x_2) \in \overline{\Omega}.$$

The above formula shows that each curve $(c, x_2, u(c, x_2)) \subset M$ is a straight line, $0 \le c \le 1$. Therefore, M is a ruled minimal surface. It is well-known that the only ruled minimal surface is a helicoid, see for example, pages 17-18 of [**18**]. □

We will prove one more criterion of parabolic surfaces. First we define that a domain Ω to be a **proper domain** if under an orthogonal coordinate (x_1, x_2),

$$\Omega = \{(x_1, x_2) \subset \mathbb{R}^2 :\ f_1(x_2) < x_1 < f_2(x_2),\ 0 < x_2 < \infty\}, \tag{9}$$

where f_1 and f_2 are continuous functions defined on $[0, \infty)$ such that $f_1(t) < f_2(t)$ for any $t \in (0, \infty)$.

Theorem 2.3. *Let Ω be a proper domain and $X : M \to \mathbb{R}^3$ be a noncompact, proper branched minimal surface such that $X(M) \subset \overline{\Omega} \times \mathbb{R}$, then M is parabolic.*

Proof. Let w be a nonnegative bounded subharmonic function vanishing on ∂M. Without loss of generality, we may assume that $0 \le w \le 1$. Define

$H_t = \{x_2 \le t\}$ for any $t \in \mathbb{R}$. Let $M_t := X^{-1}(H_t)$. For $t > 0$, define a superharmonic function

$$u_t(p) = t^{-1}X_2(p) - w(p), \quad p \in M.$$

Since $X_2(p) \ge 0$, $u_t \ge 0$ on the boundary of M_t. Since Ω is proper, u_t is bounded on M_t. By Corollary 2.1, M_t is parabolic hence by the maximum principle for bounded harmonic functions on parabolic surfaces, $u_t \ge 0$ on M_t. For $p \in M - M_t$, $X_2(p) > t$, hence $u_t(p) > 0$. We have proved that $u_t \ge 0$ on M. Since $t > 0$ was arbitrary, letting $t \to \infty$, we have $w(p) \le 0$ for any $p \in M$, thus $w \equiv 0$. Since w was an arbitrary nonnegative bounded subharmonic function vanishing on ∂M, M is parabolic. □

Theorem 2.3 has an immediate corollary. First we define a sector domain $\Omega_\alpha \subset \mathbb{R}^2$ to be the convex domain bounded by two rays issued from the same point with angle $0 < \alpha < \pi$. Therefore, Ω_α is a proper domain.

Corollary 2.5. *Let $X : M \to \mathbb{R}^3$ be a noncompact, proper branched minimal surface such that $X(M) \subset \overline{\Omega_\alpha} \times \mathbb{R}$, $0 < \alpha < \pi$, then M is parabolic.*

Finally, we would like to make a conjecture:

Conjecture 2.1. *Let $X : M \to \mathbb{R}^3$ be a proper branched minimal surface of finite genus, then M is parabolic.*

3. Proof of Theorem 1.1.

For $0 < \alpha < \pi$, let Ω_α be the sector domain defined by

$$\Omega_\alpha = \left\{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 > 0; \, |x_1| < x_2 \tan\frac{\alpha}{2}\right\}. \tag{10}$$

Then since δ is non-flat, without loss of generality we may assume that $\overline{\Omega} \subset \Omega_\alpha$. Hence by Corollary 2.5 M is parabolic. Since M is simply connected and with a connected boundary of measure greater than zero, we may assume that conformally $M = \overline{\mathbb{D}} - \{1\}$.

Let $\tau : S^2 - \{(0,0,1)\} \to \mathbb{C}$ be the stereographic project and $g := \tau \circ N$ be the Gauss map of $X : M \to \mathbb{R}^3$. It is well-known that $X : M \to \mathbb{R}^3$ is minimal if and only if g is meromorphic. Define

$$I_\alpha^\pm := \left\{e^{i\theta} \in S^1 \subset \mathbb{C} : \ \frac{\alpha}{2} < \pm\theta < \pi - \frac{\alpha}{2}\right\}, \quad I_\alpha = I_\alpha^+ \cup I_\alpha^-.$$

We claim that there is an open arc $\beta \subset I_\alpha$ such that $g(M) \cap \beta = \emptyset$. To prove this claim, we need a lemma.

Lemma 3.1. *Let $u : \overline{\mathbb{D}} - \{1\} \to \mathbb{R}$ be a continuous function such that u is harmonic in $\mathbb{D}$, $u > C$ for some constant C, and $\lim_{z\to 1,\, z\in\partial\mathbb{D}} u(z) = \infty$. Then $\lim_{z\to 1} u(z) = \infty$.*

Proof. Let $v = u - C + 1$ and $w = 1/v$, then $0 < w < 1$ and w is subharmonic in $\mathbb{D}$. Clearly $\lim_{z\to 1,\, z\in\partial\mathbb{D}} w(z) = 0$, so we can define $w(1) = 0$ such that w is continuous on $\partial\mathbb{D}$. There is an harmonic function h such that $h = w$ on $\partial\mathbb{D}$. Then $w \leq h$ in $\mathbb{D}$, thus

$$0 \leq \lim_{z\to 1} w(z) \leq \lim_{z\to 1} h(z) = 0,$$

hence we have $\lim_{z\to 1} u(z) = \infty$. □

If $p \in \mathbb{D}$ such that $g(p) \in I_\alpha$, then the tangent plane P of $X(M)$ at $X(p)$ is vertical and $P \cap \partial\Omega$ has exactly two points. Since $X(\partial M)$ is a graph over δ, we see that $P \cap X(\partial M)$ has exactly two points. Consider the harmonic function

$$u(z) = (X(z) - X(p)) \bullet N(p),$$

and the variety $V_P = X^{-1}(P) \subset M$. Clearly $u \equiv 0$ on V_P. Since $X(\partial M)$ is a graph over δ, we see that $V_P \cap \partial\mathbb{D}$ has exactly two points. Since $X(\partial M)$ is a graph over $\partial\Omega$, $\lim_{z\to 1, z\in\partial M} u(z) = \pm\infty$ depending on $g(p) \in I_\alpha^+$ or $g(p) \in I_\alpha^-$. By Lemma 3.1, $\lim_{z\to 1} u(z) = \pm\infty$. In particular, u is proper. Hence V_P is compact. It is well-known that there are at least four curves in V_P intersecting at $p \in V_P$, see, for example, [**18**]. Since $P \cap X(\partial M)$ has exactly two points and $\mathbb{D}$ is simply connected, by the Euler characteristic we know that $V_P \cup \partial\mathbb{D}$ divides $\overline{\mathbb{D}}$ into at least 4 domains. Furthermore, there is at least one domain $D \subset \mathbb{D}$ such that $\partial D \subset V_P$. This domain is a nodal domain of u, on which u is either positive or negative. Since V_P is compact, the closure of any nodal domain must be compact. Then by the maximum principle, $u \equiv 0$ and $X(M)$ is contained in P. This contradiction proves that $g(p) \notin I_\alpha$ for $p \in \mathbb{D}$.

To complete the proof of the claim, we need another lemma.

Lemma 3.2. *Let Ω be a C^1 domain and $X : M \to \mathbb{R}$ be a C^1 (up to the boundary) branched minimal surface such that $X(M) \subset \overline{\Omega} \times \mathbb{R}$ and X on ∂M is a diffeomorphism onto a C^1 graph over $\partial\Omega$, then the spherical map N satisfies $N_3 \geq 0$ or $N_3 \leq 0$ on ∂M.*

Proof. Let μ and ν be the tangent and inward normal unit vectors at a point $q \in \partial\Omega$, such that (μ, ν) has positive orientation. Let $p \in \partial M$ such that $\mathcal{P} \circ X(p) = q$ and $(\widetilde{\mu}, \widetilde{\nu})$ be the tangent and inward normal vectors of ∂M in T_pM, with the same orientation as (μ, ν). Take (u, v) to be a coordinate such that $\widetilde{\mu}X = X_u(p)$, $\widetilde{\nu}X = X_v(p)$. Since X is a diffeomorphism on ∂M to a C^1 graph over $\partial\Omega$, there is a $c_1 \neq 0$ such that

$$\mathcal{P}(X_u(p)) = c_1\mu(q). \tag{11}$$

Since $\mathcal{P} \circ X(M) \subset \overline{\Omega}$, there are $c_2 \in \mathbb{R}$ and $c_3 \geq 0$ such that

$$\mathcal{P}(X_v(p)) = c_2\mu(q) + c_3\nu(q). \tag{12}$$

Assume that X on ∂M preserves orientation, then $c_1 > 0$. If p is not a branch point, then by (2), (11) and (12),

$$N_3(p) = \frac{\det(X_u(p), X_v(p))}{|X_u(p) \wedge X_v(p)|} = \frac{c_1 c_3 \det(\mu(p), \nu(p))}{|X_u(p) \wedge X_v(p)|} \geq 0.$$

Since branch points are isolated and N is continuous on ∂M, we have $N_3 \geq 0$ on ∂M.

If $\mathcal{P} \circ X$ on ∂M reverses orientation, then $c_1 < 0$ and $N_3 \leq 0$ on ∂M. □

By the hypotheses of Theorem 1.1 and boundary regularity (see for example, §459 of [**17**] or Chapter 7 of Volume 2 of [**5**]), X is $C^{1,\alpha}$ on M. Therefore, N is continuous on M.

Since our δ is a C^2 convex curve, the inward normal ν of δ satisfying that

$$\tau \circ \nu : \partial\Omega \to S^1$$

is monotone so $\tau \circ \nu$ covers I_α^+ at most once. Moreover, since $\delta \subset \Omega_\alpha$ is noncompact complete, $\tau \circ \nu(\delta) \cap I_\alpha^- = \emptyset$.

For $z \in \partial M$, $|g(z)| = 1$ means that $N_3(z) = 0$ and $N(z) = \pm\nu(X(z))$. If there is an open arc $\gamma \subset I_\alpha^+$ such that $\gamma \subset g(\partial M)$, then $-\gamma \subset I_\alpha^-$ and $(-\gamma) \cap g(\partial M) = \emptyset$. Hence there is an nonempty open arc $\beta \subset I_\alpha$ such that $g(\partial M) \cap \beta = \emptyset$.

Without loss of generality we assume that $N_3 \geq 0$ on ∂M, then $|g| \geq 1$ on ∂M. We claim that $|g| > 1$ in $\mathrm{Int}M$.

If $M' = g^{-1}(\mathbb{D}) \neq \emptyset$, then since $X(M') \subset \Omega_\alpha \times \mathbb{R}$ is a proper branched minimal surface, by Corollary 2.5 the closure of M' in M is either compact or parabolic. Since $|g| \geq 1$ on ∂M and M is connected, $\partial M' \neq \emptyset$ and $|g| = 1$ on $\partial M'$. Since $g(M) \cap \beta = \emptyset$, the following Lemma 3.3 shows that g is a constant function on M'. This is a contradiction to the fact that $X(M)$ is non-flat. This contradiction proves that $|g| > 1$ in $\mathrm{Int}M$.

Lemma 3.3. *Let M be a compact or parabolic Riemann surface and $h : M \to \mathbb{C}$ be holomorphic and continuous up to the boundary. If $h(M) \subset \overline{\mathbb{D}}$ and $h(\partial M) \subset \partial\mathbb{D} - \Gamma$, where $\Gamma \subset \partial\mathbb{D}$ is an nonempty open arc, such that $\partial\mathbb{D} - \Gamma$ has more than one point, then h is a constant.*

Proof. Let $F : \mathbb{D} \to (0,1) \times (0,a)$ be a conformal transformation for some $a > 0$, such that F is continuous on $\overline{\mathbb{D}}$ and $F(\partial\mathbb{D} - \Gamma) = [0,1] \times \{0\}$. Then $u + iv = F \circ h$ is a holomorphic function such that v vanishes on ∂M. Since v is a bounded harmonic function vanishing on ∂M and M is compact or parabolic, v must be a constant. Thus h also has to be a constant. □

Finally let $X = (X_1, X_2, X_3)$. Since $X(\partial M)$ is a graph over $\partial\,\Omega$, $\lim_{z\to 1,\, z\in\partial\mathbb{D}} X_2 \to \infty$. By Lemma 3.1, $\lim_{z\to 1} X_2 \to \infty$, hence X_2 is proper.

We claim that X does not have interior branch points. In fact, if p is an interior branch point, then

$$DX_j(p) = (0,0), \quad j = 1,\ 2,\ 3. \tag{13}$$

Let $P'_t := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_2 = t\}$. Then since $P'_{X_2(p)}$ intersects $\partial\Omega$ in exactly two points and $X(\partial M)$ is a graph over $\partial\Omega$, $S = X^{-1}(P'_{X_2(p)})$ intersects ∂M in exactly two points. By (13), the compact variety S has at least four curves meeting at p. Since M is simply connected, there are at least one domain D such that $\overline{D}$ is compact and $\partial D \subset S$. By the maximum principle, $X_2 \equiv X_2(p)$ and $X(M)$ is a part of a plane, a contradiction.

Now we claim that if $l_{P'_t} := P'_t \cap \Omega \neq \emptyset$, then $X(M) \cap P'_t$ is a graph over $l_{P'_t}$.

Since $|g| > 1$ in $\mathbb{D}$, P'_t is transversal to $X(M)$. Since X does not have branch points, $V_{P'_t} = X^{-1}(P'_t) = X_2^{-1}(t)$ is a compact manifold with boundary. Since $V_{P'_t} \cap \partial\mathbb{D}$ has exactly two points, $V_{P'_t}$ is a closed interval. If $X(\mathrm{Int}M) \cap P'_t$ is not a graph over $l_{P'_t}$, then there will be a point $q \in X(\mathrm{Int}M) \cap P'_t$ such that $(0,0,1)$ is a tangent vector of $X(M) \cap P'_t$ at q, thus $N_3(q) = 0$, a contradiction to the fact that $|g| > 1$ in $\mathrm{Int}M$.

The proof of Theorem 1.1 is complete.

4. Proof of Theorem 1.2.

We first introduce a more general type of minimal surfaces.

Let $T_n = \cup_{j=1}^n L_j$ be the union of n parallel straight lines in $\mathbb{R}^3$. We may assume that T_n is perpendicular to the x_1x_2-plane P_0. We are interested in proper branched minimal surfaces bounded by T_n, i.e., a proper branched minimal mapping $X : M \to \mathbb{R}^3$ such that $X(\partial M) = T_n$. We say that the surface is **confined** if $\mathcal{P} \circ X(M)$ is bounded in P_0, where $\mathcal{P}$ is the perpendicular projection on P_0.

Lemma 4.1. *Suppose that M is simply connected and $X : M \to \mathbb{R}^3$ is a proper branched minimal surface such that $X : \partial M \to T_n$ is one-to-one and onto. If X is confined, then conformally,*

$$M = \overline{\mathbb{D}} - \{p_1, \dots, p_n\}, \quad p_j = e^{i\theta_j}, \quad 0 = \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi. \tag{14}$$

Furthermore, X_3 is proper and approachs $\pm\infty$ alternatively at p_i. In particular, $n = 2m$ is an even number.

Proof. Corollary 2.1 and its proof ensure that X_3 is proper and M is parabolic. Since M is simply connected and ∂M has n components, it must be that $M = \overline{\mathbb{D}} - \{p_1, \dots, p_n\}$, $|p_j| = 1$, $1 \leq j \leq n$.

Since X_3 is proper, it must be that $\lim_{z\to p_j} X_3(z) = \infty$ or $\lim_{z\to p_j} X_3(z) = -\infty$, $1 \leq j \leq n$. Otherwise, there would be $\{z_m\}_{m=1}^\infty \subset M$ such that

$\lim_{m\to\infty} z_m = p_j$ and $|X_3|(z_m) \le N$, then $X_3([-N, N])$ is not compact in M.

We may assume that $p_j = e^{i\theta_j}$ such that $\theta_1 = 0 < \theta_2 \cdots < \theta_n < 2\pi$. To be convenient, we denote $p_{n+1} = p_1$ and $\theta_{n+1} = 2\pi$. Define

$$\alpha_j = \{e^{i\theta} : \theta_j < \theta < \theta_{j+1}\}, \quad 1 \le j \le n. \tag{15}$$

Next we claim that X_3 approachs $\pm\infty$ alternatively on p_j. Otherwise, X_3 would approach ∞ at both p_j and p_{j+1}, and X_3 has a minimum in α_j, therefore X maps α_j in a ray. This is a contradiction to the fact that $X : \partial M \to T_n$ is one-to-one and onto. □

We will study only simply connected, proper, branched, confined minimal surfaces $X : M \to \mathbb{R}^3$ such that $X : \partial M \to T_n$ is one-to-one and onto. We will call such a surface T_n**-surface**. For simplicity, by (14) and (15), we may assume that

(16)

$$M = \overline{\mathbb{D}} - \{1, e^{i\theta_2}, \ldots, e^{i\theta_n}\} = \mathbb{D} \cup (\cup_{j=1}^{n} \alpha_j), \quad \mathrm{Int}M = \mathbb{D}, \quad \partial M = \cup_{j=1}^{n} \alpha_j.$$

A T_n-surface, of course, depends on the boundary T_n. Denote by $P_{i,j}$ the planes passing through L_i and L_j and $B_{i,j}$ the band in $P_{i,j}$ bounded by $L_i \cup L_j$. Let $\{q_j\} = L_j \cap P_0$ and $l_{i,j}$ be the line segments $P_0 \cap B_{i,j}$, i.e., it is the segment connecting q_i and q_j. Let $q_{i,j}$ be the middle point of $l_{i,j}$ and $\vec{n}_{i,j}$ be a unit vector normal to $P_{i,j}$. We will denote $L_{n+1} = L_1$, and denote for $1 \le j \le n$, $P_{j,j+1}$, $B_{j,j+1}$, $l_{j,j+1}$, $q_{j,j+1}$, and $\vec{n}_{j,j+1}$, accordingly.

Furthermore we fix our notation such that $X(\alpha_j) = L_j$, $1 \le j \le n$. By Lemma 4.1, without loss of generality, we will alway assume that for $1 \le j \le m$,

$$\lim_{z\to p_{2j-1}} X_3(z) = +\infty, \quad \lim_{z\to p_{2j}} X_3(z) = -\infty. \tag{17}$$

Let D_j be small disks centred at p_j such that $D_j \cap D_k = \emptyset$, $1 \le j, k \le n$, and

$$D_j \cap \partial\mathbb{D} \subset \alpha_j \cup \alpha_{j+1} \cup \{p_j\}.$$

Lemma 4.2. *Let $X : M \to \mathbb{R}^3$ be a T_n-surface, then*

$$\lim_{z\to p_j} [(X_1(z), X_2(z)) - q_{j,j+1}] \bullet \vec{n}_{j,j+1} = 0, \quad 1 \le j \le n. \tag{18}$$

Moreover, $f_j := [(X_1(z), X_2(z)) - q_{j,j+1}] \bullet \vec{n}_{j,j+1}$ can be harmonically extended to D_j. In particular, Df_j is bounded in $D_j \cap M$.

Proof. f_j is harmonic in $\mathbb{D}$ and $f_j \equiv 0$ on $\alpha_j \cup \alpha_{j+1}$. Hence $M_j = D_j \cap M$ is parabolic and f_j is a bounded harmonic function on M_j. Then f_j can be extended across $\alpha_j \cup \alpha_{j+1}$ to be a bounded harmonic function on $D_j - \{p_j\}$ and we can define $\widetilde{f}_j(p_j) = 0$ and (18) is true. □

Theorem 4.1. *The only T_2-surface is the band $B_{1,2}$.*

Proof. Let $X : M \to \mathbb{R}^3$ be a T_2 surface. We may assume that $P_{1,2}$ is parallel to the x_2x_3-plane. Then by Lemma 4.2 X_1 is a bounded harmonic function such that $X_1 \equiv 0$ on ∂M. By Corollary 2.1, M is parabolic. Hence $X_1 \equiv 0$ on M. □

From now on, we assume that $n = 2m \geq 4$.

Each T_n decides polygons (maybe more than one) such that q_j are the vertices or boundary points. When such a polygon is convex, then it is the unique polygon with q_j as vertices or boundary points, we call it $Q(T_n)$. We will denote the interior of $Q(T_n)$ as $\Omega(T_n)$. It may happen that $Q(T_n)$ is a convex k-gon with $3 \leq k \leq n$, but $\{q_j\}_{j=1}^n$ contains all the vertices.

Lemma 4.3. *Let $X : M \to \mathbb{R}^3$ be a T_n-surface. If $Q(T_n)$ is a convex n-gon then $\mathcal{P} \circ X(\mathrm{Int}M) \subset \Omega(T_n)$.*

Proof. Since M is parabolic, this is a special case of the maximum principle for bounded harmonic functions. □

Lemma 4.4. *Let $X : M \to \mathbb{R}^3$ be a T_n-surface. If $Q(T_n)$ is a convex n-gon and $\Omega(T_n) = \mathcal{P} \circ X(\mathrm{Int}M)$, then*

$$\partial Q(T_n) = \cup_{j=1}^n l_{j,j+1}.$$

Proof. If some $l_{j,j+1}$ is a diagonal of $Q(T_n)$, then there will be a segment l of $\partial Q(T_n)$ such that $l \neq l_{j,j+1}$ for any $1 \leq j \leq n$. Let Int$l$ be the interior of l. Since $\Omega(T_n) = \mathcal{P} \circ X(\mathrm{Int}M)$, there are points $z_k \in \mathbb{D} = \mathrm{Int}M$ such that $(X_1, X_2)(z_k) \to q \in \mathrm{Int}l$ as $k \to \infty$. Since $\overline{\mathbb{D}}$ is compact, passing to a subsequence if necessary, we may assume that $z_k \to z_0 \in \overline{\mathbb{D}}$ as $k \to \infty$. By Lemma 4.3 and the maximum principle for bounded harmonic functions on parabolic Riemann surfaces, $z_0 \in \partial\mathbb{D}$. But by Lemma 4.2, $(X_1, X_2)(z_k)$ approachs one of the $l_{j,j+1}$s, which is disjoint to Int$l$, a contradiction. This contradiction proves this lemma. □

Remark 4.1. Rotate a Jenkins-Serrin graph (over a convex $2m$-gon) around any of the boundary lines will give us a properly embedded minimal surfaces bounded by $2(2m-1)$ parallel lines. But the projection on $\mathbb{R}^2$ does not satisfy the condition in Lemma 4.4.

Recall that the spherical map (unit normal map) of a regular surface $X : M \to \mathbb{R}^3$ is given by

$$N(p) = \frac{X_u \wedge X_v}{|X_u \wedge X_v|}(p) \in S^2, \quad p \in M, \quad (u,v) \text{ are local coordinates.}$$

In case of branched minimal surfaces, N is also well-defined in branch points, therefore, at each point of a branched minimal surface the tangent plane is well-defined.

Lemma 4.5. *Let* $X : M \to \mathbb{R}^3$ *be a* T_n*-surface. If* $Q(T_n)$ *is a convex* n*-gon and* $\Omega(T_n) = \mathcal{P} \circ X(\mathrm{Int}M)$*, then the spherical map* N *maps* $\mathbb{D} = \mathrm{Int}M$ *into either the lower or upper hemisphere of* S^2*.*

Proof. We only need prove that for any $z \in \mathbb{D} = \mathrm{Int}M$, the tangent plane $T_{X(z)}$ is not perpendicular to the x_1x_2 plane. Consider the analytic variety $X^{-1}(T_{X(z)})$ and its closure S in $\overline{\mathbb{D}}$. Since $Q(T_n)$ is convex and $X : \partial M \to T_n$ is one-to-one and onto, a component of $S \cap \partial\mathbb{D}$ is either a single point in $\{p_1, \ldots, p_n\}$ or some $\overline{\alpha_j}$, the latter case happens when $T_{X(z)} \supset L_j$.

It is well-known that there are at least four curves in S intersecting at $z \in S$, see, for example, [**18**]. Therefore, $S \cap \partial\mathbb{D}$ must have at least four components, otherwise S will bound a domain $D \subset \mathbb{D}$ such that $\partial D \subset S$ and $X(\partial D) \subset T_{X(z)}$. By Corollary 2.1, the closure of D in M is parabolic. By the maximum principle, $X(D) \subset T_{X(z)}$. Hence $X(M)$ is flat, we have a contradiction. This contradiction proves that $S \cap D$ has at least four components.

By Lemma 4.4 $\partial Q(T_n) = \cup_{j=1}^n l_{j,j+1}$. By the maximum principle for bounded harmonic functions on parabolic surfaces, $T_{X(z)}$ cannot contain any $l_{j,j+1}$. Moreover, since $Q(T_n)$ is a convex n-gon, $T_{X(z)}$ intersects exactly two $l_{j,j+1}$, i.e., $S \cap \partial\mathbb{D}$ can have at most two components, a contradiction. This contradiction proves this lemma. □

Theorem 4.2. *Let* $X : M \to \mathbb{R}^3$ *be a* T_n*-surface. If* $Q(T_n)$ *is a convex* k*-gon,* $3 \le k \le n$*, and* $\Omega(T_n) = \mathcal{P} \circ X(\mathrm{Int}M)$*, then* $X(\mathrm{Int}M)$ *is a Jenkins-Serrin minimal graph over* $\Omega(T_n)$*. In particular,* $X : M \to \mathbb{R}^3$ *is an embedding.*

Proof. Let P be any vertical plane such that $P \cap \Omega(T_n) \neq \emptyset$. We claim that $X^{-1}(P) \cap \mathrm{Int}M$ consist of one-dimensional manifolds. In fact, by Lemma 4.5, $X(M)$ is transversal to P except at branch points, thus $X^{-1}(P) \cap \mathrm{Int}\, M$ is an analytic variety. A similar argument as in the proof of Theorem 1.1 shows that X does not have interior branch points. Hence we know that $X^{-1}(P) \cap \mathrm{Int}M$ consist of one-dimensional manifolds.

We claim that $X^{-1}(P) \cap \mathrm{Int}M$ is connected. In fact, a similar argument as in the proof of Lemma 4.5 shows that the closure of $X^{-1}(P)$ in $\overline{\mathbb{D}}$ has exactly two components in $\partial\mathbb{D}$. Denote this closure by S. If $X^{-1}(P) \cap \mathrm{Int}M$ has more than one components, then since M is simply connected, there would be a domain $D \subset \mathbb{D}$ such that $\partial D \subset S$. Therefore, $X(D) \subset P$, we have a contradiction.

Next we prove that $X(\mathrm{Int}M) \cap P$ must be a graph over $l_P = P \cap \Omega(T_n)$. Since $X^{-1}(P) \cap \mathrm{Int}M$ is a connected one-dimensional manifold, we only need prove that the tangent vector of $X(\mathrm{Int}M) \cap P$ is never vertical. In fact if $V = (0, 0, \pm 1)$ is tangent to $X(\mathrm{Int}M) \cap P$ at $X(z)$, then $N_3(z) = 0$, a contradiction to Lemma 4.5. □

Theorem 4.3. *Suppose $Q(T_n)$ is a convex k-gon, $3 \leq k \leq n$, $n = 2m \geq 4$. Then there exists a unique T_n-surface such that $\Omega(T_n) = \mathcal{P} \circ X(\mathrm{Int}M)$ if and only if*

$$\sum_{j=1}^{m} |l_{2j-1,2j}| = \sum_{j=1}^{m} |l_{2j,2j+1}|. \tag{19}$$

Proof. By Theorem 4.2 any such a T_n surface will be a Jenkins-Serrin minimal graph generated by $u : \Omega(T_n) \to \mathbb{R}$ such that $u(x_1, x_2) \to \infty$ when (x_1, x_2) approachs $l_{2j-1,2j}$ and $u(x_1, x_2) \to -\infty$ when (x_1, x_2) approachs $l_{2j,2j+1}$.

By [**10**], for each T_n there is at most one Jenkins-Serrin minimal graph and Equation (19) is the necessary and sufficient condition for the existence of such a minimal graph. □

The combination of Theorem 4.2 and 4.3 is Theorem 1.2. The proof of Theorem 1.2 is now complete.

5. Parabolicity in higher dimensions.

Let (M, g) be a Riemann manifold. We can define (M, g) to be parabolic by using the same definition as used for the Riemann surfaces, regardless $\partial M \neq \emptyset$ or $\partial M = \emptyset$.

Let δ be the Euclidean metric in $\mathbb{R}^n$. Recall that $X : M \to \mathbb{R}^n$ is an $m \leq n$ dimensional minimal submanifold if and only if under the induced Riemann metric $g = X^*(\delta)$, X is harmonic.

Then Lemma 2.1, Corollaries 2.1 and 2.2, Theorems 2.1 and 2.3 have corresponding generalizations in higher dimensions.

For example, Lemma 2.1 can be stated as follows.

Lemma 5.1. *Let S^{n-1} be the $n-1$ sphere in $\mathbb{R}^n$, $n \geq 3$, and $\vec{n} \in S^{n-1}$. Let $X : M \to \mathbb{R}^n$ be an noncompact minimal submanifold. If $X \bullet \vec{n}$ is a proper function, then $(M, X^*(\delta))$ is parabolic.*

Theorem 2.1 can be stated as follows.

We say that a system of planes $\{P_j\}_{j=1}^k$ is linear independent if the corresponding normals of $\{P_j\}_{j=1}^k$ are linear independent.

Theorem 5.1. *If $X : M \to \mathbb{R}^n$ is an noncompact proper minimal submanifold such that there is a linear independent system of planes $\{P_j\}_{j=1}^{n-1}$ such that $X(M) \cap P_j$ is compact for $1 \leq j \leq n-1$, then $(M, X^*(\delta))$ is parabolic.*

Remark 5.1. Conjecture 2.1 is not true for higher dimensional minimal surfaces in $\mathbb{R}^n$. For example, let B_1 be the open unit ball in $\mathbb{R}^3$, then $\mathbb{R}^3 - B_1 \subset \mathbb{R}^4$ is a proper minimal embedding, but $\mathbb{R}^3 - B_1$ is not parabolic.

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