PARABOLIC SUBGROUPS OF ARTIN GROUPS
OF TYPE FC

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Let \((A, S)\) be an Artin group of type FC and \(A_T\) a standard parabolic subgroup of \(A\). We use combinatorial tools to show that the normalizer of \(A_T\), the commensurator of \(A_T\), and the product of the quasi-centralizer of \(A_T\) by \(A_T\) are equal. Furthermore, we show that the centralizer and the quasi-centralizer of \(A_T\) in \(A\) are generated by their intersections with the monoid \(A^+\).

0. Introduction.

Let \(S\) be a finite set and \(M = (m_{s,t})_{s,t \in S}\) a symmetric matrix with \(m_{s,s} = 1\) for \(s \in S\) and \(m_{s,t} \in \mathbb{N} - \{0, 1\} \cup \{\infty\}\) for \(s \neq t\) in \(S\). An Artin-Tits system associated to \(M\) is the pair \((A_S, S)\) where \(A_S\) is the group defined by the presentation

\[
A_S = \left\langle S \mid \overbrace{s_t s_{\ldots s}}^{m_{s,t} \text{ terms}} = \overbrace{t s_{\ldots t}}^{m_{s,t} \text{ terms}} \quad \forall s, t \in S, s \neq t \text{ and } m_{s,t} \neq \infty \right\rangle.
\]

The group \(A_S\) is called an Artin group and relations \(\overbrace{s_t s_{\ldots s}}^{m_{s,t} \text{ terms}} = \overbrace{t s_{\ldots t}}^{m_{s,t} \text{ terms}}\) are called braid relations. For instance, if \(S = \{s_1, \ldots, s_n\}\) with \(m_{s_i, s_j} = 3\) for \(|i - j| = 1\) and \(m_{s_i, s_j} = 2\) otherwise, then the associated Artin group is the braid group. We denote by \(A_S^+\) the submonoid of \(A_S\) generated by \(S\). This monoid \(A_S^+\) has the same presentation as the group \(A_S\), considered as a monoid presentation \([11]\). When we add relations \(s_i^2 = 1\) to the presentation of \(A_S\) we obtain the Coxeter group \(W_S\) associated to \(A_S\). We say that \(A_S\) is spherical if \(W_S\) is finite. The matrix \(M\) may be represented by a graph denoted by \(\Gamma_S\), whose set of vertices is \(S\) and where an edge joins two vertices if \(m_{s,t} \geq 3\); these edges are labelled by \(m_{s,t}\) if \(m_{s,t} \geq 4\). We say that \(A_S\) (or simply \(S\)) is indecomposable if the graph \(\Gamma_S\) is connected. A subgroup \(A_T\) of \(A_S\) generated by a part \(T\) of \(S\) is called a standard parabolic subgroup, and a subgroup of \(A_S\) conjugate to a standard parabolic subgroup is called a parabolic subgroup. Van Der Lek showed \(([14])\) that \((A_T, T)\) is canonically isomorphic to the Artin-Tits system associated to the matrix \((m_{s,t})_{s,t \in T}\); its graph \(\Gamma_T\) is the full subgraph of \(\Gamma_S\) generated by \(T\). The

\[
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\]

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indecomposable components of $S$ are the maximal subsets of $S$ which are indecomposable.

One says that an Artin-Tits system $(A_S, S)$ (or simply an Artin group $A_S$) is of type FC if the following assertion is true:

$\forall T \subset S, (\forall s, t \in T, m_{s,t} \neq \infty \Rightarrow A_T$ is spherical).

For instance, the Artin group of the following graph is of type FC;

![Graph](image)

If $T$ is a subset of $S$ we call centralizer (resp. quasi-centralizer, normaliser, commensurator) of $A_T$ in $A_S$ the set

$Z_{A_S}(A_T) = \{g \in A_S \mid \forall s \in T, gs = sg\},$
$QZ_{A_S}(A_T) = \{g \in A_S \mid gT = Tg\},$
$N_{A_S}(A_T) = \{g \in A_S \mid gT \subset A_T g\},$

$\text{Com}_{A_S}(A_T) = \{g \in A_S \mid gA_T g^{-1} \cap A_T \text{ has finite index in both } A_T \text{ and } gA_T g^{-1}\}$

respectively. These sets are subgroups of $A_S$.

The first of the three main theorems we will prove is the following:

**Theorem 0.1.** Let $(A_S, S)$ be an Artin-Tits system of type FC and $X \subset S$; then

$\text{Com}_{A_S}(A_X) = N_{A_S}(A_X) = A_X \cdot QZ_{A_S}(A_X).$

This result was first proved by Rolfsen ([12]) in the case of braid groups; Paris ([10]) proved it when $A_S$ is a spherical Artin group and $T$ is indecomposable, finally in [8] we proved the result for any $T$ when $S$ is spherical.

In [6], the quasi-centralizer in the braid group of a part $T$ of $S$ is geometrically described thanks to the notion of ribbon; this notion was first generalised from a combinatorial viewpoint in [10] (and called conjugator) to a general Artin group and indecomposable part $T$, and finally generalised in [8] for any part $T$. The right viewpoint is to use the categorical language and to see the quasi-centralizer (and centralizer) as a set of morphisms in a groupoid.

Recall that $(A_S, S)$ is spherical if and only if $S$ has a lcm in $A_S^+$; in that case, this lcm is denoted by $\Delta_S$.

**Definition 0.2.** Let $(A_S, S)$ be an Artin-Tits system.

(i) We define the groupoid $\text{Conj}(S)$ as follows:

(a) Objects of $\text{Conj}(S)$ are subsets of $S$;

(b) the set $\text{Conj}(S; X, Y)$ of morphisms from $X$ to $Y$ is in bijection with the set

$\{g \in A_S \mid gX g^{-1} = Y\};$
(c) the composition of morphisms is defined by the product in $A_S$:

$$g \circ f = gf.$$ 

(ii) Let $X, Y \subset S$; we say that $w \in \text{Conj}(S; X, Y)$ is a positive elementary $Y$-ribbon-$X$ ([10, 8]) if:

(a) $w = \Delta_{X'}$ for $X'$ an indecomposable component of $X$ or,

(b) there exists $t \in S$ such that the indecomposable component $X'$ of $X \cup \{t\}$ containing $t$ is spherical and $w = \Delta_{X'} \Delta_{X' \setminus \{t\}}^{-1}$.

We say that $w \in \text{Conj}(S; X, Y)$ is an elementary $Y$-ribbon-$X$ if it is a positive elementary ribbon or $w^{-1}$ is a positive elementary $X$-ribbon-$Y$.

(iii) We denote Ribb$(S)$ the smallest subcategory of Conj$(S)$ which has the same objects and which contains the elementary ribbons; the set of morphisms from $X$ to $Y$ in Ribb$(S)$ is denoted Ribb$(S; X, Y)$ and its elements are called $Y$-ribbon-$X$.

Note that in Case (ii)(a), $X = Y$ and that in Case (ii)(b) there exists $u \in S$ such that $X \cup \{u\} = Y \cup \{t\}$.

The second main theorem of this article is the following:

**Theorem 0.3.** Let $(A_S, S)$ be an Artin-Tits system of type FC; then the category Conj$(A_S, S)$ is generated by the elementary ribbons; that is Conj$(S) = \text{Ribb}(S)$.

This result was proved by Paris in [10] for spherical Artin groups. In [8] we proved a similar result in all Artin monoids; in that case, generators are the positive elementary ribbons.

**Corollary 0.4.** Let $(A_S, S)$ be an Artin-Tits system of type FC and $X \subset S$; then

$$w \in QZ_{A_S}(A_X) \iff w = w_n \ldots w_1 \text{ with } w_i \text{ an elementary } X_i\text{-ribbon-}X_{i-1} \text{ where } X_0 = X_n = X$$

and $QZ_{A_S}(A_X)$ is the subgroup of $A_S$ generated by $QZ_{A_S}(A_X) \cap A_S^\pm$.

**Corollary 0.5.** Let $(A_S, S)$ be an Artin-Tits system of type FC and $X \subset S$; then $Z_{A_S}(A_X)$ is the subgroup of $A_S$ generated by $Z_{A_S}(A_X) \cap A_S^\pm$.

In Section 1, we recall relevant facts on Artin groups of type FC, on Artin monoids and define some useful notations; in Section 2 we look at the spherical case and in Section 3, we prove the main results.

1. Preliminaries.

In this part we assume that $(A, S)$ is an Artin-Tits system associated to the matrix $M = (m_{s,t})_{s,t \in S}$.
Lemma 1.1.

(i) ([2, 9]). \( A^+ \) is left (resp. right) cancellable and every pair \( a, b \in A \) has a left (resp. right) \( \gcd \), denoted \( a \land_l b \) (resp. \( a \land_r b \)).

(ii) ([14, Theorem II.4.13]). Let \( T \) be a nonempty subset of \( S \). Then the subgroup \((\langle T \rangle_A, T)\) of \( A \) is canonically isomorphic to the Artin group \( A_T \) associated to the matrix \( (m_{s,t})_{s,t \in T} \). Furthermore, if \( T' \) is another subset of \( S \), then \( A_T \cap A_{T'} = A_{T \cap T'} \) with the notation \( A_\emptyset = \{1\} \).

Lemma 1.2. Assume that \( A \) is spherical and let \( a, b \in A^+ \); then

(i) ([2]). \( a, b \) have a left lcm (resp. right lcm) in \( A^+ \) denoted \( a \lor_l b \) (resp. \( a \lor_r b \)).

(ii) ([5, Paragraph 4]). Let \( g \in A; g \) can be written \( g = g_1 \Delta_S^n \) with \( g_1 \in A^+ \), and \( n \in \mathbb{Z} \).

(iii) ([3, Theorem 2.6] and [4, Lemma 4.4]). Let \( g \in A; \) there exists unique \( a, b \in A^+ \) such that \( a \land_r b = 1 \) and \( g = ab^{-1} \). Furthermore, if \( c \in A^+ \) such that \( gc \in A^+ \) then \( c = bc' \) for some \( c' \in A^+ \).

We call the decomposition \( g = ab^{-1} \) of (iii) the (right) orthogonal splitting of \( g \). In a similar way one can define the left orthogonal splitting of \( g \).

Lemma 1.3 ([8, Corollary 4.4.6]). Let \( (A, S) \) be a spherical Artin group and let \( s, t \in S, g \in A \) and \( j \in \mathbb{N}^* \) be such that \( s^j g = gt^j \). Then:

(i) \( sg = gt \);

(ii) if \( w = ab^{-1} \) is the orthogonal splitting of \( g \), then

\[
\begin{align*}
  sa &= au, \\
  tb &= bu
\end{align*}
\]

for some \( u \in S \).

Notation 1.4. Let \( (A, S) \) be an Artin group and let \( X \subset S \).

(i) We denote by \( X_s \) the union of the spherical indecomposable components of \( X \) and by \( X_{as} \) the complement \( X - X_s \).

(ii) We denote by \( X^\perp \) the set \( \{ s \in S \mid \forall t \in X, \ m_{s,t} = 2 \} \); we have \( X \cap X^\perp = \emptyset \).

(iii) If \( Y \) is another subset of \( S \) we write \( X \cup Y = X \oplus Y \) if \( Y \subset X^\perp \). In particular, \( X = X_s \oplus X_{as} \).

(iv) If \( s \in X \), we denote by \( X(s) \) the indecomposable component of \( X \) which contains \( s \).

In the following we write \( X_{as}^\perp \) for \( (X_{as})^\perp \).

To prove our main results we need to introduce the following notations for a new family of subcategories of \( \text{Conj}(S) \) which generalises \( \text{Ribb}(S) \); we only give notations for their morphisms.
**Notation 1.5.** Let \((A_S, S)\) be an Artin group, and \(T \subset S\). Consider the smallest subcategory of \(\text{Conj}(S)\) which has the same objects as \(\text{Conj}(S)\) and which contains the elementary ribbons which are in \(A_T\). For \(X, Y \subset Y\), we denote by \(\text{Ribb}(T; X, Y)\) the set of morphisms from \(X\) to \(Y\) in this subcategory. They are \(Y\)-ribbons-\(X\).

**1.1. Artin-Tits system of type FC.** We assume in this section that \((A, S)\) is of type FC. Recall that Artin groups of type FC have been defined in the introduction. Most facts on Artin groups of type FC in this part are proved in [1].

**Proposition 1.6 ([13, Theorem 1]).** Let \(G = G_1 *_{H} G_2\) the amalgamated product of groups \(G_1\) and \(G_2\) over \(H\). Let \(C_1, C_2\) be transversals of \(G_1/H\) and \(G_2/H\) respectively which contain 1. For all \(x \in G\), there exists a unique sequence \((x_1, \ldots, x_n, h)\) such that \(x = x_1 \cdots x_n h\) with \(h \in H\), and where the \(x_i\) are in \(C_1 \cup C_2\) with \(x_i \neq x_{i+1}\) not in the same transversal.

We will call \((x_1, \ldots, x_n, h)\) the amalgam normal form of \(x\) relative to the amalgamated product \(G_1 *_{H} G_2\) and we set \(|x|_s = n\). We have then \(|x|_s = 0\) if and only if \(x \in H\).

**Corollary 1.7 ([1, Corollary 1]).** Let \(G = G_1 *_{H} G_2\) and \(g, c \in G\). We denote by \((g_1, \ldots, g_n, h)\) the amalgam normal form of \(g\). Assume that \(g_n \in C_1\) and \(|c|_s \leq 1\), then: The amalgam normal form of \(gc\) is

\[
\begin{cases}
(g_1, \ldots, g_n, g_{n+1}', h') & \text{if } c \in G_2 - H, \\
(g_1, \ldots, g_{n-1}', g_n, h') & \text{if } c \in G_1 - (g_n h)^{-1} H, \\
(g_1, \ldots, g_{n-1}', h') & \text{if } c \in (g_n h)^{-1} H,
\end{cases}
\]

where \((g_{n+1}', h')\) is the amalgam normal form of \(hc\) in the first case, \((g_n', h')\) is the amalgam normal form of \(g_nhc\) in the second case and \(h' = g_nhc\) in the third case.

**Corollary 1.8.** Let \(w = v_1 \cdots v_m \in G\) such that \(v_{2j} \in G_2 - H\) and \(v_{2j+1} \in G_1 - H\) for \(j \in \{0, \ldots, \lfloor m/2 \rfloor\}\). If we denote by \((w_1, \ldots, w_n, h)\) the amalgam normal form of \(w\), then one has

\[
\begin{align*}
m &= n, \\
v_1 &= w_1 h_1 \text{ with } h_1 \in H, \\
h_{i-1} v_i &= w_{i} h_i \text{ with } i \in \{2, \ldots, n\} \text{ with } h_i \in H, \\
h_n &= h.
\end{align*}
\]

**Proposition 1.9 ([1, Proposition 2]).**

(i) Let \(s_1, s_2\) be in \(S\) be such that \(m_{s_1, s_2} = \infty\). Let \(A_1 = A_{S-\{s_1\}}\), \(A_2 = A_{S-\{s_2\}}\) and \(A_{1,2} = A_{S-\{s_1,s_2\}}\), then the group \(A\) is the amalgamated product of \(A_1\) and \(A_2\) over \(A_{1,2}\), that is \(A = A_1 *_{A_{1,2}} A_2\).

(ii) The set of Artin groups of type FC is the smallest class of Artin groups which is closed under amalgamation over standard parabolic subgroups and which contains spherical Artin groups.
Proposition 1.10 ([1, Theorem 2]). Let $T \subset S$. There exists a function $m_T : A \to A$ such that for all $w \in A$ one has:

(i) $m_T(w) \in wA_T$;
(ii) for all $v \in wA_T$, $m_T(v) = m_T(w)$;
(iii) if $w \in A_U$ for $U \subset S$, then $m_T(w) \in A_U$.

The function $m_T$ gives a special representative of each coset of $A/A_T$.

Notation 1.11. Assume that $(A, S)$ is not spherical and fix $s_1, s_2 \in S$ such that $m_{s_1, s_2} = \infty$. We set $A_1 = A_{S_{1,2}}$, $A_2 = A_{S_{2}}$ with $S_{1} = S - \{s_1\}$, $S_{2} = S - \{s_2\}$ and $S_{1,2} = S - \{s_1, s_2\}$. Then we have $A = A_1 *_{A_{1,2}} A_2$. Transversals of $A_1/A_{1,2}$ and $A_2/A_{1,2}$ are transversals $C_1, C_2$ respectively induced by $m_{S_{1,2}}$.

Corollary 1.12. Assume that $(A, S)$ is not spherical and let $s_1, s_2 \in S$ with $m_{s_1, s_2} = \infty$; one has $A = A_1 *_{A_{1,2}} A_2$ with Notation 1.11. If $w \in A_T$ for $T \subset S$ then the amalgam normal form of $w$ has its terms in $A_T$.

1.2. Artin monoids.

Definition 1.13. Let $(A_S, S)$ be an Artin-Tits system.

(i) We define the small category $\text{Conj}^+(S)$ as follows:
   (a) Objects of $\text{Conj}^+(S)$ are subsets of $S$;
   (b) the set $\text{Conj}^+(S; X, Y)$ of morphisms from $X$ to $Y$ is in bijection with the set
   $$\{g \in A^+ \mid gXg^{-1} = Y\};$$
   (c) the composition of morphisms is defined by the product in $A^+$:
   $$g \circ f = gf.$$

(ii) We denote $\text{Ribb}^+(S)$ the smallest subcategory of $\text{Conj}^+(S)$ which has the same objects and which contains the positive elementary ribbons (see 0.2); the set of morphisms from $X$ to $Y$ in $\text{Ribb}^+(S)$ is denoted $\text{Ribb}^+(S; X, Y)$ and its elements are called positive $Y$-ribbon-$X$.

Categories $\text{Conj}^+(S)$ and $\text{Ribb}^+(S)$ are clearly subcategories of $\text{Conj}(S)$ and $\text{Ribb}(S)$ respectively.

In the following, we will need the following theorem in the spherical case. It is Theorem 0.3 but in the setting of the Artin monoid.

Theorem 1.14 ([10]). Let $(A_S, S)$ be an Artin-Tits system of spherical type; then

$$\text{Conj}^+(S) = \text{Ribb}^+(S).$$

In fact this theorem is true in any Artin monoid ([8]).
2. The spherical case.

As we said in the introduction, Theorems 0.1 and 0.3 are known in the spherical case. Nevertheless we need to state precise results in the spherical case to prove our theorems in type FC.

**Theorem 2.1.** Let \((A, S)\) be a spherical Artin group and \(X, Y \subset S\). Let \(k \in \mathbb{Z} - \{0\}\) and \(g \in A\). The following are equivalent:

1. \(gA_Xg^{-1} \subset A_Y\);
2. \(g\Delta_X^k g^{-1} \in A_Y\);
3. \(g = yx\) with \(y \in A_Y\), \(x \in \text{Ribb}(S; X, R)\) for \(R \subset Y\).

**Proof.** It is clear that (3) \(\Rightarrow\) (1) \(\Rightarrow\) (2). For (2) \(\Rightarrow\) (3), the proof is similar to Proposition 3.1 of [7]; thanks to Lemma 1.2(ii), we may assume that \(g \in A^+\) and is \(Y\)-reduced (i.e., not divisible by any \(s \in Y\)); then for all \(s \in X\), we have \(gsg^{-1} = t\) for some \(t \in Y\). Thus \(g \in \text{Ribb}(S; X, R)\) with \(R \subset Y\) by Theorem 1.14. \(\square\)

**Lemma 2.2.** Let \((A, S)\) be a spherical Artin group and let \(X, Y, T \subset S\). Let \(g \in A_T\) be such that \(gA_Xg^{-1} \subset A_Y\). Let \(s \in X - T\); then there exists \(x \in A_{X(s)\cap T}\) and \(y \in A_{Y\cap T}\) such that \(g = yx\). Furthermore \(X(s) \subset Y\).

**Proof.** Let \(g = a_0b_0^{-1}\) the orthogonal splitting of \(g \in A_T^+\). One has \(a_0 = a_1a\) where \(a\) is \(Y\)-reduced and \(a_1 \in A_{Y\cap T}^+\). In the same way, one has \(b_0 = b_1b\) where \(b\) is \(\{s\}^\perp\)-reduced and \(b_1 \in A_{Y\cap T}^+\). We obtain \(ab^{-1}sba^{-1} \in A_Y\); hence \(ab^{-1}sba^{-1} = u^{-1}v\) with \(u \perp_I v\) in \(A^+_Y\). Thus \(b^{-1}sb = (ua)^{-1}(va)\) with \(b \perp_I sb\), since \(b \in A_{S\cap \{s\}}^+\) and \(s \in \{s\}^\perp\)-reduced. Thus, there exists \(x\) in \(A^+\) such that \(va = asb\) and \(ua = ab\). This implies that \((b \lor_r a)^{-1}\) divides \(v\) and thus is in \(A_{Y\cap T}\). On the other hand, we have \(ba^{-1} = c^{-1}d\) with \(c = (b \lor_r a)b^{-1} \in A_T^+\) and \(d = ((b \lor_r a)a^{-1}) \in A_{T\cap Y}^+\). Thus \(csc^{-1} \in A_Y\).

Let \(c = c_2c_1\) with \(c_1 \in A_{\{s\}^\perp\cap T}^+\) and \(c_2 \text{ reduced-}\{s\}^\perp\) in \(A_T^+\). Then we have \(c_2sc_2^{-1} \in A_Y\) with \(c_2s \perp_r c_2\). Thus both \(c_2\) and \(s\) are in \(A_Y\). Then \(g = y_0x_0\) with \(x_0 = c_1b_1^{-1} \in A_{\{s\}^\perp\cap T}\) and \(y_0 = a_1d^{-1}c_2 \in A_{Y\cap T}\). We have \(x_0A_Xx_0^{-1} \subset A_Y\) with \(x_0 \in A_{\{s\}^\perp\cap T}\). If \(x_0 = 1\) or \(X(s) = \{s\}\), the result holds with \(x = x_0\) and \(y = y_0\). Assume \(x_0 \neq 1\) and \(X(s) \neq \{s\}\).

Choose \(s' \in X(s) - (\{s\}^\perp \cup \{s\})\) (it exists since \(X(s) \neq \{s\}\)). Applying the argument to \(g' = x_0\), \(T' = \{s\}^\perp \cap T\), and \(s'\), we obtain \(x_0 = y_1x_1\) with \(x_1 \in A_{\{s'\}^\perp\cap T}\) and \(y_1 \in A_{Y\cap T}\). Repeating this process yields \(g = y_0 \ldots y_nx_n\) with \(x_n \in A_{\{s'\}^\perp\cap \ldots \cap \{s^{(n)}\}^\perp\cap T}\) and \(y_0 \ldots y_n \in A_{T\cap Y}\). The process will terminate when either \(x_n = 1\) or

\[
X(s) - \left( \bigcap_{i=0}^{i=n} \{s^{(i)}\}^\perp \bigcup_{i=0}^{i=n} \{s^{(i)}\} \right) = \emptyset
\]

which means that \(X(s) = \bigcup_{i=0}^{i=n} \{s^{(i)}\}\) and \(x_n \in A_{\bigcap_{i=0}^{i=n} \{s^{(i)}\}^\perp\cap T} = A_{X(s)^\perp\cap T}\).

In either case, the result follows with \(x = x_n\) and \(y = y_0 \ldots y_n\). \(\square\)
Proposition 2.3. Under the hypotheses of Theorem 2.1, if $g \in A_T$ for $T \subset S$, then, (1), (2) and (3) are equivalent to

(3') $g = yx$ with $y \in A_{Y \cap T}$, $x \in \text{Ribb}(T \cap (\bigcup_{s \in X - T} X(s))^{\perp}; X, R)$ where $R$ is of the form $\bigcup_{s \in X - T} X(s) \oplus T_1 \subset Y$ with $T_1 \subset T \cap Y$.

Proof. It is clear that (3') $\Rightarrow$ (3). Let us show that (1) $\Rightarrow$ (3') by induction on the cardinal of $X - T$. It is enough to find $x, y$ such that $y \in A_Y$ and $x \in \text{Ribb}(T \cap (\bigcup_{s \in X - T} X(s))^{\perp}; X, R)$ since that implies $y \in A_T$ and the type of $R$. If $X - T = \emptyset$, then $g A_X g^{-1} \subset A_{Y \cap T}$ in $A_T$ and we apply Theorem 2.1 in $A_T$. Otherwise Lemma 2.2 proves that for all $\#(X - T) \geq 1$ and $s \in X - T$ then $g = y_1 x_1$ with $x_1 \in A_{X(s)^{\perp} \cap T}$ and $y_1 \in A_{Y \cap T}$. Thus, in $A_{X(s)^{\perp}}$ we have $x_1 (X - X(s)) x_1^{-1} \subset A_{Y \cap X(s)^{\perp}}$ with $x_1$ in $A_{T \cap X(s)^{\perp}}$. We apply the induction hypothesis in $A_{X(s)^{\perp}}$; after replacing $g$ by $x_1$, $X$ by $X - X(s)$, $T$ by $T \cap X(s)^{\perp}$ and $Y$ by $Y \cap X(s)^{\perp}$; we have $\#((X \cap X(s)^{\perp}) - (T \cap X(s)^{\perp})) < \#(X - T)$ since $s \notin X \cap X(s)^{\perp}$ and $s \in X - T$. We get $x_1 = y_2 x$ with $x$ in $\text{Ribb}(T \cap X(s)^{\perp} \cap \bigcup_{u \in X - X(s)^{\perp}} X(u)^{\perp}; X - X(s), R_1)$ with $R_1 \subset Y$ and $y_2 \in A_Y$. But $X(s)^{\perp} \cap \bigcup_{u \in X - X(s)^{\perp}} X(u)^{\perp} = (\bigcup_{u \in X - X(s) - T} X(u)^{\perp})$. Thus $g = yx$ with $y = y_1 y_2 \in A_Y$ and $x \in \text{Ribb}(T \cap (\bigcup_{u \in X - T} X(u)^{\perp}); X, R)$. □

3. Proof of the main results.

Proposition 3.1. Let $(A, S)$ be an Artin group of type FC. Let $X, Y, T \subset S$ with $X$ spherical. Let $k \in \mathbb{Z} - \{0\}$ and $g \in A_T$. The following are equivalent:

(1) $g A_X g^{-1} \subset A_Y$;
(2) $g A_X^{k} g^{-1} \in A_Y$;
(3) $g = yx$ with $y \in A_{Y \cap T}$ and $x \in \text{Ribb}(T \cap (\bigcup_{s \in X - T} X(s))^{\perp}; X, R)$ for some $R \subset Y$.

Proof. Implications (3) $\Rightarrow$ (1) $\Rightarrow$ (2) are clear. Let us show that (2) $\Rightarrow$ (3) by induction on the number $m$ of amalgamations; that is the number of edges in $\Gamma_S$ labelled with $\infty$. If $m = 0$ then $A$ is spherical and the result is true by Theorem 2.1 and Proposition 2.3. Assume now that $m \geq 1$ and that Proposition 3.1 is true for any Artin group of type FC with a number of amalgamation less than or equal to $m - 1$. We choose $s_1, s_2 \in S$ such that $m_{s_1, s_2} = \infty$ and we use Notation 1.11. Note that since $A_X$ is spherical, we have $X \subset S_1$ or $X \subset S_2$. Denote by $(g_1, \ldots, g_n, h)$ the amalgam normal form of $g$; elements $g_i$ and $h$ are in $A_T$ by Corollary 1.12. For $m$ fixed, let us do an induction on $n$. We have $g_1 \ldots g_n h A_X^{k} h^{-1} g_{n}^{-1} \ldots g_{1}^{-1} \in A_Y$. If $n = 0$ then the formula holds in $A_1$ or in $A_2$ and we conclude by the induction hypothesis on $m$ applied in $A_1$ or in $A_2$. Assume now that $n \geq 1$. We may assume without loss of generality that $g_n \in A_1$.

If $X \not\subset A_1$ then by Corollary 1.8, the amalgam normal form of $g A_X^{k} g^{-1}$ is of the shape $(g_1, \ldots, g_n, g_{n+1}', \ldots, g_{2n+1}', h')$ with $g'_{n+1} = m_{S_{1,2}}(h A_X^{k} h^{-1})$. If $X \subset A_1$ then...
But \( g\Delta^k_X g^{-1} \in A_Y \) thus \( g_1 \ldots g_n \in A_{Y \cap T} \) by Corollary 1.12 and \( h\Delta^k_X h^{-1} \in A_Y \). Thus, to conclude, we apply the induction hypothesis on \( n \) at rank \( n = 0 \) to \( h \).

If \( X \subset A_1 \) and \( g_n h\Delta^k_X (g_nh)^{-1} \notin A_{1,2} \) then the amalgam normal form of \( g\Delta^k_X g^{-1} \) is of shape \((g_1, \ldots, g_{n-1}, g'_n, \ldots, g'_{n-1}, h')\) with

\[ g'_n = m_{S_{1,2}}(g_n h\Delta^k_X (g_nh)^{-1}). \]

But \( g\Delta^k_X g^{-1} \in A_Y \); then we get \( g_1 \ldots g_{n-1} \in A_{Y \cap T} \) by Corollary 1.12 and thus \( g_n h\Delta^k_X (g_nh)^{-1} \in A_Y \); thus we may apply the induction hypothesis on \( m \) in \( A_1 \) at \( g_nh \) and conclude.

If \( X \subset S_1 \) and \( g_n h\Delta^k_X (g_nh)^{-1} \in A_{1,2} \) then by the induction hypothesis on \( m \) applied in \( A_1 \), we get \( g_nh = y_1 x_1 \) with

\[ x_1 \in \text{Ribb}\left( T \cap \left( \bigcup_{t \in X-T} X(t) \right)^{\perp} ; X, R \right) \]

for \( R \subset S_{1,2} \) and \( y_1 \in A_{1,2} \cap A_T \). We get \( g_1 \ldots g_{n-1} y_1^1 \Delta^k_{X} (g_{n-1} y_1)^{-1} \in A_Y \) and by the induction hypothesis on \( n \) applied at rank \( n-1 \) to \( g_1 \ldots g_{n-1} y_1 \in A_T \), we obtain \( g_1 \ldots g_{n-1} y_1 = yx_2 \) with

\[ x_2 \in \text{Ribb}\left( T \cap \left( \bigcup_{t \in X-T} X(t) \right)^{\perp} ; R, R_1 \right) \]

for some \( R_1 \subset Y \) and \( y \in A_{Y \cap T} \).

Thus \( g = yx_2 x_1 \) and \( x_2 x_1 \in \text{Ribb}\left( T \cap (\bigcup_{t \in X-T} X(t))^{\perp} ; X, R_1 \right) \) for \( R_1 \subset Y \).

\[ \square \]

**Theorem 3.2.** Let \((A, S)\) be an Artin group of type FC and let \( X, Y, T \subset S \).

Let \( g \in A_T \) and \( k \in \mathbb{Z} - \{0\} \). The following are equivalent:

1. \( g \Delta^k_X g^{-1} \subset A_Y \);
2. \( g\Delta^k_X g^{-1} \in A_Y \), \( g = yx \) with \( x \in A_{X^{\perp}} \), \( y \in A_{Y \cap T} \) and \( X_{\text{as}} \subset Y \);
3. \( g = yx \) with \( y \in A_{Y \cap T} \) and \( x \in \text{Ribb}\left( T \cap X^{\perp}_{\text{as}} ; X, R \right) \) with \( R \oplus X_{\text{as}} \subset Y \).

**Proof.** It is clear that (3) \( \Rightarrow \) (2) and that Proposition 3.1 induces (2) \( \Rightarrow \) (1). Let us show that (1) \( \Rightarrow \) (3). We are carrying out an induction on the number \( r(X) \) of edges in \( \Gamma(X) \) which are labelled with \( \infty \). If \( r(X) = 0 \), then \( X \) is spherical and the result is true by Proposition 3.1. Assume now \( X \) is not spherical (that is \( r(X) \geq 1 \)) and fix \( s_1, s_2 \) in \( X \) such that \( m_{s_1, s_2} = \infty \).

We assume that if \((A', S')\) is an Artin group of type FC and \( X', Y', T' \) are three parts of \( S' \) such that \( r(X') < r(X) \) then for all \( g' \) of \( A_{T'} \), we have:

\[ g'^{-1} A_X g' \subset A_{X'} \Rightarrow g' = yx' \text{ where } x' \in \text{Ribb}\left( T' \cap X'^{\perp}_{\text{as}} ; X', R' \right) \text{ and } y' \in A_{Y' \cap T'} \text{ with } R' \oplus X'_{\text{as}} \subset Y'. \]

We have \( A = A_1 *_{A_{1,2}} A_2 \) with Notation 1.11. Let \((g_1, \ldots, g_n, h)\) be the amalgam normal form of \( g \). The first step is to show that it is enough to prove the result for the case \( n = 0 \). Assume \( n \geq 1 \). Without loss of generality
we may assume that $g_n \in A_1$. Furthermore $g_1 \cdots g_nh_1h^{-1}g_n^{-1} \cdots g_1^{-1} \in A_Y$ since $s_1 \in X$. By Corollary 1.8 we infer that the amalgam normal form of $g_1 \cdots g_nh_1h^{-1}g_n^{-1} \cdots g_1^{-1}$ is of the shape $(g_1, \ldots, g_n, g_{n+1}', \ldots, g_{2n+1}', h')$ and has its terms in $A_Y$. Thus $g_1 \cdots g_n$ is in $A_Y \cap T$. We get that $hAXh^{-1}$ is also in $A_Y$. Thus if (1) $\Rightarrow$ (3) for any $g$ such that $n = 0$, the theorem will be proved. Assume $g = h \in A_{1,2}$. Denote by $T_1$ (resp. $T_2$) the indecomposable component of $X - \{s_1\}$ (resp. $X - \{s_2\}$) which contains $s_2$ (resp. $s_1$). In $A_1$ we have $gAX_{-{\{s_1\}}}g^{-1} \subset A_{Y-{\{s_1\}}}$ thus by the induction hypothesis, we get $g = y_1x_1$ with $y_1 \in A_{Y \cap T}$ and $x_1 \in \text{Ribb} (T \cap (X - \{s_1\})_{as} \cap (X - \{s_1\})s, R_1)$ with $R_1$ in $Y - \{s_1\}$. Furthermore, since $s_2 \notin A_{1,2}$ and $g \in A_{1,2}$, we get either by the induction hypothesis (if $T_1$ is not spherical) or by Proposition 3.1, that we can find $x_1 \in \text{Ribb} (T_1 \cap T \cap (X - \{s_1\})_{as} \cap (X - \{s_1\})s, R_1)$. From $y_1 \in A_Y$, we infer that $x_1A_{x_1}^{-1} \subset A_Y$. We can use the same argument if we replace $g$ by $x_1$ and exchange the roles of $A_1$ and $A_2$; we find, thanks to the inclusion $x_1A_{x_1}^{-1} \subset A_{1,2} - \{s_1\}$ in $A_2$, that $x_1 = y_2x_2$ with $y_2 \in A_{Y \cap T}$ and $x_2 \in \text{Ribb} (T_2 \cap T \cap (X - \{s_2\})_{as} \cap (X - \{s_2\})s, R_2)$ with $R_2$ in $Y$. Finally, since $T_1 \cup T_2 = X(s_1) = X(s_2)$, we get $T_1 \cap T \cap (X - \{s_1\})_{as} \cap (X - \{s_2\})s, R_2) \cap (X - \{s_1\})_{as} \cap (X - \{s_2\})s, R_2)$ with $R_2$ in $Y$. We get that $g = y_1y_2x_2$ with $y_1y_2 \in A_{Y \cap T}$, $x_2 \in \text{Ribb} (T \cap X_{as}^{-1}X, R)$ since $X \subset (X - \{s_2\})s$ and $R \subset R_2 \subset Y$.}

The two following lemmas are used to prove (ii) and the first equality of (i) in Theorem 3.5.

**Lemma 3.3.** Let $(A, S)$ be an Artin group of type $FC$; then $QZ_A(A) = QZ_{A_{S_i}}(A_{S_i})$.

**Proof.** It is clear that $QZ_A(A) = QZ_{A_{S_i}}(A_{S_i}) \cdot QZ_{A_{S_{as}}}(A_{S_{as}})$. Then it is enough to prove that $QZ_{A_{S_{as}}}(A_{S_{as}}) = \{1\}$ and since it is the product of quasi-centralizers of its indecomposable components, it is enough to show that if $X$ is indecomposable not spherical, then $QZ_A(A) = \{1\}$. Let $A$ be such a group and let $g \in QZ_A(A)$; choose $T \subset S$ maximal spherical and let $T' = gTg^{-1} \subset S$. Then by Proposition 3.1 applied to the equality $g^{-1}ATg = AT'$ with $T$, we get that $T = T'$ and $g \in A_T$; thus $g \in QZ_{A_T}(A_T)$. Let $(T_i)_{1 \leq i \leq k}$ be the indecomposable components of $T$ then $QZ_{A_T}(A_T) = \{\Delta_{1}^{j_1} \cdots \Delta_{k}^{j_k} ; \forall i, j_i \in \mathbb{Z}\}$. Thus $g = \Delta_{1}^{j_1} \cdots \Delta_{k}^{j_k}$ with $j_i \in \mathbb{Z}$ for all $i \in \{1, \ldots, k\}$. Assume that there exists $i \in \{1, \ldots, k\}$ such that $j_i \neq 0$. Since $T$ is maximal spherical and $S$ is indecomposable and not spherical, there exists $s \in T_i$ and $t \in S - T$ such that $m_{s,t} = \infty$. We get $A = A_{S - \{s\}, S - \{t\}}A_{S - \{t\}}$ and $gs^{-1} = s$ with $s_1 \in T$; since $j_i \neq 0$, this is impossible by Corollary 1.8. Thus for all $i$, we get $j_i = 0$ and $g = 1$. □

**Lemma 3.4.** Let $A = A_1 \ast_{A_{1,2}} A_2$ be a non-spherical Artin group of type $FC$ in the Notation of 1.11. Let $X \subset S$ be such that $\{s_1, s_2\} \subset X$, let $X_i = X \cap S_i$ for $i \in \{1, 2\}$; then $\text{Com}_A(A_X) \cap A_i \subset \text{Com}_{A_i}(A_{X_i})$ for $i \in \{1, 2\}$.

**Proof.** By symmetry, it is enough to prove the result for $i = 1$. Let $g \in \text{Com}_A(A_X) \cap A_1$. We have to show that $A_{X_1} \cap (gA_{X_1}g^{-1})$ has finite index
in $A_X \cap gA_X \cdot g^{-1}$. For this, we show that if $x, y \in A_X$ (resp. $x, y \in gA_X \cdot g^{-1}$) have the same image in $A_X/(A_X \cap gA_X \cdot g^{-1})$ (resp. $gA_X \cdot g^{-1}/(A_X \cap gA_X \cdot g^{-1})$) then they have the same image in $A_X/(A_X \cap gA_X \cdot g^{-1})$ (resp. $gA_X \cdot g^{-1}/(A_X \cap gA_X \cdot g^{-1})$).

Let $x, y \in A_X$ be such that $x \alpha = y$ for some $\alpha \in A_X \cap (gA_X \cdot g^{-1})$. We have $\alpha = x \gamma^{-1} \in A_X \cap A_X \cdot (gA_X \cdot g^{-1}) = A_X \cap (gA_X \cdot g^{-1}) = A_X \cap (gA_X \cdot g^{-1})$. The last equality come from the fact that $g$ is in $A_S$ and that $A_X = A_S \cap A_X$. Let $x, y \in gA_X \cdot g^{-1}$ be such that $x \alpha = y$ for some $\alpha \in A_X \cap (gA_X \cdot g^{-1})$. We have by the same arguments that $\alpha \in A_X \cap (gA_X \cdot g^{-1})$. Thus $g \in \text{Com}_A(A_X)$. \hfill $\blacksquare$

**Theorem 3.5.** Let $(A, S)$ be an Artin group of type FC. Let $X \subset S$; then:

(i) $\text{Com}_A(A_X) = N_A(A_X) = A_X \cdot QZ_A(A_X)$;

(ii) $\text{Conj}(S; X, Y) = \text{Ribb}(S; X, Y) \subset \text{Ribb}(X_\alpha, X_s, X_\alpha)$.

Furthermore, if $\text{Conj}(S; X, Y) \neq \emptyset$ then the inclusion is a equality and $X_\alpha = X_s$.

**Proof.** Thanks to Theorem 3.2 we get the inclusions $A_X \cdot QZ_A(A_X) \subset N_A(A_X) \subset A_X \cdot \text{Ribb}(X_\alpha, X_s, X_\alpha) \subset A_X \cdot QZ_A(A_X)$. That proves the second equality of (i): $N_A(A_X) = A_X \cdot QZ_A(A_X)$. If $\text{Conj}(S; X, Y) = \emptyset$ then (ii) is clear since $\text{Ribb}(S; X, Y) \subset \text{Conj}(S; X, Y)$. Assume now that $\text{Conj}(S; X, Y) \neq \emptyset$. Since $\text{Conj}(S; X, Y) \subset A_X \cdot \text{Ribb}(X_\alpha, X_s, X_\alpha)$, one has $X_\alpha = X_s$, and $\text{Ribb}(X_\alpha, X_s, X_\alpha) \subset \text{Conj}(S; X, Y)$; then we get $\text{Conj}(S; X, Y) = QZ_A(A_X) \cdot \text{Ribb}(X_\alpha, X_s, X_\alpha)$. Now, by Lemma 3.3 $QZ_A(A_X) = QZ_{A_X}(A_X)$ and since $X_\alpha$ is spherical, we get $QZ_{A_X}(A_X) = \text{Ribb}(X_\alpha, X_s, X_\alpha)$. Then, we proved (ii). We have now to prove the first equality of (i). We have clearly $\text{Com}_A(A_X) \supset N_A(A_X)$; if $X$ is spherical, we prove the other inclusion as in [10] thanks to the implication (2) $\Rightarrow$ (1) of Proposition 3.1. Let us show that $\text{Com}_A(A_X) \subset N_A(A_X)$ for $X$ not spherical. In order to do this, we proceed by induction on the number $n$ of edges labelled with $\infty$ in the graph $\Gamma_X$. Note that for $r = 0$, we have that $X$ is spherical and the result is true in that case. Assume $r \geq 1$ and write $A = A_1 \ast A_2$ such that $\{s_1, s_2\} \in X$ following Notation 1.11. By the induction hypothesis, we have $\text{Com}_A(A_X \cap S_i) = N_A(A_X \cap S_i)$ for $i \in \{1, 2\}$. Let $g \in \text{Com}_A(A_X)$ and $(g_1, \ldots, g_n, h)$ its amalgam normal form. There exists $p \in \mathbb{N} - \{0\}$ such that $gs_1^p g^{-1}$ and $gs_2^p g^{-1}$ are in $A_X$. If $n \neq 0$, we infer from Corollary 1.8 that $g_1 \ldots g_n \in A_X$ and thus $h \in \text{Com}_A(A_X)$. Now, for $g = h \in \text{Com}_A(A_X) \cap A_1 \ast A_2$, we can apply Lemma 3.4; we get that for $i \in \{1, 2\}$, we have $h \in \text{Com}_A(A_X \cap S_i)$. But by the induction hypothesis, $\text{Com}_A(A_X \cap S_i) = N_A(A_X \cap S_i)$. Then $h \in N_A(A_X \cap S_i) \cap N_A(A_X \cap S_2) \subset N_A(A_X)$. \hfill $\blacksquare$

**Corollary 3.6.** Let $(A, S)$ be an Artin groups of type FC. Let $X \subset S$; Then:

(i) $QZ_A(A_X) = \text{Ribb}(X_\alpha, X_s, X_\alpha)$;

(ii) $Z_A(A_X) = (Z_A(A_X) \cap A^+)A$;

(iii) $QZ_A(A_X) = (QZ_A(A_X) \cap A^+)A$. 

Proof. The (i) is a particular case of 3.5(ii); (ii) and (iii) are equivalent by [8] Theorem 4.1.2; furthermore, it is clear that 3.5(ii) implies (iii) thanks to Lemma 1.3.

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