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EMBEDDINGS OF  $S^p \times S^q \times S^r$  IN  $S^{p+q+r+1}$

LAÉRCIO APARECIDO LUCAS AND OSAMU SAEKI



## EMBEDDINGS OF $S^p \times S^q \times S^r$ IN $S^{p+q+r+1}$

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Let  $f : S^p \times S^q \times S^r \rightarrow S^{p+q+r+1}$ ,  $2 \leq p \leq q \leq r$ , be a smooth embedding. In this paper we show that the closure of one of the two components of  $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$ , denoted by  $C_1$ , is diffeomorphic to  $S^p \times S^q \times D^{r+1}$  or  $S^p \times D^{q+1} \times S^r$  or  $D^{p+1} \times S^q \times S^r$ , provided that  $p + q \neq r$  or  $p + q = r$  with  $r$  even. We also show that when  $p + q = r$  with  $r$  odd, there exist infinitely many embeddings which do not satisfy the above property. We also define standard embeddings of  $S^p \times S^q \times S^r$  into  $S^{p+q+r+1}$  and, using the above result, we prove that if  $C_1$  has the homology of  $S^p \times S^q$ , then  $f$  is standard, provided that  $q < r$ .

### 1. Introduction.

In [A], Alexander has shown that a piecewise linearly embedded torus in the three sphere  $S^3$  bounds a solid torus in  $S^3$ , which is known as Alexander's torus theorem. This theorem holds also for smooth embeddings.

Let  $f : S^p \times S^q \rightarrow S^{p+q+1}$  be a codimension one smooth embedding with  $p, q \geq 1$ . Then the closure of one of the two components of  $S^{p+q+1} - f(S^p \times S^q)$  is diffeomorphic to  $D^{p+1} \times S^q$  if  $1 \leq p \leq q$  with  $p + q \neq 3$ , and is homeomorphic to  $D^2 \times S^2$  if  $p = 1$  and  $q = 2$ . This is a generalization of Alexander's torus theorem and has been obtained in [K], [Wa], [G], [R] and [LNS]. An important consequence of this result is that for  $2 \leq p \leq q$ , embeddings of  $S^p \times S^q$  into  $S^{p+q+1}$  are unique up to isotopy. In [LNS], some applications of this result to the study of codimension two smooth embeddings of  $S^p \times S^q$  into  $S^{p+q+2}$  have been given.

The purpose of this paper is to study codimension one smooth embeddings of  $S^p \times S^q \times S^r$  into  $S^{p+q+r+1}$ . More precisely, we completely determine the conditions on  $p, q$  and  $r$  in order that the closure of one of the two components of  $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$  is diffeomorphic to the product of two spheres and a disk. Our first result is the following.

**Theorem 1.1.** *Let  $f : S^p \times S^q \times S^r \rightarrow S^{p+q+r+1}$  be a smooth embedding with  $2 \leq p \leq q \leq r$ . We suppose  $p + q \neq r$ , or  $p + q = r$  and  $r$  is even. Then the closure of one of the two components of  $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$  is diffeomorphic to  $S^p \times S^q \times D^{r+1}$  or  $S^p \times D^{q+1} \times S^r$  or  $D^{p+1} \times S^q \times S^r$ .*

It is surprising that the above condition on  $p, q$  and  $r$  is essential: i.e., if it is not satisfied, then there exist infinitely many counter-examples, which can be called exotic embeddings. In §9, we will show the following by explicitly constructing such embeddings.

**Theorem 1.2.** *If  $p, q \geq 1$  and  $p+q = r$  with  $r$  odd, then there exist mutually distinct embeddings  $f_n : S^p \times S^q \times S^r \rightarrow S^{p+q+r+1}$ ,  $n \in \mathbf{Z} - \{0\}$ , such that the closure of neither of the two components of  $S^{p+q+r+1} - f_n(S^p \times S^q \times S^r)$  is homotopy equivalent to the product of two spheres and a disk.*

However, if we put some more conditions, then we have the following theorem, which will be proved in §7. In the following, homology groups are always with integer coefficients.

**Theorem 1.3.** *Let  $f : S^p \times S^q \times S^r \rightarrow S^{p+q+r+1}$  be a smooth embedding with  $2 \leq p \leq q \leq r$ . Then the closure of one of the two components of  $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$ , denoted by  $C_1$ , has the same homology as  $S^p \times S^q$  or  $S^p \times S^r$  or  $S^q \times S^r$ . Furthermore, if  $C_1$  is homotopy equivalent to  $S^p \times S^q$  or  $S^p \times S^r$  or  $S^q \times S^r$ , then it is diffeomorphic to  $S^p \times S^q \times D^{r+1}$  or  $S^p \times D^{q+1} \times S^r$  or  $D^{p+1} \times S^q \times S^r$  respectively.*

Using the above results, we can obtain more precise information about the embedding  $f$  in some cases. In fact, we will prove, in §8, that if  $2 \leq p \leq q < r$  and  $C_1$  has the homology of  $S^p \times S^q$  in Theorem 1.3, then  $f$  is *standard* (see Definition 8.1 and Corollary 8.2).

The proof of Theorem 1.1 will be divided into five cases according to the homology group structure of  $S^p \times S^q \times S^r$  as follows:

- |   |                               |
|---|-------------------------------|
| (A) $p < q < r$ and $r \neq p + q$ ,                  | (B) $p = q = r$ ,             |
| (C) $p = q < r$ and $p + q \neq r$ , or $p < q = r$ , | (D) $p = q$ and $p + q = r$ , |
| (E) $p < q$ and $p + q = r$ with $r$ even.            |                               |

These cases will be treated in §2–§6 respectively. Our technique for the proof of Theorem 1.1 is based on the standard homology theory and the  $h$ -cobordism theorem [Sm, Mi], which is essentially the same as in [K], [Wa], [G] or in [LNS]. The main difficulty lies in the construction of an embedding of the product of two spheres into  $C_1$  which induces a homotopy equivalence.

Throughout the paper, all manifolds and maps are assumed to be differentiable of class  $C^\infty$  and all homology and cohomology groups are with coefficients in  $\mathbf{Z}$ . The symbol “ $\cong$ ” denotes a diffeomorphism between manifolds or an appropriate isomorphism between algebraic objects. The symbol “[\*]” denotes the homology class represented by \*. The notation “id” denotes the identity map.

**2. Case (A)  $p < q < r$  and  $r \neq p + q$ .**

First let us introduce some notations which will be used throughout the proofs of Theorems 1.1 and 1.3 (§2–§7). Let  $f : S^p \times S^q \times S^r \rightarrow S^{p+q+r+1}$  be a smooth embedding with  $2 \leq p \leq q \leq r$ . Then, by Alexander duality,  $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$  consists exactly of two components and they are simply connected by van Kampen’s theorem. We denote the two components by  $C'_1$  and  $C'_2$  and their closures in  $S^{p+q+r+1}$  by  $C_1$  and  $C_2$  respectively. We identify  $C_1 \cap C_2 = \partial C_1 = \partial C_2$  with  $S^p \times S^q \times S^r$  by the embedding  $f$ . Furthermore,  $i : \partial C_1 \rightarrow C_1$  will denote the inclusion map.

From now on, we assume  $p < q < r$  and  $r \neq p + q$  in this section.

**Lemma 2.1.** *Either  $C_1$  or  $C_2$  has the same homology as  $S^p \times S^q \times D^{r+1}$  or  $S^p \times D^{q+1} \times S^r$  or  $D^{p+1} \times S^q \times S^r$ .*

*Proof.* By Alexander duality, we see easily that there are eight possibilities for the homology groups  $(H_p(C_1), H_q(C_1), H_r(C_1), H_{q+r}(C_1), H_{r+p}(C_1), H_{p+q}(C_1))$ :

- (1)  $(\mathbf{Z}, \mathbf{Z}, 0, 0, 0, \mathbf{Z}), (\mathbf{Z}, 0, \mathbf{Z}, 0, \mathbf{Z}, 0), (0, \mathbf{Z}, \mathbf{Z}, \mathbf{Z}, 0, 0),$
- (2)  $(0, 0, \mathbf{Z}, \mathbf{Z}, \mathbf{Z}, 0), (0, \mathbf{Z}, 0, \mathbf{Z}, 0, \mathbf{Z}), (\mathbf{Z}, 0, 0, 0, \mathbf{Z}, \mathbf{Z}),$
- (3)  $(\mathbf{Z}, \mathbf{Z}, \mathbf{Z}, 0, 0, 0), (0, 0, 0, \mathbf{Z}, \mathbf{Z}, \mathbf{Z}).$

In Case (1)  $C_1$  has the desired homology, and in Case (2)  $C_2$  has the desired homology.

Suppose that  $C_1$  has the homology  $(\mathbf{Z}, \mathbf{Z}, \mathbf{Z}, 0, 0, 0)$ . Since we have  $H^{p+1}(C_1, \partial C_1) \cong H_{q+r}(C_1) = 0$  and  $H^{q+1}(C_1, \partial C_1) \cong H_{p+r}(C_1) = 0$ , the homomorphisms  $i^* : H^p(C_1) \rightarrow H^p(\partial C_1)$  and  $i^* : H^q(C_1) \rightarrow H^q(\partial C_1)$  are surjective, and hence so is  $i^* \otimes i^* : H^p(C_1) \otimes H^q(C_1) \rightarrow H^p(\partial C_1) \otimes H^q(\partial C_1)$ . Then the commutative diagram of cup products

$$\begin{array}{ccc} H^p(C_1) \otimes H^q(C_1) & \xrightarrow{i^* \otimes i^*} & H^p(\partial C_1) \otimes H^q(\partial C_1) \\ \downarrow \smile & & \downarrow \smile \\ H^{p+q}(C_1) & \xrightarrow{i^*} & H^{p+q}(\partial C_1) \end{array}$$

leads to a contradiction, since  $H^{p+q}(C_1) = 0$  and the second column is nonzero. We see that the case  $(0, 0, 0, \mathbf{Z}, \mathbf{Z}, \mathbf{Z})$  cannot happen either by using the same argument for  $C_2$ . □

We may assume that  $C_1$  has the same homology as  $S^p \times S^q \times D^{r+1}$  without loss of generality. Note that we do not have  $p \leq q \leq r$  any more, although  $p, q, r, p + q, q + r$  and  $r + p$  are all distinct.

**Lemma 2.2.** *The composite*

$$\varphi : S^p \times S^q \times \{*\} \xrightarrow{j} S^p \times S^q \times S^r = \partial C_1 \xrightarrow{i} C_1$$

*is a homotopy equivalence, where  $j$  and  $i$  are the inclusion maps.*

*Proof.* Since  $i_*[\{*\} \times S^q \times S^r] = 0$  in  $H_{q+r}(C_1) = 0$ , there exists a  $(q+r+1)$ -chain  $(\Gamma, \partial\Gamma)$  in  $(C_1, \partial C_1)$  such that  $\partial\Gamma$  is homologous to  $\{*\} \times S^q \times S^r$  in  $\partial C_1$ . The intersection number  $[\Gamma, \partial\Gamma] \cdot i_*[S^p \times \{*\} \times \{*\}]$  in  $C_1$  is equal to the intersection number  $[\{*\} \times S^q \times S^r] \cdot [S^p \times \{*\} \times \{*\}] = \pm 1$  in  $\partial C_1$ . This implies that  $\varphi_* : H_p(S^p \times S^q \times \{*\}) \rightarrow H_p(C_1)$  is an isomorphism, since  $H_p(S^p \times S^q \times \{*\}) \cong \mathbf{Z}$  is generated by  $[S^p \times \{*\} \times \{*\}]$  and  $\varphi_*[S^p \times \{*\} \times \{*\}]$  must be a primitive homology class in  $H_p(C_1) \cong \mathbf{Z}$ .

Since  $H_{p+r}(C_1) = 0$  and  $H_r(C_1) = 0$ , we see that  $\varphi_* : H_k(S^p \times S^q \times \{*\}) \rightarrow H_k(C_1)$  is an isomorphism also for  $k = q$  and  $p+q$  by using similar arguments. Then the result follows from Whitehead's theorem.  $\square$

*Proof of Theorem 1.1 for Case (A).* Set  $\Sigma_1 = S^p \times S^q \times \{*\} \subset \partial C_1 = S^p \times S^q \times S^r$ . We can push  $\Sigma_1$  into the interior of  $C_1$  by using an inward normal vector field of  $\partial C_1$  in  $C_1$  and obtain a submanifold  $\Sigma'_1$ . Let  $G$  be a sufficiently small closed tubular neighborhood of  $\Sigma'_1$  in  $\text{Int } C_1$ . We see easily that  $G$  is diffeomorphic to  $S^p \times S^q \times D^{r+1}$ .

By excision and Lemma 2.2, we see that the manifold  $V = C_1 - \text{Int } G$  is an  $h$ -cobordism between  $\partial G$  and  $\partial C_1$ . Since  $\dim V = p + q + r + 1 \geq 6$ , we see by the  $h$ -cobordism theorem [Sm, Mi] that  $V \cong \partial G \times [0, 1]$ . Then we have  $C_1 = V \cup G \cong \partial G \times [0, 1] \cup G \cong G$ , which is diffeomorphic to  $S^p \times S^q \times D^{r+1}$ .  $\square$

### 3. Case (B) $p = q = r$ .

The main tool used in this section is the result about automorphisms of  $H_p(S^p \times S^p \times S^p)$  which can be realized by self-diffeomorphisms of  $S^p \times S^p \times S^p$  (for details, see [LS]).

First, by the same argument as in the proof of Lemma 2.1, we may assume that  $H_*(C_1) \cong H_*(S^p \times S^p \times D^{p+1})$  without loss of generality. As in the previous section, in order to prove Theorem 1.1 for this case, we have only to show the following.

**Lemma 3.1.** *There exists an embedding  $j : S^p \times S^p \rightarrow S^p \times S^p \times S^p$  with trivial normal bundle such that the embedding*

$$(3.1) \quad \varphi : S^p \times S^p \xrightarrow{j} S^p \times S^p \times S^p = \partial C_1 \xrightarrow{i} C_1$$

*is a homotopy equivalence.*

*Proof.* (B1) *When  $p$  is even.* Consider the exact sequence

$$\begin{aligned} 0 = H^p(C_1, \partial C_1) &\rightarrow H^p(C_1) \xrightarrow{i^*} H^p(\partial C_1) \\ &\rightarrow H^{p+1}(C_1, \partial C_1) \rightarrow H^{p+1}(C_1) = 0. \end{aligned}$$

Since  $H^{p+1}(C_1, \partial C_1) \cong \mathbf{Z}$  is free,  $\text{Im } i^*$  is a direct summand of  $H^p(\partial C_1)$ . Let  $\xi$  and  $\eta$  be generators of  $H^p(C_1) \cong \mathbf{Z} \oplus \mathbf{Z}$  and  $\{\alpha^*, \beta^*, \gamma^*\}$  the basis of

$H^p(\partial C_1) \cong \text{Hom}(H_p(\partial C_1), \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$  dual to the basis

$$(3.2) \quad \{\alpha = [S^p \times \{*\} \times \{*\}], \beta = [\{*\} \times S^p \times \{*\}], \gamma = [\{*\} \times \{*\} \times S^p]\}$$

of  $H_p(\partial C_1)$ . Then we have  $i^*\xi = a\alpha^* + b\beta^* + c\gamma^*$  and  $i^*\eta = d\alpha^* + e\beta^* + g\gamma^*$  for some integers  $a, b, c, d, e$  and  $g$ . Since  $\text{Im } i^*$  is a direct summand of  $H^p(\partial C_1)$ , there exist integers  $h, l, m$  such that

$$\det \begin{pmatrix} a & d & h \\ b & e & l \\ c & g & m \end{pmatrix} = \pm 1.$$

By the commutative diagram

$$\begin{array}{ccc} H^p(C_1) \otimes H^p(C_1) & \xrightarrow{i^* \otimes i^*} & H^p(\partial C_1) \otimes H^p(\partial C_1) \\ \downarrow \smile & & \downarrow \smile \\ \mathbf{Z} \cong H^{2p}(C_1) & \xrightarrow{i^*} & H^{2p}(\partial C_1) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}, \end{array}$$

we have that the subgroup generated by

$$\begin{aligned} i^*\xi \smile i^*\xi &= 2ab(\alpha^* \smile \beta^*) + 2bc(\beta^* \smile \gamma^*) + 2ca(\gamma^* \smile \alpha^*), \\ i^*\xi \smile i^*\eta &= (ae + bd)(\alpha^* \smile \beta^*) + (bg + ce)(\beta^* \smile \gamma^*) \\ &\quad + (cd + ag)(\gamma^* \smile \alpha^*), \\ i^*\eta \smile i^*\eta &= 2de(\alpha^* \smile \beta^*) + 2eg(\beta^* \smile \gamma^*) + 2gd(\gamma^* \smile \alpha^*) \end{aligned}$$

has rank at most one. Using the fact that  $\{\alpha^* \smile \beta^*, \beta^* \smile \gamma^*, \gamma^* \smile \alpha^*\}$  is a basis of  $H^{2p}(\partial C_1)$ , we see easily that  $abc = deg = 0$ . Then, we can show that for an embedding  $j : S^p \times S^p \rightarrow S^p \times S^p \times S^p$  such that  $j(S^p \times S^p) = \{*\} \times S^p \times S^p, S^p \times \{*\} \times S^p$ , or  $S^p \times S^p \times \{*\}$ , the composite  $\varphi$  as in (3.1) induces an isomorphism on the  $p$ -th cohomology groups. Then by the universal coefficient theorem,  $\varphi_* : H_p(S^p \times S^p) \rightarrow H_p(C_1)$  is also an isomorphism. Consider the commutative diagram

$$\begin{array}{ccc} H^p(C_1) \otimes H^p(C_1) & \xrightarrow{\varphi^* \otimes \varphi^*} & H^p(S^p \times S^p) \otimes H^p(S^p \times S^p) \\ \downarrow \smile & & \downarrow \smile \\ \mathbf{Z} \cong H^{2p}(C_1) & \xrightarrow{\varphi^*} & H^{2p}(S^p \times S^p) \cong \mathbf{Z}, \end{array}$$

where  $k_2 \circ (\varphi^* \otimes \varphi^*)$  is an epimorphism, since  $k_2$  is unimodular. This implies that  $\varphi^* : H^{2p}(C_1) \rightarrow H^{2p}(S^p \times S^p)$  is also an epimorphism. Since  $H^{2p}(C_1) \cong H^{2p}(S^p \times S^p) \cong \mathbf{Z}$ , we see that  $\varphi^* : H^{2p}(C_1) \rightarrow H^{2p}(S^p \times S^p)$  is an isomorphism, which implies that  $\varphi_* : H_{2p}(S^p \times S^p) \rightarrow H_{2p}(C_1)$  is also an isomorphism. Then by Whitehead's theorem, the result follows.

In order to prove Lemma 3.1 when  $p$  is odd, we need the following, which can be easily proved by examining the exact sequence

$$0 \rightarrow \ker i_* \rightarrow H_p(\partial C_1) \xrightarrow{i_*} H_p(C_1) \rightarrow H_p(C_1, \partial C_1) = 0.$$

**Lemma 3.2.** *For every  $p$ , there exists a basis  $\{\zeta_1, \zeta_2, \zeta\}$  of  $H_p(\partial C_1)$  such that  $\{i_*\zeta_1, i_*\zeta_2\}$  is a basis of  $H_p(C_1)$  and  $i_*\zeta = 0$ .*

(B2) *When  $p = 3$  or  $p = 7$ . The following is a direct consequence of [LS, Theorem 2.2].*

**Lemma 3.3.** *If  $p = 3$  or  $p = 7$ , then there exists a diffeomorphism  $\phi : S^p \times S^p \times S^p \rightarrow S^p \times S^p \times S^p$  such that  $\phi_*H_p(S^p \times S^p \times \{*\}) = \langle \zeta_1, \zeta_2 \rangle$ , where  $\langle \zeta_1, \zeta_2 \rangle$  is the subgroup of  $H_p(S^p \times S^p \times S^p)$  generated by  $\zeta_1$  and  $\zeta_2$ .*

By putting  $j = \phi|_{S^p \times S^p \times \{*\}}$ , we see that Lemma 3.1 holds for  $p = 3, 7$ .

(B3) *When  $p$  is odd with  $p \neq 3, 7$ . Let  $\eta_n : GL(n; \mathbf{Z}) \rightarrow GL(n; \mathbf{Z}_2)$  be the natural homomorphism. Note that  $\eta_n$  is an epimorphism (see, for example, [Mc, Proposition I.14]). We define the subgroup  $G_1$  of  $GL(n; \mathbf{Z})$  by  $G_1 = \eta_n^{-1}(\eta_n(\mathfrak{S}_n))$ , where we naturally identify the symmetric group  $\mathfrak{S}_n$  with the corresponding subgroup of  $GL(n; \mathbf{Z})$ . Note that  $G_1$  corresponds to the set of those automorphisms which can be realized by diffeomorphisms of the product of  $n$  copies of  $S^p$  for  $p$  odd with  $p \neq 3, 7$  (see [LS]).*

We define the matrix  $A \in GL(3; \mathbf{Z})$  by  $(\zeta_1, \zeta_2, \zeta) = (\alpha, \beta, \gamma)A$ , where  $\{\alpha, \beta, \gamma\}$  is the canonical basis of  $H_p(S^p \times S^p \times S^p)$  as in (3.2). Note that  $A$  may not lie in  $G_1$ .

**Lemma 3.4.** *There exists a matrix  $A' \in G_1 \subset GL(3; \mathbf{Z})$  such that  $\{i_*\zeta'_1, i_*\zeta'_2\}$  is a basis of  $H_p(C_1)$ , where  $(\zeta'_1, \zeta'_2, \zeta') = (\alpha, \beta, \gamma)A'$ .*

Note that  $i_*\zeta'$  may not be zero in  $H_p(C_1)$  any more.

*Proof of Lemma 3.4.* By changing the order of  $\alpha, \beta, \gamma$  and by adding  $\zeta$  to  $\zeta_1, \zeta_2$  if necessary, we may assume that  $A_2 = \eta_3(A)$  is of the form

$$\begin{pmatrix} a & b & * \\ c & d & * \\ 0 & 0 & 1 \end{pmatrix} \in GL(3; \mathbf{Z}_2).$$

Since  $\eta_2 : GL(2; \mathbf{Z}) \rightarrow GL(2; \mathbf{Z}_2)$  is surjective, there exists a matrix  $B \in GL(2; \mathbf{Z})$  with

$$\eta_2(B) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Using  $B$ , we can change  $\zeta_1, \zeta_2$  so that  $A_2$  is of the form

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally, adding  $\zeta_1, \zeta_2$  to  $\zeta$ , we get the desired basis  $\{\zeta'_1, \zeta'_2, \zeta'\}$ . □

Now by using the same argument as in Case (B2) together with [LS, Theorem 2.2], we see that Lemma 3.1 holds for this case as well. This completes the proof of Lemma 3.1, and hence Theorem 1.1 for Case (B). □

**4. Case (C)  $p = q < r$  and  $p + q \neq r$ , or  $p < q = r$ .**

We will assume that  $p = q \neq r \neq p + q$  throughout this section, although  $r$  can be smaller than  $p = q$ . By the same argument as in the proof of Lemma 2.1, we see that it suffices to study the cases

$$(C1) \quad H_*(C_1) \cong H_*(S^p \times S^p \times D^{r+1}),$$

$$(C2) \quad H_*(C_1) \cong H_*(S^p \times D^{p+1} \times S^r).$$

(C1) When  $H_*(C_1) \cong H_*(S^p \times S^p \times D^{r+1})$ . Let  $j : S^p \times S^p \times \{*\} \rightarrow S^p \times S^p \times S^r = \partial C_1$  be the inclusion and set  $\varphi = i \circ j$ . By carefully examining the exact sequence of the triple  $(C_1, \partial C_1, S^p \times S^p \times \{*\})$

$$\begin{aligned} \cdots \rightarrow H_k(\partial C_1, S^p \times S^p \times \{*\}) &\rightarrow H_k(C_1, S^p \times S^p \times \{*\}) \rightarrow H_k(C_1, \partial C_1) \\ &\rightarrow H_{k-1}(\partial C_1, S^p \times S^p \times \{*\}) \rightarrow \cdots \end{aligned}$$

and by applying an argument similar to that in the proof of Lemma 3.1, we can show that the inclusion map  $\varphi : S^p \times S^p \times \{*\} \rightarrow C_1$  gives a homotopy equivalence. Then as in §2, we see that  $C_1 \cong S^p \times S^p \times D^{r+1}$ .

(C2) When  $H_*(C_1) \cong H_*(S^p \times D^{p+1} \times S^r)$ .

**Lemma 4.1.** *There exists an embedding  $\psi : S^p \rightarrow S^p \times S^p$  such that the embedding*

$$(4.1) \quad \varphi : S^p \times S^r \xrightarrow{\psi \times \text{id}} (S^p \times S^p) \times S^r = \partial C_1 \xrightarrow{i} C_1$$

*is a homotopy equivalence.*

*Proof.* As in Lemma 3.2, there exists a basis  $\{\zeta, \zeta_1\}$  of  $H_p(\partial C_1) \cong \mathbf{Z} \oplus \mathbf{Z}$  such that  $i_*\zeta = 0$  and  $i_*\zeta_1$  is a generator of  $H_p(C_1) \cong \mathbf{Z}$ . By the isomorphism  $H_p(\partial C_1) \cong H_p(S^p \times S^p \times S^r) \cong H_p(S^p \times S^p \times \{*\})$ ,  $\zeta_1 \in H_p(\partial C_1)$  corresponds to an element  $\zeta'_1 \in H_p(S^p \times S^p \times \{*\}) \cong \pi_p(S^p \times S^p)$ . When  $p \geq 3$ , by [H] or [Wh], we can represent  $\zeta'_1$  by an embedding  $\psi : S^p \rightarrow S^p \times S^p$ . Then the composite  $\varphi$  as in (4.1) is an embedding such that  $\varphi_* : H_p(S^p \times S^r) \rightarrow H_p(C_1)$  is an isomorphism.

When  $p = 2$ , we cannot use the above argument (see, for example, [KM]). However, since  $p = 2$  is even, we can show that the embedding  $\psi : S^p \rightarrow S^p \times S^p$  such that  $\psi(S^p) = \{*\} \times S^p$  or  $S^p \times \{*\}$  satisfies the same property, by using an argument similar to that in (B1) of the proof of Lemma 3.1.

Then by the same arguments as in the proofs of Lemmas 2.2 and 3.1, we see that  $\varphi : S^p \times S^r \rightarrow C_1$  is a homotopy equivalence. This completes the proof of Lemma 4.1. □

Let  $\Sigma'_1$  denote the submanifold of  $C_1$  which is obtained by pushing  $\Sigma_1 = \varphi(S^p \times S^r)$  into the interior of  $C_1$  using a normal vector field of  $\partial C_1$  pointing toward  $\text{Int } C_1$ .

**Lemma 4.2.** *The normal bundle of  $\Sigma'_1$  in  $C_1$  is trivial.*

*Proof.* Let  $\psi : S^p \rightarrow S^p \times S^p$  be the embedding as above. It suffices to show that the normal bundle  $\nu_{\tilde{\varphi}}$  of  $\tilde{\varphi} = f \circ (\psi \times \text{id}) : S^p \times S^r \rightarrow S^{2p+r+1}$  is trivial. We have

$$\nu_{\tilde{\varphi}} \cong \pi^*(\nu_\psi) \oplus \varepsilon_{S^p \times S^r}^1 \cong \pi^*(\nu_\psi) \oplus \pi^*(\varepsilon_{S^p}^1) \cong \pi^*(\nu_\psi \oplus \varepsilon_{S^p}^1),$$

where  $\pi : S^p \times S^r \rightarrow S^p$  is the projection to the first factor,  $\nu_\psi$  denotes the normal bundle of  $\psi$ , and  $\varepsilon_X^1$  denotes the trivial line bundle over a space  $X$ . On the other hand, using the embedding

$$\tilde{\psi} : S^p \xrightarrow{\psi} S^p \times S^p \hookrightarrow S^{2p+1},$$

we see that  $\nu_\psi \oplus \varepsilon_{S^p}^1$  is trivial. Thus the result follows. □

Finally, by the same argument as in Case (A), we see that Theorem 1.1 holds for Case (C2) as well. □

### 5. Case (D) $p = q$ and $p + q = r$ .

By the same argument as in the proof of Lemma 2.1, we see that either  $C_1$  or  $C_2$  has the same homology as  $S^p \times S^p \times D^{r+1}$  or  $S^p \times D^{p+1} \times S^r$ .

When  $H_*(C_1) \cong H_*(S^p \times S^p \times D^{r+1})$  or  $H_*(C_2) \cong H_*(S^p \times S^p \times D^{r+1})$ , by using arguments similar to those in Case (C1), we see that  $C_1$  or  $C_2$  is diffeomorphic to  $S^p \times S^p \times D^{r+1}$ .

When  $H_*(C_1) \cong H_*(S^p \times D^{p+1} \times S^r)$ , we see easily that  $H_*(C_2) \cong H_*(S^p \times D^{p+1} \times S^r) (\cong H_*(D^{p+1} \times S^p \times S^r))$ . First we prepare the following lemmas. Note that  $r = 2p$  is even and that  $\dim \partial C_1$  is equal to  $2r$ .

**Lemma 5.1.** *There exists a basis  $\{\zeta, \zeta'\}$  of  $H_r(\partial C_1) \cong \mathbf{Z} \oplus \mathbf{Z}$  such that  $i_*\zeta = 0$ ,  $\zeta \cdot \zeta = 0$ ,  $\zeta \cdot \zeta' = 1$  and  $\zeta' \cdot \zeta' = 0$ .*

The above lemma can be proved by using an argument similar to that in [LNS, Lemmas 3.2-3.4].

Set  $\alpha = [S^p \times S^p \times \{*\}]$  and  $\beta = [\{*\} \times \{*\} \times S^r]$ , which generate  $H_r(\partial C_1) = H_r(S^p \times S^p \times S^r)$ . We have  $\alpha \cdot \alpha = \beta \cdot \beta = 0$  and may assume  $\alpha \cdot \beta = 1$ , choosing suitable orientations for  $S^p \times S^p \times \{*\}$  and  $\{*\} \times \{*\} \times S^r$ .

**Lemma 5.2.** *If  $p \geq 3$ , then for some embedding  $\psi : S^p \rightarrow S^p \times S^p$ , the composite*

$$\begin{aligned} \varphi_1 : S^p \times S^r &\xrightarrow{\psi \times \text{id}} (S^p \times S^p) \times S^r = \partial C_1 \xrightarrow{i} C_1 \quad \text{or} \\ \varphi_2 : S^p \times S^r &\xrightarrow{\psi \times \text{id}} (S^p \times S^p) \times S^r = \partial C_2 \xrightarrow{j} C_2 \end{aligned}$$

*is a homotopy equivalence, where  $i$  and  $j$  are the inclusion maps.*

*Proof.* We have  $H_*(C_1) \cong H_*(S^p \times D^{p+1} \times S^r) \cong H_*(C_2)$ . Consider the endomorphism  $\Theta : H_r(S^p \times S^p \times S^r) \rightarrow H_r(S^p \times S^p \times S^r)$  defined by  $\Theta(\alpha) = \zeta'$  and  $\Theta(\beta) = \zeta$ . Since  $\zeta \cdot \zeta = \zeta' \cdot \zeta' = 0$  and  $\zeta \cdot \zeta' = 1$ , we see that  $\Theta$  is an automorphism of  $(H_r(S^p \times S^p \times S^r), \cdot)$ , where “ $\cdot$ ” denotes the intersection form. By an argument similar to that of [LNS, Lemma 3.5], we have that  $\zeta = \pm\alpha$  and  $\zeta' = \pm\beta$ , or  $\zeta = \pm\beta$  and  $\zeta' = \pm\alpha$ .

When  $\zeta = \pm\alpha$  and  $\zeta' = \pm\beta$ , we have  $i_*[S^p \times S^p \times \{*\}] = 0$ , and as in Lemma 2.2,  $i_{1*} : H_r(\{*\} \times \{*\} \times S^r) \rightarrow H_r(C_1)$  is an isomorphism, where  $i_1$  is the inclusion map. Similarly, when  $\zeta = \pm\beta$  and  $\zeta' = \pm\alpha$ ,  $j_{1*} : H_r(\{*\} \times \{*\} \times S^r) \rightarrow H_r(C_2)$  is an isomorphism for the inclusion map  $j_1$ . Then, since  $p \geq 3$ , by arguments similar to those in the proofs of Lemmas 4.1 and 3.1, we have the desired result.  $\square$

When  $p = 2$ , we cannot apply the same argument. Nevertheless, as in the proof of Lemma 5.2, we may assume that  $i_{1*} : H_r(\{*\} \times \{*\} \times S^r) \rightarrow H_r(C_1)$  is an isomorphism.

Consider a collar neighborhood  $c : \partial C_1 \times [0, 1] \rightarrow C_1$  of  $\partial C_1$  in  $C_1$ , where  $c(x, 0) = x$  for every  $x \in \partial C_1$ . We will use the identification

$$\partial C_1 \times [0, 1] \xrightarrow{f^{-1} \times \text{id}} S^p \times S^p \times S^r \times [0, 1] \cong (S^p \times S^p \times [0, 1]) \times S^r.$$

**Lemma 5.3.** *If  $p = 2$ , then there exists an embedding  $\psi_1 : S^p \rightarrow S^p \times S^p \times [0, 1]$  such that the embedding  $\varphi : S^p \times S^r \rightarrow C_1$  defined by*

$$\begin{aligned} \varphi : S^p \times S^r &\xrightarrow{\psi_1 \times \text{id}} (S^p \times S^p \times [0, 1]) \times S^r \\ &\cong S^p \times S^p \times S^r \times [0, 1] \xrightarrow{f \times \text{id}} \partial C_1 \times [0, 1] \xrightarrow{c} C_1 \end{aligned}$$

*is a homotopy equivalence.*

*Proof.* As in the proof of Lemma 4.1, there exists a continuous map  $\psi' : S^p \rightarrow S^p \times S^p$  which represents  $\zeta'_1 \in H_p(S^p \times S^p) (\cong \pi_p(S^p \times S^p)) \cong H_p(S^p \times S^p \times S^r)$  with  $i_*\zeta'_1$  being a generator of  $H_p(C_1) \cong \mathbf{Z}$ . Consider the composite

$$\psi'' : S^p \xrightarrow{\psi'} S^p \times S^p \xrightarrow{i'} S^p \times S^p \times [0, 1],$$

where  $i' : S^p \times S^p = S^p \times S^p \times \{0\} \rightarrow S^p \times S^p \times [0, 1]$  is the inclusion map. By [H, Theorem 1(a)], there exists a differentiable embedding  $\psi_1 : S^p \rightarrow S^p \times S^p \times [0, 1]$  homotopic to  $\psi''$ . Then,  $\varphi$  is a differentiable embedding such that  $\varphi_* : H_p(S^p \times S^r) \rightarrow H_p(C_1)$  is an isomorphism. The rest of the proof is the same as before.  $\square$

As in Lemma 4.2, if  $p \geq 3$ , then the normal bundles of  $\varphi_1$  and  $\varphi_2$  of Lemma 5.2 are trivial. When  $p = 2$ , by embedding  $S^p \times S^p \times [0, 1]$  in  $S^{2p+1}$ , we see that  $\psi_1$  as above has trivial normal bundle  $\nu_{\psi_1}$ . Furthermore, we have  $\nu_\varphi \cong \pi^*(\nu_{\psi_1})$ , where  $\pi : S^p \times S^r \rightarrow S^p$  is the projection to the first factor. Hence  $\nu_\varphi$  is trivial.

Then as in the previous sections, we see that  $C_1 \cong S^p \times D^{p+1} \times S^r$  or  $C_2 \cong S^p \times D^{p+1} \times S^r$ . This completes the proof of Theorem 1.1 for Case (D).  $\square$

**6. Case (E)  $p < q$  and  $p + q = r$  with  $r$  even.**

By the same argument as in the proof of Lemma 2.1, we see that either  $C_1$  or  $C_2$  has the same homology as  $S^p \times S^q \times D^{r+1}$  or  $S^p \times D^{q+1} \times S^r$  or  $D^{p+1} \times S^q \times S^r$ .

$$(E1) \quad H_*(C_1) \cong H_*(S^p \times S^q \times D^{r+1}) \text{ or } H_*(C_2) \cong H_*(S^p \times S^q \times D^{r+1}).$$

We may assume that  $H_*(C_1) \cong H_*(S^p \times S^q \times D^{r+1})$ . Then by arguments similar to those in the proofs of Lemmas 2.2 and 3.1, we see that the inclusion map

$$\varphi : S^p \times S^q \times \{*\} \rightarrow S^p \times S^q \times S^r = \partial C_1 \xrightarrow{i} C_1$$

is a homotopy equivalence. Then the rest of the proof for this case is the same as before.

**Remark 6.1.** Even when  $p + q = r$  with  $r$  odd, if  $H_*(C_1) \cong H_*(S^p \times S^q \times D^{r+1})$ , then we can prove that  $C_1 \cong S^p \times S^q \times D^{r+1}$  by using the above argument.

$$(E2) \quad H_*(C_1) \cong H_*(S^p \times D^{q+1} \times S^r) (\Leftrightarrow H_*(C_2) \cong H_*(D^{p+1} \times S^q \times S^r)).$$

Let  $i_1 : S^p \times \{*\} \times S^r \rightarrow S^p \times S^q \times S^r, j : \partial C_2 \rightarrow C_2$  and  $j_1 : \{*\} \times S^q \times S^r \rightarrow S^p \times S^q \times S^r$  be the inclusion maps. By using arguments similar to the previous ones, we can show the following.

**Lemma 6.2.** *The inclusion*

$$\begin{aligned} \varphi_1 : S^p \times \{*\} \times S^r &\xrightarrow{i_1} S^p \times S^q \times S^r = \partial C_1 \xrightarrow{i} C_1 \quad \text{or} \\ \varphi_2 : \{*\} \times S^q \times S^r &\xrightarrow{j_1} S^p \times S^q \times S^r = \partial C_2 \xrightarrow{j} C_2 \end{aligned}$$

*is a homotopy equivalence.*

Thus, Theorem 1.1 holds for Case (E2). This completes the proof of Theorem 1.1 for all the cases.  $\square$

**7. Case (F)  $p + q = r$  with  $r$  odd.**

In this section, let us consider the case where  $r = p + q$  with  $r$  odd. The main result of this section is the following:

**Proposition 7.1.** *Let  $f : S^p \times S^q \times S^r \rightarrow S^{p+q+r+1}$  be a smooth embedding and  $C_1$  the closure of one of the two components of  $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$  with  $p, q \geq 2, r = p + q$  and  $r$  odd.*

- (1) *If  $H_*(C_1) \cong H_*(S^p \times S^q)$ , then  $C_1$  is diffeomorphic to  $S^p \times S^q \times D^{r+1}$ .*

- (2) If  $C_1$  has the same cohomology ring as  $S^p \times S^r$  or  $S^q \times S^r$ , then  $C_1$  is diffeomorphic to  $S^p \times D^{q+1} \times S^r$  or  $D^{p+1} \times S^q \times S^r$  respectively.

**Lemma 7.2.** *If  $C_1$  has the same cohomology ring as  $S^p \times S^r$  or  $S^q \times S^r$ , then  $H_r(C_1)$  is generated by  $i_*[\{*\} \times \{*\} \times S^r]$ .*

*Proof.* Suppose that  $C_1$  has the same cohomology ring as  $S^p \times S^r$ . The other case can be proved similarly. It is not difficult to show that  $i_* : H_k(\partial C_1) \rightarrow H_k(C_1)$  is an isomorphism for  $k = p$  and  $p + r$  as in the proof of Lemma 2.2. Set

$$\begin{aligned} \alpha^* &= [S^p \times S^q \times \{*\}]^*, \beta^* = [\{*\} \times \{*\} \times S^r]^* \in H^r(\partial C_1) \cong \mathbf{Z} \oplus \mathbf{Z}, \\ \gamma^* &= [S^p \times \{*\} \times \{*\}]^* \in H^p(\partial C_1) \cong \mathbf{Z}, \\ \delta^* &= [S^p \times \{*\} \times S^r]^* \in H^{p+r}(\partial C_1) \cong \mathbf{Z}, \end{aligned}$$

and let  $\xi_p \in H^p(C_1) \cong \mathbf{Z}$  and  $\xi_r \in H^r(C_1) \cong \mathbf{Z}$  be generators, where each  $[*]^*$  means a dual basis. Note that we have  $\gamma^* \smile \beta^* = \pm \delta^*$  and  $\gamma^* \smile \alpha^* = 0$ .

Let us consider the commutative diagram

$$\begin{array}{ccc} H^p(C_1) \otimes H^r(C_1) & \xrightarrow{i^* \otimes i^*} & H^p(\partial C_1) \otimes H^r(\partial C_1) \\ \downarrow \smile & & \downarrow \smile \\ H^{p+r}(C_1) & \xrightarrow{i^*} & H^{p+r}(\partial C_1). \end{array}$$

The cohomology class  $\xi_p \smile \xi_r$  generates  $H^{p+r}(C_1)$ , since  $C_1$  has the same cohomology ring as  $S^p \times S^r$ . On the other hand, we have  $i^* \xi_p = \pm [S^p \times \{*\} \times \{*\}]^* = \pm \gamma^*$ , since  $i^* : H^p(C_1) \rightarrow H^p(\partial C_1)$  is an isomorphism. Furthermore, the cohomology class  $i^*(\xi_p \smile \xi_r)$  generates  $H^{p+r}(\partial C_1)$ , since  $i^*$  in the second row is an isomorphism. We can put  $i^* \xi_r = a\alpha^* + b\beta^*$  for some integers  $a$  and  $b$ . We see easily that  $i^*(\xi_p \smile \xi_r) = \pm b\delta^*$ . This implies that  $b = \pm 1$ .

Then we see that  $\langle \xi_r, i_*[\{*\} \times \{*\} \times S^r] \rangle = \pm 1$ , where  $\langle *, * \rangle$  denotes the Kronecker product. Thus,  $i_*[\{*\} \times \{*\} \times S^r]$  generates  $H_r(C_1)$ .  $\square$

*Proof of Proposition 7.1.* (1) This follows from Remark 6.1.

(2) We may assume that  $C_1$  has the same cohomology ring as  $S^p \times S^r$ . Consider the inclusion map

$$\varphi : S^p \times \{*\} \times S^r \rightarrow S^p \times S^q \times S^r = \partial C_1 \xrightarrow{i} C_1.$$

By Lemma 7.2,  $\varphi_* : H_r(S^p \times \{*\} \times S^r) \rightarrow H_r(C_1)$  is an isomorphism. Furthermore, as has been seen in the proof of Lemma 7.2,  $\varphi_* : H_k(S^p \times \{*\} \times S^r) \rightarrow H_k(C_1)$  is an isomorphism for  $k = p$  and  $p + r$ . Thus,  $\varphi$  is a homotopy equivalence. Then by arguments similar to those in §2, we see that  $C_1 \cong S^p \times D^{q+1} \times S^r$ .  $\square$

*Proof of Theorem 1.3.* The theorem follows from Theorem 1.1 and Proposition 7.1.  $\square$

## 8. Standard embeddings.

As a consequence of Theorem 1.1, we have the following Corollary 8.2. This result is important, since it gives a sufficient condition for an embedding to be standard. The characterization of standard embeddings is fundamental in the study of embeddings.

Let us begin by defining standard embeddings.

**Definition 8.1.** Let  $g : S^p \times S^q \rightarrow S^{p+q+r+1}$  be a smooth embedding. We say that  $g$  is *standard* or that  $S^p \times S^q$  is *standardly embedded* in  $S^{p+q+r+1}$ , if  $g(S^p \times S^q)$  is isotopic to the boundary of a tubular neighborhood of  $S^p$  or  $S^q$  standardly embedded in  $S^{p+q+r+1}$  in the usual sense. We say that a smooth embedding  $f : S^p \times S^q \times S^r \rightarrow S^{p+q+r+1}$  is *standard* if  $f(S^p \times S^q \times S^r)$  is isotopic to the boundary of a tubular neighborhood of  $S^p \times S^q$  or  $S^q \times S^r$  or  $S^p \times S^r$  standardly embedded in  $S^{p+q+r+1}$ .

**Corollary 8.2.** Let  $f : S^p \times S^q \times S^r \rightarrow S^{p+q+r+1}$  be a smooth embedding with  $2 \leq p \leq q < r$  and  $C_1$  the closure of one of the two components of  $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$ . If  $H_*(C_1) \cong H_*(S^p \times S^q)$ , then  $f$  is standard.

*Proof.* By Theorem 1.1 and Proposition 7.1 (1) together with our dimensional assumptions,  $f(S^p \times S^q \times S^r)$  bounds in  $S^{p+q+r+1}$  an embedded manifold  $T$  diffeomorphic to  $S^p \times S^q \times D^{r+1}$ . Note that  $T$  is a tubular neighborhood of  $S$ , where  $S$  is the product of two spheres  $S^p \times S^q$  embedded in  $S^{p+q+r+1}$  which corresponds to  $S^p \times S^q \times \{0\} \subset S^p \times S^q \times D^{r+1} \cong T \subset S^{p+q+r+1}$ .

By [H] together with our hypothesis on  $p, q$  and  $r$ , there exists a diffeomorphism  $h : S^{p+q+r+1} \rightarrow S^{p+q+r+1}$  isotopic to the identity such that  $h(S)$  is the product of spheres  $S^p \times S^q$  standardly embedded in  $S^{p+q+r+1}$ . Then  $h(T)$  is a tubular neighborhood of  $S^p \times S^q$ . Thus,  $f(S^p \times S^q \times S^r)$  bounds the tubular neighborhood  $h^{-1}(h(T)) = T$  of  $h^{-1}(S^p \times S^q)$ , which is standardly embedded in  $S^{p+q+r+1}$ . Therefore,  $f$  is standard.  $\square$

**Remark 8.3.** Compare the above corollary with [LNS, Theorem 1.3] about codimension one embeddings of product of two spheres.

## 9. Exotic embeddings.

In this section, let us consider the case where  $r = p + q$  with  $r$  odd and prove Theorem 1.2, which insures the existence of exotic embeddings under this dimensional assumption. The result is surprising when compared with Theorem 1.1 and the results obtained in [A], [K], [Wa], [G], [R] and [LNS] about codimension one embeddings of product of two spheres. In the following, we will construct the embeddings  $f_n$  so that the complements  $S^{p+q+r+1} - f_n(S^p \times S^q \times S^r)$  are not homotopy equivalent to each other.

Let us write  $S^{2r+1}$  as the union

$$(9.1) \quad S^{2r+1} = (D_{-2}^{r+1} \times S^r) \cup_{\varphi_-} (S^r \times S^r \times I) \cup_{\varphi_+} (S^r \times D_2^{r+1}),$$

where  $I = [-1, 1]$ ,  $D_{\pm 2}^{r+1}$  are  $(r+1)$ -disks, and  $\varphi_- : \partial(D_{-2}^{r+1} \times S^r) \rightarrow S^r \times S^r \times \{-1\}$  and  $\varphi_+ : \partial(S^r \times D_2^{r+1}) \rightarrow S^r \times S^r \times \{1\}$  are the standard identification maps. Since  $S^r \times I$  is diffeomorphic to the closure of the complement of two disjoint  $(r+1)$ -disks in  $S^{r+1} = (S^p \times D^{q+1}) \cup (D^{p+1} \times S^q)$ , we can write  $S^r \times S^r \times I$  as the union of

$$\begin{aligned} X_- &= ((S^p \times D^{q+1}) - \text{Int } D_{-1}^{r+1}) \times S^r \quad \text{and} \\ X_+ &= ((D^{p+1} \times S^q) - \text{Int } D_1^{r+1}) \times S^r \end{aligned}$$

attached along  $S^p \times S^q \times S^r$ , which is a boundary component of each, where  $D_{\pm 1}^{r+1}$  are interior disks. Note that the embedding  $S^p \times S^q \times S^r = X_- \cap X_+ \subset X_- \cup X_+ = S^r \times S^r \times I \subset S^{2r+1}$  defined via (9.1) is standard. In the following, we will modify this embedding by changing the identification maps  $\varphi_{\pm}$  in (9.1).

Let  $\psi : S^r \times S^r \rightarrow S^r \times S^r$  be an arbitrary diffeomorphism. By (9.1), we still have

$$S^{2r+1} \cong (D_{-2}^{r+1} \times S^r) \cup_{\varphi_- \circ \psi} (S^r \times S^r \times I) \cup_{\varphi_+ \circ \psi} (S^r \times D_2^{r+1}).$$

Put

$$\tilde{X}_- = (D_{-2}^{r+1} \times S^r) \cup_{\varphi_- \circ \psi} X_-, \quad \tilde{X}_+ = X_+ \cup_{\varphi_+ \circ \psi} (S^r \times D_2^{r+1}),$$

and consider  $S^p \times S^q \times S^r = \tilde{X}_- \cap \tilde{X}_+ \subset \tilde{X}_- \cup \tilde{X}_+ = S^{2r+1}$ . We will show that, for a suitable diffeomorphism  $\psi$ ,  $\tilde{X}_{\pm}$  are not homotopy equivalent to the product of two spheres.

Suppose that  $\psi_* : H_r(S^r \times S^r) \rightarrow H_r(S^r \times S^r)$  is given by  $\psi_*\alpha = k\alpha + l\beta$  and  $\psi_*\beta = m\alpha + n\beta$ , where

$$A = \begin{pmatrix} k & m \\ l & n \end{pmatrix} \in GL(2; \mathbf{Z}), \quad \text{and}$$

$$\alpha = [\partial D_{-2}^{r+1} \times \{*\}] = [S^r \times \{*\}], \quad \beta = [\{*\} \times \partial D_2^{r+1}] = [\{*\} \times S^r]$$

are the generators of  $H_r(S^r \times S^r)$ . Then by using standard techniques in homology theory, we can show the following:

**Lemma 9.1.**

- (1) *The homology group  $H_r(\tilde{X}_-)$  is isomorphic to  $\mathbf{Z}$  and is generated by  $\xi = m[\partial D_{-1}^{r+1} \times \{*\}] + n[\{*\} \times S^r]$ , where  $\partial D_{-1}^{r+1} \times \{*\}, \{*\} \times S^r \subset X_-$  and we can identify  $[\partial D_{-1}^{r+1} \times \{*\}]$  with  $[S^p \times S^q \times \{*\}]$  ( $S^p \times S^q \times \{*\} \subset \partial \tilde{X}_-$ ). Furthermore, we have  $[\{*\} \times S^r] = \pm k\xi$ .*

- (2) The homology group  $H_r(\tilde{X}_+)$  is isomorphic to  $\mathbf{Z}$  and is generated by  $\xi' = k[\partial D_1^{r+1} \times \{*\}] + l[\{*\} \times S^r]$ , where  $\partial D_1^{r+1} \times \{*\}, \{*\} \times S^r \subset X_+$  and we can identify  $[\partial D_1^{r+1} \times \{*\}]$  with  $[S^p \times S^q \times \{*\}]$  ( $S^p \times S^q \times \{*\} \subset \partial \tilde{X}_+$ ). Furthermore, we have  $[\{*\} \times S^r] = \mp m \xi'$ .

Then we have the following:

**Lemma 9.2.**

- (1) The manifold  $\tilde{X}_-$  is not homotopy equivalent to  $S^p \times S^q \times D^{r+1}$  nor to  $D^{p+1} \times S^q \times S^r$ .
- (2) The manifold  $\tilde{X}_+$  is not homotopy equivalent to  $S^p \times D^{q+1} \times S^r$  nor to  $S^p \times S^q \times D^{r+1}$ .
- (3) If  $k \neq \pm 1$  and  $m \neq \pm 1$ , then the manifolds  $\tilde{X}_\pm$  are not homotopy equivalent to  $S^p \times D^{q+1} \times S^r$  or  $D^{p+1} \times S^q \times S^r$  respectively.

*Proof.* We see easily that  $H_q(\tilde{X}_-) = 0 = H_p(\tilde{X}_+)$ , from which (1) and (2) follow. Part (3) follows from Lemmas 9.1 and 7.2. □

Using the above lemma, we can easily show the following.

**Proposition 9.3.** *If the diffeomorphism  $\psi : S^r \times S^r \rightarrow S^r \times S^r$  satisfies  $k \neq \pm 1$  and  $m \neq \pm 1$ , then the embedding  $\tilde{f} : S^p \times S^q \times S^r = \tilde{X}_- \cap \tilde{X}_+ \rightarrow \tilde{X}_- \cup \tilde{X}_+ = S^{p+q+r+1}$  has the property that the closure of neither of the two components of  $S^{p+q+r+1} - \tilde{f}(S^p \times S^q \times S^r)$  is homotopy equivalent to the product of two spheres and a disk.*

By [G, Proposition 2.5] or [LS, Theorem 2.2], for each matrix

$$\kappa_n = \begin{pmatrix} 4n + 1 & 2n \\ 2 & 1 \end{pmatrix} \in GL(2; \mathbf{Z})$$

with  $n \neq 0$ , the automorphism of  $H_r(S^r \times S^r)$  given by the matrix  $\kappa_n$  is realized by a diffeomorphism  $\psi_n : S^r \times S^r \rightarrow S^r \times S^r$ , since  $r$  is odd. In this way, we can construct infinitely many embeddings  $f_n : S^p \times S^q \times S^r \rightarrow S^{p+q+r+1}$  which satisfy the property of Proposition 9.3 by setting  $f_n = \tilde{f}$  with  $\psi = \psi_n$ , since  $4n + 1 \neq \pm 1$  and  $2n \neq \pm 1$ .

The following lemma is important in showing that the embeddings  $f_n : S^p \times S^q \times S^r \rightarrow S^{p+q+r+1}$  constructed from the matrices  $\kappa_n$  are mutually distinct.

**Lemma 9.4.** *Let  $W$  be a compact manifold such that  $\partial W = S^p \times S^q \times S^r$ ,  $p, q \geq 1$ ,  $r = p+q$  with  $r$  odd, and  $H_*(W) \cong H_*(S^p \times S^r)$ . Let  $\xi_p \in H^p(W) \cong \mathbf{Z}$ ,  $\xi_r \in H^r(W) \cong \mathbf{Z}$ ,  $\xi_{p+r} \in H^{p+r}(W) \cong \mathbf{Z}$  and  $\eta \in H_r(W) \cong \mathbf{Z}$  be respective generators. If  $\xi_p \smile \xi_r = k \xi_{p+r}$  ( $k \in \mathbf{Z}$ ), then  $i_*[\{*\} \times \{*\} \times S^r] = \pm k \eta$ , where  $i : \partial W \rightarrow W$  is the inclusion.*

The above lemma can be proved by an argument similar to that in the proof of Lemma 7.2.

**Definition 9.5.** We call the number  $|k| \in \mathbf{Z}$  in Lemma 9.4 the *cup product invariant* of  $W$ . Note that  $|k| \in \mathbf{Z}$  is well-defined: More precisely,  $|k|$  is a homotopy invariant of  $W$ .

*Proof of Theorem 1.2.* Consider the embeddings  $f_n : S^p \times S^q \times S^r \rightarrow S^{p+q+r+1}$  constructed from the matrices  $\kappa_n$  with  $n \neq 0$ . These satisfy the property of Proposition 9.3. We will show that the embeddings  $f_n$  are mutually distinct.

Let  $f_{n_1} : S^p \times S^q \times S^r \rightarrow S^{p+q+r+1}$  be the embedding constructed from the matrix  $\kappa_{n_1}$  with  $n_1 \neq 0$ . We may assume that  $H_*(C_1) \cong H_*(S^p \times S^r)$  and  $H_*(C_2) \cong H_*(S^q \times S^r)$ , where  $C_1$  and  $C_2$  are the closures of the two components of  $S^{p+q+r+1} - f_{n_1}(S^p \times S^q \times S^r)$ . The cup product invariants of  $C_1$  and  $C_2$  are equal to  $|4n_1 + 1|$  and  $|2n_1|$  respectively by Lemma 9.1.

Similarly, for the embedding  $f_{n_2} : S^p \times S^q \times S^r \rightarrow S^{p+q+r+1}$  constructed from the matrix  $\kappa_{n_2}$  with  $n_1 \neq n_2 \neq 0$ , we may assume that  $H_*(C_3) \cong H_*(S^p \times S^r)$  and  $H_*(C_4) \cong H_*(S^q \times S^r)$ , where  $C_3$  and  $C_4$  are the closures of the two components of  $S^{p+q+r+1} - f_{n_2}(S^p \times S^q \times S^r)$ . Suppose that there exists a diffeomorphism  $h : S^{p+q+r+1} \rightarrow S^{p+q+r+1}$  such that  $h(f_{n_1}(S^p \times S^q \times S^r)) = f_{n_2}(S^p \times S^q \times S^r)$ . Then we have  $h(C_1) = C_3$ , which implies that  $|4n_1 + 1| = |4n_2 + 1|$ . This contradicts the assumption that  $n_1 \neq n_2$ . Therefore,  $f_{n_1}$  and  $f_{n_2}$  are distinct if  $n_1 \neq n_2$ . This completes the proof of Theorem 1.2.  $\square$

**Remark 9.6.** When  $n = 0$ , for the embedding  $f_0$ , if  $p, q \geq 2$ , then it follows from Proposition 7.1 that  $C_1$  is diffeomorphic to  $S^p \times D^{q+1} \times S^r$ .

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ACADEMIA DA FORÇA AÉREA  
 RUA D-3 CASA 9 – AFA  
 13630-000, PIRASSUNUNGA, SP  
 BRAZIL  
*E-mail address:* llucas@siteplanet.com.br

DEPARTMENT OF MATHEMATICS  
 HIROSHIMA UNIVERSITY  
 HIGASHI-HIROSHIMA, 739-8526  
 JAPAN  
*E-mail address:* saeki@math.sci.hiroshima-u.ac.jp

FACULTY OF MATHEMATICS  
 KYUSHU UNIVERSITY  
 HAKOZAKI, FUKUOKA 812-8581  
 JAPAN