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EMBEDDINGS OF $S^p \times S^q \times S^r$ IN $S^{p+q+r+1}$

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Let $f : S^p \times S^q \times S^r \to S^{p+q+r+1}, 2 \leq p \leq q \leq r$, be a smooth embedding. In this paper we show that the closure of one of the two components of $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$, denoted by C_1, is diffeomorphic to $S^p \times S^q \times D^{r+1}$ or $S^p \times D^{q+1} \times S^r$ or $D^{p+1} \times S^q \times S^r$, provided that $p + q \neq r$ or $p + q = r$ with r even. We also show that when $p + q = r$ with r odd, there exist infinitely many embeddings which do not satisfy the above property. We also define standard embeddings of $S^p \times S^q \times S^r$ into $S^{p+q+r+1}$ and, using the above result, we prove that if C_1 has the homology of $S^p \times S^q$, then f is standard, provided that $q < r$.

1. Introduction.

In [**A**], Alexander has shown that a piecewise linearly embedded torus in the three sphere S^3 bounds a solid torus in S^3, which is know as Alexander's torus theorem. This theorem holds also for smooth embeddings.

Let $f : S^p \times S^q \to S^{p+q+1}$ be a codimension one smooth embedding with $p, q \geq 1$. Then the closure of one of the two components of $S^{p+q+1} - f(S^p \times S^q)$ is diffeomorphic to $D^{p+1} \times S^q$ if $1 \leq p \leq q$ with $p + q \neq 3$, and is homeomorphic to $D^2 \times S^2$ if $p = 1$ and $q = 2$. This is a generalization of Alexander's torus theorem and has been obtained in [**K**], [**Wa**], [**G**], [**R**] and [**LNS**]. An important consequence of this result is that for $2 \leq p \leq q$, embeddings of $S^p \times S^q$ into S^{p+q+1} are unique up to isotopy. In [**LNS**], some applications of this result to the study of codimension two smooth embeddings of $S^p \times S^q$ into S^{p+q+2} have been given.

The purpose of this paper is to study codimension one smooth embeddings of $S^p \times S^q \times S^r$ into $S^{p+q+r+1}$. More precisely, we completely determine the conditions on p, q and r in order that the closure of one of the two components of $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$ is diffeomorphic to the product of two spheres and a disk. Our first result is the following.

Theorem 1.1. *Let $f : S^p \times S^q \times S^r \to S^{p+q+r+1}$ be a smooth embedding with $2 \leq p \leq q \leq r$. We suppose $p + q \neq r$, or $p + q = r$ and r is even. Then the closure of one of the two components of $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$ is diffeomorphic to $S^p \times S^q \times D^{r+1}$ or $S^p \times D^{q+1} \times S^r$ or $D^{p+1} \times S^q \times S^r$.*

It is surprising that the above condition on p, q and r is essential: i.e., if it is not satisfied, then there exist infinitely many counter-examples, which can be called exotic embeddings. In §9, we will show the following by explicitly constructing such embeddings.

Theorem 1.2. *If $p, q \geq 1$ and $p+q = r$ with r odd, then there exist mutually distinct embeddings $f_n : S^p \times S^q \times S^r \to S^{p+q+r+1}$, $n \in \mathbf{Z} - \{0\}$, such that the closure of neither of the two components of $S^{p+q+r+1} - f_n(S^p \times S^q \times S^r)$ is homotopy equivalent to the product of two spheres and a disk.*

However, if we put some more conditions, then we have the following theorem, which will be proved in §7. In the following, homology groups are always with integer coefficients.

Theorem 1.3. *Let $f : S^p \times S^q \times S^r \to S^{p+q+r+1}$ be a smooth embedding with $2 \leq p \leq q \leq r$. Then the closure of one of the two components of $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$, denoted by C_1, has the same homology as $S^p \times S^q$ or $S^p \times S^r$ or $S^q \times S^r$. Furthermore, if C_1 is homotopy equivalent to $S^p \times S^q$ or $S^p \times S^r$ or $S^q \times S^r$, then it is diffeomorphic to $S^p \times S^q \times D^{r+1}$ or $S^p \times D^{q+1} \times S^r$ or $D^{p+1} \times S^q \times S^r$ respectively.*

Using the above results, we can obtain more precise information about the embedding f in some cases. In fact, we will prove, in §8, that if $2 \leq p \leq q < r$ and C_1 has the homology of $S^p \times S^q$ in Theorem 1.3, then f is *standard* (see Definition 8.1 and Corollary 8.2).

The proof of Theorem 1.1 will be divided into five cases according to the homology group structure of $S^p \times S^q \times S^r$ as follows:

(A) $p < q < r$ and $r \neq p + q$,
(B) $p = q = r$,
(C) $p = q < r$ and $p + q \neq r$, or $p < q = r$,
(D) $p = q$ and $p + q = r$,
(E) $p < q$ and $p + q = r$ with r even.

These cases will be treated in §2–§6 respectively. Our technique for the proof of Theorem 1.1 is based on the standard homology theory and the h-cobordism theorem [**Sm**, **Mi**], which is essentially the same as in [**K**], [**Wa**], [**G**] or in [**LNS**]. The main difficulty lies in the construction of an embedding of the product of two spheres into C_1 which induces a homotopy equivalence.

Throughout the paper, all manifolds and maps are assumed to be differentiable of class C^∞ and all homology and cohomology groups are with coefficients in $\mathbf{Z}$. The symbol "$\cong$" denotes a diffeomorphism between manifolds or an appropriate isomorphism between algebraic objects. The symbol "$[*]$" denotes the homology class represented by $*$. The notation "id" denotes the identity map.

2. Case (A) $p < q < r$ and $r \neq p + q$.

First let us introduce some notations which will be used throughout the proofs of Theorems 1.1 and 1.3 (§2–§7). Let $f : S^p \times S^q \times S^r \to S^{p+q+r+1}$ be a smooth embedding with $2 \leq p \leq q \leq r$. Then, by Alexander duality, $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$ consists exactly of two components and they are simply connected by van Kampen's theorem. We denote the two components by C_1' and C_2' and their closures in $S^{p+q+r+1}$ by C_1 and C_2 respectively. We identify $C_1 \cap C_2 = \partial C_1 = \partial C_2$ with $S^p \times S^q \times S^r$ by the embedding f. Furthermore, $i : \partial C_1 \to C_1$ will denote the inclusion map.

From now on, we assume $p < q < r$ and $r \neq p + q$ in this section.

Lemma 2.1. *Either C_1 or C_2 has the same homology as $S^p \times S^q \times D^{r+1}$ or $S^p \times D^{q+1} \times S^r$ or $D^{p+1} \times S^q \times S^r$.*

Proof. By Alexander duality, we see easily that there are eight possibilities for the homology groups $(H_p(C_1), H_q(C_1), H_r(C_1), H_{q+r}(C_1), H_{r+p}(C_1), H_{p+q}(C_1))$:

$$(\mathbf{Z}, \mathbf{Z}, 0, 0, 0, \mathbf{Z}), (\mathbf{Z}, 0, \mathbf{Z}, 0, \mathbf{Z}, 0), (0, \mathbf{Z}, \mathbf{Z}, \mathbf{Z}, 0, 0), \tag{1}$$

$$(0, 0, \mathbf{Z}, \mathbf{Z}, \mathbf{Z}, 0), (0, \mathbf{Z}, 0, \mathbf{Z}, 0, \mathbf{Z}), (\mathbf{Z}, 0, 0, 0, \mathbf{Z}, \mathbf{Z}), \tag{2}$$

$$(\mathbf{Z}, \mathbf{Z}, \mathbf{Z}, 0, 0, 0), (0, 0, 0, \mathbf{Z}, \mathbf{Z}, \mathbf{Z}). \tag{3}$$

In Case (1) C_1 has the desired homology, and in Case (2) C_2 has the desired homology.

Suppose that C_1 has the homology $(\mathbf{Z}, \mathbf{Z}, \mathbf{Z}, 0, 0, 0)$. Since we have $H^{p+1}(C_1, \partial C_1) \cong H_{q+r}(C_1) = 0$ and $H^{q+1}(C_1, \partial C_1) \cong H_{p+r}(C_1) = 0$, the homomorphisms $i^* : H^p(C_1) \to H^p(\partial C_1)$ and $i^* : H^q(C_1) \to H^q(\partial C_1)$ are surjective, and hence so is $i^* \otimes i^* : H^p(C_1) \otimes H^q(C_1) \to H^p(\partial C_1) \otimes H^q(\partial C_1)$. Then the commutative diagram of cup products

$$\begin{array}{ccc} H^p(C_1) \otimes H^q(C_1) & \xrightarrow{i^* \otimes i^*} & H^p(\partial C_1) \otimes H^q(\partial C_1) \\ \downarrow \smile & & \downarrow \smile \\ H^{p+q}(C_1) & \xrightarrow{i^*} & H^{p+q}(\partial C_1) \end{array}$$

leads to a contradiction, since $H^{p+q}(C_1) = 0$ and the second column is nonzero. We see that the case $(0, 0, 0, \mathbf{Z}, \mathbf{Z}, \mathbf{Z})$ cannot happen either by using the same argument for C_2. □

We may assume that C_1 has the same homology as $S^p \times S^q \times D^{r+1}$ without loss of generality. Note that we do not have $p \leq q \leq r$ any more, although $p, q, r, p + q, q + r$ and $r + p$ are all distinct.

Lemma 2.2. *The composite*

$$\varphi : S^p \times S^q \times \{*\} \xrightarrow{j} S^p \times S^q \times S^r = \partial C_1 \xrightarrow{i} C_1$$

is a homotopy equivalence, where j and i are the inclusion maps.

Proof. Since $i_*[\{*\} \times S^q \times S^r] = 0$ in $H_{q+r}(C_1) = 0$, there exists a $(q+r+1)$-chain $(\Gamma, \partial\Gamma)$ in $(C_1, \partial C_1)$ such that $\partial\Gamma$ is homologous to $\{*\} \times S^q \times S^r$ in ∂C_1. The intersection number $[\Gamma, \partial\Gamma] \cdot i_*[S^p \times \{*\} \times \{*\}]$ in C_1 is equal to the intersection number $[\{*\} \times S^q \times S^r] \cdot [S^p \times \{*\} \times \{*\}] = \pm 1$ in ∂C_1. This implies that $\varphi_* : H_p(S^p \times S^q \times \{*\}) \to H_p(C_1)$ is an isomorphism, since $H_p(S^p \times S^q \times \{*\}) \cong \mathbf{Z}$ is generated by $[S^p \times \{*\} \times \{*\}]$ and $\varphi_*[S^p \times \{*\} \times \{*\}]$ must be a primitive homology class in $H_p(C_1) \cong \mathbf{Z}$.

Since $H_{p+r}(C_1) = 0$ and $H_r(C_1) = 0$, we see that $\varphi_* : H_k(S^p \times S^q \times \{*\}) \to H_k(C_1)$ is an isomorphism also for $k = q$ and $p+q$ by using similar arguments. Then the result follows from Whitehead's theorem. □

Proof of Theorem 1.1 for Case (A). Set $\Sigma_1 = S^p \times S^q \times \{*\} \subset \partial C_1 = S^p \times S^q \times S^r$. We can push Σ_1 into the interior of C_1 by using an inward normal vector field of ∂C_1 in C_1 and obtain a submanifold Σ_1'. Let G be a sufficiently small closed tubular neighborhood of Σ_1' in $\operatorname{Int} C_1$. We see easily that G is diffeomorphic to $S^p \times S^q \times D^{r+1}$.

By excision and Lemma 2.2, we see that the manifold $V = C_1 - \operatorname{Int} G$ is an h-cobordism between ∂G and ∂C_1. Since $\dim V = p + q + r + 1 \geq 6$, we see by the h-cobordism theorem [**Sm**, **Mi**] that $V \cong \partial G \times [0, 1]$. Then we have $C_1 = V \cup G \cong \partial G \times [0, 1] \cup G \cong G$, which is diffeomorphic to $S^p \times S^q \times D^{r+1}$. □

3. Case (B) $p = q = r$.

The main tool used in this section is the result about automorphisms of $H_p(S^p \times S^p \times S^p)$ which can be realized by self-diffeomorphisms of $S^p \times S^p \times S^p$ (for details, see [**LS**]).

First, by the same argument as in the proof of Lemma 2.1, we may assume that $H_*(C_1) \cong H_*(S^p \times S^p \times D^{p+1})$ without loss of generality. As in the previous section, in order to prove Theorem 1.1 for this case, we have only to show the following.

Lemma 3.1. *There exists an embedding $j : S^p \times S^p \to S^p \times S^p \times S^p$ with trivial normal bundle such that the embedding*

$$\varphi : S^p \times S^p \xrightarrow{j} S^p \times S^p \times S^p = \partial C_1 \xrightarrow{i} C_1 \tag{3.1}$$

is a homotopy equivalence.

Proof. (B1) *When p is even.* Consider the exact sequence

$$0 = H^p(C_1, \partial C_1) \to H^p(C_1) \xrightarrow{i^*} H^p(\partial C_1) \to H^{p+1}(C_1, \partial C_1) \to H^{p+1}(C_1) = 0.$$

Since $H^{p+1}(C_1, \partial C_1) \cong \mathbf{Z}$ is free, $\operatorname{Im} i^*$ is a direct summand of $H^p(\partial C_1)$. Let ξ and η be generators of $H^p(C_1) \cong \mathbf{Z} \oplus \mathbf{Z}$ and $\{\alpha^*, \beta^*, \gamma^*\}$ the basis of

$H^p(\partial C_1) \cong \mathrm{Hom}(H_p(\partial C_1), \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ dual to the basis

(3.2) $\{\alpha = [S^p \times \{*\} \times \{*\}],\ \beta = [\{*\} \times S^p \times \{*\}],\ \gamma = [\{*\} \times \{*\} \times S^p]\}$

of $H_p(\partial C_1)$. Then we have $i^*\xi = a\alpha^* + b\beta^* + c\gamma^*$ and $i^*\eta = d\alpha^* + e\beta^* + g\gamma^*$ for some integers a, b, c, d, e and g. Since $\mathrm{Im}\, i^*$ is a direct summand of $H^p(\partial C_1)$, there exist integers h, l, m such that

$$\det \begin{pmatrix} a & d & h \\ b & e & l \\ c & g & m \end{pmatrix} = \pm 1.$$

By the commutative diagram

$$\begin{array}{ccc} H^p(C_1) \otimes H^p(C_1) & \xrightarrow{i^* \otimes i^*} & H^p(\partial C_1) \otimes H^p(\partial C_1) \\ \downarrow \smile & & \downarrow \smile \\ \mathbf{Z} \cong H^{2p}(C_1) & \xrightarrow{i^*} & H^{2p}(\partial C_1) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}, \end{array}$$

we have that the subgroup generated by

$$\begin{aligned} i^*\xi \smile i^*\xi &= 2ab(\alpha^* \smile \beta^*) + 2bc(\beta^* \smile \gamma^*) + 2ca(\gamma^* \smile \alpha^*), \\ i^*\xi \smile i^*\eta &= (ae + bd)(\alpha^* \smile \beta^*) + (bg + ce)(\beta^* \smile \gamma^*) \\ &\quad + (cd + ag)(\gamma^* \smile \alpha^*), \\ i^*\eta \smile i^*\eta &= 2de(\alpha^* \smile \beta^*) + 2eg(\beta^* \smile \gamma^*) + 2gd(\gamma^* \smile \alpha^*) \end{aligned}$$

has rank at most one. Using the fact that $\{\alpha^* \smile \beta^*, \beta^* \smile \gamma,^* \gamma^* \smile \alpha^*\}$ is a basis of $H^{2p}(\partial C_1)$, we see easily that $abc = deg = 0$. Then, we can show that for an embedding $j : S^p \times S^p \to S^p \times S^p \times S^p$ such that $j(S^p \times S^p) = \{*\} \times S^p \times S^p, S^p \times \{*\} \times S^p$, or $S^p \times S^p \times \{*\}$, the composite φ as in (3.1) induces an isomorphism on the p-th cohomology groups. Then by the universal coefficient theorem, $\varphi_* : H_p(S^p \times S^p) \to H_p(C_1)$ is also an isomorphism. Consider the commutative diagram

$$\begin{array}{ccc} H^p(C_1) \otimes H^p(C_1) & \xrightarrow{\varphi^* \otimes \varphi^*} & H^p(S^p \times S^p) \otimes H^p(S^p \times S^p) \\ k_1 \downarrow \smile & & k_2 \downarrow \smile \\ \mathbf{Z} \cong H^{2p}(C_1) & \xrightarrow{\varphi^*} & H^{2p}(S^p \times S^p) \cong \mathbf{Z}, \end{array}$$

where $k_2 \circ (\varphi^* \otimes \varphi^*)$ is an epimorphism, since k_2 is unimodular. This implies that $\varphi^* : H^{2p}(C_1) \to H^{2p}(S^p \times S^p)$ is also an epimorphism. Since $H^{2p}(C_1) \cong H^{2p}(S^p \times S^p) \cong \mathbf{Z}$, we see that $\varphi^* : H^{2p}(C_1) \to H^{2p}(S^p \times S^p)$ is an isomorphism, which implies that $\varphi_* : H_{2p}(S^p \times S^p) \to H_{2p}(C_1)$ is also an isomorphism. Then by Whitehead's theorem, the result follows.

In order to prove Lemma 3.1 when p is odd, we need the following, which can be easily proved by examining the exact sequence

$$0 \to \ker i_* \to H_p(\partial C_1) \xrightarrow{i_*} H_p(C_1) \to H_p(C_1, \partial C_1) = 0.$$

Lemma 3.2. *For every p, there exists a basis $\{\zeta_1, \zeta_2, \zeta\}$ of $H_p(\partial C_1)$ such that $\{i_*\zeta_1, i_*\zeta_2\}$ is a basis of $H_p(C_1)$ and $i_*\zeta = 0$.*

(B2) *When $p = 3$ or $p = 7$.* The following is a direct consequence of [**LS**, Theorem 2.2].

Lemma 3.3. *If $p = 3$ or $p = 7$, then there exists a diffeomorphism $\phi : S^p \times S^p \times S^p \to S^p \times S^p \times S^p$ such that $\phi_* H_p(S^p \times S^p \times \{*\}) = \langle \zeta_1, \zeta_2 \rangle$, where $\langle \zeta_1, \zeta_2 \rangle$ is the subgroup of $H_p(S^p \times S^p \times S^p)$ generated by ζ_1 and ζ_2.*

By putting $j = \phi | S^p \times S^p \times \{*\}$, we see that Lemma 3.1 holds for $p = 3, 7$.

(B3) *When p is odd with $p \neq 3, 7$.* Let $\eta_n : GL(n; \mathbf{Z}) \to GL(n; \mathbf{Z}_2)$ be the natural homomorphism. Note that η_n is an epimorphism (see, for example, [**Mc**, Proposition I.14]). We define the subgroup G_1 of $GL(n; \mathbf{Z})$ by $G_1 = {\eta_n}^{-1}(\eta_n(\mathfrak{S}_n))$, where we naturally identify the symmetric group $\mathfrak{S}_n$ with the corresponding subgroup of $GL(n; \mathbf{Z})$. Note that G_1 corresponds to the set of those automorphisms which can be realized by diffeomorphisms of the product of n copies of S^p for p odd with $p \neq 3, 7$ (see [**LS**]).

We define the matrix $A \in GL(3; \mathbf{Z})$ by $(\zeta_1, \zeta_2, \zeta) = (\alpha, \beta, \gamma)A$, where $\{\alpha, \beta, \gamma\}$ is the canonical basis of $H_p(S^p \times S^p \times S^p)$ as in (3.2). Note that A may not lie in G_1.

Lemma 3.4. *There exists a matrix $A' \in G_1 \subset GL(3; \mathbf{Z})$ such that $\{i_*\zeta_1', i_*\zeta_2'\}$ is a basis of $H_p(C_1)$, where $(\zeta_1', \zeta_2', \zeta') = (\alpha, \beta, \gamma)A'$.*

Note that $i_*\zeta'$ may not be zero in $H_p(C_1)$ any more.

Proof of Lemma 3.4. By changing the order of α, β, γ and by adding ζ to ζ_1, ζ_2 if necessary, we may assume that $A_2 = \eta_3(A)$ is of the form

$$\begin{pmatrix} a & b & * \\ c & d & * \\ 0 & 0 & 1 \end{pmatrix} \in GL(3; \mathbf{Z}_2).$$

Since $\eta_2 : GL(2; \mathbf{Z}) \to GL(2; \mathbf{Z}_2)$ is surjective, there exists a matrix $B \in GL(2; \mathbf{Z})$ with

$$\eta_2(B) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Using B, we can change ζ_1, ζ_2 so that A_2 is of the form

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally, adding ζ_1, ζ_2 to ζ, we get the desired basis $\{\zeta_1', \zeta_2', \zeta'\}$. □

Now by using the same argument as in Case (B2) together with [**LS**, Theorem 2.2], we see that Lemma 3.1 holds for this case as well. This completes the proof of Lemma 3.1, and hence Theorem 1.1 for Case (B). □

4. Case (C) $p = q < r$ and $p + q \neq r$, or $p < q = r$.

We will assume that $p = q \neq r \neq p + q$ throughout this section, although r can be smaller than $p = q$. By the same argument as in the proof of Lemma 2.1, we see that it suffices to study the cases

$$H_*(C_1) \cong H_*(S^p \times S^p \times D^{r+1}), \tag{C1}$$

$$H_*(C_1) \cong H_*(S^p \times D^{p+1} \times S^r). \tag{C2}$$

(C1) *When* $H_*(C_1) \cong H_*(S^p \times S^p \times D^{r+1})$. Let $j : S^p \times S^p \times \{*\} \to S^p \times S^p \times S^r = \partial C_1$ be the inclusion and set $\varphi = i \circ j$. By carefully examining the exact sequence of the triple $(C_1, \partial C_1, S^p \times S^p \times \{*\})$

$$\cdots \to H_k(\partial C_1, S^p \times S^p \times \{*\}) \to H_k(C_1, S^p \times S^p \times \{*\}) \to H_k(C_1, \partial C_1) \\ \to H_{k-1}(\partial C_1, S^p \times S^p \times \{*\}) \to \cdots$$

and by applying an argument similar to that in the proof of Lemma 3.1, we can show that the inclusion map $\varphi : S^p \times S^p \times \{*\} \to C_1$ gives a homotopy equivalence. Then as in §2, we see that $C_1 \cong S^p \times S^p \times D^{r+1}$.

(C2) *When* $H_*(C_1) \cong H_*(S^p \times D^{p+1} \times S^r)$.

Lemma 4.1. *There exists an embedding* $\psi : S^p \to S^p \times S^p$ *such that the embedding*

$$\varphi : S^p \times S^r \xrightarrow{\psi \times \mathrm{id}} (S^p \times S^p) \times S^r = \partial C_1 \xrightarrow{i} C_1 \tag{4.1}$$

is a homotopy equivalence.

Proof. As in Lemma 3.2, there exists a basis $\{\zeta, \zeta_1\}$ of $H_p(\partial C_1) \cong \mathbf{Z} \oplus \mathbf{Z}$ such that $i_*\zeta = 0$ and $i_*\zeta_1$ is a generator of $H_p(C_1) \cong \mathbf{Z}$. By the isomorphism $H_p(\partial C_1) \cong H_p(S^p \times S^p \times S^r) \cong H_p(S^p \times S^p \times \{*\})$, $\zeta_1 \in H_p(\partial C_1)$ corresponds to an element $\zeta_1' \in H_p(S^p \times S^p \times \{*\}) \cong \pi_p(S^p \times S^p)$. When $p \geq 3$, by [**H**] or [**Wh**], we can represent ζ_1' by an embedding $\psi : S^p \to S^p \times S^p$. Then the composite φ as in (4.1) is an embedding such that $\varphi_* : H_p(S^p \times S^r) \to H_p(C_1)$ is an isomorphism.

When $p = 2$, we cannot use the above argument (see, for example, [**KM**]). However, since $p = 2$ is even, we can show that the embedding $\psi : S^p \to S^p \times S^p$ such that $\psi(S^p) = \{*\} \times S^p$ or $S^p \times \{*\}$ satisfies the same property, by using an argument similar to that in (B1) of the proof of Lemma 3.1.

Then by the same arguments as in the proofs of Lemmas 2.2 and 3.1, we see that $\varphi : S^p \times S^r \to C_1$ is a homotopy equivalence. This completes the proof of Lemma 4.1. □

Let Σ_1' denote the submanifold of C_1 which is obtained by pushing $\Sigma_1 = \varphi(S^p \times S^r)$ into the interior of C_1 using a normal vector field of ∂C_1 pointing toward $\mathrm{Int}\, C_1$.

Lemma 4.2. *The normal bundle of Σ_1' in C_1 is trivial.*

Proof. Let $\psi : S^p \to S^p \times S^p$ be the embedding as above. It suffices to show that the normal bundle $\nu_{\widetilde{\varphi}}$ of $\widetilde{\varphi} = f \circ (\psi \times \mathrm{id}) : S^p \times S^r \to S^{2p+r+1}$ is trivial. We have

$$\nu_{\widetilde{\varphi}} \cong \pi^*(\nu_\psi) \oplus \varepsilon^1_{S^p \times S^r} \cong \pi^*(\nu_\psi) \oplus \pi^*(\varepsilon^1_{S^p}) \cong \pi^*(\nu_\psi \oplus \varepsilon^1_{S^p}),$$

where $\pi : S^p \times S^r \to S^p$ is the projection to the first factor, ν_ψ denotes the normal bundle of ψ, and ε^1_X denotes the trivial line bundle over a space X. On the other hand, using the embedding

$$\widetilde{\psi} : S^p \xrightarrow{\psi} S^p \times S^p \hookrightarrow S^{2p+1},$$

we see that $\nu_\psi \oplus \varepsilon^1_{S^p}$ is trivial. Thus the result follows. □

Finally, by the same argument as in Case (A), we see that Theorem 1.1 holds for Case (C2) as well. □

5. Case (D) $p = q$ and $p + q = r$.

By the same argument as in the proof of Lemma 2.1, we see that either C_1 or C_2 has the same homology as $S^p \times S^p \times D^{r+1}$ or $S^p \times D^{p+1} \times S^r$.

When $H_*(C_1) \cong H_*(S^p \times S^p \times D^{r+1})$ or $H_*(C_2) \cong H_*(S^p \times S^p \times D^{r+1})$, by using arguments similar to those in Case (C1), we see that C_1 or C_2 is diffeomorphic to $S^p \times S^p \times D^{r+1}$.

When $H_*(C_1) \cong H_*(S^p \times D^{p+1} \times S^r)$, we see easily that $H_*(C_2) \cong H_*(S^p \times D^{p+1} \times S^r) (\cong H_*(D^{p+1} \times S^p \times S^r))$. First we prepare the following lemmas. Note that $r = 2p$ is even and that $\dim \partial C_1$ is equal to $2r$.

Lemma 5.1. *There exists a basis $\{\zeta, \zeta'\}$ of $H_r(\partial C_1) \cong \mathbf{Z} \oplus \mathbf{Z}$ such that $i_*\zeta = 0$, $\zeta \cdot \zeta = 0$, $\zeta \cdot \zeta' = 1$ and $\zeta' \cdot \zeta' = 0$.*

The above lemma can be proved by using an argument similar to that in [**LNS**, Lemmas 3.2-3.4].

Set $\alpha = [S^p \times S^p \times \{*\}]$ and $\beta = [\{*\} \times \{*\} \times S^r]$, which generate $H_r(\partial C_1) = H_r(S^p \times S^p \times S^r)$. We have $\alpha \cdot \alpha = \beta \cdot \beta = 0$ and may assume $\alpha \cdot \beta = 1$, choosing suitable orientations for $S^p \times S^p \times \{*\}$ and $\{*\} \times \{*\} \times S^r$.

Lemma 5.2. *If $p \geq 3$, then for some embedding $\psi : S^p \to S^p \times S^p$, the composite*

$$\varphi_1 : S^p \times S^r \xrightarrow{\psi \times \mathrm{id}} (S^p \times S^p) \times S^r = \partial C_1 \xrightarrow{i} C_1 \quad or$$

$$\varphi_2 : S^p \times S^r \xrightarrow{\psi \times \mathrm{id}} (S^p \times S^p) \times S^r = \partial C_2 \xrightarrow{j} C_2$$

is a homotopy equivalence, where i and j are the inclusion maps.

Proof. We have $H_*(C_1) \cong H_*(S^p \times D^{p+1} \times S^r) \cong H_*(C_2)$. Consider the endomorphism $\Theta : H_r(S^p \times S^p \times S^r) \to H_r(S^p \times S^p \times S^r)$ defined by $\Theta(\alpha) = \zeta'$ and $\Theta(\beta) = \zeta$. Since $\zeta \cdot \zeta = \zeta' \cdot \zeta' = 0$ and $\zeta \cdot \zeta' = 1$, we see that Θ is an automorphism of $(H_r(S^p \times S^p \times S^r), \cdot)$, where "$\cdot$" denotes the intersection form. By an argument similar to that of [**LNS**, Lemma 3.5], we have that $\zeta = \pm\alpha$ and $\zeta' = \pm\beta$, or $\zeta = \pm\beta$ and $\zeta' = \pm\alpha$.

When $\zeta = \pm\alpha$ and $\zeta' = \pm\beta$, we have $i_*[S^p \times S^p \times \{*\}] = 0$, and as in Lemma 2.2, $i_{1*} : H_r(\{*\} \times \{*\} \times S^r) \to H_r(C_1)$ is an isomorphism, where i_1 is the inclusion map. Similarly, when $\zeta = \pm\beta$ and $\zeta' = \pm\alpha$, $j_{1*} : H_r(\{*\} \times \{*\} \times S^r) \to H_r(C_2)$ is an isomorphism for the inclusion map j_1. Then, since $p \geq 3$, by arguments similar to those in the proofs of Lemmas 4.1 and 3.1, we have the desired result. □

When $p = 2$, we cannot apply the same argument. Nevertheless, as in the proof of Lemma 5.2, we may assume that $i_{1*} : H_r(\{*\} \times \{*\} \times S^r) \to H_r(C_1)$ is an isomorphism.

Consider a collar neighborhood $c : \partial C_1 \times [0,1] \to C_1$ of ∂C_1 in C_1, where $c(x, 0) = x$ for every $x \in \partial C_1$. We will use the identification

$$\partial C_1 \times [0,1] \xrightarrow{f^{-1} \times \mathrm{id}} S^p \times S^p \times S^r \times [0,1] \cong (S^p \times S^p \times [0,1]) \times S^r.$$

Lemma 5.3. *If $p = 2$, then there exists an embedding $\psi_1 : S^p \to S^p \times S^p \times [0,1]$ such that the embedding $\varphi : S^p \times S^r \to C_1$ defined by*

$$\begin{aligned}\varphi : S^p \times S^r &\xrightarrow{\psi_1 \times \mathrm{id}} (S^p \times S^p \times [0,1]) \times S^r \\ &\cong S^p \times S^p \times S^r \times [0,1] \xrightarrow{f \times \mathrm{id}} \partial C_1 \times [0,1] \xrightarrow{c} C_1\end{aligned}$$

is a homotopy equivalence.

Proof. As in the proof of Lemma 4.1, there exists a continuous map $\psi' : S^p \to S^p \times S^p$ which represents $\zeta_1' \in H_p(S^p \times S^p) (\cong \pi_p(S^p \times S^p)) \cong H_p(S^p \times S^p \times S^r)$ with $i_*\zeta_1'$ being a generator of $H_p(C_1) \cong \mathbf{Z}$. Consider the composite

$$\psi'' : S^p \xrightarrow{\psi'} S^p \times S^p \xrightarrow{i'} S^p \times S^p \times [0,1],$$

where $i' : S^p \times S^p = S^p \times S^p \times \{0\} \to S^p \times S^p \times [0,1]$ is the inclusion map. By [**H**, Theorem 1(a)], there exists a differentiable embedding $\psi_1 : S^p \to S^p \times S^p \times [0,1]$ homotopic to ψ''. Then, φ is a differentiable embedding such that $\varphi_* : H_p(S^p \times S^r) \to H_p(C_1)$ is an isomorphism. The rest of the proof is the same as before. □

As in Lemma 4.2, if $p \geq 3$, then the normal bundles of φ_1 and φ_2 of Lemma 5.2 are trivial. When $p = 2$, by embedding $S^p \times S^p \times [0,1]$ in S^{2p+1}, we see that ψ_1 as above has trivial normal bundle ν_{ψ_1}. Furthermore, we have $\nu_\varphi \cong \pi^*(\nu_{\psi_1})$, where $\pi : S^p \times S^r \to S^p$ is the projection to the first factor. Hence ν_φ is trivial.

Then as in the previous sections, we see that $C_1 \cong S^p \times D^{p+1} \times S^r$ or $C_2 \cong S^p \times D^{p+1} \times S^r$. This completes the proof of Theorem 1.1 for Case (D). □

6. Case (E) $p < q$ and $p + q = r$ with r even.

By the same argument as in the proof of Lemma 2.1, we see that either C_1 or C_2 has the same homology as $S^p \times S^q \times D^{r+1}$ or $S^p \times D^{q+1} \times S^r$ or $D^{p+1} \times S^q \times S^r$.

(E1) $H_*(C_1) \cong H_*(S^p \times S^q \times D^{r+1})$ or $H_*(C_2) \cong H_*(S^p \times S^q \times D^{r+1})$.

We may assume that $H_*(C_1) \cong H_*(S^p \times S^q \times D^{r+1})$. Then by arguments similar to those in the proofs of Lemmas 2.2 and 3.1, we see that the inclusion map

$$\varphi : S^p \times S^q \times \{*\} \to S^p \times S^q \times S^r = \partial C_1 \xrightarrow{i} C_1$$

is a homotopy equivalence. Then the rest of the proof for this case is the same as before.

Remark 6.1. Even when $p + q = r$ with r odd, if $H_*(C_1) \cong H_*(S^p \times S^q \times D^{r+1})$, then we can prove that $C_1 \cong S^p \times S^q \times D^{r+1}$ by using the above argument.

(E2) $H_*(C_1) \cong H_*(S^p \times D^{q+1} \times S^r)(\Leftrightarrow H_*(C_2) \cong H_*(D^{p+1} \times S^q \times S^r))$.

Let $i_1 : S^p \times \{*\} \times S^r \to S^p \times S^q \times S^r$, $j : \partial C_2 \to C_2$ and $j_1 : \{*\} \times S^q \times S^r \to S^p \times S^q \times S^r$ be the inclusion maps. By using arguments similar to the previous ones, we can show the following.

Lemma 6.2. *The inclusion*

$$\varphi_1 : S^p \times \{*\} \times S^r \xrightarrow{i_1} S^p \times S^q \times S^r = \partial C_1 \xrightarrow{i} C_1 \quad \textit{or}$$

$$\varphi_2 : \{*\} \times S^q \times S^r \xrightarrow{j_1} S^p \times S^q \times S^r = \partial C_2 \xrightarrow{j} C_2$$

is a homotopy equivalence.

Thus, Theorem 1.1 holds for Case (E2). This completes the proof of Theorem 1.1 for all the cases. □

7. Case (F) $p + q = r$ with r odd.

In this section, let us consider the case where $r = p + q$ with r odd. The main result of this section is the following:

Proposition 7.1. *Let $f : S^p \times S^q \times S^r \to S^{p+q+r+1}$ be a smooth embedding and C_1 the closure of one of the two components of $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$ with $p, q \geq 2, r = p + q$ and r odd.*

(1) *If $H_*(C_1) \cong H_*(S^p \times S^q)$, then C_1 is diffeomorphic to $S^p \times S^q \times D^{r+1}$.*

(2) *If C_1 has the same cohomology ring as $S^p \times S^r$ or $S^q \times S^r$, then C_1 is diffeomorphic to $S^p \times D^{q+1} \times S^r$ or $D^{p+1} \times S^q \times S^r$ respectively.*

Lemma 7.2. *If C_1 has the same cohomology ring as $S^p \times S^r$ or $S^q \times S^r$, then $H_r(C_1)$ is generated by $i_*[\{*\} \times \{*\} \times S^r]$.*

Proof. Suppose that C_1 has the same cohomology ring as $S^p \times S^r$. The other case can be proved similarly. It is not difficult to show that $i_* : H_k(\partial C_1) \to H_k(C_1)$ is an isomorphism for $k = p$ and $p+r$ as in the proof of Lemma 2.2. Set

$$\alpha^* = [S^p \times S^q \times \{*\}]^*,\ \beta^* = [\{*\} \times \{*\} \times S^r]^* \in H^r(\partial C_1) \cong \mathbf{Z} \oplus \mathbf{Z},$$
$$\gamma^* = [S^p \times \{*\} \times \{*\}]^* \in H^p(\partial C_1) \cong \mathbf{Z},$$
$$\delta^* = [S^p \times \{*\} \times S^r]^* \in H^{p+r}(\partial C_1) \cong \mathbf{Z},$$

and let $\xi_p \in H^p(C_1) \cong \mathbf{Z}$ and $\xi_r \in H^r(C_1) \cong \mathbf{Z}$ be generators, where each $[*]^*$ means a dual basis. Note that we have $\gamma^* \smile \beta^* = \pm\delta^*$ and $\gamma^* \smile \alpha^* = 0$.

Let us consider the commutative diagram

$$\begin{array}{ccc} H^p(C_1) \otimes H^r(C_1) & \xrightarrow{i^* \otimes i^*} & H^p(\partial C_1) \otimes H^r(\partial C_1) \\ \downarrow \smile & & \downarrow \smile \\ H^{p+r}(C_1) & \xrightarrow{i^*} & H^{p+r}(\partial C_1). \end{array}$$

The cohomology class $\xi_p \smile \xi_r$ generates $H^{p+r}(C_1)$, since C_1 has the same cohomology ring as $S^p \times S^r$. On the other hand, we have $i^*\xi_p = \pm[S^p \times \{*\} \times \{*\}]^* = \pm\gamma^*$, since $i^* : H^p(C_1) \to H^p(\partial C_1)$ is an isomorphism. Furthermore, the cohomology class $i^*(\xi_p \smile \xi_r)$ generates $H^{p+r}(\partial C_1)$, since i^* in the second row is an isomorphism. We can put $i^*\xi_r = a\alpha^* + b\beta^*$ for some integers a and b. We see easily that $i^*(\xi_p \smile \xi_r) = \pm b\delta^*$. This implies that $b = \pm 1$.

Then we see that $\langle \xi_r, i_*[\{*\} \times \{*\} \times S^r] \rangle = \pm 1$, where $\langle *, * \rangle$ denotes the Kronecker product. Thus, $i_*[\{*\} \times \{*\} \times S^r]$ generates $H_r(C_1)$. □

Proof of Proposition 7.1. (1) This follows from Remark 6.1.

(2) We may assume that C_1 has the same cohomology ring as $S^p \times S^r$. Consider the inclusion map

$$\varphi : S^p \times \{*\} \times S^r \to S^p \times S^q \times S^r = \partial C_1 \xrightarrow{i} C_1.$$

By Lemma 7.2, $\varphi_* : H_r(S^p \times \{*\} \times S^r) \to H_r(C_1)$ is an isomorphism. Furthermore, as has been seen in the proof of Lemma 7.2, $\varphi_* : H_k(S^p \times \{*\} \times S^r) \to H_k(C_1)$ is an isomorphism for $k = p$ and $p + r$. Thus, φ is a homotopy equivalence. Then by arguments similar to those in §2, we see that $C_1 \cong S^p \times D^{q+1} \times S^r$. □

Proof of Theorem 1.3. The theorem follows from Theorem 1.1 and Proposition 7.1. □

8. Standard embeddings.

As a consequence of Theorem 1.1, we have the following Corollary 8.2. This result is important, since it gives a sufficient condition for an embedding to be standard. The characterization of standard embeddings is fundamental in the study of embeddings.

Let us begin by defining standard embeddings.

Definition 8.1. Let $g : S^p \times S^q \to S^{p+q+r+1}$ be a smooth embedding. We say that g is *standard* or that $S^p \times S^q$ is *standardly embedded* in $S^{p+q+r+1}$, if $g(S^p \times S^q)$ is isotopic to the boundary of a tubular neighborhood of S^p or S^q standardly embedded in $S^{p+q+r+1}$ in the usual sense. We say that a smooth embedding $f : S^p \times S^q \times S^r \to S^{p+q+r+1}$ is *standard* if $f(S^p \times S^q \times S^r)$ is isotopic to the boundary of a tubular neighborhood of $S^p \times S^q$ or $S^q \times S^r$ or $S^p \times S^r$ standardly embedded in $S^{p+q+r+1}$.

Corollary 8.2. *Let $f : S^p \times S^q \times S^r \to S^{p+q+r+1}$ be a smooth embedding with $2 \leq p \leq q < r$ and C_1 the closure of one of the two components of $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$. If $H_*(C_1) \cong H_*(S^p \times S^q)$, then f is standard.*

Proof. By Theorem 1.1 and Proposition 7.1 (1) together with our dimensional assumptions, $f(S^p \times S^q \times S^r)$ bounds in $S^{p+q+r+1}$ an embedded manifold T diffeomorphic to $S^p \times S^q \times D^{r+1}$. Note that T is a tubular neighborhood of S, where S is the product of two spheres $S^p \times S^q$ embedded in $S^{p+q+r+1}$ which corresponds to $S^p \times S^q \times \{0\} \subset S^p \times S^q \times D^{r+1} \cong T \subset S^{p+q+r+1}$.

By [**H**] together with our hypothesis on p, q and r, there exists a diffeomorphism $h : S^{p+q+r+1} \to S^{p+q+r+1}$ isotopic to the identity such that $h(S)$ is the product of spheres $S^p \times S^q$ standardly embedded in $S^{p+q+r+1}$. Then $h(T)$ is a tubular neighborhood of $S^p \times S^q$. Thus, $f(S^p \times S^q \times S^r)$ bounds the tubular neighborhood $h^{-1}(h(T)) = T$ of $h^{-1}(S^p \times S^q)$, which is standardly embedded in $S^{p+q+r+1}$. Therefore, f is standard. □

Remark 8.3. Compare the above corollary with [**LNS**, Theorem 1.3] about codimension one embeddings of product of two spheres.

9. Exotic embeddings.

In this section, let us consider the case where $r = p + q$ with r odd and prove Theorem 1.2, which insures the existence of exotic embeddings under this dimensional assumption. The result is surprising when compared with Theorem 1.1 and the results obtained in [**A**], [**K**], [**Wa**], [**G**], [**R**] and [**LNS**] about codimension one embeddings of product of two spheres. In the following, we will construct the embeddings f_n so that the complements $S^{p+q+r+1} - f_n(S^p \times S^q \times S^r)$ are not homotopy equivalent to each other.

Let us write S^{2r+1} as the union

$$S^{2r+1} = (D^{r+1}_{-2} \times S^r) \cup_{\varphi_-} (S^r \times S^r \times I) \cup_{\varphi_+} (S^r \times D^{r+1}_2), \tag{9.1}$$

where $I = [-1, 1]$, $D^{r+1}_{\pm 2}$ are $(r+1)$-disks, and $\varphi_- : \partial(D^{r+1}_{-2} \times S^r) \to S^r \times S^r \times \{-1\}$ and $\varphi_+ : \partial(S^r \times D^{r+1}_2) \to S^r \times S^r \times \{1\}$ are the standard identification maps. Since $S^r \times I$ is diffeomorphic to the closure of the complement of two disjoint $(r+1)$-disks in $S^{r+1} = (S^p \times D^{q+1}) \cup (D^{p+1} \times S^q)$, we can write $S^r \times S^r \times I$ as the union of

$$X_- = ((S^p \times D^{q+1}) - \operatorname{Int} D^{r+1}_{-1}) \times S^r \qquad \text{and}$$
$$X_+ = ((D^{p+1} \times S^q) - \operatorname{Int} D^{r+1}_{1}) \times S^r$$

attached along $S^p \times S^q \times S^r$, which is a boundary component of each, where $D^{r+1}_{\pm 1}$ are interior disks. Note that the embedding $S^p \times S^q \times S^r = X_- \cap X_+ \subset X_- \cup X_+ = S^r \times S^r \times I \subset S^{2r+1}$ defined via (9.1) is standard. In the following, we will modify this embedding by changing the identification maps $\varphi_\pm$ in (9.1).

Let $\psi : S^r \times S^r \to S^r \times S^r$ be an arbitrary diffeomorphism. By (9.1), we still have

$$S^{2r+1} \cong (D^{r+1}_{-2} \times S^r) \cup_{\varphi_- \circ \psi} (S^r \times S^r \times I) \cup_{\varphi_+ \circ \psi} (S^r \times D^{r+1}_2).$$

Put

$$\widetilde{X}_- = (D^{r+1}_{-2} \times S^r) \cup_{\varphi_- \circ \psi} X_-, \quad \widetilde{X}_+ = X_+ \cup_{\varphi_+ \circ \psi} (S^r \times D^{r+1}_2),$$

and consider $S^p \times S^q \times S^r = \widetilde{X}_- \cap \widetilde{X}_+ \subset \widetilde{X}_- \cup \widetilde{X}_+ = S^{2r+1}$. We will show that, for a suitable diffeomorphism ψ, $\widetilde{X}_\pm$ are not homotopy equivalent to the product of two spheres.

Suppose that $\psi_* : H_r(S^r \times S^r) \to H_r(S^r \times S^r)$ is given by $\psi_* \alpha = k\alpha + l\beta$ and $\psi_* \beta = m\alpha + n\beta$, where

$$A = \begin{pmatrix} k & m \\ l & n \end{pmatrix} \in GL(2; \mathbf{Z}), \quad \text{and}$$

$$\alpha = [\partial D^{r+1}_{-2} \times \{*\}] = [S^r \times \{*\}], \quad \beta = [\{*\} \times \partial D^{r+1}_2] = [\{*\} \times S^r]$$

are the generators of $H_r(S^r \times S^r)$. Then by using standard techniques in homology theory, we can show the following:

Lemma 9.1.

(1) *The homology group $H_r(\widetilde{X}_-)$ is isomorphic to $\mathbf{Z}$ and is generated by $\xi = m[\partial D^{r+1}_{-1} \times \{*\}] + n[\{*\} \times S^r]$, where $\partial D^{r+1}_{-1} \times \{*\}, \{*\} \times S^r \subset X_-$ and we can identify $[\partial D^{r+1}_{-1} \times \{*\}]$ with $[S^p \times S^q \times \{*\}]$ ($S^p \times S^q \times \{*\} \subset \partial \widetilde{X}_-$). Furthermore, we have $[\{*\} \times S^r] = \pm k\xi$.*

(2) *The homology group* $H_r(\widetilde{X}_+)$ *is isomorphic to* $\mathbf{Z}$ *and is generated by* $\xi' = k[\partial D_1^{r+1} \times \{*\}] + l[\{*\} \times S^r]$, *where* $\partial D_1^{r+1} \times \{*\}, \{*\} \times S^r \subset X_+$ *and we can identify* $[\partial D_1^{r+1} \times \{*\}]$ *with* $[S^p \times S^q \times \{*\}]$ $(S^p \times S^q \times \{*\} \subset \partial \widetilde{X}_+)$. *Furthermore, we have* $[\{*\} \times S^r] = \mp m\xi'$.

Then we have the following:

Lemma 9.2.

(1) *The manifold* $\widetilde{X}_-$ *is not homotopy equivalent to* $S^p \times S^q \times D^{r+1}$ *nor to* $D^{p+1} \times S^q \times S^r$.

(2) *The manifold* $\widetilde{X}_+$ *is not homotopy equivalent to* $S^p \times D^{q+1} \times S^r$ *nor to* $S^p \times S^q \times D^{r+1}$.

(3) *If* $k \neq \pm 1$ *and* $m \neq \pm 1$, *then the manifolds* $\widetilde{X}_\pm$ *are not homotopy equivalent to* $S^p \times D^{q+1} \times S^r$ *or* $D^{p+1} \times S^q \times S^r$ *respectively.*

Proof. We see easily that $H_q(\widetilde{X}_-) = 0 = H_p(\widetilde{X}_+)$, from which (1) and (2) follow. Part (3) follows from Lemmas 9.1 and 7.2. □

Using the above lemma, we can easily show the following.

Proposition 9.3. *If the diffeomorphism* $\psi : S^r \times S^r \to S^r \times S^r$ *satisfies* $k \neq \pm 1$ *and* $m \neq \pm 1$, *then the embedding* $\widetilde{f} : S^p \times S^q \times S^r = \widetilde{X}_- \cap \widetilde{X}_+ \to \widetilde{X}_- \cup \widetilde{X}_+ = S^{p+q+r+1}$ *has the property that the closure of neither of the two components of* $S^{p+q+r+1} - \widetilde{f}(S^p \times S^q \times S^r)$ *is homotopy equivalent to the product of two spheres and a disk.*

By [**G**, Proposition 2.5] or [**LS**, Theorem 2.2], for each matrix

$$\kappa_n = \begin{pmatrix} 4n+1 & 2n \\ 2 & 1 \end{pmatrix} \in GL(2; \mathbf{Z})$$

with $n \neq 0$, the automorphism of $H_r(S^r \times S^r)$ given by the matrix κ_n is realized by a diffeomorphism $\psi_n : S^r \times S^r \to S^r \times S^r$, since r is odd. In this way, we can construct infinitely many embeddings $f_n : S^p \times S^q \times S^r \to S^{p+q+r+1}$ which satisfy the property of Proposition 9.3 by setting $f_n = \widetilde{f}$ with $\psi = \psi_n$, since $4n+1 \neq \pm 1$ and $2n \neq \pm 1$.

The following lemma is important in showing that the embeddings $f_n : S^p \times S^q \times S^r \to S^{p+q+r+1}$ constructed from the matrices κ_n are mutually distinct.

Lemma 9.4. *Let* W *be a compact manifold such that* $\partial W = S^p \times S^q \times S^r$, $p, q \geq 1$, $r = p+q$ *with* r *odd, and* $H_*(W) \cong H_*(S^p \times S^r)$. *Let* $\xi_p \in H^p(W) \cong \mathbf{Z}, \xi_r \in H^r(W) \cong \mathbf{Z}, \xi_{p+r} \in H^{p+r}(W) \cong \mathbf{Z}$ *and* $\eta \in H_r(W) \cong \mathbf{Z}$ *be respective generators. If* $\xi_p \smile \xi_r = k\xi_{p+r}$ $(k \in \mathbf{Z})$, *then* $i_*[\{*\} \times \{*\} \times S^r] = \pm k\eta$, *where* $i : \partial W \to W$ *is the inclusion.*

The above lemma can be proved by an argument similar to that in the proof of Lemma 7.2.

Definition 9.5. We call the number $|k| \in \mathbf{Z}$ in Lemma 9.4 the *cup product invariant* of W. Note that $|k| \in \mathbf{Z}$ is well-defined: More precisely, $|k|$ is a homotopy invariant of W.

Proof of Theorem 1.2. Consider the embeddings $f_n : S^p \times S^q \times S^r \to S^{p+q+r+1}$ constructed from the matrices κ_n with $n \neq 0$. These satisfy the property of Proposition 9.3. We will show that the embeddings f_n are mutually distinct.

Let $f_{n_1} : S^p \times S^q \times S^r \to S^{p+q+r+1}$ be the embedding constructed from the matrix κ_{n_1} with $n_1 \neq 0$. We may assume that $H_*(C_1) \cong H_*(S^p \times S^r)$ and $H_*(C_2) \cong H_*(S^q \times S^r)$, where C_1 and C_2 are the closures of the two components of $S^{p+q+r+1} - f_{n_1}(S^p \times S^q \times S^r)$. The cup product invariants of C_1 and C_2 are equal to $|4n_1 + 1|$ and $|2n_1|$ respectively by Lemma 9.1.

Similarly, for the embedding $f_{n_2} : S^p \times S^q \times S^r \to S^{p+q+r+1}$ constructed from the matrix κ_{n_2} with $n_1 \neq n_2 \neq 0$, we may assume that $H_*(C_3) \cong H_*(S^p \times S^r)$ and $H_*(C_4) \cong H_*(S^q \times S^r)$, where C_3 and C_4 are the closures of the two components of $S^{p+q+r+1} - f_{n_2}(S^p \times S^q \times S^r)$. Suppose that there exists a diffeomorphism $h : S^{p+q+r+1} \to S^{p+q+r+1}$ such that $h(f_{n_1}(S^p \times S^q \times S^r)) = f_{n_2}(S^p \times S^q \times S^r)$. Then we have $h(C_1) = C_3$, which implies that $|4n_1 + 1| = |4n_2 + 1|$. This contradicts the assumption that $n_1 \neq n_2$. Therefore, f_{n_1} and f_{n_2} are distinct if $n_1 \neq n_2$. This completes the proof of Theorem 1.2. □

Remark 9.6. When $n = 0$, for the embedding f_0, if $p, q \geq 2$, then it follows from Proposition 7.1 that C_1 is diffeomorphic to $S^p \times D^{q+1} \times S^r$.

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