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AMPLE FAMILIES, MULTIHOMOGENEOUS SPECTRA, AND ALGEBRAIZATION OF FORMAL SCHEMES

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Generalizing homogeneous spectra for rings graded by natural numbers, we introduce multihomogeneous spectra for rings graded by abelian groups. Such homogeneous spectra have the same completeness properties as their classical counterparts, but are possibly nonseparated. We relate them to ample families of invertible sheaves and simplicial toric varieties. As an application, we generalize Grothendieck's Algebraization Theorem and show that formal schemes with certain ample families are algebraizable.

Introduction.

A powerful method to study algebraic varieties is to embed them, if possible, into some projective space $\mathbb{P}^n$. Such an embedding $X \subset \mathbb{P}^n$ allows you to view points $x \in X$ as homogeneous prime ideals in some $\mathbb{N}$-graded ring. The purpose of this paper is to extend this to *divisorial varieties*, which are not necessarily quasiprojective.

The notion of divisorial varieties is due to Borelli [**2**]. The class of divisorial varieties contains all quasiprojective schemes, smooth varieties, and locally $\mathbb{Q}$-factorial varieties. Roughly speaking, divisoriality means that there is a finite collection of invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_r$ so that the whole collection behaves like an ample invertible sheaf. Such collections are called *ample families.*

Our main idea is to define *homogeneous spectra* $\operatorname{Proj}(S)$ for multigraded rings, that is, for rings graded by an abelian group of finite type. We obtain $\operatorname{Proj}(S)$ by patching affine pieces $D_+(f) = \operatorname{Spec}(S_{(f)})$, where $f \in S$ are certain homogeneous elements. Roughly speaking, we demand that f has many homogeneous divisors. Multihomogeneous spectra share many properties of classical homogeneous spectra. For example, they are universally closed, and the intersection of two affine open subsets is affine. They are, however, not necessarily separated.

Roberts [**23**] gave a similar construction for $\mathbb{N}^r$-graded rings S satisfying certain conditions on homogeneous generators. He used it to study Hilbert functions in several variables and local multiplicities. Roberts' homogeneous spectrum is an open subset of ours.

It turns out that a scheme is divisorial if and only if it admits an embedding into suitable multihomogeneous spectra. More precisely, we shall characterize ample families of invertible sheaves in terms of $\operatorname{Proj}(S)$ for the multigraded ring of global sections $S = \bigoplus_{d\in\mathbb{N}^r} \Gamma(X, \mathcal{L}_1^{d_1} \otimes \cdots \otimes \mathcal{L}_r^{d_r})$. Generalizing *Grauert's Criterion* for ample sheaves, we characterize ample families also in terms of affine hulls and contractions. Furthermore, we give a cohomological characterization which is analogous to *Serre's Criterion* for ample sheaves. We also relate homogeneous spectra to Cox's homogeneous coordinate rings for toric varieties [**6**].

As an application, we shall generalize *Grothendieck's Algebraization Theorem.* The result is that a proper formal scheme $\mathfrak{X} \to \operatorname{Spf}(R)$ is algebraizable if there is a finite collection of invertible formal sheaves restricting to an ample family on the closed fiber and satisfying an additional condition.

1. Grauert's criterion for ample families.

In this section, we shall generalize Grauert's criterion for ample sheaves to ample families. Given a collection of invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_r$, we use multiindices and set $\mathcal{L}^d = \mathcal{L}_1^{d_1} \otimes \cdots \otimes \mathcal{L}_r^{d_r}$ for each $d = (d_1, \ldots, d_r) \in \mathbb{Z}^r$. Let us start with the defining property of ample families:

Proposition 1.1. *Let X be a quasicompact and quasiseparated scheme. For a family $\mathcal{L}_1, \ldots, \mathcal{L}_r$ of invertible sheaves, the following are equivalent:*

(i) *The open sets X_f with $f \in \Gamma(X, \mathcal{L}^d)$ and $d \in \mathbb{N}^r$ form a base of the topology.*

(ii) *For each $x \in X$, there is some $d \in \mathbb{N}^r$ and $f \in \Gamma(X, \mathcal{L}^d)$ so that X_f is an affine neighborhood of x.*

(iii) *For each point $x \in X$, there is a $\mathbb{Q}$-basis $d_1, \ldots, d_r \in \mathbb{N}^r$ and global sections $f_i \in \Gamma(X, \mathcal{L}^{d_i})$ so that X_{f_i} are affine neighborhoods of x.*

Proof. To see (ii) $\Rightarrow$ (iii), choose a degree $d \in \mathbb{N}^r$ and a section $f \in \Gamma(X, \mathcal{L}^d)$ so that $U = X_f$ is an affine neighborhood of the point x. Choose a $\mathbb{Q}$-basis $e_1, \ldots, e_r \in \mathbb{N}^r$. Since U is affine, we find sections $f_i' \in \Gamma(U, \mathcal{L}^{e_i})$ satisfying $f_i'(x) \neq 0$. According to [**10**] Theorem 6.8.1, the sections $f_i = f_i' \otimes f^n$ extend from U to X for $n \gg 0$. The sections f and f_i have degrees d and $d_i = nd + e_i$, respectively. Skipping one of them we have a basis and the desired sections. The other implications are clear. □

Following Borelli [**2**], we call a finite collection $\mathcal{L}_1, \ldots, \mathcal{L}_r$ of invertible $\mathcal{O}_X$-modules an *ample family* if the scheme X is quasicompact and quasiseparated, and the equivalent conditions in Proposition 1.1 hold. A scheme is called *divisorial* if it admits an ample family of invertible sheaves.

Recall that a scheme is *separated* if the diagonal embedding $X \subset X \times X$ is closed, and *quasiseparated* if the diagonal is quasicompact. Note that, in contrast to the definition of ample sheaves ([**11**] Def. 4.5.3), we do not require separateness for ample families. However, the possible nonseparatedness is rather mild:

Proposition 1.2. *The diagonal embedding of a divisorial scheme is affine.*

Proof. Let X be a divisorial scheme, $\mathcal{L}$ be an invertible sheaf, $f \in \Gamma(X, \mathcal{L})$ a global section, and $U = \mathrm{Spec}(A)$ an affine open subset. Then $U \cap X_f = \mathrm{Spec}(A_f)$ is affine. Since X is covered by affine open subsets of the form X_f, this ensures that the diagonal embedding $X \subset X \times X$ is affine. □

Obviously, schemes admitting ample invertible sheaves are divisorial. The following gives another large class of divisorial schemes:

Proposition 1.3. *Normal noetherian locally $\mathbb{Q}$-factorial schemes with affine diagonal are divisorial schemes.*

Proof. As in [**14**] II 2.2.6, the complement of an affine dense open subset is a Weil divisor. By assumption it is $\mathbb{Q}$-Cartier, so X is divisorial. Using quasi-compactness, we find finitely many effective Cartier divisors $D_1, \ldots, D_r \subset X$ with $\bigcap D_i = \emptyset$. By Proposition 1.1, the invertible sheaves $\mathcal{O}_X(D_i)$ form an ample family. □

Here are two useful properties of divisorial schemes:

Proposition 1.4. *For divisorial noetherian schemes, the following hold:*

(i) *Each coherent $\mathcal{O}_X$-module admits a resolution with locally free $\mathcal{O}_X$-modules of finite rank.*
(ii) *There is a noetherian ring A together with a smooth surjective affine morphism* $\mathrm{Spec}(A) \to X$.

Proof. The first assertion is due to Borelli [**3**] Theorem 3.3. The second statement is called the Jouanolou–Thomason trick. For a proof, see [**24**] Proposition 4.4. □

Grauert's Criterion states that a line bundle is ample if and only if its zero section contracts to a point (see [**9**] p. 341 and [**11**] Theorem 8.9.1). The task now is to generalize this to families of line bundles. To do so, we shall use vector bundles. Recall that the category of locally free $\mathcal{O}_X$-modules $\mathcal{E}$ is antiequivalent to the category of vector bundles $\pi : B \to Z$ via $B = \mathrm{Spec}\, S(\mathcal{E})$ and $\mathcal{E} = \pi_*(\mathcal{O}_B)_1$. Under this correspondence, the sections $f \in \Gamma(X, S(\mathcal{E}))$ correspond to functions $f \in \Gamma(B, \mathcal{O}_B)$. For $f \in \Gamma(X, S^n(\mathcal{E}))$ define $X_f = \{x \in X : f(x) \neq 0\}$. Then $B_f \subset \pi^{-1}(X_f)$ and $X_f = \pi(B_f)$.

A locally free sheaf $\mathcal{E}$ is called *ample* if the invertible sheaf $\mathcal{O}_P(1)$ is ample on $P = \mathbb{P}(\mathcal{E})$ (see [**15**]). This easily implies that the open subsets $X_f \subset X$ with $f \in \Gamma(X, S^n(\mathcal{E}))$ generate the topology. In contrast to line bundles, the latter condition is not sufficient for ampleness. Let us characterize this condition:

Theorem 1.5. *Suppose X is quasicompact and quasiseparated. Let $\pi : B \to X$ be a vector bundle, $\mathcal{E} = \pi_*(\mathcal{O}_B)_1$ the corresponding locally free sheaf, and $Z \subset B$ the zero section. Then the following are equivalent:*

(i) *The open subsets $X_f \subset X$ with $f \in \Gamma(X, S^n(\mathcal{E}))$ generate the topology.*
(ii) *For every point $x \in X$ there is a function $f \in \Gamma(B, \mathcal{O}_B)$ so that B_f is affine and $\pi^{-1}(x) \cap B_f \neq \emptyset$.*
(iii) *There is a scheme B', and a morphism $q : B \to B'$, and an open subset $U' \subset B'$ so that the following holds: The image $q(Z) \subset B'$ admits a quasiaffine open neighborhood, $B \to B'$ induces an isomorphism $q^{-1}(U') \cong U'$ and the projection $q^{-1}(U') \to X$ is surjective.*
(iv) *There exists an open subset $Z \subset W \subset B$ such that for every $x \in X$ there is a function $f \in \Gamma(W, \mathcal{O}_B)$ so that W_f is affine and $W_f \cap \pi^{-1}(x)$ is nonempty.*

Proof. We shall prove the implications (i) $\Rightarrow \cdots \Rightarrow$ (iv) $\Rightarrow$ (i). First assume (i). Fix a point $x \in X$. Choose an affine neighborhood $x \in V$ and a section $f \in \Gamma(X, S^n(\mathcal{E}))$ so that $x \in X_f \subset V$. Then $B_f = \pi^{-1}(V)_f$ is affine and $\pi^{-1}(x) \cap B_f \neq \emptyset$.

Assume (ii) holds. Set $B' = B^{\text{aff}} = \operatorname{Spec} \Gamma(B, \mathcal{O}_B)$. According to [**10**] Corollary 6.8.3, the map $\Gamma(B, \mathcal{O}_B)_f \to \Gamma(B_f, \mathcal{O}_B)$ is bijective, so the affine hull $q : B \to B^{\text{aff}}$ induces an isomorphism $B_f = q^{-1}(D(f)) \to D(f)$ for B_f affine. Then take $U' = \bigcup_f D(f)$ and $U = \bigcup_f B_f$, where the union runs over f, B_f affine.

Assume that (iii) holds. Let $W' \subseteq B'$ be a quasiaffine neighborhood of $q(Z)$. Then $W = q^{-1}(W')$ is an open neighborhood of the zero section. Fix a point $x \in X$ and let $\eta \in \pi^{-1}(x)$ be the generic point, such that $\eta \in q^{-1}(U') \cap W$. Then we find $f \in \Gamma(W', \mathcal{O}_{B'})$ so that $q(\eta) \in W'_f \subset U' \cap W'$ is affine and $\eta \in W_f = q^{-1}(W'_f) \cong W'_f$ is an affine neighborhood.

Now suppose (iv) holds. Let $x \in V \subset X$. Then there exists $f \in \Gamma(W, \mathcal{O}_B)$ so that $W_f \subset \pi^{-1}(V) \cap W$ is affine and $\pi^{-1}(x) \cap W_f \neq \emptyset$. Let $\mathcal{I} \subset \mathcal{O}_W$ be the ideal of the zero section $Z \subset W$. Then $\mathcal{O}_W / \mathcal{I}^{n+1} = \bigoplus_{d=0}^n S^d(\mathcal{E})$, and $f \in \Gamma(W, \mathcal{O}_B)$ has a Taylor series expansion $f = \sum_{d=0}^\infty f_d$ with $f_d \in \Gamma(X, S^d(\mathcal{E}))$. Choose a degree $d \geq 0$ with $f_d(x) \neq 0$. Then $x \in X_{f_d} \subset V$ is the desired open neighborhood of x. □

Remark 1.6. If X is connected and proper over a base field, the image of the zero section $Z \subset B$ in any quasiaffine scheme is a closed point. In this case, the assumption in Condition (iii) implies that q contracts $Z \cong X$ to a point.

The preceding result yields a characterization of ample families $\mathcal{L}_1, \ldots, \mathcal{L}_r$ in terms of the corresponding locally free sheaf $\mathcal{E} = \bigoplus \mathcal{L}_i$.

Corollary 1.7. *Suppose X is quasicompact and quasiseparated. A family of invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_r$ is ample if and only if the vector bundle*

$\pi : B \to X$ *with* $\pi_*(\mathcal{O}_B)_1 = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$ *satisfies the equivalent conditions in Theorem* 1.5.

Proof. If the family is ample, then Condition (i) of Theorem 1.5 holds. For the converse, fix a point $x \in X$, and choose a section $f \in \Gamma(X, S^n(\mathcal{E}))$ so that $x \in X_f \subset V$ where V is affine. Write $f = \sum f_d$ according to the decomposition $S^n(\mathcal{E}) = \bigoplus_d \mathcal{L}^d$, where the sum runs over all degrees $d \in \mathbb{N}^r$ with $n = \sum d_i$. Pick a summand f_d with $f_d(x) \neq 0$. Then $X_{f_d} \subset X_f \subset V$ is the desired affine open neighborhood. $\square$

Remark 1.8. In the situation of 1.7, the vector bundle $B = B_1 \times_X \cdots \times_X B_r$ decomposes into $B_i = \operatorname{Spec} S(\mathcal{L}_i)$. The affine hull $B \to B^{\mathrm{aff}}$ is an isomorphism outside the coordinate hyperplanes. The following examples illustrate what may happen on the union of the coordinate hyperplanes:

Example 1.9. Set $X = \mathbb{P}^1 \times \mathbb{P}^1$. First, let $\mathcal{L}_1$ and $\mathcal{L}_2$ be invertible sheaves of bidegree $(1,0)$ and $(0,1)$, respectively. This is an ample family because $\mathcal{L}_1 \otimes \mathcal{L}_2$ is ample. Set $B_i = \operatorname{Spec} S(\mathcal{L}_i)$ and consider the corresponding rank two vector bundle $B = B_1 \times_X B_2$. On each summand B_i, the affine hull $B \to B^{\mathrm{aff}}$ restrict to the morphism

$$B_i = \widetilde{\mathbb{A}}^2 \times \mathbb{P}^1 \xrightarrow{\mathrm{pr}_1} \widetilde{\mathbb{A}}^2 \xrightarrow{g} \mathbb{A}^2$$

where $g : \widetilde{\mathbb{A}}^2 \to \mathbb{A}^2$ is the blowing-up of the origin.

Now let $\mathcal{L}_1$ and $\mathcal{L}_2$ be invertible sheaves of bidegree $(2,-1)$ and $(-1,1)$, respectively. This is an ample family, because $\mathcal{L}_1^2 \otimes \mathcal{L}_2^3$ is an ample sheaf. Here the affine hull $B \to B^{\mathrm{aff}}$ contracts the union of the coordinate hyperplane $B_1 \cup B_2$ to a point.

In the next examples we consider the following situation: Let k denote a field and let A be an $\mathbb{N}$-graded k-algebra of finite type with $A = k[A_1]$. Let $n_1, \ldots, n_r \in \mathbb{Z}$ and consider the family $\mathcal{O}_X(n_1), \ldots, \mathcal{O}_X(n_r)$ on X. Now we describe the corresponding vector bundle and its ring of global sections. (The second description uses Proj of a $\mathbb{Z}$-graded ring, which we introduce in the next section.)

Proposition 1.10. *Let* A, X *and* $n_1, \ldots, n_r$ *as above. Let* $S = A[T_1, \ldots, T_r]$ *be* $\mathbb{Z}$*-graded by degrees* $\deg(T_j) = -n_j$*. Then the vector bundle* $B = \operatorname{Spec} S(\mathcal{O}_X(n_1) \oplus \cdots \oplus \mathcal{O}_X(n_r))$ *is*

$$B \cong \bigcup_{f \in A_1} D_+(f) \subset \operatorname{Proj} S_{\geq 0} \quad \textit{and} \quad B \cong \bigcup_{f \in A_1} D_+(f) \subset \operatorname{Proj} S.$$

If A *is normal and* $\dim X \geq 1$*, then* $\Gamma(B, \mathcal{O}_B) = S_0$*.*

Proof. Let $R = S$ or $= S_{\geq 0}$. R_0 has also a $\mathbb{Z}^r$-graduation with $(R_0)_d = \{aT^d : a \in A_{d_1 n_1 + \cdots + d_r n_r}\}$, $d = (d_1, \ldots, d_r)$. There is a natural rational mapping $\operatorname{Proj} R \dashrightarrow \operatorname{Proj} A$ which is defined on $\bigcup D_+(f)$, where the union runs over $f \in A_1$. This is an affine morphism, thus we can check the identities by looking at the rings of global sections and at the restriction maps. We have

$$\begin{aligned}\Gamma(\pi^{-1}(D_+(f)), \mathcal{O}_B)_d &= \Gamma(D_+(f), S(\mathcal{O}_X(n_1) \oplus \cdots \oplus \mathcal{O}_X(n_r)))_d \\ &= \Gamma(D_+(f), \mathcal{O}_X(d_1 n_1 + \cdots + d_r n_r)) \\ &= (A_f)_{d_1 n_1 + \cdots + d_r n_r} \\ &= ((R_f)_0)_d \\ &= \Gamma(D_+(f), \mathcal{O}_{\operatorname{Proj} R})_d\end{aligned}$$

and the restriction maps respect these identities. The last statement follows if we replace $D_+(f)$ by $U = \bigcup_{f \in A_1} D_+(f)$. Then

$$\begin{aligned}\Gamma(U, \mathcal{O}_X(d_1 n_1 + \cdots + d_r n_r)) &= \Gamma(D(A_1), (\mathcal{O}_{\operatorname{Spec}(A)})_{d_1 n_1 + \cdots + d_r n_r}) \\ &= A_{d_1 n_1 + \cdots + d_r n_r},\end{aligned}$$

since A is normal and $V(A_1)$ has codimension ≥ 2. □

Example 1.11. Let $X = \mathbb{P}^m$ be the homogeneous spectrum of $k[X_0, ..., X_m]$, $m \geq 1$. Let $\mathcal{L}_1 = \mathcal{O}_X(1)$ and $\mathcal{L}_2 = \mathcal{O}_X(-1)$. Then $B_1 = \operatorname{Spec} S(\mathcal{O}(1))$ is the blowing up of the vertex point in $\mathbb{A}^{m+1}$ and $B_2 = \operatorname{Spec} S(\mathcal{O}(-1))$ is the projection from a point in $\mathbb{P}^{m+1}$. The ring of global sections of the rank two vector bundle $B = B_1 \times_X B_2$ is the polynomial algebra

$$\Gamma(B, \mathcal{O}_B) = k[X_0 S, \ldots, X_m S, ST] \subset k[X_0, \ldots, X_m][S, T]$$

where S, T are indeterminates with degrees $\deg(S) = -1$ and $\deg(T) = 1$. The affine hull $B \to B^{\text{aff}}$ contracts B_2 to a point and is an isomorphism on the complement $B - B_2$. We have

$$\Gamma(B|D_+(X_i), \mathcal{O}_B) = k[X_0/X_i, \ldots, X_m/X_i, X_i S, T/X_i],$$

and the affine hull is given by

$$X_j S \mapsto \frac{X_j}{X_i} X_i S \quad \text{and} \quad ST \mapsto \frac{T}{X_i} X_i S.$$

Example 1.12. Again let $A = k[X_0, \ldots, X_m]$ $(m \geq 1)$ and consider the family $\mathcal{L}_1 = \mathcal{O}_X(1), \ldots, \mathcal{L}_r = \mathcal{O}_X(1)$. Then $\Gamma(B, \mathcal{O}_B)$ is the determinantial algebra

$$k[X_i T_j \mid 0 \leq i \leq m, 1 \leq j \leq r] \subset k[X_0, \ldots, X_m, T_1, \ldots, T_r],$$

generated by the entries of the $(m+1) \times r$-matrix $(X_i T_j)$, with relations given by the 2×2-minors $(X_i T_j)(X_k T_l) - (X_i T_l)(X_k T_j)$. The affine hull $\varphi : B \to B^{\text{aff}}$ contracts exactly the zero section to a point. The affine hull φ may also be described as the blowing-up of the column ideal $(X_0T_1, \ldots, X_mT_1)$. For this blowing up is given by Proj $k[X_iT_j][X_0T_1U, \ldots, X_mT_1U]$, and

$$\begin{aligned} k[X_iT_j][X_0T_1U, \ldots, X_mT_1U] &\cong k[X_iT_j][X_0, \ldots, X_m] \\ &= k[X_0, \ldots, X_m, T_1, \ldots, T_r]_{\geq 0}. \end{aligned}$$

2. The homogeneous spectrum of a multigraded ring.

Generalizing the classical notion of homogeneous coordinates, Grothendieck defined homogeneous spectra for $\mathbb{N}$-graded rings ([**11**] §2). In this section, we shall generalize his approach to multigraded rings. Let D be a finitely generated abelian group and let $S = \bigoplus_{d\in D} S_d$ be a D-graded ring. Note that by [**13**] I 4.7.3, such gradings correspond to actions of the diagonizable group scheme $\operatorname{Spec}(S_0[D])$ on the affine scheme $\operatorname{Spec}(S)$.

According to geometric invariant theory (see [**22**] Thm. 1.1), the projection $\operatorname{Spec}(S) \to \operatorname{Spec}(S_0)$ is a categorical quotient in the category of schemes. There is a quotient $\operatorname{Spec}(S) \to \operatorname{Quot}(S)$ in the category of ringed spaces as well. In general, the latter is quite different from the first. However, we have the following favorable situation: Call the ring S *periodic* if the degrees of the homogeneous units $f \in S^\times$ form a subgroup $D' \subset D$ of finite index. In this case, we may choose a free subgroup $D' \subset D$ of finite index, such that $S' = \bigoplus_{d\in D'} S_d$ is a Laurent polynomial algebra $S_0[T_1^{\pm 1}, \ldots, T_r^{\pm 1}]$.

Lemma 2.1. *For periodic rings S, the projection* $\operatorname{Spec}(S) \to \operatorname{Spec}(S_0)$ *is a geometric quotient in the sense of geometric invariant theory.*

Proof. Choose a free subgroup $D' \subset D$ of finite index as above. The corresponding inclusion of the Veronese subring $S' \subset S$ is an integral ring extension, because D/D' is torsion, such that $\operatorname{Spec}(S) \to \operatorname{Spec}(S')$ is a closed morphism. By [**22**] Amplification 1.3, this morphism is a geometric quotient.

Since S' is a Laurent polynomial ring, $\operatorname{Spec}(S')$ is a principal homogeneous space for the induced action of $\operatorname{Spec}(S_0[\mathbb{Z}^r])$ and the projection $\operatorname{Spec}(S') \to \operatorname{Spec}(S_0)$ is a geometric quotient. Being the composition of two geometric quotients, $\operatorname{Spec}(S) \to \operatorname{Spec}(S_0)$ is a geometric quotient as well. $\square$

In light of this, we seek to pass from a given graded ring S to periodic rings via localization. An element $f \in S$ is called *relevant* if it is homogeneous and the localization S_f is periodic. Equivalently, the degrees of all homogeneous divisors $g|f^n$, $n \geq 0$ generate a subgroup $D' \subset D$ of finite index. Since geometric quotients are quotients in the category of ringed spaces, localization of relevant elements yields open subschemes

$$D_+(f) = \operatorname{Spec}(S_{(f)}) \subset \operatorname{Quot}(S)$$

inside the ringed space $\mathrm{Quot}(S)$. Here $S_{(f)} \subset S_f$ is the degree zero part of the localization. This leads to the following definition:

Definition 2.2. Let D be a finitely generated abelian group and let $S = \bigoplus_{d\in D} S_d$ be a D-graded ring. We define the scheme

$$\mathrm{Proj}(S) = \bigcup_{f\in S \text{ relevant}} D_+(f) \subset \mathrm{Quot}(S),$$

and call it the *homogeneous spectrum* of the graded ring S.

For $\mathbb{N}$-gradings, this coincides with the usual definition. As in the classical situation, we define $S_+ \subset S$ to be the ideal generated by all relevant $f \in S$. The corresponding invariant closed subscheme $V(S_+)$ is called the *irrelevant subscheme*. The complementary invariant open subset $\mathrm{Spec}(S) - V(S_+)$ is called the *relevant locus*. Obviously, we obtain an affine projection

$$\mathrm{Spec}(S) - V(S_+) \longrightarrow \mathrm{Proj}(S),$$

which is a geometric quotient for the induced action.

Remark 2.3. The points $x \in \mathrm{Proj}(S)$ correspond to graded (not necessarily prime) ideals $\mathfrak{p} \subset S$ not containing S_+ such that the subset of homogeneous elements $H \subset S - \mathfrak{p}$ is closed under multiplication. The stalk of the structure sheaf at $x \in \mathrm{Proj}(S)$ is canonically isomorphic to $(H^{-1}S)_0$.

To proceed, we need a finiteness condition for multigraded rings. In the special case $D = \mathbb{Z}$, the following is due to Bruns and Herzog ([**5**] Thm. 1.5.5):

Lemma 2.4. *Let S be a ring graded by a finitely generated abelian group D. Then the following are equivalent:*

(i) *The homogeneous ideals of S satisfy the ascending chain condition.*
(ii) *The ring S is noetherian.*
(iii) *S_0 is noetherian and S is an S_0-algebra of finite type.*

If S is noetherian and $M \subset D$ is a finitely generated submonoid, then S_M is also noetherian.

Proof. The implications (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$(i) are trivial, so assume that (i) holds. We start with some preparations. For a submonoid $M \subset D$, let $S_M = \bigoplus_{d\in M} S_d$ be the corresponding *Veronese subring*. Let $D' \subset D$ denote a subgroup. Then the ring $S_{D'} \subset S$ is a direct summand. Therefore $S_{D'}$ satisfies also the ascending chain condition for homogeneous ideals. In particular, S_0 is noetherian. It follows at once that S_e are noetherian S_0-modules. Let $D \cong D' \oplus T$, where T is finite and D' free. Then S is finite over $S_{D'}$ and $S_{D'}$ fulfills (i). Thus we may assume that D is free. Let us call a free submonoid $M \subset D$ a *quadrant* if $M = \bigoplus \mathbb{N}d_i$, where the $d_i \in D$ is a subset of a $\mathbb{Z}$-basis for D.

Claim. For each quadrant $M \subset D$ and each degree $e \in D$, the S_M-module $S_{M+e} = \bigoplus_{d \in M} S_{d+e}$ is finitely generated.

We prove this by induction on $r = \operatorname{rank} M$. Since S_e are noetherian S_0-modules, this holds for $r = 0$. Fix a quadrant M of rank r and suppose the Claim is true for each quadrant of rank $r-1$. Choose $d_1, \ldots, d_r \in D$ with $M = \bigoplus_{i=1}^r \mathbb{N} d_i$, and let $M_i = \bigoplus_{j \neq i} \mathbb{N} d_j$ be the i-th boundary quadrant.

Condition (i) implies that the graded S-ideal $S \cdot S_{M+e}$ is finitely generated. Choose homogeneous generators $f_1, \ldots, f_s \in S_{M+e}$. Then $S_d \subset \sum S_M f_i$ for each $d \in M+e$ with $d \geq \max\{\deg(f_1), \ldots, \deg(f_s)\}$. Fix such a degree $d \in M+e$. Clearly, there are finitely many $d_{ij} \in M+e$ with

$$M + e = (M + d) + \sum_i \sum_j (M_i + d_{ij}).$$

By induction, each $S_{M_i + d_{ij}}$ is a finitely generated S_{M_i}-module, and we conclude that S_{M+e} is a finitely generated S_M-module. This proves the claim.

Fix a quadrant $M = \sum \mathbb{N} d_i$ and set $M^* = M - 0 = \sum (M + d_i)$. Then S_{M+d_i} is a finitely generated S_M-module and thus $S_{M^*} = \sum S_{M+d_i}$ is finitely generated.

Let $S_{M^*} = (f_1, \ldots, f_n)$, where f_i are homogeneous of degree > 0. We show by induction on $M \cong \mathbb{N}^r$ that $S_M = S_0[f_1, \ldots, f_n]$. Let $g \in S_d$, $d \in M$ and suppose that $S_e \subset S_0[f_1, \ldots, f_n]$ for all $e < d$. We have $g = h_1 f_1 + \cdots + h_n f_n$, where $\deg(h_j) = \deg(g) - \deg(f_j) < d$ and the result follows.

Since D is a finite union of quadrants with $\operatorname{rank}(M) = \operatorname{rank}(D)$, we conclude that S is a finitely generated S_0-algebra.

To prove the additional statement, let $g_i, i \in I$ be a generating system for S with degrees d_i. We may assume that M is saturated, thus we may describe M with finitely many linear forms $\psi_k : D \to \mathbb{Z}$, say $M = \bigcap_k \psi_k^{-1}(\mathbb{N})$. Consider the mapping $\varphi : \mathbb{N}^I \to D$, $(n_i)_{i \in I} \mapsto \sum_i n_i d_i$. Then $\varphi^{-1}(M) = \bigcap_k (\psi_k \circ \varphi)^{-1}(\mathbb{N})$ and therefore $\varphi^{-1}(M)$ is finitely generated, say by $r_j, j \in J$. Set $e_j = \varphi(r_j)$ in M. We claim that S_M is generated by elements of degree $e_j, j \in J$. An element $f \in S_d$, $d \in M$ can be written as the sum of products $\prod_{i \in I} g_i^{n_i}$, where $d = \sum_{i \in I} n_i d_i$. But then $(n_i)_{i \in I} = \sum_j m_j r_j$ and

$$\prod_{i \in I} g_i^{n_i} = \prod_{i \in I} g_i^{\sum_j m_j r_{ji}} = \prod_{j \in J} \left(\prod_{i \in I} g_i^{r_{ji}} \right)^{m_j}.$$

Thus $\prod_i g_i^{n_i}$ is a product of elements in S_{e_j}. □

We have the following finiteness condition for homogeneous spectra:

Proposition 2.5. *The morphism* $\operatorname{Proj}(S) \to \operatorname{Spec}(S_0)$ *is universally closed and of finite type, provided that* S *is noetherian.*

Proof. By Lemma 2.4, the ring S_0 is noetherian and the S_0-algebra S is of finite type. The relevant locus $\mathrm{Spec}(S) - V(S_+)$ is quasicompact and surjects onto $\mathrm{Proj}(S)$, so the homogeneous spectrum is quasicompact.

Next, we check that the projection $\mathrm{Proj}(S) \to \mathrm{Spec}(S_0)$ is locally of finite type. To do so, fix a relevant element $f \in S$, and let $D' \subset D$ be a free subgroup of finite index such that for each $d \in D'$, there is a homogeneous unit $g \in S_f^\times$ with $\deg(g) = d$. Let $S' = \bigoplus_{d \in D'} S_d$ be the corresponding Veronese subring. By Lemma 2.4, the ring extension $S' \subset S$ is finite, so Artin–Tate [**1**] tells us that the ring extension $S_0 \subset S'$ is of finite type. Clearly, the localization S'_f is isomorphic to a Laurent polynomial ring $S_{(f)}[T_1^{\pm 1}, \ldots, T_r^{\pm 1}]$. Setting $T_i = 1$, we deduce that $S_0 \subset S_{(f)}$ is of finite type.

Finally, we verify universal closedness. Let $S_0 \to R_0$ be a base change and set $R = S \otimes_{S_0} R_0$. A direct argument gives $R_+ = S_+ \otimes_{S_0} R_0$. Hence we have to check that $h : \mathrm{Proj}(S) \to \mathrm{Spec}(S_0)$ is closed under the new hypothesis that S is an S_0-algebra of finite type, that $S_+ \subset S$ is an ideal of finite type, and that each S_d is a finitely generated S_0-module. Each closed subset of $\mathrm{Proj}(S)$ is of the form $\mathrm{Proj}(S/\mathfrak{b}) = V_+(\mathfrak{b})$ for some graded ideal $\mathfrak{b} \subset S$, and we have $h(V_+(\mathfrak{b})) \subset V(\mathfrak{b}_0)$. Consequently, it suffices to show that $h : \mathrm{Proj}(S) \to \mathrm{Spec}(S_0)$ has closed image.

Fix a point $x \in \mathrm{Spec}(S_0)$ with $h^{-1}(x) = \emptyset$, and let $\mathfrak{p} \subset S_0$ be the corresponding prime ideal. We have to construct $g \in S_0 - \mathfrak{p}$ with $h^{-1}(\mathrm{Spec}(S_0)_g) = \emptyset$. The condition $h^{-1}(x) = \emptyset$ signifies that each relevant $f \in S$ is nilpotent in $S/\mathfrak{p}S$. Hence $S_+^k \subset \mathfrak{p}S_+$ for some integer $k > 0$. Choose finitely many homogeneous $g_1, \ldots, g_n \in S$ with $S = S_0[g_1, \ldots, g_n]$, and set $d_i = \deg(g_i)$. Call a degree $d \in D$ *generic* if for each linear combination $d = \sum n_i d_i$ with nonnegative coefficients, the set $\{d_i \mid n_i \geq k\}$ generates a subgroup of finite index. Then the set of nongeneric degrees is a union of a finite set and finitely many affine hyperplanes.

Let $d = \sum n_i d_i$ be generic and consider $g = \prod g_i^{n_i} \in S_d$. Then one may write $g = g_1^k \cdot \ldots \cdot g_r^k \cdot g'$ such that $g_1 \cdot \ldots \cdot g_r$ is relevant. Using $S_+^k \subset \mathfrak{p}S_+$, we see $g \in \mathfrak{p}S_d$. This gives $\mathfrak{p}S_d = S_d$ for all generic degrees $d \in D$, and the Nakayama Lemma gives $(S_d)_\mathfrak{p} = 0$.

Next, choose finitely many relevant $f_1, ..., f_m \in S$ with $S_+ = \sqrt{(f_1, ..., f_m)}$. We may assume $f_i = f_{i1} \ldots f_{ir}$ such that each sequence $\deg(f_{i1}), ..., \deg(f_{ir}) \in D$ generates a subgroup of finite index. For each f_i, choose a linear combination with positive coefficients $e_i = \sum_j n_j \deg(f_{ij}) \in D$ that is generic. Then some $g \in S_0 - \mathfrak{p}$ annihilates all S_{e_i}. Consequently, each $f_i \in S[1/g]$ is nilpotent, hence the preimage $h^{-1}(\mathrm{Spec}(S_0)_g)$ is empty. □

Remark 2.6. Roberts ([**23**], Sect. 8.2) introduced multihomogeneous spectra for certain $\mathbb{N}^r$-graded rings. To explain Roberts' conditions, let $S_{[j]} \subset S$, $0 \leq j \leq r$ be the graded subring generated by all homogeneous elements whose degrees are of the form $(i_1, \ldots, i_j, 0, \ldots, 0)$. Then Roberts assumes that each $S_{[j+1]}$ is generated over $S_{[j]}$ by finitely many elements of degree $(i_1, \ldots, i_j, 1, 0, \ldots, 0)$. Now Roberts' homogeneous spectrum is the subset $\bigcup D_+(f) \subset \mathrm{Proj}(S)$, where the union runs over all $f \in S$ admitting a factorization $f = g_1 \ldots g_r$ so that $\deg(g_j)$ has the form $(i_1, \ldots, i_{j-1}, 1, 0, \ldots, 0)$, compare [**23**], Proposition 8.2.5.

3. Separation criteria.

In Proposition 2.5, we may say that $\mathrm{Proj}(S)$ is a *complete* S_0-scheme. We cannot, however, infer that it is proper. For example, set $S = k[X, Y]$ with degrees in $D = \mathbb{Z}$ given by $\deg(X) = 1$ and $\deg(Y) = -1$. Then $S_0 = k[XY]$ yields the affine line $\mathbb{A}^1_k$, and $\mathrm{Proj}(S)$ is the affine line with double origin, which is nonseparated. However, the following holds:

Proposition 3.1. *The diagonal embedding of a homogeneous spectrum is affine.*

Proof. The intersection $D_+(f) \cap D_+(g) = D_+(fg)$ is affine. □

Here is a criterion for separatedness, which trivially holds for $\mathbb{N}$-gradings:

Proposition 3.2. *If for each pair $x, y \in \mathrm{Proj}(S)$ there is a relevant $f \in S$ with $x, y \in D_+(f)$, then $\mathrm{Proj}(S)$ is separated.*

Proof. Under the assumption the affine open subsets $U \times U$, where $U \subset \mathrm{Proj}(S)$ is affine, cover $\mathrm{Proj}(S) \times \mathrm{Proj}(S)$. Clearly, the diagonal embedding of $\mathrm{Proj}(S)$ is closed over each of these open subsets, hence it is closed. □

The next task is to recognize large separated open subsets in $\mathrm{Proj}(S)$. Given a homogeneous element $f \in S$, let $H_f \subset S$ be the set of homogeneous divisors $g | f^n$, $n \geq 0$, and $C_f \subset D \otimes \mathbb{R}$ the closed convex cone generated by the degrees $\deg(g)$, $g \in H_f$. Note that a homogeneous element is relevant if and only if its cone has nonempty interior.

Proposition 3.3. *Let $f_i \in S$ be a collection of relevant elements so that each closed convex cone $C_{f_i} \cap C_{f_j} \subset D \otimes \mathbb{R}$ has nonempty interior. Then $\bigcup D_+(f_i) \subset \mathrm{Proj}(S)$ is a separated open subset.*

Proof. According to [**10**] Proposition 5.3.6, it suffices to show that the multiplication map $S_{(f)} \otimes S_{(g)} \to S_{(fg)}$ is surjective for each pair of relevant elements $f, g \in S$ such that $C_f \cap C_g$ has nonempty interior. Note that, for each factorization $g^n = g_1 \ldots g_m$, we may replace g by $g_1^{n_1} \ldots g_m^{n_m}$, $n_i > 0$ without changing the localization $S_{(g)}$. Thus we may assume $\deg(g) \in C_f$. Passing to a suitable power of g, we may assume $\deg(g) = \sum n_i \deg(f_i)$, $n_i \geq 0$ with $f_i \in H_f$. Each element in $S_{(fg)}$ has the form $a/(fg)^k$ with

$a \in S$ homogeneous, so

$$\frac{a}{(fg)^k} = \frac{a}{f^k \prod f_i^{kn_i}} \cdot \frac{\prod f_i^{kn_i}}{g^k}$$

is contained in the image of $S_{(f)} \otimes S_{(g)}$. □

Next, we shall relate homogeneous spectra of multigraded polynomial algebras to toric varieties. Fix a ground ring R and a free abelian group M of finite rank. A *simplicial torus embedding* of the torus $T = \operatorname{Spec}(R[M])$ is an equivariant open embedding $T \subset X$ that is locally given by $R[M \cap \sigma^\vee] \subset R[M]$ for some strongly convex, simplicial cone $\sigma \subset N_\mathbb{R}$ in the dual lattice $N = \operatorname{Hom}(M, \mathbb{Z})$. Here *simplicial cone* means that the cone is generated by a linear independent set. In contrast to the usual definition, we do not require that our torus embeddings are separated.

Simplicial torus embeddings occur in the following context, which is related to a construction of Cox [**6**]: Let $S = R[T_1, \ldots, T_k]$ be a D-graded polynomial algebra, such that the grading is given by a linear map $d : \mathbb{Z}^k \to D$ sending the i-th base vector to $\deg(T_i) \in D$. Let $M \subset \mathbb{Z}^k$ be the kernel.

Proposition 3.4. *Notation as above. Then* $\operatorname{Proj}(S)$ *is a* (*possibly nonseparated*) *simplicial torus embedding of the torus* $\operatorname{Spec}(R[M])$.

Proof. Let $I = \{1, \ldots, k\}$ be the index set for the indeterminates. Fix a relevant monomial $T^n = T_1^{n_1} \ldots T_k^{n_k}$ and let $J = \{i \in I \mid n_i > 0\}$ be its support. A direct argument gives $S_{(T^n)} = R[M_J]$ for the monoid

$$M_J = (\mathbb{Z}^J \oplus \mathbb{N}^{I-J}) \cap M \subset \mathbb{Z}^k.$$

Clearly, the submonoid $M_J \subset M$ is the intersection of $\operatorname{Card}(I - J) \leq \operatorname{rank}(M)$ half spaces. Therefore, its dual cone $\sigma \subset N_\mathbb{R}$ is simplicial.

It remains to check $M = M_J + (-M_J)$. Let $D' \subset D$ be the subgroup generated by $\deg(T_i)$ with $i \in J$, and $m = \operatorname{ord}(D/D')$ be its index. Then there are integers $\lambda_j \in \mathbb{Z}$, $j \in J$ solving the equation $\sum_{i \in J} \lambda_i \deg(T_i) = -m \sum_{i \in I-J} \deg(T_i)$. So the element $g \in \mathbb{Z}^k$ defined by

$$g_i = \begin{cases} \lambda_i & \text{for } i \in J \\ m & \text{for } i \in I - J \end{cases}$$

lies in M_J. For each $f \in M$, we have $f + ng \in M_J$ for $n \gg 0$, hence $f = (f + ng) - ng$ is contained in $M_J + (-M_J)$. □

Corollary 3.5. *If S is finitely generated as S_0-algebra, then* $\operatorname{Proj}(S)$ *is divisorial.*

Proof. We may choose a D-graded polynomial R-algebra S' with $S'_0 = S_0$, together with a graded surjection $S' \to S$. This induces a closed embedding

$\mathrm{Proj}(S) \subset \mathrm{Proj}(S')$, because for every relevant element $f \in S$ we may find a relevant element $f' \in S'$ mapping to it.

By Proposition 3.4, the scheme $\mathrm{Proj}(S')$ is a simplicial torus embedding, which has affine diagonal by Proposition 3.1, hence by Proposition 1.3 it is a divisorial scheme. Consequently, the closed subscheme $\mathrm{Proj}(S)$ is divisorial as well. □

Corollary 3.6. *Suppose that S is finitely generated over S_0. If each finite subset of* $\mathrm{Proj}(S)$ *admits an affine neighborhood, then* $\mathrm{Proj}(S)$ *is projective.*

Proof. We already know that $\mathrm{Proj}(S)$ is of finite type, universally closed, separated, and divisorial (Prop. 2.5, Prop. 3.2, and Cor. 3.5). Since each finite subset admits an affine neighborhood, the generalized Chevalley Conjecture ([**18**], Thm. 3) applies, and we conclude that $\mathrm{Proj}(S) \to \mathrm{Spec}(S_0)$ is projective. □

Remark 3.7. Let us make the torus embedding in Proposition 3.4 more explicit. For each subset $J \subset I = \{1, \dots, k\}$, let $\sigma_J \subset N_{\mathbb{R}}$ be the convex cone generated by the projections $\mathrm{pr}_i : \mathbb{Z}^k \to \mathbb{Z}$ restricted to M, $i \in J$. You easily check that $J \mapsto \sigma_{I-J}$ gives a bijection between the subsets $J \subset I$ with $\prod_{j \in J} T_j$ relevant, and the strongly convex simplicial cones $\sigma_{I-J} \subset N_{\mathbb{R}}$. Let us call such subsets *relevant.* Then the torus embedding is given by

$$\mathrm{Proj}(S) = \bigcup_{J \subset I \text{ relevant}} \mathrm{Spec}(R[\sigma_{J-I}^{\vee} \cap M]).$$

The (possibly nonseparated) union is taken with respect to the inclusions $J \subset J'$.

Example 3.8. Let $S = R[T_1, \dots, T_k]$ be a polynomial algebra graded by $D = \mathbb{Z}$ so that all indeterminates have positive degree. Then $\mathrm{Proj}(S)$ is the weighted projective space studied by Delorme [**7**], Mori [**21**], and Dolgachev [**8**].

Example 3.9. Here we construct a separated non-quasiprojective scheme defined by a single equation inside a multihomogeneous spectrum. Let

$$S = k[X_1, \dots, X_4, Y_1, \dots, Y_4, Z]$$

be a $\mathbb{Z}^2$-graded polynomial ring over a field k, with degrees $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (0, 1)$, and $\deg(Z) = (1, 1)$. Set $P = \mathrm{Proj}(S)$ and consider the open subset $U = D_+(X_1 Z) \cup D_+(Y_1 Z)$. This is not separated: We have

$$\Gamma(D_+(X_1 Z), \mathcal{O}_P) = k\left[\frac{X_i}{X_1}, \frac{X_1 Y_m}{Z}\right] \quad \text{and} \tag{1}$$

$$\Gamma(D_+(Y_1 Z), \mathcal{O}_P) = k\left[\frac{Y_j}{Y_1}, \frac{X_m Y_1}{Z}\right],$$

with $1 \leq i, j, m \leq 4$. On the intersection $D_+(X_1Y_1Z) = D_+(X_1Z) \cap D_+(Y_1Z)$, these algebras generate the subalgebra

$$k\left[\frac{X_i}{X_1}, \frac{Y_j}{Y_1}, \frac{X_1Y_1}{Z}\right] \subset k\left[\frac{X_i}{X_1}, \frac{Y_j}{Y_1}, \frac{X_1Y_1}{Z}, \frac{Z}{X_1Y_1}\right] = \Gamma(D_+(X_1Y_1Z), \mathcal{O}_P),$$

which does not contain Z/X_1Y_1. To obtain separated subschemes, we have to kill Z/X_1Y_1. Consider the homogeneous polynomials of degree $(2,2)$

$$g = X_1Y_1Z + X_2^2Y_1^2 + X_1^2Y_2^2 \quad \text{and} \quad f = X_1Y_1Z + X_2^2Y_1^2 + X_1^2Y_2^2 + X_3X_4Y_3Y_4.$$

Let $S' = S/(f)$, $P' = \operatorname{Proj}(S')$ and $U' = U \cap P'$. Modulo f, the element Z/X_1Y_1 is generated by the algebras in (1), thus U' is a separated scheme. It is, however, not quasiprojective. First observe that S' is a factorial domain: $S'_{X_3X_4Y_3} \cong k[X_0, \ldots, X_4, Y_1, Y_2, Y_3, Z]_{X_3X_4Y_3}$ is factorial and X_3, X_4, Y_3 are prime in S', because $g \in k[X_1, X_2, Y_1, Y_2, Z]$ is prime.

Choose points $x \in V_+(X_1, \ldots, X_4) \cap U'$ and $y \in V_+(Y_1, \ldots, Y_4) \cap U'$ (such points exist), and assume that they admit a common affine neighborhood $W \subset U'$. Then the preimage $V \subset \operatorname{Spec}(S')$ is affine as well. By factoriality, $V = D(h)$ for some homogeneous $h \in S$ with $h \in (X_1Z, Y_1Z)$. Write $h = pX_1Z + qY_1Z$. Let $\deg(h) = (d_1, d_2)$ and suppose $d_1 \geq d_2$. Since Y_1Z has degree $(1,2)$, it follows that $q \in (X_1, \ldots, X_4)$. But then $h(x) = 0$, contradiction.

4. Ample families and mappings to homogeneous spectra.

In this section, we shall relate homogeneous spectra to ample families. Let X be a scheme, D a finitely generated abelian group, and $\mathcal{B} = \bigoplus_{d \in D} \mathcal{B}_d$ a quasicoherent D-graded $\mathcal{O}_X$-algebra. We say that $\mathcal{B}$ is *periodic* if each stalk $\mathcal{B}_x$ is a periodic $\mathcal{O}_{X,x}$-algebra. For each homogeneous $f \in \Gamma(X, \mathcal{B})$, let $X_f \subset X$ be the largest open subset such that all multiplication maps $f^n : \mathcal{O}_X \to \mathcal{B}_{nd}$ with $n \geq 0, d = \deg(f)$ are bijective.

Proposition 4.1. *Let S be a D-graded ring, $X = \operatorname{Proj}(S)$ its homogeneous spectrum, $Y = \operatorname{Spec}(S) - V(S_+)$ the relevant locus, and $\pi : Y \to X$ the natural projection. Then $\pi_*(\mathcal{O}_Y)$ is a periodic $\mathcal{O}_X$-algebra. Furthermore, $X_f = D_+(f)$ for each relevant $f \in S$.*

Proof. By definition, S_g is a periodic $S_{(g)}$-algebra for each relevant $g \in S$, so $\pi_*(\mathcal{O}_Y)$ is a periodic $\mathcal{O}_X$-algebra with $\pi_*(\mathcal{O}_Y)_0 = \mathcal{O}_X$. The inclusion $D_+(f) \subset X_f$ is obvious. To verify $X_f \subset D_+(f)$, it suffices to check that $f \in S_g$ is invertible for each relevant $g \in S$ with $D_+(g) \subset X_f$. Replacing f by a positive multiple and S_g by a suitable Veronese subring, the ring S_g becomes isomorphic to the Laurent polynomial algebra $S_{(g)}[T_1^{\pm 1}, \ldots, T_r^{\pm 1}]$, and f corresponds to a monomial λT^d with $\lambda \in S_{(g)}^\times$. Hence f is invertible. $\square$

Next, we extend Grothendieck's description ([**11**] Prop. 3.7.3) of mappings into homogeneous spectra to the multigraded case:

Proposition 4.2. *Let X be a scheme, $\mathcal{B}$ a quasicoherent D-graded $\mathcal{O}_X$-algebra, S a D-graded ring, and $\varphi : S \to \Gamma(X, \mathcal{B})$ a graded homomorphism. Set $U = \bigcup X_{\varphi(f)}$, where the union runs over all relevant $f \in S$. Then there is a natural morphism $r_{\mathcal{B},\varphi} : U \to \operatorname{Proj}(S)$ and a commutative diagram*

$$\begin{tikzcd} U \arrow[d, "r_{\mathcal{B},\varphi}"'] & \operatorname{Spec}(\mathcal{B}_U) \arrow[l] \arrow[r] \arrow[d, "\varphi"] & \operatorname{Spec}(\mathcal{B}) \arrow[d, "\varphi"] \\ \operatorname{Proj}(S) & \operatorname{Spec}(S) - V(S_+) \arrow[l] \arrow[r] & \operatorname{Spec}(S). \end{tikzcd}$$

Proof. Each relevant $f \in S$ gives a homomorphism $S_{(f)} \to \Gamma(X, \mathcal{B})_{(f)}$, where we write f instead of $\varphi(f)$. Furthermore, there is a homomorphism

$$\Gamma(X, \mathcal{B})_{(f)} \longrightarrow \Gamma(X_f, \mathcal{O}_X), \quad g/f^n \mapsto (f^n|X_f)^{-1}(g),$$

where $(f^n|X_f)^{-1}$ is the inverse mapping for the bijective multiplication mapping $f^n|X_f : \mathcal{O}_X|X_f \to \mathcal{B}_{nd}|X_f$. The composition defines a morphism $X_f \to D_+(f)$. You easily check that these morphisms coincide on the overlaps, and we obtain the desired morphism $r_{\mathcal{B},\varphi} : U \to \operatorname{Proj}(S)$. $\square$

We write $r_{\mathcal{B},\varphi} : X \dashrightarrow \operatorname{Proj}(S)$ for the morphism $r_{\mathcal{B},\varphi} : U \to \operatorname{Proj}(S)$ and call it a *rational map*. Saying that a rational map is *everywhere defined* means $U = X$. In this case, we have a honest morphism $r_{\mathcal{B},\varphi} : X \to \operatorname{Proj}(S)$.

Corollary 4.3. *Let S be a D-graded ring. For each morphism $r : X \to \operatorname{Proj}(S)$, there is a quasicoherent periodic D-graded $\mathcal{O}_X$-algebra $\mathcal{B}$ and a homomorphism $\varphi : S \to \Gamma(X, \mathcal{B})$ such that the rational map $r_{\mathcal{B},\varphi} : X \dashrightarrow \operatorname{Proj}(S)$ is everywhere defined and coincides with $r : X \to \operatorname{Proj}(S)$.*

Proof. Let $Y = \operatorname{Spec}(S) - V(S_+)$ be the irrelevant locus, $\pi : Y \to \operatorname{Proj}(S)$ the canonical projection, and set $\mathcal{B} = r^*(\pi_*(\mathcal{O}_X))$. $\square$

We come to the characterization of ample families in terms of homogeneous spectra:

Theorem 4.4. *Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be a family of invertible sheaves on a quasi-compact and quasiseparated scheme X. Then the following conditions are equivalent:*

(i) *The family $\mathcal{L}_1, \ldots, \mathcal{L}_r$ is ample.*
(ii) *The canonical rational map $X \dashrightarrow \operatorname{Proj}\Gamma(X, \bigoplus_{d \in \mathbb{N}^r} \mathcal{L}^d)$ is everywhere defined and an open embedding.*

If X is of finite type over a noetherian ring R, this is also equivalent with:

(iii) *There is a finite family of sections* $f_i \in \Gamma(X, \mathcal{L}^{d_i}), i \in I$ *and a* D*-graded polynomial algebra* $A = R[T_i]_{i\in I}$ *such that the rational map* $X \dashrightarrow \mathrm{Proj}(A)$ *induced by* $T_i \mapsto f_i$ *is everywhere defined and an embedding.*

Proof. Set $S = \Gamma(X, \bigoplus_{d\in\mathbb{N}^r} \mathcal{L}^d)$. We start with the implication (i) $\Rightarrow$ (ii). According to Proposition 1.1, for each point $x \in X$, there is a $\mathbb{Q}$-basis $d_i \in \mathbb{N}^r$ and sections $f_i \in \Gamma(X, \mathcal{L}^{d_i})$ so that X_{f_i} are affine neighborhoods of x. Consequently, $f = f_1 \dots f_r \in S$ is relevant, so the rational map $X \dashrightarrow \mathrm{Proj}(S)$ is everywhere defined. Fix a relevant $f \in S$ so that X_f is affine. According to [**10**], the canonical map $\Gamma(X, \bigoplus \mathcal{L}^d)_f \to \Gamma(X_f, \bigoplus \mathcal{L}^d)$ is bijective. Consequently, $X_f \to D_+(f)$ is an isomorphism, so $X \to \mathrm{Proj}(S)$ is an open embedding. The reverse implication is trivial.

For the rest of the proof, suppose that X is of finite type over a noetherian ring R. Assume that $\mathcal{L}_1, \dots, \mathcal{L}_r$ is ample. Choose finitely many relevant $f_i \in S$ so that $X_{f_i} \subset X$ form an affine open cover. Due to the assumption we may write $\Gamma(X_{f_i}, \mathcal{O}_X) = R[h_{i1}, \dots, h_{im_i}]$. For suitable n we have $f_{ij} := f_i^n h_{ij} \in S$. Let $f_i \in S, i \in I$ be these elements all together.

Let $R[T_i]$ be graded by $d(T_i) = \deg(f_i)$ such that the natural mapping $R[T_i] \to S$ is homogeneous. Then $X_{f_i} \to D_+(f_i)$ are closed embeddings, since the ring morphisms are surjective, and so $X \to \mathrm{Proj}(A)$ is an embedding. Finally, the implication (iii) $\Rightarrow$ (ii) is trivial. $\square$

Corollary 4.5. *Let* $\mathcal{L}_1, \dots, \mathcal{L}_r$ *an ample family on* X*. For each relevant element* $f \in \Gamma(X, \bigoplus_{d\in\mathbb{N}^r} \mathcal{L}^d)$*, the open subset* X_f *is quasiaffine.*

Proof. According to Theorem 4.4, we have an open embedding $X_f \subset D_+(f)$. $\square$

Corollary 4.6. *Let* $\mathcal{L}_1, \dots, \mathcal{L}_r$ *an ample family of invertible sheaves on* X*. Set* $S = \Gamma(X, \bigoplus_{d\in\mathbb{N}^r} \mathcal{L}^d)$*. Then the following conditions are equivalent:*

(i) *The open embedding* $r : X \to \mathrm{Proj}(S)$ *is an isomorphism.*
(ii) *For each relevant* $f \in S$*, the quasiaffine open subset* $X_f \subset X$ *is affine.*

If the affine hull $X \to X^{\mathrm{aff}}$ *is proper, this is also equivalent to:*

(iii) *The homogeneous spectrum* $\mathrm{Proj}(S)$ *is separated.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from $D_+(f) = X_f^{\mathrm{aff}}$. Now assume that $X \to X^{\mathrm{aff}}$ is proper. The implication (i) $\Rightarrow$ (iii) is trivial. To see the converse, we apply [**11**] Corollary 5.4.3 and infer that the open dense embedding $X_f \to D_+(f)$ is proper, hence an isomorphism. $\square$

Finally, we generalize Hausen's [**17**] characterization of divisorial varieties:

Corollary 4.7. *Let* X *be a scheme of finite type over a noetherian ring* R*. Then the following are equivalent:*

(i) *The scheme* X *is divisorial.*

(ii) *There is an embedding of X into the homogeneous spectrum of a multi-graded R-algebra of finite type.*
(iii) *X is embeddable into a simplicial torus embedding with affine diagonal.*

Proof. If X is divisorial, Theorem 4.4 ensures the existence of an embedding $X \subset \mathrm{Proj}(S)$ with S finitely generated. The implication (ii) $\Rightarrow$ (iii) follows from Proposition 3.4, and (iii) $\Rightarrow$ (i) is trivial. $\square$

5. Cohomological characterization of ample families.

Throughout this section, R is a noetherian ring and X is a proper R-scheme. According to Serre's Criterion ([**12**] Prop. 2.6.1), an invertible $\mathcal{O}_X$-module $\mathcal{L}$ is ample if and only if for each coherent $\mathcal{O}_X$-module $\mathcal{F}$ there is an integer n_0 so that $H^p(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$ for all $p > 0$, $n > n_0$. The task now is to generalize this to ample families.

Let $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$ be a graded quasicoherent $\mathcal{O}_X$-algebra of finite type generated by $\mathcal{A}_1$. Then $P = \mathrm{Proj}(\mathcal{A})$ is a projective X-scheme and $\mathcal{O}_P(1)$ is an X-ample invertible sheaf. In general, however, $\mathcal{O}_P(1)$ is not ample in the absolute sense. More precisely:

Proposition 5.1. *With the preceding notation, the invertible sheaf $\mathcal{O}_P(1)$ is ample if and only if for each coherent $\mathcal{O}_X$-module $\mathcal{F}$, there is an integer n_0 so that $H^p(X, \mathcal{F} \otimes \mathcal{A}_n) = 0$ for $p > 0$ and $n > n_0$.*

Proof. Let $h : P \to X$ be the canonical projection. First, suppose that $\mathcal{M} = \mathcal{O}_P(1)$ is ample. Choose $n_0 > 0$ so that the canonical map $\mathcal{A}_n \to h_*(\mathcal{M}^n)$ is bijective and that $R^q h_*(h^*(\mathcal{F}) \otimes \mathcal{M}^n) = 0$ and $H^p(P, h^*(\mathcal{F}) \otimes \mathcal{M}^n) = 0$ holds for $p, q > 0$, $n > n_0$. Using the Leray–Serre spectral sequence we infer $H^p(X, h_*(h^*(\mathcal{F}) \otimes \mathcal{M}^n)) = 0$ for all $p > 0$, $n > n_0$.

We claim that the adjunction map $\mathcal{F} \otimes h_*(\mathcal{M}^n) \to h_*(h^*(\mathcal{F}) \otimes \mathcal{M}^n)$ is bijective for $n \gg 0$. Fix a point $x \in X$ and choose a finite presentation

$$\mathcal{O}_{X,x}^{\oplus k} \longrightarrow \mathcal{O}_{X,x}^{\oplus l} \longrightarrow \mathcal{F}_x \longrightarrow 0.$$

Then we have an exact sequence

$$\mathcal{O}_{X,x}^{\oplus k} \otimes h_*(\mathcal{M}^n) \longrightarrow \mathcal{O}_{X,x}^{\oplus l} \otimes h_*(\mathcal{M}^n) \longrightarrow \mathcal{F}_x \otimes h_*(\mathcal{M}^n) \longrightarrow 0,$$

and another exact sequence

$$h_*(\mathcal{O}_P^{\oplus k} \otimes \mathcal{M}^n) \longrightarrow h_*(\mathcal{O}_P^{\oplus l} \otimes \mathcal{M}^n) \longrightarrow h_*(h^*(\mathcal{F}) \otimes \mathcal{M}^n) \longrightarrow R^1 h_*(\mathcal{G} \otimes \mathcal{M}^n)$$

on $\mathrm{Spec}(\mathcal{O}_{X,x})$, where $\mathcal{G}$ is the kernel of $\mathcal{O}_P^{\oplus l} \to h^*(\mathcal{F})$. But $R^1 h_*(\mathcal{G} \otimes \mathcal{M}^n)_x = 0$ for $n \gg 0$. By the 5-Lemma, the Claim is true locally around x. Using quasicompactness, we infer that the Claim holds globally. Enlarging n_0 if necessary, we have $H^p(X, \mathcal{F} \otimes \mathcal{A}_n) = 0$ for $p > 0$ and $n > n_0$ as desired. The converse is similar. $\square$

Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The idea now is to consider coherent submodules $\mathcal{K} \subset \mathcal{L}$. Note that such submodules correspond to quasicoherent graded subalgebras $\bigoplus_{n\geq 0} \mathcal{K}^n \subset S(\mathcal{L})$ locally generated by terms of degree one. In turn, we obtain a projective X-scheme $P = \operatorname{Proj}(\bigoplus_{n\geq 0} \mathcal{K}^n)$ endowed with an X-ample invertible sheaf $\mathcal{O}_P(1)$. Locally, $P \to X$ looks like the blowing-up of an ideal.

Proposition 5.2. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, and $x \in X$ a closed point. Then the following are equivalent:*

(i) *For some $n > 0$, there is a section $f \in H^0(X, \mathcal{L}^n)$ so that X_f is an affine neighborhood of x.*

(ii) *For some $d > 0$, there is a coherent submodule $\mathcal{K} \subset \mathcal{L}^d$ with $\mathcal{K}_x \subset \mathcal{L}^d_x$ bijective, so that the graded $\mathcal{O}_X$-algebra $\mathcal{A} = \bigoplus_{n\geq 0} \mathcal{K}^n$ satisfies the equivalent conditions of Proposition* 5.1.

Proof. First, we check (i) $\Rightarrow$ (ii). Set $U = X_f$. According to [**19**] Proposition 5.4, there is a blowing-up $h : P \to X$ with center disjoint from U, together with an effective ample Cartier divisor $D \subset P$ satisfying $\operatorname{Supp}(D) = P - U$. Set $\mathcal{M} = \mathcal{O}_P(D)$, such that $P = \operatorname{Proj}(\bigoplus_{n\geq 0} h_*(\mathcal{M}^n))$. Replacing D by a suitable multiple, we may assume that $\bigoplus_{n\geq 0} h_*(\mathcal{M}^n)$ is generated by terms of degree one.

The identity $g_U : h_*(\mathcal{M})|U \to \mathcal{O}_X|U$ extends to a mapping $g : h_*(\mathcal{M}) \to \mathcal{L}^d$ for $d \gg 0$. This map is injective because a nonzero section

$$t \in \Gamma(P, \mathcal{O}_P(D)) = \Gamma(X, h_*(\mathcal{M}))$$

does not vanish on U. Let $\mathcal{K} \subset \mathcal{L}^d$ be the image of $g : h_*(\mathcal{M}) \to \mathcal{L}^d$. Then $\mathcal{K}^n = h_*(\mathcal{M}^n)$, because $\bigoplus_{n\geq 0} h_*(\mathcal{M}^n)$ is generated by terms of degree one. On $P = \operatorname{Proj}(\bigoplus \mathcal{K}^n)$ we have $\mathcal{O}_P(1) = \mathcal{M}$, which is ample as desired.

Now we check (ii) $\Rightarrow$ (i). Set $P = \operatorname{Proj}(\mathcal{A})$ and let $h : P \to X$ be the canonical morphism. Choose an affine open neighborhood $U \subset X$ of x so that the induced projection $h^{-1}(U) \to U$ is an isomorphism. Since $\mathcal{O}_P(1)$ is ample, there is an integer $m > 0$ and a section $g \in H^0(X, \mathcal{K}^m)$ so that the induced section $g' \in H^0(P, \mathcal{O}_P(m))$ vanishes on $P - h^{-1}(U)$ and is nonzero on the point $h^{-1}(x)$. Let f be the image of g under the inclusion $H^0(X, \mathcal{K}^m) \hookrightarrow H^0(X, \mathcal{L}^{dm})$. Let $y \in X_f$. Then $\mathcal{K}^m_y = \mathcal{L}^{dm}_y$ and X_f lies inside the locus where $h : P \to X$ is an isomorphism. Therefore $P_g \cong X_f \subset U$ and X_f is affine. $\square$

The preceding result leads to a characterization of ample families in terms of cohomology:

Theorem 5.3. *Let R be a noetherian ring and X a proper R-scheme. A family $\mathcal{L}_1, \ldots, \mathcal{L}_r$ of invertible $\mathcal{O}_X$-modules is ample if and only if for each closed point $x \in X$, there is $d \in \mathbb{N}^r$ and a coherent subsheaf $\mathcal{K} \subset \mathcal{L}^d$ with*

$\mathcal{K}_x = \mathcal{L}_x^d$ so that for each coherent $\mathcal{O}_X$-module $\mathcal{F}$ there is an integer n_0 with $H^p(X, \mathcal{F} \otimes \mathcal{K}^n) = 0$ for all $p > 0$, $n > n_0$.

Proof. This follows directly from Proposition 5.2. □

6. Algebraization of formal schemes via ample families.

Throughout this section, $(R, \mathfrak{m}, k)$ denotes a complete local noetherian ring, and $\mathfrak{X} \to \operatorname{Spf}(R)$ is a proper formal scheme. Such formal schemes frequently occur as formal solutions for problems related to moduli spaces and deformation theories. A natural question is whether such a formal scheme is *algebraizable.* This means that there is a proper scheme $X \to \operatorname{Spec}(R)$ whose $\mathfrak{m}$-adique completion is isomorphic to $\mathfrak{X}$. Grothendieck's Algebraization Theorem ([**12**] Thm. 5.4.5) asserts that $\mathfrak{X}$ is algebraizable if there is an invertible $\mathcal{O}_{\mathfrak{X}}$-module whose restriction to the closed fiber $X_0 = \mathfrak{X} \otimes k$ is ample. Here is a generalization:

Theorem 6.1. *Let $\mathfrak{X}$ be a proper formal scheme as above, and $\mathcal{L}_1, \ldots, \mathcal{L}_r$ a family of invertible $\mathcal{O}_{\mathfrak{X}}$-modules. Suppose that for each closed point $x \in \mathfrak{X}$, there is a degree $d \in \mathbb{N}^r$ and a coherent submodule $\mathcal{K} \subset \mathcal{L}^d$ with $\mathcal{K}_x \subset \mathcal{L}_x^d$ bijective, so that $\mathcal{O}_{P_0}(1)$ is ample on $P_0 = \operatorname{Proj}(\bigoplus_{m \geq 0} \mathcal{K}^m / \mathcal{I}\mathcal{K}^m)$. Then the formal scheme $\mathfrak{X}$ is algebraizable.*

Proof. Set $X_n = \mathfrak{X} \otimes R/\mathfrak{m}^{n+1}$, such that $\mathfrak{X} = \varinjlim X_n$, and let $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}}$ be the ideal of the closed fiber $X_0 \subset \mathfrak{X}$. As in the Proof of Proposition 5.2, there is an integer $m > 0$ and a global section $s_0 \in H^0(X_0, \mathcal{K}^m / \mathcal{I}\mathcal{K}^m)$ so that the induced section $t_0 \in \Gamma(X_0, \mathcal{L}^{dm} / \mathcal{I}\mathcal{L}^{dm})$ defines an affine open neighborhood $x \in (X_0)_{t_0}$. We seek to extend such sections to formal sections.

For each $m \geq 0$, set $\mathcal{A}_m = \bigoplus_{n \geq 0} \mathcal{I}^n \mathcal{K}^m / \mathcal{I}^{n+1} \mathcal{K}^m$, and let $\mathcal{A} = \bigoplus_{m \geq 0} \mathcal{A}_m$ be the corresponding $\mathbb{N}$-graded quasicoherent $\mathcal{O}_{X_0}$-algebra. Consider its homogeneous spectrum $P = \operatorname{Proj}(\mathcal{A})$. We claim that the invertible sheaf $\mathcal{O}_P(1)$ is ample. To see this, let X' be the affine X_0-scheme defined by $\mathcal{O}_{X'} = \bigoplus_{n \geq 0} \mathcal{I}^n / \mathcal{I}^{n+1}$. Note that X' is proper over the noetherian ring $R' = \bigoplus \mathfrak{m}^n / \mathfrak{m}^{n+1}$. Set

$$P' = \operatorname{Proj}\left(\bigoplus_{m \geq 0}\left(\bigoplus_{n \geq 0} \mathcal{I}^n / \mathcal{I}^{n+1} \otimes \mathcal{K}^m / \mathcal{I}\mathcal{K}^m\right)\right),$$

where the homogeneous spectrum is taken with respect to the grading $m \in \mathbb{Z}$. We have $P' = X' \times_{X_0} P_0$ and conclude that $\mathcal{O}_{P'}(1)$ is ample.

The surjective mapping $\mathcal{I}^n \otimes_{\mathfrak{X}} \mathcal{K}^m \to \mathcal{I}^n \mathcal{K}^m$ induces a surjective mapping

$$\begin{aligned} \mathcal{I}^n / \mathcal{I}^{n+1} \otimes_{X_0} \mathcal{K}^m / \mathcal{I}\mathcal{K}^m = \mathcal{I}^n \otimes_{\mathfrak{X}} \mathcal{K}^m \otimes_{\mathfrak{X}} \mathcal{O}_{X_0} &\longrightarrow \mathcal{I}^n \mathcal{K}^m \otimes_{\mathfrak{X}} \mathcal{O}_{X_0} \\ &= \mathcal{I}^n \mathcal{K}^m / \mathcal{I}^{n+1} \mathcal{K}^m. \end{aligned}$$

This yields a closed embedding $P \subset P'$, showing that $\mathcal{O}_P(1)$ is ample. To proceed, consider the exact sequence

$$H^0(X_n, \mathcal{K}^m/\mathcal{I}^{n+1}\mathcal{K}^m) \longrightarrow H^0(X_{n-1}, \mathcal{K}^m/\mathcal{I}^n\mathcal{K}^m) \longrightarrow H^1(X_0, \mathcal{I}^n\mathcal{K}^m/\mathcal{I}^{n+1}\mathcal{K}^m).$$

By Proposition 5.1, there is an integer $m_0 > 0$ so that the group on the right is zero for all $m \geq m_0$ and all $n \geq 0$. Passing to a suitable multiple if necessary, we can lift our section $s_0 \in H^0(X_0, \mathcal{K}^m/\mathcal{I}\mathcal{K}^m)$ to a formal section $s \in H^0(\mathfrak{X}, \mathcal{K}^m)$. Therefore, the section $t_0 \in \Gamma(X_0, \mathcal{L}^{dm}/\mathcal{I}\mathcal{L}^{dm})$ lifts to a formal section $t \in \Gamma(\mathfrak{X}, \mathcal{L}^{dm})$.

Using such formal sections, you construct as in the Proof of Theorem 4.4 a finitely generated $\mathbb{N}^r$-graded polynomial R-algebra S and a compatible sequence of embeddings $X_n \subset \mathrm{Proj}(S)$. Choose an open neighborhood $U \subset \mathrm{Proj}(S)$ so that $X_n \subset U$ are closed embeddings. Let $I_n \subset S$ be the graded ideal of the closed embedding $\overline{X}_n \subset \mathrm{Proj}(S)$, and set $I = \bigcap_{n \geq 0} I_n$. Then $X = \mathrm{Proj}(S/I) \cap U$ is the desired algebraization of the formal scheme $\mathfrak{X}$. □

Remark 6.2. If we have $\mathcal{K} = \mathcal{L}^d$, then $P_0 = X_0$, such that the formal sheaf $\mathcal{L}$ is ample on the closed fiber X_0. In this case, Grothendieck's Algebraization Theorem ensures that $\mathfrak{X}$ is algebraizable.

Question 6.3. The assumptions in Theorem 6.1 imply that the restriction of the family $\mathcal{L}_1, \ldots, \mathcal{L}_r$ to the closed fiber is ample. A natural question to ask: Given a proper formal scheme with a family of invertible sheaves whose restriction to the closed fiber is ample – is the formal scheme algebraizable?

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