CUBIC SINGULAR MODULI, RAMANUJAN’S CLASS INVARIANTS $\lambda_n$ AND THE EXPLICIT SHIMURA RECIPROCITY LAW

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In this paper, we use the explicit Shimura Reciprocity Law to compute the cubic singular moduli $\alpha_n^*$, which are used in the constructions of new rapidly convergent series for $1/\pi$. We also complete a table of values for the class invariant $\lambda_n$ initiated by S. Ramanujan on page 212 of his Lost Notebook.

1. Introduction.

In his famous paper [26], S. Ramanujan offers several beautiful series representations for $1/\pi$, one of which is

\[
\frac{4}{\pi} = \sum_{k=0}^{\infty} \frac{(6k + 1) \left(\frac{1}{2}\right)^3_k}{(k!)^3 4^k},
\]

where $(a)_0 = 1$ and for each positive integer $k$,
\[
(a)_k = (a)(a + 1)(a + 2)\ldots(a + k - 1).
\]

Motivated by Ramanujan’s series, J.M. Borwein and P.B. Borwein [10] obtained many general representations for $1/\pi$. One generalization of (1.1) takes the form

\[
\frac{1}{\pi} = \sum_{k=0}^{\infty} \left\{ \frac{(1/2)_k}{k!} \right\}^3 (a_n + b_n k)(G_n^{12})^{2k},
\]

where $n$ is a positive integer (usually odd) and $a_n$, $b_n$ and $G_n$ are certain special values of modular forms. It turns out that these special values can be expressed in terms of the singular modulus $\alpha_n$, which is defined to be the unique positive number between 0 and 1 satisfying

\[
\frac{2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \alpha_n\right)}{2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha_n\right)} = \sqrt{n}, \quad n \in \mathbb{Q},
\]

where

\[
2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1.
\]
In his Notebooks, S. Ramanujan recorded many values of $\alpha_n$, one of which is

$$\alpha_3 = \frac{2 - \sqrt{3}}{4}.$$  

This value, when substituted into the Borweins’ formula (1.2) yields (1.1). The proofs for all the singular moduli recorded in Ramanujan’s Notebooks can now be found in [9] and [6].

Recently, B.C. Berndt, S. Bhargava and F.G. Garvan [3] succeeded in developing theories of elliptic functions to alternative bases vaguely mentioned by Ramanujan in [26]. As indicated in [3], Ramanujan’s elliptic functions to alternative base 3 turns out to be the most interesting case of his theories. For this particular base, an analogue of the singular modulus, which we shall call “cubic singular modulus”, is defined as the unique positive number $\alpha_n^*$ between 0 and 1 such that

$$\frac{2\,{}_{2}F_{1}(\frac{1}{3}, \frac{2}{3}; 1; 1 - \alpha_n^*)}{2\,{}_{2}F_{1}(\frac{1}{3}, \frac{2}{3}; 1; \alpha_n^*)} = \sqrt{n}, \quad n \in \mathbb{Q}.$$  

Although Ramanujan did not record any cubic singular moduli in his Notebooks or Lost Notebook, he must have computed some of them since these values (see [10]) are essential in his derivations of the series [26]

$$\frac{27}{4\pi} = \sum_{k=0}^{\infty} (2 + 15k) \left( \frac{1}{2} \right)_k \left( \frac{1}{3} \right)_k \left( \frac{2}{3} \right)_k \left( \frac{2}{27} \right)^k$$

and

$$\frac{15\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty} (4 + 33k) \left( \frac{1}{4} \right)_k \left( \frac{1}{7} \right)_k \left( \frac{2}{7} \right)_k \left( \frac{4}{125} \right)^k.$$

The first discussion of the computations of the cubic singular moduli was given by the Borweins [10]. They determined $\alpha_n^*$ for $n = 2, 3, 5$ and 6 from known values of Ramanujan-Weber class invariants $G_{3n}$ and $g_{6n}$, and deduced three new series for $1/\pi$ corresponding to $n = 2, 3$ and 6. Recently, Chan and Liaw [16] succeeded in evaluating $\alpha_n^*$ for $n = 2, 5, 7, 11$, and 23 using cubic Russell-type modular equations. From the values of $\alpha_7^*$ and $\alpha_{11}^*$, they discovered that when $3n$ is an Euler convenient number, $\alpha_n^*$ can be determined using Kronecker’s Limit Formula. Using these new $\alpha_n^*$’s, they derived many new series for $1/\pi$. Their method, however, cannot be extended to include the computations of $\alpha_n^*$ when $3n$ is not convenient.

In Sections 2 and 3, we use an explicit version of the Shimura Reciprocity Law to extend the list of $\alpha_n^*$. We show that when the class group of $\mathbb{Q}(\sqrt{-3n})$
takes the form $\mathbb{Z}_2^t \oplus \mathbb{Z}_k$, with $t \in \mathbb{N}$ and $k = 4, 6$ and 8, $\alpha_n^*$ can be determined explicitly.

On page 212 of his Lost Notebook, Ramanujan defined a certain function $\lambda_n$ (see (3.1)) and recorded its values for $n = 1, 9, 17, 25, 33, 41, 49, 73, 97,$ and 121. He also indicated that he could compute $\lambda_n$ when $n = 57, 65, 81, 89, 169, 193, 217, 241, 265, 289,$ and 361 but did not supply any values for these $n$’s. Using cubic Russell-type modular equations, Kronecker’s Limit Formulas and other techniques, Berndt, Chan, S.-Y. Kang and L.-C. Zhang [7] provided proofs of all these values except for $n = 73, 97, 193, 217,$ and 241. In Section 4, we modify our method in Sections 2 and 3 and determine rigorously these remaining values of $\lambda_n$.

2. Some properties of $\alpha_n^*$.

Let

\begin{equation}
\eta(\tau) := q^{1/24} \prod_{k=1}^{\infty} (1 - q^k), \quad \text{where} \quad q = e^{2\pi i \tau} \quad \text{with} \quad \text{Im} \tau > 0,
\end{equation}

and

\begin{equation}
\mu_n = \frac{1}{3\sqrt{3}} \left\{ \frac{\eta\left(\sqrt{-n/3}\right)}{\eta\left(\sqrt{-3n}\right)} \right\}^6, \quad n \in \mathbb{Q}.
\end{equation}

The relation between $\mu_n$ and the cubic singular moduli $\alpha_n^*$ is given by [17]

\begin{equation}
\frac{1}{\alpha_n^*} = 1 + \mu_n^2.
\end{equation}

Identity (2.3) shows that in order to determine $\alpha_n^*$, it suffices to compute $\mu_n^2$. First, we need the following:

**Theorem 2.1.** Suppose that $n$ is squarefree so that $-12n$ is a fundamental imaginary quadratic discriminant. Then $\mu_n^2$ is a real unit contained in $K_1$, the Hilbert class field of $K := \mathbb{Q}(\sqrt{-3n})$.

To prove Theorem 2.1, we need the following lemmas:

**Lemma 2.2** ([23, p. 159, Corollary]). Let $K$ be as defined in Theorem 2.1, and let $\mathcal{O}_K$ be the ring of integers of $K$. Let $a = [\tau_1, \tau_2]$ be an $\mathcal{O}_K$-ideal and define

\begin{equation}
\Delta(a) := \tau_2^{-12} \eta^{24}(\tau),
\end{equation}

where $\tau = \tau_1/\tau_2$ with $\text{Im} \tau > 0$. Then the value $\Delta(a)/\Delta(\mathcal{O}_K)$ lies in $K_1$, where $K_1$ is the Hilbert class field of $K$. 
Lemma 2.3 ([23, p. 166, Corollary]). Let $N(a)$ denote the index $(\mathfrak{O}_K : a)$ where $a$ is an $\mathfrak{O}_K$-ideal. Then the number

$$N(a)^{1/2} \frac{|\Delta(a)|^2}{|\Delta(\mathfrak{O}_K)|^2}$$

is a unit.

Lemma 2.4. Recall that for $\tau \in \mathbb{C}$, with $\text{Im}\, \tau > 0$, the $j$-function is defined by

$$j(\tau) = 1728 \frac{g_3^2(\tau)}{g_2^4(\tau) - 27g_2^6(\tau)},$$

with $g_2$ and $g_3$ given by

$$g_2(\tau) = 60 \sum_{m,n=-\infty}^{\infty} \frac{1}{(m+n\tau)^4}, \quad \text{and} \quad g_3(\tau) = 140 \sum_{m,n=-\infty}^{\infty} \frac{1}{(m+n\tau)^6}.$$

If

$$g(\tau) := \frac{1}{3\sqrt{3}} \left( \frac{\eta(\tau)}{\eta(3\tau)} \right)^6,$$

then

$$(2.5) \quad j(\tau) = \frac{27(1 + g^2(\tau))(9 + g^2(\tau))^3}{g^6(\tau)}.$$

Lemma 2.4 follows from the fact that $g^2(\tau)$ generates the function field associated with the group $\Gamma_0(3)$, which implies the $j(\tau)$ is a rational function of $g^2(\tau)$. For a more elementary proof of this lemma using Ramanujan’s identities, see [14] and [5].

Proof of Theorem 2.1. Let $a = [3, \sqrt{-3n}]$ with $n \equiv 3 \pmod{4}$. By (2.4),

$$(2.6) \quad \mu_n^4 = 3^{-12} \frac{\eta^{24}(\sqrt{-n/3})}{\eta^{24}(\sqrt{-3n})} = \Delta(a) \frac{\Delta(a)}{\Delta(\mathfrak{O}_K)} = N(a)^6 \frac{|\Delta(a)|}{|\Delta(\mathfrak{O}_K)|}.$$

From the second equality of (2.6) and Lemma 2.2, we find that $\mu_n^4$ belongs to $K_1$ and from the last equality of (2.6) and Lemma 2.3, we conclude that $\mu_n^2$ is a real unit. To complete the proof of Theorem 2.1, it remains to show that $\mu_n^2$ is in $K_1$.

Now, when $\tau = \sqrt{-n/3}$, $g(\tau) = \mu_n$ and

$$\mu_n^8 + 270\mu_n^4 + 3^6 = ((j(\sqrt{-n/3})/27 - 28)\mu_n^4 - 972)\mu_n^2,$$
by Lemma 2.4. Since both \( \mu_n^2 \) and \( j(\sqrt{-n/3}) \) are in \( K_1 \), we can conclude that \( \mu_n^2 \in K_1 \) unless \( (j(\sqrt{-n/3})/27 - 28)\mu_n^4 - 972 = 0 \). If this is the case, we can deduce that \( j(\sqrt{-n/3}) \) satisfies the quadratic equation

\[
X^2 + 8208X - 5832000 = 0.
\]

But the two roots of this equation has numerical values 657.8 and -8865.8. This contradicts the fact that \( j(\sqrt{-m}) \geq 1728 \) for any \( m \geq 1 \). This completes the Proof of Theorem 2.1.

\[\square\]

A class invariant \( \gamma \) of a field \( K \) is defined to be a generator for the Hilbert class field of \( K \), i.e., \( K_1 = K(\gamma) \). Theorem 2.1, Lemma 2.4 and the fact that \( j(\sqrt{-n/3}) \) is a class invariant \([20]\) imply that \( \mu_n^2 \) is a class invariant of \( \mathbb{Q}(\sqrt{-3n}) \) when \( n \equiv 3 \pmod{4} \). Hence, we conclude that \( \alpha_n^* \) is also a class invariant of \( \mathbb{Q}(\sqrt{-3n}) \) by Theorem 2.1 and (2.3).

We remark here that our result given in this section is not “optimal”. We have shown that \( \mu_n^2 \) is a class invariant whenever \( 3 \nmid n \) and \( n \) squarefree. It is possible to show further that smaller powers of the \( \eta \)-quotients given in the definition of \( \mu_n^2 \), namely, \( \frac{\eta(\sqrt{-n/3})^s}{\eta(\sqrt{-3n})} \), with \( s | 12 \) and \( s < 12 \), is a class invariant if we impose further congruence conditions on \( n \). This can be established using Gee’s results \([21, \text{Section 5}]\).

3. The explicit Shimura reciprocity law and new values of \( \alpha_n^* \).

We have seen in Section 2 that \( \mu_n^2 \) is a class invariant whenever \( n \) satisfies the hypothesis of Theorem 2.1. In this section, we identify \( \mu_n^2 \) as a value of a modular function, construct the explicit action of \( \text{Gal}(K_1|K) \) on \( \mu_n^2 \) and as a result, evaluate \( \mu_n^2 \).

Let \( \mathbb{M}_2^+(\mathbb{Z}) \) denote the set of \( 2 \times 2 \) matrices with integer coefficients and positive determinant. For each \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2^+(\mathbb{Z}) \), define the function

\[
\eta \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \eta : \tau \mapsto \frac{\eta(a\tau + b)}{\eta(c\tau + d)}.
\]

It is easy to see that \( \mu_n \) is the value of \( g_0(\tau)^6/(3\sqrt{3}) \) at \( \tau = \sqrt{-3n} \) where

\[
g_0(\tau) := \frac{\eta \circ \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}}{\eta}(\tau).
\]

The function \( g_0(\tau) \) is an element of \( F_{72} \), the modular function field of level 72 defined over \( \mathbb{Q}(\zeta_{72}) \). This means that it is meromorphic on the completed upper half plane \( \mathbb{H} \cup \mathbb{Q} \cup \{\infty\} \), admits a Laurent series expansion in the
variable \( q^{1/72} = e^{2\pi i \tau/72} \) centered at \( q = 0 \) having coefficients in \( \mathbb{Q}(\zeta_{72}) \) and invariant with respect to the matrix group

\[
\Gamma(72) := \ker[SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/72\mathbb{Z})].
\]

From (2.5), we find that the minimal polynomial for \( g_{12} \) over the modular function field \( \mathbb{Q}(j) \) is

\[
X^4 + 36 X^3 + 270 X^2 + (756 - j)X + 36.
\]

Over \( \mathbb{Q}(j) \), the conjugates of \( g_{12} \) are \( g_1^{12}, g_2^{12} \) and \( g_3^{12} \) defined by

\[
g_1 := \zeta_{24}^{-1} \eta \frac{(1 1)}{\eta}, \quad g_2 := \eta \frac{(1 2)}{\eta}, \quad \text{and} \quad g_3 := \sqrt{3} \eta \frac{(3 0)}{\eta}.
\]

If \( K \) is an imaginary quadratic field of discriminant \( D \), class field theory gives an isomorphism

\[
\text{Gal}(K_1/K) \simeq C(D)
\]

between the Galois group for \( K \subset K_1 \) and the form class group of discriminant \( D \). Among the primitive forms \([a, b, c]\) having discriminant \( D = b^2 - 4ac \), one obtains a complete set of representatives in \( C(D) \) by choosing the reduced forms

\[
| b | \leq a \leq c \quad \text{and} \quad b \geq 0 \quad \text{if either} \quad | b | = a \quad \text{or} \quad a = c.
\]

The class of \([a, -b, c]\) is the inverse of \([a, b, c]\) in \( C(D) \), and the elements having order 2 in \( C(D) \) correspond to \textit{ambiguous forms}. These are the reduced forms \([a, b, c]\) for which \( a = b \), \( a = c \) or \( b = 0 \) occurs.

Given \( h \in F_m \), if \( h(\sqrt{D}) \in K_1 \) where \( \theta \) is the generator of \( \mathcal{O}_K \) over \( \mathbb{Z} \) (we assume here the algebraic closure of \( K \) is embedded in the complex plane such that \( \theta \) lies in the upper half plane \( \mathbb{H} \)), there is an explicit formula for computing the action of \( C(D) \) on \( h(\theta) \) which is a consequence of the Shimura Reciprocity Law. This is given as follows:

**Lemma 3.1.** Let \( K \) be an imaginary quadratic number field of discriminant \( D \) and let \( h \in F_m \) be such that \( h(\sqrt{D}) \in K_1 \). Given a primitive quadratic form \([a, b, c]\) of discriminant \( D \), let \( M = M_{[a,b,c]} \in GL_2(\mathbb{Z}/m\mathbb{Z}) \) be the matrix that satisfies the congruences

\[
M \equiv \begin{cases} 
\begin{pmatrix} a & b \\ \frac{a}{p} & 1 \\
\end{pmatrix} & (\text{mod } p^r) \quad \text{if } p \nmid a, \\
\begin{pmatrix} -b & -c \\ 1 & 0 \\
\end{pmatrix} & (\text{mod } p^r) \quad \text{if } p \mid a \text{ and } p \nmid c, \\
\begin{pmatrix} -b - a & -b - c \\ 1 & -1 \\
\end{pmatrix} & (\text{mod } p^r) \quad \text{if } p \mid a \text{ and } p \mid c.
\end{cases}
\]
at all prime power factors \( p^r | m \). The Galois action of the class of \([a, -b, c]\) in \( C(D)\) with respect to the Artin map is given by

\[
\left( h \left( \frac{\sqrt{D}}{2} \right) \right)^{[a,-b,c]} = h^M \left( \frac{-b + \sqrt{D}}{2a} \right),
\]

where \( h^M \) denote the image of \( h \) under the action of \( M \).

For a proof of Lemma 3.1 and the description of the action of \( M \) on \( h \), see [21].

In view of Lemma 3.1, we first need to discuss the action of \( M \in \text{GL}_2(\mathbb{Z}/m\mathbb{Z}) \) on functions \( h \in F_m \). The action of such an \( M \) depends only on \( M_{p^r} \) for all prime factors \( p | m \) where \( M_N \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \) is the reduction modulo \( N \) of \( M \) and \( r_p \) is the largest power of \( p \) such that \( p^{r_p} \) divides \( m \).

Now every \( M_N \) with determinant \( x \) decomposes as

\[
M_N = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_N \begin{pmatrix} a & b \\ c & d \end{pmatrix}_N
\]

for some \( \begin{pmatrix} a & b \\ c & d \end{pmatrix}_N \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \). Since \( \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \) is generated by \( S_N \) and \( T_N \) where \( S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), it suffices to find the action of \( \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_{p^r} \), \( S_{p^r} \) and \( T_{p^r} \) on \( h \) for all \( p | m \).

For \( \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_{p^r} \), the action on \( F_m \) is given by lifting the automorphism of \( \mathbb{Q}(\zeta_m) \) determined by

\[
\zeta_{p^r} \mapsto \zeta_{x^{p^r}} \quad \text{and} \quad \zeta_{q^{r_q}} \mapsto \zeta_{q^{r_q}}
\]

for all prime factors \( q | m \) such that \( q \neq p \).

In order that the actions of the matrices at different primes commute with each other, we have to lift \( S_{p^r} \) and \( T_{p^r} \) to matrices in \( \text{SL}_2(\mathbb{Z}/m\mathbb{Z}) \) such that they reduce to the identity matrix in \( \text{SL}_2(\mathbb{Z}/q^{r_q}\mathbb{Z}) \) for all \( q \neq p \). In our case for \( m = 72 \), the prime powers are 8 and 9 and we have

\[
S_8 \mapsto \begin{pmatrix} -8 & 9 \\ -9 & -8 \end{pmatrix}_{72}, \quad T_8 \mapsto \begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix}_{72},
\]

\[
S_9 \mapsto \begin{pmatrix} 9 & -8 \\ 8 & 9 \end{pmatrix}_{72}, \quad T_9 \mapsto \begin{pmatrix} 1 & -8 \\ 0 & 1 \end{pmatrix}_{72}.
\]

When \( h \in F_m \) is an \( \eta \)-quotient, we can use the transformation rule

\[
\eta \circ S_m(\tau) = \sqrt{-i\tau}\eta(\tau) \quad \text{and} \quad \eta \circ T_m(\tau) = \zeta_{24}\eta(\tau)
\]

to determine the action of any \( M_m \in \text{SL}_2(\mathbb{Z}/m\mathbb{Z}) \). In particular, we have

\[
(\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3) \circ S_{72} = (\mathfrak{g}_3, \zeta_{24}^0 \mathfrak{g}_2, \zeta_{24}^1 \mathfrak{g}_1, \mathfrak{g}_0),
\]
and 

\((g_0, g_1, g_2, g_3) \circ T_{72} = (g_1, \zeta_{24}^2 g_2, g_0, \zeta_{24}^2 g_3)\).

Consequently, we derive the following actions:

<table>
<thead>
<tr>
<th>Action</th>
<th>(g_0^{12} \quad g_1^{12} \quad g_2^{12} \quad g_3^{12})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_8)</td>
<td>(g_0^{12} \quad g_1^{12} \quad g_2^{12} \quad g_3^{12})</td>
</tr>
<tr>
<td>(T_8)</td>
<td>(g_0^{12} \quad g_1^{12} \quad g_2^{12} \quad g_3^{12})</td>
</tr>
<tr>
<td>(S_9)</td>
<td>(g_0^{12} \quad g_1^{12} \quad g_2^{12} \quad g_3^{12})</td>
</tr>
<tr>
<td>(T_9)</td>
<td>(g_0^{12} \quad g_1^{12} \quad g_2^{12} \quad g_3^{12})</td>
</tr>
</tbody>
</table>

Using this, together with Lemma 3.1, we have:

**Theorem 3.2.** The action of a reduced primitive quadratic form \([a, b, c]\) with discriminant \(D\) in \(C(D)\) on \(g_0(\sqrt{2D}/2)^{12}\) is given by

\[
\begin{cases}
  g_0(\sqrt{2D}/2)^{12} & \text{if } b \equiv 0, a \neq 0 \pmod{3}, \\
  g_1(\sqrt{2D}/2)^{12} & \text{if } ab \equiv -1 \pmod{3}, \\
  g_2(\sqrt{2D}/2)^{12} & \text{if } ab \equiv 1 \pmod{3}, \\
  g_3(\sqrt{2D}/2)^{12} & \text{if } a \equiv 0 \pmod{3}.
\end{cases}
\]

**Proof.** The above result follows from the observation that the action of \(M_8\) on \(g_0^{12}\) is trivial. Hence, it suffices to consider the action of \(M_9\) on \(g_0^{12}\). When \(3 \nmid a\),

\[
M_9 = \begin{pmatrix} a & \frac{b-1}{2} \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b-1}{2} \\ 0 & 1 \end{pmatrix} = S_9 \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} S_9 T_{\frac{b-1}{2}}.
\]

When \(3 \mid a\), then \(3 \nmid c\), so

\[
M_9 = \begin{pmatrix} \frac{b-1}{2} & -c \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} c & -\frac{b-1}{2} \\ 0 & 1 \end{pmatrix} S_9 \equiv \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b-1}{2c} \\ 0 & 1 \end{pmatrix} S_9
\equiv S_9 \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} S_9 T_{\frac{b-1}{2c}} S_9.
\]

\(\square\)
Theorem 3.2 should be viewed as a cubic analogue of the results of N. Yui and D. Zagier [28, Proposition, Section 2] and it indicates that all the conjugates of $\mu_n^2$ can be computed numerically once we determine the class group of $\mathbb{Q}(\sqrt{-3n}), n \equiv 3 \pmod{4}$. Using these numerical values, we could then determine the minimal polynomial satisfied by $\mu_n^2$. If the degree of the minimal polynomial is at most 4, we could solve the minimal polynomial and determine $\mu_n^2$ explicitly. In order to calculate $\mu_n^2$ for which the class number of $\mathbb{Q}(\sqrt{-3n})$ is greater than 4, we need the following lemma, which essentially tells us the action of the ambiguous forms (see the remarks before Lemma 3.1 for the definition of ambiguous forms) on $\mu_n^2$.

**Lemma 3.3.** Let $n \equiv 3 \pmod{4}$ and $K = \mathbb{Q}(\sqrt{-3n})$, where $n$ is squarefree. Then

$$\left(\mu_n^2\right)^{[2, 2, \frac{3n+1}{2}]} = -\lambda_n^2,$$

where

$$\lambda_n = \frac{1}{3\sqrt{3}} \left\{ \frac{\eta(1+\sqrt{-n/3})}{\eta(1+\sqrt{-3n})} \right\}^6. \tag{3.1}$$

If $n = p_1p_2\ldots p_k$ then

$$\left(\mu_n^2\right)^{[p_1p_2\ldots p_j, 0, \frac{3n}{p_1p_2\ldots p_j}]} = \mu_{n/(p_1p_2\ldots p_j)^2},$$

where $j \leq k$.

**Proof.** We apply Theorem 3.2 with $ab \equiv 1 \pmod{3}$ and $b \equiv 0 \pmod{3}$, respectively and note that

$$\lambda_n^2 = -\frac{1}{27} \mathfrak{g}_1^2 \left( \frac{1 + \sqrt{-3n}}{2} \right)$$

and

$$\mu_{n/(p_1p_2\ldots p_j)^2} = \frac{1}{27} \mathfrak{g}_0^2 \left( \sqrt{\frac{3n}{(p_1p_2\ldots p_j)^2}} \right).$$

We can now explicitly determine $\mu_n^2$ by first collecting in a symmetric way the products of the real conjugates of $\mu_n^2$.

For example, when $n = 23$, $C(-276) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_4$, generated by $a = [2, 2, 35]$ and $b = [5, 2, 14]$. Now define

$$P_{23} := (\mu_{23}\lambda_{23})^2 + (\mu_{23}\lambda_{23})^{-2} \quad \text{and} \quad Q_{23} := (\mu_{23}/\lambda_{23})^2 + (\mu_{23}/\lambda_{23})^{-2}.$$

These numbers are fixed by the Galois action of $a^2$ and $b$ and since $P_{23}$ and $Q_{23}$ are algebraic integers, one concludes that $P_{23} + P_{23}^a$, $P_{23}P_{23}^a$, $Q_{23} + Q_{23}^a$, and $Q_{23}Q_{23}^a$ are integers. These integers can be found by approximating
the numerical values of the $g_i^{12}$ at the corresponding arguments given by Theorem 3.2. Hence, we obtain

\[ P_{23} + P_{23}^a = 296143772, \]
\[ P_{23}P_{23}^a = -389054012, \]
\[ Q_{23} + Q_{23}^a = 5980, \]
\[ Q_{23}Q_{23}^a = -17852. \]

Solving the quadratic polynomials satisfied by $P_{23}$ and $Q_{23}$ and simplifying, we deduce that

\[ \mu_{23}^2 = (5\sqrt{3} + 24)^{1/2}(13\sqrt{23} + 36\sqrt{3})^{1/2} \left( \sqrt{84 + 48\sqrt{3} + \sqrt{83 + 48\sqrt{3}}} \right)^3. \]

Substituting the value $\mu_{23}^2$ into (2.3), we easily determine $\alpha_{23}^*$, which is crucial in the derivation of the following series.

\[ \frac{1}{\pi} = \sum_{m=0}^{\infty} (a_{23} + b_{23}m) \left( \frac{1}{2} \right)_m \left( \frac{1}{3} \right)_m \left( \frac{2}{3} \right)_m \frac{m}{(m!)^3} H_{23}^m, \]
\[ z_{23} = \frac{1}{23} \left( \sqrt{-83 + 48\sqrt{3}(444 + 252\sqrt{3}) - 56 + 54\sqrt{3}} \right), \]
\[ a_{23} = -\frac{1}{6\sqrt{3}} \left( z_{23} + (8\alpha_{23}^* - 4)\sqrt{23} \right), \]
\[ b_{23} = \frac{2\sqrt{23} \mu_{23}^2 - 1}{\sqrt{3} \mu_{23}^2 + 1}, \quad \text{and} \]
\[ H_{23} = \frac{1}{24^2\sqrt{23}^3} \left( 6\sqrt{-83 + 48\sqrt{3} + 9\sqrt{3}\sqrt{-83 + 48\sqrt{3} - 2 - 3\sqrt{3}}} \right)^3. \]

For methods of deriving series of the above type, and the relation between $\mu_n^2$ and series for $1/\pi$, see [17] and [18].

**Remarks.**

(a) The method illustrated above for the case $n = 23$ works for any $n$ such that $C(-12n)$ is of the type $Z_s \oplus Z_{2s}$, where $s = 1, 2, 3$, or 4.

(b) If $C(-12n)$ is of the type $Z_s^t \oplus Z_{2s}$ with $s = 1, 2, 3$ or 4 and $t \in \mathbb{N}$, we need to construct more numbers analogous to $P_n$ and $Q_n$. Examples of such constructions can be found in [15] and Section 4.

(c) One can modify the method in [15] to evaluate the corresponding $\mu_n^2$ whenever the class group is of the form $Z_s^t \oplus Z_{2s}$, where $s = 2, 3$, or 4. The method used there avoids the use of the explicit Shimura Reciprocity Law but it cannot be extended to compute $\mu_n^2$ when the associated class groups are different from those mentioned above.
(d) Gee and M. Honsbeek [22] have recently devised a method of computing class invariants without solving their minimal polynomials. Their method involves determining the Lagrange resolvents of these minimal polynomials by determining the conjugates of the corresponding class invariants explicitly.

4. The class invariant $\lambda_n^2$ and the missing entries in the Lost Notebook.

We first note that

$$\lambda_n^2 = -\frac{1}{27} g_2^{12} \left( \frac{-1 + \sqrt{-3n}}{2} \right).$$

To compute $\lambda_n$, it suffices to determine the action of the elements in the corresponding class groups. This is given by the following analogue of Theorem 3.2:

**Theorem 4.1.** The action of a reduced primitive quadratic form $[a, b, c]$ with discriminant $D$ in $C(D)$ on $g_2 \left( \frac{-1 + \sqrt{D}}{2} \right)^{12}$ is given by

$$\begin{cases} g_2 \left( \frac{-1 + \sqrt{D}}{2} \right)^{12} & \text{if } b \equiv 0, a \not\equiv 0 \pmod{3}, \\
\frac{g_0(-b+\sqrt{D})^{12}}{2a} & \text{if } ab \equiv -1 \pmod{3}, \\
\frac{g_1(-b+\sqrt{D})^{12}}{2a} & \text{if } ab \equiv 1 \pmod{3}, \\
\frac{g_2(-b+\sqrt{D})^{12}}{2a} & \text{if } a \equiv 0 \pmod{3}. \\
\end{cases}$$

To facilitate the computations of $\lambda_n$ we need the analogue of Lemma 3.3.

**Lemma 4.2.** Let $n \equiv 1 \pmod{4}$ and $K = \mathbb{Q}(\sqrt{-3n})$, where $n$ is squarefree. If $n = p_1 p_2 \ldots p_k$ then

$$\lambda_n^2 \left[ p_1 p_2 \ldots p_j, p_1 p_2 \ldots p_j, \frac{3n + (p_1 p_2 \ldots p_j)^2}{(p_1 p_2 \ldots p_j)^2} \right] = \lambda_n^2 / (p_1 p_2 \ldots p_j)^2,$$

where $j \leq k$.

Lemma 4.2 indicates that instead of calculating the conjugates of $\lambda_n^2$ directly, it suffices to calculate the conjugates of symmetric combinations of all the real conjugates of $\lambda_n^2$.

We may now proceed to complete the table of $\lambda_n$ initiated by Ramanujan on page 212 of his Lost Notebook. For $p = 73, 97, \text{ and } 241$, all of which are primes, set

$$(4.1) \quad P_p = \lambda_p^2 + \frac{1}{\lambda_p^2}.$$

Since the class groups corresponding to these $p$'s are of the form $\mathbb{Z}_4$, we conclude that $P_p$ each satisfies a quadratic polynomial. We now derive the polynomial satisfied by $P_{73}$. 

Now the class group of $\mathbb{Q}(\sqrt{-219})$ is generated by the form $[5, 1, 11]$. By Theorem 4.1, we easily deduce that

$$P_{73} + P_{73}^{[5,1,11]} = 199044,$$

and

$$P_{73}P_{73}^{[5,1,11]} = 287491,$$

where $P_{73}^{[5,1,11]}$ denotes the image of $P_{73}$ under the action of $[5, 1, 11]$. Hence, $P_{73}$ satisfies the quadratic polynomial

$$x^2 - 199044x + 287491 = 0.$$

Solving and simplifying, we deduce that

$$\lambda_{73} = \left(\sqrt{11 + \sqrt{73}}\right)^6.$$

The cases for $n = 97$ and 241 are similar.

We now turn to the case $n = 217$. Here 217 is divisible by two primes, namely, 7 and 31. In this case we consider two numbers $Q_{217}$ and $R_{217}$ defined by

$$Q_{217} = \lambda_{217}^2 \lambda_{31}^2 / 8 + 1 / \lambda_{217} \lambda_{31}^2 / 8$$

and

$$R_{217} = \lambda_{217}^2 \lambda_{31} / \lambda_{217}.$$

Note that the class group of $\mathbb{Q}(\sqrt{-651})$ is generated by $a := [5, 3, 33]$ and $b := [3, 3, 55]$. The order of $a$ is 4 and the group generated by $a^2$ and $b$ fixes $Q_{217}$ and $R_{217}$. Hence it suffices to determine the action of $a$ on $Q_{217}$ and $R_{217}$, which can be easily done by Theorem 4.1. The value of $\lambda_{217}$ which results from this consideration is a product of two units, given by

$$\lambda_{217} = \left(\sqrt{\frac{1901 + 129\sqrt{217}}{8}} + \sqrt{\frac{1893 + 129\sqrt{217}}{8}}\right)^{3/2} \cdot \left(\sqrt{\frac{1597 + 108\sqrt{217}}{4}} + \sqrt{\frac{1593 + 108\sqrt{217}}{4}}\right)^{3/2}.$$

Finally, consider the case $n = 193$. This is the case which we cannot evaluate using the previous method given in [15]. Here the class group of $\mathbb{Q}(\sqrt{-579})$ is generated by $a := [5, 1, 29]$ and it is of order 8. We consider the
expression $P_{193}$ where $P_p$ is given by (4.1). To determine $P_{193}$ we compute the image of $P_{193}$ under $a, a^2,$ and $a^3$. Our computations show that if

$$\alpha := P_{193},$$

$$\beta := P_{193}^a = -\frac{1}{27} g_2^{12} \left( \frac{1 + \sqrt{-579}}{10} \right) - 27 g_2^{-12} \left( \frac{1 + \sqrt{-579}}{10} \right),$$

$$\gamma := P_{193}^{a^2} = -\frac{1}{27} g_0^{12} \left( \frac{3 + \sqrt{-579}}{14} \right) - 27 g_0^{-12} \left( \frac{3 + \sqrt{-579}}{14} \right),$$

and

$$\delta := P_{193}^{a^3} = -\frac{1}{27} g_0^{12} \left( \frac{-9 + \sqrt{-579}}{22} \right) - 27 g_0^{-12} \left( \frac{-9 + \sqrt{-579}}{22} \right),$$

then

$$\alpha + \beta + \gamma + \delta = 3251132424,$$

$$\alpha \beta + \alpha \gamma + \alpha \delta + \beta \gamma + \beta \delta + \gamma \delta = 82707128352,$$

$$\alpha \beta \gamma + \alpha \beta \delta + \beta \gamma \delta + \alpha \gamma \delta = 9465475096,$$

and

$$\alpha \beta \gamma \delta = 176664526832.$$

Solving the quartic polynomial satisfied by $P_{193}$ and simplifying, we deduce that

$$\lambda_{193}^{1/3} + \frac{1}{\lambda_{193}^{1/3}} = \frac{1}{4} \left( \frac{39 + 3\sqrt{193} + \sqrt{2690 + 194\sqrt{193}}}{3} \right).$$

It was not clear to us what motivated Ramanujan to construct the table of values for $\lambda_n$. Perhaps he intended to set up a table for $\lambda_n$ similar to that for the Ramanujan-Weber class invariants $G_n$ and $g_{2n}$ (see [26]). Recently, Chan, Liaw and Tan offered another reason for the existence of Ramanujan’s table. They succeeded in deriving a new class of series for $1/\pi$ associated with $\lambda_n$. Two of such series are

$$\frac{4}{\pi \sqrt{3}} = \sum_{k=0}^{\infty} \left( 5k + 1 \right) \frac{\left( \frac{1}{3} \right)_k \left( \frac{2}{3} \right)_k \left( \frac{1}{2} \right)_k}{(k!)^3} \left( \frac{-9}{16} \right)^k,$$

and

$$\frac{12 \sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \left( 51k + 7 \right) \frac{\left( \frac{1}{3} \right)_k \left( \frac{2}{3} \right)_k \left( \frac{1}{2} \right)_k}{(k!)^3} \left( \frac{-1}{16} \right)^k.$$

These simple series came as a surprise as it was thought that all the possible simple series should have been exhausted after the work of Ramanujan, the Chudnovskys [19] and the Borweins.
References


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