Let $K$ be any field of characteristic zero. We show that there are at least four ideals in the group algebra $KG$ of every simple locally finite group $G$ of 1-type, thus providing the final step in solving an old question of I. Kaplansky’s for locally finite groups. We also determine the ideal lattice in $KG$ for those 1-type groups $G$ which are a direct limit of finite direct products of alternating groups.

1. Introduction.

It is an old question due to I. Kaplansky [K] for which groups $G$ and which fields $K$ the augmentation ideal $\omega(KG)$ is the only nonzero proper two-sided ideal in $KG$. Every such group $G$ must necessarily be simple. Kaplansky’s question was the starting point for a research programme begun by A.E. Zalesskii to determine the ideal lattices of complex group algebras of infinite simple locally finite groups $G$. Here, a group is said to be locally finite if it is a direct limit of finite groups. The investigation of this topic leads naturally into the realm of character theory of finite groups, since ideals in $CG$ correspond naturally with certain systems of irreducible complex characters of the finite subgroups of $G$ ([Z2], Proposition 1.2). Thus one can use the highly developed theory of characters to obtain information about ideals in group algebras of locally finite groups. And conversely results about such ideals can be interpreted in the spirit of the asymptotic theory of finite groups.

At present, every infinite simple locally finite group $G$ can be sorted into one of the following classes: Finitary linear groups, groups of 1-type, groups of $p$-type (where $p$ is a prime), and groups of $\infty$-type. The definition of these classes is given in Section 2. For any field $\mathbb{K}$ of characteristic zero it is known that the augmentation ideal is the only proper ideal in $\mathbb{K}G$ when $G$ is finitary linear with natural representation over an infinite field ([HZ1], Theorem B, [HZ3], Theorem 1.1, [LP], Theorem 4.5), or when $G$ is of $p$- or $\infty$-type ([LP], Corollary 2.2). On the other hand, if $G$ is a finitary linear group defined over a finite field, then many ideals can be expected ([HZ2]; [Z2], Proposition 2.6; [LP]). Thus, as far as infinite locally finite groups are
concerned, Kaplansky’s question is just open for groups of 1-type. In the present paper we shall prove the following result.

**Theorem A.** Let $G$ be an infinite simple locally finite group of 1-type, and let $K$ be any field of characteristic zero. Then $K[G]$ has at least four two-sided ideals.

In general, it seems to be quite difficult to describe the full ideal lattice of $K[G]$ for groups of 1-type. However, we shall do so for an interesting subclass, namely for those simple locally finite groups of 1-type which are direct limits of finite direct products of alternating groups. We shall call these groups $\text{LD}\mathfrak{A}$-groups. Note that Theorem A was established for $\text{LD}\mathfrak{A}$-groups in [HZ3], Lemma 5.8.

**Theorem B.** Let $G$ be an $\text{LD}\mathfrak{A}$-group, and let $K$ be any field of characteristic zero. Then the lattice of two-sided ideals in $K[G]$ has the form $K[G] = I_0 \supset \omega(K[G]) = I_1 \supset I_2 \supset I_3 \supset \cdots \supset \bigcap_{n \in \omega} I_n = 0$.

Both theorems will be proved for countable groups first, and then be extended to groups of arbitrary cardinality by means of the Löwenheim-Skolem Theorem. One of the key results used in the proof of Theorem A is due to U. Meierfrankenfeld (see Proof of Lemma 3.1). Theorem B generalizes the corresponding result for direct limits of finite alternating groups ([Z1], Theorem 1). The proof of the countable version closely follows the arguments of [Z1]. In particular it relies heavily on the representation theory of finite alternating groups.

2. Preliminaries.

For the convenience of the reader, we shall collect some basic notions in this section. First of all recall that a *Kegel cover* in the simple locally finite group $G$ is a family $\{(G_i, M_i) \mid i \in I\}$, such that for each $i$, $M_i$ is a maximal normal subgroup in the finite subgroup $G_i$ of $G$, and such that for every finite subgroup $F$ of $G$ there exists some $i \in I$ satisfying $F \leq G_i$ and $F \cap M_i = 1$.

The sections $G_i/M_i$ are called the *Kegel factors* of the Kegel cover. It was proved in [KW] that every simple locally finite group has a Kegel cover. When $G$ is countable, the subgroups $G_i$ can be chosen in such a way that they form an ascending chain $G_0 \subset G_1 \subset G_2 \subset \ldots$ with $G_i \cap M_{i+1} = 1$ for all $i$. A Kegel cover of this kind is called a *Kegel sequence*.

A simple locally finite group $G$ is said to be:

- *Finitary linear* if it acts faithfully on a (possibly infinite-dimensional) vector space $V$ in such a way that every $g \in G$ fixes a subspace of finite codimension in $V$ pointwise,

- *of 1-type* if it is not finitary linear, and if every Kegel cover of $G$ has an alternating Kegel factor,
• of \(p\)-type if it is not finitary linear, and if every Kegel cover of \(G\) has a Kegel factor which is a classical group over a field of characteristic \(p\),
• of \(\infty\)-type if it is not finitary linear, and neither of \(1\)-type nor of \(p\)-type for any prime \(p\).

It is clear that every countable simple locally finite group of \(1\)-type has a Kegel sequence with alternating factors.

The above classification can be found in S. Delcroix’ Ph.D. thesis [D], Section 3.3. There in fact a simple locally finite group of \(\infty\)-type is characterized by the property that for every finite subgroup \(F\) of \(G\) there exist a finite subgroup \(H\) of \(G\) and a maximal normal subgroup \(M\) in \(H\) such that:

1. \(H/M \cong \text{Alt}(\Omega)\) for some finite set \(\Omega\),
2. \(F \leq H\) and \(F \cap M = 1\), and
3. \(F\) has a regular orbit on \(\Omega\).

In contrast to this, the following notion of diagonal embedding is introduced in [PZ]. Suppose that \(H_i\) (\(i = 1, 2\)) are finite groups with maximal normal subgroups \(M_i\) and alternating quotients \(H_i/M_i \cong \text{Alt}(\Omega_i)\). Then an embedding \(H_1 \rightarrow H_2\) is said to be diagonal if in the natural action of \(H_2\) on \(\Omega_2\), the index \(|H_1 : (H_1)_{\omega} M_1|\) is either 1 or \(|\Omega_1|\) for each point \(\omega \in \Omega_2\). Here \((H_1)_{\omega}\) denotes the point stabilizer of \(\omega\) in \(H_1\). The following proposition is a straight consequence of [PZ], Theorem 1.7 and of the above characterization of simple locally finite groups of \(\infty\)-type.

**Proposition 2.1.** Let \(\{(H_i, M_i) \mid i \in \omega\}\) be a Kegel sequence with alternating Kegel factors in the countable simple locally finite group \(G\) of \(1\)-type. Then there exist positive integers \(i_0 < i_1 < i_2 < \ldots\) such that the inclusion \(H_{i_k} \rightarrow H_{i_\ell}\) is a diagonal embedding for all \(k < \ell\).

An important tool in the study of the (two-sided) ideals of the group algebra \(\mathbb{K}G\) of a locally finite group \(G\), are the so-called inductive systems with respect to a fixed local system \(\{G_i \mid i \in I\}\) of finite subgroups in \(G\). By definition, an inductive system \(\Phi = \{\Phi_i \mid i \in I\}\) is a set where:

1. Each \(\Phi_i\) is a subset of the set \(\text{Irr}_{\mathbb{K}}(G_i)\) of all irreducible \(\mathbb{K}G_i\)-representations, and
2. whenever \(G_i \leq G_j\), then \(\Phi_i\) consists precisely of the irreducible \(\mathbb{K}G_i\)-constituents of \(\mathbb{K}G_j\)-representations from \(\Phi_j\).

Inductive systems are linked with ideals via the following result of A.E. Zalesskii.

**Proposition 2.2 ([Z2], Proposition 1.2).** Let \(G\) be a locally finite group, and let \(\mathbb{K}\) be a field of characteristic zero. Then the inductive systems with respect to a fixed local system of finite subgroups in \(G\) are in order reversing one-to-one correspondence with the ideals in \(\mathbb{K}G\).

Here an ideal \(J\) corresponds to the inductive system \(\Phi = \{\Phi_i \mid i \in I\}\), where \(\Phi_i\) consists of the irreducible \(\mathbb{K}G_i\)-representations corresponding to...
the irreducible $\mathbb{K}G_i$-modules occurring in $\mathbb{K}G_i/(J \cap \mathbb{K}G_i)$. In particular, the zero ideal corresponds to $\{\text{Irr}_{\mathbb{K}}(G_i) \mid i \in I\}$, the augmentation ideal corresponds to $\{1_{G_i} \mid i \in I\}$, and $\mathbb{K}G$ corresponds to $\{\emptyset \mid i \in I\}$.

3. Groups of 1-type.

This section is devoted to the proof of Theorem A for countable groups $G$. We therefore suppose throughout that $G$ is a countable simple locally finite group of 1-type with a Kegel sequence $\mathcal{H} = \{(H_i, M_i) \mid i \in \omega\}$ such that $H_i/M_i \cong \text{Alt}(\Omega_i)$. From Proposition 2.1 we may assume further that all the inclusions $H_i \to H_k$ ($i < k$) are diagonal.

**Lemma 3.1.** For each $i \geq 1$ there is some $j > i$ such that the following hold:

(a) There exist an $H_i$-orbit $\Theta_i$ in $\Omega_j$ and a maximal $H_i$-system of imprimitivity $D_i$ in $\Theta_i$ such that $D_i$ is equivalent to $\Omega_i$ as an $H_i$-set.

(b) The stabilizer in $H_i$ of any point $\omega \in \Omega_i$ contains the stabilizer of every point $\delta \in \Theta_i$ which is contained in the block $\Delta \in D_i$ corresponding to $\omega$ with respect to the above equivalence.

**Proof.** (a) Because all the inclusions $H_{i-1} \to H_k$ ($i \leq k$) are diagonal, $H_{i-1}$ does not have a regular orbit on $\Omega_k$. Since $G$ is not finitary, we must have $\lim \inf_{k \to \infty} |\text{supp}_{\Omega_k} h| = \infty$ for every nontrivial $h \in H_{i-1}$ — otherwise an ultraproduct argument would yield a finitary permutation representation for $G$ (see [H1]). Therefore [M], Lemma 2.14 provides some $j > i$ and a positive integer $t \leq |H_{i-1}| - 2$ such that $H_i$ has a $t$-pseudo natural orbit $\Theta_i$ on $\Omega_j$ with respect to $M_j$. By definition this means that $M_j$ is the stabilizer of a maximal $H_i$-system of imprimitivity $D_i$ in $\Theta_i$, and that the action of $H_i$ on $D_i$ is equivalent to the action of $H_i$ on the set of $t$-subsets of $\Omega_i$.

Consider some $\Delta \in D_i$ and $\delta \in \Delta$. Clearly $S = N_{H_i}(\Delta)$ is a point-stabilizer in the action of $H_i$ on $D_i$, and $(H_i)_{\delta}M_i \leq S$. In particular, the index $|H_i : (H_i)_{\delta}M_i|$ exceeds 1 and must therefore be equal to $|\Omega_i|$. But then $S = (H_i)_{\delta}M_i$, because there are no proper subgroups of index smaller than $|\Omega_i|$ in $\text{Alt}(\Omega_i)$. Thus $|D_i| = |\Omega_i|$, and $H_i/M_i$ acts naturally on $D_i$, that is, $D_i$ and $\Omega_i$ are equivalent as $H_i$-sets.

(b) follows immediately from (a), since $(H_i)_{\delta} \leq N_{H_i}(\Delta)$ whenever $\delta \in \Delta \in D_i$. \qed

From passing to a subsequence of the Kegel sequence $\mathcal{H}$ we may clearly assume in the sequel that $j = i + 1$ for all $i$ in Lemma 3.1. We are now well-prepared to construct a permutation representation of the group $G$.

To this end, we consider the subset $\Omega$ of the Cartesian product $\prod_{i \in \omega} \Omega_i$, which consists of all tuples $(\omega_i)_{i \in \omega}$ with the following property: There exists $m$ (depending on the particular tuple) such that for all $i \geq m$ the point $\omega_{i+1} \in \Omega_{i+1}$ is contained in the block in $D_i$ which corresponds to the point
\( \omega_i \in \Omega_i \) with respect to the \( H_i \)-equivalence between \( \mathcal{D}_i \) and \( \Omega_i \) furnished by Lemma 3.1. Such a tuple is said to be nice from \( m \) onwards. We define an equivalence relation \( \sim \) on \( \Omega \) via

\[(\omega_i)_{i \in \omega} \sim (\nu_i)_{i \in \omega} \quad \text{if and only if} \quad \omega_i = \nu_i \quad \text{for all but finitely many} \quad i.
\]

Let \( \overline{\Omega} = \Omega/\sim \) denote the set of equivalence classes \([\omega_i]_{i \in \omega}\) of tuples \((\omega_i)_{i \in \omega} \in \Omega \) modulo \( \sim \). Since every \( H_i \) acts on \( \Omega_j \) for all \( j \geq i \), there is a well-defined componentwise action of \( G \) on \( \overline{\Omega} \).

**Lemma 3.2.** For each \( \omega = [\omega_i]_{i \in \omega} \in \overline{\Omega} \) and every \( k \geq 1 \) there exists \( n \geq k \) such that the orbit \( \omega^{H_k} \) in \( \overline{\Omega} \) is \( H_k \)-equivalent to the orbit \( \omega_n^{H_k} \) in \( \Omega_n \). Moreover, if \( \omega \) is nice from \( k \) onwards, then \( M_k \) acts intransitively on \( \omega^{H_k} \).

**Proof.** The tuple \( \omega \) is nice from some \( m \geq k \) onwards. By Lemma 3.1 the point stabilizers \( (H_k)_{\omega^\ell} \) (\( \ell \geq m \)) form a descending chain. By finiteness of \( H_k \) the chain must become stationary. But then \( (H_k)_{\omega^\ell} = \bigcap_{n \geq m} (H_k)_{\omega^\ell} = (H_k)_{\omega_n} \) for some \( n \geq m \). Suppose now that \( m = k \). As \( H_k \) acts naturally on \( \Omega_k \), it is clear that \( M_k(H_k)\omega_n \subseteq (H_k)\omega_k \neq H_k \). Hence \( M_k \) acts intransitively on \( \omega_n^{H_k} \).

We are now in a position to prove the main result of this section.

**Theorem 3.3.** Let \( G \) be a countable simple locally finite group of 1-type. For any field \( \mathbb{K} \) of characteristic zero, the group algebra \( \mathbb{K}G \) has at least four two-sided ideals.

**Proof.** Consider the permutation module \( V = \mathbb{K}\Gamma \) for some \( G \)-orbit \( \Gamma \) in \( \overline{\Omega} \) containing a tuple which is nice from \( m = 1 \) onwards. Let \( \Phi = \{ \Phi_i \mid i \in \omega \} \) be the inductive system with respect to \( \mathcal{H} \) where \( \Phi_i \) is the set of irreducible \( \mathbb{K}H_i \)-constituents of the representation of \( G \) on \( V \). For each \( i \) we shall show that \( \Psi_i = \{ \phi \in \Phi_i \mid M_i \subseteq \ker \phi \} \) equals \( \{ 1_{H_i}, \eta_i \} \) where \( \eta_i \) denotes the nontrivial irreducible constituent of the natural permutation representation of \( H_i \) on \( \Omega_i \).

Consider some \( \omega = [\omega_j]_{j \in \omega} \in \Gamma \). From Lemma 3.2 there exists \( n > i \) such that the orbit \( \omega^{H_i} \) is \( H_i \)-equivalent to the orbit \( \Delta = \omega_n^{H_i} \) in \( \Omega_n \). If \( M_i \) acts transitively on \( \Delta \), then \( \text{fix}_{\mathbb{K}\Delta}(M_i) \) is spanned by the sum of the elements of \( \mathbb{K}\Delta \), whence \( \text{fix}_{\mathbb{K}\Delta}(M_i) \) is 1-dimensional and the trivial \( H_i \)-module. However, \( \Gamma \) also contains a tuple \( \omega \) which is nice from \( m = 1 \) onwards, and in this situation Lemma 3.2 ensures that \( M_i \) acts intransitively on \( \Delta \). The \( M_i \)-orbits in \( \Delta \) form an \( H_i \)-system of imprimitivity \( \mathcal{S} = \{ \Sigma_1, \ldots, \Sigma_r \} \). Clearly \( M_i \) is the stabilizer of \( \mathcal{S} \). Since the inclusion \( H_i \to H_n \) is diagonal, it follows as in the proof of Lemma 3.1 that the alternating group \( H_i/M_i \) acts naturally on \( \mathcal{S} \). It is readily seen that \( \text{fix}_{\mathbb{K}\Delta}(M_i) = \mathbb{K}s_1 \oplus \cdots \oplus \mathbb{K}s_r \) with \( s_k = \sum_{\sigma \in \Sigma_k} \sigma \). But the alternating group \( H_i/M_i \) acts naturally on the set \( \{ s_1, \ldots, s_r \} \). We conclude that \( \text{fix}_{\mathbb{K}\Delta}(M_i) \) is isomorphic to the natural \( \mathbb{K}H_i \)-module \( \Omega_i \). By
Maschke’s Theorem, $\text{fix}_{K\Delta}(M_i)$ contains every irreducible $H_i$-submodule of $K\Delta$ on which $M_i$ acts trivially. Therefore $\{1_{H_i}, \eta_i\} = \Psi_i$.

Since every representation of the alternating quotient $H_i/M_i$ lifts to a representation of $H_i$, it is now clear that $\Psi_i = \{1_{H_i}, \eta_i\}$ is a proper subset of $\{\varphi \in \text{Irr}_K(H_i) \mid M_i \subseteq \ker \varphi\}$, whence $\Phi_i$ is a proper subset of $\text{Irr}_K(H_i)$. Therefore Proposition 2.2 links $\Phi_i$ to a proper ideal in the augmentation ideal of $K\mathbb{G}$. □

4. $\text{ldA}$-groups.

In this section we shall completely describe the ideal lattice of countable $\text{ldA}$-groups. Again the generalization to uncountable groups will be deferred to Section 5. Since the proof of our result follows the ideas contained in [Z1], the reader is referred to that paper for all the definitions related to the representation theory of alternating groups.

Suppose then that $G$ is a countable $\text{ldA}$-group. $G$ is the union of an ascending chain $G_0 < G_1 < G_2 < \ldots$ of finite subgroups where

$$G_i = A_{i,1} \times A_{i,2} \times \cdots \times A_{i,d_i}$$

with $A_{i,j} = \text{Alt}(\Omega_{i,j})$ for suitable finite sets $\Omega_{i,j}$. Note that $G_i$ acts on the disjoint union $\Omega_i = \Omega_{i,1} \cup \cdots \cup \Omega_{i,d_i}$ such that $A_{i,\nu}$ acts naturally on $\Omega_{i,\nu}$ while

$$M_{i,\nu} = A_{i,1} \times \cdots \times A_{i,\nu-1} \times A_{i,\nu+1} \times \cdots \times A_{i,d_i}$$

fixes $\Omega_{i,\nu}$ pointwise. The above chain of finite subgroups $G_i$ in $G$ is said to be of strongly diagonal type if for all pairs $i \leq j$ and every nontrivial $G_i$-orbit $\Delta$ on $\Omega_j = \Omega_{j,1} \cup \cdots \cup \Omega_{j,d_j}$ there exists $\nu \in \{1, \ldots, d_i\}$ such that $A_{i,\nu}$ acts naturally on $\Delta$ while $M_{i,\nu}$ fixes $\Delta$ pointwise. On the other hand the chain is said to be of regular type if $G_i$ has a regular orbit on $\Omega_j$ whenever $i < j$.

It was proved in [HZ3], Theorem 5.4 that any chain of the above kind has a subchain of either strongly diagonal or regular type. In view of this fact we readily have that our countable $\text{ldA}$-group, which is of course not of $\infty$-type, is the union of a chain of strongly diagonal type. We can still improve this chain slightly.

Lemma 4.1. Every countable $\text{ldA}$-group $G$ is the union of an ascending chain $G_0 < G_1 < G_2 < \ldots$ of finite direct products of finite alternating groups as above such that the chain is of strongly diagonal type, such that $G_i \cap M_{i+1,j} = 1$ for all $i$ and all $j \in \{1, \ldots, d_{i+1}\}$, and such that $A_{i,\nu}$ has at least $i + 2$ nontrivial orbits on $\Omega_{i+1,j}$ for all $i$, all $\nu \in \{1, \ldots, d_i\}$, and all $j \in \{1, \ldots, d_{i+1}\}$.

Proof. $G$ is already the union of a chain $\mathcal{H} = \{H_i \mid i \in \omega\}$ of strongly diagonal type. The terms $G_i$ of the desired chain are constructed recursively from $\mathcal{H}$ as follows. Let $G_0 = H_0$ and assume that $G_i$ has been found for some
Consider $F = \langle H_i, G_i \rangle$. Since $G$ is simple and locally finite, there exists a positive integer $n$ such that $x \in \langle y^{H_n} \rangle$ for all $x \in F$ and every nontrivial $y \in F$. Let $G_{i+1}$ be the normal closure of $F$ in $H_n$. Because $H_n$ is a direct product of simple groups, every normal subgroup of $G_{i+1}$ is normal in $H_n$ too. Hence any proper normal subgroup of $G_{i+1}$ does not contain $F$. By choice of $H_n$, it must then have trivial intersection with $F$ and in particular with $G_i$. This completes the construction of a chain satisfying the first two requirements.

If there would exist integers $\nu_j (j \geq i)$ such that the number of orbits of $A_i, \nu_i$ in $\Omega_j, \nu_j$ is uniformly bounded for all $j \geq i$, then an ultraproduct argument as in $[H1]$ would show that the group $G$ is finitary linear, a contradiction. Therefore an infinite subchain of the chain constructed so far also satisfies the third requirement. □

In the sequel we shall consider all representations over a fixed field $K$ of characteristic zero. The following facts can be extracted from $[Z1]$.

To each irreducible representation $\varphi$ of $\text{Alt}(n)$ we can associate a Young diagram which is completely described by a finite descending sequence $(l_1, l_2, \ldots, l_k)$ of positive integers with sum $n$; here $l_i$ is the number of cells in the $i$-th row of the Young diagram. This diagram will be denoted by $D(\varphi)$. The depth $\delta(\varphi)$ of the representation $\varphi$ is defined as $\delta(\varphi) = l_2 + \cdots + l_k$. It is clear that the trivial representation is the unique irreducible representation of depth zero. The unique irreducible representation of depth one has the diagram described by the sequence $(n-1, 1)$ and will be denoted by $\eta_n$.

Lemma 4.2 ([Z1], Lemma 4). Let $n = n_0 + \cdots + n_k$ with $n_0 \geq 0$ and $n_i > 3$ for all $i \geq 1$. Assume that $\psi$ is an irreducible representation of $\text{Alt}(n)$ of depth $k$. If $H = \text{Alt}(n_0) \times \text{Alt}(n_1) \times \cdots \times \text{Alt}(n_k)$ is embedded naturally in $\text{Alt}(n)$, then $\psi|_H$ contains the irreducible component $1_{\text{Alt}(n_0)} \otimes \eta_{n_1} \otimes \cdots \otimes \eta_{n_k}$.

Lemma 4.3 ([Z1], Lemma 5). Let $m > 1$. Then

$$\{ \psi \in \text{Irr}_K(\text{Alt}(n)) \mid \delta(\psi) \leq m \}$$

is the set of all irreducible components of the representation $\bigotimes_{i=1}^{m} \eta_n$ of $\text{Alt}(n)$.

Lemma 4.4 ([Z1], Lemma 11). Let $\text{Alt}(k) \to \text{Alt}(n)$ be a strongly diagonal embedding, and suppose that $\varphi$ is an irreducible representation of $\text{Alt}(n)$ of depth $m$. If $k > \max\{2m, 4\}$, then every irreducible constituent of $\varphi|_{\text{Alt}(k)}$ has degree at most $m$.

We shall now combine the preceding three lemmata to establish the following result.

Proposition 4.5. Suppose that the group $H = \text{Alt}(n_1) \times \cdots \times \text{Alt}(n_k)$ is strongly diagonally embedded in $\text{Alt}(n)$. Assume further that there exists
$m \in \omega$ such that $n_i > 2m + 2$ for each $i$, and such that each $\text{Alt}(n_i)$ has at least $m + 2$ nontrivial orbits on $\Omega = \{1, \ldots, n\}$, the natural set for $\text{Alt}(n)$.

(a) Let $\varphi$ be any irreducible representation of $\text{Alt}(n)$ of depth $m$, and choose irreducible representations $\sigma_i$ of $\text{Alt}(n_i)$ in such a way that $\delta(\sigma_1) + \cdots + \delta(\sigma_k) \leq m$. Then $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ is a component of the restriction $\varphi|_H$.

(b) Conversely, if $\varphi$ is any irreducible representation of $\text{Alt}(n)$ of depth $m$ and $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ is an irreducible $H$-constituent of $\varphi$, then $\delta(\sigma_1) + \cdots + \delta(\sigma_k) \leq m$.

Proof. Since the result is trivial for $m = 0$, we assume in the sequel that $m \geq 1$.

(a) For each $i \in \{1, \ldots, k\}$, select $m_i = \delta(\sigma_i) + 2$ nontrivial $\text{Alt}(n_i)$-orbits $\Delta_{i,j}$ ($1 \leq j \leq m_i$) in $\Omega$. Since $H$ is embedded strongly diagonally in $\text{Alt}(n)$, all these orbits are pairwise distinct $H$-orbits. Let $\Delta_0 = \Omega \setminus \bigcup_{i,j} \Delta_{i,j}$. The group $H$ is contained in the subgroup $K = \text{Alt}(\Delta_0) \times K_1 \times \cdots \times K_k$ of $\text{Alt}(n)$, where $K_i = \text{Alt}(\Delta_{i,1}) \times \cdots \times \text{Alt}(\Delta_{i,m_i})$. And the canonical projection $\pi_i : K \to K_i$ embeds $\text{Alt}(n_i)$ onto the diagonal subgroup of $K_i$.

By Lemma 4.2, the restriction $\varphi|_K$ contains the irreducible component $\psi = 1_{\text{Alt}(\Delta_0)} \otimes \psi_1 \otimes \cdots \otimes \psi_k$, where $\psi_i = \bigotimes_{j = 1}^{m_i} \eta_{n_i}$. And from Lemma 4.3, the restriction of $\psi_i$ to $\text{Alt}(n_i)$ contains all the irreducible representations of $\text{Alt}(n_i)$ of depth at most $m_i$. In particular, $\sigma_i$ is a component of $\psi_i|_{\text{Alt}(n_i)}$. It follows that $\sigma$ is a component of $\varphi|_H$.

(b) Conversely, let $\Delta_i$ be the support of $\text{Alt}(n_i)$ in $\Omega$, and let $\Delta_0 = \Omega \setminus \text{supp} \Omega H$. Then $H$ is contained in the subgroup $K = \text{Alt}(\Delta_0) \times \text{Alt}(\Delta_1) \times \cdots \times \text{Alt}(\Delta_k)$ of $\text{Alt}(n)$ in such a way that the inclusion $\text{Alt}(n_i) \to \text{Alt}(\Delta_i)$ is strongly diagonal for each $i \geq 1$. Let $\rho = \rho_0 \otimes \cdots \otimes \rho_k$ be an irreducible component of $\varphi|_K$. It follows from the Littlewood-Richardson rule (see [Z1], (IV)) and from Frobenius reciprocity that $\delta(\rho_1) + \cdots + \delta(\rho_k) \leq m$. We now apply Lemma 4.4 which shows that the irreducible $\text{Alt}(n_i)$-components of $\rho_i$ all have depth at most $\delta(\rho_i)$. This proves that the irreducible $H$-constituent $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ of $\rho$ satisfies $\delta(\sigma_1) + \cdots + \delta(\sigma_k) \leq \delta(\rho_1) + \cdots + \delta(\rho_k) \leq m$, as required.

We are now prepared to describe inductive systems for our countable $\text{LD}\mathfrak{A}$-group $G$.

Theorem 4.6. Let $\mathbb{K}$ be a field of characteristic zero, and let $G$ be a countable $\text{LD}\mathfrak{A}$-group which is the union of an ascending chain of finite direct products $G_i = A_{i,1} \times \cdots \times A_{i,d_i}$ of finite alternating groups $A_{i,j} = \text{Alt}(\Omega_{i,j})$ as in Lemma 4.1. Then for every $m \in \omega$, an inductive system $\Phi_m = \{\Phi_{m,i} \mid i \in \omega\}$ with respect to $\{G_i \mid i \in \omega\}$ is given by

$$\Phi_{m,i} = \{\varphi \in \text{Irr}_\mathbb{K}(G_i) \mid \varphi = \varphi_1 \otimes \cdots \otimes \varphi_{d_i} \text{ with } \delta(\varphi_1) + \cdots + \delta(\varphi_{d_i}) \leq m\}$$
for all \( i \geq m \).

**Proof.** For \( i \geq m \), choose an irreducible representation \( \varphi_1 \) of \( A_{i+1,1} \) with depth \( m \), and let \( \varphi = \varphi_1 \otimes 1 \otimes \cdots \otimes 1 \in \Phi_{m,i+1} \). It follows straight away from Part (a) of Proposition 4.5, that every representation in \( \Phi_{m,i} \) is a component of \( \varphi|_{G_i} \). Hence \( \Phi_{m,i} \subseteq \Phi_{m,i+1}|_{G_i} \).

Conversely, consider \( \varphi = \varphi_1 \otimes \cdots \otimes \varphi_{d_{i+1}} \in \Phi_{m,i+1} \). Let \( \pi_j : G_{i+1} \to A_{i+1,j} \) denote the canonical projection. The restriction \( \varphi|_{G_i} \) is a sum of representations of the form \( \sigma = \sigma_1 \otimes \cdots \otimes \sigma_{d_{i+1}} \), where each \( \sigma_j \) is an irreducible constituent of \( \varphi_j|_{G_i} \). And \( \sigma_j = \rho_{j,1} \otimes \cdots \otimes \rho_{j,d_i} \), for certain irreducible representations \( \rho_{j,k} \) of \( A_{i,k} \). Proposition 4.5(b) yields \( \delta(\varphi_j) \leq \delta(\varphi_j) \) for each \( j \in \{1, \ldots, d_{i+1}\} \). Every irreducible constituent \( \chi \) of \( \sigma \) has the form \( \chi = \chi_1 \otimes \cdots \otimes \chi_{d_i} \), for certain irreducible representations \( \chi_k \) of \( A_{i,k} \), and \( \chi_k \) is a component of \( \varphi_j|_{A_{i,k}} = \rho_{1,k} \otimes \cdots \otimes \rho_{d_{i+1},k} \). From [Z1], (III) we see that \( \delta(\chi_k) \leq \delta(\rho_{1,k}) + \cdots + \delta(\rho_{d_{i+1},k}) \). Thus

\[
\sum_{k=1}^{d_i} \delta(\chi_k) \leq \sum_{k=1}^{d_i} \sum_{j=1}^{d_{i+1}} \delta(\rho_{j,k}) = \sum_{j=1}^{d_{i+1}} \sum_{k=1}^{d_i} \delta(\rho_{j,k}) \leq \sum_{j=1}^{d_{i+1}} \delta(\varphi_j) \leq m.
\]

This shows that \( \Phi_{m,i+1}|_{G_i} \subseteq \Phi_{m,i} \). \( \square \)

It remains to prove that every inductive system of \( G \) coincides with \( \Phi_m \) for some \( m \).

**Proposition 4.7.** In the notation of Theorem 4.6, suppose that \( i \geq m \). Then every representation in \( \Phi_{m,i} \) is a constituent of the restriction to \( G_i \) of every representation in \( \tau = \tau_1 \otimes \cdots \otimes \tau_{d_{i+2}} \) of \( G_{i+2} \) with \( \delta(\tau_1) + \cdots + \delta(\tau_{d_{i+2}}) = m \).

**Proof.** Since the canonical projection \( G_{i+2} \to A_{i+2,j} \) embeds \( A_{i+1,1} \) strongly diagonally into \( A_{i+2,j} \), Proposition 4.5(a) ensures that, for each \( j \in \{1, \ldots, d_{i+2}\} \), some irreducible representation \( \theta_j \) of \( A_{i+1,1} \) of depth \( m_j = \delta(\tau_j) \) is a constituent of \( \tau_j|_{A_{i+1,1}} \). In particular, the representation \( \theta = \theta_1 \otimes \cdots \otimes \theta_{d_{i+2}} \) is a component of \( \tau|_{A_{i+1,1}} \). From [Z1], (III), \( \theta \) contains an irreducible representation \( \sigma_1 \) of depth \( \delta(\sigma_1) = m_1 \) of \( G_{i+1} \). And so, when restricting \( \tau \) to \( G_{i+1} \), we find an irreducible component of the form \( \sigma = \sigma_1 \otimes \cdots \otimes \sigma_{d_{i+1}} \) where \( \sigma_k \) is an irreducible representation of \( A_{i+k} \) also for \( k \geq 2 \). On the other hand, Proposition 4.5(b) implies that \( \delta(\sigma_1) + \cdots + \delta(\sigma_{d_{i+1}}) \leq m \), whence \( \delta(\sigma_k) = 0 \) and \( \sigma_k = 1 \) for all \( k \geq 2 \). But now Proposition 4.5(b) yields that every representation in \( \Phi_{m,i} \) is a constituent of \( \sigma|_{G_i} \) and hence of \( \tau|_{G_i} \). \( \square \)

**Theorem 4.8.** In the situation of Theorem 4.6, every inductive system \( \Psi = \{ \Psi_i \mid i \in \omega \} \) of \( G \) with respect to \( \{ G_i \mid i \in \omega \} \) is either \( \{ \emptyset \mid i \in \omega \} \), or one of the \( \Phi_m \ (m \in \omega) \), or \( \{ \text{Irr}_G(G_i) \mid i \in \omega \} \).
Proof. Let the $\Psi_i$ be nonempty. Every representation $\sigma \in \Psi_i$ has the form $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_d$ for certain irreducible representations $\sigma_j$ of $A_{i,j}$. Define $\delta(\sigma) = \delta(\sigma_1) + \cdots + \delta(\sigma_d)$. Consider the function $\mu: \omega \to \omega$, given by $\mu(i) = \max\{\delta(\sigma) \mid \sigma \in \Psi_i\}$.

Suppose first that $\lim \sup_{i \to \infty} \mu(i)$ takes a finite value $m$. Then, for every $n \geq m$, there exists $k \geq n + 2$ such that $\mu(k) = m$, and Propositions 4.7 and 4.6 imply that $\Phi_{m,i} \subseteq \Psi_i$ for all $i \leq n$. It follows that $\Phi_{m,i} \subseteq \Psi_i$ for all $i$.

The converse inclusion holds trivially. Hence $\Psi = \Phi_m$.

Suppose now that $\lim \sup_{i \to \infty} \mu(i) = \infty$. Then, for every pair $n \geq m$, there exists $k \geq n + 2$ such that $\mu(k) \geq m$, and Propositions 4.7 and 4.6 imply that $\Phi_{m,i} \subseteq \Psi_i$ for all $i \leq n$. It follows that $\Phi_{m,i} \subseteq \Psi_i$ for all $i$ and all $m$. Hence $\Psi = \operatorname{Irr}_K(G_i)$. □

Clearly $\Phi_{m,i} \subset \Phi_{m+1,i}$ for all $i$ and every $m$. Therefore the final result of this section is a direct consequence of Proposition 2.2 and Theorem 4.8.

**Theorem 4.9.** Let $G$ be a countable LD$\mathfrak{A}$-group, and let $K$ be any field of characteristic zero. Then the lattice of two-sided ideals in $K G$ has the form $K G = I_0 \supset \omega(KG) = I_1 \supset I_2 \supset I_3 \supset \cdots \supset \bigcap_{n \in \omega} I_n = 0$.

5. Uncountable groups.

It remains to extend Theorems A and B to uncountable groups. This will be accomplished by an application of the Theorem of L"owenheim and Skolem (see [EFT], Theorem IX.2.4).

**5.1. The extension of Theorem A.** Because of Theorem 3.3 it suffices to show the following two facts.

**Proposition 5.1.**

(a) The class of simple locally finite groups of 1-type is axiomatizable by a sentence in the infinitary language $L_{\omega_1 \omega}$.

(b) The assertion in Theorem A is an $L_{\omega_1 \omega}$-sentence.

**Proof.** We use a 2-sorted language with variables for field elements and for group elements.

(a) In view of the characterization of simple locally finite groups of $\infty$-type mentioned in Section 2, we just need to formalize the two sentences:

Every finite subset $X$ of $G$ is contained in a finite subgroup $H$ of $G$, such that $X \cap N \subseteq \{1\}$ for some maximal normal subgroup $N$ of $H$ with alternating quotient,

and:

There exists a finite subgroup $F$ in $G$ such that, whenever $H$ is a finite subgroup of $G$ with $F \leq H$ and $F \cap N = \{1\}$ for some maximal normal subgroup $N$ of $H$ with alternating quotient $H/N \cong \operatorname{Alt}(\Omega)$, then $F$ does not have a regular orbit on $\Omega$. 

In $L_{\omega_1\omega}$ we can easily quantify over all finite subsets or all finite subgroups of $G$ (by writing down group tables). Therefore the first sentence is expressible in $L_{\omega_1\omega}$. We can similarly scope with the second sentence provided we can express that $F$ does not have a regular orbit on $\Omega$. But the point stabilizers of the action of $H$ on $\Omega$ are precisely the subgroups of index $|\Omega|$ in $H$ which contain $N$. And so we just need to write down that every such maximal subgroup of $H$ contains a nontrivial element from $F$.

(b) Let $K$ be a field and $G$ be a group. It suffices to formalize the sentence:

If $\text{char } K = 0$, then there exist elements $0 \neq x, y \in K[G]$ with coefficient sum zero, such that $y$ does not lie in the ideal generated by $x$.

The condition “$\text{char } K = 0$” is clearly encoded by the infinite conjunction $\bigwedge_{n \in \omega} \psi_n$ where the sentence $\psi_n$ expresses that the $(n+1)$-fold sum of $1 \in K$ is nonzero. The membership “there exists $x \in K[G]$” where $x$ is considered to be of the form $x = k_0g_0 + \cdots + k_ng_n$ is expressible by the infinite disjunction $\bigvee_{n \in \omega} \exists k_0, \ldots, k_n \in K \exists g_0, \ldots, g_n \in G$.

And finally, “$y \in K[G]$ does not lie in the ideal generated by $x \in K[G]$” becomes $\bigwedge_{n \in \omega} \forall u_0, \ldots, u_n, v_0, \ldots, v_n \in K[G] \ y \neq u_0xv_0 + \cdots + u_nxv_n$.

\[ \Box \]

5.2. The extension of Theorem B. We shall use the Löwenheim-Skolem technique to deduce the following result from Theorem 4.9.

Proposition 5.2. Let $K$ be a field of characteristic zero, and let $G$ be any $\mathsf{LD\Delta}$-group. Then there are ideals $K[G] = I_0 \supset I_1 \supset I_2 \supset \ldots$ such that every further ideal of $K[G]$ is contained in $I_\omega = \bigcap_{n \in \omega} I_n$.

To this end it suffices to establish the following two facts.

Proposition 5.3.

(a) The class of $\mathsf{LD\Delta}$-groups is axiomatizable by a sentence in $L_{\omega_1\omega}$.
(b) The assertion in Proposition 5.2 is an $L_{\omega_1\omega}$-sentence.

Proof. (a) This follows as in the proof of Proposition 5.1(a), since every $\mathsf{LD\Delta}$-group is a simple locally finite group of 1-type such that every finite subset is contained in a finite subgroup which is a direct product of alternating groups.

(b) Simply consider $\bigwedge_{n \in \omega} \psi_n \longrightarrow \bigwedge_{n \in \omega} \varphi_n$ where the sentence $\psi_n$ expresses that the $(n + 1)$-fold sum of $1 \in K$ is nonzero, and where $\varphi_n$ is the
There exist elements $x_0, \ldots, x_n \in KG$ such that:

- Every element of $KG$ lies in the ideal generated by $x_0$,
- for every $i \leq n-1$, the element $x_i$ does not lie in the ideal generated by $x_{i+1}$, but $x_{i+1}$ lies in the ideal generated by $x_i$, and
- for every $1 \leq i \leq n$, whenever an element $y \in KG$ does not lie in the ideal generated by $x_i$, then $x_{i-1}$ is contained in the ideal generated by $y$. □

Proof of Theorem B. In the notation of Proposition 5.2, it suffices to show that $I_\omega = 0$. Choose elements $u_n \in I_n \setminus I_{n+1}$ ($n \in \omega$). By Proposition 5.3(a) and [EFT], Theorem IX.2.4, every countable subset of $G$ is contained in a countable ld2i-subgroup $H$ of $G$. By Theorem 4.9, the assertion of Theorem B holds already for every such subgroup $H$. We may also assume without loss that $u_0, u_1, \cdots \in KH$ for every such group $H$. Hence $KH \cap I_\omega = 0$. And this implies that $I_\omega = 0$, as desired. □

References


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