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Ulrich Clarenz

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In this paper we consider parametric integrals with elliptic integrands depending on the surface normal. The main result is a stability criterion for extremal immersions of those functionals, containing a result of Barbosa and do Carmo for minimal surfaces as a special case. Using similar techniques we are also able to show a condition for instability. The last section contains a simple proof of the fact that the surface normal of extremals of parametric integrals is topologically equivalent to a holomorphic function.

1. Introduction.

Since the last century, mathematicians have tried to find criteria for stability and instability of a given minimal surface. First important criteria of stability and instability were proven by Schwarz (for a survey of his results concerning stability of minimal surfaces see [**9**, pp. 90-116]).

He showed that a minimal surface is stable if its spherical image is fully contained in a half-sphere [**9**, p. 99].

In 1976, Barbosa and do Carmo were able to generalize this criterion considerably. They proved that if the spherical image of a given minimal surface has area less than 2π, the surface is stable [**1**].

Barbosa and do Carmo generalized this result in several directions as e.g., for different target manifolds and for general dimensions [**2**].

Ruchert proved a similar criterion for the case of surfaces of constant mean curvature [**11**].

In [**7**], Fischer-Colbrie and Schoen were able to give a shorter proof of the result of Barbosa and do Carmo [**1**]. Here we will follow the idea of Fischer-Colbrie and Schoen to prove a more general theorem for critical points of parametric functionals. In our context a parametric functional is defined for an immersion $X : M \to \mathbb{R}^3$ with normal mapping $N : M \to S^2$ by

$$\mathcal{F}(X) := \int_M F(N)\, dA,$$

where M is a two-dimensional, orientable manifold.

The integrand $F : \mathbb{R}^3 \to \mathbb{R}$ is a 1-homogeneous function, i.e., $F(tz) = tF(z)$ for all $t > 0$. Immersions satisfying the Euler equation of $\mathcal{F}$ are called *$\mathcal{F}$-extremals* or *$\mathcal{F}$-minimal surfaces.*

Furthermore we call $D \subset M$ a *stable domain* of an $\mathcal{F}$-extremal X if for all $\varphi \in C_0^\infty(D)$ with $\varphi \not\equiv 0$ the second variation

$$\delta^2\mathcal{F}(X,\varphi) = \frac{d^2}{d\epsilon^2}\mathcal{F}(X_\epsilon)_{|\epsilon=0},\ X_\epsilon = X + \epsilon\varphi N,$$

is positive (Definition 4.1).

In this paper we only deal with *elliptic integrands.* Ellipticity means that

$$F_{zz}(z) : z^\perp \to z^\perp$$

is a positive definite endomorphism of $z^\perp$ for all $z \in \mathbb{R}^3 - \{0\}$.

Here we state our main result:

Theorem 1.1. *Let $X : M \to \mathbb{R}^3$ be an $\mathcal{F}$-extremal immersion of an elliptic integrand F. If the area of the spherical image of a smooth domain $D \subset M$ is smaller than a positive number a_F depending on F, then D is a stable domain of X.*

In the case of $F(z) = |z|$, the number a_F is 2π. Furthermore, a_F depends continuously on F_{zz} w.r.t. $C^0(S^2)$.

Using similar techniques as in the proof of Theorem 1.1 we can also prove Theorem 4.5, a generalization for $\mathcal{F}$-extremals of a Schwarz criterion for instability (see [**1**, Theorem 2.7]).

The paper is organized as follows:

In Section 2 we define the notion of a *degenerate metric* $\widetilde{g}$ on a Riemannian manifold (M, g) and give important properties of them. Degenerate metrics $\widetilde{g}$ are allowed to have isolated singularities and g and $\widetilde{g}$ are related by the equivalence given in Definition 2.1. One example of such a degenerate metric is $g_N(V,W) := g(DN(V), DN(W))$ for a minimal immersion X with canonical metric g and normal mapping N (in fact $g_N = -Kg$, where K is the Gauß-curvature).

In Section 3 we will study basic properties of parametric functionals. Especially we will see that for $\mathcal{F}$-extremals, g_N generally is not conformal as in the minimal surface case but still degenerate in the sense of Definition 2.1.

In Section 4 we prove the main result Theorem 1.1. First we give an estimate of the second variation $\delta^2\mathcal{F}$, showing a connection between stability and eigenvalue problems for the Laplacian Δ_{S^2} on the sphere S^2 (Proposition 4.2):

$$\delta^2\mathcal{F}(X,\varphi) \geq C_F \int_M |grad_N\varphi|_N^2 - 2c_F\varphi^2\, dA_N. \tag{1}$$

Here $c_F \geq 1$ and C_F are constants depending on F. The subscript N indicates the relation to g_N. Now we use a Faber-Krahn argument to show that the right-hand side of (1) is positive. To this aim we take a first (positive) eigenfunction u of Δ_{S^2} of the spherical image of a domain D, i.e.,

$$\Delta_{S^2} u + \mu_1 u = 0.$$

Lifting this equation onto M we obtain a positive function v on M (because N is an open mapping) satisfying outside the singularities of g_N:

$$\Delta_N v + \mu_1 v = 0. \tag{2}$$

At this point it is crucial that g_N is a degenerate metric. Using Proposition 2.4, a generalization of [**7**, Corollary 1], Equation (2) implies the inequality $\int |\mathrm{grad}_N \varphi|_N^2 - \mu_1 \varphi^2 \, dA_N > 0$ for all $\varphi \in C_0^\infty(D)$ and $\varphi \not\equiv 0$. Now we can choose a_F dependent on c_F such that $\mu_1 > 2c_F$.

In the above reasoning we have used that N is an open mapping. This was proven by Sauvigny [**12**, p. 94]. As a by-product of our considerations we can give a simple proof of the fact that for $\mathcal{F}$-extremals N is a local branched covering. This is the content of Section 5.

2. Degenerate metrics.

Let (M, g) be a two-dimensional Riemannian manifold. The key notion for our considerations is given in the following:

Definition 2.1. A symmetric bilinear form $\widetilde{g}$ on (M, g) is a *degenerate metric* if for all $V \in T_pM$

$$\Gamma\, h_1\, g(V, V) \leq \widetilde{g}(V, V) \leq \Gamma\, h_2\, g(V, V),$$

where h_1, $h_2 > 0$, $\Gamma \geq 0$ are smooth functions on M. Furthermore, the set $\{\Gamma = 0\}$ consists of isolated points only.

Lemma 2.2. *Let g_1, g_2 be two metrics on a manifold M. If for all $V \in T_pM$*

$$g_1(V, V) \leq g_2(V, V) \tag{3}$$

then we have for all smooth functions φ:

$$|\mathrm{grad}_1 \varphi|_1 \geq |\mathrm{grad}_2 \varphi|_2,$$

where the index indicates the corresponding metric.

Proof. We have the identity

$$g_1(\mathrm{grad}_1 \varphi, V) = g_2(\mathrm{grad}_2 \varphi, V),$$

for all $V \in T_pM$. Using the Cauchy-Schwarz inequality, this leads to

$$\begin{aligned} |g_2(\mathrm{grad}_2 \varphi, V)| &\leq |\mathrm{grad}_1 \varphi|_1 \, |V|_1 \\ &\leq |\mathrm{grad}_1 \varphi|_1 \, |V|_2, \end{aligned}$$

because of inequality (3). Setting $V = \mathrm{grad}_2 \varphi$ gives the assertion. □

As a direct consequence of the above lemma we obtain:

Corollary 2.3. *If $\widetilde{g}$ is a degenerate metric on a Riemannian manifold (M, g), the following inequality holds on $\{\Gamma \neq 0\}$:*

$$\Gamma\, h_1\, |\mathrm{grad}_{\widetilde{g}}\, \varphi|^2_{\widetilde{g}} \leq |\mathrm{grad}_g\, \varphi|^2_g \leq \Gamma\, h_2\, |\mathrm{grad}_{\widetilde{g}}\, \varphi|^2_{\widetilde{g}}.$$

The next proposition is a generalization of [7, Corollary 1]:

Proposition 2.4. *Let (M, g) be a Riemannian manifold with degenerate metric $\widetilde{g}$. Furthermore let q be a smooth function on M. Assume that there is a smooth function v on M, $v > 0$ on a domain D, with*

$$\Delta_{\widetilde{g}} v - qv = 0 \tag{4}$$

on $\{\Gamma > 0\} \cap D$. Then for all $\varphi \in C_0^\infty(D)$ we have

$$\int_D |\mathrm{grad}_{\widetilde{g}}\, \varphi|^2_{\widetilde{g}} + q\varphi^2\, dA_{\widetilde{g}} \geq 0.$$

Note that one can estimate the form $|\mathrm{grad}_{\widetilde{g}}\varphi|_{\widetilde{g}} dA_{\widetilde{g}}$ by $|\mathrm{grad}_g\varphi|^2_g h_2/h_1\, dA_g$. Therefore the integral $\int_D |\mathrm{grad}_{\widetilde{g}}\, \varphi|^2_{\widetilde{g}}\, dA_{\widetilde{g}}$ is well-defined.

Proof of Proposition 2.4. The zeroes of Γ are isolated, i.e., $\mathrm{supp}\, \varphi \cap \overline{D} \cap \{\Gamma > 0\}$ is a finite set $\{p_1, \ldots, p_k\}$. In the following reasoning we may assume $p_i \notin \partial D$. Let $B_\epsilon(p_i)$ be a geodesic ball in M (measured in the g-metric) of radius $0 < \epsilon << 1$, such that

$$V_\epsilon := \bigcup_{i=1}^{k} B_\epsilon(p_i)$$

is a disjoint union. Because of (4) we have with $w = \log v$

$$\Delta_{\widetilde{g}} w = q - |\mathrm{grad}_{\widetilde{g}}\, w|^2_{\widetilde{g}}.$$

Integration by parts leads to

$$\begin{aligned}
&\int_{D - V_\epsilon} \varphi^2 |\mathrm{grad}_{\widetilde{g}}\, w|^2_{\widetilde{g}} - q\varphi^2\, dA_{\widetilde{g}} \\
&= -\int_{D - V_\epsilon} \varphi^2\, \Delta_{\widetilde{g}} w\, dA_{\widetilde{g}} \\
&= \int_{D - V_\epsilon} 2\varphi\, \widetilde{g}(\mathrm{grad}_{\widetilde{g}}\, \varphi, \mathrm{grad}_{\widetilde{g}}\, w)\, dA_{\widetilde{g}} \\
&\quad + \sum_{i=1}^{k} \int_{\partial B_\epsilon(p_i)} \varphi^2\, \widetilde{g}(\mathrm{grad}_{\widetilde{g}}\, w, \widetilde{\nu})\, d\sigma_{\widetilde{g}},
\end{aligned}$$

with $\varphi \in C_0^\infty(D)$ and $\widetilde{\nu}$ being a suitable unit normal on $\partial B_\epsilon(p_i)$. Using the fact that $\widetilde{g}$ is a degenerate metric on (M, g), we obtain the following estimate:

$$\begin{aligned}
&\left| \int_{\partial B_\epsilon(p_i)} \varphi^2 \, \widetilde{g}(\operatorname{grad}_{\widetilde{g}} w, \widetilde{\nu}) \, d\sigma_{\widetilde{g}} \right| \\
&\leq \int_{\partial B_\epsilon(p_i)} \varphi^2 \, |\operatorname{grad}_{\widetilde{g}} w|_{\widetilde{g}} \, d\sigma_{\widetilde{g}} \\
&\leq \int_{\partial B_\epsilon(p_i)} \varphi^2 |\operatorname{grad}_g w|_g \sqrt{h_2/h_1} \, d\sigma_g = O(\epsilon), \quad \text{with } \epsilon \to 0.
\end{aligned}$$

Thus we have

(5)

$$\int_{D-V_\epsilon} \varphi^2 |\operatorname{grad}_{\widetilde{g}} w|^2_{\widetilde{g}} - q\varphi^2 \, dA_{\widetilde{g}} = \int_{D-V_\epsilon} 2\varphi \, \widetilde{g}(\operatorname{grad}_{\widetilde{g}} w, \operatorname{grad}_{\widetilde{g}} \varphi) \, dA_{\widetilde{g}} + O(\epsilon).$$

This implies

$$O(\epsilon) \leq \int_{D-V_\epsilon} |\operatorname{grad}_{\widetilde{g}} \varphi|^2_{\widetilde{g}} + q\varphi^2 \, dA_{\widetilde{g}} \, .$$

$\square$

3. Parametric functionals.

Now we consider immersed surfaces $X : M \to \mathbb{R}^3$, where M is a two-dimensional and oriented manifold, equipped with the metric $g(V, W) = \langle DX(V), DX(W) \rangle$ for V, $W \in T_pM$ and a normal mapping

$$N : M \to S^2.$$

A parametric functional $\mathcal{F}$ is given by a smooth 1-homogeneous integrand

$$F : \mathbb{R}^3 \to \mathbb{R}.$$

The corresponding functional $\mathcal{F}$ is defined by $\mathcal{F}(X) := \int_M F(N) \, dA$.

As a version of F_{zz} on TM we define

$$\begin{aligned}
A_F : T_pM &\longrightarrow T_pM \\
V &\longmapsto DX^{-1} \, F_{zz}(N(p)) \, DX(V).
\end{aligned}$$

Definition 3.1. An integrand F is called *elliptic* if the linear mapping

$$F_{zz}(z) : z^\perp \to z^\perp$$

is positive definite for all $z \in \mathbb{R}^3 - \{0\}$.

Using the endomorphism A_F the Euler equation of $\mathcal{F}$ is: $-\operatorname{tr}(A_F \circ S) = 0$, where S is the shape operator $DX \circ S := DN$ (for the corresponding computation see, e.g., [4] or [5]). The trace $H_F = -\operatorname{tr}(A_F \circ S)$ is called the *F-mean curvature* H_F generalizing the classical mean curvature $H = -\operatorname{tr} S$.

Now we want to prove for extremals X of $\mathcal{F}$ that the bilinear form

$$g_N(V,W) := \langle DN(V), DN(W)\rangle \ ,\ V,W \in T_pM$$

is a degenerate metric on (M,g).

First, we apply the Cayley-Hamilton Theorem and get

$$(A_F S)^2 + K_F\,\mathrm{id}_{T_pM} = 0, \tag{6}$$

where $K_F := \det(A_F \circ S) = (\det A_F)\,K$. The so called *$F$-Gauß curvature K_F* generalizes the classical *Gauß curvature* $K = \det S$. Equation (6) implies for all $V \in T_pM$ the identity

$$g(A_F S\,V, S\,V) = -K_F\,g(A_F^{-1}\,V, V). \tag{7}$$

In the following, the eigenvalues of A_F are denoted by $\lambda_1 \le \lambda_2$. Now we estimate the left- and right-hand side of (7) for elliptic integrands.

$$\begin{aligned} \lambda_1\,g_N(V,V) \le& g(A_F S\,V, S\,V) \le \lambda_2\,g_N(V,V),\\ -K_F\,\frac{1}{\lambda_2}\,g(V,V) \le& -K_F\,g(A_F^{-1}V,V) \le -K_F\,\frac{1}{\lambda_1}\,g(V,V). \end{aligned}$$

Because of (7) we arrive at

$$-K_F\,\frac{1}{\lambda_2^2}\,g(V,V) \le g_N(V,V) \le -K_F\,\frac{1}{\lambda_1^2}\,g(V,V)$$

and keeping in mind $K_F = \lambda_1\lambda_2\,K$ we state:

Proposition 3.2. *Let F be an elliptic integrand. For an $\mathcal{F}$-minimal surface $X : M \to \mathbb{R}^3$ the bilinear form g_N is a degenerate metric on (M,g). More precisely we have*

$$-K\,\frac{\lambda_1}{\lambda_2}\,g(V,V) \le g_N(V,V) \le -K\,\frac{\lambda_2}{\lambda_1}\,g(V,V).$$

For the proof of the above proposition it remains to show that $\{K = 0\}$ is a set of isolated points but this is a result of Sauvigny [**13**, p. 53].

4. Stability.

In this part we assume that F is an elliptic integrand. First we give a definition of stability.

Definition 4.1. Let $X : M \to \mathbb{R}^3$ be an $\mathcal{F}$-extremal immersion. The surface X is called stable if for all $\varphi \in C_0^\infty(M)$ with $\varphi \not\equiv 0$ the relation

$$\int_M g(A_F\mathrm{grad}\varphi, \mathrm{grad}\varphi) + \mathrm{tr}A_F K\varphi^2\,dA > 0, \tag{8}$$

is satisfied, where K is the Gauß-curvature of X. We say that $D \subset M$ is a stable domain if (8) is fulfilled for all $\varphi \in C_0^\infty(D)$ with $\varphi \not\equiv 0$.

This definition generalizes the notion of stable minimal surfaces (see e.g., [**6**, p. 84] or [**9**, p. 96]; note that in this case $A_F = \mathrm{id}_{T_pM}$ and $\mathrm{tr}A_F = 2$). Its motivation is as follows:

If we consider $\varphi \in C_0^\infty(M)$ and the related disturbed surface $X_\epsilon = X + \epsilon\varphi N$, then the second variation $\delta^2\mathcal{F}(X,\varphi) = \frac{d^2}{d\epsilon^2}\mathcal{F}(X_\epsilon)_{|\epsilon=0}$ in direction φ is given by the quadratic form in (8). The corresponding computation can be found in [**10**] or in [**13**].

In the following, all notions with subscript N are related to the degenerate metric g_N. For the proof of the main result, we start with:

Proposition 4.2. *Let $X : M \to \mathbb{R}^3$ be an $\mathcal{F}$-critical immersion. Then we can conclude that $D \subset M$ is a stable domain if for all $\varphi \in C_0^\infty(D)$ and $\varphi \not\equiv 0$*

$$\int_M |\mathrm{grad}_N\varphi|_N^2 - 2c_F\varphi^2\, dA_N > 0,$$

where $c_F := \max_{S^2}\left(\frac{\lambda_1+\lambda_2}{2}\right) \Big/ \min_{S^2}\left(\frac{\lambda_1^2}{\lambda_2}\right)$.

Proof. Using the ellipticity of F and the fact that g_N is a degenerate metric we obtain because of Lemma 2.2 and Proposition 3.2

$$|\mathrm{grad}\varphi|^2 \geq -K\frac{\lambda_1}{\lambda_2}|\mathrm{grad}_N\varphi|_N^2.$$

This leads to

$$\begin{aligned}
\delta^2\mathcal{F}(X,\varphi) &= \int_M g(A_F\mathrm{grad}\varphi, \mathrm{grad}\varphi) + (\lambda_1+\lambda_2)K\varphi^2\, dA \\
&\geq \int_M \lambda_1|\mathrm{grad}\varphi|^2 + (\lambda_1+\lambda_2)K\varphi^2\, dA \\
&\geq \int_M -K\frac{\lambda_1^2}{\lambda_2}|\mathrm{grad}_N\varphi|_N^2 + (\lambda_1+\lambda_2)K\varphi^2\, dA.
\end{aligned}$$

The equation $\mathrm{tr}(A_FS) = 0$ characterizing $\mathcal{F}$-extremals implies the inequality $\det(A_FS) \leq 0$ and the ellipticity of F gives $K = \det S \leq 0$. Therefore we have $dA_N = -K\, dA$. This completes the proof because of

$$\begin{aligned}
\delta^2\mathcal{F}(X,\varphi) &\geq \int_M \frac{\lambda_1^2}{\lambda_2}|\mathrm{grad}_N\varphi|_N^2 - (\lambda_1+\lambda_2)\varphi^2\, dA_N, \\
&\geq \min_{S^2}\left(\frac{\lambda_1^2}{\lambda_2}\right)\int_M |\mathrm{grad}_N\varphi|_N^2 - 2c_F\varphi^2\, dA_N.
\end{aligned}$$

□

The above proposition shows a connection between the stability of $\mathcal{F}$-extremals and eigenvalue problems for the Laplacian on the sphere S^2.

We will denote the first eigenvalue w.r.t. the Laplacian of a domain $D \subset S^2$ by $\mu_1(D)$. For a proof of the following proposition, we refer to [2, pp. 19, 20] and [3, pp. 50]:

Proposition 4.3. *Assume that $D \subset S^2$ is a domain of area $A(D) = A$. Then we have:*

(i) $$A \leq 2\pi \ \textit{implies} \ \mu_1(D) \geq \frac{4\pi}{A},$$

(ii) $$A \geq 2\pi \ \textit{implies} \ \mu_1(D) \geq 2\frac{4\pi - A}{A}.$$

If in addition D is a geodesic disc on S^2, then

(iii) $$\mu_1(D) \leq \frac{4\pi}{A} \ \textit{for} \ A \geq 2\pi,$$

(iv) $$\lim_{A \to 4\pi} \mu_1(D) = 0.$$

We see that for all $\mu \geq 2$ there is a spherical cap in S^2 whose first eigenvalue of the Laplacian is exactly μ. The area of such a spherical cap is denoted by $a(\mu)$. For elliptic integrands we define

$$a_F := a(2c_F).$$

This enables us to prove the main result Theorem 1.1.

Proof of Theorem 1.1. Let Δ_{S^2} be the Laplacian on S^2. On $\{K \neq 0\}$ the normal N is a local isometry between S^2 and (M, g_N). We assume that μ_1 is the first eigenvalue of $N(D)$ on S^2. Then there is a positive function u in the interior of $N(D)$ satisfying $u_{|\partial N(D)} = 0$ and

$$\Delta_{S^2} u + \mu_1 u = 0.$$

In [13] it is proven that N is an open mapping (see also Section 5). Therefore we can conclude that $v := u \circ N$ is positive in D and satisfies on $K \neq 0$

$$\Delta_N v + \mu_1 v = 0.$$

Because of Proposition 2.4 we have for all $\varphi \in C_0^\infty(D)$

$$\int_D |\mathrm{grad}_N \varphi|_N^2 - \mu_1 \varphi^2 \, dA_N \geq 0.$$

It is a well-known fact that the spherical caps on S^2 are minimizers of the first eigenvalue of Δ_{S^2} among all domains in S^2 of the same area [2, p. 18] and therefore $\mu_1 > 2c_F$. Thus we can conclude:

$$\begin{aligned}
&\int_D |\mathrm{grad}_N \varphi|_N^2 - 2c_F \varphi^2 \, dA_N \\
&= \int_D |\mathrm{grad}_N \varphi|_N^2 - \mu_1 \varphi^2 \, dA_N + (\mu_1 - 2c_F) \int_D \varphi^2 \, dA_N > 0.
\end{aligned}$$

Using Proposition 4.2, stability is proven.

In case of the area-functional, i.e., $F(z) = |z|$, the constant c_F is exactly 1. Therefore, for this functional we have $a_F = a(2) = 2\pi$ and Theorem 1.1 contains the main result of [**1**] as a special case. $\square$

The proof of the following proposition is very similar to the proof of Proposition 4.2:

Proposition 4.4. *Assume that $X : M \to \mathbb{R}^3$ is an $\mathcal{F}$-extremal and that there is a $\varphi \in C^\infty(D) \cap C^0(\overline{D})$, $D \subset S^2$, satisfying $\varphi_{|\partial D} = 0$ and*

$$\int_D |\mathrm{grad}_N \varphi|_N^2 - 2d_F\varphi^2\, dA_N < 0,$$

where $d_F := \min_{S^2}(\frac{\lambda_1+\lambda_2}{2})/\max_{S^2}(\frac{\lambda_2^2}{\lambda_1})$. Then X cannot be stable in D.

This proposition leads to the following generalization of the Schwarz criterion [**1**, Theorem 2.7] for instability of minimal surfaces:

Theorem 4.5. *Let $X : M \to \mathbb{R}^3$ be an $\mathcal{F}$-minimal surface. If $N : \overline{D} \to S^2$ is a branched covering of $N(\overline{D})$ and if the first eigenvalue of $N(D)$ for Δ_{S^2} is smaller than $2d_F$, then D cannot be a stable domain of X.*

Proof. As in the Proof of Theorem 1.1, we consider an eigenfunction $u > 0$ of $N(D)$ for Δ_{S^2}, i.e.,

$$\begin{aligned} \Delta_{S^2} u + \mu_1 u &= 0 \text{ in } N(D) \\ u &= 0 \text{ on } \partial N(D). \end{aligned}$$

Lifting u on D we obtain the equation

$$\int_D |\mathrm{grad}_N v|_N^2 - 2d_F v^2\, dA_N = (\mu_1 - 2d_F)\int_D v^2\, dA_N$$

for the function $v := u \circ N$. By assumption we have $\mu_1 < 2d_F$ and Proposition 4.4 completes the proof. $\square$

Let us conclude this section applying the main result Theorem 1.1 to a certain class of integrands F^β, where

$$F^\beta(z) = \sqrt{\beta|z_1|^2 + |z_2|^2 + |z_3|^2},$$

and $\beta > 0$.[1]

The positive eigenvalues of $F^\beta_{zz}(z)$ for $z \in S^2$ are given by $1/F^\beta(z)$ and $\beta/[F^\beta(z)]^3$. For $\beta \geq 1$ one has $\lambda_1 = 1/F^\beta(z)$ and $\lambda_2 = \beta/[F^\beta(z)]^3$. In case $\beta < 1$ we get $\lambda_1 = \beta/[F^\beta(z)]^3$ and $\lambda_2 = 1/F^\beta(z)$. This leads to

$$\max_{S^2}\left(\frac{\lambda_1+\lambda_2}{2}\right) = \begin{cases} (1+\beta)/2 & : \quad \beta \geq 1 \\ 1/\sqrt{\beta} & : \quad \beta < 1 \end{cases}$$

[1]This example was added in proof.

$$\min_{S^2}\left(\frac{\lambda_1^2}{\lambda_2}\right)=\begin{cases} 1/\beta & : \quad \beta \geq 1 \\ \beta^2 & : \quad \beta < 1. \end{cases}$$

The constant c_{F^β} (see Proposition 4.2) is dependent on β in the following way:

$$c_{F^\beta}=\begin{cases} \beta\,(1+\beta)/2 & : \quad \beta \geq 1 \\ 1/(\beta^2\sqrt{\beta}) & : \quad \beta < 1. \end{cases}$$

Thus we see $\lim_{\beta\to 0} c_{F^\beta} = \lim_{\beta\to\infty} c_{F^\beta} = \infty$ and therefore the corresponding area a_{F^β} (see Theorem 1.1) tends to zero in both cases. Thus for extreme anisotropic integrands the condition for stability is very strong. This is also true for the instability criterion Theorem 4.5 because in this case $\lim_{\beta\to 0} d_{F^\beta} = \lim_{\beta\to\infty} d_{F^\beta} = 0$.

5. A topological property of the normal of *F*-minimal surfaces.

In the Proof of Theorem 1.1 we used the open-mapping property of the normal N of an $\mathcal{F}$-minimal surface. In fact, more is true:

Theorem 5.1. *The Gauß-map of an $\mathcal{F}$-minimal surface $X : M \to \mathbb{R}^3$ is a local branched covering for elliptic integrands F.*

The above theorem follows from:

Proposition 5.2. *Let $\omega : B_1 \to \mathbb{C}$, B_1 the unit disc in $\mathbb{C}$, be a bounded solution of*

$$|\nabla\omega|^2 \leq 2cJ_\omega,\ c \geq 1, \tag{9}$$

where J_ω is the Jacobian of ω. The mapping ω is a local branched covering if $\#\{|\nabla\omega| = 0\} < \infty$.

For the proof of Theorem 5.1 we have to justify, that for all $p \in M$ there is a chart $x : U(p) \to B_1(0)$ such that $\{K = 0\} \cap U$ consists of only one point and that the stereographic projection of $N \circ x^{-1}$ is a solution of (9). This fact is a result of Sauvigny [**13**]. Now we can apply Proposition 5.2 and for the completion of the proof of Theorem 5.1 it remains to show Proposition 5.2.

Proof of Proposition 5.2. First we see that $|\omega_z(z)|^2 - |\omega_{\bar z}(z)|^2 > 0$ a.e., where $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar z}$ are the Wirtinger-derivatives and $z = x + iy \in B_1$. This inequality is true because of (9). Now, the differential inequality shows that

$$\mu := \frac{\omega_{\bar z}}{\omega_z}$$

is an L^∞-function with $||\mu||_\infty \leq k < 1$, where $\frac{1+k^2}{1-k^2} := c$. By a result of Morrey [**8**, p. 204], there is a homeomorphism $\lambda : \overline{B_1} \to \overline{B_1}$ of class $H^{1,2}$

satisfying

$$\lambda(0) = 0$$
$$\lambda_{\bar{z}} = \mu\, \lambda_z \text{ a.e. on } B_1.$$

Now we want to show that $\varphi := \omega \circ \lambda^{-1}$ is a holomorpic mapping.

To this aim, we chose a point $z_0 \in \{|\nabla\omega| \neq 0\}$ and an open set U, $z_0 \in U$, such that $\omega_{|U}$ is a diffeomorphism. Then one can define

$$\Phi := \lambda \circ (\omega_{|U})^{-1} : \omega(U) \to \lambda(U).$$

Setting $V := \omega(U)$ we see that $\Phi \in H^{1,2}(V, \mathbb{C})$. With $\tau := (\omega_{|U})^{-1}$ the following equation holds a.e.:

$$\Phi_{\bar{\zeta}} = \lambda_z \tau_{\bar{\zeta}} + \lambda_{\bar{z}} \bar{\tau}_{\bar{\zeta}} = \lambda_z(\tau_{\bar{\zeta}} + \mu \bar{\tau}_{\bar{\zeta}}),\ \zeta \in V.$$

By differentiation of the identity $\zeta = \omega(\tau(\zeta))$ we obtain:

$$1 = \omega_z(\tau_\zeta + \mu\bar{\tau}_\zeta),$$
$$0 = \omega_z(\tau_{\bar{\zeta}} + \mu\bar{\tau}_{\bar{\zeta}}).$$

These equations show $\tau_{\bar{\zeta}} + \mu\bar{\tau}_{\bar{\zeta}} = 0$ and therefore $\Phi_{\bar{\zeta}}(\zeta) = 0$ for almost all $\zeta \in V$. Thus Φ is a holomorphic and injective mapping. For this reason, Φ is a diffeomorphism and φ is holomorphic on $\lambda(\{|\nabla\omega| \neq 0\})$. The proof is complete because the set $\lambda(\{|\nabla\omega| = 0\})$ consists only of removable singularities. $\square$

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Gerhard-Mercator-Universiät Duisburg
Fachbereich Mathematik
Lotharstrasse 63/65
47048 Duisburg
Germany
E-mail address: clarenz@math.uni-duisburg.de