Pacific Journal of Mathematics

PARABOLIC SUBGROUPS OF ARTIN GROUPS OF TYPE FC

Eddy Godelle

Volume 208 No. 2 February 2003

PACIFIC JOURNAL OF MATHEMATICS
Vol. 208, No. 2, 2003

PARABOLIC SUBGROUPS OF ARTIN GROUPS OF TYPE FC

Eddy Godelle

Let (A, S) be an Artin group of type FC and A_T a standard parabolic subgroup of A. We use combinatorial tools to show that the normalizer of A_T, the commensurator of A_T, and the product of the quasi-centralizer of A_T by A_T are equal. Furthermore, we show that the centralizer and the quasi-centralizer of A_T in A are generated by their intersections with the monoid A^+.

0. Introduction.

Let S be a finite set and $M = (m_{s,t})_{s,t\in S}$ a symmetric matrix with $m_{s,s} = 1$ for $s \in S$ and $m_{s,t} \in \mathbb{N} - \{0, 1\} \cup \{\infty\}$ for $s \neq t$ in S. An Artin-Tits system associated to M is the pair (A_S, S) where A_S is the group defined by the presentation

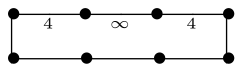
$$A_S = \left\langle S \;\middle|\; \underbrace{sts\ldots}_{m_{s,t}\text{ terms}} = \underbrace{tst\ldots}_{m_{s,t}\text{ terms}}\ ;\ \forall s,t \in S, s \neq t \text{ and } m_{s,t} \neq \infty \right\rangle.$$

The group A_S is called an Artin group and relations $\underbrace{sts\ldots}_{m_{s,t}\text{ terms}} = \underbrace{tst\ldots}_{m_{s,t}\text{ terms}}$ are called braid relations. For instance, if $S = \{s_1, \ldots, s_n\}$ with $m_{s_i,s_j} = 3$ for $|i - j| = 1$ and $m_{s_i,s_j} = 2$ otherwise, then the associated Artin group is the braid group. We denote by A_S^+ the submonoid of A_S generated by S. This monoid A_S^+ has the same presentation as the group A_S, considered as a monoid presentation ([**11**]). When we add relations $s^2 = 1$ to the presentation of A_S we obtain the Coxeter group W_S associated to A_S. We say that A_S is spherical if W_S is finite. The matrix M may be represented by a graph denoted by Γ_S, whose set of vertices is S and where an edge joins two vertices if $m_{s,t} \geq 3$; these edges are labelled by $m_{s,t}$ if $m_{s,t} \geq 4$. We say that A_S (or simply S) is indecomposable if the graph Γ_S is connected. A subgroup A_T of A_S generated by a part T of S is called a standard parabolic subgroup, and a subgroup of A_S conjugate to a standard parabolic subgroup is called a parabolic subgroup. Van Der Lek showed ([**14**]) that (A_T, T) is canonically isomorphic to the Artin-Tits system associated to the matrix $(m_{s,t})_{s,t\in T}$; its graph Γ_T is the full subgraph of Γ_S generated by T. The indecomposable components of S are the maximal subsets of S which are indecomposable.

One says that an Artin-Tits system (A_S, S) (or simply an Artin group A_S) is of type FC if the following assertion is true:

$$\forall T \subset S,\ (\forall s, t \in T,\ m_{s,t} \neq \infty \Rightarrow A_T \text{ is spherical}).$$

For instance, the Artin group of the following graph is of type FC;



If T is a subset of S we call centralizer (*resp.* quasi-centralizer, normaliser, commensurator) of A_T in A_S the set

$$Z_{A_S}(A_T) = \{g \in A_S \mid \forall s \in T, gs = sg\},$$
$$QZ_{A_S}(A_T) = \{g \in A_S \mid gT = Tg\},$$
$$N_{A_S}(A_T) = \{g \in A_S \mid gT \subset A_T g\},$$

$$\mathrm{Com}_{A_S}(A_T) = \{g \in A_S \mid gA_Tg^{-1} \cap A_T \text{ has finite index in both } A_T \text{ and } gA_Tg^{-1}\}$$

respectively. These sets are subgroups of A_S.

The first of the three main theorems we will prove is the following:

Theorem 0.1. *Let (A_S, S) be an Artin-Tits system of type* FC *and $X \subset S$; then*

$$\mathrm{Com}_{A_S}(A_X) = N_{A_S}(A_X) = A_X \cdot QZ_{A_S}(A_X).$$

This result was first proved by Rolfsen ([**12**]) in the case of braid groups; Paris ([**10**]) proved it when A_S is a spherical Artin group and T is indecomposable, finally in [**8**] we proved the result for any T when S is spherical.

In [**6**], the quasi-centralizer in the braid group of a part T of S is geometrically described thanks to the notion of ribbon; this notion was first generalised from a combinatorial viewpoint in [**10**] (and called conjugator) to a general Artin group and indecomposable part T, and finally generalised in [**8**] for any part T. The right viewpoint is to use the categorical language and to see the quasi-centralizer (and centralizer) as a set of morphisms in a groupoid.

Recall that (A_S, S) is spherical if and only if S has a lcm in A_S^+; in that case, this lcm is denoted by Δ_S.

Definition 0.2. Let (A_S, S) be an Artin-Tits system.

(i) We define the groupoid $\mathrm{Conj}\,(S)$ as follows:
  (a) Objects of $\mathrm{Conj}\,(S)$ are subsets of S;
  (b) the set $\mathrm{Conj}\,(S; X, Y)$ of morphisms from X to Y is in bijection with the set
$$\{g \in A_S \mid gXg^{-1} = Y\};$$

(c) the composition of morphisms is defined by the product in A_S:

$$g \circ f = gf.$$

(ii) Let $X, Y \subset S$; we say that $w \in \mathrm{Conj}\,(S; X, Y)$ is a positive elementary Y-ribbon-X ([**10**, **8**]) if:

(a) $w = \Delta_{X'}$ for X' an indecomposable component of X or,

(b) there exists $t \in S$ such that the indecomposable component X' of $X \cup \{t\}$ containing t is spherical and $w = \Delta_{X'}\Delta^{-1}_{X'-\{t\}}$.

We say that $w \in \mathrm{Conj}\,(S; X, Y)$ is an elementary Y-ribbon-X if it is a positive elementary ribbon or w^{-1} is a positive elementary X-ribbon-Y.

(iii) We denote $\mathrm{Ribb}\,(S)$ the smallest subcategory of $\mathrm{Conj}\,(S)$ which has the same objects and which contains the elementary ribbons; the set of morphisms from X to Y in $\mathrm{Ribb}\,(S)$ is denoted $\mathrm{Ribb}\,(S; X, Y)$ and its elements are called Y-ribbon-X.

Note that in Case (ii)(a), $X = Y$ and that in Case (ii)(b) there exists $u \in S$ such that $X \sqcup \{u\} = Y \sqcup \{t\}$.

The second main theorem of this article is the following:

Theorem 0.3. *Let (A_S, S) be an Artin-Tits system of type* FC; *then the category* $\mathrm{Conj}\,(A_S, S)$ *is generated by the elementary ribbons; that is* $\mathrm{Conj}\,(S) = \mathrm{Ribb}\,(S)$.

This result was proved by Paris in [**10**] for spherical Artin groups. In [**8**] we proved a similar result in all Artin monoids; in that case, generators are the positive elementary ribbons.

Corollary 0.4. *Let (A_S, S) be an Artin-Tits system of type* FC *and $X \subset S$; then*

$$w \in QZ_{A_S}(A_X) \iff w = w_n \dots w_1 \ \ \textit{with } w_i \textit{ an elementary } \\ X_i\textit{-ribbon-}X_{i-1} \textit{ where } X_0 = X_n = X$$

and $QZ_{A_S}(A_X)$ is the subgroup of A_S generated by $QZ_{A_S}(A_X) \cap A_S^+$.

Corollary 0.5. *Let (A_S, S) be an Artin-Tits system of type* FC *and $X \subset S$; then $Z_{A_S}(A_X)$ is the subgroup of A_S generated by $Z_{A_S}(A_X) \cap A_S^+$.*

In Section 1, we recall relevant facts on Artin groups of type FC, on Artin monoids and define some useful notations; in Section 2 we look at the spherical case and in Section 3, we prove the main results.

1. Preliminaries.

In this part we assume that (A, S) is an Artin-Tits system associated to the matrix $M = (m_{s,t})_{s,t \in S}$.

Lemma 1.1.

(i) ([**2**, **9**]). *A^+ is left (resp. right) cancellable and every pair $a, b \in A$ has a left (resp. right) gcd, denoted $a \wedge_l b$ (resp. $a \wedge_r b$).*

(ii) ([**14**, Theorem II.4.13]). *Let T be a nonempty subset of S. Then the subgroup $(\langle T \rangle_A, T)$ of A is canonically isomorphic to the Artin group A_T associated to the matrix $(m_{s,t})_{s,t \in T}$. Furthermore, if T' is another subset of S, then $A_T \cap A_{T'} = A_{T \cap T'}$ with the notation $A_\emptyset = \{1\}$.*

Lemma 1.2. *Assume that A is spherical and let $a, b \in A^+$; then*

(i) ([**2**]). *a, b have a left lcm (resp. right lcm) in A^+ denoted $a \vee_l b$ (resp. $a \vee_r b$).*

(ii) ([**5**, Paragraph 4]). *Let $g \in A$; g can be written $g = g_1 \Delta_S^n$ with $g_1 \in A^+$, and $n \in \mathbb{Z}$.*

(iii) ([**3**, Theorem 2.6] and [**4**, Lemma 4.4]). *Let $g \in A$; there exists unique $a, b \in A^+$ such that $a \wedge_r b = 1$ and $g = ab^{-1}$. Furthermore, if $c \in A^+$ such that $gc \in A^+$ then $c = bc'$ for some $c' \in A^+$.*

We call the decomposition $g = ab^{-1}$ of (iii) the (right) orthogonal splitting of g. In a similar way one can define the left orthogonal splitting of g.

Lemma 1.3 ([**8**, Corollary 4.4.6])**.** *Let (A, S) be a spherical Artin group and let $s, t \in S$, $g \in A$ and $j \in \mathbb{N}^*$ be such that $s^j g = g t^j$. Then:*

(i) *$sg = gt$;*

(ii) *if $w = ab^{-1}$ is the orthogonal splitting of g, then*

$$\begin{cases} sa = au, \\ tb = bu \end{cases}$$

for some $u \in S$.

Notation 1.4. Let (A, S) be an Artin group and let $X \subset S$.

(i) We denote by X_s the union of the spherical indecomposable components of X and by X_{as} the complement $X - X_s$.

(ii) We denote by $X^\perp$ the set $\{s \in S \mid \forall t \in X,\ m_{s,t} = 2\}$; we have $X \cap X^\perp = \emptyset$.

(iii) If Y is another subset of S we write $X \cup Y = X \oplus Y$ if $Y \subset X^\perp$. In particular, $X = X_s \oplus X_{as}$.

(iv) If $s \in X$, we denote by $X(s)$ the indecomposable component of X which contains s.

In the following we write $X_{as}^\perp$ for $(X_{as})^\perp$.

To prove our main results we need to introduce the following notations for a new family of subcategories of $\mathrm{Conj}\,(S)$ which generalises $\mathrm{Ribb}\,(S)$; we only give notations for their morphisms.

Notation 1.5. Let (A_S, S) be an Artin group, and $T \subset S$. Consider the smallest subcatgory of $\mathrm{Conj}\,(S)$ which has the same objects as $\mathrm{Conj}\,(S)$ and which contains the elementary ribbons which are in A_T. For $X, Y \subset Y$, we denote by $\mathrm{Ribb}\,(T; X, Y)$ the set of morphisms from X to Y in this subcategory. They are Y-ribbons-X.

1.1. Artin-Tits system of type FC. We assume in this section that (A, S) is of type FC. Recall that Artin groups of type FC have been defined in the introduction. Most facts on Artin groups of type FC in this part are proved in [**1**].

Proposition 1.6 ([**13**, Theorem 1])**.** *Let $G = G_1 *_H G_2$ the amalgamated product of groups G_1 and G_2 over H. Let C_1, C_2 be transversals of G_1/H and G_2/H respectively which contain 1. For all $x \in G$, there exists a unique sequence $(x_1, \dots, x_n, h)$ such that $x = x_1 \dots x_n h$ with $h \in H$, and where the x_i are in $C_1 \cup C_2$ with x_i and x_{i+1} not in the same transversal.*

We will call $(x_1, \dots, x_n, h)$ the amalgam normal form of x relative to the amalgamated product $G_1 *_H G_2$ and we set $|x|_* = n$. We have then $|x|_* = 0$ if and only if $x \in H$.

Corollary 1.7 ([**1**, Corollary 1])**.** *Let $G = G_1 *_H G_2$ and $g, c \in G$. We denote by $(g_1, \dots, g_n, h)$ the amalgam normal form of g. Assume that $g_n \in C_1$ and $|c|_* \leq 1$, then: The amalgam normal form of gc is*

$$\begin{cases} (g_1, \dots, g_n, g_{n+1}, h') & \text{if } c \in G_2 - H, \\ (g_1, \dots, g_{n-1}, g'_n, h') & \text{if } c \in G_1 - (g_n h)^{-1} H, \\ (g_1, \dots, g_{n-1}, h') & \text{if } c \in (g_n h)^{-1} H, \end{cases}$$

where (g_{n+1}, h') is the amalgam normal form of hc in the first case, (g'_n, h') is the amalgam normal form of $g_n hc$ in the second case and $h' = g_n hc$ in the third case.

Corollary 1.8. *Let $w = v_1 \dots v_m \in G$ such that $v_{2j} \in G_2 - H$ and $v_{2j+1} \in G_1 - H$ for $j \in \{0, \dots, [\frac{m}{2}]\}$. If we denote by $(w_1, \dots, w_n, h)$ the amalgam normal form of w, then one has*

$$\begin{cases} m = n, \\ v_1 = w_1 h_1 \text{ with } h_1 \in H, \\ h_{i-1} v_i = w_i h_i \text{ with } i \in \{2, \dots, n\} \text{ with } h_i \in H, \\ h_n = h. \end{cases}$$

Proposition 1.9 ([**1**, Proposition 2])**.**

(i) *Let s_1, s_2 be in S be such that $m_{s_1,s_2} = \infty$. Let $A_1 = A_{S-\{s_1\}}$, $A_2 = A_{S-\{s_2\}}$ and $A_{1,2} = A_{S-\{s_1,s_2\}}$, then the group A is the amalgamated product of A_1 and A_2 over $A_{1,2}$, that is $A = A_1 *_{A_{1,2}} A_2$.*

(ii) *The set of Artin groups of type* FC *is the smallest class of Artin groups which is closed under amalgamation over standard parabolic subgroups and which contains spherical Artin groups.*

Proposition 1.10 ([**1**, Theorem 2])**.** *Let $T \subset S$. There exists a function $m_T : A \to A$ such that for all $w \in A$ one has:*

(i) $m_T(w) \in wA_T$;
(ii) *for all* $v \in wA_T$, $m_T(v) = m_T(w)$;
(iii) *if* $w \in A_U$ *for* $U \subset S$, *then* $m_T(w) \in A_U$.

The function m_T gives a special representative of each coset of A/A_T.

Notation 1.11. Assume that (A, S) is not spherical and fix $s_1, s_2 \in S$ such that $m_{s_1,s_2} = \infty$. We set $A_1 = A_{S_1}$, $A_2 = A_{S_2}$ with $S_1 = S - \{s_1\}$, $S_2 = S - \{s_2\}$ and $S_{1,2} = S - \{s_1, s_2\}$. Then we have $A = A_1 *_{A_{1,2}} A_2$. Transversals of $A_1/A_{1,2}$ and $A_2/A_{1,2}$ are transversals C_1, C_2 respectively induced by $m_{S_{1,2}}$.

Corollary 1.12. *Assume that (A, S) is not spherical and let $s_1, s_2 \in S$ with $m_{s_1,s_2} = \infty$; one has $A = A_1 *_{A_{1,2}} A_2$ with Notation* 1.11. *If $w \in A_T$ for $T \subset S$ then the amalgam normal form of w has its terms in A_T.*

1.2. Artin monoids.

Definition 1.13. Let (A_S, S) be an Artin-Tits system.

(i) We define the small category $\mathrm{Conj}^+(S)$ as follows:
 (a) Objects of $\mathrm{Conj}^+(S)$ are subsets of S;
 (b) the set $\mathrm{Conj}^+(S; X, Y)$ of morphisms from X to Y is in bijection with the set
$$\{g \in A^+ \mid gXg^{-1} = Y\};$$
 (c) the composition of morphisms is defined by the product in A^+:
$$g \circ f = gf.$$
(ii) We denote $\mathrm{Ribb}^+(S)$ the smallest subcategory of $\mathrm{Conj}^+(S)$ which has the same objects and which contains the positive elementary ribbons (see 0.2); the set of morphisms from X to Y in $\mathrm{Ribb}^+(S)$ is denoted $\mathrm{Ribb}^+(S; X, Y)$ and its elements are called positive Y-ribbon-X.

Categories $\mathrm{Conj}^+(S)$ and $\mathrm{Ribb}^+(S)$ are clearly subcategories of $\mathrm{Conj}(S)$ and $\mathrm{Ribb}(S)$ respectively.

In the following, we will need the following theorem in the spherical case. It is Theorem 0.3 but in the setting of the Artin monoid.

Theorem 1.14 ([**10**])**.** *Let (A_S, S) be an Artin-Tits system of spherical type; then*
$$\mathrm{Conj}^+(S) = \mathrm{Ribb}^+(S).$$

In fact this theorem is true in any Artin monoid ([**8**]).

2. The spherical case.

As we said in the introduction, Theorems 0.1 and 0.3 are known in the spherical case. Nevertheless we need to state precise results in the spherical case to prove our theorems in type FC.

Theorem 2.1. *Let (A,S) be a spherical Artin group and $X,Y\subset S$. Let $k\in\mathbb{Z}-\{0\}$ and $g\in A$. The following are equivalent:*
(1) $gA_Xg^{-1}\subset A_Y$;
(2) $g\Delta_X^k g^{-1}\in A_Y$;
(3) $g=yx$ *with* $y\in A_Y$, $x\in \mathrm{Ribb}\,(S;X,R)$ *for* $R\subset Y$.

Proof. It is clear that (3) ⇒ (1) ⇒ (2). For (2) ⇒ (3), the proof is similar to Proposition 3.1 of [**7**]; thanks to Lemma 1.2(ii), we may assume that $g\in A^+$ and is Y-reduced (i.e., not divisible by any $s\in Y$); then for all $s\in X$, we have $gsg^{-1}=t$ for some $t\in Y$. Thus $g\in\mathrm{Ribb}\,(S;X,R)$ with $R\subset Y$ by Theorem 1.14. □

Lemma 2.2. *Let (A,S) be a spherical Artin group and let $X,Y,T\subset S$. Let $g\in A_T$ be such that $gA_Xg^{-1}\subset A_Y$. Let $s\in X-T$; then there exists $x\in A_{X(s)^\perp\cap T}$ and $y\in A_{Y\cap T}$ such that $g=yx$. Furthermore $X(s)\subset Y$.*

Proof. Let $g=a_0b_0^{-1}$ the orthogonal splitting of g in A_T^+. One has $a_0=a_1a$ where a is Y-reduced and $a_1\in A^+_{Y\cap T}$. In the same way, one has $b_0=b_1b$ where b is $\{s\}^\perp$-reduced and $b_1\in A^+_{\{s\}^\perp\cap T}$. We obtain $ab^{-1}sba^{-1}\in A_Y$; hence $ab^{-1}sba^{-1}=u^{-1}v$ with $u\perp_l v$ in A_Y^+. Thus $b^{-1}sb=(ua)^{-1}(va)$ with $b\perp_l sb$, since $b\in A^+_{S-\{s\}}$ and is $\{s\}^\perp$-reduced. Thus, there exists α in A^+ such that $va=\alpha sb$ and $ua=\alpha b$. This implies that $(b\vee_r a)a^{-1}$ divides v and thus is in $A_{Y\cap T}$. On the other hand, we have $ba^{-1}=c^{-1}d$ with $c=(b\vee_r a)b^{-1}\in A_T^+$ and $d=((b\vee_r a)a^{-1})\in A^+_{T\cap Y}$. Thus $csc^{-1}\in A_Y$. Let $c=c_2c_1$ with $c_1\in A^+_{\{s\}^\perp\cap T}$ and c_2 reduced-$\{s\}^\perp$ in A_T^+. Then we have $c_2sc_2^{-1}\in A_Y$ with $c_2s\perp_r c_2$. Thus both c_2 and s are in A_Y. Then $g=y_0x_0$ with $x_0=c_1b_1^{-1}\in A_{\{s\}^\perp\cap T}$ and $y_0=a_1d^{-1}c_2\in A_{Y\cap T}$. We have $x_0A_Xx_0^{-1}\subset A_Y$ with $x_0\in A_{\{s\}^\perp\cap T}$. If $x_0=1$ or $X(s)=\{s\}$, the result holds with $x=x_0$ and $y=y_0$. Assume $x_0\neq 1$ and $X(s)\neq\{s\}$. Choose $s'\in X(s)-(\{s\}^\perp\cup\{s\})$ (it exists since $X(s)\neq\{s\}$). Applying the argument to $g'=x_0$, $T'=\{s\}^\perp\cap T$, and s', we obtain $x_0=y_1x_1$ with $x_1\in A_{\{s'\}^\perp\cap\{s\}^\perp\cap T}$ and $y_1\in A_{Y\cap T}$. Repeating this process yields $g=y_0\dots y_nx_n$ with $x_n\in A_{\{s\}^\perp\cap\{s'\}^\perp\cap\dots\cap\{s^{(n)}\}^\perp\cap T}$ and $y_0\dots y_n\in A_{T\cap Y}$. The process will terminate when either $x_n=1$ or

$$X(s)-\left(\bigcap_{i=0}^{i=n}\{s^{(i)}\}^\perp\cup\bigcup_{i=0}^{i=n}\{s^{(i)}\}\right)=\emptyset$$

which means that $X(s)=\bigcup_{i=0}^{i=n}\{s^{(i)}\}$ and $x_n\in A_{\bigcap_{i=0}^{i=n}\{s^{(i)}\}^\perp\cap T}=A_{X(s)^\perp\cap T}$. In either case, the result follows with $x=x_n$ and $y=y_0\dots y_n$. □

Proposition 2.3. *Under the hypotheses of Theorem* 2.1*, if* $g \in A_T$ *for* $T \subset S$*, then, (1), (2) and (3) are equivalent to*

(3′) $g = yx$ *with* $y \in A_{Y\cap T}$*,* $x \in \mathrm{Ribb}\,(T \cap (\bigcup_{s\in X-T} X(s))^{\perp}; X, R)$ *where* R *is of the form* $\bigcup_{s\in X-T} X(s) \oplus T_1 \subset Y$ *with* $T_1 \subset T \cap Y$.

Proof. It is clear that $(3') \Rightarrow (3)$. Let us show that $(1) \Rightarrow (3')$ by induction on the cardinal of $X - T$. It is enough to find x, y such that $y \in A_Y$ and $x \in \mathrm{Ribb}\,(T\cap(\bigcup_{s\in X-T} X(s))^{\perp}; X, R)$ since that implies $y \in A_T$ and the type of R. If $X-T = \emptyset$, that is $X \subset T$, then $gA_Xg^{-1} \subset A_{Y\cap T}$ in A_T and we apply Theorem 2.1 in A_T. Otherwise Lemma 2.2 proves that for all $\#(X-T) \geq 1$ and $s \in X - T$ then $g = y_1x_1$ with x_1 in $A_{X(s)^{\perp}\cap T}$ and y_1 in $A_{Y\cap T}$. Thus, in $A_{X(s)^{\perp}}$ we have $x_1(X - X(s))x_1^{-1} \subset A_{Y\cap X(s)^{\perp}}$ with x_1 in $A_{T\cap X(s)^{\perp}}$. We apply the induction hypothesis in $A_{X(s)^{\perp}}$, after replacing g by x_1, X by $X-X(s)$, T by $T\cap X(s)^{\perp}$ and Y by $Y\cap X(s)^{\perp}$; we have $\#((X\cap X(s)^{\perp})-(T\cap X(s)^{\perp})) < \#(X-T)$ since $s \notin X\cap X(s)^{\perp}$ and $s \in X-T$. We get $x_1 = y_2x$ with x in $\mathrm{Ribb}\,(T\cap X(s)^{\perp}\cap\bigcap_{u\in X-X(s)-T} X(u)^{\perp}; X-X(s), R_1)$ with $R_1 \subset Y$ and $y_2 \in A_Y$. But $X(s)^{\perp} \cap \bigcap_{u\in X-X(s)-T} X(u)^{\perp} = (\bigcup_{u\in X-T} X(u))^{\perp}$. Thus $g = yx$ with $y = y_1y_2 \in A_Y$ and $x \in \mathrm{Ribb}\,(T \cap (\bigcup_{u\in X-T} X(u))^{\perp}; X, R)$. □

3. Proof of the main results.

Proposition 3.1. *Let* (A, S) *be an Artin group of type* FC*. Let* $X, Y, T \subset S$ *with* X *spherical. Let* $k \in \mathbb{Z}-\{0\}$ *and* $g \in A_T$*. The following are equivalent:*

1. $gA_Xg^{-1} \subset A_Y$;
2. $g\Delta_X^k g^{-1} \in A_Y$;
3. $g = yx$ *with* $y \in A_{Y\cap T}$ *and* $x \in \mathrm{Ribb}\,(T \cap (\bigcup_{s\in X-T} X(s))^{\perp}; X, R)$ *for some* $R \subset Y$.

Proof. Implications $(3) \Rightarrow (1) \Rightarrow (2)$ are clear. Let us show that $(2) \Rightarrow (3)$ by induction on the number m of amalgamations; that is the number of edges in Γ_S labelled with ∞. If $m = 0$ then A is spherical and the result is true by Theorem 2.1 and Proposition 2.3. Assume now that $m \geq 1$ and that Proposition 3.1 is true for any Artin group of type FC with a number of amalgamation less than or equal to $m - 1$. We choose $s_1, s_2 \in S$ such that $m_{s_1,s_2} = \infty$ and we use Notation 1.11. Note that since A_X is spherical, we have $X \subset S_1$ or $X \subset S_2$. Denote by $(g_1, \ldots, g_n, h)$ the amalgam normal form of g ; elements g_i and h are in A_T by Corollary 1.12. For m fixed, let us do an induction on n. We have $g_1 \ldots g_nh\Delta_X^kh^{-1}g_n^{-1} \ldots g_1^{-1} \in A_Y$. If $n = 0$ then the formula holds in A_1 or in A_2 and we conclude by the induction hypothesis on m applied in A_1 or in A_2. Assume now that $n \geq 1$. We may assume without loss of generality that $g_n \in A_1$.

If $X \not\subset A_1$ then by Corollary 1.8, the amalgam normal form of $g\Delta_X^kg^{-1}$ is of the shape $(g_1, \ldots, g_n, g'_{n+1}, \ldots, g'_{2n+1}, h')$ with $g'_{n+1} = m_{S_{1,2}}(h\Delta_X^kh^{-1})$.

But $g\Delta_X^k g^{-1} \in A_Y$ thus $g_1 \dots g_n \in A_{Y\cap T}$ by Corollary 1.12 and $h\Delta_X^k h^{-1} \in A_Y$. Thus, to conclude, we apply the induction hypothesis on n at rank $n = 0$ to h.

If $X \subset A_1$ and $g_n h\Delta_X^k (g_n h)^{-1} \notin A_{1,2}$ then the amalgam normal form of $g\Delta_X^k g^{-1}$ is of shape $(g_1, \dots, g_{n-1}, g'_n, \dots, g'_{2n-1}, h')$ with

$$g'_n = m_{S_{1,2}}(g_n h\Delta_X^k (g_n h)^{-1}).$$

But $g\Delta_X^k g^{-1} \in A_Y$; then we get $g_1 \dots g_{n-1} \in A_{Y\cap T}$ by Corollary 1.12 and thus $g_n h\Delta_X^k (g_n h)^{-1} \in A_Y$; thus we may apply the induction hypothesis on m in A_1 at $g_n h$ and conclude.

If $X \subset S_1$ and $g_n h\Delta_X^k (g_n h)^{-1} \in A_{1,2}$ then by the induction hypothesis on m applied in A_1, we get $g_n h = y_1 x_1$ with

$$x_1 \in \mathrm{Ribb}\left(T \cap \left(\bigcup_{t\in X-T} X(t)\right)^{\perp}; X, R\right)$$

for $R \subset S_{1,2}$ and $y_1 \in A_{1,2} \cap A_T$. We get $g_1 \dots g_{n-1} y_1 \Delta_R^k (g_1 \dots g_{n-1} y_1)^{-1} \in A_Y$ and by the induction hypothesis on n applied at rank $n-1$ to $g_1 \dots g_{n-1} y_1 \in A_T$, we obtain $g_1 \dots g_{n-1} y_1 = y x_2$ with

$$x_2 \in \mathrm{Ribb}\left(T \cap \left(\bigcup_{t\in X-T} X(t)\right)^{\perp}; R, R_1\right)$$

for some $R_1 \subset Y$ and $y \in A_{Y\cap T}$.

Thus $g = y x_2 x_1$ and $x_2 x_1 \in$ Ribb $(T \cap (\bigcup_{t\in X-T} X(t))^{\perp}; X, R_1)$ for $R_1 \subset Y$. □

Theorem 3.2. *Let (A, S) be an Artin group of type* FC *and let $X, Y, T \subset S$. Let $g \in A_T$ and $k \in \mathbb{Z} - \{0\}$. The following are equivalent:*

(1) *$gA_X g^{-1} \subset A_Y$;*
(2) *$g\Delta_{X_s}^k g^{-1} \in A_Y$, $g = yx$ with $x \in A_{X_{as}^{\perp}\cap T}$, $y \in A_{Y\cap T}$ and $X_{as} \subset Y$;*
(3) *$g = yx$ with $y \in A_{Y\cap T}$ and $x \in$ Ribb $(T \cap X_{as}^{\perp}; X_s, R)$ with $R \oplus X_{as} \subset Y$.*

Proof. It is clear that (3) ⇒ (2) and that Proposition 3.1 induces (2) ⇒ (1). Let us show that (1) ⇒ (3). We are carrying out an induction on the number $r(X)$ of edges in $\Gamma(X)$ which are labelled with ∞. If $r(X) = 0$, then X is spherical and the result is true by Proposition 3.1. Assume now X is not spherical (that is $r(X) \geq 1$) and fix s_1, s_2 in X such that $m_{s_1,s_2} = \infty$. We assume that if (A', S') is an Artin group of type FC and X', Y', T' are three parts of S' such that $r(X') < r(X)$ then for all g' of $A'_{T'}$ we have: ${g'}^{-1} A_{X'} g' \subset A'_{Y'} \Rightarrow g' = y'x'$ where $x' \in$ Ribb $(T' \cap {X'}_{as}^{\perp}; X'_s, R')$ and $y' \in A'_{Y'\cap T'}$ with $R' \oplus X'_{as} \subset Y'$.

We have $A = A_1 *_{A_{1,2}} A_2$ with Notation 1.11. Let $(g_1, \dots, g_n, h)$ be the amalgam normal form of g. The first step is to show that it is enough to prove the result for the case $n = 0$. Assume $n \geq 1$. Without loss of generality we may assume that $g_n \in A_1$. Furthermore $g_1 \dots g_n h s_1 h^{-1} g_n^{-1} \dots g_1^{-1} \in A_Y$ since $s_1 \in X$. By Corollary 1.8 we infer that the amalgam normal form of $g_1 \dots g_n h s_1 h^{-1} g_n^{-1} \dots g_1^{-1}$ is of the shape $(g_1, \dots, g_n, g'_{n+1}, \dots, g'_{2n+1}, h')$ and has its terms in A_Y. Thus $g_1 \dots g_n$ is in $A_{Y\cap T}$. We get that $hA_X h^{-1}$ is also in A_Y. Thus if (1) $\Rightarrow$ (3) for any g such that $n = 0$, the theorem will be proved. Assume $g = h \in A_{1,2}$. Denote by T_1 (resp. T_2) the indecomposable component of $X - \{s_1\}$ (resp. $X - \{s_2\}$) which contains s_2 (*resp.* s_1). In A_1 we have $gA_{X-\{s_1\}}g^{-1} \subset A_{Y-\{s_1\}}$ thus by the induction hypothesis,we get $g = y_1 x_1$ with y_1 in $A_{Y\cap T}$ and $x_1 \in \mathrm{Ribb}\,(T \cap (X - \{s_1\})^{\perp}_{as}; (X - \{s_1\})_s, R_1)$ with R_1 in $Y - \{s_1\}$. Furthermore, since $s_2 \notin A_{1,2}$ and $g \in A_{1,2}$, we get, either by the induction hypothesis (if T_1 is not spherical) or by Proposition 3.1, that we can find $x_1 \in \mathrm{Ribb}\,(T_1^{\perp} \cap T \cap (X - \{s_1\})^{\perp}_{as}; (X - \{s_1\})_s, R_1)$. From $y_1 \in A_Y$, we infer that $x_1 A_X x_1^{-1}$ is in A_Y. We can use the same argument if we replace g by x_1 and exchange the roles of A_1 and A_2; we find, thanks to the inclusion $x_1 A_{X-\{s_2\}} x_1^{-1} \subset A_{Y-\{s_2\}}$ in A_2, that $x_1 = y_2 x$ with $y_2 \in A_{Y\cap T}$ and $x \in \mathrm{Ribb}\,(T_1^{\perp} \cap T_2^{\perp} \cap T \cap (X - \{s_1\})^{\perp}_{as} \cap (X - \{s_2\})^{\perp}_{as}; (X - \{s_2\})_s, R_2)$ with R_2 in Y. Finally, since $T_1 \cup T_2 = X(s_1) = X(s_2)$, we get $T_1^{\perp} \cap T_2^{\perp} \cap T \cap (X - \{s_1\})^{\perp}_{as} \cap (X - \{s_2\})^{\perp}_{as} = T \cap X^{\perp}_{as}$. We get that $g = y_1 y_2 x$ with $y_1 y_2 \in A_{Y\cap T}$, x in $\mathrm{Ribb}\,(T \cap X^{\perp}_{as}; X_s, R)$ since $X_s \subset (X - \{s_2\})_s$ and $R \subset R_2 \subset Y$. $\square$

The two following lemmas are used to prove (ii) and the first equality of (i) in Theorem 3.5.

Lemma 3.3. *Let (A, S) be an Artin group of type* FC*; then $QZ_A(A) = QZ_{A_{S_s}}(A_{S_s})$.*

Proof. It is clear that $QZ_A(A) = QZ_{A_{S_s}}(A_{S_s}) \cdot QZ_{A_{S_{as}}}(A_{S_{as}})$. Then it is enough to prove that $QZ_{A_{S_{as}}}(A_{S_{as}}) = \{1\}$ and since it is the product of quasi-centralizers of its indecomposable components, it is enough to show that if X is indecomposable not spherical, then $QZ_A(A) = \{1\}$. Let A be such a group and let $g \in QZ_A(A)$; choose $T \subset S$ maximal spherical and let $T' = gTg^{-1} \subset S$. Then by Proposition 3.1 applied to the equality $g^{-1} A_T g = A_{T'}$ with T, we get that $T = T'$ and $g \in A_T$; thus $g \in QZ_{A_T}(A_T)$. Let $(T_i)_{1\le i\le k}$ be the indecomposable components of T then $QZ_{A_T}(A_T) = \{\Delta_{T_1}^{j_1} \dots \Delta_{T_k}^{j_k}; \forall i,\ j_i \in \mathbb{Z}\}$. Thus $g = \Delta_{T_1}^{j_1} \dots \Delta_{T_k}^{j_k}$ with $j_i \in \mathbb{Z}$ for all $i \in \{1, \dots, k\}$. Assume that there exists $i \in \{1, \dots, k\}$ such that $j_i \neq 0$. Since T is maximal spherical and S is indecomposable and not spherical, there exists $s \in T_i$ and $t \in S - T$ such that $m_{s,t} = \infty$. We get $A = A_{S-\{s\}} *_{A_{S-\{s,t\}}} A_{S-\{t\}}$ and $gsg^{-1} = s_1$ with $s_1 \in T$; since $j_i \neq 0$, this is impossible by Corollary 1.8. Thus for all i, we get $j_i = 0$ and $g = 1$. $\square$

Lemma 3.4. *Let $A = A_1 *_{A_{1,2}} A_2$ be a non-spherical Artin group of type* FC *in the Notation of* 1.11*. Let $X \subset S$ be such that $\{s_1, s_2\} \subset X$, let $X_i = X \cap S_i$ for $i \in \{1, 2\}$; then $\mathrm{Com}_A(A_X) \cap A_i \subset \mathrm{Com}_{A_i}(A_{X_i})$ for $i \in \{1, 2\}$.*

Proof. By symmetry, it is enough to prove the result for $i = 1$. Let $g \in \mathrm{Com}_A(A_X) \cap A_1$. We have to show that $A_{X_1} \cap (gA_{X_1}g^{-1})$ has finite index in A_{X_1} and in $gA_{X_1}g^{-1}$. For this, we show that if $x, y \in A_{X_1}$ (resp. $x, y \in gA_{X_1}g^{-1}$) have the same image in $A_X/(A_X \cap gA_Xg^{-1})$ (resp. $gA_Xg^{-1}/(A_X \cap gA_Xg^{-1})$) then they have the same image in $A_{X_1}/(A_{X_1} \cap gA_{X_1}g^{-1})$ (resp. $gA_{X_1}g^{-1}/(A_{X_1} \cap gA_{X_1}g^{-1})$).

Let $x, y \in A_{X_1}$ be such that $x\alpha = y$ for some $\alpha \in A_X \cap (gA_Xg^{-1})$. We have $\alpha = xy^{-1} \in A_{X_1} \cap A_X \cap (gA_Xg^{-1}) = A_{X_1} \cap (gA_Xg^{-1}) = A_{X_1} \cap (gA_{X_1}g^{-1})$. The last equality come from the fact that g is in A_{S_1} and that $A_{X_1} = A_{S_1 \cap X} = A_{S_1} \cap A_X$. Let $x, y \in gA_{X_1}g^{-1}$ be such that $x\alpha = y$ for some $\alpha \in A_X \cap (gA_Xg^{-1})$; we have by the same arguments that $\alpha \in A_{X_1} \cap (gA_{X_1}g^{-1})$. Thus $g \in \mathrm{Com}_{A_1}(A_{X_1})$. □

Theorem 3.5. *Let (A, S) be an Artin group of type* FC*. Let $X \subset S$; then:*

(i) $\mathrm{Com}_A(A_X) = N_A(A_X) = A_X \cdot QZ_A(A_X)$;

(ii) $\mathrm{Conj}\,(S; X, Y) = \mathrm{Ribb}\,(S; X, Y) \subset \mathrm{Ribb}\,(X_{as}^{\perp}, X_s, Y_s)$. *Furthermore, if* $\mathrm{Conj}\,(S; X, Y) \neq \emptyset$ *then the inclusion is a equality and* $X_{as} = Y_{as}$.

Proof. Thanks to Theorem 3.2 we get the inclusions $A_X \cdot QZ_A(A_X) \subset N_A(A_X) \subset A_X \cdot \mathrm{Ribb}\,(X_{as}^{\perp}; X_s, X_s) \subset A_X \cdot QZ_A(A_X)$. That proves the second equality of (i): $N_A(A_X) = A_X \cdot QZ_A(A_X)$. If $\mathrm{Conj}\,(S; X, Y) = \emptyset$ then (ii) is clear since $\mathrm{Ribb}\,(S; X, Y) \subset \mathrm{Conj}\,(S; X, Y)$. Assume now that $\mathrm{Conj}\,(S; X, Y) \neq \emptyset$. Since $\mathrm{Conj}\,(S; X, Y) \subset A_X \cdot \mathrm{Ribb}\,(X_{as}^{\perp}; X_S, X_S)$, one has $X_{as} = Y_{as}$, and $\mathrm{Ribb}\,(X_{as}^{\perp}; X_s, Y_s) \subset \mathrm{Conj}\,(S; X, Y)$; then we get $\mathrm{Conj}\,(S; X, Y) = QZ_{A_X}(A_X) \cdot \mathrm{Ribb}\,(X_{as}^{\perp}; X_s, Y_s)$. Now, by Lemma 3.3 $QZ_{A_X}(A_X) = QZ_{A_{X_s}}(A_{X_s})$ and since X_s is spherical, we get $QZ_{A_{X_s}}(A_{X_s}) = \mathrm{Ribb}\,(X_s; X_s, X_s)$. Then, we proved (ii). We have now to prove the first equality of (i). We have clearly $\mathrm{Com}_A(A_X) \supset N_A(A_X)$; if X is spherical, we prove the other inclusion as in [**10**] thanks to the implication (2) $\Rightarrow$ (1) of Proposition 3.1. Let us show that $\mathrm{Com}_A(A_X) \subset N_A(A_X)$ for X not spherical. In order to do this, we proceed by induction on the number r of edges labelled with ∞ in the graph Γ_X. Note that for $r = 0$, we have that X is spherical and the result is true in that case. Assume $r \geq 1$ and write $A = A_1 *_{A_{1,2}} A_2$ such that $\{s_1, s_2\} \in X$ following Notation 1.11. By the induction hypothesis, we have $\mathrm{Com}_A(A_{X \cap S_i}) = N_{A_i}(A_{X \cap S_i})$ for $i \in \{1, 2\}$. Let $g \in \mathrm{Com}_A(A_X)$ and $(g_1, \ldots, g_n, h)$ its amalgam normal form. There exists $p \in \mathbb{N} - \{0\}$ such that $gs_1^pg^{-1}$ and $gs_2^pg^{-1}$ are in A_X. If $n \neq 0$, we infer from Corollary 1.8 that $g_1 \ldots g_n \in A_X$ and thus $h \in \mathrm{Com}_A(A_X)$. Now, for $g = h \in \mathrm{Com}_A(A_X) \cap A_{1,2}$, we can apply Lemma 3.4; we get that for $i \in \{1, 2\}$, we have $h \in \mathrm{Com}_{A_i}(A_{X \cap S_i})$. But by the induction hypothesis, $\mathrm{Com}_{A_i}(A_{X \cap S_i}) = N_{A_i}(A_{X \cap S_i})$. Then $h \in N_{A_1}(A_{X \cap S_1}) \cap N_{A_2}(A_{X \cap S_2}) \subset N_A(A_X)$. □

Corollary 3.6. *Let (A, S) be an Artin groups of type* FC*. Let $X \subset S$; Then:*

(i) $QZ_A(A_X) = \mathrm{Ribb}\,(X_{as}^{\perp}; X_s, X_s)$;

(ii) $Z_A(A_X) = \langle Z_A(A_X) \cap A^+ \rangle_A$;

(iii) $QZ_A(A_X) = \langle QZ_A(A_X) \cap A^+ \rangle_A$.

Proof. The (i) is a particular case of 3.5(ii); (ii) and (iii) are equivalent by [**8**] Theorem 4.1.2; furthermore, it is clear that 3.5(ii) implies (iii) thanks to Lemma 1.3. □

Aknowledgements. This work was begun while I was visiting the Laboratory of Topology UMR 5584 CNRS, University of Bourgogne; I am grateful for the attention I received during my stay. Special thanks are due to John Crisp and Luis Paris for their hospitality and useful discussions and remarks. My travel was supported by the GDR Tresses 2105 CNRS.

I thank the referee for his editing work.

References

[1] J.A. Altobelli, *The word problem for Artin groups of* FC *type*, J. Pure Appl. Algebra, **129** (1998), 1-22, MR 99f:20064, Zbl 0937.20015.

[2] E. Brieskorn and K. Saito, *Artin gruppen und Coxeter gruppen*, Invent. Math., **17** (1972), 245-271, MR 48 #2263, Zbl 0243.20037.

[3] R. Charney, *Geodesic automation and growth function for Artin groups of finite type*, Math. Ann., **301** (1995), 307-324, MR 95k:20055, Zbl 0813.20042.

[4] ______, *Injectivity of the positive monoid for some infinite type Artin groups*, in 'Geometric Group Theory Down Under' (J. Cossey, C. Miller, W. Neumann and M. Shapiro eds.), Walter de Gruyter, Berlin, 1996, 103-118, MR 2000h:20073.

[5] P. Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math., **17** (1972), 273-302, MR 54 #10659, Zbl 0238.20034.

[6] R. Fenn, D. Rolfsen and J. Zhu, *Centralisers in the braid group and singular braid monoid*, Enseign. Math., **42** (1996), 75-96, MR 97h:20047, Zbl 0869.20024.

[7] E. Godelle, *Centralisateur d'un sous-groupe parabolique dans un groupe d'Artin, partie* I, publication du LAMFA CNRS FRE 2270, May 2000.

[8] ______, *Normalisateurs et centralisateurs des sous-groupes paraboliques dans les groupes d'Artin-Tits*, Université de Picardie-Jules Verne, 2001.

[9] J. Michel, *A note on words in braid monoids*, J. Algebra, **215** (1999), 366-377, MR 2000b:20046, Zbl 0937.20017.

[10] L. Paris, *Parabolic subgroups of Artin groups*, J. Algebra, **196** (1997), 369-399, MR 98k:2006, Zbl 0926.20022.

[11] ______, *Artin monoids inject in their groups*. Comment. Math. Helv., **77** (2002), 609-637.

[12] D. Rolfsen, *Braid subgroup normalisers and commensurators and induced representations*, Invent. Math., **130** (1997), 575-587, MR 98j:20048, Zbl 0897.20031.

[13] J.-P. Serre, *Arbres, Amalgames, SL_2*, Astérisque, **26**, Soc. Math. France, Paris, 1977, MR 57 #16426, Zbl 0369.20013.

[14] H. Van der Lek, *The homotopy type of complex hyperplane complements*, Nijmegen, 1993.

School of Mathematics & Statistics
University of Sydney
NSW 2006
Australia
E-mail address: eddyg@maths.usyd.edu.au