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Let K be any field of characteristic zero. We show that there are at least four ideals in the group algebra KG of every simple locally finite group G of 1-type, thus providing the final step in solving an old question of I. Kaplansky's for locally finite groups. We also determine the ideal lattice in KG for those 1-type groups G which are a direct limit of finite direct products of alternating groups.

1. Introduction.

It is an old question due to I. Kaplansky [**K**] for which groups G and which fields $\mathbb{K}$ the augmentation ideal $\omega(\mathbb{K}G)$ is the only nonzero proper two-sided ideal in $\mathbb{K}G$. Every such group G must necessarily be simple. Kaplansky's question was the starting point for a research programme begun by A.E. Zalesskiĭ to determine the ideal lattices of complex group algebras of infinite simple locally finite groups G. Here, a group is said to be *locally finite* if it is a direct limit of finite groups. The investigation of this topic leads naturally into the realm of character theory of finite groups, since ideals in $\mathbb{C}G$ correspond naturally with certain systems of irreducible complex characters of the finite subgroups of G ([**Z2**], Proposition 1.2). Thus one can use the highly developed theory of characters to obtain information about ideals in group algebras of locally finite groups. And conversely results about such ideals can be interpreted in the spirit of the asymptotic theory of finite groups.

At present, every infinite simple locally finite group G can be sorted into one of the following classes: Finitary linear groups, groups of 1-type, groups of p-type (where p is a prime), and groups of ∞-type. The definition of these classes is given in Section 2. For any field $\mathbb{K}$ of characteristic zero it is known that the augmentation ideal is the only proper ideal in $\mathbb{K}G$ when G is finitary linear with natural representation over an infinite field ([**HZ1**], Theorem B, [**HZ3**], Theorem 1.1, [**LP**], Theorem 4.5), or when G is of p- or ∞-type ([**LP**], Corollary 2.2). On the other hand, if G is a finitary linear group defined over a finite field, then many ideals can be expected ([**HZ2**]; [**Z2**], Proposition 2.6; [**LP**]). Thus, as far as infinite locally finite groups are concerned, Kaplansky's question is just open for groups of 1-type. In the present paper we shall prove the following result.

Theorem A. *Let G be an infinite simple locally finite group of* 1*-type, and let $\mathbb{K}$ be any field of characteristic zero. Then $\mathbb{K}G$ has at least four two-sided ideals.*

In general, it seems to be quite difficult to describe the full ideal lattice of $\mathbb{K}G$ for groups of 1-type. However, we shall do so for an interesting subclass, namely for those simple locally finite groups of 1-type which are direct limits of finite direct products of alternating groups. We shall call these groups LD$\mathfrak{A}$*-groups.* Note that Theorem A was established for LD$\mathfrak{A}$-groups in [**HZ3**], Lemma 5.8.

Theorem B. *Let G be an* LD$\mathfrak{A}$*-group, and let $\mathbb{K}$ be any field of characteristic zero. Then the lattice of two-sided ideals in $\mathbb{K}G$ has the form $\mathbb{K}G = I_0 \supset \omega(\mathbb{K}G) = I_1 \supset I_2 \supset I_3 \supset \cdots \supset \bigcap_{n\in\omega} I_n = 0$.*

Both theorems will be proved for countable groups first, and then be extended to groups of arbitrary cardinality by means of the Löwenheim-Skolem Theorem. One of the key results used in the proof of Theorem A is due to U. Meierfrankenfeld (see Proof of Lemma 3.1). Theorem B generalizes the corresponding result for direct limits of finite alternating groups ([**Z1**], Theorem 1). The proof of the countable version closely follows the arguments of [**Z1**]. In particular it relies heavily on the representation theory of finite alternating groups.

2. Preliminaries.

For the convenience of the reader, we shall collect some basic notions in this section. First of all recall that a *Kegel cover* in the simple locally finite group G is a family $\{(G_i, M_i) \mid i \in I\}$, such that for each i, M_i is a maximal normal subgroup in the finite subgroup G_i of G, and such that for every finite subgroup F of G there exists some $i \in I$ satisfying $F \leq G_i$ and $F \cap M_i = 1$. The sections G_i/M_i are called the *Kegel factors* of the Kegel cover. It was proved in [**KW**] that every simple locally finite group has a Kegel cover. When G is countable, the subgroups G_i can be chosen in such a way that they form an ascending chain $G_0 \subset G_1 \subset G_2 \subset \dots$ with $G_i \cap M_{i+1} = 1$ for all i. A Kegel cover of this kind is called a *Kegel sequence.*

A simple locally finite group G is said to be:

- *Finitary linear* if it acts faithfully on a (possibly infinite-dimensional) vector space V in such a way that every $g \in G$ fixes a subspace of finite codimension in V pointwise,
- of 1*-type* if it is not finitary linear, and if every Kegel cover of G has an alternating Kegel factor,

- of *p-type* if it is not finitary linear, and if every Kegel cover of G has a Kegel factor which is a classical group over a field of characteristic p,
- of ∞-*type* if it is not finitary linear, and neither of 1-type nor of p-type for any prime p.

It is clear that every countable simple locally finite group of 1-type has a Kegel sequence with alternating factors.

The above classification can be found in S. Delcroix' Ph.D. thesis [**D**], Section 3.3. There in fact a simple locally finite group of ∞-type is characterized by the property that for every finite subgroup F of G there exist a finite subgroup H of G and a maximal normal subgroup M in H such that:

(1) $H/M \cong \mathrm{Alt}(\Omega)$ for some finite set Ω,
(2) $F \leq H$ and $F \cap M = 1$, and
(3) F has a regular orbit on Ω.

In contrast to this, the following notion of diagonal embedding is introduced in [**PZ**]. Suppose that H_i $(i = 1, 2)$ are finite groups with maximal normal subgroups M_i and alternating quotients $H_i/M_i \cong \mathrm{Alt}(\Omega_i)$. Then an embedding $H_1 \to H_2$ is said to be *diagonal* if in the natural action of H_2 on Ω_2, the index $|H_1 : (H_1)_\omega M_1|$ is either 1 or $|\Omega_1|$ for each point $\omega \in \Omega_2$. Here $(H_1)_\omega$ denotes the point stabilizer of ω in H_1. The following proposition is a straight consequence of [**PZ**], Theorem 1.7 and of the above characterization of simple locally finite groups of ∞-type.

Proposition 2.1. *Let $\{(H_i, M_i) \mid i \in \omega\}$ be a Kegel sequence with alternating Kegel factors in the countable simple locally finite group G of 1-type. Then there exist positive integers $i_0 < i_1 < i_2 < \dots$ such that the inclusion $H_{i_k} \to H_{i_\ell}$ is a diagonal embedding for all $k < \ell$.*

An important tool in the study of the (two-sided) ideals of the group algebra $\mathbb{K}G$ of a locally finite group G, are the so-called *inductive systems* with respect to a fixed local system $\{G_i \mid i \in I\}$ of finite subgroups in G. By definition, an inductive system $\Phi = \{\Phi_i \mid i \in I\}$ is a set where:

(1) Each Φ_i is a subset of the set $\mathrm{Irr}_{\mathbb{K}}(G_i)$ of all irreducible $\mathbb{K}G_i$-representations, and
(2) whenever $G_i \leq G_j$, then Φ_i consists precisely of the irreducible $\mathbb{K}G_i$-constituents of $\mathbb{K}G_j$-representations from Φ_j.

Inductive systems are linked with ideals via the following result of A.E. Zalesskiĭ.

Proposition 2.2 ([**Z2**], Proposition 1.2)**.** *Let G be a locally finite group, and let $\mathbb{K}$ be a field of characteristic zero. Then the inductive systems with respect to a fixed local system of finite subgroups in G are in order reversing one-to-one correspondence with the ideals in $\mathbb{K}G$.*

Here an ideal J corresponds to the inductive system $\Phi = \{\Phi_i \mid i \in I\}$, where Φ_i consists of the irreducible $\mathbb{K}G_i$-representations corresponding to the irreducible $\mathbb{K}G_i$-modules occurring in $\mathbb{K}G_i/(J \cap \mathbb{K}G_i)$. In particular, the zero ideal corresponds to $\{\mathrm{Irr}_{\mathbb{K}}(G_i) \mid i \in I\}$, the augmentation ideal corresponds to $\{\{1_{G_i}\} \mid i \in I\}$, and $\mathbb{K}G$ corresponds to $\{\emptyset \mid i \in I\}$.

3. Groups of 1-type.

This section is devoted to the proof of Theorem A for countable groups G. *We therefore suppose throughout that G is a countable simple locally finite group of 1-type with a Kegel sequence $\mathcal{H} = \{(H_i, M_i) \mid i \in \omega\}$ such that $H_i/M_i \cong \mathrm{Alt}(\Omega_i)$.* From Proposition 2.1 we may assume further that all the inclusions $H_i \to H_k$ $(i < k)$ are diagonal.

Lemma 3.1. *For each $i \geq 1$ there is some $j > i$ such that the following hold:*

- (a) *There exist an H_i-orbit Θ_i in Ω_j and a maximal H_i-system of imprimitivity $\mathcal{D}_i$ in Θ_i such that $\mathcal{D}_i$ is equivalent to Ω_i as an H_i-set.*
- (b) *The stabilizer in H_i of any point $\omega \in \Omega_i$ contains the stabilizer of every point $\delta \in \Theta_i$ which is contained in the block $\Delta \in \mathcal{D}_i$ corresponding to ω with respect to the above equivalence.*

Proof. (a) Because all the inclusions $H_{i-1} \to H_k$ $(i \leq k)$ are diagonal, H_{i-1} does not have a regular orbit on Ω_k. Since G is not finitary, we must have $\liminf_{k\to\infty} |\mathrm{supp}_{\Omega_k} h| = \infty$ for every nontrivial $h \in H_{i-1}$ – otherwise an ultraproduct argument would yield a finitary permutation representation for G (see [**H1**]). Therefore [**M**], Lemma 2.14 provides some $j > i$ and a positive integer $t \leq |H_{i-1}| - 2$ such that H_i has a *t-pseudo natural orbit* Θ_i on Ω_j with respect to M_i. By definition this means that M_i is the stabilizer of a maximal H_i-system of imprimitivity $\mathcal{D}_i$ in Θ_i, and that the action of H_i on $\mathcal{D}_i$ is equivalent to the action of H_i on the set of t-subsets of Ω_i.

Consider some $\Delta \in \mathcal{D}_i$ and $\delta \in \Delta$. Clearly $S = N_{H_i}(\Delta)$ is a point-stabilizer in the action of H_i on $\mathcal{D}_i$, and $(H_i)_\delta M_i \leq S$. In particular, the index $|H_i : (H_i)_\delta M_i|$ exceeds 1 and must therefore be equal to $|\Omega_i|$. But then $S = (H_i)_\delta\, M_i$, because there are no proper subgroups of index smaller than $|\Omega_i|$ in $\mathrm{Alt}(\Omega_i)$. Thus $|\mathcal{D}_i| = |\Omega_i|$, and H_i/M_i acts naturally on $\mathcal{D}_i$, that is, $\mathcal{D}_i$ and Ω_i are equivalent as H_i-sets.

(b) follows immediately from (a), since $(H_i)_\delta \leq N_{H_i}(\Delta)$ whenever $\delta \in \Delta \in \mathcal{D}_i$. □

From passing to a subsequence of the Kegel sequence $\mathcal{H}$ we may clearly assume in the sequel that $j = i + 1$ for all i in Lemma 3.1. We are now well-prepared to construct a permutation representation of the group G.

To this end, we consider the subset Ω of the Cartesian product $\prod_{i\in\omega} \Omega_i$, which consists of all tupels $(\omega_i)_{i\in\omega}$ with the following property: There exists m (depending on the particular tuple) such that for all $i \geq m$ the point $\omega_{i+1} \in \Omega_{i+1}$ is contained in the block in $\mathcal{D}_i$ which corresponds to the point

$\omega_i \in \Omega_i$ with respect to the H_i-equivalence between $\mathcal{D}_i$ and Ω_i furnished by Lemma 3.1. Such a tuple is said to be *nice from m onwards.* We define an equivalence relation $\sim$ on Ω via

$$(\omega_i)_{i\in\omega} \sim (\nu_i)_{i\in\omega} \quad \text{if and only if} \quad \omega_i = \nu_i \quad \text{for all but finitely many } i.$$

Let $\overline{\Omega} = \Omega/\sim$ denote the set of equivalence classes $[\omega_i]_{i\in\omega}$ of tupels $(\omega_i)_{i\in\omega} \in \Omega$ modulo $\sim$. Since every H_i acts on Ω_j for all $j \geq i$, there is a well-defined componentwise action of G on $\overline{\Omega}$.

Lemma 3.2. *For each $\omega = [\omega_i]_{i\in\omega} \in \overline{\Omega}$ and every $k \geq 1$ there exists $n \geq k$ such that the orbit ω^{H_k} in $\overline{\Omega}$ is H_k-equivalent to the orbit $\omega_n^{H_k}$ in Ω_n. Moreover, if ω is nice from k onwards, then M_k acts intransitively on ω^{H_k}.*

Proof. The tuple ω is nice from some $m \geq k$ onwards. By Lemma 3.1 the point stabilizers $(H_k)_{\omega_\ell}$ $(\ell \geq m)$ form a descending chain. By finiteness of H_k the chain must become stationary. But then $(H_k)_\omega = \bigcap_{\ell\geq m}(H_k)_{\omega_\ell} = (H_k)_{\omega_n}$ for some $n \geq m$. Suppose now that $m = k$. As H_k acts naturally on Ω_k, it is clear that $M_k(H_k)_{\omega_n} \subseteq (H_k)_{\omega_k} \neq H_k$. Hence M_k acts intransitively on $\omega_n^{H_k}$. □

We are now in a position to prove the main result of this section.

Theorem 3.3. *Let G be a countable simple locally finite group of 1-type. For any field $\mathbb{K}$ of characteristic zero, the group algebra $\mathbb{K}G$ has at least four two-sided ideals.*

Proof. Consider the permutation module $V = \mathbb{K}\Gamma$ for some G-orbit Γ in $\overline{\Omega}$ containing a tuple which is nice from $m = 1$ onwards. Let $\Phi = \{\Phi_i \mid i \in \omega\}$ be the inductive system with respect to $\mathcal{H}$ where Φ_i is the set of irreducible $\mathbb{K}H_i$-constituents of the representation of G on V. For each i we shall show that $\Psi_i = \{\varphi \in \Phi_i \mid M_i \subseteq \ker\varphi\}$ equals $\{1_{H_i}, \eta_i\}$ where η_i denotes the nontrivial irreducible constituent of the natural permutation representation of H_i on Ω_i.

Consider some $\omega = [\omega_j]_{j\in\omega} \in \Gamma$. From Lemma 3.2 there exists $n > i$ such that the orbit ω^{H_i} is H_i-equivalent to the orbit $\Delta = \omega_n^{H_i}$ in Ω_n. If M_i acts transitively on Δ, then $\mathrm{fix}_{\mathbb{K}\Delta}(M_i)$ is spanned by the sum of the elements of $\mathbb{K}\Delta$, whence $\mathrm{fix}_{\mathbb{K}\Delta}(M_i)$ is 1-dimensional and the trivial H_i-module. However, Γ also contains a tuple ω which is nice from $m = 1$ onwards, and in this situation Lemma 3.2 ensures that M_i acts intransitively on Δ. The M_i-orbits in Δ form an H_i-system of imprimitivity $\mathcal{S} = \{\Sigma_1, \ldots, \Sigma_r\}$. Clearly M_i is the stabilizer of $\mathcal{S}$. Since the inclusion $H_i \to H_n$ is diagonal, it follows as in the proof of Lemma 3.1 that the alternating group H_i/M_i acts naturally on $\mathcal{S}$. It is readily seen that $\mathrm{fix}_{\mathbb{K}\Delta}(M_i) = \mathbb{K}s_1 \oplus \cdots \oplus \mathbb{K}s_r$ with $s_k = \sum_{\sigma\in\Sigma_k}\sigma$. But the alternating group H_i/M_i acts naturally on the set $\{s_1, \ldots, s_r\}$. We conclude that $\mathrm{fix}_{\mathbb{K}\Delta}(M_i)$ is isomorphic to the natural $\mathbb{K}H_i$-module Ω_i. By

Maschke's Theorem, $\mathrm{fix}_{\mathbb{K}\Delta}(M_i)$ contains every irreducible H_i-submodule of $\mathbb{K}\Delta$ on which M_i acts trivially. Therefore $\{1_{H_i}, \eta_i\} = \Psi_i$.

Since every representation of the alternating quotient H_i/M_i lifts to a representation of H_i, it is now clear that $\Psi_i = \{1_{H_i}, \eta_i\}$ is a proper subset of $\{\varphi \in \mathrm{Irr}_{\mathbb{K}}(H_i) \mid M_i \subseteq \ker\varphi\}$, whence Φ_i is a proper subset of $\mathrm{Irr}_{\mathbb{K}}(H_i)$. Therefore Proposition 2.2 links Φ_i to a proper ideal in the augmentation ideal of $\mathbb{K}G$. □

4. LD$\mathfrak{A}$-groups.

In this section we shall completely describe the ideal lattice of countable LD$\mathfrak{A}$-groups. Again the generalization to uncountable groups will be deferred to Section 5. Since the proof of our result follows the ideas contained in [**Z1**], the reader is referred to that paper for all the definitions related to the representation theory of alternating groups.

Suppose then that G is a countable LD$\mathfrak{A}$-group. G is the union of an ascending chain $G_0 < G_1 < G_2 < \dots$ of finite subgroups where

$$G_i = A_{i,1} \times A_{i,2} \times \cdots \times A_{i,d_i}$$

with $A_{i,j} = \mathrm{Alt}(\Omega_{i,j})$ for suitable finite sets $\Omega_{i,j}$. Note that G_i acts on the disjoint union $\Omega_i = \Omega_{i,1} \,\dot\cup\, \cdots \,\dot\cup\, \Omega_{i,d_i}$ such that $A_{i,\nu}$ acts naturally on $\Omega_{i,\nu}$ while

$$M_{i,\nu} = A_{i,1} \times \cdots \times A_{i,\nu-1} \times A_{i,\nu+1} \times \cdots \times A_{i,d_i}$$

fixes $\Omega_{i,\nu}$ pointwise. The above chain of finite subgroups G_i in G is said to be *of strongly diagonal type* if for all pairs $i \le j$ and every nontrivial G_i-orbit Δ on $\Omega_j = \Omega_{j,1} \,\dot\cup\, \cdots \,\dot\cup\, \Omega_{j,d_j}$ there exists $\nu \in \{1, \dots, d_i\}$ such that $A_{i,\nu}$ acts naturally on Δ while $M_{i,\nu}$ fixes Δ pointwise. On the other hand the chain is said to be *of regular type* if G_i has a regular orbit on Ω_j whenever $i < j$.

It was proved in [**HZ3**], Theorem 5.4 that any chain of the above kind has a subchain of either strongly diagonal or regular type. In view of this fact we readily have that our countable LD$\mathfrak{A}$-group, which is of course not of ∞-type, is the union of a chain of strongly diagonal type. We can still improve this chain slightly.

Lemma 4.1. *Every countable* LD$\mathfrak{A}$*-group G is the union of an ascending chain $G_0 < G_1 < G_2 < \dots$ of finite direct products of finite alternating groups as above such that the chain is of strongly diagonal type, such that $G_i \cap M_{i+1,j} = 1$ for all i and all $j \in \{1, \dots, d_{i+1}\}$, and such that $A_{i,\nu}$ has at least $i+2$ nontrivial orbits on $\Omega_{i+1,j}$ for all i, all $\nu \in \{1, \dots, d_i\}$, and all $j \in \{1, \dots, d_{i+1}\}$.*

Proof. G is already the union of a chain $\mathcal{H} = \{H_i \mid i \in \omega\}$ of strongly diagonal type. The terms G_i of the desired chain are constructed recursively from $\mathcal{H}$ as follows. Let $G_0 = H_0$ and assume that G_i has been found for some

i. Consider $F = \langle H_i, G_i \rangle$. Since G is simple and locally finite, there exists a positive integer n such that $x \in \langle y^{H_n} \rangle$ for all $x \in F$ and every nontrivial $y \in F$. Let G_{i+1} be the normal closure of F in H_n. Because H_n is a direct product of simple groups, every normal subgroup of G_{i+1} is normal in H_n too. Hence any proper normal subgroup of G_{i+1} does not contain F. By choice of H_n, it must then have trivial intersection with F and in particular with G_i. This completes the construction of a chain satisfying the first two requirements.

If there would exist integers ν_j $(j \geq i)$ such that the number of orbits of A_{i,ν_i} in Ω_{j,ν_j} is uniformly bounded for all $j \geq i$, then an ultraproduct argument as in [**H1**] would show that the group G is finitary linear, a contradiction. Therefore an infinite subchain of the chain constructed so far also satisfies the third requirement. $\square$

In the sequel we shall consider all representations over a fixed field $\mathbb{K}$ *of characteristic zero.* The following facts can be extracted from [**Z1**].

To each irreducible representation φ of $\mathrm{Alt}(n)$ we can associate a Young diagram which is completely described by a finite descending sequence $(l_1, l_2, \ldots, l_k)$ of positive integers with sum n; here l_i is the number of cells in the i-th row of the Young diagram. This diagram will be denoted by $D(\varphi)$. The *depth* $\delta(\varphi)$ of the representation φ is defined as $\delta(\varphi) = l_2 + \cdots + l_k$. It is clear that the trivial representation is the unique irreducible representation of depth zero. The unique irreducible representation of depth one has the diagram described by the sequence $(n-1, 1)$ and will be denoted by η_n.

Lemma 4.2 ([**Z1**], Lemma 4)**.** *Let* $n = n_0 + \cdots + n_k$ *with* $n_0 \geq 0$ *and* $n_i > 3$ *for all* $i \geq 1$. *Assume that* ψ *is an irreducible representation of* $\mathrm{Alt}(n)$ *of depth* k. *If* $H = \mathrm{Alt}(n_0) \times \mathrm{Alt}(n_1) \times \cdots \times \mathrm{Alt}(n_k)$ *is embedded naturally in* $\mathrm{Alt}(n)$, *then* $\psi|_H$ *contains the irreducible component* $1_{\mathrm{Alt}(n_0)} \otimes \eta_{n_1} \otimes \cdots \otimes \eta_{n_k}$.

Lemma 4.3 ([**Z1**], Lemma 5)**.** *Let* $m > 1$. *Then*

$$\{\psi \in \mathrm{Irr}_{\mathbb{K}}(\mathrm{Alt}(n)) \mid \delta(\psi) \leq m\}$$

is the set of all irreducible components of the representation $\bigotimes_{i=1}^{m} \eta_n$ *of* $\mathrm{Alt}(n)$.

Lemma 4.4 ([**Z1**], Lemma 11)**.** *Let* $\mathrm{Alt}(k) \to \mathrm{Alt}(n)$ *be a strongly diagonal embedding, and suppose that* φ *is an irreducible representation of* $\mathrm{Alt}(n)$ *of depth* m. *If* $k > \max\{2m, 4\}$, *then every irreducible constituent of* $\varphi|_{\mathrm{Alt}(k)}$ *has degree at most* m.

We shall now combine the preceding three lemmata to establish the following result.

Proposition 4.5. *Suppose that the group* $H = \mathrm{Alt}(n_1) \times \cdots \times \mathrm{Alt}(n_k)$ *is strongly diagonally embedded in* $\mathrm{Alt}(n)$. *Assume further that there exists*

$m \in \omega$ such that $n_i > 2m + 2$ for each i, and such that each $\mathrm{Alt}(n_i)$ *has at least $m + 2$ nontrivial orbits on $\Omega = \{1, \ldots, n\}$, the natural set for* $\mathrm{Alt}(n)$.

(a) *Let φ be any irreducible representation of* $\mathrm{Alt}(n)$ *of depth m, and choose irreducible representations σ_i of* $\mathrm{Alt}(n_i)$ *in such a way that $\delta(\sigma_1) + \cdots + \delta(\sigma_k) \leq m$. Then $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ is a component of the restriction $\varphi|_H$.*

(b) *Conversely, if φ is any irreducible representation of* $\mathrm{Alt}(n)$ *of depth m and $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ is an irreducible H-constituent of φ, then $\delta(\sigma_1) + \cdots + \delta(\sigma_k) \leq m$.*

Proof. Since the result is trivial for $m = 0$, we assume in the sequel that $m \geq 1$.

(a) For each $i \in \{1, \ldots, k\}$, select $m_i = \delta(\sigma_i) + 2$ nontrivial $\mathrm{Alt}(n_i)$-orbits $\Delta_{i,j}$ $(1 \leq j \leq m_i)$ in Ω. Since H is embedded strongly diagonally in $\mathrm{Alt}(n)$, all these orbits are pairwise distinct H-orbits. Let $\Delta_0 = \Omega \backslash \bigcup_{i,j} \Delta_{i,j}$. The group H is contained in the subgroup $K = \mathrm{Alt}(\Delta_0) \times K_1 \times \cdots \times K_k$ of $\mathrm{Alt}(n)$, where $K_i = \mathrm{Alt}(\Delta_{i,1}) \times \cdots \times \mathrm{Alt}(\Delta_{i,m_i})$. And the canonical projection $\pi_i \colon K \to K_i$ embeds $\mathrm{Alt}(n_i)$ onto the diagonal subgroup of K_i.

By Lemma 4.2, the restriction $\varphi|_K$ contains the irreducible component $\psi = 1_{\mathrm{Alt}(\Delta_0)} \otimes \psi_1 \otimes \cdots \otimes \psi_k$, where $\psi_i = \bigotimes_{j=1}^{m_i} \eta_{n_i}$. And from Lemma 4.3, the restriction of ψ_i to $\mathrm{Alt}(n_i)$ contains all the irreducible representations of $\mathrm{Alt}(n_i)$ of depth at most m_i. In particular, σ_i is a component of $\psi_i|_{\mathrm{Alt}(n_i)}$. It follows that σ is a component of $\varphi|_H$.

(b) Conversely, let Δ_i be the support of $\mathrm{Alt}(n_i)$ in Ω, and let $\Delta_0 = \Omega \backslash \mathrm{supp}_\Omega H$. Then H is contained in the subgroup $K = \mathrm{Alt}(\Delta_0) \times \mathrm{Alt}(\Delta_1) \times \cdots \times \mathrm{Alt}(\Delta_k)$ of $\mathrm{Alt}(n)$ in such a way that the inclusion $\mathrm{Alt}(n_i) \to \mathrm{Alt}(\Delta_i)$ is strongly diagonal for each $i \geq 1$. Let $\rho = \rho_0 \otimes \cdots \otimes \rho_k$ be an irreducible component of $\varphi|_K$. It follows from the Littlewood-Richardson rule (see [**Z1**], (IV)) and from Frobenius reciprocity that $\delta(\rho_1) + \cdots + \delta(\rho_k) \leq m$. We now apply Lemma 4.4 which shows that the irreducible $\mathrm{Alt}(n_i)$-components of ρ_i all have depth at most $\delta(\rho_i)$. This proves that the irreducible H-constituent $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ of ρ satisfies $\delta(\sigma_1) + \cdots + \delta(\sigma_k) \leq \delta(\rho_1) + \cdots + \delta(\rho_k) \leq m$, as required. □

We are now prepared to describe inductive systems for our countable LD$\mathfrak{A}$-group G.

Theorem 4.6. *Let $\mathbb{K}$ be a field of characteristic zero, and let G be a countable* LD$\mathfrak{A}$*-group which is the union of an ascending chain of finite direct products $G_i = A_{i,1} \times \cdots \times A_{i,d_i}$ of finite alternating groups $A_{i,j} = \mathrm{Alt}(\Omega_{i,j})$ as in Lemma 4.1. Then for every $m \in \omega$, an inductive system $\Phi_m = \{\Phi_{m,i} \mid i \in \omega\}$ with respect to $\{G_i \mid i \in \omega\}$ is given by*

$$\Phi_{m,i} = \{\varphi \in \mathrm{Irr}_{\mathbb{K}}(G_i) \mid \varphi = \varphi_1 \otimes \cdots \otimes \varphi_{d_i} \text{ with } \delta(\varphi_1) + \cdots + \delta(\varphi_{d_i}) \leq m\}$$

for all $i \geq m$.

Proof. For $i \geq m$, choose an irreducible representation φ_1 of $A_{i+1,1}$ with depth m, and let $\varphi = \varphi_1 \otimes 1 \otimes \cdots \otimes 1 \in \Phi_{m,i+1}$. It follows straight away from Part (a) of Proposition 4.5, that every representation in $\Phi_{m,i}$ is a component of $\varphi|_{G_i}$. Hence $\Phi_{m,i} \subseteq \Phi_{m,i+1}|_{G_i}$.

Conversely, consider $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_{d_{i+1}} \in \Phi_{m,i+1}$. Let $\pi_j \colon G_{i+1} \to A_{i+1,j}$ denote the canonical projection. The restriction $\varphi|_{G_i}$ is a sum of representations of the form $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_{d_{i+1}}$, where each σ_j is an irreducible constituent of $\varphi_j|_{G_i\pi_j}$. And $\sigma_j = \rho_{j,1} \otimes \cdots \otimes \rho_{j,d_i}$, for certain irreducible representations $\rho_{j,k}$ of $A_{i,k}\pi_j$. Proposition 4.5(b) yields $\delta(\rho_{j,1}) + \cdots + \delta(\rho_{j,d_i}) \leq \delta(\varphi_j)$ for each $j \in \{1, \ldots, d_{i+1}\}$. Every irreducible constituent χ of σ has the form $\chi = \chi_1 \otimes \cdots \otimes \chi_{d_i}$ for certain irreducible representations χ_k of $A_{i,k}$, and χ_k is a component of $\sigma|_{A_{i,k}} = \rho_{1,k} \otimes \cdots \otimes \rho_{d_{i+1},k}$. From [**Z1**], (III) we see that $\delta(\chi_k) \leq \delta(\rho_{1,k}) + \cdots + \delta(\rho_{d_{i+1},k})$. Thus

$$\sum_{k=1}^{d_i} \delta(\chi_k) \leq \sum_{k=1}^{d_i} \sum_{j=1}^{d_{i+1}} \delta(\rho_{j,k}) = \sum_{j=1}^{d_{i+1}} \sum_{k=1}^{d_i} \delta(\rho_{j,k}) \leq \sum_{j=1}^{d_{i+1}} \delta(\varphi_j) \leq m.$$

This shows that $\Phi_{m,i+1}|_{G_i} \subseteq \Phi_{m,i}$. □

It remains to prove that every inductive system of G coincides with Φ_m for some m.

Proposition 4.7. *In the notation of Theorem* 4.6*, suppose that $i \geq m$. Then every representation in $\Phi_{m,i}$ is a constituent of the restriction to G_i of every representation in $\tau = \tau_1 \otimes \cdots \otimes \tau_{d_{i+2}}$ of G_{i+2} with $\delta(\tau_1) + \cdots + \delta(\tau_{d_{i+2}}) = m$.*

Proof. Since the canonical projection $G_{i+2} \to A_{i+2,j}$ embeds $A_{i+1,1}$ strongly diagonally into $A_{i+2,j}$, Proposition 4.5(a) ensures that, for each $j \in \{1, \ldots, d_{i+2}\}$, some irreducible representation θ_j of $A_{i+1,1}$ of depth $m_j = \delta(\tau_j)$ is a constituent of $\tau_j|_{A_{i+1,1}}$. In particular, the representation $\theta = \theta_1 \otimes \cdots \otimes \theta_{d_{i+2}}$ is a component of $\tau|_{A_{i+1,1}}$. From [**Z1**], (III), θ contains an irreducible representation σ_1 of depth $\delta(\sigma_1) = m_1 + \cdots + m_{d_{i+2}} = m$. And so, when restricting τ to G_{i+1}, we find an irreducible component of the form $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_{d_{i+1}}$ where σ_k is an irreducible representation of $A_{i+1,k}$ also for $k \geq 2$. On the other hand, Proposition 4.5(b) implies that $\delta(\sigma_1) + \cdots + \delta(\sigma_{d_{i+1}}) \leq m$, whence $\delta(\sigma_k) = 0$ and $\sigma_k = 1$ for all $k \geq 2$. But now Proposition 4.5(b) yields that every representation in $\Phi_{m,i}$ is a constituent of $\sigma|_{G_i}$ and hence of $\tau|_{G_i}$. □

Theorem 4.8. *In the situation of Theorem* 4.6*, every inductive system $\Psi = \{\Psi_i \mid i \in \omega\}$ of G with respect to $\{G_i \mid i \in \omega\}$ is either $\{\emptyset \mid i \in \omega\}$, or one of the Φ_m $(m \in \omega)$, or $\{\mathrm{Irr}_{\mathbb{K}}(G_i) \mid i \in \omega\}$.*

Proof. Let the Ψ_i be nonempty. Every representation $\sigma \in \Psi_i$ has the form $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_{d_i}$ for certain irreducible representations σ_j of $A_{i,j}$. Define $\delta(\sigma) = \delta(\sigma_1) + \cdots + \delta(\sigma_{d_i})$. Consider the function $\mu \colon \omega \to \omega$, given by $\mu(i) = \max\{\delta(\sigma) \mid \sigma \in \Psi_i\}$.

Suppose first that $\limsup_{i\to\infty} \mu(i)$ takes a finite value m. Then, for every $n \geq m$, there exists $k \geq n+2$ such that $\mu(k) = m$, and Propositions 4.7 and 4.6 imply that $\Phi_{m,i} \subseteq \Psi_i$ for all $i \leq n$. It follows that $\Phi_{m,i} \subseteq \Psi_i$ for all i. The converse inclusion holds trivially. Hence $\Psi = \Phi_m$.

Suppose now that $\limsup_{i\to\infty} \mu(i) = \infty$. Then, for every pair $n \geq m$, there exists $k \geq n+2$ such that $\mu(k) \geq m$, and Propositions 4.7 and 4.6 imply that $\Phi_{m,i} \subseteq \Psi_i$ for all $i \leq n$. It follows that $\Phi_{m,i} \subseteq \Psi_i$ for all i and all m. Hence $\Psi = \mathrm{Irr}_{\mathbb{K}}(G_i)$. □

Clearly $\Phi_{m,i} \subset \Phi_{m+1,i}$ for all i and every m. Therefore the final result of this section is a direct consequence of Proposition 2.2 and Theorem 4.8.

Theorem 4.9. *Let G be a countable* LD$\mathfrak{A}$*-group, and let $\mathbb{K}$ be any field of characteristic zero. Then the lattice of two-sided ideals in $\mathbb{K}G$ has the form* $\mathbb{K}G = I_0 \supset \omega(\mathbb{K}G) = I_1 \supset I_2 \supset I_3 \supset \cdots \supset \bigcap_{n\in\omega} I_n = 0$.

5. Uncountable groups.

It remains to extend Theorems A and B to uncountable groups. This will be accomplished by an application of the Theorem of Löwenheim and Skolem (see [**EFT**], Theorem IX.2.4).

5.1. The extension of Theorem A. Because of Theorem 3.3 it suffices to show the following two facts.

Proposition 5.1.

(a) *The class of simple locally finite groups of* 1*-type is axiomatizable by a sentence in the infinitary language $L_{\omega_1\omega}$.*

(b) *The assertion in Theorem* A *is an $L_{\omega_1\omega}$-sentence.*

Proof. We use a 2-sorted language with variables for field elements and for group elements.

(a) In view of the characterization of simple locally finite groups of ∞-type mentioned in Section 2, we just need to formalize the two sentences:

Every finite subset X of G is contained in a finite subgroup H of G, such that $X \cap N \subseteq \{1\}$ for some maximal normal subgroup N of H with alternating quotient,

and:

There exists a finite subgroup F in G such that, whenever H is a finite subgroup of G with $F \leq H$ and $F \cap N = \{1\}$ for some maximal normal subgroup N of H with alternating quotient $H/N \cong \mathrm{Alt}(\Omega)$, then F does not have a regular orbit on Ω.

In $L_{\omega_1\omega}$ we can easily quantify over all finite subsets or all finite subgroups of G (by writing down group tables). Therefore the first sentence is expressible in $L_{\omega_1\omega}$. We can similarly scope with the second sentence provided we can express that F does not have a regular orbit on Ω. But the point stabilizers of the action of H on Ω are precisely the subgroups of index $|\Omega|$ in H which contain N. And so we just need to write down that every such maximal subgroup of H contains a nontrivial element from F.

(b) Let $\mathbb{K}$ be a field and G be a group. It suffices to formalize the sentence:

If $\operatorname{char}\mathbb{K} = 0$, *then there exist elements* $0 \neq x, y \in \mathbb{K}G$ *with coefficient sum zero, such that* y *does not lie in the ideal generated by* x.

The condition "$\operatorname{char}\mathbb{K} = 0$" is clearly encoded by the infinite conjunction $\bigwedge_{n\in\omega}\psi_n$ where the sentence ψ_n expresses that the $(n+1)$-fold sum of $1 \in \mathbb{K}$ is nonzero. The membership *"there exists* $x \in \mathbb{K}G$*"* where x is considered to be of the form $x = k_0g_0 + \cdots + k_ng_n$ is expressible by the infinite disjunction

$$\bigvee_{n\in\omega} \quad \exists\, k_0, \ldots, k_n \in \mathbb{K} \quad \exists\, g_0, \ldots, g_n \in G.$$

And finally, *"*$y \in \mathbb{K}G$ *does not lie in the ideal generated by* $x \in \mathbb{K}G$*"* becomes

$$\bigwedge_{n\in\omega} \quad \forall\, u_0, \ldots, u_n, v_0, \ldots, v_n \in \mathbb{K}G \quad y \neq u_0xv_0 + \cdots + u_nxv_n.$$

□

5.2. The extension of Theorem B. We shall use the Löwenheim-Skolem technique to deduce the following result from Theorem 4.9.

Proposition 5.2. *Let* $\mathbb{K}$ *be a field of characteristic zero, and let* G *be any* LD$\mathfrak{A}$*-group. Then there are ideals* $\mathbb{K}G = I_0 \supset I_1 \supset I_2 \supset \ldots$ *such that every further ideal of* $\mathbb{K}G$ *is contained in* $I_\omega = \bigcap_{n\in\omega} I_n$.

To this end it suffices to establish the following two facts.

Proposition 5.3.

(a) *The class of* LD$\mathfrak{A}$*-groups is axiomatizable by a sentence in* $L_{\omega_1\omega}$.
(b) *The assertion in Proposition* 5.2 *is an* $L_{\omega_1\omega}$*-sentence.*

Proof. (a) This follows as in the proof of Proposition 5.1(a), since every LD$\mathfrak{A}$-group is a simple locally finite group of 1-type such that every finite subset is contained in a finite subgroup which is a direct product of alternating groups.

(b) Simply consider $\bigwedge_{n\in\omega}\psi_n \longrightarrow \bigwedge_{n\in\omega}\varphi_n$ where the sentence ψ_n expresses that the $(n+1)$-fold sum of $1 \in \mathbb{K}$ is nonzero, and where φ_n is the sentence:

There exist elements $x_0, \dots, x_n \in \mathbb{K}G$ *such that:*

- *Every element of* $\mathbb{K}G$ *lies in the ideal generated by* x_0*,*
- *for every* $i \le n-1$*, the element* x_i *does not lie in the ideal generated by* x_{i+1}*, but* x_{i+1} *lies in the ideal generated by* x_i*, and*
- *for every* $1 \le i \le n$*, whenever an element* $y \in \mathbb{K}G$ *does not lie in the ideal generated by* x_i*, then* x_{i-1} *is contained in the ideal generated by* y*.*

□

Proof of Theorem B. In the notation of Proposition 5.2, it suffices to show that $I_\omega = 0$. Choose elements $u_n \in I_n \setminus I_{n+1}$ $(n \in \omega)$. By Proposition 5.3(a) and [**EFT**], Theorem IX.2.4, every countable subset of G is contained in a countable LD$\mathfrak{A}$-subgroup H of G. By Theorem 4.9, the assertion of Theorem B holds already for every such subgroup H. We may also assume without loss that $u_0, u_1, \dots \in \mathbb{K}H$ for every such group H. Hence $\mathbb{K}H \cap I_\omega = 0$. And this implies that $I_\omega = 0$, as desired. □

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