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If G is a Banach space valued measure whose range range is convex, we consider the space of G-integrable functions. We also establish a factorization result for operators from $L_p(\mu)$ through $L_1(G)$. We apply these results in order to obtain a description of the range of p'-summing operators from L_p-spaces.

1. Introduction and notation.

Let (Ω, Σ) be a measurable space and let X be a Banach space. Throughout this paper G will be a countably additive vector measure $G : \Sigma \to X$. Consider the space $L_1(G)$ of (classes of real) G-integrable functions, following the definition of Bartle, Dunford and Schwartz [**1**] and Lewis [**9**]. The properties of this space have been studied by Kluvánek and Knowles [**7**], Okada [**10**] and Curbera [**2**]. In the first part of this paper (Section 1) we investigate the relation between the convexity of the range of G and the structure of the space $L_1(G)$. We use an alternate definition of the norm of $L_1(G)$ that is obtained by identifying each function $f \in L_1(G)$ with an operator from an L_∞ space. In the second part (Section 2), we apply these ideas to obtain several properties of operators from an L_p space to a Banach space. In particular, we use the results of Dinculeanu that relates vector measures and operators from L_p spaces (see [**6**]) to obtain a factorization theorem. As an application, we also show a description of the range of p'-summing operators from L_p spaces.

We use well-known results about general Vector Measure Theory (see [**5**]). The notation is standard. If $x' \in X'$, $|x'G|$ is the variation of the scalar measure $x'G$ defined by $x'G(A) := \langle G(A), x' \rangle$. The semivariation of G in a set $A \in \Sigma$ is given by $\|G\|(A) = \sup\{|x'G|(A) : x' \in B_{X'}\}$. We write χ_A for the characteristic function of $A \in \Sigma$ and A^c for $\Omega \setminus A$. If $1 \le p < \infty$, p' is the (extended) real number that satisfies $1/p + 1/p' = 1$. The space of linear and continuous operators between the Banach spaces Y and X is denoted by $L(Y, X)$.

A measurable real function defined on Ω is *G-integrable* if it is $x'G$-integrable for each $x' \in X'$, and for every $A \in \Sigma$ there is an element $\int_A f dG$

of X such that $\langle \int_A f dG, x' \rangle = \int_A f dx'G$ for every $x' \in X'$ ([**9**]). The Banach lattice $L_1(G)$ is the space of all the (classes of) G-integrable functions with the $\|G\|$-almost everywhere order, endowed with the norm

$$\|f\|_G := \sup \left\{ \int_\Omega |f| d|x'G| : x' \in B_{X'} \right\}, \qquad f \in L_1(G).$$

The expression $|||f|||_G := \sup_{A \in \Sigma} \| \int_A f dG\|$, $f \in L_1(G)$, provides an equivalent norm that satisfies the inequalities $|||f|||_G \leq \|f\|_G \leq 2|||f|||_G$. However, although the norm $|||.|||_G$ appears in the literature, it does not seem to be a natural one. The integration operator $I_G : L_1(G) \to X$ is defined by $I_G(f) := \int_\Omega f dG$, $f \in L_1(G)$ (see [**11**]). Let μ be a finite measure that controls G (see [**5**] Ch. IX). Integration with respect to G also gives a continuous linear map $T_G \in L(L_\infty(\mu), X)$. Let $S(\mu)$ be the set of (classes of) simple functions (that are equal μ-a.e.).

Lemma 1. *The following formula gives the L_∞-norm on $S(\mu)$;*

$$\|f\|_\infty = \inf \left\{ \sum_{i=1}^n |\lambda_i| : f = \sum_{i=1}^n \lambda_i(\chi_{A_i} - \chi_{A_i^c}), \quad A_i \in \Sigma \right\}, \qquad f \in S(\mu).$$

Proof. Let $f \in S(\mu)$. Then we can find a finite family of disjoint measurable subsets $(A_i)_{i=1}^n$ and real numbers $(\lambda)_{i=1}^n$ such that $f = \sum_{i=1}^n \lambda_i \chi_{A_i}$. Thus, the function f belongs to the subspace of $L_\infty(\mu)$ that is obtained by the restriction of this space to the finite subalgebra generated by $\{A_i : i = 1, \ldots, n\}$. This is a finite dimensional L_∞ space, and then is isometric to l_∞^n. Since the extreme points of the unit ball on l_∞^n are the elements of $\{-1, 1\}^n$, we obtain the result as a direct consequence of the Krein-Milman Theorem. □

Definition 2. Let μ be a finite control measure for G. For every $f \in L_1(G)$, we define $\|f\|_{G\infty} := \sup_{g \in B_{L_\infty(\mu)}} \| \int_\Omega f g dG\|$.

Note that the definition of $\|.\|_{G\infty}$ does not depend on the particular control measure μ. This is a consequence of the following simple argument. Each function $f \in L_1(G)$ defines the countably additive vector measure G_f and the semivariation of this measure gives an equivalent norm for the space $L_1(\lambda)$ (see [**9**] or [**2**]). Theorem 13 of [**5**, Ch. I] for the case of a finite measure μ defined over the σ-algebra Σ establishes the equality between the norm of the operator T_{G_f} and the semivariation of G_f. A direct application of the particular representation of the norm of the functions of $L_\infty(\mu)$ given in Lemma 1 leads to the following result, that simplifies the arguments related to the convexity of G.

Proposition 3. *For every $f \in L_1(G)$,*

$$\|f\|_G = \|f\|_{G\infty} = \sup_{A \in \Sigma} \left\| \int_\Omega f(\chi_A - \chi_{A^c}) dG \right\|.$$

Proof. First note that if $x' \in X'$ and $A \in \Sigma$ the variation of the scalar measure $x'G$ on A is given by

$$|x'G|(A) = \sup\{\langle G(A \cap B), x'\rangle - \langle G(A \cap B^c), x'\rangle : B \in \Sigma\}.$$

Let us denote $\sup_{A\in\Sigma} \| \int_\Omega f(\chi_A - \chi_{A^c})dG\|$ by $\|f\|_s$ for every $f \in L_1(G)$. Let f be a simple function and let $\epsilon > 0$. Then a direct calculation using the above representation of $|x'G|$ shows that there is a set $B \in \Sigma$ such that the function $h := \chi_B - \chi_{B^c}$ satisfies $\|f\|_G \le \langle \int_\Omega fhdG, x'\rangle + \epsilon$. This and the density of the simple functions in $L_1(G)$ imply the inequalities $\|f\|_G \le \|f\|_s \le \|f\|_{G\infty}$ for every $f \in L_1(G)$. Since the inequality $\|f\|_s \le \|f\|_G$ is obvious and Lemma 1 implies $\|f\|_{G\infty} \le \|f\|_s$ for every $f \in L_1(G)$, we obtain the result. □

Let μ be a Rybakov control measure for G (see [**5**, Ch. IX.2]). The main known result that relates the structure of the space $L_\infty(\mu)$ with the convexity and weak compactness of the class of sets $\{G(A \cap B) : B \in \Sigma\}$, $A \in \Sigma$, is the following theorem due to Knowles (see Theorem 4 and Corollary 7 of [**5**, Ch. IX.1]).

Theorem 4. *The following are equivalent:*

1) *If* $0 \neq f \in L_\infty(\mu)$, *there is a function* $g \in L_\infty(\mu)$ *such that* $\|fg\|_\infty > 0$ *but* $\int_\Omega fgdG = 0$.
2) *For each* $A \in \Sigma$, $\{G(A \cap B) : B \in \Sigma\}$ *is a weakly compact convex set in* X.
3) *For each* $A \in \Sigma$ *there is a set* $B \in \Sigma$ *such that* $G(A \cap B) = \frac{G(A)}{2}$.

We will say that a vector measure G that satisfies any one of the statements of Theorem 4 is a Σ-*weakly compact convex vector measure.* If G is such a measure, $A \in \Sigma$ and $G(A) \neq 0$, we can define the following tree of subsets. An application of Statement 3) of Theorem 4 gives a set $A_1^1 \in \Sigma$ such that $A_1^1 \subset A$ and $G(A_1^1) = \frac{G(A)}{2}$. Then, if we define the set $A_1^2 \in \Sigma$ as $A_1^2 = A - A_1^1$, we obtain $G(A_1^1) = \frac{G(A)}{2} = G(A_1^2)$. Following the same procedure for $G(A_1^1)$ and $G(A_1^2)$, we obtain *four* subsets $G(A_2^i)$, $i = 1, \dots, 4$, such that $A_2^1 \cup A_2^2 = A_1^1$, $A_2^3 \cup A_2^4 = A_1^2$ and $G(A_2^i) = \frac{G(A)}{4}$ for each i. We may continue in this way in order to obtain a tree of sets $\{A_n^i\}$ such that:

1) For each natural number n, $\{A_n^i\}$ is a partition of A and for every $k = 1, \dots, 2^{n-1}$, $A_n^{2k-1} \cup A_n^{2k} = A_{n-1}^k$.
2) For each natural number n and every $i = 1, \dots, 2^n$, $G(A_n^i) = \frac{G(A)}{2^n}$.

This sequence of subsets leads to a σ-subalgebra Σ_A of Σ, the restriction of G to which is one dimensional and equivalent to the Lebesgue measure on $[0, 1]$. We thus have:

Corollary 5. *In the notation above, denote by G_A the restriction of G to Σ_A. Then $(L_1(G_A), \|.\|_{G_A})$ is isometric to $L_1(0,1)$.*

Remark 6. It is clear that $\|f\|_{G_A} \leq \|f\|_G$ for every function $f \in L_1(G_A)$. If $A \in \Sigma$ and there is a constant K_A such that $\|f\|_G \leq K_A \|f\|_{G_A}$ for every function $f \in L_1(G_A)$ we can identify the space $L_1(G_A)$ with a subspace of $L_1(G)$. For instance, this is the case when G is defined on a Banach lattice and integration with respect to G gives a positive operator. The above construction yields a lot of information about the structure of the function spaces $L_1(G_A)$, since the structure of $L_1(0,1)$ is very well-known (see Th. 7 of [**8**, Ch. 6.17]).

2. A factorization theorem.

Let μ be a (countably additive purely nonatomic) finite measure. In this section we obtain several properties of the range of the operators of $L(L_p(\mu), X)$ related to the Uhl Theorem about the relative compactness of the range of a vector measure G. Through this section we consider the norm $\| \ \|_{G\infty}$ for $L_1(G)$. The results of Dinculeanu about vector measure integration (see [**6**]) allow to a factorization theorem for these operators. Let $P(\Sigma)$ be the class of all the finite partitions of Ω in Σ. The following definition can be found in [**6**, Ch. II.13].

Definition 7. If μ be a finite control measure for G and $1 \leq p < \infty$, the *p-semi-variation* of G in $A \in \Sigma$ is defined by

$$\|G\|_p^s(A) := \sup_{\|x'\| \leq 1, \{A_i\}_{i=1}^n \in P(\Sigma)} \left(\sum_{i=1}^n \frac{|\langle G(A_i \cap A), x' \rangle|^p}{\mu(A_i \cap A)^{\frac{p}{p'}}} \right)^{\frac{1}{p}}.$$

We denote by $M_p^s(\mu, X)$ the space of all the X-valued vector measures of finite p-semi-variation $\|G\|_p^s := \|G\|_p^s(\Omega)$. If $T \in L(L_p(\mu), X)$, we write G_T for the vector measure given by $G_T(A) := T(\chi_A)$, $A \in \Sigma$. Conversely, if $G : \Sigma \to X$ is a vector measure and μ controls G, we denote by T_G the linear map defined on $S(\mu)$ by $T_G(\chi_A) := G(A)$. We will use the same notation for this map if it is possible to extend it to the space $L_p(\mu)$. In our context the general result given in [**6**, Ch. II.13, Th. 1] can be written as follows. We obtain Theorem 11 as a consequence.

Theorem 8. *$M_{p'}^s(\mu, X)$ is isomorphic to the space $L(L_p(\mu), X)$.*

Theorem 9. *Let $1 \leq p < \infty$ and $T \in L(L_p(\mu), X)$. Then it can be factored through $L_1(G_T)$ as*

$$L_p(\mu) \xrightarrow{I} L_1(G_T) \xrightarrow{I_{G_T}} X$$

where $I(f) := \bar{f}$, if $\bar{f}$ is the class of f in $L_1(G_T)$, and $I_{G_T}(\bar{f}) = \int_\Omega \bar{f} dG_T$. Moreover, $\|G_T\|_{p'}^s \leq \|T\| \leq \|I\| \leq 2\|G_T\|_{p'}^s$.

Proof. We need to show that $\|f\|_{G\infty} \leq \|T\|$ if $\|f\|_{L_p} \leq 1$. By Proposition 3, it is enough to estimate $\| \int_\Omega fgdG_T\|$, where $\|g\|_{L_\infty} \leq 1$. But this integral is just $T(fg)$. Since $\|fg\|_{L_p} \leq 1$, we obtain $\|T(fg)\| \leq \|T\|$, as required. The inequalities between the norms of the operators and the p'-semi-variation can be obtained by mean of a direct calculation following the arguments given above and the ones that can be found in [**6**]. □

The properties of the operator $T : L_p(\mu) \to X$ and the space X determine the structure of the factorization space $L_1(T_G)$. For example, if the measure G_T is Σ-weakly compact convex and $A \in \Sigma$, the restriction of the operator to the subspace of $L_p(\mu)$ generated by Σ_A factorizes through $L_1(0,1)$, as a consequence of the arguments given in Section 1. Nowadays, a lot of properties of the space $L_1(G)$ that may be applied in our context are known (see [**10**], [**2**] and [**3**]). We present an example that is a consequence of [**2**, Th. 3].

Corollary 10. *Let $1 \leq p < \infty$ and let X be a cotype 2 Banach space. Consider an operator $T : L_p(\mu) \to X$. If for every partition $(A_n)_1^\infty$ the sequence $\left(\frac{T(\chi_{A_n})}{\|G_T\|(A_n)}\right)_1^\infty$ is 2-lacunary in X, then T factors through a space that is order isomorphic to a Hilbert space.*

To finish this paper, we apply the above results to obtain a description of the range of the p'-summing operators from $L_p(\mu)$ to X. The reader can find information about the operator ideals of p-summing operators (P_p, Π_p) in [**4**] and [**12**]. Let C be a subset of a Banach space Y and let $T \in L(Y,X)$. We say that the range of T is *approximable by C* if the set $\{\lambda T(y) : \lambda \in R,\ y \in C\}$ is dense in the range of T. Let $B_0 := \{\chi_A - \chi_{A^c} : A \in \Sigma\}$.

Theorem 11. *Let $1 < p < \infty$ and μ be a nonatomic probability measure. Let X be a Banach space with the Radon-Nikodym property and let $R \in L(L_p(\mu), X)$. If R is p'-summing, then the range of R is approximable by B_0.*

Proof. First we claim that the composition of the operators $T : L_p(\mu) \to Y$ and $S : Y \to X$, if T is continuous and S is p'-summing, defines a countably additive vector measure G_{ST} of bounded variation. Since $G_{ST}(A) = ST(\chi_A)$, we get $\|G_{ST}(A)\| \leq \|ST\|\mu(A)^{\frac{1}{p}}$ and then G_{ST} is countably additive. Let $\{A_i\}_{i=1}^n \in P(\Sigma)$. Then

$$\begin{aligned}\sum_{i=1}^n \|ST(\chi_{A_i})\| &= \sum_{i=1}^n \mu(A_i)^{\frac{1}{p}} \frac{\|ST(\chi_{A_i})\|}{\mu(A_i)^{\frac{1}{p}}} \\ &\leq \left(\sum_{i=1}^n \frac{\|ST(\chi_{A_i})\|^{p'}}{\mu(A_i)^{\frac{p'}{p}}} \right)^{\frac{1}{p'}}\end{aligned}$$

$$\leq \Pi_{p'}(S) \sup_{\|x'\|_Y \leq 1} \left(\sum_{i=1}^{n} \frac{|\langle T(\chi_{A_i}), x' \rangle|^{p'}}{\mu(A_i)^{\frac{p'}{p}}} \right)^{\frac{1}{p'}}$$
$$\leq \Pi_{p'}(S) \|G_T\|_{p'}^{s}.$$

Since T is a continuous map, Theorem 9 gives that $\|G_T\|_{p'}^{s} < \infty$. Now, suppose that X has the Radon-Nikodym property. Since μ is nonatomic, so is G_{ST}. An application of Uhl Theorem (see [**5**] Ch. IX.1) gives the convexity of the closure of the range of G_{ST}. Since

$$\overline{\{G_{ST}(A) : A \in \Sigma\}} = \frac{1}{2} G_{ST}(\Omega) + \frac{1}{2} \overline{ST(B_0)},$$

and $B_{L_\infty(\mu)} = \overline{co(B_0)}$ as a consequence of Lemma 1, we obtain $\overline{ST(B_0)} = \overline{ST(B_{L_\infty(\mu)})}$. This obviously proves that the range of ST is approximable by B_0. To finish the proof it is enough to consider the factorization of R as $R = RI$, where I is the identity map in $L_p(\mu)$. □

It is possible to find several applications of this theorem in the context of the Operator Ideals Theory. Obviously, it also holds for each p'-integral operator $T \in L(L_p(\mu), X)$. The same argument can be applied for (p,q)-summing operators. In the context of the Hilbert spaces, we may obtain that each Hilbert-Schmidt operator satisfies that its range is approximable by B_0.

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