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HASSE PRINCIPLES FOR THE BRAUER GROUPS OF ALGEBRAIC FUNCTION FIELDS OF GENUS ZERO OVER GLOBAL FIELDS

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Let F be a global field with $\mathrm{char}(F) \neq 2$ and K an algebraic function field in one variable of genus zero over F. In this paper, we investigate two kinds of Hasse principles for Brauer classes on K. If $\mathrm{Br}(K)$ is the Brauer group of K and $\mathrm{Br}(K)'$ is the subgroup of $\mathrm{Br}(K)$ whose elements have order relatively prime to $\mathrm{char}(F)$, then we precisely determine the kernels of the maps

$$h_1 : \mathrm{Br}(K)' \to \prod_{\mathfrak{p}} \mathrm{Br}(\widehat{F}_{\mathfrak{p}}K) \quad \text{and} \quad h_2 : \mathrm{Br}(K) \to \prod_{P} \mathrm{Br}(\widehat{K_P}),$$

where $\mathfrak{p}$ runs over the prime spots of F and P runs over the places of K which are trivial over F, and $\widehat{F}_{\mathfrak{p}}$, $\widehat{K_P}$ are the completions at $\mathfrak{p}$, P respectively. To facilitate the determination of these kernels, we compute the kernel of the map $h : \mathrm{Br}(K) \to \prod_P \mathrm{Br}(K\overline{V}_P)$ where $\overline{V}_P$ is the residue field with respect to P and show that the kernels of these three maps coincide. We then consider a more general version of the maps above by describing the 2-torsion subgroup of the kernel of h_1 when a finite number of prime spots in the product are omitted.

1. Introduction.

Let F be a global field with $\mathrm{char}(F) \neq 2$. By a *prime spot* on F, we mean an equivalence class of discrete valuations on F or an equivalence class of archimedean absolute values on F. Define

$$P(F) = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime spot of } F\}.$$

For $\mathfrak{p} \in P(F)$ let $\widehat{F}_{\mathfrak{p}}$ denote the corresponding completion of F. These fields $\widehat{F}_{\mathfrak{p}}$ are the local objects, with relatively easy arithmetic properties. Local global principles allow us to understand properties of F in terms of those over all the $\widehat{F}_{\mathfrak{p}}$.

Let K be an algebraic function field in one variable over a field F. By a *place* of K/F, we mean a normalized discrete valuation on K which is trivial on $F^* = F - \{0\}$. Define

$$\mathbb{P}(K/F) = \{P \mid P \text{ is a place of } K/F\}.$$

We denote by $\widehat{K}_P$ the completion of K with respect to $P \in \mathbb{P}(K/F)$. Another type of local global principle is to get information about K via the family of $\widehat{K}_P$ for $P \in \mathbb{P}(K/F)$. We will be particularly interested in the case where K has genus 0. Then K has the form $K = F(x, \sqrt{ax^2+b})$ where $a, b \in F^*$ and x is transcendental over F. Since this K is determined up to isomorphism by the quaternion algebra $Q = (a, b/F)$, we will write $F(Q)$ for K.

If $\mathrm{Br}(k)$ denotes the Brauer group of a field k, then the classical Hasse principle for the Brauer group of a global field F states that the map

$$\mathrm{Br}(F) \longrightarrow \prod_{\mathfrak{p} \in P(F)} \mathrm{Br}(\widehat{F}_{\mathfrak{p}})$$

is injective. This is a local global principle, since it says that a central simple algebra A over F is determined by its extensions $A \otimes_F \widehat{F}_{\mathfrak{p}}$ over $\widehat{F}_{\mathfrak{p}}$ as $\mathfrak{p}$ ranges over $P(F)$.

One could ask whether there is a corresponding Hasse principle using the $K \otimes_F \widehat{F}_{\mathfrak{p}}$, $\mathfrak{p} \in P(F)$, for the Brauer group of an algebraic function field K over a global field F. However, such a Hasse principle no longer holds. The first counterexample was given by Witt (see [**Wi**, p. 466]) in 1934, taken from an algebraic function field K of genus 0 over $\mathbb{Q}$. He showed that if Q and Q' are quaternion algebras over $\mathbb{Q}$, which are nonsplit just at $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4 \in P(\mathbb{Q})$ and $\mathfrak{p}_1, \mathfrak{p}_2$ respectively, then $Q' \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}}_{\mathfrak{p}}(Q)$ is split for every $\mathfrak{p} \in P(\mathbb{Q})$ although $Q' \otimes_{\mathbb{Q}} \mathbb{Q}(Q)$ is nonsplit. Further, he pointed out that $Q' \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}}_P$ is split for every $P \in \mathbb{P}(\mathbb{Q}(Q)/\mathbb{Q})$.

Let $\mathcal{C}$ be an irreducible nonsingular projective curve over a global field F. Considering the function field $K = F(\mathcal{C})$ of the curve $\mathcal{C}$ over F, one can ask about two kinds of possible Hasse principles for the Brauer group of K. One corresponds to the map h_1 and the other corresponds to the map h_2 below:

$$h_1 : \mathrm{Br}(K)' \longrightarrow \prod_{\mathfrak{p} \in P(F)} \mathrm{Br}\big(\widehat{F}_{\mathfrak{p}}(\mathcal{C})\big)$$

$$h_2 : \mathrm{Br}(K) \longrightarrow \prod_{P \in \mathbb{P}(K/F)} \mathrm{Br}(\widehat{K}_P).$$

Here, $\mathrm{Br}(K)'$ denotes the subgroup of $\mathrm{Br}(K)$ consisting of those $[B]$ with the exponent of B relatively prime to $\mathrm{char}(F)$. So, if $\mathrm{char}(F) = 0$, then $\mathrm{Br}(K)' = \mathrm{Br}(K)$.

In this paper, we consider the case of K having genus 0, putting $K = F(Q)$, and give explicit description of the kernels of the maps h_1 and h_2 above, which in fact coincide as shown in Theorem 4.4 and Theorem 4.5.

Nontrivial kernels of these maps could be called the obstruction to a Hasse principle.

In order to facilitate the determination of these kernels, we compute, in Theorem 3.7, the kernel of the map

$$h : \mathrm{Br}(K) \longrightarrow \prod_{P \in \mathbb{P}(K/F)} \mathrm{Br}(K\overline{V}_P)$$

where $\overline{V}_P$ is the residue field with respect to $P \in \mathbb{P}(K/F)$. The kernel of h turns out to coincide with those of h_1 and h_2, and it has a direct description in terms of the quaternion algebra Q.

Finally, we give a relative version of the facts shown above. In Theorem 5.8, we completely describe the 2-torsion subgroup of the kernel of h_1 when a finite number of prime spots are omitted from $P(F)$. In the case that the prime spots deleted from $P(F)$ contain those $\mathfrak{p} \in P(F)$ such that $Q \otimes_F \widehat{F}_{\mathfrak{p}}$ is nonsplit, this part of $\ker(h_1)$ can be expressed in terms of quaternion algebras over K together with the images in $\mathrm{Br}(K)$ of cyclic algebras of exponent 4 over F. Otherwise, we can describe the kernel as the intersection of the relative Brauer groups of some quadratic residue fields over K so that it can be all expressed in terms of quaternion algebras over K.

Parimala and Sujatha considered in [**PS**] the kernels of similar maps for the function field of a curve with a rational point. These results do not apply for nonrational function fields of genus 0 since these are function fields of anisotropic conics, which have no rational points. On the other hand, they pointed out that the kernel of h_1 is nontrivial for the function field of genus 0 associated with a quaternion algebra which is locally split at 4 prime spots or more.

Kato considered in [**Ka**, Theorem 0.8(2)] a Hasse principle for H^3 cohomology groups analogous to h_1 for H^2 cohomology groups; he showed that the corresponding map from $H^3(K, \mathbb{Z}/2\mathbb{Z})$ is injective. However, we will see that the map h_1 is not injective in general, even when restricted to the 2-torsion of $\mathrm{Br}(K)$, ${}_2\mathrm{Br}(K) = H^2(K, \mathbb{Z}/2\mathbb{Z})$.

This work was motivated by and arose in connection with the author's work on tractability of algebraic function fields in one variable of genus 0 over global fields (see Remark 5.10). Especially, Chapter 3 is based on a part of the author's Ph.D. thesis work. We would like to thank Prof. A. Wadsworth at UCSD for providing invaluable guidance and the referee for making helpful comments, which improved the final version of this paper.

2. Preliminaries.

In this section, we will briefly review basic facts on the algebraic function fields in one variable of genus 0, which are associated with quaternion algebras.

Let F be a field with $\mathrm{char}(F) \neq 2$. (Throughout, all fields are assumed to have characteristic not equal to 2.) Let K be an algebraic function field in one variable of genus 0 over F. It is known (cf. [**Ar**, Theorem 6, p. 302]) that such a K has a genus 0 if and only if K is of the form $F(x, \sqrt{ax^2+b})$, where $a, b \in F^* = F - \{0\}$ and x is transcendental over F. Let $(a, b/F)$ denote the 4-dimensional quaternion algebra over F with F-base $1, i, j, k$, such that $i^2 = a$, $j^2 = b$, and $ij = -ji = k$. Then $K = F(x, \sqrt{ax^2+b})$ is determined by a quaternion algebra $Q = (a, b/F)$ since K is isomorphic to the function field of a conic determined by norm form on the pure part of Q. Thus K will be also denoted by $F(Q)$. Recall (cf. [**Wi**, Satz, p. 464 and Satz, p. 465]) that for two quaternion algebras Q and Q', $Q \cong Q'$ as algebras if and only if $F(Q) \cong F(Q')$ as fields.

For each place $P \in \mathbb{P}(K/F)$, let V_P be the associated discrete valuation ring. Recall (e.g., [**DI**, Lemma 2.2, p. 136]) that the restriction map $\mathrm{Br}(V_P) \to \mathrm{Br}(K)$ induced by the inclusion $V_P \hookrightarrow K$ is injective. If $\overline{V}_P$ denotes the residue field of V_P, then $\overline{V}_P$ is a finite degree extension of F. The *degree* of P is defined by $\deg(P) = [\overline{V}_P : F]$. Now, let F_{sep} be the separable closure of F. We will denote by $G_P = \mathcal{G}al(F_{\mathrm{sep}}/\overline{V}_P)$ the absolute Galois group of $\overline{V}_P$ and by $X(G_P) = \mathrm{Hom}_c(G_P, \mathbb{Q}/\mathbb{Z})$ the (continuous) character group of G_P. The following result, due to Scharlau, will be essential to our study of algebraic function fields of genus 0.

Proposition 2.1 (Scharlau)**.** *Let F be any field. For a quaternion algebra $Q = (a, b/F)$, let $K = F(Q) = F(x, \sqrt{ax^2+b})$. Then the following sequence is exact:*

$$(1) \quad 0 \longrightarrow \{\, [F], [Q] \,\} \longrightarrow \mathrm{Br}(F) \xrightarrow{\alpha} \mathrm{Br}(F_{\mathrm{sep}} \cdot K/K) \xrightarrow{\beta} \bigoplus_{P \in \mathbb{P}(K/F)} X(G_P).$$

For details of Proposition 2.1, see [**Sc**, p. 5].

The following well-known lemma will be useful to determine when an algebraic function field of genus 0 is a rational function field.

Lemma 2.2. *Let F be any field. For a quaternion algebra Q over F, let $K = F(Q)$. Then the following conditions are equivalent:*

(i) *Q is split over F.*
(ii) *There exists $P \in \mathbb{P}(K/F)$ with* $\deg(P) = 1$.
(iii) *K is purely transcendental over F.*

A proof can be found, e.g., in [**Wa**, p. 747] or in [**Ha**, Lemma 3.2].

Note that if the quaternion algebra Q is split over F, then Proposition 2.1 reduces to the Auslander-Brumer-Faddeev Theorem (cf. [**AB**], [**Fa**] or [**FS**]) in view of Lemma 2.2. This lemma provides a corollary (see [**Ha**, Corollary 3.3] for proof):

Corollary 2.3. *Let F be any field. For a quaternion algebra Q over F, let $K = F(Q)$. For any field $E \supseteq F$ with $[E : F] < \infty$, E splits Q if and only if $E \supseteq \overline{V}_P$ for some $P \in \mathbb{P}(K/F)$.*

Let F be a global field, that is, F is either an algebraic number field (i.e., a finite extension of $\mathbb{Q}$) or an algebraic function field in one variable over a finite field. If Q is a quaternion algebra over F, we define the *support* of Q as follows:

$$\operatorname{supp}(Q) = \{\mathfrak{p} \in P(F) \mid Q \otimes_F \widehat{F}_{\mathfrak{p}} \text{ is nonsplit}\}.$$

Recall that if $\mathfrak{p} \in P(F)$ is an archimedean prime spot, then the field $\widehat{F}_{\mathfrak{p}}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. If $\mathfrak{p}$ is non-archimedean, then $\widehat{F}_{\mathfrak{p}}$ is a *local field*, i.e., a field with complete discrete valuation and with finite residue field.

Lemma 2.4 below can be easily shown (or see [**Ha**, Remark 4.4]).

Lemma 2.4. *Let Q and Q' be quaternion algebras over a global field F, and let E be a finite degree field extension of F. If $\operatorname{supp}(Q') \subseteq \operatorname{supp}(Q)$, then $\operatorname{supp}(Q' \otimes_F E) \subseteq \operatorname{supp}(Q \otimes_F E)$.*

The next two theorems are well-known and are of fundamental importance.

Hasse Principle (special case) 2.5. For a global field F, let Q and Q' be quaternion algebras over F. Then $Q \cong Q'$ if and only if $Q \otimes_F \widehat{F}_{\mathfrak{p}} \cong Q' \otimes_F \widehat{F}_{\mathfrak{p}}$ for all $\mathfrak{p} \in P(F)$. In particular, Q is split if and only if $\operatorname{supp}(Q)$ is the empty set.

Hilbert's Reciprocity Law 2.6. Let F be a global field. For a quaternion algebra Q over F, the set $\operatorname{supp}(Q)$ is finite with even cardinality. Further, given any finite subset $\mathcal{S}$ of $P(F)$ with $|\mathcal{S}|$ even, there is a unique quaternion algebra Q over F with $\operatorname{supp}(Q) = \mathcal{S}$.

If Q is split over a global field F, all the assertions we will make are vacuously true. Thus we will exclude this trivial case.

Proposition 2.7. *Let F be any field. Suppose that Q is a quaternion division algebra over F. Let $K = F(Q)$. For $r \in F^* - F^{*2}$, the following are equivalent:*

(i) *$F(\sqrt{r})$ splits Q.*
(ii) *$F(\sqrt{r})$ is the residue field of a place of K/F.*

Further, if F is a global field and $\operatorname{supp}(Q) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$, then (i) and (ii) are also equivalent to:

(iii) *$r \notin \widehat{F}_{\mathfrak{p}_1}^{*2} \cup \widehat{F}_{\mathfrak{p}_2}^{*2} \cup \cdots \cup \widehat{F}_{\mathfrak{p}_n}^{*2}$.*

Proof.

(i) $\Rightarrow$ (ii) Since $Q \otimes_F F(\sqrt{r})$ is split over $F(\sqrt{r})$, $K(\sqrt{r})$ is a rational function field over $F(\sqrt{r})$ by Lemma 2.2. Thus we can take a place P' of

$K(\sqrt{r})/F(\sqrt{r})$ of $\deg(P')=1$ and a place P of K/F with $P'|P$. For the respective residue fields of P' and P , we have $F(\sqrt{r}) = \overline{V}_{P'} \supseteq \overline{V}_P \supseteq F$. However, since Q is not split over F, there exists no place P of $\deg(P)=1$ by Lemma 2.2 again and so $\overline{V}_P \neq F$. Since $[F(\sqrt{r}):F]=2$, we have $\overline{V}_P = F(\sqrt{r})$.

(ii) $\Rightarrow$ (i) This is immediate as a consequence of Corollary 2.3.

(i) $\Rightarrow$ (iii) For each $\mathfrak{p} \in \operatorname{supp}(Q)$, $Q \otimes_F \widehat{F}_{\mathfrak{p}}$ is nonsplit. However, $Q \otimes_F \widehat{F}_{\mathfrak{p}}(\sqrt{r})$ is split since $Q \otimes_F F(\sqrt{r})$ is split. Hence $\widehat{F}_{\mathfrak{p}}(\sqrt{r}) \neq \widehat{F}_{\mathfrak{p}}$. In other words, $r \notin \widehat{F}^{*2}_{\mathfrak{p}_1} \cup \widehat{F}^{*2}_{\mathfrak{p}_2} \cup \cdots \cup \widehat{F}^{*2}_{\mathfrak{p}_n}$.

(iii) $\Rightarrow$ (i) By the Hasse Principle 2.5, it suffices to show that $Q \otimes_F \widehat{F(\sqrt{r})}_{\mathfrak{P}}$ is split for every prime spot $\mathfrak{P}$ of $F(\sqrt{r})$. To see this, assume that $\mathfrak{p}$ is the restriction of $\mathfrak{P}$ to F. We have two possibilities for this $\mathfrak{p}$: If $\mathfrak{p} \notin \operatorname{supp}(Q)$, then $Q \otimes_F \widehat{F(\sqrt{r})}_{\mathfrak{P}}$ is clearly split since $Q \otimes_F \widehat{F}_{\mathfrak{p}}$ is already split. On the other hand, if $\mathfrak{p} \in \operatorname{supp}(Q)$, then $Q \otimes_F \widehat{F}_{\mathfrak{p}}$ is nonsplit. However, by (iii) $\widehat{F}_{\mathfrak{p}}(\sqrt{r}) = \widehat{F(\sqrt{r})}_{\mathfrak{P}}$ is a quadratic extension of $\widehat{F}_{\mathfrak{p}}$. Now, recall the well-known results that any local field L or $L=\mathbb{R}$ has a unique nonsplit quaternion algebra, which is split by each quadratic extension of L. (For this, use e.g., [**Re**, Theorems 31.4, 31.8, and 31.9] and the facts that $\operatorname{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ and $\operatorname{Br}(\mathbb{C}) \cong 0$.) Thus, $Q \otimes_F \widehat{F}_{\mathfrak{p}}(\sqrt{r})$ must be split. Hence, $Q \otimes_F \widehat{F}_{\mathfrak{p}}(\sqrt{r}) = Q \otimes_F \widehat{F(\sqrt{r})}_{\mathfrak{P}}$ is split for every prime spot $\mathfrak{P}$ of $F(\sqrt{r})$. □

3. Computation of $\bigcap_{P \in \mathbb{P}(K/F)} \operatorname{Br}(K\overline{V}_P/K)$.

Let F be a global field and let $Q = (a, b/F)$ be a quaternion division algebra over F with $\operatorname{supp}(Q) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$. Recall that n is finite and even by Hilbert's Reciprocity Law 2.6. Let $K = F(Q) = F(x, \sqrt{ax^2+b})$. Consider the map

$$h : \operatorname{Br}(K) \longrightarrow \prod_{P \in \mathbb{P}(K/F)} \operatorname{Br}(K\overline{V}_P). \tag{2}$$

The purpose of this section is to explicitly compute

$$\ker(h) = \bigcap_{P \in \mathbb{P}(K/F)} \operatorname{Br}(K\overline{V}_P/K)$$

(see Theorem 3.7) which will be used to prove our main theorems in Chapter 4. For this, we will first compute $\bigcap_{P \in \mathbb{P}(K/F)} \operatorname{Br}(\overline{V}_P/F)$ by utilizing the local-global principle. This intersection can be directly described, in Proposition 3.2, in terms of the quaternion algebra Q.

To begin with, we want to give explicit calculation of $\bigcap_{\deg(P)=2} \mathrm{Br}(\overline{V}_P/F)$ over all the places $P \in \mathbb{P}(K/F)$ with $\deg(P)=2$. For $\mathrm{supp}(Q) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$, we define

$$\mathcal{F}_Q = \{r \in F \mid r \notin \widehat{F}^{*2}_{\mathfrak{p}_1} \cup \widehat{F}^{*2}_{\mathfrak{p}_2} \cup \cdots \cup \widehat{F}^{*2}_{\mathfrak{p}_n}\}. \tag{3}$$

Proposition 2.7 yields another description of $\mathcal{F}_Q$:

$$\mathcal{F}_Q = \{r \in F^* - F^{*2} \mid F(\sqrt{r}) \text{ is the residue field of a place of } K/F\}.$$

Therefore, we have

$$\bigcap_{\deg(P)=2} \mathrm{Br}(\overline{V}_P/F) = \bigcap_{r \in \mathcal{F}_Q} \mathrm{Br}(F(\sqrt{r})/F). \tag{4}$$

We will use a number of times below the following well-known combinatorial fact:

Lemma 3.1. *Let $\mathcal{T}$ be a set with $|\mathcal{T}| = n$ where $n \in \mathbb{N}$. Then the number of subsets with an even number of elements of $\mathcal{T}$ is 2^{n-1}.*

We next define

$$\mathcal{I}_Q = \left\{ [Q'] \;\middle|\; \begin{array}{l} Q' \text{ is a quaternion algebra over } F \\ \text{with } \mathrm{supp}(Q') \subseteq \mathrm{supp}(Q) \end{array} \right\} \subseteq \mathrm{Br}(F). \tag{5}$$

Proposition 3.2. *Let F be a global field. Suppose that Q is a quaternion division algebra over F. Let $K = F(Q)$. For the $\mathcal{F}_Q$ in (3) and $\mathcal{I}_Q$ in (5), we have*

$$\bigcap_{P \in \mathbb{P}(K/F)} \mathrm{Br}(\overline{V}_P/F) = \bigcap_{\deg(P)=2} \mathrm{Br}(\overline{V}_P/F) = \bigcap_{r \in \mathcal{F}_Q} \mathrm{Br}(F(\sqrt{r})/F) = \mathcal{I}_Q. \tag{6}$$

The cardinality of this set is 2^{n-1} where $n = \big|\mathrm{supp}(Q)\big|$.

Proof. We will prove the last equality in (6) first. Any element in the relative Brauer group of $F(\sqrt{r})/F$ is the class of a quaternion algebra over F (cf. [**Dr**, Corollary 1, p. 79]). Thus, any element in the intersection is the class of a quaternion algebra over F. If Q' is a nonsplit quaternion algebra such that $\mathrm{supp}(Q') \subseteq \mathrm{supp}(Q)$, then $\mathcal{F}_Q \subseteq \mathcal{F}_{Q'}$. Thus, for each $r \in \mathcal{F}_Q$, $F(\sqrt{r})$ splits Q' by Proposition 2.7 applied to Q'. Hence, $[Q'] \in \bigcap_{r \in \mathcal{F}_Q} \mathrm{Br}(F(\sqrt{r})/F)$. On the other hand, if $\mathrm{supp}(Q') \nsubseteq \mathrm{supp}(Q)$, we can take $\mathfrak{p} \in \mathrm{supp}(Q') - \mathrm{supp}(Q)$. Assume that $\mathrm{supp}(Q) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$. By the Weak Approximation Theorem (cf. [**Ws**, p. 8]), there exists $r \in F$ such that

$$r \in \widehat{F}^{*2}_{\mathfrak{p}} \text{ but } r \notin \widehat{F}^{*2}_{\mathfrak{p}_1} \cup \widehat{F}^{*2}_{\mathfrak{p}_2} \cup \cdots \cup \widehat{F}^{*2}_{\mathfrak{p}_n}$$

(for this, recall that if two elements are $\mathfrak{q}$-adically close enough, then they lie in the same $\mathfrak{q}$-adic square class). In other words, $r \in \mathcal{F}_Q$ but $r \notin \mathcal{F}_{Q'}$. Thus,

$[Q'] \notin \mathrm{Br}(F(\sqrt{r})/F)$ which clearly implies that $[Q'] \notin \bigcap_{r\in\mathcal{F}_Q} \mathrm{Br}(F(\sqrt{r})/F)$.
This assertion gives the last equality in (6).

The middle equality in (6) was already given in (4). Now, we show that

$$\bigcap_{P\in\mathbb{P}(K/F)} \mathrm{Br}(\overline{V}_P/F) = \bigcap_{r\in\mathcal{F}_Q} \mathrm{Br}(F(\sqrt{r})/F).$$

One inclusion ($\subseteq$) is clear from (4). For the other inclusion ($\supseteq$), take $[Q'] \in \bigcap_{r\in\mathcal{F}_Q} Br(F(\sqrt{r})/F)$. Then, we may assume that Q' is a quaternion algebra over F with $\mathrm{supp}(Q') \subseteq \mathrm{supp}(Q)$ by the last equality of (6). If we put $\overline{V} = \overline{V}_P$ for $P \in \mathbb{P}(K/F)$, then $[\overline{V} : F] < \infty$ and $\overline{V}$ splits Q (as seen in the proof (ii) $\Rightarrow$ (i) of Proposition 2.7). By Lemma 2.4,

$$\mathrm{supp}(Q' \otimes_F \overline{V}) \subseteq \mathrm{supp}(Q \otimes_F \overline{V}) = \varnothing.$$

Hence, $\overline{V}$ splits Q'.

Finally, Hilbert's Reciprocity Law 2.6 implies that the number of elements in $\mathcal{I}_Q$ equals the number of subsets with an even number of elements of a set with n elements. By Lemma 3.1, the cardinality of this set is 2^{n-1}. □

Example 3.3. Let $Q = (-1, -1/\mathbb{Q})$ be a quaternion algebra over $\mathbb{Q}$ and let $K = \mathbb{Q}(Q) = \mathbb{Q}(x, \sqrt{-x^2-1})$. It is easy to see that $\mathrm{supp}(Q) = \{2, \infty\}$ where 2 is the dyadic spot and ∞ is the real infinite spot of $\mathbb{Q}$. Thus, we have

$$\bigcap_{P\in\mathbb{P}(K/F)} \mathrm{Br}(\overline{V}_P/F) = \{0, [Q]\}.$$

In this example, we can explicitly describe all the quadratic residue fields (see (iii) and (vi) below). For $a_1, \cdots, a_n \in F^*$, let $\langle a_1, \cdots, a_n\rangle$ be the diagonal quadratic form $a_1x_1^2 + \cdots + a_nx_n^2$. Assume that r is a square-free integer. Then the following are equivalent:

(i) $\mathbb{Q}(\sqrt{r})$ is a residue field of a place of $K/\mathbb{Q}$.
(ii) $r \in \mathcal{F}_Q$, i.e., $r \notin \widehat{\mathbb{Q}}_2^2 \cup \mathbb{R}^2$.
(iii) $r < 0$ and $r \not\equiv 1 \pmod 8$.
(iv) $r < 0$ and the quadratic form $\langle 1, 1, 1, r\rangle$ is isotropic over $\widehat{\mathbb{Q}}_2$.
(v) $r < 0$ and the quadratic form $\langle 1, 1, 1, r\rangle$ is isotropic over $\mathbb{Q}$.
(vi) $r < 0$ and $-r$ is the sum of three squares in $\mathbb{Q}$.

Indeed, (i) $\Leftrightarrow$ (ii) is given by Proposition 2.7 since $\mathrm{supp}(Q) = \{2, \infty\}$. For (ii) $\Leftrightarrow$ (iii), clearly $r \in \mathbb{R}^2 \Leftrightarrow r > 0$. The fact that $r \in \widehat{\mathbb{Q}}_2^2 \Leftrightarrow r \equiv 1 \pmod 8$ is well-known (see [**La**, Corollary 2.24, p. 162]). For (ii) $\Leftrightarrow$ (iv), recall (cf. [**OM**, 63:17, p. 169]) that over a local field, a 4-dimensional anisotropic quadratic form has determinant 1 modulo squares. Because $\mathrm{supp}(Q) = \{2, \infty\}$, the quadratic form $\langle 1, 1, 1, 1\rangle$ is anisotropic over $\widehat{\mathbb{Q}}_2$ as it is the norm form of $(-1, -1/\widehat{\mathbb{Q}}_2)$. To show (iv) $\Leftrightarrow$ (v), $\langle 1, 1, 1, r\rangle$ is certainly isotropic over $\mathbb{R}$ and for any prime spot p corresponding to an odd prime p, observe that $\langle 1,1,1\rangle$ is already isotropic over $\widehat{\mathbb{Q}}_p$ and so is $\langle 1,1,1,r\rangle$. Then, this equivalence is an immediate consequence of the Hasse-Minkowski Theorem (cf. [**La**, p. 168]). (v) $\Leftrightarrow$ (vi) is obvious.

For two central simple algebras A and B over F, we will write $A \sim B$ if $[A] = [B]$ in $\mathrm{Br}(F)$. The following examples will be used later in the sequel.

Example 3.4. Let $n > 0$ be an odd integer. Suppose that $p_1, p_2, \dots, p_n$ are distinct odd prime numbers such that each $p_i \equiv 3 \pmod 4$. Let $Q = (-1, m/\mathbb{Q})$ where $m = p_1 p_2 \dots p_n$ and let $K = \mathbb{Q}(Q) = \mathbb{Q}(x, \sqrt{-x^2+m})$. Then

$$\mathrm{supp}(Q) = \{\, 2, p_1, \dots, p_n \,\},$$

where 2 is the dyadic spot and each p_i denotes the prime spot corresponding to the odd prime p_i. To see this, let $P_i = (-1, p_i/\mathbb{Q})$. For a fixed i, we observe that $\mathbb{R}$ splits P_i since $p_i > 0$ and that $\widehat{\mathbb{Q}}_p$ splits P_i for p any odd prime different from p_i since -1 and p_i are both p-adic units. However, $P_i \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}}_{p_i}$ is not split since -1 is not a square in $\mathbb{Z}/p_i\mathbb{Z}$ as $p_i \equiv 3 \pmod 4$. By Hilbert's Reciprocity Law 2.6, we have $\mathrm{supp}(P_i) = \{2, p_i\}$. Since

$$Q \sim P_1 \otimes_{\mathbb{Q}} P_2 \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} P_n$$

and n is odd, it follows that $\mathrm{supp}(Q) = \{2, p_1, \dots, p_n\}$ as claimed. Then, Proposition 3.2 yields

$$\bigcap_{P \in \mathbb{P}(K/\mathbb{Q})} \mathrm{Br}(\overline{V}_P/\mathbb{Q}) = \{\, 0,\ [P_{i_1}],\ [P_{i_1} \otimes_{\mathbb{Q}} P_{i_2}],\ \dots,\ [P_{i_1} \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} P_{i_{n-1}}],\ [Q] \,\},$$

where the i_j range over all distinct numbers in $\{1, 2, \dots, n\}$, and the cardinality of this set is $2^{|\mathrm{supp}(Q)|-1} = 2^n$. In particular, if $n = 1$ (so $m = p_1$), then

$$\bigcap_{P \in \mathbb{P}(K/\mathbb{Q})} \mathrm{Br}(\overline{V}_P/\mathbb{Q}) = \{0, [Q]\}.$$

Example 3.5. Let $n > 0$ be an even integer. Suppose that $p_1, p_2, \dots, p_n$ are as in Example 3.4. Let $Q = (-1, m/\mathbb{Q})$ where $m = p_1 p_2 \dots p_n$ and let $K = \mathbb{Q}(Q) = \mathbb{Q}(x, \sqrt{-x^2+m})$. If we let $P_i = (-1, p_i/\mathbb{Q})$, then we have $Q \sim P_1 \otimes_{\mathbb{Q}} P_2 \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} P_n$. Applying the similar arguments in Example 3.4, we have

$$\mathrm{supp}(Q) = \{p_1, p_2, \dots, p_n\}$$

since n is even. Then, Proposition 3.2 yields

$$\bigcap_{P \in \mathbb{P}(K/\mathbb{Q})} \mathrm{Br}(\overline{V}_P/\mathbb{Q}) = \big\{0,\ [P_{i_1} \otimes_{\mathbb{Q}} P_{i_2}],\ [P_{i_1} \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} P_{i_4}],\ \dots,$$
$$[P_{i_1} \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} P_{i_{n-2}}],\ [Q]\big\},$$

where the i_j range over all distinct numbers in $\{1, 2, \ldots, n\}$, and the cardinality of this set is 2^{n-1}. In particular, if $n = 2$ (so $m = p_1 p_2$), then

$$\bigcap_{P \in \mathbb{P}(K/\mathbb{Q})} \mathrm{Br}(\overline{V}_P/\mathbb{Q}) = \{0, [Q]\}.$$

Let $Q = (a, b/F)$ and $K = F(Q) = F(x, \sqrt{ax^2 + b})$. Fix $r \in \mathcal{F}_Q$. For $P \in \mathbb{P}(K/F)$ and $P' \in \mathbb{P}\big(K(\sqrt{r})/F(\sqrt{r})\big)$, denote by G_P and $G_{P'}$ the absolute Galois groups of the residue fields $\overline{V}_P$ and $\overline{V}_{P'}$, respectively. Note that P' depends on $r \in \mathcal{F}_Q$. If P' is a place above P, note that $G_{P'}$ is a subgroup of G_P and $\big|\, G_P \colon G_{P'} \big| = \big[\, \overline{V}_{P'} \colon \overline{V}_P \big] = 1$ or 2.

Lemma 3.6. *Keeping the notation as above, let $\chi_{_P} \in X(G_P)$, the character group of G_P. Suppose that $\chi_{_P}\big|_{G_{P'}} = 0$ for all $P' \in \mathbb{P}\big(K(\sqrt{r})/F(\sqrt{r})\big)$ with $P'\big|P$ for all $r \in \mathcal{F}_Q$. Then $\chi_{_P} = 0$.*

Proof. Consider the map $\varphi : F^*\big/F^{*2} \to \overline{V}_P^*\big/\overline{V}_P^{*2}$. By Kummer theory, we have

$$\big|\ker(\varphi)\big| = \big[\, F\big(\{\, \sqrt{a} \,\big|\, aF^{*2} \in \ker(\varphi)\,\}\big) : F \,\big] \leq \big[\, \overline{V}_P : F \,\big] < \infty.$$

Now, we show that $\mathcal{F}_Q F^{*2}\big/F^{*2}$ is infinite. To see this, let

$$\mathcal{A} = \{\mathfrak{p} \in P(F) \;\big|\; \mathfrak{p} \;\text{ is finite and }\; \mathfrak{p} \notin \mathrm{supp}(Q)\}$$

and let $\mathcal{B}$ be any finite subset of $\mathcal{A}$. Note that for any $\mathfrak{p} \in \mathcal{B}$, by the Weak Approximation Theorem (cf. [**Ws**, p. 8]), there is $r \in \mathcal{F}_Q$ with $v_{\mathfrak{p}}(r)$ odd. From the surjective map

$$\psi : F^*/F^{*2} \to \bigoplus_{\mathfrak{p} \in \mathcal{B}} \mathbb{Z}/2\mathbb{Z},$$

given by $aF^{*2} \mapsto (\cdots, v_{\mathfrak{p}}(a) + 2\mathbb{Z}, \cdots)$, we have $\psi(\mathcal{F}_Q F^{*2}/F^{*2}) \geq 2^{|\mathcal{B}|}$. Since this is true for any finite subset $\mathcal{B}$ of $\mathcal{A}$, it follows that $\psi(\mathcal{F}_Q F^{*2}/F^{*2})$ and thus $\mathcal{F}_Q F^{*2}\big/F^{*2}$ is infinite. This implies that $\varphi(\mathcal{F}_Q F^{*2}\big/F^{*2})$ is infinite, since $\ker(\varphi) < \infty$. Thus, it is possible to take $r_1, r_2 \in \mathcal{F}_Q$ such that $\overline{V}_P(\sqrt{r_1}) \neq \overline{V}_P(\sqrt{r_2})$. For $i = 1$ or 2, suppose that $P_i \in \mathbb{P}\big(K(\sqrt{r_i})/F(\sqrt{r_i})\big)$ with $\deg(P_i) = 1$. If $r_i \in \overline{V}_P$ for some i, then $G_{P_i} = G_P$ and so $\chi_{_P} = \chi_{_P}\big|_{G_{P_i}} = 0$, as desired. Thus, we assume that $r_1, r_2 \notin \overline{V}_P$. Clearly, $\overline{V}_{P_i} = \overline{V}_P(\sqrt{r_i})$. If we let $G_{P_i} = \mathcal{G}al(F_{\mathrm{sep}}/\,\overline{V}_{P_i})$, then

$$\big|\, G_P : G_{P_1} \big| = \big|\, G_P : G_{P_2} \big| = 2 \;\text{ and }\; G_{P_1} \neq G_{P_2}.$$

Since $G_P = G_{P_1} \cdot G_{P_2}$, we have $\chi_{_P} = \chi_{_P}\big|_{G_{P_1} \cdot G_{P_2}} = 0$. □

Before we discuss the main theorem of this section, it will be convenient to define some new terminology. Let K be an algebraic function field in one variable (of genus 0) over a constant field F. For the scalar extension map $\alpha : \mathrm{Br}(F) \to \mathrm{Br}(K)$, a class $[B] \in \mathrm{Br}(K)$ is called a *constant class* if $[B] \in \mathrm{im}(\alpha)$.

For the map h in (2) above, we compute the kernel of h. The elements in $\ker(h)$ turn out to be all constant classes of quaternion algebras over F. As in Proposition 3.2, it suffices to take the intersection over all the quadratic residue fields.

Theorem 3.7. *Let F be a global field. Suppose that Q is a quaternion division algebra over F. Let $K = F(Q)$. For the map h in (2), the $\mathcal{F}_Q$ in (3), and $\mathcal{I}_Q$ in (5), we have*

$$\begin{aligned}(7)\qquad \ker(h) &= \bigcap_{P\in\mathbb{P}(K/F)} \mathrm{Br}(K\overline{V}_P/K) = \bigcap_{\deg(P)=2} \mathrm{Br}(K\overline{V}_P/K)\\ &= \bigcap_{r\in\mathcal{F}_Q} \mathrm{Br}(K(\sqrt{r})/K) = \big\{[Q'\otimes_F K]\,\big|\,[Q']\in\mathcal{I}_Q\big\}.\end{aligned}$$

The cardinality of this set is 2^{n-2} where $n = \big|\mathrm{supp}(Q)\big|$.

Proof. Let us show the last equality of (7) first. For each $r \in \mathcal{F}_Q$, we have the following diagram with exact rows:

(8)

$$\begin{array}{ccccccccc}
0 \longrightarrow \mathbb{Z}_2 \longrightarrow & \mathrm{Br}(F) & \xrightarrow{\alpha} & \mathrm{Br}(F_{\mathrm{sep}}\cdot K/K) & \xrightarrow{\beta} & \bigoplus_{P\in\mathbb{P}(K/F)} X(G_P) \\
& \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \oplus e_P\cdot\mathrm{res}} \\
0 \longrightarrow & \mathrm{Br}(F(\sqrt{r})) & \xrightarrow{\gamma} & \mathrm{Br}(F_{\mathrm{sep}}\cdot K(\sqrt{r})/K(\sqrt{r})) & \xrightarrow{\delta} & \bigoplus_{P'|P} X(G_{P'}).
\end{array}$$

Here the top sequence is the sequence (1) and the bottom sequence comes also from (1) since $F(\sqrt{r})$ splits Q (or, from the Auslander-Brumer-Fadeev Theorem (cf. [**AB**], [**Fa**] or [**FS**])). Each component $e_P\cdot\mathrm{res}$ in the right vertical map in (8) is the ramification index $e_P = e(v_{P'}/v_P)$ times the natural restriction map from $X(G_P)$ to $X(G_{P'})$. The right square in (8) is commutative (see [**Sa**, Theorem 10.4]). The left square is clearly commutative, since all the maps are restriction maps. Note that for each $P \in \mathbb{P}(K/F)$, $e_P = 1$, since $r \in F$ and so $v_P(r) = 0$ (cf. [**St**, Theorem III.6.3, p. 103]).

Let $[B] \in \bigcap_{r\in\mathcal{F}_Q} \mathrm{Br}(K(\sqrt{r})/K)$. Since $\exp(B) \le 2$ and $\mathrm{char}(F) \ne 2$, $[B]$ is split by $F_{\mathrm{sep}}\cdot K$. Obviously, for any $r \in \mathcal{F}_Q$, $\beta([B]) \in \ker(\oplus\, e_P\cdot\mathrm{res})$ from the commutativity of the right square. Put

$$\beta([B]) = \sum_{\text{finite}} \chi_P.$$

Applying Lemma 3.6 to each χ_P, we have $\beta([B]) = 0$ and thus $[B] \in \mathrm{im}(\alpha)$ by the exactness at $\mathrm{Br}(F_{\mathrm{sep}}\cdot K/K)$. Suppose that $[A]$ is a preimage of $[B]$ in $\mathrm{Br}(F)$. Since γ is injective, we have $[A] \in \mathrm{Br}(F(\sqrt{r})/F)$ for each $r \in \mathcal{F}_Q$ by the commutativity of the left square in (8). Thus $[B]$ is of the form $[Q' \otimes_F K]$ where

$$[Q'] \in \bigcap_{r \in \mathcal{F}_Q} \mathrm{Br}(F(\sqrt{r})/F) = \mathcal{I}_Q$$

by Proposition 3.2. Conversely, for each $[Q'] \in \mathcal{I}_Q$ we clearly have $[Q' \otimes_F K] \in \bigcap_{r \in \mathcal{F}_Q} \mathrm{Br}(K(\sqrt{r})/K)$.

Moreover, by Proposition 3.2, there are 2^{n-1} elements in $\mathcal{I}_Q$. Since $\mathrm{Br}(K/F) = \{0, [Q]\}$ from Proposition 2.1, two different elements $[Q']$ and $[Q' \otimes_F Q]$ in $\mathrm{Br}(F)$ have the same image $[Q' \otimes_F K]$ in $\mathrm{Br}(F_{\mathrm{sep}} \cdot K/K)$. This observation shows that the set in (7) contains 2^{n-2} elements of the form $[Q' \otimes_F K]$.

The first equality of (7) is clear from the definition of the map h, and the third one is also clear by Proposition 2.7.

Finally, we verify the second equality of (7). One inclusion ($\subseteq$) is clear. In order to show the other inclusion ($\supseteq$), we use the fact just proved that

$$\bigcap_{\deg(P)=2} \mathrm{Br}(K\overline{V}_P/K) = \big\{[Q' \otimes_F K] \big| \, [Q'] \in \mathcal{I}_Q\big\}.$$

For each $[Q'] \in \mathcal{I}_Q$ and $P \in \mathbb{P}(K/F)$, we apply Proposition 3.2 to have $[Q'] \in \mathrm{Br}(\overline{V}_P/F)$ and therefore $[Q' \otimes_F K] \in \mathrm{Br}(K\overline{V}_P/K)$. This completes the proof. □

Example 3.8. For an odd integer $n > 0$, let $p_1, p_2, \ldots, p_n$ be distinct odd prime numbers with each $p_i \equiv 3 \pmod 4$. Let $Q = (-1, m/\mathbb{Q})$ where $m = p_1 p_2 \ldots p_n$ and let $K = \mathbb{Q}(Q) = \mathbb{Q}(x, \sqrt{-x^2 + m})$ as in Example 3.4. Recall that $\mathrm{supp}(Q) = \{2, p_1, \ldots, p_n\}$. For each $P_i = (-1, p_i/\mathbb{Q})$, let

$$B_i = P_i \otimes_{\mathbb{Q}} K \cong (-1, p_i/K).$$

If we set $k = \frac{n-1}{2} \in \mathbb{Z}$, then Theorem 3.7 and Example 3.4 yield

$$\bigcap_{P \in \mathbb{P}(K/\mathbb{Q})} \mathrm{Br}(K\overline{V}_P/K) = \big\{0,\ [B_{i_1}],\ [B_{i_1} \otimes_K B_{i_2}],\ \ldots,\ [B_{i_1} \otimes_K \cdots \otimes_K B_{i_k}]\big\},$$

where the i_j range over all distinct numbers in $\{1, 2, \ldots, n\}$, and the cardinality of this set is $2^{\mathrm{supp}(Q)-2} = 2^{n-1}$. In particular, if $n = 1$ (so $m = p_1$), then we have

$$\bigcap_{P \in \mathbb{P}(K/\mathbb{Q})} \mathrm{Br}(K\overline{V}_P/K) = \{0\}.$$

Example 3.9. For an even integer $n > 0$, let $p_1, p_2, \ldots, p_n$ be as in Example 3.5. Let $Q = (-1, m/\mathbb{Q})$ where $m = p_1 p_2 \ldots p_n$ and let $K = \mathbb{Q}(Q) = \mathbb{Q}(x, \sqrt{-x^2 + m})$. Recall that $\mathrm{supp}(Q) = \{p_1, p_2, \ldots, p_n\}$. For each

$P_i = (-1, p_i/\mathbb{Q})$, let $B_i = P_i \otimes_{\mathbb{Q}} K \cong (-1, p_i/K)$. If we set $k = \frac{n}{2} \in \mathbb{Z}$, then Theorem 3.7 and Example 3.5 yield

$$\bigcap_{P \in \mathbb{P}(K/\mathbb{Q})} \mathrm{Br}(K\overline{V}_P/K) = \big\{0,\ [B_{i_1} \otimes_K B_{i_2}],\ [B_{i_1} \otimes_K \cdots \otimes_K B_{i_4}], \ldots, \\ [B_{i_1} \otimes_K \cdots \otimes_K B_{i_k}]\big\},$$

where the i_j range over all distinct numbers in $\{1, 2, \ldots, n\}$, and the cardinality of this set is 2^{n-2}. In particular, if $n = 2$ (so $m = p_1 p_2$), then we have

$$\bigcap_{P \in \mathbb{P}(K/\mathbb{Q})} \mathrm{Br}(K\overline{V}_P/K) = \{0\}.$$

4. Obstruction to Hasse principle for the Brauer group of a function field of genus 0.

The fundamental and profound result on the Brauer group of a global field F provides an exact sequence (cf. [**We**, Theorem 2, p. 206, and Theorem 4, p. 164])

$$(9) \qquad 0 \longrightarrow \mathrm{Br}(F) \xrightarrow{\ i\ } \bigoplus_{\mathfrak{p} \in P(F)} \mathrm{Br}(\widehat{F}_{\mathfrak{p}}) \xrightarrow{\mathrm{inv}} \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

where the map i is the direct sum of scalar extension maps $\mathrm{Br}(F) \to \mathrm{Br}(\widehat{F}_{\mathfrak{p}})$ for $\mathfrak{p} \in P(F)$ and the map inv is the invariant map, computed locally on each component $\mathfrak{p}$, that is, $\mathrm{inv} = \oplus_{\mathfrak{p}} \mathrm{inv}_{\widehat{F}_{\mathfrak{p}}}$. In particular, the injectivity of the natural map i in (9) asserts the classical Hasse principle for $\mathrm{Br}(F)$ (cf. (2.5) for the case of quaternion algebras). In this section, we consider the analogous possible Hasse principles for the Brauer groups of algebraic function fields in one variable of genus 0 over global fields.

We begin with the case of rational function fields. For a field k, let $k(x)$ be the rational function field over k.

Lemma 4.1. *Let F be a global field and F_{sep} the separable closure of F. Then the map*

$$(10) \qquad j : \mathrm{Br}(F_{\mathrm{sep}}(x)/F(x)) \longrightarrow \prod_{\mathfrak{p} \in P(F)} \mathrm{Br}\big(\widehat{F}_{\mathfrak{p}}(x)\big)$$

is injective.

Proof. We have the following commutative diagram with exact rows from the Auslander-Brumer-Faddeev Theorem (cf. [**AB**], [**Fa**] or [**FS**]):

(11)
$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Br}(F) & \longrightarrow & \mathrm{Br}(F_{\mathrm{sep}}(x)/F(x)) & \longrightarrow & \bigoplus_{P\in\mathbb{P}(F(x)/F)} X(G_P) \\
& \big\downarrow i & & \big\downarrow j_0 & & \big\downarrow \psi \\
0 \longrightarrow & \prod_{\mathfrak{p}\in P(F)} \mathrm{Br}(\widehat{F}_{\mathfrak{p}}) & \longrightarrow & \prod_{\mathfrak{p}\in P(F)} \mathrm{Br}\big((\widehat{F}_{\mathfrak{p}})_{\mathrm{sep}}(x)/\widehat{F}_{\mathfrak{p}}(x)\big) & \longrightarrow & \prod_{\mathfrak{p}\in P(F)} \bigoplus_{P'|P} X(G_{P'}),
\end{array}$$

where $P' \in \mathbb{P}(\widehat{F}_{\mathfrak{p}}(x)/\widehat{F}_{\mathfrak{p}})$ is a place above P.

We claim that the map ψ in (11) is injective. For this, take a nontrivial character $\chi \in X(G_P)$. Observe first that $\chi(G_P) \subseteq \mathbb{Q}/\mathbb{Z}$ is finite and so cyclic. If $M \subseteq F_{\mathrm{sep}}$ is the fixed field of $\ker(\chi)$, then M is a cyclic Galois extension of $\overline{V}_P =: E$ and the order of χ equals the degree $[M : E]$. For $\mathfrak{p} \in P(F)$, take $P' \in \mathbb{P}(\widehat{F}_{\mathfrak{p}}(x)/\widehat{F}_{\mathfrak{p}})$ with $P'|P$. Then for the image χ' of χ in $X(G_{P'})$, the order of χ' equals the degree $[M\overline{V}_{P'} : \overline{V}_{P'}]$. By Tchebotarev density (cf. [**FJ**, Theorem 5.6]), there exists $\mathfrak{P} \in P(E)$ such that for every $\mathfrak{P}' \in P(M)$ with $\mathfrak{P}'|\mathfrak{P}$, we have $[M : E] = [\widehat{M}_{\mathfrak{P}'} : \widehat{E}_{\mathfrak{P}}]$. Choose $\mathfrak{p} \in P(F)$ as the restriction of $\mathfrak{P}'$ to F; then there is $P' \in \mathbb{P}(\widehat{F}_{\mathfrak{p}}(x)/\widehat{F}_{\mathfrak{p}})$ with $P'|P$ and $\overline{V}_{P'} = \widehat{E}_{\mathfrak{P}}$. Then we have $M\overline{V}_{P'} = \widehat{M}_{\mathfrak{P}'}$ and therefore the order of χ' equals that of χ. Hence, the map ψ is injective as claimed.

Now, since the map i in (11) is also injective by (9), we conclude that the map j_0 in (11) and so the map j in (10) is injective. □

For a field k, let $\mathrm{Br}(k)'$ denote the subgroup of $\mathrm{Br}(k)$ consisting of all elements of order relatively prime to p if $\mathrm{char}(k) = p > 0$. If $\mathrm{char}(k) = 0$, set $\mathrm{Br}(k)' = \mathrm{Br}(k)$. Notice then that

$$\mathrm{Br}(F_{\mathrm{sep}}(x)/F(x))' = \mathrm{Br}(F(x))'$$

since $\mathrm{Br}(F_{\mathrm{sep}}(x))' = 0$ (see [**FS**, Lemma 2, p. 51]). Thus, we have:

Corollary 4.2. *Let F be a global field and let $F(x)$ be the rational function field over F. Then, the map*

$$j : \mathrm{Br}(F(x))' \longrightarrow \prod_{\mathfrak{p}\in P(F)} \mathrm{Br}\big(\widehat{F}_{\mathfrak{p}}(x)\big) \tag{12}$$

is injective.

Remark 4.3. Corollary 4.2 can be also proved by specialization argument using the fact that every global field is Brauer-Hilbertian in the sense of Fein, Saltman, and Schacher. For details on Brauer-Hilbertian fields, see [**FSS**].

Let $Q = (a, b/F)$ be a quaternion division algebra over a global field F and let $K = F(Q) = F(x, \sqrt{ax^2+b})$. We want to describe the kernel of the map

$$h_1 : \mathrm{Br}(K)' \longrightarrow \prod_{\mathfrak{p}\in P(F)} \mathrm{Br}\big(\widehat{F}_{\mathfrak{p}}(Q)\big). \tag{13}$$

Theorem 4.4. *Let F be a global field and $K = F(Q)$ as above. For the map h_1 in (13) and the set $\mathcal{I}_Q$ in (5), we have*

$$\ker(h_1) = \ \big\{[Q' \otimes_F K] \,\big|\, [Q'] \in \mathcal{I}_Q\big\}.$$

Proof. For each $\mathfrak{p} \in \mathbb{P}(K/F)$ and $Q' \in \mathcal{I}_Q$, we first show that $Q' \otimes_F \widehat{F}_{\mathfrak{p}}(Q)$ is split. For this, it suffices to consider $\mathfrak{p} \in \mathrm{supp}(Q')$. Then, $Q' \otimes_F \widehat{F}_{\mathfrak{p}}$ is nonsplit as is $Q \otimes_F \widehat{F}_{\mathfrak{p}}$. It follows that $Q' \otimes_F \widehat{F}_{\mathfrak{p}} \cong Q \otimes_F \widehat{F}_{\mathfrak{p}}$ since there exists a unique (up to isomorphism) quaternion division algebra over $\widehat{F}_{\mathfrak{p}}$. Hence, $\widehat{F}_{\mathfrak{p}}(Q)$ splits Q'.

We show the other inclusion. Fix $P \in \mathbb{P}(K/F)$ and write the corresponding residue field as $\overline{V}$, instead of $\overline{V}_P$ for convenience. For each extension $\mathfrak{P}$ of $\mathfrak{p}$ to $\overline{V}$, let $\widehat{\overline{V}}_{\mathfrak{P}}$ be the completion of $\overline{V}$ at $\mathfrak{P}$. Then we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(F(Q))' & \xrightarrow{h_1} & \prod_{\mathfrak{p}\in P(F)} \mathrm{Br}(\widehat{F}_{\mathfrak{p}}(Q)) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Br}(\overline{V}(Q))' & \xrightarrow{j} & \prod_{\mathfrak{P}\in P(\overline{V})} \mathrm{Br}(\widehat{\overline{V}}_{\mathfrak{P}}(Q)). \end{array} \tag{14}$$

Since $\overline{V}$ splits Q by Corollary 2.3, $\overline{V}(Q)$ is a rational function field over $\overline{V}$. By Corollary 4.2, the map j in the diagram (14) is injective. Therefore, any $[B] \in \ker(h_1)$ becomes trivial in $\mathrm{Br}(\overline{V}(Q))$, which implies that $[B] \in \ker(h)$ by Theorem 3.7. □

Let V be a discrete valuation ring with its quotient field K. Let $\widehat{K}$ be the completion of K with respect to V and $\widehat{K}_{\mathrm{nr}}$ the maximal unramified extension of $\widehat{K}$. Denote by $X(G_{\overline{V}})$ the character group of the absolute Galois group of the residue field $\overline{V}$. There is a short exact sequence:

$$0 \longrightarrow \mathrm{Br}(V) \longrightarrow \mathrm{Br}(\widehat{K}_{\mathrm{nr}}/K) \xrightarrow{\mathrm{ram}} X(G_{\overline{V}}) \longrightarrow 0. \tag{15}$$

In order to define the ramification map ram, recall that for $[B] \in \mathrm{Br}(\widehat{K}_{\mathrm{nr}}/K)$, $\mathrm{ram}([B])$ is computed by first extending scalars to the completion $\widehat{K}$ and then applying the map

$$H^2(G_{\overline{V}}, \widehat{K}_{\mathrm{nr}}^*) \to H^2(G_{\overline{V}}, \mathbb{Z}) \cong X(G_{\overline{V}})$$

induced by the valuation. For details, see [**Sa**, Theorem 10.3].

Let $\widehat{V}$ be the completion of V, which is a discrete valuation ring with quotient field $\widehat{K}$; note that the residue field of $\widehat{V}$ is isomorphic to $\overline{V}$ and further we have

$$\mathrm{Br}(\overline{V}) \cong \mathrm{Br}(\widehat{V}) \hookrightarrow \mathrm{Br}(\widehat{K}). \tag{16}$$

See [**JW**, Theorem 2.8 (b) and Theorem 5.6 (a)] for the isomorphism. These facts will be used in proving Theorem 4.5 below.

Making use of the results in Section 3, we now describe the kernel of the map:

$$h_2 : \mathrm{Br}(K) \longrightarrow \prod_{P \in \mathbb{P}(K/F)} \mathrm{Br}(\widehat{K}_P). \tag{17}$$

Theorem 4.5. *Let F be a global field. Suppose that $Q = (a, b/F)$ is a quaternion division algebra over F. Let $K = F(Q) = F(x, \sqrt{ax^2 + b})$. For the map h_2 in* (17) *and the set $\mathcal{I}_Q$ in* (5)*, we have*

$$\ker(h_2) = \big\{[Q' \otimes_F K] \,\big|\, [Q'] \in \mathcal{I}_Q\big\}.$$

Proof. Since $[Q'] \in \mathcal{I}_Q$, $Q' \otimes_F \overline{V}_P$ is split by Proposition 3.2. Then, each class $[Q' \otimes_F K]$ obviously lies in $\ker(h_2)$ from the commutative diagram:

$$\begin{array}{ccc} \mathrm{Br}(F) & \xrightarrow{\text{res}} & \mathrm{Br}(K) \\ {\scriptstyle\text{res}}\downarrow & & \downarrow{\scriptstyle\text{res}} \\ \mathrm{Br}(\overline{V}_P) & \longrightarrow & \mathrm{Br}(\widehat{K}_P) \end{array} \tag{18}$$

for each $P \in \mathbb{P}(K/F)$. Here, the bottom map is the composition of the maps in (16). This proves one inclusion.

To verify the other inclusion, take any $[B] \in \ker(h_2)$. We claim that $[B]$ is a constant class. For each $P \in \mathbb{P}(K/F)$, the class $[B]$ becomes trivial in $\mathrm{Br}(\widehat{K}_P)$. By the definition of the ramification map in (15) associated to $\overline{V} = \overline{V}_P$, we have $\mathrm{ram}([B]) = 0$. Recall (cf. [**Ha**, Lemma 2.11]) that the map β in (1) is the direct sum of these ramification maps where P ranges over all $P \in \mathbb{P}(K/F)$. We then have $\beta([B]) = 0$. It follows that $[B]$ is a constant class from the exact sequence (1). Let $[Q'] \in \mathrm{Br}(F)$ such that $[Q' \otimes_F K] = [B]$. Since the bottom map in (18) is injective from (16), the class $[Q']$ becomes trivial in $\mathrm{Br}(\overline{V}_P)$. Replacing $\mathrm{Br}(\widehat{K}_P)$ by $\mathrm{Br}(K\overline{V}_P)$ in (18), we still have a commutative diagram with all restriction maps. Consequently, $B \otimes_K K\overline{V}_P$ is split for each P, which implies that $[B] \in \ker(h)$. Hence, Theorem 3.7 shows that $[Q'] \in \mathcal{I}_Q$. □

Putting Theorem 3.7, Theorem 4.4 and Theorem 4.5 all together, we have

$$\ker(h) = \ker(h_1) = \ker(h_2). \tag{19}$$

For $K = F(Q)$ as above, we say that the Hasse principle for $\mathrm{Br}(K)$ holds (in the sense of h_1) if $\ker(h_1) = 0$. The following corollary is immediate from (19) since $|\ker(h)| = 2^{|\mathrm{supp}(Q)|-2}$ by Theorem 3.7.

Corollary 4.6. *Let F be a global field. Suppose that Q is a quaternion division algebra over F. Let $K = F(Q)$. Then, the Hasse principle for* $\mathrm{Br}(K)$ *holds if and only if* $\big|\mathrm{supp}(Q)\big| = 2$.

Example 4.7. Let $p_1, p_2, \ldots, p_n$ be distinct odd prime numbers such that each $p_i \equiv 3 \pmod 4$. According to Examples 3.3, 3.8, and 3.9, the Hasse principle holds for $\mathrm{Br}\big(\mathbb{Q}(x, \sqrt{-x^2-1}\,)\big)$, for $\mathrm{Br}\big(\mathbb{Q}(x, \sqrt{-x^2+p_1}\,)\big)$, and for $\mathrm{Br}\big(\mathbb{Q}(x, \sqrt{-x^2+p_1p_2}\,)\big)$. On the other hand, for $n \geq 3$, let $K = \mathrm{Br}\big(\mathbb{Q}(x, \sqrt{-x^2+m}\,)\big)$ where $m = p_1p_2\ldots p_n$. Then the nontrivial elements in

$$\ker(h_1) = \bigcap_{P \in \mathbb{P}(K/\mathbb{Q})} \mathrm{Br}(K\overline{V}_P/K)$$

(see Examples 3.8 and 3.9) are the obstruction to the Hasse principle.

Remark 4.8. For $K = F(Q)$, assume that the Hasse principle for $\mathrm{Br}(K)$ (in the sense of h_1) holds. However, there is no analogue to Hilbert's Reciprocity Law in K even for the constant classes. That is, there exist quaternion division algebras B over $F(Q)$ such that $\widehat{F}_{\mathfrak{p}}(Q)$ splits Q for all $\mathfrak{p} \in P(F)$ but an odd number of prime spots. To see this, suppose $\mathrm{supp}(Q) = \{p_1, p_2\}$. Take $q \in P(F)$ such that $q \neq p_i$ for $p = 1, 2$. By Hilbert's Reciprocity Law 2.6, there exists a quaternion algebra Q' over F with $\mathrm{supp}(Q') = \{q, p_i\}$. Put $B = Q' \otimes_F F(Q)$. Then, it is easy to check that $B \otimes_K \widehat{F}_{\mathfrak{p}}(Q)$ is split for all $\mathfrak{p} \in P(F)$ except q.

5. The cases when a finite number of prime spots are omitted.

The purpose of this section is to generalize the results given in the previous sections. Specifically, we investigate the kernel of the map h_1 in (13) when a finite number of prime spots are dropped from $P(F)$. The need to consider what happens when finitely many primes are omitted arises often in algebraic number theory. In particular, when the author considered tractability of algebraic function fields of genus 0 over global fields, it was necessary to delete the dyadic prime spots (cf. Remark 5.10). In considering this relative h_1, we restrict our attention to the 2-torsion of the kernel of h_1 since otherwise $\ker(h_1)$ contains infinitely many elements in most cases as we will see in Remark 5.2. This part of $\ker(h_1)$ can be expressed in terms of quaternion algebras over K possibly together with the images in $\mathrm{Br}(K)$ of cyclic algebras of exponent 4 over F (cf. Theorem 5.8).

Let us start this section with recalling the exact sequence (9) and the (local) results that

$$\mathrm{Br}(\widehat{F}_{\mathfrak{p}}) \cong \begin{cases} \mathbb{Q}/\mathbb{Z} & \text{if } \mathfrak{p} \text{ is a finite prime spot,} \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } \mathfrak{p} \text{ is a real infinite,} \\ 0 & \text{if } \mathfrak{p} \text{ is a complex infinite.} \end{cases}$$

Thus, if F is an algebraic number field, we will consider only finite prime spots and real infinite spots on F. (If F is an algebraic function field in one variable over a finite field, then every prime spot is finite.)

Now, we want to see what happens to $\ker(i)$ in the exact sequence (9) if a finite number of prime spots in the direct sum are removed. For this, let $\mathcal{S}$ denote a finite subset of $P(F)$. Define the map

$$i_{\mathcal{S}} : \mathrm{Br}(F) \longrightarrow \prod_{\mathfrak{p} \in P(F) - \mathcal{S}} \mathrm{Br}(\widehat{F}_{\mathfrak{p}}). \tag{20}$$

Lemma 5.1. *For the map $i_{\mathcal{S}}$ in (20), we have the following:*

(i) *If $|\mathcal{S}| \leq 1$, then $i_{\mathcal{S}}$ is injective.*

(ii) *If $|\mathcal{S}| \geq 2$ and $\mathcal{S}$ contains only one finite prime spot (together with some real infinite spots), then we have*

$$\ker(i_{\mathcal{S}}) = \{[A] \,|\, A \text{ is a quaternion algebra over } F \text{ with } \mathrm{supp}(A) \subseteq \mathcal{S}\}.$$

The cardinality of this set is 2^{n-1} where $n = |\mathcal{S}|$.

(iii) *If $\mathcal{S}$ contains at least two finite prime spots, then* $\ker(i_{\mathcal{S}})$ *is infinite.*

Proof. (i) If $\mathcal{S} = \varnothing$, then $i_{\mathcal{S}} = i$ in (9). If $|\mathcal{S}| = 1$, $i_{\mathcal{S}}$ is still injective from the exactness at $\bigoplus_{\mathfrak{p}} \mathrm{Br}(\widehat{F}_{\mathfrak{p}})$ in (9).

(ii) Let A be a central division algebra over F with $[A] \in \ker(i_{\mathcal{S}})$. The exactness at $\bigoplus_{\mathfrak{p}} \mathrm{Br}(\widehat{F}_{\mathfrak{p}})$ in (9) assures that $[A]$ has a local invariant either 0 or $\frac{1}{2} + \mathbb{Z}$ at each $\mathfrak{p} \in \mathcal{S}$ and further $\mathrm{supp}(A)$ has nonzero even cardinality. By Hilbert's Reciprocity Law 2.6, A is a quaternion division algebra over F. This shows one inclusion and the other inclusion is clear. For the cardinality, apply Lemma 3.1.

(iii) It suffices to consider the case that

$$\mathcal{S} = \{\mathfrak{p}_1, \mathfrak{p}_2 \,|\, \mathfrak{p}_1 \text{ and } \mathfrak{p}_2 \text{ are finite prime spots in } P(F)\}.$$

Utilizing the exactness at $\bigoplus_{\mathfrak{p}} \mathrm{Br}(\widehat{F}_{\mathfrak{p}})$ in (9) again, we observe that each n-torsion subgroup of $\ker(i_{\mathcal{S}})$ has a class (of a cyclic algebra) of order n whose local invariant is $\frac{k}{n} + \mathbb{Z}$ at $\mathfrak{p}_1$, $\frac{n-k}{n} + \mathbb{Z}$ at $\mathfrak{p}_2$, and 0 otherwise, where $n \geq 2$ and $1 \leq k < n$. $\square$

Remark 5.2. Let Q be a quaternion algebra over F and let $K = F(Q)$. Assume that $\mathcal{S}$ contains at least two finite prime spots as in Lemma 5.1 (iii). Define the map

$$h_{\mathcal{S}} : \mathrm{Br}(F(Q))' \longrightarrow \prod_{\mathfrak{p}\in P(F)-\mathcal{S}} \mathrm{Br}(\widehat{F}_{\mathfrak{p}}(Q)).$$

Then for any $[A] \in \ker(i_{\mathcal{S}})$, each class $[A \otimes_F K]$ obviously lies in $\ker(h_{\mathcal{S}})$. This implies that $\ker(h_{\mathcal{S}})$ is infinite since $\mathrm{Br}(K/F) \cong \{[F], [Q]\}$ by Proposition 2.1.

From now on, we focus on the 2-torsion subgroups of the Brauer groups above. For an abelian group G, let ${}_2G$ denote the 2-torsion subgroup of G. For a global field F, recall (cf. [**Re**, Theorem 32.19] and [**Pi**, Theorem, p. 236]) that any element in ${}_2\mathrm{Br}(F)$ is the class of a quaternion algebra over F. Using (9) and the definition of the support, we obviously have the following exact sequence:

$$(21)\quad 0 \longrightarrow \left\{ [Q'] \;\middle|\; \begin{array}{l} Q' \text{ is a quaternion algebra} \\ \text{over } F \text{ with } \mathrm{supp}(Q') \subseteq \mathcal{S} \end{array} \right\} \longrightarrow {}_2\mathrm{Br}(F) \xrightarrow{\widetilde{i_{\mathcal{S}}}} \prod_{\mathfrak{p}\in P(F)-\mathcal{S}} {}_2\mathrm{Br}(\widehat{F}_{\mathfrak{p}}).$$

Let F be any field. For any $a \in F$, there is an associated $(x-a)$-adic discrete valuation ring

$$F[x]_a := \{f/g \mid f, g \in F[x],\ g(a) \neq 0\},$$

whose residue field is $\overline{F[x]}_a = F$. Since there is a natural injection $\mathrm{Br}(F[x]_a) \hookrightarrow \mathrm{Br}(F(x))$, we can view $\mathrm{Br}(F[x]_a)$ as a subgroup of $\mathrm{Br}(F(x))$. The residue map $F[x]_a \to F$, called specialization at a, is given by $f(x)/g(x) \mapsto f(a)/g(a)$. This ring homomorphism induces the specialization map $\mathrm{Br}(F[x]_a) \to \mathrm{Br}(F)$, a group homomorphism. Recall that for any $[A] \in \mathrm{Br}(F(x))$, we have $[A] \in \mathrm{Br}(F[x]_a)$ for all but finitely many $a \in F$ (cf. [**FSS**, p. 924]).

Lemma 5.3. *Let F be a global field. Let $\mathcal{S}$ be a finite subset of $P(F)$. If $[B] \in \mathrm{Br}(F(x))'$ is not a constant class, then $[B]$ has a specialization $[A]$ in $\mathrm{Br}(F)$ with $\mathfrak{p} \in \mathrm{supp}(A)$ for some $\mathfrak{p} \in P(F) - \mathcal{S}$.*

Proof. There exist a finite degree extension field E of F and a prime spot $\mathfrak{P} \in P(E)$ such that for $\mathfrak{p} = \mathfrak{P}\big|_F \in P(F)$ we have $\mathfrak{p} \notin \mathcal{S}$ and that there is a specialization to $\mathrm{Br}(F)$ such that the class $[B]$ specializes to some $[A]$ (which has the same order as $[B]$, and) whose support contains $\mathfrak{p}$ (see the proof of Theorem 2.4 and Theorem 2.5 in [**FSS**]). □

Proposition 5.4. *Let F be a global field and let $F(x)$ be the rational function field over F. Let $\mathcal{S}$ be a finite subset of $P(F)$. Then, the following sequence is exact:*

$$0 \longrightarrow \left\{ [Q' \otimes_F F(x)] \,\middle|\, \begin{array}{l} Q' \text{ is a quaternion algebra} \\ \text{over } F \text{ with } \operatorname{supp}(Q') \subseteq \mathcal{S} \end{array} \right\} \longrightarrow {}_2\mathrm{Br}(F(x)) \xrightarrow{\widetilde{j_{\mathcal{S}}}} \prod_{\mathfrak{p} \in P(F) - \mathcal{S}} {}_2\mathrm{Br}\big(\widehat{F}_{\mathfrak{p}}(x)\big). \tag{22}$$

Proof. Let $[B] \in \ker(\widetilde{j_{\mathcal{S}}})$ with B nonsplit. We first show that $[B]$ is in fact a constant class. Otherwise, by Lemma 5.3, $[B] \in {}_2\mathrm{Br}(F(x))$ has a specialization $[A] \in {}_2\mathrm{Br}(F)$ and there exists $\mathfrak{p} \in P(F)$ with $\mathfrak{p} \in \operatorname{supp}(A) - \mathcal{S}$. Thus, for this $\mathfrak{p}$, $A \otimes_F \widehat{F}_{\mathfrak{p}}$ is nonsplit and therefore $B \otimes_{F(x)} \widehat{F}_{\mathfrak{p}}(x)$ is also nonsplit from the commutative diagram:

$$\begin{array}{ccc} {}_2\mathrm{Br}(F(x)) & \xrightarrow{\text{res}} & {}_2\mathrm{Br}\big(\widehat{F}_{\mathfrak{p}}(x)\big) \\ \big\downarrow \rho & & \big\downarrow \rho \\ {}_2\mathrm{Br}(F) & \xrightarrow{\text{res}} & {}_2\mathrm{Br}(\widehat{F}_{\mathfrak{p}}) \end{array}$$

where the vertical maps ρ are specialization maps. Hence, $[B] \notin \ker(\widetilde{j_{\mathcal{S}}})$ if $[B]$ is not a constant class; so $[B]$ must be a constant class. Now, consider the following commutative diagram:

$$\begin{array}{ccc} {}_2\mathrm{Br}(F) & \xrightarrow{\widetilde{i_{\mathcal{S}}}} & \displaystyle\prod_{\mathfrak{p} \in P(F) - \mathcal{S}} {}_2\mathrm{Br}(\widehat{F}_{\mathfrak{p}}) \\ \big\downarrow & & \big\downarrow \\ {}_2\mathrm{Br}(F(x)) & \xrightarrow{\widetilde{j_{\mathcal{S}}}} & \displaystyle\prod_{\mathfrak{p} \in P(F) - \mathcal{S}} {}_2\mathrm{Br}\big(\widehat{F}_{\mathfrak{p}}(x)\big). \end{array}$$

Notice that the vertical maps are both injective. From the description of $\ker(\widetilde{i_{\mathcal{S}}})$ in (21), it follows that $\ker(\widetilde{j_{\mathcal{S}}}) = \big\{ [Q' \otimes_F F(x)] \,\big|\, Q' \text{ is a quaternion algebra over } F \text{ with } \operatorname{supp}(Q') \subseteq \mathcal{S} \big\}$. □

For a global field F, let Q be a quaternion division algebra over F with

$$\operatorname{supp}(Q) = \{ \mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n \}.$$

Let $K = F(Q)$. Consider another set

$$\mathcal{S} = \{ \mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_m \} \subseteq P(F)$$

with $|\mathcal{S}| = m$. We do not assume that $\mathcal{S}$ is disjoint from $\operatorname{supp}(Q)$. Define

$$\mathcal{F}_{Q,\mathcal{S}} = \big\{ r \in F \,\big|\, r \notin \widehat{F}_{\mathfrak{p}_1}^{*2} \cup \cdots \cup \widehat{F}_{\mathfrak{p}_n}^{*2} \cup \widehat{F}_{\mathfrak{q}_1}^{*2} \cup \cdots \cup \widehat{F}_{\mathfrak{q}_n}^{*2} \big\}. \tag{23}$$

Recall from Proposition 2.7 that for each $r \in \mathcal{F}_{Q,\mathcal{S}}$, $F(\sqrt{r})$ is the residue field of a place of K/F. We also define

$$\mathcal{I}_{Q,\mathcal{S}} = \left\{ [Q'] \;\middle|\; \begin{array}{l} Q' \text{ is a quaternion algebra over } F \\ \text{with } \operatorname{supp}(Q') \subseteq \operatorname{supp}(Q) \cup \mathcal{S} \end{array} \right\} \subseteq \operatorname{Br}(F). \tag{24}$$

The following lemma is a generalization of the last equality in Proposition 3.2.

Lemma 5.5. *With notations as above, we have*

$$\bigcap_{r \in \mathcal{F}_{Q,\mathcal{S}}} \operatorname{Br}(F(\sqrt{r})/F) = \mathcal{I}_{Q,\mathcal{S}}.$$

The cardinality of this set is 2^{l-1} where $l = \left|\operatorname{supp}(Q) \cup \mathcal{S}\right|$.

Lemma 5.6. *Let $K = F(Q)$ as above. If $[B]$ is a nonconstant class in $\operatorname{Br}(K)'$, then $[B]$ is a nonconstant class in $\operatorname{Br}\big(F(\sqrt{r})(Q)\big)$ for some $r \in \mathcal{F}_{Q,\mathcal{S}}$.*

Proof. We recall the commutative diagram (8) with exact rows. If $[B]$ is a nonconstant class in $\operatorname{Br}(K)'$, so in $\operatorname{Br}(F_{\text{sep}} \cdot K/K)$, then $\beta([B])$ is nontrivial in $\bigoplus_{P \in \mathbb{P}(K/F)} X(G_P)$. A little modification of Lemma 3.6 gives that if $\chi_P \neq 0$, then $\chi_P\big|_{G_{P'}} \neq 0$ for some $r \in \mathcal{F}_{Q,\mathcal{S}}$ and some $P' \in \mathbb{P}\big(K(\sqrt{r})/F(\sqrt{r})\big)$ with $P'\big|P$. For this $r \in \mathcal{F}_{Q,\mathcal{S}}$, $(\oplus\, e_P \cdot \operatorname{res}) \circ \beta([B]) \neq 0$ in $\bigoplus_{P'|P} X(G_{P'})$. Note that for each $P \in \mathbb{P}(K/F)$, $e_P = 1$ since $v_P(r) = 0$ as in the proof of Theorem 3.7. From the commutativity of the right square of (8), $\operatorname{res}([B])$ is a nonconstant class in $\operatorname{Br}(K(\sqrt{r}))$. $\square$

For $K = F(Q)$, we define a map

$$\widetilde{h_{\mathcal{S}}} : {}_2\operatorname{Br}(K) \longrightarrow \prod_{\mathfrak{p} \in P(F) - \mathcal{S}} {}_2\operatorname{Br}\big(\widehat{F}_{\mathfrak{p}}(Q)\big). \tag{25}$$

Lemma 5.7. *Let $K = F(Q)$ as above. Assume that $\mathcal{S}$ is a finite subset of $P(F)$ and that $\operatorname{supp}(Q) \not\subseteq S$. Let B be a nonsplit division algebra over K. If $[B]$ is a constant class in $\ker(\widetilde{h_{\mathcal{S}}})$, then B is a quaternion algebra over K. In fact, $[B] = [A \otimes_F K]$ where A is a quaternion algebra over F.*

Proof. Since $[B] \in \ker(\widetilde{h_{\mathcal{S}}})$ is a constant class, we can choose a central division algebra A over F such that $[A \otimes_F K] = [B]$. Since $\exp(B) = 2$, the function field K splits $A^{\otimes 2} := A \otimes_F A$. We claim that $[A^{\otimes 2}]$ is trivial. Suppose that $[A^{\otimes 2}]$ were nontrivial. Note then that $A^{\otimes 2} \sim Q$ by Proposition 2.1. Take $\mathfrak{p} \in \operatorname{supp}(Q) - S$. For this $\mathfrak{p}$, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Br}(F) & \xrightarrow{\text{res}} & \mathrm{Br}(\widehat{F}_{\mathfrak{p}}) \\ {\scriptstyle\text{res}}\big\downarrow & & \big\downarrow{\scriptstyle\text{res}} \\ \mathrm{Br}(F(Q)) & \xrightarrow{\text{res}} & \mathrm{Br}(\widehat{F}_{\mathfrak{p}}(Q)) \end{array} \tag{26}$$

Observe that $\exp(A \otimes_F \widehat{F}_{\mathfrak{p}}) = 4$ since $Q \otimes_F \widehat{F}_{\mathfrak{p}} \sim A^{\otimes 2} \otimes_F \widehat{F}_{\mathfrak{p}}$ has exponent 2. It follows that $A \otimes_F \widehat{F}_{\mathfrak{p}}(Q)$ is nonsplit over $\widehat{F}_{\mathfrak{p}}(Q)$ by Proposition 2.1. However, the diagram (26) shows that this fact contradicts our assumption that $[B] \in \ker(\widetilde{h_{\mathcal{S}}})$. Hence, $[A^{\otimes 2}]$ must be trivial. This shows that $\exp(A) \leq 2$. Since F is a global field, we have $\mathrm{ind}(A) = \exp(A)$ and so A is a quaternion algebra over F (cf. [**Pi**, Theorem, p. 236]). Therefore B is a quaternion algebra over K. □

We need to generalize the notion of the support to central simple algebras, not restricting to quaternion algebras, for our main theorem in this section. For a central simple algebra A over a global field F, define

$$\mathrm{supp}(A) = \{\mathfrak{p} \in P(F) \mid A \otimes_F \widehat{F}_{\mathfrak{p}} \text{ is nonsplit}\}.$$

Theorem 5.8. *Let F be a global field and $K = F(Q)$ where Q is a quaternion division algebra over F. Assume that $\mathcal{S}$ is a finite subset of $P(F)$. For the $\widetilde{h_{\mathcal{S}}}$ in (25), we have the following:*

(i) *If $\mathrm{supp}(Q) \not\subseteq S$, then*

$$\ker(\widetilde{h_{\mathcal{S}}}) = \bigcap_{r \in \mathcal{F}_{Q,\mathcal{S}}} \mathrm{Br}(K(\sqrt{r})/K) = \left\{ [Q' \otimes_F K] \,\middle|\, Q' \in \mathcal{I}_{Q,\mathcal{S}} \right\} \tag{27}$$

for the $\mathcal{F}_{Q,\mathcal{S}}$ in (23), and the $\mathcal{I}_{Q,\mathcal{S}}$ in (24). The cardinality of $\ker(\widetilde{h_{\mathcal{S}}})$ is 2^{n-2} where $n = \left|\mathrm{supp}(Q) \cup \mathcal{S}\right|$.

(ii) *If $\mathrm{supp}(Q) \subseteq S$, then*

$$\ker(\widetilde{h_{\mathcal{S}}}) = \left\{ [Q' \otimes_F K] \,\middle|\, \begin{array}{l} Q' \text{ is a quaternion algebra} \\ \text{over } F \text{ with } \mathrm{supp}(Q') \subseteq \mathcal{S} \end{array} \right\} \cup \left\{ [A \otimes_F K] \,\middle|\, \begin{array}{l} A^{\otimes 2} \sim Q \text{ and} \\ \mathrm{supp}(A) \subseteq \mathcal{S} \end{array} \right\}. \tag{28}$$

The cardinality of $\ker(\widetilde{h_{\mathcal{S}}})$ is 2^{n-1} where $n = |\mathcal{S}|$.

Proof. (i) Using a similar argument to that in the proof of Theorem 3.7, we have the second equality in (27). We now want to show the first equality. Let

$$\mathcal{I} = \left\{ [Q' \otimes_F K] \,\middle|\, Q' \in \mathcal{I}_{Q,\mathcal{S}} \right\}.$$

Plainly, $\mathcal{I} \subseteq \ker(\widetilde{h_{\mathcal{S}}})$. For the other inclusion, assume that $[B] \in \ker(\widetilde{h_{\mathcal{S}}})$.

We first assert that $[B]$ is a constant class. To see this, suppose that $[B]$ were a nonconstant class. Then, $[B] \notin \mathcal{I} = \bigcap_{r \in \mathcal{F}_{Q,\mathcal{S}}} \mathrm{Br}(K(\sqrt{r})/K)$. By Lemma 5.6, we can find $r \in \mathcal{F}_{Q,\mathcal{S}}$ so that $[B \otimes_K K(\sqrt{r})]$ is still a nonconstant class in $\mathrm{Br}(K(\sqrt{r}))$. Put $E = F(\sqrt{r})$. Let $\mathcal{T}$ be the set of the extensions of the prime spots in $\mathcal{S}$ to the quadratic extension E. Consider the following diagram:

$$
\begin{array}{ccc}
{}_2\mathrm{Br}(F(Q)) & \xrightarrow{\widetilde{h_{\mathcal{S}}}} & \prod_{\mathfrak{p} \in P(F)-\mathcal{S}} {}_2\mathrm{Br}(\widehat{F}_{\mathfrak{p}}(Q)) \\
\downarrow & & \downarrow \\
{}_2\mathrm{Br}(E(Q)) & \xrightarrow{\widetilde{h_{\mathcal{T}}}} & \prod_{\mathfrak{P} \in P(E)-\mathcal{T}} {}_2\mathrm{Br}(\widehat{E}_{\mathfrak{P}}(Q))
\end{array}
\tag{29}
$$

Note that this diagram is commutative (cf. (14)). Since E splits Q, the function fields $E(Q)$ and $\widehat{E}_{\mathfrak{P}}(Q)$ are both rational over E and $\widehat{E}_{\mathfrak{P}}$ respectively. Then,

$$\ker(\widetilde{h_{\mathcal{T}}}) = \left\{[Q' \otimes_K E(Q)] \,\middle|\, \mathrm{supp}(Q') \subseteq \mathcal{T}\right\}$$

by the exact sequence (22) (replacing F by E). Since $\ker(\widetilde{h_{\mathcal{T}}})$ contains only constant classes, it follows that $[B \otimes E(Q)] \notin \ker(\widetilde{h_{\mathcal{T}}})$ and so $[B] \notin \ker(\widetilde{h_{\mathcal{S}}})$ from the commutativity of (29). This contradicts our assumption. Therefore $[B]$ is a constant class.

Since $[B]$ is a constant class, Lemma 5.7 shows that $[B] = [Q' \otimes_F K]$ for some quaternion algebra Q' over F. We now claim that $[B] \in \mathcal{I}$. Otherwise, we can choose a prime spot $\mathfrak{p} \in \mathrm{supp}(Q') - (\mathrm{supp}(Q) \cup \mathcal{S})$. For this $\mathfrak{p}$, $Q' \otimes \widehat{F}_{\mathfrak{p}}$ is nonsplit and so is $Q' \otimes \widehat{F}_{\mathfrak{p}}(Q)$ since Q is split over $\widehat{F}_{\mathfrak{p}}$ (cf. Lemma 2.2). Hence, $[B] \notin \ker(\widetilde{h_{\mathcal{S}}})$, which is a contradiction. This proves the first equality in (27).

From the definition of $\mathcal{I}_{Q,\mathcal{S}}$ in (24) together with Proposition 2.1, it is clear that $\left|\ker(\widetilde{h_{\mathcal{S}}})\right| = 2^{n-2}$ where $n = \left|\mathrm{supp}(Q) \cup \mathcal{S}\right|$.

(ii) One inclusion ($\supseteq$) in (28) is clear. We show the other inclusion ($\subseteq$). By the same argument as in the proof of (i), any $[B] \in \ker(\widetilde{h_{\mathcal{S}}})$ is a constant class so that we can find $[A] \in \mathrm{Br}(F)$ with $[B] = [A \otimes_F K]$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Br}(F) & \xrightarrow{i_{\mathcal{S}}} & \prod_{\mathfrak{p} \in P(F)-\mathcal{S}} \mathrm{Br}(\widehat{F}_{\mathfrak{p}}) \\
\downarrow & & \downarrow \\
\mathrm{Br}(F(Q)) & \xrightarrow{h_{\mathcal{S}}} & \prod_{\mathfrak{p} \in P(F)-\mathcal{S}} \mathrm{Br}(\widehat{F}_{\mathfrak{p}}(Q)).
\end{array}
\tag{30}
$$

Notice that for each $\mathfrak{p} \notin \mathcal{S}$, $\widehat{F}_{\mathfrak{p}}(Q)$ is purely transcendental over $\widehat{F}_{\mathfrak{p}}$ and so the right vertical map in (30) is injective. This implies that $[A] \in \ker(i_{\mathcal{S}})$. Since $\exp(B) = 2$, we have $\exp(A) = 2$ or 4 because K is a quadratic extension of a purely transcendental extension of F. This implies that $\mathrm{ind}(A) = 2$ or 4 since F is a global field. If $\mathrm{ind}(A) = 2$, then A is a quaternion algebra with $[A] \in \ker(i_{\mathcal{S}})$, so $\mathrm{supp}(A) \subseteq \mathcal{S}$ by (21). If $\mathrm{ind}(A) = 4$, then $A^{\otimes 2}$ is nonsplit over F, but split over $F(Q)$. Therefore, $A^{\otimes 2} \sim Q$ by Proposition 2.1. The injectivity of the right vertical map in (30) shows that $\mathrm{supp}(A) \subseteq \mathcal{S}$.

Finally, for the cardinality of $\ker(\widetilde{h_{\mathcal{S}}})$, let H_1 denote the set of preimages in $\mathrm{Br}(F)$ of the first set of the union in (28) and H_2 denote that of the second set. Notice that H_1 is a subgroup of $\mathrm{Br}(F)$. We show that H_2 is a coset relative to H_1 generated by any element $[A]$ in H_2. In fact, for any two $[A], [A'] \in H_2$ with $A \ncong A'$, we have

$$(A' \otimes_F A^{\mathrm{op}})^{\otimes 2} \sim Q \otimes_F Q^{\mathrm{op}} \sim F,$$

where A^{op} is the opposite algebra of A. It follows that $A' \otimes_F A^{\mathrm{op}}$ has exponent 2. Evidently, $\mathrm{supp}(A' \otimes_F A^{\mathrm{op}}) \subseteq \mathcal{S}$. Thus, $A' \otimes_F A^{\mathrm{op}} \sim Q'$ for some $[Q'] \in H_1$. That is, $A' \sim A \otimes_F Q'$. On the other hand, it is obvious that for any $[Q'] \in H_1$, $[A \otimes_F Q'] \in H_2$. Hence, $|H_1| = |H_2| = 2^{n-1}$ where $n = |\mathcal{S}|$. Consequently,

$$|\ker(\widetilde{h_{\mathcal{S}}})| = \frac{1}{2}(2^{n-1} + 2^{n-1}) = 2^{n-1}.$$

This completes the proof. □

Remark 5.9. Suppose that $\mathrm{supp}(Q) \nsubseteq S$. If the cardinality of the set $\mathrm{supp}(Q) \cup \mathcal{S}$ is even, there exists a quaternion algebra Q' such that $\mathrm{supp}(Q') = \mathrm{supp}(Q) \cup \mathcal{S}$ by Hilbert's Reciprocity Law 2.6. Then,

$$\ker(\widetilde{h_{\mathcal{S}}}) = \ker(h_1),$$

where h_1 is a map in (13) for $K = F(Q')$.

Remark 5.10. Suppose that Q is a quaternion division algebra with $\mathrm{supp}(Q) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ and $\mathcal{S} = \{\mathfrak{q}\}$. For the map

$$h_{\mathcal{S}} : \mathrm{Br}(F(Q))' \longrightarrow \prod_{\mathfrak{p} \in P(F) - \mathcal{S}} \mathrm{Br}(\widehat{F}_{\mathfrak{p}}(Q)), \tag{31}$$

Proposition 2.1 and Lemma 5.1 (i) allow us to conclude that

$$\ker(h_{\mathcal{S}}) = \ker(\widetilde{h_{\mathcal{S}}}) = \left\{ 0, [Q' \otimes_F K] \;\middle|\; \begin{array}{l} Q' \text{ is a quaternion algebra} \\ \text{over } F \text{ with } \mathrm{supp}(Q') = \{\mathfrak{p}_1, \mathfrak{q}\} \end{array} \right\}.$$

Observe that if $\mathfrak{q}$ is equal to one of $\mathfrak{p}_i$, then $\ker(h_{\mathcal{S}})$ is trivial. This situation arose in [**Ha**] when we studied the tractability of function fields of genus 0 over global fields as follows: Let F be an algebraic number field with exactly one dyadic spot $\mathfrak{d}$. Assume that $\mathrm{supp}(Q) = \{\, \mathfrak{p}_1, \mathfrak{d} \,\}$ and $\mathcal{S} = \{\, \mathfrak{d} \,\}$. Then, the map $h_{\mathcal{S}}$ is injective. On the other hand, let F be an algebraic function field in one variable over a finite field with $\mathcal{S} = \varnothing$. Then

$$\ker(h_{\mathcal{S}}) = \ker(h_1) = 0,$$

where h_1 is the map in (13), since $\operatorname{supp}(Q)$ contains exactly 2 prime spots. In both cases, the injectivity of $h_{\mathcal{S}}$ guarantees that the function field $F(Q)$ is tractable since $\widehat{F}_{\mathfrak{p}}(Q)$ is tractable for each nondyadic local field $\widehat{F}_{\mathfrak{p}}$ (see [**Ha**, Theorem 4.9] for details).

Example 5.11. Let $Q = (-1, p_1p_2/\mathbb{Q})$, where p_1 and p_2 are distinct odd primes with $p_i \equiv 3 \pmod 4$. Let $K = \mathbb{Q}(Q) = \mathbb{Q}(x, \sqrt{-x^2 + p_1p_2}\,)$. Then $\operatorname{supp}(Q) = \{\, p_1, p_2 \,\}$ as in Example 3.5. Assume that $\mathcal{S} = \{2, \infty\}$ where 2 is the dyadic spot and ∞ is the real infinite spot of $\mathbb{Q}$. Then,

$$\ker(\widetilde{h_{\mathcal{S}}}) = \big\{0,\ \big[(-1, -1/\mathbb{Q})\big],\ \big[(-1, p_1/\mathbb{Q})\big],\ \big[(-1, p_2/\mathbb{Q})\big]\big\}.$$

We can also describe the kernel of the map $h_{\mathcal{S}}$ in (31). In fact, since ∞ is the real infinite spot, any Brauer class in $\ker(h_{\mathcal{S}})$ has local invariant either 0 or $\frac{1}{2} + \mathbb{Z}$ at the dyadic spot 2 (see Lemma 5.1 (ii)). It follows that

$$\ker(h_{\mathcal{S}}) = \ker(\widetilde{h_{\mathcal{S}}}).$$

Moreover, by Remark 5.9 we have $\ker(\widetilde{h_{\mathcal{S}}}) = \ker(h_1)$ where h_1 is the map in (13) for $K = \mathbb{Q}(x, \sqrt{-x^2 - p_1p_2}\,)$.

In case that $\operatorname{supp}(Q) \subseteq \mathcal{S}$, the following remark provides us with concrete descriptions of the inverse images in $\operatorname{Br}(F)$ of the elements in $\ker(\widetilde{h_{\mathcal{S}}})$.

Remark 5.12. For $[A \otimes_F K] \in \ker(\widetilde{h_{\mathcal{S}}})$ in Theorem 5.8 (ii), the local invariant of A at $\mathfrak{p} \in P(F)$ is as follows:

$$\operatorname{inv}_{\mathfrak{p}}(A) = \begin{cases} \frac{1}{4} \ \text{ or } \ \frac{-1}{4} + \mathbb{Z} & \text{if } \ \mathfrak{p} \in \operatorname{supp}(Q) \\ 0 \ \text{ or } \ \frac{1}{2} + \mathbb{Z} & \text{if } \ \mathfrak{p} \in \mathcal{S} - \operatorname{supp}(Q) \\ 0 & \text{if } \ \mathfrak{p} \notin \mathcal{S}. \end{cases}$$

Note that $A \otimes_F K$ has exponent 2 but index 4. In fact, by the index reduction formula since K is a generic splitting field of Q (cf. [**SV**, Theorem 2.3, p. 735]), we obtain

$$\operatorname{ind}(A \otimes_F K) = \min\big(\operatorname{ind}(A), \operatorname{ind}(A \otimes_F Q)\big) = \min\big(\exp(A), \exp(A \otimes_F Q)\big) = 4.$$

This tells us that $A \otimes_F K$ is a product of two quaternion algebras over K by a theorem of Albert (cf. [**KMRT**, Theorem 16.1, p. 233]). Moreover, we observe that neither of these quaternion algebras is of constant class. For, if one of them were of constant class then so would be the other. This would imply that A is a tensor product of quaternion algebras over F contradicting the fact that $\exp(A) = 4$.

Example 5.13. Let $K = F(Q)$ as above. Suppose that $\mathrm{supp}(Q) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ and $\mathcal{S} = \{\mathfrak{p}_1, \mathfrak{p}_2\ \mathfrak{p}_3, \mathfrak{p}_4\}$. Then $|\ker(\widetilde{h_\mathcal{S}})| = 8$. To see this, let $A := (n_1, n_2, n_3, n_4)$ denote a cyclic algebra over F with local invariants

$$\mathrm{inv}_\mathfrak{p}(A) = \begin{cases} n_i + \mathbb{Z} & \text{if } \mathfrak{p} = \mathfrak{p}_i \ (i = 1, \ldots, 4), \\ 0 & \text{if } \mathfrak{p} \notin \mathcal{S}. \end{cases}$$

Obviously, we have $\mathrm{supp}(A) \subseteq \mathcal{S}$. The preimages in $\mathrm{Br}(F)$ of $\ker(\widetilde{i_\mathcal{S}})$ are the classes of the following algebras:

$$\begin{aligned}
Q_1 &= (0, 0, 0, 0) & Q_5 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0) = Q \\
Q_2 &= (0, \tfrac{1}{2}, \tfrac{1}{2}, 0) & Q_6 &= (\tfrac{1}{2}, 0, \tfrac{1}{2}, 0) \\
Q_3 &= (0, \tfrac{1}{2}, 0, \tfrac{1}{2}) & Q_7 &= (\tfrac{1}{2}, 0, 0, \tfrac{1}{2}) \\
Q_4 &= (0, 0, \tfrac{1}{2}, \tfrac{1}{2}) & Q_8 &= (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \\
A_1 &= (\tfrac{1}{4}, \tfrac{-1}{4}, 0, 0) & A_5 &= (\tfrac{-1}{4}, \tfrac{1}{4}, 0, 0) \\
A_2 &= (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}, 0) & A_6 &= (\tfrac{-1}{4}, \tfrac{-1}{4}, \tfrac{1}{2}, 0) \\
A_3 &= (\tfrac{1}{4}, \tfrac{1}{4}, 0, \tfrac{1}{2}) & A_7 &= (\tfrac{-1}{4}, \tfrac{-1}{4}, 0, \tfrac{1}{2}) \\
A_4 &= (\tfrac{1}{4}, \tfrac{-1}{4}, \tfrac{1}{2}, \tfrac{1}{2}) & A_8 &= (\tfrac{-1}{4}, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2})
\end{aligned}$$

Here, Q_i are quaternion algebras over F, and A_i are cyclic algebras over F of exponent 4 satisfying $A_i^{\otimes 2} \sim Q$. Notice that

$$Q_{i+4} \sim Q_i \otimes_F Q \text{ and } A_{i+4} \sim A_i \otimes_F Q \text{ for } i = 1, \ldots, 4.$$

By Theorem 5.8, we have

$$\ker(\widetilde{h_\mathcal{S}}) = \big\{[Q_i \otimes_F K],\ [A_i \otimes_F K] \,\big|\, i = 1, \ldots, 4\big\}.$$

References

[Ar] E. Artin, *Algebraic Numbers and Algebraic Functions*, Gordon and Breach, New York, 1967, MR 38 #5742, Zbl 0194.3530.

[AB] M. Auslander and A. Brumer, *Brauer groups of discrete valuation rings*, Nederl. Akad. Wetensch. Proc. Ser A, **71** (1968), 286-296, MR 37 #4051, Zbl 0182.07601.

[DI] F. Demeyer and E. Ingraham, *Separable Algebras over Commutative Rings*, Lecture Notes in Math., **181**, Springer, Berlin, 1971, MR 43 #6199, Zbl 0215.36602.

[Dr] P. Draxl, *Skew Fields*, Cambridge University Press, Cambridge, 1983, MR 85a:16022, Zbl 0498.16015.

[Fa] D. Faddeev, *Simple algebras over a field of algebraic functions of one variable*, Trudy Mat. Inst. Steklov, **38** (1951), 321-344; Amer. Math. Soc. Trans. Ser. II, **3** (1956), 15-38, MR 13,905c, MR 17,1046e, Zbl 0075.02901.

[FSS] B. Fein, D. Saltman and M. Schacher, *Brauer-Hilbertian fields*, Trans. Amer. Math. Soc., **334** (1992), 915-928, MR 93b:12006, Zbl 0767.12003.

[FS] B. Fein and M. Schacher, *Brauer groups of rational function fields over global fields*, in 'Groupe de Brauer' (eds. M. Kervaire and M. Ojanguren), 46-74, Lecture Notes in Math., **844**, Springer, Berlin, 1981, MR 82h:12025, Zbl 0455.12011.

[FJ] M. Fried and M. Jarden, *Field Arithmetic*, Springer, Berlin, 1986, MR 89b:12010, Zbl 0625.12001.

[Ha] I. Han, *Tractability of algebraic function fields in one variable of genus zero over global fields*, J. Algebra, **244** (2001), 217-235, MR 2002k:12003.

[JW] B. Jacob and A. Wadsworth, *Division algebras over Henselian fields*, J. Algebra, **128** (1990), 126-179, MR 91d:12006, Zbl 0692.16011.

[Ka] K. Kato, *A Hasse principle for two dimensional global fields* (with an appendix by J.-L. Colliot-Thélène), J. Reine Angew. Math., **366** (1986), 142-183, MR 88b:11036, Zbl 0576.12012.

[KMRT] M.-A. Knus, A.S. Merkurjev, M. Rost and J.-P. Tignol, *The Book of Involutions*, Amer. Math. Soc., Providence, RI, 1998, MR 2000a:16031, Zbl 0955.16001.

[La] T.-Y. Lam, *The Algebraic Theory of Quadratic Forms*, rev. ed., Benjamin, Reading, Mass., 1980, MR 83d:10022, Zbl 0437.10006.

[OM] O.T. O'Meara, *Introduction to Quadratic Forms*, Springer, Berlin, 1963, MR 27 #2485, Zbl 0107.03301.

[PS] R. Parimala and R. Sujatha, *Hasse principle for Witt groups of function fields with special reference to elliptic curves*, Duke Math. J., **85** (1996), 555-582, MR 98f:11067, Zbl 0876.11018.

[Pi] R. Pierce, *Associative Algebras*, Springer, New York, 1982, MR 84c:16001, Zbl 0497.16001.

[Re] I. Reiner, *Maximal Orders*, Academic Press, London, 1975, MR 52 #13910, Zbl 0305.16001.

[Sa] D. Saltman, *Lectures on Division Algebras*, Amer. Math. Soc., Providence, RI, 1999, MR 2000f:16023, Zbl 0934.16013.

[Sc] W. Scharlau, *Über die Brauer-gruppe eines algebraischen funktionenkörpers in einer variablen*, J. Reine Angew. Math., **239/240** (1969), 1-6, MR 40 #4270, Zbl 0184.24502.

[SV] A. Schofield and M. Van den Bergh, *The index of a Brauer class on a Brauer-Severi variety*, Trans. Amer. Math. Soc., **333** (1992), 729-739, MR 92m:12003, Zbl 0778.12004.

[St] H. Stichtenoth, *Algebraic Function Fields and Codes*, Springer, Berlin, 1993, MR 94k:14016, Zbl 0816.14011.

[Wa] A.R. Wadsworth, *Merkurjev's elementary proof of Merkurjev's theorem*, in 'Applications of Algebraic K-theory to Algebraic Geometry and Number Theory' (eds. S. Bloch et al.), 741-776, Contemp. Math., **55**, Part II, 1986, MR 88b:11078, Zbl 0604.16022.

[We] A. Weil, *Basic Number Theory*, Springer, New York, 1967, MR 38 #3244, Zbl 0176.33601.

[Ws] E. Weiss, *Algebraic Number Theory*, McGraw-Hill, New York, 1963, MR 28 #3021, Zbl 0115.03601.

[Wi] E. Witt, *Über ein gegenbeispiel zum normensatz*, Math. Zeitschrift, **39** (1935), 462-467, Zbl 0010.14901.

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