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CONNECTIONS ON PRINCIPAL BUNDLES OVER CURVES IN POSITIVE CHARACTERISTICS

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Let X be an irreducible smooth projective curve over an algebraically closed field k of characteristic p, with $p > 5$. Let G be a connected reductive algebraic group over k. Let H be a Levi factor of some parabolic subgroup of G and χ a character of H. Given a reduction E_H of the structure group of a G-bundle E_G to H, let E_χ be the line bundle over X associated to E_H for the character χ. If G does not contain any $\mathrm{SL}(n)/Z$ as a simple factor, where Z is a subgroup of the center of $\mathrm{SL}(n)$, we prove that a G-bundle E_G over X admits a connection if and only if for every such triple (H, χ, E_H), the degree of the line bundle E_χ is a multiple of p. If G has a factor of the form $\mathrm{SL}(n)/Z$, then this result is valid if n is not a multiple of p. If G is a classical group but not of the form $\mathrm{SL}(n)/Z$, then this criterion for the existence of connection remains valid even if $p \geq 3$.

1. Introduction.

Let X be an irreducible smooth projective curve over an algebraically closed field k. Take a vector bundle E over X. A subbundle V of E is called a *direct summand* if the quotient homomorphism $E \longrightarrow E/V$ splits. For $k = \mathbb{C}$, a theorem of Andre Weil says that E admits a connection if and only if every direct summand of E is of degree zero [**10**].

Let G be a connected reductive algebraic group over k. Let E_G be a principal G-bundle over X. Our aim is to give a criterion for the existence of a connection on E_G. Note that since the dimension of X is one, the curvature of a connection on E_G must vanish. In other words, any connection on E_G is automatically flat.

Let $H \subseteq G$ be a Levi factor of a parabolic subgroup of G. Let $E_H \subseteq E_G$ be a reduction of structure group of E_G to H. Take a character

$$\chi : H \longrightarrow k^*$$

of H and consider the line bundle $E_\chi := (E_H \times k)/H$ associated to E_H for the character χ. A connection on E_G induces a connection on E_H (Proposition 2.2), which, in turn, induces a connection on the line bundle E_χ.

It is well-known that a line bundle ξ over X admits a connection if and only if the degree of ξ is a multiple of the characteristic of k (see also Corollary 2.1). Therefore, if E_G admits a connection then the degree of any line bundle E_χ of the above type must be a multiple of the characteristic of k.

Let p denote the characteristic of k. We will assume that $p > 5$.

Let $Z(\mathrm{SL}(n)) \subset \mathrm{SL}(n)$ be the center. First assume that G does not have a simple factor of the form $\mathrm{SL}(n)/Z$, where $Z \subseteq Z(\mathrm{SL}(n))$. We prove that if the degree of any E_χ of the above type is a multiple of p, then E_G admits a connection (Theorem 2.3).

If G contains a simple factor of the form $\mathrm{SL}(n)/Z$, where $Z \subseteq Z(\mathrm{SL}(n))$, and p does not divide n, then E_G admits a connection if and only if the degree of any line bundle E_χ of the above type is a multiple of p.

If G is a classical group but not $\mathrm{SL}(n)/Z$, where $Z \subseteq Z(\mathrm{SL}(n))$, then the condition $p > 5$ can be relaxed to $p > 2$.

2. The Atiyah bundle.

Let H be an algebraic group over k. Take a principal H-bundle E_H over X. The projection of the total space of E_H to X will be denoted by ψ. For any open subset U of X, consider the space of H-invariant vector fields on $\psi^{-1}(U)$ for the natural action of H on the fibers of ψ. This gives rise to a vector bundle $\mathrm{At}(E_H)$ on X known as the *Atiyah bundle*.

Let $\mathfrak{h}$ denote the Lie algebra of H. Consider the adjoint action of H on $\mathfrak{h}$. The associated vector bundle $(E_H \times \mathfrak{h})/H$, known as the *adjoint bundle*, will be denoted by $\mathrm{ad}(E_H)$. Note that $\mathrm{ad}(E_H)$ corresponds to the sheaf of H-invariant vertical vector fields on E_H. Therefore, we have an exact sequence

$$0 \longrightarrow \mathrm{ad}(E_H) \longrightarrow \mathrm{At}(E_H) \longrightarrow TX \longrightarrow 0 \tag{2.1}$$

of vector bundles over X. This sequence is known as the *Atiyah exact sequence*.

A *connection* on E_H is a splitting of the exact sequence (2.1) [**1**], [**9**]. See Section 5 of [**7**] for connections on vector bundles in positive characteristics.

Note that both the sheaves $\mathrm{At}(E_H)$ and TX are equipped with a Lie algebra structure induced by the Lie bracket operation of vector fields. Given a splitting

$$\sigma : TX \longrightarrow \mathrm{At}(E_H)$$

of the Atiyah exact sequence, consider the homomorphism

$$\overline{\sigma} : TX \otimes TX \longrightarrow \mathrm{ad}(E_H)$$

defined by $s \otimes t \longmapsto [\sigma(s), \sigma(t)] - \sigma([s,t])$, where s and t are local sections of TX, which is known as the *curvature*. Since $p \neq 2$, $\overline{\sigma}$ is skew-symmetric and $\dim X = 1$, we have $\overline{\sigma} = 0$. In other words, any connection on X is flat.

Set $H = \mathrm{GL}(n)$. So, using the standard representation of $\mathrm{GL}(n)$, E_H corresponds to a rank n vector bundle V over X. The exact sequence (2.1) becomes

$$0 \longrightarrow \mathrm{End}(V) \longrightarrow \mathrm{At}(V) \longrightarrow TX \longrightarrow 0, \tag{2.2}$$

where $\mathrm{At}\,(V)$ is the subbundle of the sheaf of differential operators $\mathrm{Diff}^1_X(V,V)$ defined by the condition that the image by the symbol homomorphism $\mathrm{Diff}^1_X(V,V) \longrightarrow TX \bigotimes \mathrm{End}(V)$ is contained in the subbundle $TX \bigotimes \mathrm{Id}_V$.

Consider the extension class

$$\tau \in H^1(X,\, K_X \otimes \mathrm{End}(V))$$

for the exact sequence (2.2), where K_X is the canonical bundle of X. Using the trace homomorphism $\mathrm{tr} : \mathrm{End}(V) \longrightarrow \mathcal{O}_X$, we have

$$\mathrm{tr}(\tau) \in H^1(X,\, K_X) = k$$

(k is the base field), where the identification $H^1(X,\, K_X) = k$ is the one given by Serre duality.

Let $d \in \mathbb{Z}$ be the degree V, which is same as the degree of the line bundle $\bigwedge^n V$. The image of d in k by the obvious homomorphism $\mathbb{Z} \longrightarrow k$ coincides with $\mathrm{tr}(\tau)$.

Consequently, if a $\mathrm{GL}(n)$-bundle admits a connection, then the degree of the corresponding rank n vector bundle is a multiple of p, the characteristic of k.

This observation and the above identity $d = \mathrm{tr}(\tau)$ together have the following corollary:

Corollary 2.1. *A line bundle ξ over X admits a connection if and only if the degree of ξ is a multiple of p (possibly zero).*

The above corollary is well-known [**7**, p. 190, Theorem 5.1] (in [**7**] this Theorem 5.1 is attributed to P. Cartier), [**8**].

As in the introduction, let G be a connected reductive algebraic group over k. Let P be a parabolic subgroup of G. Let $R_u(P)$ denote the unipotent radical of P. The quotient group $P/R_u(P)$ is called the *Levi factor* of P [**4**]. The projection $P \longrightarrow P/R_u(P)$ splits in the sense that there is a connected closed reductive subgroup H of P which projects isomorphically to $P/R_u(P)$. However, there may be more than one such subgroup. We will call a subgroup H of P with this property a *Levi factor* of P.

Take a G-bundle E_G over X. Suppose

$$\sigma : X \longrightarrow E_G/H$$

be a reduction of structure group of E_G to a Levi factor H. So, the inverse image $q^{-1}(\sigma(X))$, where $q : E_G \longrightarrow E_G/H$ is the obvious quotient map, is an H-bundle. This H-bundle will be denoted by E_H.

Fix a character $\chi : H \longrightarrow k^*$ of H. Consider the quotient

$$E_\chi := (E_H \times k)/H$$

for the diagonal action of H, where H acts on k through χ, which is a line bundle over X. We recall that the diagonal action of any $g \in H$ sends a point $(z, t) \in E_H \times k$ to $(zg, \chi(g^{-1})t)$.

Proposition 2.2. *If E_G admits a connection, then the degree of the line bundle E_χ is a multiple of p.*

Proof. Any connection on E_H induces a connection on E_χ. Therefore, in view of Corollary 2.1 it suffices to show that any connection on E_G induces a connection on E_H.

Let $\mathfrak{g}$ (respectively, $\mathfrak{h}$) denote the Lie algebra of G (respectively, H). Since H is a Levi factor, there exists a H-equivariant splitting

$$f : \mathfrak{g} \longrightarrow \mathfrak{h} \tag{2.3}$$

of the inclusion homomorphism of $\mathfrak{h}$ in $\mathfrak{g}$. Indeed, if $\mathfrak{p}$ and $\mathfrak{q}$ are two opposite parabolics containing $\mathfrak{h}$ as the common Levi factor, then the direct sum of the radicals of $\mathfrak{p}$ and $\mathfrak{q}$ is a H-invariant complement of $\mathfrak{h}$.

We recall that a connection on E_G is a $\mathfrak{g}$-valued 1-form ω on E_G satisfying the two conditions:

1) For any $v \in \mathfrak{g}$, the evaluation of ω on the vector field corresponding to v coincides with the constant function v;
2) the form ω is equivariant for the action of G on E_G and the adjoint action of G on its Lie algebra $\mathfrak{g}$.

The kernel of such a form ω defines a splitting of the Atiyah exact sequence (2.1). To explain this, let ψ denote the projection of E_G to X. Given a tangent vector $v \in T_xX$, where $x \in X$, and a point $z \in \psi^{-1}(x) \subset E_G$, there is a unique tangent vector $w \in T_zE_G$ projecting to v that is contained in the kernel of the form ω. This way we get a section v' of TE_G over $\psi^{-1}(x)$. This section v' is clearly G-invariant. In other words, v' gives n element v'' of the fiber $\mathrm{At}(E_G)_x$. Sending any v to v'' we obtain a splitting of the Atiyah exact sequence (2.1). Conversely, given a splitting $\sigma : TX \longrightarrow \mathrm{At}(E_G)$ of the Atiyah exact sequence, it is easy to see that there is a unique one-form ω satisfying the above two conditions such that the kernel of ω is the image of σ.

Given a connection on E_G defined by a one-form ω, let ω' denote the restriction of ω to $E_H \subseteq E_G$. Now consider the $\mathfrak{h}$-valued one-form

$$\overline{\omega} := f \circ \omega'$$

on E_H, where f is defined in (2.3).

It is easy to check that the form $\overline{\omega}$ satisfies the two conditions needed to define a connection on E_H. Consequently, existence of a connection on E_G

ensures the existence of a connection on E_H. This completes the proof of the proposition. □

As before, let $Z(\mathrm{SL}(n))$ denote the center of $\mathrm{SL}(n)$.

Following is the main result proved here:

Theorem 2.3. *Let $p > 5$ and assume that G does not contain $\mathrm{SL}(n)/Z$ as a simple factor, where $Z \subseteq Z(\mathrm{SL}(n))$. A G-bundle E_G over X admits a connection if and only if for every pair (H, χ), where χ is a character of the Levi factor H of some parabolic subgroup, the degree of the line bundle E_χ is a multiple of p. If there is a subgroup $Z \subseteq Z(\mathrm{SL}(n))$ such that G contains $\mathrm{SL}(n)/Z$ as a simple factor, then same criterion is valid if $p > 5$ and p does not divide n.*

Since H is reductive, Proposition 2.2 says that if E_G admits a connection, then the line bundle E_χ is a multiple of p. We will complete the proof of the theorem in Section 4. In the next section we will show that it suffices to prove for simple groups.

3. Reduction to the case of simple groups.

Let $Z(G) \subset G$ denote the *reduced* center of G. Let

$$G' := G/Z(G)$$

be the quotient. Consider the commutator $[G, G]$, and let

$$Z = G/[G, G]$$

be the quotient. So G' is a semisimple quotient of G and Z is an abelian quotient of G.

For a principal G-bundle E_G on X, let $E_{G'}$ (respectively, E_Z) denote the principal G'-bundle (respectively, principal Z-bundle) obtained by extending the structure group of E_G using the obvious projection of G to G' (respectively, Z).

Lemma 3.1. *The G-bundle E_G admits a connection if and only if both $E_{G'}$ and E_Z admit connection.*

Proof. Since $E_{G'}$ and E_Z are extensions of structure group of E_G, any connection on E_G induces connection on $E_{G'}$ and E_Z.

Note that the fiber product $E_{G'} \times_X E_Z$ is a principal $(G' \times Z)$-bundle. Let

$$\rho : G \longrightarrow G' \times Z$$

be the diagonal homomorphism induced by the projections of G to G' and Z. Since the kernel of ρ is finite and it induces an isomorphism of Lie algebras, the natural map $E_G \longrightarrow E_{G'} \times_X E_Z$ is an étale covering map. Consequently, the Atiyah exact sequence for E_G and $E_{G'} \times_X E_Z$ coincide.

It is easy to see that if

$$0 \longrightarrow \mathrm{ad}\,(E_{G'}) \longrightarrow \mathcal{A} \xrightarrow{f_1} TX \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{ad}(E_Z) \longrightarrow \mathcal{B} \xrightarrow{f_2} TX \longrightarrow 0$$

are the Atiyah exact sequences for $E_{G'}$ and E_Z respectively, and p (respectively, q) is the obvious projection of $\mathcal{A}\bigoplus\mathcal{B}$ to $\mathcal{A}$ (respectively, $\mathcal{B}$), then the exact sequence

$$0 \longrightarrow \mathrm{ad}(E_{G'}) \oplus \mathrm{ad}(E_Z) \longrightarrow \mathrm{kernel}(f_1 \circ p - f_2 \circ q) \subset \mathcal{A} \oplus \mathcal{B} \longrightarrow TX \longrightarrow 0$$

obtained by combining the above two exact sequences is the Atiyah exact sequence for $E_{G'} \times_X E_Z$. From this it follows that if the Atiyah exact sequences for $E_{G'}$ and E_Z split, then the Atiyah exact sequence for $E_{G'} \times_X E_Z$ also splits. Indeed, if

$$\sigma_1 : TX \longrightarrow \mathcal{A}$$

and $\sigma_2 : TX \longrightarrow \mathcal{B}$ are splittings of Atiyah exact sequences for $E_{G'}$ and E_Z respectively, then the diagonal homomorphism

$$(\sigma_1, \sigma_2) : TX \longrightarrow \mathcal{A} \oplus \mathcal{B}$$

is the splitting of the Atiyah exact sequence for $E_{G'} \times_X E_Z$. Therefore, if both $E_{G'}$ and E_Z admit connections then the $(G' \times Z)$-bundle $E_{G'} \times_X E_Z$ also admits a connection. This completes the proof of the lemma. $\square$

The group Z is a product of copies of k^*, and Z has exactly one parabolic subgroup which is Z itself. Therefore, Theorem 2.3 is valid for Z.

The image of a parabolic subgroup P of G by the projection $G \longrightarrow G'$ is a parabolic subgroup of G'. Moreover, all parabolic subgroups of G' arise this way. The image, in G', of a Levi factor $H \subset P$ is a Levi factor of the corresponding parabolic subgroup of G'.

Consequently, to establish Theorem 2.3 for G, it suffices to prove it for the semisimple group G'.

Any parabolic subgroup of $G_1 \times G_2$, where G_1 and G_2 are semisimple, is of the form $P_1 \times P_2$, where P_i is a parabolic subgroup of G_i. Furthermore, from the Proof of Proposition 2.2 it follows immediately that if we have G_i-bundle E_{G_i}, $i = 1, 2$, over X, then both E_{G_1} and E_{G_2} admit a connection if and only if the principal $(G_1 \times G_2)$-bundle $E_{G_1} \times_X E_{G_2}$ admits a connection. Therefore, it suffices to prove Theorem 2.3 under the assumption that G is simple.

If $H \subset G$ is a Levi factor of a parabolic subgroup of G, and if $H_1 \subset H$ is a Levi factor of a parabolic subgroup of H, then H_1, as a subgroup of G, is a Levi factor of some parabolic subgroup of G. Since H_1 is a Levi factor of G, using reverse induction, we may reduce the structure group of G to such a situation where it does not admit any further reduction to some

Levi factor. A Levi factor H of some parabolic subgroup of G will be called *nontrivial* if H is a proper subgroup of G. Theorem 2.3 follows from the following theorem:

Theorem 3.2. *Let $p > 5$ and G simple. Assume that either of the following two is valid:*

1) *G is not isomorphic to $\mathrm{SL}(n)/Z$ for some subgroup Z of the center $Z(\mathrm{SL}(n))$ of $\mathrm{SL}(n)$;*
2) *if G is isomorphic to $\mathrm{SL}(n)/Z$, where $Z \subseteq Z(\mathrm{SL}(n))$, then p does not divide n.*

Let E_G be a G-bundle over X such that E_G does not admit any reduction of structure group to any nontrivial Levi factor. Such a G-bundle E_G admits a connection.

This theorem will be proved in the next section.

4. Obstruction for connection.

Let G be a simple algebraic group over k. As in Theorem 3.2, assume that either of the following two is valid:

1) G is not isomorphic to $\mathrm{SL}(n)/Z$ for some subgroup Z of the center $Z(\mathrm{SL}(n))$ of $\mathrm{SL}(n)$;
2) if G is isomorphic to $\mathrm{SL}(n)/Z$, where $Z \subseteq Z(\mathrm{SL}(n))$, then p does not divide n.

This assumption ensures that the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathfrak{g}^*$ as a G-module. Indeed, from [**5**, 0.13] we know that $\mathfrak{g}$ is simple. Now, as $\mathfrak{g}$ and $\mathfrak{g}^*$ are simple modules of same highest-weight, they are isomorphic [**6**, p. 200, Proposition 2.4(a)]. In other words, $\mathfrak{g}$ is self-dual.

Consequently, for G-bundle E_G we have $\mathrm{ad}(E_G) = \mathrm{ad}(E_G)^*$. Now the Serre duality gives

$$H^1(X,\, K_X \otimes \mathrm{ad}(E_G)) = H^0(X,\, \mathrm{ad}(E_G))^*. \tag{4.1}$$

Assume that E_G satisfies the conditions in Theorem 3.2. Let

$$\tau \in H^1(X,\, K_X \otimes \mathrm{ad}(E_G)) \tag{4.2}$$

be the extension class for the Atiyah exact sequence for E_G. Let

$$\theta \in H^0(X,\, \mathrm{ad}(E_G))^* \tag{4.3}$$

be the functional that corresponds to τ by the isomorphism (4.1). Theorem 3.2 will be proved by showing that the functional θ vanishes identically. For this we need to study the section of $\mathrm{ad}(E_G)$.

For any $x \in X$ the fiber of $\mathrm{ad}(E_G)$ over x will be denoted by $\mathrm{ad}(E_G)_x$. Note that $\mathrm{ad}(E_G)_x$ is isomorphic, as a Lie algebra, with $\mathfrak{g}$.

Lemma 4.1. *Let ϕ be a section of* $\mathrm{ad}(E_G)$ *such that for some point $x \in X$, the evaluation $\phi(x)$ is a nilpotent element of the Lie algebra* $\mathrm{ad}(E_G)_x$. *Then we have $\theta(\phi) = 0$.*

Proof. We noted earlier that the assumption on G (stated at the beginning of this section) ensures that $\mathfrak{g}$ is simple. Therefore, an element v of the simple Lie algebra $\mathfrak{g}$ is nilpotent if $\mathrm{ad}(v)$ is nilpotent. If f is a G-invariant function on $\mathfrak{g}$, then evaluating f on ϕ we get a function on X. Note that an element v of $\mathfrak{g}$ is nilpotent if and only if all G-invariant functions on $\mathfrak{g}$ vanishing at $0 \in \mathfrak{g}$ also vanishes at v. Since X is connected and complete there are no nonconstant functions on X. Consequently, if ϕ is nilpotent over some point, this observation implies that $\phi(y)$ is a nilpotent element of $\mathrm{ad}(E_G)_y$ for every point $y \in X$.

Using ϕ we will construct a reduction of structure group of E_G to a parabolic subgroup of G. For that we will first construct a parabolic subalgebra bundle of $\mathrm{ad}(E_G)$.

Take any point $y \in X$ such that $\phi(y) \neq 0$. Let V_y be the line in $\mathrm{ad}(E_G)_y$ generated by $\phi(y)$. Let $\mathfrak{n}^1_y \subset \mathrm{ad}(E_G)_y$ be the normalizer of V_y and $\mathfrak{r}^1_y \subset \mathfrak{n}^1_y$ be the nilpotent radical.

Now inductively define $\mathfrak{n}^{i+1}_y$ to be the normalizer of $\mathfrak{r}^i_y$ in $\mathrm{ad}(E_G)_y$ and $\mathfrak{r}^{i+1}_y$ to be the nilpotent radical of $\mathfrak{n}^{i+1}_y$.

Let $\mathfrak{n}_y := \lim \mathfrak{n}^i_y$ and $\mathfrak{r}_y := \lim \mathfrak{r}^i_y$ be the limits of these two increasing sequences. From the construction of the two sequences it is obvious that $\mathfrak{n}_y$ is the normalizer, in $\mathrm{ad}(E_G)_y$, of $\mathfrak{r}_y$. Also, $\mathfrak{r}_y$ is the nilpotent radical of $\mathfrak{n}_y$. Therefore, $\mathfrak{n}_y$ is a parabolic subalgebra of $\mathrm{ad}(E_G)_y$. See [4, 30.3] for the details of this construction.

Consider the action of G on itself by inner conjugation. Let $\mathrm{Ad}(E_G) := (E_G \times G)/G$ be the *gauge bundle* (adjoint bundle) constructed using this action. Let $P_y \subset \mathrm{Ad}(E_G)_y$ be the parabolic subgroup of the fiber $\mathrm{Ad}(E_G)_y$ whose Lie algebra coincides with the parabolic subalgebra $\mathfrak{n}_y$ constructed above.

Since there are only finitely many conjugacy classes of parabolic subalgebras of G, there is a nonempty Zariski open subset U of X such that the conjugacy class of $\mathfrak{n}_z$ is independent of $z \in U$. Fix a parabolic subgroup P of G whose Lie algebra is in the same conjugacy class as $\mathfrak{n}_z$, where $z \in U$. To explain this with more details, we observe that the variety of nilpotent elements in $\mathfrak{g}$ is irreducible. Indeed, it is the image of $\mathfrak{g}$ by the Jordan decomposition. The variety of nilpotent elements in $\mathfrak{g}$ is filtered by conjugacy classes. Therefore, on some nonempty Zariski open subset of X, the evaluation of ϕ must lie in some particular stratum of this filtered variety.

Consider the obvious projection

$$q(y) : (E_G)_y \times G \longrightarrow \mathrm{Ad}(E_G)_y. \tag{4.4}$$

Let $(E_P)_y \subset (E_G)_y$ be the subvariety consisting all elements z such that $q(z, g) \in P_z$ for every $g \in P$. It is easy to check that $E_P \subset E_G$ is a reduction of structure group over U of E_G to the parabolic subgroup P of G. Indeed, this is an immediate consequence of the fact that the normalizer of P in G is P itself.

Since G/P is a complete variety and $\dim X = 1$, the reduction over U extends to a reduction of structure group over X of E_G to P. Let $E_P \subset E_G$ denote this reduction of structure group.

Let $\mathrm{ad}(E_P) \subset \mathrm{ad}(E_G)$ be the adjoint bundle. The commutativity of the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{ad}(E_P) & \longrightarrow & \mathrm{At}(E_P) & \longrightarrow & TX & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\| & & \\
0 & \longrightarrow & \mathrm{ad}(E_G) & \longrightarrow & \mathrm{At}(E_G) & \longrightarrow & TX & \longrightarrow & 0
\end{array}$$

ensures that the cohomology class $\tau \in H^1(X,\, K_X \bigotimes \mathrm{ad}(E_G))$ defined in (4.2) lies inside the image of $H^1(X,\, K_X \bigotimes \mathrm{ad}(E_P))$ for the homomorphism defined by the inclusion of $\mathrm{ad}(E_P)$ in $\mathrm{ad}(E_G)$.

Now, since $\phi(y)$ is in the nilpotent radical $\mathfrak{r}_y$ of the parabolic subalgebra $\mathfrak{n}_y$, it follows that

$$\theta(\phi) = 0,$$

where θ is defined in (4.3). Indeed, the subalgebra $\mathfrak{r}_y$ is contained in the annihilator of $\mathfrak{n}_y$ for each $y \in X$. Now, τ defined in (4.2) is in the image of $H^1(X,\, K_X \bigotimes \mathrm{ad}(E_P))$. This immediately implies that for the nondegenerate pairing

$$H^1(X,\, K_X \otimes \mathrm{ad}(E_P)) \otimes H^0(X,\, \mathrm{ad}(E_P)) \longrightarrow k$$

defining the Serre duality, we have $\tau \otimes \phi \longmapsto 0$. In other words, $\theta(\phi) = 0$. This completes the proof of the lemma. $\square$

In view of Lemma 4.1, to complete the proof of Theorem 3.2 it suffices to show that $\theta(\phi) = 0$, where ϕ is a everywhere semisimple section of $\mathrm{ad}(E_G)$. Indeed, by the Jordan decomposition theorem, any section ϕ of $\mathrm{ad}(E_G)$ decomposes uniquely as $\phi_s + \phi_n$, where ϕ_s is everywhere semisimple and ϕ_n is everywhere nilpotent. So, to prove that the $\theta(\phi) = 0$, it is enough to show that $\theta(\phi_s) = 0$ and $\theta(\phi_n) = 0$.

We will show that $\mathrm{ad}(E_G)$ does not admit any nonzero section which is semisimple everywhere. This is the content of the following lemma:

Lemma 4.2. *Let E_G be as in* Theorem 3.2*. Let $\phi \in H^0(X\, \mathrm{ad}\,(E_G))$ be such that $\phi(y)$ is a semisimple vector of* $\mathrm{ad}(E_G)_y$ *for every $y \in X$. Then the section ϕ vanishes identically.*

Proof. Let ϕ be a nonzero section of $\mathrm{ad}(E_G)$ which is semisimple everywhere. Since X is connected and complete, the characteristic polynomial for the adjoint action of $\phi(y)$ on $\mathrm{ad}(E_G)_y$ is independent of y. So ϕ does not vanish at point of X.

We have a decomposition

$$\mathrm{ad}(E_G)_y = \bigoplus_{\lambda\in\Lambda} V_y^{\lambda}, \tag{4.5}$$

where V^{λ} is the eigenspace for the eigenvalue λ for the adjoint action of $\phi(y)$ on $\mathrm{ad}(E_G)_y$. So, V_y^0 coincides with the subalgebra of $\mathrm{ad}(E_G)_y$ that centralizes $\phi(y)$.

Let $\mathcal{V}_y \subset \mathrm{ad}(E_G)_y$ denote the direct sum of all eigenspaces in (4.2) with eigenvalue less than or equal to zero in the lexicographic ordering. Since

$$[V_y^{\lambda_1}, V_y^{\lambda_2}] \subset V_y^{\lambda_1+\lambda_2},$$

unless $[V_y^{\lambda_1}, V_y^{\lambda_2}] = 0$, we have $\mathcal{V}_y$ as a subalgebra of $\mathrm{ad}(E_G)_y$.

Note that the direct sum of all eigenspaces in (4.5) with eigenvalue strictly positive (in the lexicographic ordering) is a nilpotent subalgebra. In other words, $\mathcal{V}_y$ has a complement which is nilpotent. Using [**2**, p. 473, Corollary 4.10], [**3**, p. 747, Lemma 4], it now follows that $\mathcal{V}_y$ is a parabolic subalgebra of $\mathrm{ad}(E_G)_y$ and V_y^0 its Levi factor.

Let $P_y \subset \mathrm{Ad}(E_G)_y$ be the parabolic subgroup with $\mathcal{V}_y$ as its Lie algebra. Let H_y be the Levi factor of P_y whose Lie algebra is V_y^0.

Let $\mathcal{W}_y$ denote the direct sum of all eigenspaces in (4.5) with eigenvalue greater than or equal to zero. Just as before, $\mathcal{W}_y$ is a parabolic subalgebra of $\mathrm{ad}(E_G)_y$ with V_y^0 as its Levi factor.

Let $Q_y \subset \mathrm{Ad}(E_G)_y$ be the parabolic subgroup whose Lie algebra is the direct sum of all eigenspaces in (4.5) with nonnegative eigenvalues. For the same reason as for P_y, the subgroup H_y is a Levi factor of Q_y. Clearly we have $P_y \bigcap Q_y = H_y$.

It is easy to see that the conjugacy classes of P_y and H_y are independent of y. Fix subgroups P, Q and H in G such that some identification of G with $\mathrm{Ad}(E_G)_y$ takes them to P_y, Q_y and H_y respectively. Note that P, Q and H have been fixed independent of y. So $P \bigcap Q = H$ and H is a Levi factor for both P and Q.

As in the Proof of Lemma 4.1, using P_y we have a reduction of structure group $E_P \subset E_G$ to P. More precisely, let $(E_P)_y \subset (E_G)_y$ be the subvariety consists of all z with such that $q(y)(z) \in P_x$ for every $g \in P$, where $q(y)$ is the projection in (4.4). Similarly, we have a reduction of structure group $E_Q \subset E_G$ of E_G to $Q \subset G$. Let

$$E_H := E_P \cap E_Q \subset E_G$$

be the intersection of the two subvarieties E_P and E_Q of E_G. Clearly E_H defines a reduction of structure group of E_G to $P \bigcap Q = H$.

Recall the assumption in Theorem 3.2 that E_G does not admit any reduction to a nontrivial Levi. Therefore, we have $H = G$. This immediately implies that $\phi = 0$. This completes the proof of the lemma. $\square$

Lemma 4.1 and Lemma 4.2 together complete the proof of Theorem 3.2. It was noted in Section 3 that Theorem 3.2 completes the proof of Theorem 2.3. Therefore, the proof of Theorem 2.3 is complete.

If G is a classical group but not isomorphic to $\mathrm{SL}(n)/Z$ for some subgroup Z of the center $Z(\mathrm{SL}(n))$ of $\mathrm{SL}(n)$, then $\mathfrak{g} = \mathfrak{g}^*$ as a G-module if $p > 2$ [**5**, 0.13]. Therefore, Theorem 2.3 remains valid in this case.

We note that for $G = E_6$, the G-module $\mathfrak{g}$ fails to be isomorphic to $\mathfrak{g}^*$ if $p = 3$ [**5**, 0.13, p. 9]. For $G = E_7$, the G-module $\mathfrak{g}$ fails to be isomorphic to $\mathfrak{g}^*$ if $p = 2$ [**5**, 0.13, p. 9]. For classical groups of type B_r and C_r, we have $\mathfrak{g} \neq \mathfrak{g}^*$ if $p = 2$ [**5**, 0.13].

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