ORBIFOLD HODGE NUMBERS OF CALABI–YAU HYPERSURFACES

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We identify the twisted sectors of a compact simplicial toric variety. We do the same for a generic nondegenerate Calabi-Yau hypersurface of an \( n \)-dimensional simplicial Fano toric variety and then explicitly compute \( h_{orb}^{1,1} \) and \( h_{orb}^{n-2,1} \) for the hypersurface. We give applications to the orbifold string theory conjecture and orbifold mirror symmetry.

1. Introduction.

The \( K\)-Orbifold string theory conjecture states that there is a natural isomorphism between the Orbifold \( K\)-theory of a Gorenstein orbifold and the ordinary \( K\)-theory of its crepant resolution (see [AR], [Ru]). To construct a natural isomorphism as the conjecture demands, is a very hard problem. But weaker versions of the conjecture that compare Euler numbers, Hodge numbers, etc. have been studied extensively in the literature in the case of orbifolds that are global-quotients. Batyrev [B2], and Batyrev and Dais [BD] proved, in particular, the equality of orbifold Hodge numbers and the Hodge numbers of smooth crepant resolutions for Gorenstein global-quotient orbifolds. But there were no results for nonglobal-quotient orbifolds.

In this paper, we show that the orbifold Hodge numbers of a generic Calabi-Yau hypersurface in a complex 4-dimensional simplicial Fano toric variety coincide with the Hodge numbers of its smooth crepant resolution. Besides being the first nonglobal-quotient example, this is also an important example in mirror symmetry. An immediate corollary of this is the pairing of orbifold Hodge numbers of Calabi-Yau 3-folds and their Batyrev mirrors.

While this paper was being refereed, extensive generalisations and related results were reported in [BoM], [P] and [Y]. [BoM] and [P] use the characterisation of twisted sectors presented here. [Y] uses the theory of algebraic stacks and achieves a deep result. The survey article [Re] nicely explains the heart of the matter.

Now we briefly describe how this article is organised. In Sections 2 and 3 we review relevant facts from orbifold cohomology and toric geometry respectively. In Section 4 we find characterisations for the twisted sectors of complete simplicial toric varieties and nondegenerate Calabi-Yau hypersurfaces of simplicial Fano toric varieties. In Section 5 we compute formulas for
some orbifold Hodge numbers of these hypersurfaces, state some corollaries
and then give an example.

2. Orbifolds.

2.1. Orbifold structure. Let $U$ be a connected topological space, $V$ a
connected $n$-dimensional smooth manifold and $G$ a finite group acting smoothly
on $V$. Then an $n$-dimensional uniformising system of $U$ is a triple $(V, G, \pi)$,
where $\pi: V \to U$ is a continuous map inducing a homeomorphism between
the quotient space $V/G$ and $U$. Two uniformising systems $(V_i, G_i, \pi_i)$, $i = 1, 2$, are isomorphic if there is a diffeomorphism $\phi: V_1 \to V_2$ and an isomor-
phism $\lambda: G_1 \to G_2$ such that $\phi$ is $\lambda$-equivariant, and $\pi_2 \circ \phi = \pi_1$.

If $(\phi, \lambda)$ is an automorphism of $(V, G, \pi)$, then there is a $g \in G$ such that
$\phi(x) = g.x$ and $\lambda(a) = gag^{-1}$, for any $x \in V$ and $a \in G$.

Let $i : U' \to U$ be a connected open subset of $U$. An uniformising
system $(V', G', \pi')$ of $U'$ is said to be induced from $(V, G, \pi)$ if there is a
monomorphism $\lambda: G' \to G$ and a $\lambda$-equivariant open embedding $\phi: V' \to
V$ such that $i \circ \pi' = \pi \circ \phi$. The pair $(\phi, \lambda): (V', G', \pi') \to (V, G, \pi)$ is
called an injection. Two uniformising systems $(V_1, G_1, \pi_1)$ and $(V_2, G_2, \pi_2)$
of neighbourhoods $U_1$ and $U_2$ of a point $p$ are equivalent at $p$ if they induce
isomorphic uniformising systems for a neighbourhood $U_3$ of $p$.

Let $X$ be a Hausdorff, second countable topological space. An $n$-dimen-
sional orbifold structure on $X$ is given by the following data: For every point
$p \in X$, there is an assigned neighbourhood $U_p$ of $p$ and an $n$-dimensional
uniformising system $(V_p, G_p, \pi_p)$ of $U_p$. The assignment satisfies the condi-
tion that for any point $q \in U_p$, $(V_p, G_p, \pi_p)$ and $(V_q, G_q, \pi_q)$ are equivalent
at $q$.

Two orbifold structures $\{(V_p, G_p, \pi_p) : p \in X\}$ and $\{(V'_p, G'_p, \pi'_p) : p \in X\}$
are equivalent if for any $p \in X$, $(V_p, G_p, \pi_p)$ and $(V'_p, G'_p, \pi'_p)$ are equivalent
at $p$. With a given equivalence class of orbifold structures on it, $X$ is called
an orbifold.

We call each $U_p$ a uniformised neighbourhood of $p$, and $(V_p, G_p, \pi_p)$ a chart
at $p$. In fact we choose $U_p$ to be small enough that $G_p$ has the minimum
possible order; that is, every element of $G_p$ fixes the preimage of $p$ in $V_p$.
In what follows, this choice is assumed. Then a point $p$ is called smooth if
$G_p$ is trivial; otherwise, it is called singular. $X$ is called a global-quotient
orbifold if $X$ itself is an uniformised open set.

An orbifold $X$ is called reduced if $G_p$ acts effectively on $V_p$. Furthermore
if a group element acts nontrivially, we require the fixed-point set to be of at
least (real) codimension two, so that the complement is locally connected.
We will deal with reduced orbifolds only. Note that even a reduced non-
smooth orbifold can have a smooth underlying variety because of examples
with complex reflections. Gorenstein orbifolds do not present this problem as they do not admit such complex reflections.

2.2. Orbifold (Chen-Ruan) cohomology. First we will describe the so-called twisted sectors. Consider the set of pairs:

\[ \tilde{X} = \{(p, (g)_{G_p}) | p \in X, g \in G_p \}, \]

where \((g)_{G_p}\) denotes the conjugacy class of \(g\) in \(G_p\). Then Kawasaki showed (see [CR]) that \(\tilde{X}\) has a natural orbifold structure. We will describe the connected components of \(\tilde{X}\). Recall that each point \(p\) has a local chart \((V_p, G_p, \pi_p)\) which gives a local uniformised neighbourhood \(U_p = \pi_p(V_p)\). If \(q \in U_p\), up to conjugation, there is an injective homomorphism \(G_q \rightarrow G_p\). For \(g \in G_q\), the conjugacy class \((g)_{G_p}\) is well-defined. We define an equivalence relation \((g)_{G_q} \sim (g)_{G_p}\). Let \(T\) denote the set of equivalence classes. By an abuse of notation, we use \((g)\) to denote the equivalence class to which \((g)_{G_q}\) belongs. \(\tilde{X}\) is decomposed as a disjoint union of connected components

\[ \tilde{X} = \bigsqcup_{(g) \in T} X_{(g)}, \]

where

\[ X_{(g)} = \{(p, (g')_{G_p}) | g' \in G_p, (g')_{G_p} \in (g)\}. \]

**Definition 1.** \(X_{(g)}\) is called a **twisted sector** if \(g \neq 1\). We call \(X_{(1)} = X\) the **nontwisted sector**.

Assume that \(X\) is an almost complex orbifold with an almost complex structure \(J\) (see [CR]). Then for a singular point \(p\), \(J\) gives rise to an effective representation \(\rho_p : G_p \rightarrow GL(n, \mathbb{C})\). For any \(g \in G_p\) we write \(\rho_p(g)\), up to conjugation, as a diagonal matrix \(\text{diag}(e^{2\pi i m_{1,g} / m_g}, \ldots, e^{2\pi i m_{n,g} / m_g})\), where \(m_g\) is the order of \(g\) in \(G_p\), and \(0 \leq m_{i,g} < m_g\). Define a function \(\iota : \tilde{X} \rightarrow \mathbb{Q}\) by

\[ \iota(p, (g)_{G_p}) = \sum_{i=1}^{n} \frac{m_{i,g}}{m_g}. \]

This function \(\iota : \tilde{X} \rightarrow \mathbb{Q}\) is locally constant. Denote its value on \(X_{(g)}\) by \(\iota_{(g)}\). \(\iota_{(g)}\) is called the **degree shifting number** of \(X_{(g)}\). It has the following properties:

1. \(\iota_{(g)}\) is integral iff \(\rho_p(g) \in SL(n, \mathbb{C})\).
2. \(\iota_{(g)} + \iota_{(g^{-1})} = \text{rank}(\rho_p(g) - I) = n - \dim(X_{(g)})\).

**Definition 2.** An almost complex orbifold is called **Gorenstein** if \(\iota_{(g)}\) is integral for all \((g)\).
Remark. An almost complex, complex or Kähler structure on $X$ induces a corresponding similar structure on each $X_{(g)}$.

**Definition 3.** Let $F$ be any field containing $\mathbb{Q}$ as a subfield. We define the orbifold cohomology groups of $X$ with coefficients in $F$ by

$$H^d_{\text{orb}}(X; F) = \oplus_{(g) \in T} H^{d - 2\iota(g)}(X_{(g)}; F).$$

**Definition 4.** Let $X$ be a closed complex orbifold. We define, for $0 \leq p, q \leq \dim \mathbb{C} X$, orbifold Dolbeault cohomology groups

$$H^{p, q}_{\text{orb}}(X; \mathbb{C}) = \oplus_{(g)} H^{p - \iota(g), q - \iota(g)}(X_{(g)}; \mathbb{C}).$$

**Remark.** When $X$ is a closed Kähler orbifold (so is each $X_{(g)}$), these Dolbeault groups are related to the singular cohomology groups of $X$ and $X_{(g)}$ as in the manifold case, and the Hodge decomposition theorem holds for these cohomology groups.

**Definition 5.** We define orbifold Hodge numbers by

$$h^{p, q}_{\text{orb}}(X) = \dim H^{p, q}_{\text{orb}}(X; \mathbb{C}).$$

3. Facts from toric geometry.

3.1. Orbits, divisors and polytopes. A complex $n$-dimensional toric variety $X_\Xi$ is constructed from an $n$-dimensional lattice $N$ and a fan $\Xi$ in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. We will write $X$ for $X_\Xi$ when there is no confusion. Let $M = \text{Hom}(N, \mathbb{Z})$ denote the dual lattice, with dual pairing denoted by $\langle , \rangle$.

If $\sigma$ is a cone in $N$, the dual cone $\check{\sigma}$ in $M_{\mathbb{R}}$ determines a finitely generated commutative semigroup $R_\sigma = \check{\sigma} \cap M$. $\mathbb{C}[R_\sigma]$ is the $\mathbb{C}$-algebra with generators $\chi^m$ for each $m \in R_\sigma$ and relations $\chi^m \chi^{m'} = \chi^{m + m'}$. It gives an open affine subset $U_\sigma := \text{spec}(\mathbb{C}[R_\sigma])$ of $X$. A face $\tau$ of $\sigma$ gives an inclusion $U_\tau \rightarrow U_\sigma$.

$\Xi(d)$ denotes the set of $d$-dimensional cones of $\Xi$. We reserve the letter $\eta$ to denote elements of $\Xi(1)$. For each $\eta$, let $v_\eta$ denote the unique generator of the semigroup $\eta \cap N$. The $v_\eta \in \sigma$ are the generators of $\sigma$. If $r = |\Xi(1)|$ is the number of 1-dimensional cones, we sometimes write the $v_\eta$’s as $v_1, \ldots, v_r$.

$X$ is nonsingular iff for every cone in $\Xi$, its generators are part of a $\mathbb{Z}$-basis of $N$. Such a fan is called smooth. $X$ is an orbifold iff the generators for every cone in $\Xi$ are linearly independent over $\mathbb{R}$; and we say $X$ and $\Xi$ are simplicial.

The action of the torus $T_N = U_{\{0\}} = N \otimes \mathbb{C}$ on $X$ has exactly one orbit $O_\tau$ corresponding to each cone $\tau \in \Xi$. Each $O_\eta$ is an irreducible $T_N$-invariant Weil divisor denoted $D_\eta$. If $X$ is complete, these generate the Chow group $A_{n-1}(X)$. Two $T_N$-invariant Weil divisors are linearly equivalent iff they differ by $\text{div}(\chi^m) = \sum_\eta \langle m, v_\eta \rangle D_\eta$ for some $m \in M$. A Weil divisor $D = \sum_\eta a_\eta D_\eta$ is Cartier iff for each $\sigma \in \Xi$, there is $m_\sigma \in M$ such that
\[ \langle m_\sigma, v_\eta \rangle = -a_\eta \text{ whenever } \eta \subset \sigma. \]  
A Cartier divisor \( D \) is ample iff \( \langle m_\sigma, v_\eta \rangle > -a_\eta \) whenever \( \eta \) is not in \( \sigma \) and \( \sigma \) is \( n \)-dimensional.

If \( X \) is complete and \( D = \sum a_\eta D_\eta \) is Cartier, then \( \Delta_D = \{ m \in M_\mathbb{R} : \langle m, v_\eta \rangle \geq -a_\eta \forall \eta \} \) is a polytope. A polytope is called integral if its vertices are integral. \( \Delta_D \) is integral if \( D \) is ample. Conversely, given any \( n \)-dimensional integral polytope \( \Delta \) one can canonically associate a projective toric variety \( \mathbb{P}_\Delta \) to it. See [CK], Section 3.2.2 for details. It comes with a specific choice of ample divisor \( D_\Delta \) such that \( \Delta_D = \Delta \). The \( T_N \) orbit closures of \( \mathbb{P}_\Delta \) are in one-to-one correspondence with the nonempty faces \( F \) of \( \Delta \). There is a natural inclusion of toric varieties \( \mathbb{P}_F \hookrightarrow \mathbb{P}_\Delta \).

Choose a basis for \( M \). This corresponds to picking coordinates \( t_1, \ldots, t_n \) for the torus \( T_N \). Then, if \( m \in M \) is written \( m = (a_1, \ldots, a_n) \), we have \( \chi^m = \prod_{i=1}^n t_i^{a_i} \), so we can write \( t^m \) instead of \( \chi^m \). For any \( k \geq 0 \), we have the space of Laurent polynomials \( L(k\Delta) = \{ f : f = \sum_{m \in k\Delta \cap M} \lambda_m t^m, \lambda_m \in \mathbb{C} \} \). Each \( f \in L(k\Delta) \) gives the affine hypersurface \( Z_f \subset T_N \) defined by \( f = 0 \). There is a \( T_N \)-equivariant map \( H^0(X, \mathcal{O}(D)) \cong \bigoplus_{m \in k\Delta \cap M} \mathbb{C} \chi^m \). Under this isomorphism, \( f \) corresponds to an effective divisor \( Z_f \subset \mathbb{P}_\Delta \). \( Z_f \) is a hypersurface. It is a compactification of \( Z_f \) for generic \( f \).

We will use the following notation:

(a) \( l(k\Delta) = \dim(L(k\Delta)) \),

(b) \( l^*(k\Delta) = \{ m \in k\Delta \cap M : m \text{ is not in any facet of } k\Delta \cap M \} \).

### 3.2. Homogeneous coordinate ring.

We introduce a variable \( x_\eta \) for each \( \eta \in \Xi(1) \) and consider the polynomial ring \( S = \mathbb{C}[x_\eta : \eta \in \Xi(1)] \). A monomial in \( S \) is written \( x^D = \prod_{\eta} x_\eta^{a_\eta} \), where \( D = \sum a_\eta D_\eta \) is an effective torus-invariant divisor on \( X \). We say that \( x^D \) has degree \( \deg(x^D) = [D] \in A_{n-1}(X) \). Thus, \( S \) is graded by \( A_{n-1}(X) \). Given a divisor class \( \alpha \in A_{n-1}(X) \), \( S_\alpha \) denotes the graded piece of \( S \) of degree \( \alpha \). We often write the variables as \( x_1, \ldots, x_r \), where \( x_i \) corresponds to the cone generator \( v_i \) and \( r = |\Xi(1)| \). The ring \( S \), together with the grading defined above is called the homogeneous coordinate ring of \( X \). See [CK], Chapter 3.2 for more details.

If \( \tau \) is any cone of \( \Xi \) then the orbit closure \( \overline{O}_\tau \) is given by the ideal \( (x_i : v_i) \) is a generator of \( \tau \) of \( S \). Also the graded pieces of \( S \) have nice cohomological interpretation. We noted that \( L(\Delta) \cong H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D_\Delta)) \). Now the map sending the Laurent monomial \( t^m \) to \( \prod_{\eta} x_\eta^{\langle m, v_\eta \rangle + \alpha_\eta} \) induces an isomorphism \( H^0(X, \mathcal{O}_X(D)) \cong S_\alpha \), where \( \alpha = [D] \in A_{n-1}(X) \).

### 3.3. Fano toric varieties.

For any toric variety \( X \), the anticanonical divisor \( -K_X = \sum D_\eta \). A complete toric variety \( X \) is called Fano if \( -K_X \) is Cartier and ample.

The anticanonical divisor of a Fano toric variety \( X \) determines a reflexive polytope \( \Delta \). An integral polytope is called reflexive if:
(a) All facets \( \Gamma \) of \( \Delta \) are supported by an affine hyperplane of the form
\[ \{ m \in M_{\mathbb{R}} : \langle m, v_{\Gamma} \rangle = -1 \} \] for some \( v_{\Gamma} \in N \).

(b) \( \text{Int}(\Delta) \cap M = \{ 0 \} \).

The polar polytope \( \Delta^\circ \) of the reflexive polytope \( \Delta \) is obtained by
\[ \Delta^\circ = \{ v \in N_{\mathbb{R}} : \langle m, v \rangle \geq -1 \forall m \in \Delta \} \subset N_{\mathbb{R}}. \]

The fan \( \Xi \) of \( X \) can be retrieved by coning over the proper faces of \( \Delta^\circ \). This fan is called the normal fan of \( \Delta \) and \( X = P_{\Delta^\circ}. \)

4. Twisted sectors.

We claim that the twisted sectors of a toric variety or a Calabi-Yau hypersurface, up to reduction of orbifold structure, can be identified with subvarieties. Note that in general a twisted sector could be a multiple cover of the corresponding singular locus even if the group actions are all Abelian.

4.1. Twisted sectors in simplicial toric variety. Let \( \Xi \) be any complete simplicial fan. Then the orbifold structure of the toric variety \( X_\Xi \) can be described as follows. Let \( \sigma \) be any \( n \)-dimensional cone of \( \Xi \). Let \( v_1, \ldots, v_n \) be the generators of \( \sigma \). These are linearly independent in \( N_{\mathbb{R}} \). Let \( N_\sigma \) be the sublattice of \( N \) generated by \( v_1, \ldots, v_n \). Let \( G_\sigma := N/N_\sigma \) be the quotient group. \( G_\sigma \) is finite and abelian.

Let \( \sigma' \) be the cone \( \sigma \) regarded in \( N_\sigma \). Let \( \sigma' \) be the dual lattice of \( N_\sigma \). \( U_{\sigma'} = \text{spec}(C[\sigma' \cap M_\sigma]) \). Note that \( \sigma' \) is a smooth cone in \( N_\sigma \). So \( U_{\sigma'} \cong \mathbb{C}^n \).

There is a canonical dual pairing \( M_\sigma/M \times N/N_\sigma \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{C}^* \), the first map by the pairing \( \langle , \rangle \) and the second by \( q \mapsto \exp(2\pi iq) \). Now \( G_\sigma \) acts on
\( \mathbb{C}[M_{\sigma}] \), the group ring of \( M_{\sigma} \), by \( v(\chi^u) = \exp(2\pi i(u, v))\chi^u \), for \( v \in N \) and \( u \in M_{\sigma} \). Note that

\[
(4.1.1) \quad (\mathbb{C}[M_{\sigma}])^{G_{\sigma}} = \mathbb{C}[M].
\]

Thus \( G_{\sigma} \) acts on \( U_{\sigma'} \). Let \( \pi_{\sigma} \) be the quotient map. Then \( U_{\sigma} = U_{\sigma'}/G_{\sigma} \). So \( U_{\sigma} \) is uniformised by \( (U_{\sigma'}, G_{\sigma}, \pi_{\sigma}) \). For any \( \tau < \sigma \), the orbifold structure on \( U_{\tau} \) is same as the one induced from the uniformising system on \( U_{\sigma} \). Then in the absence of complex reflections, toric gluing implies that \( \{ (U_{\sigma'}, G_{\sigma}, \pi_{\sigma}) : \sigma \in \Xi(n) \} \) defines a reduced orbifold structure on \( X \). We show this in the general case.

Let \( B \) be the nonsingular matrix with generators \( v_1, \ldots, v_n \) of \( \sigma \) as rows. Then \( \sigma' \) is generated in \( M_{\sigma} \) by the column vectors \( v_1, \ldots, v_n \) of the matrix \( B^{-1} \). So \( \chi^{v_1}, \ldots, \chi^{v_n} \) are the coordinates of \( U_{\sigma'} \). For any \( \kappa = (k_1, \ldots, k_n) \in N \), the corresponding coset \([\kappa] \in G_{\sigma} \) acts on \( U_{\sigma'} \) in these coordinates as a diagonal matrix: \( \text{diag}(e^{2\pi ik_1}, \ldots, e^{2\pi ik_n}) \) where \( c_i = (\kappa, v^i) \). Such a matrix is uniquely represented by an \( n \)-tuple \( a = (a_1, \ldots, a_n) \) where \( a_i \in \{0, 1\} \) and \( c_i = a_i + b_i, b_i \in \mathbb{Z} \). In matrix notation, \( \kappa B^{-1} = a + b \iff \kappa = aB + bB \). We denote the integral vector \( aB \) in \( N \) by \( \kappa_a \) and the diagonal matrix corresponding to \( a \) by \( g_a \). \( \kappa_a \mapsto g_a \) gives a one to one correspondence between the elements of \( G_{\sigma} \) and the integral vectors in \( N \) that are linear combinations of the generators of \( \sigma \) with coefficients in \( \{0, 1\} \).

Now let us examine the orbifold chart induced by \( (U_{\sigma'}, G_{\sigma}, \pi_{\sigma}) \) at any point \( x \in U_{\sigma} \). By the orbit decomposition of \( U_{\sigma} \), there is a unique face \( \tau \) of \( \sigma \) such that \( x \in O_{\tau} \). Without loss of generality assume that \( \tau \) is generated by \( v_1, \ldots, v_j \). Then any preimage of \( x \) with respect to \( \pi_{\sigma} \) has coordinates \( \chi^{v_i} = 0 \) iff \( i \leq j \). Let \( z = (0, \ldots, 0, z_{j+1}, \ldots, z_n) \) be one such preimage. Let \( G_{\tau} := \{ g_a \in G_{\sigma} : a_i = 0 \text{ if } j + 1 \leq i \leq n \} \). We can find a small neighbourhood \( W \subset (\mathbb{C}^*)^{n-j} \) of \( (z_{j+1}, \ldots, z_n) \) such that the inclusions \( \mathbb{C}^j \times W \hookrightarrow U_{\sigma'} \) and \( G_{\tau} \hookrightarrow G_{\sigma} \) induces an injection of uniformising systems \( (\mathbb{C}^j \times W, G_{\tau}, \pi_{\tau}) \hookrightarrow (U_{\sigma'}, G_{\sigma}, \pi_{\sigma}) \) on some small open neighbourhood \( U_{\delta} \) of \( x \). So we have \( G_x = G_{\tau} \) and an orbifold chart \( (\mathbb{C}^j \times W, G_{\tau}, \pi_{\tau}) \). Note that \( G_{\tau} \) can be constructed from the set \( \{ \kappa_a = \sum_{i=1}^{j} a_i v_i : \kappa_a \in N, a_i \in \{0, 1\} \} \) which is completely determined by \( \tau \) and hence is independent of \( \sigma \).

Now we determine the twisted sectors. Take any \( x \in X \). \( x \) belongs to a unique \( O_{\tau} \). Assume the generators of \( \tau \) are \( v_1, \ldots, v_j \). Consider any \( n \)-dimensional \( \sigma > \tau \). Assume \( v_1, \ldots, v_n \) generate \( \sigma \). First suppose there is a \( g_a \) in \( G_x \) such that \( a_i \neq 0 \), \( \forall i \leq j \). Assume any \( \kappa_a \) lies in the interior of \( \tau \). We want to find the twisted sector \( X_{(g_a)} \). Consider \( g_a \) as an element of \( G_{\sigma} \). It is clear that \( g_a \) fixes \( z \in U_{\sigma'} \) iff \( z_1 = \cdots = z_{j+s} = 0 \), for some \( s \geq 0 \). Hence \( \pi_{\sigma}(z) \in O_{\tau} \) or \( \pi_{\sigma}(z) \in O_{\delta} \) for some \( \delta > \tau \). So \( X_{(g_a)} \cap U_{\sigma} = O_{\tau} \cap U_{\sigma} \). Since a twisted sector is connected, \( X_{(g_a)} = O_{\tau} \). If \( g_a \in G_x \) is such that \( (\text{without loss of generality}) \) only \( a_1, \ldots, a_k \neq 0 \), \( k < j \), then \( g_a \in G_{\delta} \) where \( \delta \) is the
cone generated by \(v_1, \ldots, v_k\) and by the above argument \(X_{(g_a)} = \overline{O}_a\). Thus we have proved the following theorem:

**Theorem 1.** A twisted sector of any complete simplicial toric variety \(X_\Xi\) is isomorphic to a subvariety \(\overline{O}_\tau\) of \(X_\Xi\) for some cone \(\tau \in \Xi\). Moreover, there is a one-to-one correspondence between the set of twisted sectors of the type \(\overline{O}_\tau\) and the set of integral vectors in the interior of \(\tau\) which are linear combinations of the 1-dimensional generators of \(\tau\) with coefficients in \((0,1)\).

Note that the degree shifting number \(\nu_{(g_a)} = \sum a_i\). Now if \(X_\Xi\) is Fano, i.e., \(\Xi\) is obtained by coning over the faces of a reflexive polytope \(\Delta^0\), then the twisted sectors with \(\nu = 1\) are in one to one correspondence with the integral interior points of faces of \(\Delta^0\).

### 4.2. Twisted sectors of a hypersurface of a Fano variety.

We identify the twisted sectors of a generic nondegenerate anticanonical (Calabi-Yau) hypersurface \(V\) of a simplicial Fano toric variety \(X = \mathbb{P}_\Delta\). Nondegenerate means that \(V \cap O_\tau\) is either empty or a smooth subvariety of codimension one in \(O_\tau\), for each torus orbit \(O_\tau\) in \(X\). Then \(V\) turns out to be a suborbifold of \(X\). Also nondegeneracy is a generic condition. We show that \(V = \widetilde{F}_f\), for a generic \(f \in L(\Delta)\), is nondegenerate and a suborbifold of \(X\). For a different treatment of this, see [BC].

Consider any \(n\)-dimensional cone \(\sigma\) with generators \(v_1, \ldots, v_n\). For notational simplicity set \(\chi^v_i = z_i\). Then \(z_1, \ldots, z_n\) are the coordinates of \(U_{\sigma'}\). Let \(Y\) be the preimage of \(V \cap U_{\sigma'}\) in \(U_{\sigma'}\) with respect to \(\pi_{\sigma'}\). Then \(Y\) is defined by the equation \(\sum_{m \in \Delta \cap M} \lambda_m \prod_{i=1}^n z_i^{(m,v_i)+1} = 0\). This is because, \(t^m = \prod_{i=1}^n z_i^{q_i} \iff m = \sum q_i v_i = B^{-1} q \iff q = m B \iff q_i = (m,v_i)\). The one is added to ensure that \(V\) is anticanonical. Note that by definition of \(\Delta\), \((m,v_i) + 1 \geq 0\). If \(\lambda_{m_0} \neq 0\) then \(Y\) does not pass through the origin. It can be checked from this description using Bertini’s theorem that for generic values of the coefficients \(\lambda_m, Y\) is a smooth submanifold of \(U_{\sigma'}\) that intersects the coordinate planes \(z_i = \cdots = z_j = 0\) transversely.

\(Y\) is \(G_{\sigma}\)-stable by (4.1.1). When \(Y\) is smooth, all singularities of \(V \cap U_{\sigma}\) are quotient singularities induced by action of \(G_{\sigma}\) on \(Y\). Since there are only finitely many \(n\)-dimensional cones, \(V\) is nondegenerate and a suborbifold of \(X\). \((Y, G_{\sigma}, \pi_{\sigma})\) is an uniformising system for \(V \cap U_{\sigma}\). Let \(\tau\) be the face of \(\sigma\) obtained by coning over the face \(\widetilde{F}^0\) of \(\Delta^0\). Without loss of generality let \(v_1, \ldots, v_j : j \geq 2\) be the generators of \(\tau\). (We remarked in Section 3.3 that there is no codimension one singularity.) We want to find a chart for any point \(x \in V \cap O_\tau\). By our earlier remark that \(Y\) misses the origin of \(U_{\sigma'}\), \(V \cap O_\tau\) is empty. So we need only consider proper faces \(\tau\) of \(\sigma\). First assume that \(\widetilde{F}^0\) has codimension 2. This means that \(\overline{O}_\tau\) has dimension 1. Then the corollary on page 112 of [F] implies that the number of points in \(V \cap \overline{O}_\tau\) is the normalised volume of \(\widetilde{F}^0\), which equals \(l^*(\widetilde{F}^0) + 1\) since \(\widetilde{F}^0\)
has dimension 1. Since the only other points in \( \overline{O_\tau} \) in this case are \( O_\sigma \) for \( n \)-dimensional cones \( \sigma > \tau \), all the intersection points actually lie in \( O_\tau \). If codimension \( F^o \) is bigger than 2, then \( V \cap \overline{O_\tau} \) is irreducible by Bertini.

Following the same notation as before, \( x \) has a small neighbourhood \( U_x \cap Y \) such that \( (\mathbb{C}^l \times W) \cap Y, G_\tau, \pi \) is a chart for \( V \) at \( x \). \( \mathbb{C}^l \times W \) is, as before, a suitable neighbourhood of some preimage \( z \) of \( x \) in \( U_{\sigma'} \). The tangent space \( TY_z \) is a \( G_\tau \)-stable subspace of \( T\mathbb{C}^n \). Any \( g_\alpha \in G_\tau \) acts trivially on \( TW_z = \text{span}\{\partial/\partial z_i, i = j + 1, \ldots, n\} \). By transversality, we can choose basis \( \{\xi_1, \ldots, \xi_n\} \) of \( T\mathbb{C}^n \) such that \( \xi_i \in TY_z \) \( \forall i \leq n - 1 \) and \( \xi_n \in TW_z \). \( g_\alpha \) acts trivially on \( \xi_n \). This implies that the degree shifting number of \( g_\alpha|_{TY_z} \) is still \( \sum_{i=1}^l a_i \).

From the description of the charts, it is clear that twisted sectors of \( V \) are isomorphic to \( V \cap \overline{O_\tau} \) where \( 2 \leq \dim(\tau) \leq n - 1 \). Recall that \( \overline{O_\tau} = \mathbb{P}_{F^o} \) where \( F^o \) is the face of \( \Delta \) dual to \( F^o \). In particular we have the following theorem:

**Theorem 2.** Let \( V \) be a generic nondegenerate anticanonical hypersurface of an \( n \)-dimensional simplicial Fano toric variety \( \mathbb{P}_\Delta \). Then the twisted sectors of \( V \) are isomorphic to \( V \cap \overline{O_\tau} \) for some face \( F^o \) of \( \Delta \) such that \( 1 \leq \dim(F^o) \leq n - 2 \). There is exactly one twisted sector of this type having \( i = 1 \), corresponding to each integral interior point of \( F^o \) if \( \dim(F^o) < n - 2 \). If \( \dim(F^o) = n - 2 \), then there are exactly \( l^*(\tilde{F^o}) + 1 \) twisted sectors of this type having \( i = 1 \), corresponding to each integral interior point of \( F^o \).

### 5. Orbifold Hodge numbers.

#### 5.1. \( h_{\text{orb}}^{1,1}(V) \)

Let \( V(g) \) denote a twisted sector of the hypersurface \( V \) and \( \nu(g) \) its degree shifting number. \( h_{\text{orb}}^{1,1}(V) = h^{1,1}(V) + \sum_{\nu(g) = 1} h^{0,0}(V(g)) \). Since \( h^{0,0}(V(g)) = 1 \) for each twisted sector, by Theorem 2 we obtain

\[
\sum_{\nu(g) = 1} h^{0,0}(V(g)) = \sum_{1 \leq \dim(F^o) \leq n - 2} l^*(F^o) + \sum_{\dim(F^o) = n - 2} l^*(F^o) l^*(\tilde{F^o})
\]

\[
= l(\Delta^o) - r - 1 - \sum_{\dim(F^o) = n - 1} l^*(F^o) + \sum_{\dim(F^o) = n - 2} l^*(F^o) l^*(\tilde{F^o}).
\]

To compute \( h^{1,1}(V) \) we invoke the following Lefschetz hyperplane theorem ([BC, Proposition 10.8]):

**Lemma 1.** Let \( V \) be a nondegenerate ample hypersurface of an \( n \)-dimensional complete simplicial toric variety \( X \). Then the natural map induced by inclusion \( j^*: H^i(X) \to H^i(V) \), is an isomorphism for \( i < n - 1 \) and an injection for \( i = n - 1 \).
In our case $V$ is anticanonical, and since the anticanonical divisor of a Fano variety is ample, $V$ is ample. Also it is well-known ([F, Section 5.1]) that for any simplicial toric variety $X$, $H^2(X, \mathbb{R}) = H^{1,1}(X, \mathbb{R}) = A_{n-1}(X) \otimes \mathbb{R}$. So for $n \geq 4$, $h^{1,1}(V) = h^{1,1}(\mathbb{P}_\Delta) = \text{rank}A_{n-1}(\mathbb{P}_\Delta) = r - n$. Thus we have the following theorem:

**Theorem 3.** For any generic nondegenerate anticanonical hypersurface $V$ of an $n$-dimensional simplicial Fano toric variety $\mathbb{P}_\Delta$, $n \geq 4$, 

$$h^{1,1}_{\text{orb}}(V) = l(\Delta^o) - n - 1 - \sum_{\dim(F^o) = n-1} l^*(F^o) + \sum_{\dim(F^o) = n-2} l^*(F^o)l^*(\widetilde{F^o}).$$

### 5.2. $h^{n-2,1}(V)$. Next we compute $h^{n-2,1}(V)$ for $n \geq 4$. For this we use the homogeneous coordinate ring $S$ of $X = \mathbb{P}_\Delta$. Let $v_1, \ldots, v_r$ be the one-dimensional cones of the normal fan of $\Delta$. Let $x_1, \ldots, x_r$ be the corresponding homogeneous coordinates. Let $\beta_0 = [-K_X] = [\sum_{i=1}^r D_i] \in A_{n-1}(X)$. Then $S_{\beta_0} \simeq L(\Delta)$. And the divisor $f \in L(\Delta)$ corresponding to $V$ can be written in the homogeneous coordinates as $f = \sum_{m \in M \cap \Delta} \lambda_m \prod_{i=1}^r x_i^{(m, v_i)+1}$. For notational simplicity, we will denote $\prod_{i=1}^r x_i^{(m, v_i)+1}$ by $x^m$ for any $m \in M$. Define the Jacobian ideal of $f$ to be $J(f) = \langle \partial f / \partial x_1, \ldots, \partial f / \partial x_r \rangle$.

First we quote the following theorem ([BC, Theorem 10.13]):

**Lemma 2.** Let $X$ be an $d$-dimensional complete simplicial toric variety and $V \subset X$ be a quasi-smooth (i.e., suborbifold) ample hypersurface defined by $f \in S_{\beta}$. Then for $k \neq (d/2) + 1$, there exists a canonical isomorphism 

$$(S/J(f))_{k\beta-\beta_0} \simeq PH^{d-k,k-1}(V).$$

**Remark.** The primitive cohomology $PH^{d-1}(V) := H^{d-1}(V)/(\text{im}H^{d-1}(X))$. This coincides with the usual cohomology if $d$ is even.

For our application of Lemma 2, set $d = n$, $k = 2$ and $\beta = \beta_0$. With $n \geq 4$ these choices imply that $k \neq (d/2) + 1$ in the lemma. Also, $H^{n-2,1}(X) = 0$ if $n \geq 4$, so that $PH^{n-2,1}(V) = H^{n-2,1}(V)$. Thus $h^{n-2,1}(V) = \text{rank}(S/J(f))_{\beta_0}$.

**Lemma 3.** $x_i \partial f / \partial x_i \in J(f)_{\beta_0}, i = 1, \ldots, r$, and the space of complex linear relations among these has dimension $r - (n+1)$.

**Proof.**

$$x_i \frac{\partial f}{\partial x_i} = \sum_{m} \lambda_m (\langle m, v_i \rangle + 1)x^m,$$

$$\sum_{i} c_i x_i \frac{\partial f}{\partial x_i} = \sum_{m} \lambda_m \left( \langle m, \sum_{i} c_i v_i \rangle + \sum_{i} c_i \right) x^m.$$

For a generic $f$ we can assume that $\lambda_m \neq 0$ for each $m \in \Delta \cap M$. Hence $\sum_{i} c_i x_i \partial f / \partial x_i \equiv 0 \iff \langle m, \sum_{i} c_i v_i \rangle + \sum_{i} c_i = 0 \forall m \in \Delta \cap M$. 


In particular, taking $m = 0$ we get $\sum_i c_i = 0$. Therefore $\langle m, \sum_i c_i v_i \rangle = 0 \forall m \in \Delta \cap M$ and since $\Delta$ is $n$-dimensional we have $\sum_i c_i v_i = 0$.

Thus $\sum_i c_i x_i \partial f / \partial x_i = 0 \iff \sum_i c_i v_i = 0$ and $\sum_i c_i = 0$.

Now let $\overline{v}_i = (v_i, 1) \in \mathbb{R}^{n+1} \subset \mathbb{C}^{n+1} = \mathbb{R}^{n+1} \otimes \mathbb{C}$.

Since the $v_i$'s are vertices of $\Delta$, the $\overline{v}_i$'s are generators of the $(n + 1)$-dimensional cone $\{ q \in \mathbb{R}^{n+1} : qt \in \Delta \times \{1\} \mbox{ for some } t \in \mathbb{R}_{>0}\}$. So the $\overline{v}_i$'s span $\mathbb{R}^{n+1}$ over $\mathbb{R}$, and hence they span $\mathbb{R}^{n+1} \otimes \mathbb{C}$ over $\mathbb{C}$. Note that $(\sum c_i v_i, \sum c_i) = \sum c_i \overline{v}_i$. Hence the lemma follows. \qed

Without loss of generality assume that the $x_k \partial f / \partial x_k, k = 1, \ldots, n + 1$, are linearly independent. In other words $\overline{v}_k, k = 1, \ldots, n + 1$ are linearly independent. We will consider monomials $\prod_{j \neq i} x_j^{p_j}$ that have same degree in $S$ as $x_i$. So we want $m^* \in M$ such that $\langle m^*, v_j \rangle = p_j \geq 0 > 1$ if $j \neq i$ and $\langle m^*, v_i \rangle = -1$. Such $m^*$ is given by interior lattice points of $F_i$, the $(n - 1)$-dimensional face of $\Delta$ that is dual to the 0-dimensional face $\{v_i\}$ of $\Delta$. Then for each $m^* \in \text{Int}(F_i) \cap M$, $\prod_{j \neq i} x_j^{\langle m^*, v_i \rangle} \partial f / \partial x_i$ belongs to $J(f)_{\overline{v}_0}$. Together with the $x_i \partial f / \partial x_i$, these generate $J(f)_{\overline{v}_0}$ as we vary over all $i$.

In the following computation, we denote the characteristic function of a set by $I(\cdot)$. For instance, $I(m' \in \Delta)$ is 1 when $m' \in \Delta$ and 0 otherwise. Also, recall that $f = \sum_{m' \in M \cap \Delta} \lambda_{m'} x^{m'}$.

$$\begin{align*}
\left( \prod_{j \neq i} x_j^{\langle m^*, v_j \rangle} \right) \partial f / \partial x_i \\
= \left( \prod_{j=1}^{r} x_j^{\langle m^*, v_j \rangle} \right) x_i \partial f / \partial x_i \\
= \sum_{m' \in \Delta \cap M} \lambda_{m'} (\langle m', v_i \rangle + 1) x^{m' + m^*} \\
= \sum_{m' + m^* \in \Delta \cap M} \lambda_{m'} (\langle m', v_i \rangle + 1) I(m' \in \Delta) x^{m' + m^*} \\
= \sum_{m \in \Delta \cap M} \lambda_{m - m^*} (\langle m - m^*, v_i \rangle + 1) I(m - m^* \in \Delta) x^{m}.
\end{align*}$$

To justify the fourth line in the above calculation, note that given $m' \in \Delta \cap M$, either $m' + m^* \in \Delta \cap M$ or $\langle m', v_i \rangle + 1 = 0$. Then setting $m = m' + m^*$ leads to the last line.

Let $\text{Int}(F_i) \cap M = \{ m_{i,s} : 1 \leq s \leq t_i; t_i \geq 0 \}$. Then $J(f)_{\overline{v}_0} = \text{span}\{ x_k \partial f / \partial x_k, \prod_{j=1}^{r} x_j^{\langle m_{i,s}, v_j \rangle} x_i \partial f / \partial x_i : 1 \leq k \leq n+1, 1 \leq i_s \leq t_i, i = 1, \ldots, r \}$. We want to find the dimension of this complex vector
space. So we study the space of linear relations:

\[
\sum_k c_k x_k \frac{\partial f}{\partial x_k} + \sum_{i, is} d_{i, is} \prod_{j=1}^r x_j^{(m_{i, is}, v_i)} x_i \frac{\partial f}{\partial x_i} \equiv 0
\]

\[\iff\]

\[
\sum_m \left\{ \sum_k c_k \lambda_m (\langle v_k, m \rangle + 1) + \sum_{i, is} d_{i, is} \lambda_{m-i, is} I(m - m_{i, is}, v_i) + 1 \right\} x^m \equiv 0
\]

\[\iff\]

\[
\sum_k c_k \lambda_m (\langle v_k, m \rangle + 1) + \sum_{i, is} d_{i, is} \lambda_{m-i, is} I(m - m_{i, is}, v_i) + 1 \equiv 0
\]

for each \( m \in \Delta \cap M \), [note: \( \langle m_{i, is}, v_i \rangle = -1 \)].

This is a system of \( l(\Delta) \) number of linear equations in \( \gamma = n + 1 + \sum_{i=1}^r l^r(F_i) \) variables namely \( c_k, d_{i, is} \). Note that \( l(\Delta) \geq \gamma \). We shall find a nonsingular subsystem of rank \( \gamma \).

To do so pick \( n \) linearly independent vertices \( m_1, \ldots, m_n \) of \( \Delta \) and let \( m_{n+1} = 0 \), the origin. Then from the above system we pick the equations corresponding to \( m = m_1, \ldots, m_{n+1} \) and \( m = m_{i, is} : i = 1, \ldots, r; 0 \leq i_s \leq t_i \). Denote this \( \gamma \times \gamma \) system by \((*)\). It can be written as:

\[
\begin{bmatrix}
P & A \\
B & Q
end{bmatrix}
\begin{bmatrix}
c \\
d
end{bmatrix}
= \begin{bmatrix}
0 \\
0
end{bmatrix}
\]

where

\[
P = \begin{bmatrix}
\lambda_{m_1}(\langle m_1, v_1 \rangle + 1) & \ldots & \lambda_{m_1}(\langle m_1, v_{n+1} \rangle + 1) \\
\vdots & \ddots & \vdots \\
\lambda_{m_n}(\langle m_n, v_1 \rangle + 1) & \ldots & \lambda_{m_n}(\langle m_n, v_{n+1} \rangle + 1)
end{bmatrix}
\]

\[
Q = \begin{bmatrix}
\lambda_{m_{1,1}-m_{1,1}} I(\langle m_{1,1}, m_{1,1}, v_1 \rangle + 1) & \ldots & \lambda_{m_{1,1}-m_r t_r} I(\langle m_{1,1}, m_r t_r, v_r \rangle + 1) \\
\vdots & \ddots & \vdots \\
\lambda_{m_{1,1}-m_{1,1}} I(\langle m_{r t_r}, m_{1,1}, v_1 \rangle + 1) & \ldots & \lambda_{m_{1,1}-m_r t_r} I(\langle m_{r t_r}, m_r t_r, v_r \rangle + 1)
end{bmatrix}
\]

Observe that all the diagonal entries of \( Q \) are \( \lambda_0 \), and none of its off-diagonal entries has \( \lambda_0 \). Also any entry of \( A \) is of the form \( \lambda_{m-k, i_s} I(\cdot) \) and hence does not involve \( \lambda_0 \). Similarly an entry of \( B \) is of the form \( \lambda_{m_{i, is}} (\langle m_{i, is}, v_k \rangle + 1) \) and so does not have \( \lambda_0 \).

Consider the determinant of the coefficient matrix \( \begin{bmatrix}
P & A \\
B & Q
end{bmatrix} \) as a polynomial in the \( \lambda \)'s. Then the term of this determinant having the highest power of \( \lambda_0 \) is \( (\lambda_0)^{\sum_r l^r(F_i)} \det P \). We will show below that \( \det P = \) nonzero

constant times \( \lambda_1 \ldots \lambda_n \lambda_0 \). Thus the determinant of the coefficient matrix of the system \((**)\) is a nontrivial polynomial in the \( \lambda \)'s and is therefore nonzero for generic choice of the \( \lambda \)'s. Hence \( J(f) \gamma_0 \) has rank \( \gamma \) as a complex vector space, for a generic \( f \in L(\Delta) \). Since \( S_{\gamma_0} \cong L(\Delta) \), so \((S/J(f)) \gamma_0 \) has rank \( l(\Delta) - \gamma \), for a generic \( f \).

**Lemma 4.** The \((n + 1) \times (n + 1)\) matrix \( P = ((P_{i,j} = \lambda_{m_i} \langle m_i, v_j \rangle + 1)) \) is nonsingular for generic choice of \( \lambda \)'s.

**Proof.** Let \( E \) be the \((n + 1) \times (n + 1)\) matrix \(( (E_{i,j} = \langle m_i, v_j \rangle + 1)) \). Then \( \det E = \lambda_{m_1} \ldots \lambda_{m_{n+1}} \det E \). We claim that \( E \) is nonsingular. Otherwise there exists a nontrivial vector \((c_1, \ldots, c_{n+1})\) such that \( \sum_{k=1}^{n+1} c_k \langle m_i, v_k \rangle + 1 = 0 \) for all \( i = 1, \ldots, n+1 \). In particular, for \( i = n+1 \) we get \( \sum_{k=1}^{n+1} c_k = 0 \). This implies \( \sum_{k=1}^{n+1} c_k \langle m_i, v_k \rangle = 0 \) for all \( i = 1, \ldots, n \). Since \( m_1, \ldots, m_n \) are linearly independent, this would imply that \( \sum_{k=1}^{n+1} c_k \langle m, v_k \rangle = 0 \) for all \( m \in \Delta \). Therefore \( \sum_{k=1}^{n+1} c_k v_k = 0 \). This combined with \( \sum c_k = 0 \) implies that \( \sum_{k=1}^{n+1} c_k \tilde{v}_k = 0 \) which contradicts the linear independence of \( \tilde{v}_1, \ldots, \tilde{v}_{n+1} \). Thus the lemma holds.

So we have the following theorem:

**Theorem 4.** For any generic nondegenerate anticanonical hypersurface \( V \) of an \( n \)-dimensional simplicial Fano toric variety \( \mathbb{P}_\Delta \), and \( n \geq 4 \),

\[
h^{n-2,1}(V) = l(\Delta) - n - 1 - \sum_{\text{dim}(F) = n-1} l'(F).
\]

### 5.3. Cohomology of the twisted sectors of \( V \)

We now want to compute \( h^{n-3,0}(V(g)) \) for any twisted sector \( V(g) \cong V \cap \mathbb{P}_{\mathcal{F}_0} \). This is obviously zero if \( \dim(F^0) > 1 \), since \( \dim(\mathbb{P}_{\mathcal{F}_0}) = n - 1 - \dim(F^0) \). So we will only consider the case \( \dim(F^0) = 1 \). Let \( \tau \) be the 2-dimensional cone obtained by coning over \( F^0 \). As noted earlier \( \overline{O}_\tau = \mathbb{P}_{\mathcal{F}_0} \). The restriction of \( V \) to \( \overline{O}_\tau \) gives a quasi-smooth ample hypersurface of \( \overline{O}_\tau \), which we shall identify with \( V(g) \). So we are again in a situation where we can invoke Lemma 2.

For this we need to understand the homogeneous coordinate ring \( S' \) of \( \overline{O}_\tau \). According to Fulton [F], Section 3.1, a fan for \( \overline{O}_\tau \) can be constructed from the fan \( \Xi \) of \( X \) as follows.

Let \( N_\tau \) be the sublattice of \( N \) generated by the primitive one dimensional generators of \( \tau \). Let \( N(\tau) = N/N_\tau \). The dual lattice of \( N(\tau) \) is given by \( M(\tau) = \tau^* \cap M \). The star of a cone \( \tau \) can be defined abstractly as the set of cones \( \sigma \) in \( \Xi \) that contain \( \tau \) as a face. Such cones \( \sigma \) are determined by their images in \( N(\tau) \) i.e., by \( \overline{\sigma} = (\sigma + (N_\tau))_\mathbb{R} / (N_\tau) \subset N_\mathbb{R} / (N_\tau) = N(\tau)_\mathbb{R} \). These cones \( \{ \overline{\sigma} : \tau < \sigma \} \) form a fan \( \text{Star}(\tau) \) in \( N(\tau) \). \( \overline{O}_\tau \) is the toric variety corresponding to this fan. Without loss of generality, let \( v_1, v_2 \) be the generators of \( \tau \). The corresponding Weil divisors in \( X \) are \( D_1 \) and \( D_2 \).
Assume that \( v_j, j = 3, \ldots, l \) are the 1-dimensional cones of \( \Xi \) such that \( \{v_1, v_2, v_j\} \) generate a 3-dimensional cone of \( \Xi \). In other words, \( \overline{v}_j, j = 3, \ldots, l \) are the 1-dimensional cones of \( \text{Star}(\tau) \). Let \( \widetilde{D}_j := D_1D_2D_j \) for \( j = 3, \ldots, r \). Note that \( \widetilde{D}_j = 0 \) if \( j > l \). The divisor of \( \overline{O}_\tau \) corresponding to \( \overline{v}_j \) is \( \widetilde{D}_j \) for \( j = 3, \ldots, l \). So the homogeneous coordinate ring \( S' \) is generated by variables \( y_j \) corresponding to \( \widetilde{D}_j \) for \( j = 3, \ldots, l \). Denote by \( \alpha_0 \) the anticanonical class \( \sum_{j=3}^l \widetilde{D}_j \) in \( S' \).

On the other hand the divisor \( V \) restricts to \( -K_XD_1D_2 = (D_1 + \cdots + D_r)D_1D_2 = \sum_{j=3}^l \widetilde{D}_j + (D_1 + D_2)D_1D_2 \). To see what \( (D_1 + D_2)D_1D_2 \) is in terms of the \( \widetilde{D}_j \), we can pick a point \( m \in \widehat{F}_0 \cap M \) and let \( b_i := \langle m, v_i \rangle, 1 \leq i \leq r \). Then \( \sum_{i=1}^r b_iD_i \) is linearly equivalent to zero. Note that \( b_1 = b_2 = -1 \).

Hence we have \( D_1 + D_2 = \sum_{i=3}^l b_iD_i \). So, \( (D_1 + D_2)D_1D_2 = \sum_{i=3}^l b_iD_i \). Let \( \alpha \) be the class in \( S' \) representing the effective ample divisor \(-K_XD_1D_2 \). Let \( f' \) be the associated homogeneous polynomial in the \( y_j \)s. Now we can apply Lemma 2 to the \((n-2)\)-dimensional variety \( \overline{O}_\tau \) and the ample hypersurface \( V(g) \). Choose \( k = 1 \) in the lemma to get

\[
(S'/J(f'))_{\alpha-\alpha_0} \simeq PH^{n-3,0}(V(g)) = H^{n-3,0}(V(g))
\]

since \( H^{n-3,0}(\overline{O}_\tau) = 0 \).

Now \( \alpha - \alpha_0 = [\alpha - \sum_{j=3}^l \widetilde{D}_j] \). A typical generator \( \partial f'/\partial y_i \in J(f') \) has degree \( [\alpha - \widetilde{D}_j] \). There are no nonconstant regular functions on the projective variety \( \overline{O}_\tau \). So any nontrivial effective divisor, and in particular \( \sum_{j=3}^l b_j \widetilde{D}_j \), is not linearly equivalent to zero. This implies that \( J(f')_{\alpha-\alpha_0} = 0 \). Hence we obtain that

\[
(S'/J(f'))_{\alpha-\alpha_0} \simeq H^{n-3,0}(V(g)).
\]

Now \( \alpha - \alpha_0 = [\sum_{j=3}^l b_j \widetilde{D}_j] \). We want to identify the effective divisors in this class. So we want \( m_* \in M(\tau) \) such that \( \sum_{j=3}^l (b_j + \langle m_*, \overline{v}_j \rangle) \overline{D}_j \) is effective. This is if and only if \( \langle b_j + \langle m_*, \overline{v}_j \rangle \rangle \geq 0 \) for all \( j = 3, \ldots, l \) \( \iff \langle (m + m_*, v_j) \rangle \geq 0 \iff m + m_* \in \text{Int} (\widehat{F}_0) \cap M \).

To justify the last step note that \( \langle m + m_*, v_i \rangle = -1 \) for \( i = 1, 2 \).

Since \( m \) is fixed, the required effective divisors are in one-to-one correspondence with the interior lattice points of \( \widehat{F}_0 \). Hence \( H^{n-3,0}(V(g)) = l^*(\widehat{F}_0) \). Since there are \( l^*(\widehat{F}_0) \) twisted sectors isomorphic to \( V \cap \mathbb{P}^{\widehat{F}_0} \) we have the following:

\[
h_{\text{orb}}^{n-2,1}(V) = h_{\text{orb}}^{n-2,1}(V) + \sum_{i(g)=1} h^{n-3,0}(V(g))
\]
\[
= l(\Delta) - n - 1 - \sum_{\dim(F) = n-1} l^* (F) + \sum_{\dim(F^\circ) = 1} l^* (F^\circ) l^* (\hat{F}^\circ)
\]

\[
= l(\Delta) - n - 1 - \sum_{\dim(F) = n-1} l^* (F) + \sum_{\dim(F) = n-2} l^* (F) l^* (\hat{F}).
\]

For the last step we used the one-to-one correspondence between faces of \(\Delta\) and \(\Delta^\circ\).

5.4. Main results. Combining the above formula with Theorem 4 we have the following theorem:

**Theorem 5.** For any generic nondegenerate anticanonical hypersurface \(V\) of an \(n\)-dimensional simplicial Fano toric variety \(\mathbb{P}_\Delta\), \(n \geq 4\),

\[
h_{n-2,1}^{\text{orb}} (V) = l(\Delta) - n - 1 - \sum_{\dim(F) = n-1} l^* (F) + \sum_{\dim(F) = n-2} l^* (F) l^* (\hat{F}).
\]

**Corollary 1.** If \(\hat{V}\) is an MPCP desingularisation of any generic nondegenerate anticanonical hypersurface \(V\) of an \(n\)-dimensional simplicial Fano toric variety \(\mathbb{P}_\Delta\), \(n \geq 4\), then \(h_{p,1}^{\text{orb}} (V) = h_{p,1}^{\text{orb}} (\hat{V})\) for \(p = 1\) and \(p = n - 2\).

**Proof.** The formulas for \(h_{p,1}^{\text{orb}} (\hat{V})\) for \(p = 1, n - 2\) computed in [B1] by Batyrev match the orbifold Hodge numbers for \(V\) obtained in Theorem 3 and Theorem 5. \(\square\)

**Corollary 2.** In the case \(n = 4\), \(h_{n-2,1}^{\text{orb}} (V) = h_{3,0}^{\text{orb}} (\hat{V})\) for any \(p\) and \(q\).

**Proof.** We need only consider \(p, q \leq 3\). Also \(h_{0,0}^{\text{orb}} = h_{p,0}^{\text{orb}}\) by definition since \(\iota\) is nonnegative. So, by Serre duality for ordinary and orbifold cohomologies, it is enough to consider just the cases \(p = 1, q = 1\) and \(p = 2, q = 1\). These are addressed by Corollary 1. (We should remark here that in this case \(\hat{V}\) is actually smooth.) \(\square\)

**Corollary 3.** If \(\mathbb{P}_{\Delta^\circ}\) is also simplicial, and \(V^\circ\) is a generic nondegenerate anticanonical hypersurface of \(\mathbb{P}_{\Delta^\circ}\), then \(h_{1,1}^{\text{orb}} (V) = h_{n-2,1}^{\text{orb}} (V^\circ)\) and vice versa.

**Proof.** Follows from interchanging the roles of \(\Delta\) and \(\Delta^\circ\) in the formulas. \(\square\)

**Remark.** In particular, for the \(n = 4\) case, we have \(h_{1,1}^{\text{orb}} (V) = h_{3,0}^{\text{orb}} (V^\circ)\). This is an example of ‘mirror symmetry’ of orbifold hodge numbers.

5.5. An example. This example first appeared in the Greene-Plesser mirror construction [GP] and was also studied in [COFKM] in the context of mirror symmetry.

Consider the complex 4-dimensional weighted projective space \(X = \mathbb{P}(1,1,2,2,2)\). It is a simplicial Fano toric variety. Its fan \(\Xi\) has the following
1-dimensional cones in $N \cong \mathbb{Z}^4$: $v_1 = (-1, -2, -2, -2)$, $v_2 = (1, 0, 0, 0)$, $v_3 = (0, 1, 0, 0)$, $v_4 = (0, 0, 1, 0)$, $v_5 = (0, 0, 0, 1)$. $\Xi$ has five 4-dimensional cones, obtained by dropping one of the $v_i$'s at a time and taking the cone generated by the remaining four.

Let $D_i$ denote the torus-invariant divisor given by the orbit closure $\overline{O_{v_i}}$. It is easy to check that in $A_3(X)$, $[D_2] = [D_1]$ and $[D_i] = 2[D_1]$ for $i \geq 3$.

Construct the homogeneous coordinate ring of $X$ by introducing variables $x_i$ corresponding to $v_i$. Then $\deg(x_1) = \deg(x_2) = 1$ (= $[D_1]$) and $\deg(x_i) = 2$ for $i \geq 3$.

This leads to the more familiar description of $\mathbb{P}(1,1,2,2,2)$ as $(\mathbb{C}^5 - \{0\})/\mathbb{C}^*$. The action of any $\alpha \in \mathbb{C}^*$ on $\mathbb{C}^5 - \{0\}$ is as follows:

$$\alpha. [x_1 : x_2 : x_3 : x_4 : x_5] = [\alpha x_1 : \alpha x_2 : \alpha^2 x_3 : \alpha^2 x_4 : \alpha^2 x_5].$$

In this description, $D_i$ corresponds to the hyperplane $\{x_i = 0\}$ and the 4-dimensional cones of $\Xi$ correspond to the open sets $\{x_i \neq 0\}$. It is also easily seen that the singular locus of $X$ is precisely the surface $\{x_1 = x_2 = 0\}$. In fact, this represents the only twisted sector of $X$. The $v_i$'s are the vertices of a reflexive polytope $\Delta^\circ$. The faces of $\Delta^\circ$ have only one interior lattice point: $(0, -1, -1, -1) = \frac{1}{2}(v_1 + v_2)$. This lattice point corresponds to the twisted sector and the local isotropy group is $\mathbb{Z}_2$.

The dual reflexive polytope $\Delta$ in $M_\mathbb{R}$ has the following vertices:

$$w_1 = (-1, -1, -1, 1), \quad w_2 = (7, -1, -1, -1), \quad w_3 = (-1, 3, -1, 1), \quad w_4 = (-1, -1, 3, -1), \quad w_5 = (-1, -1, -1, 3).$$

$\Delta$ is the polytope corresponding to the anticanonical divisor $-K_X = \sum_{i=1}^5 D_i$ of $X$, and $X = \mathbb{P}_\Delta$. If $V$ is a generic nondegenerate Calabi-Yau (anticanonical) hypersurface of $X$, then $V$ has just one twisted sector namely $C = V \cap \{x_1 = x_2 = 0\}$. One can directly compute the genus of this curve $C$ by using the Riemann-Hurwitz formula. It turns out to be 3. $V$ has the Hodge numbers: $h^{1,0} = h^{2,0} = 0$, $h^{3,0} = 1$, $h^{1,1} = 1$, $h^{2,1} = 83$. Since the degree shifting number of the twisted sector $C$ is 1, we compute $h^{1,1}_{\text{orb}}(V) = h^{1,1}(V) + h^{0,0}(C) = 1 + 1 = 2$, and $h^{2,1}_{\text{orb}}(V) = h^{2,1}(V) + h^{1,0}(C) = 83 + 3 = 86$.

The dual Fano variety $\mathbb{P}_\Delta^\circ$ is also simplicial. This is easily checked since its fan is is obtained by coning over the faces of $\Delta$. In fact, $\mathbb{P}_\Delta^\circ = \mathbb{P}_\Delta / \mathbb{Z}_2^4$. This is also shown easily. First, observe that $w_1 = -w_2 - 2w_3 - 2w_4 - 2w_5$. Secondly, if $\overline{M}$ is the sublattice of $M$ generated by $w_2, w_3, w_4, w_5$, then $M / \overline{M} = \mathbb{Z}_4^3$.

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