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We prove that the standard double bubble is the minimizing double bubble in $\mathbf{R}^4$ and in certain higher dimensional cases, extending the recent work in $\mathbf{R}^3$ of Hutchings, Morgan, Ritoré and Ros.

1. Introduction.

1.1. The Double Bubble Conjecture.

Conjecture 1.1 (Double Bubble Conjecture)**.** *The least-area hypersurface enclosing and separating two given volumes in* $\mathbf{R^n}$ *is the standard double soap bubble of Figure* 1*, consisting of three* $(n-1)$*-dimensional spherical caps intersecting at* 120 *degree angles.* (*For the case of equal volumes, the middle sphere is a flat disk.*)

In 1990, Foisy et al. [**F**] proved the Double Bubble Conjecture in $\mathbf{R^2}$. In 1995, Hass, Hutchings and Schlafly [**HHS**], [**HS**] used a computer to prove the conjecture for the case of equal volumes in $\mathbf{R^3}$. Most recently, in 2000, Hutchings, Morgan, Ritoré and Ros [**HMRR**] have used stability arguments to prove the conjecture for all cases in $\mathbf{R^3}$. Morgan's reference [**M**] discusses these results.

Here, we extend the methods of Hutchings et al. to higher dimensions. Component bounds after Hutchings [**H**] guarantee that the "$1+k$" double

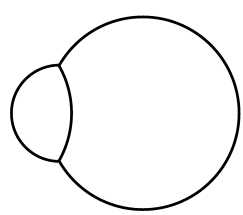

Figure 1. The standard double bubble, consisting of three spherical caps meeting at 120 degree angles, is the conjectured least-area hypersurface that encloses two given volumes in $\mathbf{R^n}$.

bubble — a double bubble in which one region is connected and the other region has k components — is the only alternative to the standard double bubble as the minimizing hypersurface in $\mathbf{R^4}$ and for sufficiently unequal volumes in $\mathbf{R^n}$, and that the larger region is connected (2.2, 2.5). By showing such bubbles unstable, we prove the Double Bubble Conjecture in $\mathbf{R^4}$ (Theorem 9.1) and, when the larger region has more than 2/3 the total volume, in $\mathbf{R^n}$ (Theorem 9.2).

1.2. The Instability Argument. An area-minimizing double bubble Σ exists and has an axis of rotational symmetry L. Consider small rotations about a line M orthogonal to L, chosen such that the points of tangency between the rotation vector field v and Σ separate the bubble into at least four pieces. Then we can linearly combine the restrictions of v to each piece to obtain a vector field which vanishes on one piece and preserves volume. By regularity for eigenfunctions, v is tangent to certain related parts of Σ, implying that they are spheres centered on $L \cap M$. This is the instability argument of [**HMRR**] behind Theorem 4.1. The corollaries to the theorem use the spherical pieces of Σ to show that it must be the standard double bubble.

We consider Σ a nonstandard double bubble with the larger region connected, and assume that Σ is a minimizer. In §6 and §7, we look at isolated parts of Σ — its "root" and its "leaves" according to an associated tree structure — and we classify all possible root and leaf configurations in which no useful perturbation axis M can be found. In §8 we combine our local classification results to nevertheless find a suitable M. By the above argument, Σ cannot in fact be a minimizer. So, for example, the double bubble with cross-section as in Figure 2 cannot be a minimizer.

Having eliminated all nonstandard double bubbles from consideration, the only possible minimizer left is the standard double bubble.

1.3. Open Questions. It can be shown that the leaf classification of Proposition 7.1 remains valid without the restriction that the larger region be connected. The instability of all nonstandard double bubbles in which one region is connected follows.

However, our component bounds are not strong enough to assure that one region must always be connected; for higher dimensions, they in general only establish that the larger region has at most 3 components and the smaller region has a finite number of components [**HLRS**]. But we have not been able to prove the instability of all double bubbles where both regions are disconnected.

Indeed, our methods fail to prove unstable the $2+2$ double bubble generated by rotating the curves of Figure 3 about the symmetry axis, in $\mathbf{R^5}$ or higher dimensions (§5 explains the rotation numbers attached to the vertices in the figure). Showing that this configuration is not minimizing, together with our bounds in [**HLRS**] and the results here, would prove the Double Bubble Conjecture in $\mathbf{R^5}$.

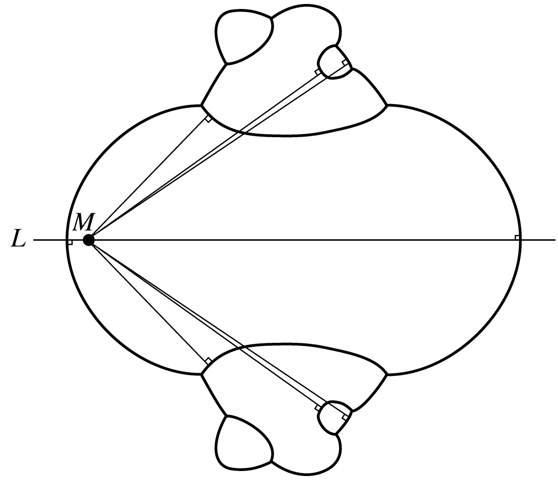

Figure 2. The lines orthogonal to Σ through the points of the separating set all pass through M. Σ cannot be a minimizer.

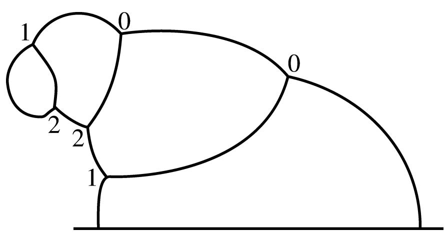

Figure 3. A $2+2$ double bubble might not have any disallowed interior separating sets, thus cannot be eliminated as unstable by our methods.

2. Double bubbles and component bounds.

A double bubble is a piecewise smooth oriented hypersurface $\Sigma \subset \mathbf{R^n}$ consisting of three compact pieces Σ_1, Σ_2 and Σ_0 (smooth on their interiors), with a common boundary such that $\Sigma_1 \cup \Sigma_0$, $\Sigma_2 \cup \Sigma_0$ enclose two regions of given volumes. Let $A_n(v,w)$ be the least area of a double bubble enclosing regions R of volume v and S of volume w. Let $\widetilde{A}_n(v,w) \geq A_n(v,w)$ be the area of the standard double bubble. Let $A_n(v) = n\pi^{1/2}v^{\frac{n-1}{n}}/(n/2)!^{1/n}$ be the area of a sphere of volume v.

[**H**] shows strict concavity of the least area function $A_n(v,w)$ and uses it to find bounds on the number of components of minimizing double bubbles. We will list some of his results in $\mathbf{R^n}$ and numerically compute them in $\mathbf{R^4}$. (See [**HLRS**] for more extensive numerical computations.)

Theorem 2.1 ([**H**, Theorem 3.2])**.** *If $n \geq 3$, if (v_1, w_1), (v_2, w_2) are two pairs of nonnegative volumes, and if $0 < t < 1$, then*

$$A_n(tv_1 + (1-t)v_2, tw_1 + (1-t)w_2) > tA_n(v_1, w_1) + (1-t)A_n(v_2, w_2).$$

Theorem 2.1 yields a dimension-independent component bound for unequal volumes:

Corollary 2.2 ([**H**, Theorem 3.5])**.** *If $v > 2w$, then in any least-area enclosure of volumes v, w in $\mathbf{R^n}$, R the region of volume v is connected.*

A slightly more sophisticated decomposition argument, together with the pigeonhole principle, gives a better component bound:

Theorem 2.3 ([**H**, Theorem 4.2])**.** *Consider a minimal enclosure of volumes v,w in $\mathbf{R^n}$. Then the number of components k of R the region of volume v satisfies*

$$2A_n(v,w) \geq A_n(w) + A_n(v+w) + A_n(v) \cdot k^{1/n}.$$

Clearly, k in Theorem 2.3 is finite:

Corollary 2.4 ([**H**, Corollary 4.3])**.** *A minimal enclosure of two volumes in $\mathbf{R^n}$ separates $\mathbf{R^n}$ into finitely many components.*

Proposition 2.5. *In a minimizing double bubble in $\mathbf{R^4}$, a region of at least half the total volume is connected.*

Proof. By concavity Theorem 2.1, for $v \in [0,1]$,

$$A_4(v, 1-v) \leq A_4(.5,.5) \leq \widetilde{A}_4(.5,.5) = \left(\frac{4}{3} + \frac{3\sqrt{3}}{4\pi}\right)^{1/4},$$

by computation. Hence, letting $w = 1-v$ in Theorem 2.3, we obtain

$$\begin{aligned} k^{1/4} &\leq \frac{1}{v^{3/4}}\left[\left(\frac{64}{3} + \frac{12\sqrt{3}}{\pi}\right)^{1/4} - 1\right] - \left(\frac{1}{v} - 1\right)^{3/4} \\ &< \frac{1.3}{v^{3/4}} - \left(\frac{1}{v} - 1\right)^{3/4} =: b(v). \end{aligned}$$

Differentiating $b(v)$ gives that $b'(v)$ has the same sign as $1 - 1.3(1-v)^{1/4}$, which is increasing. Hence $b'(v)$ passes from having negative sign to having positive sign, so on a closed interval $b(v)$ attains its maximum at an endpoint. Now,

$$b(.5)^4 = (1.3 \cdot 2^{3/4} - 1)^4 < 1.99$$

$$b(2/3)^4 = (1.3 \cdot (3/2)^{3/4} - (1/2)^{3/4})^4 < 1.86.$$

Hence $b(v)^4 < 2$ for $v \in [0.5, 2/3]$, implying that $k = 1$; R is connected. By Corollary 2.2, R is connected for $v > 2/3$. Hence R is connected for all $v \in [0.5, 1]$. □

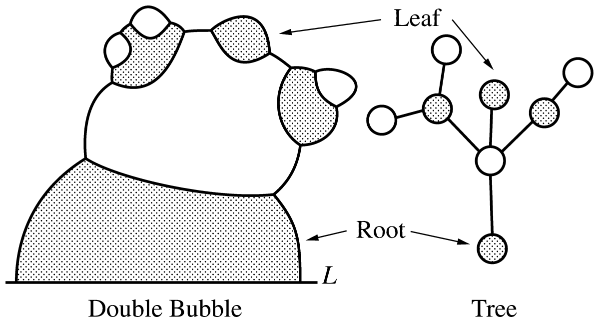

Figure 4. A nonstandard minimal double bubble must be a hypersurface of revolution consisting of a central bubble with layers of toroidal bands. Here we show the generating curves of a typical $4+4$ double bubble, together with the associated tree T.

3. Structure of minimal double bubbles.

The work of Almgren ([**A**] and see [**M**, Chapt. 13]) tells us that an area-minimizing double bubble enclosing any two given volumes in $\mathbf{R}^{\mathbf{n}}$ exists and is almost everywhere regular, if we allow disconnected regions. It is both stationary and stable. Hence, each region has a well-defined pressure, positive by [**H**, Corollary 3.3].

Lemma 3.1 ([**HMRR**, Lemma 6.4]). *In a minimizing double bubble for unequal volumes, the smaller region has larger pressure.*

This result follows easily from Hutchings concavity Theorem 2.1. Hutchings further classifies possible nonstandard minimizing double bubbles:

Theorem 3.2 ([**H**, Theorem 5.1]). *Any nonstandard minimal double bubble is a hypersurface of revolution about some line L, composed of pieces of constant mean curvature hypersurfaces meeting in threes at* 120 *degree angles. The bubble is a topological sphere with a tree T of annular bands attached, as in Figure* 4. *The two caps of the bottom component are pieces of spheres, and the root of the tree has just one branch.*

Hence, any minimal double bubble is determined by an upper half-planar diagram of arcs of generating curves which, when rotated about L, generate the double bubble. By studying these generating curves, we will eliminate as unstable nonstandard double bubbles.

[**HY**] shows that the only constant mean-curvature hypersurfaces of revolution are Delaunay hypersurfaces (Figure 5 and see [**D**], [**E**]):

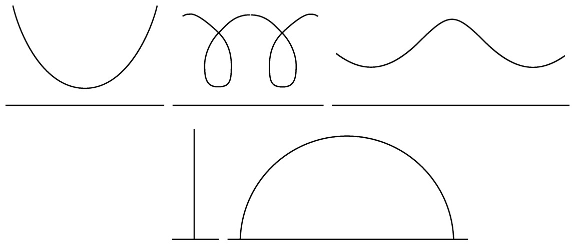

Figure 5. Smooth regions of the cluster are parts of Delaunay hypersurfaces: Catenoid, nodoid, unduloid, vertical plane, sphere.

Theorem 3.3 ([**HMRR**, Proposition 4.3])**.** *Let Γ be a complete upper half-planar generating curve which, when rotated about L, generates a hypersurface Σ with constant mean curvature. Then exactly one of the following statements holds:*

1) *Γ is a curve of catenary type and Σ is a hypersurface of catenoid type.*
2) *Γ is a locally convex curve and Σ is a nodoid.*
3) *Γ is a periodic graph over L and Σ is an unduloid or a cylinder.*
4) *Γ is a ray orthogonal to L and Σ is a vertical hyperplane.*
5) *Γ is a semi-circle and Σ is a sphere.*

The Delaunay hypersurfaces with nonzero mean curvature are the sphere, unduloid and nodoid. If Σ has positive mean curvature upward then it must be a nodoid. If Γ is not graph, then Σ must be either a nodoid or a hyperplane.

4. Instability by separation.

Let $\Sigma \subset \mathbf{R^n}$ be a regular stationary double bubble of revolution about axis L, with upper half planar generating curves Γ consisting of arcs $\bar{\Gamma}_i$, with interiors Γ_i, ending either at the axis or in threes at vertices v_{ijk}.

We consider the map $f : \cup\Gamma_i \longrightarrow L \cup \{\infty\} \equiv [-\infty, +\infty]/(-\infty \sim +\infty)$ which maps each $p \in \cup\Gamma_i$ to the point $L(p) \cap L$, where $L(p)$ denotes the normal line to Γ at p. Later we will denote the limiting values of f on the left and right endpoints of Γ_i by $iA, iB \in [-\infty, +\infty]$, respectively; for consistency, we will often simply consider $f(p)$ as its preimage in $[-\infty, +\infty]$. (With this notation, if $iA \in f(\Gamma_j)$ and Γ_j is not a circle or hyperplane, then for all $p \in \Gamma_i$ sufficiently close to the left endpoint, $f(p) \in f(\Gamma_j)$.)

Theorem 4.1 ([**HMRR**, Proposition 5.2])**.** *Consider a stable double bubble of revolution $\Sigma \subset \mathbf{R^n}$, $n \geq 3$, with axis L. Assume that there is a minimal set of points $\{p_1, \dots, p_k\}$ in $\cup\Gamma_i$ with $f(p_1) = \cdots = f(p_k)$ which separates Γ.*

Then every connected component of Σ *which contains one of the points* p_i *is part of a sphere centered at* x *(if* $x \in L$*) or part of a hyperplane orthogonal to* L *(in the case* $x = \infty$*).*

We sketched the proof of this theorem in our introduction §1.2.

Corollary 4.2. *No generating arc which turns downward past the vertical can have an internal separating set, i.e., two points* $p_1 \neq p_2$ *in the arc, with* $f(p_1) = f(p_2)$*.*

Proof. Otherwise, by Theorem 4.1, the arc would have to be part of either a circle with center on the axis L or a line perpendicular to L. But neither turns past the vertical. □

Corollary 4.3. *No generating arc which is not part of a vertical line can go vertical twice, including at least once in its interior.*

Proof. Such an arc (a nodoid by Theorem 3.3) has a separating set $\{f^{-1}(x)\}$ for some x with $|x|$ large enough, contradicting Corollary 4.2. □

Corollary 4.4. *Consider a nonstandard minimizing double bubble. Then there is no* $x \in L \cup \{\infty\}$ *such that* $f^{-1}(x) - \{$*two circular caps*$\}$ *contains points in the interiors of distinct* Γ_i *which separate* Γ*.*

Proof. For $x \in L$, the statement is [**HMRR**, Proposition 5.7]. Arguments using "force balancing" show that more pieces of the minimizer are spherical and hence the bubble is the standard double bubble. For $x = \infty$, note that a separating set crosses at least one outer boundary. By Theorem 4.1, this boundary is a vertical line, contradicting positive pressure of the regions. □

We will consider various nonstandard double bubbles, and show that they violate one of the above corollaries of Theorem 4.1, hence cannot be minimizing.

5. Rotation notation.

A nonstandard minimal double bubble's generating curves can be further classified by how many notches m a vertex has been rotated from the standard position of Figure 6 in which all the generating curves are graphs (unlike Figure 4). A positive rotation notch about a vertex corresponds to an arc passing the vertical counterclockwise, as occurs for Γ_3 from Figure 6 to Figure 7, and from Figure 8(a) to 8(b). The extreme position with an arc leaving the vertex at the vertical divides two consecutive m cases. If the limiting value of f along the vertical arc is $+\infty$ (or the arc is a straight line), the position is assigned the smaller m rotation number, and if the limiting value is $-\infty$, the position is given the larger m value.

The rotation numbers for our earlier $4+4$ example are indicated in Figure 9.

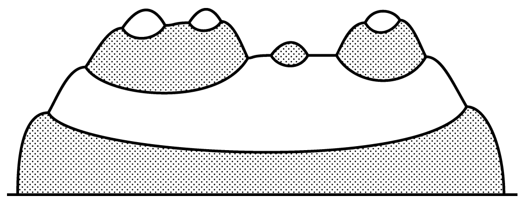

Figure 6. If all the generating curves are graphs, then $m = 0$ for each vertex.

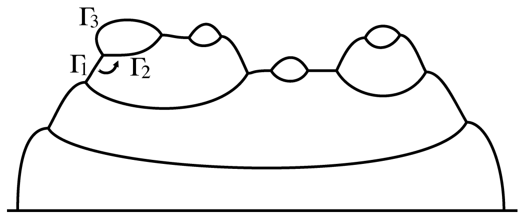

Figure 7. From the curves of Figure 6, vertex $v_{123} \equiv \overline{\Gamma}_1 \cap \overline{\Gamma}_2 \cap \overline{\Gamma}_3$ has turned one counterclockwise "notch," since Γ_3 has passed the vertical. Hence $m = 1$ for v_{123}.

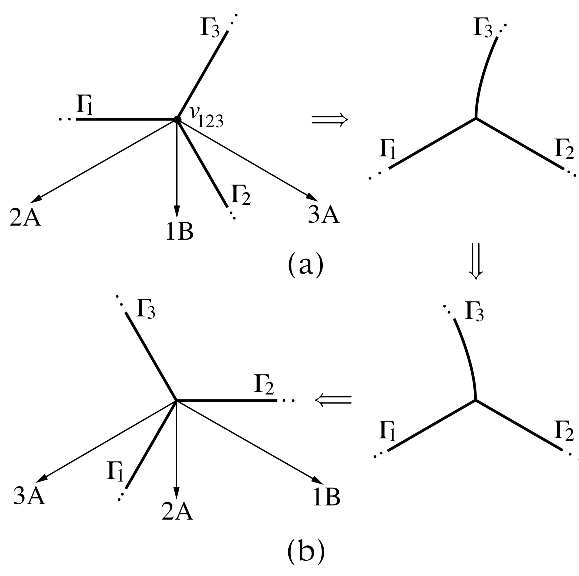

Figure 8. A close-up of v_{123} as it turns one notch counterclockwise. In (a), $m = 0$ and $2A < 1B < 3A$; on the right, $3A = +\infty$. In (b), $m = 1$ and $3A < 2A < 1B$; $3A = -\infty$ on the right.

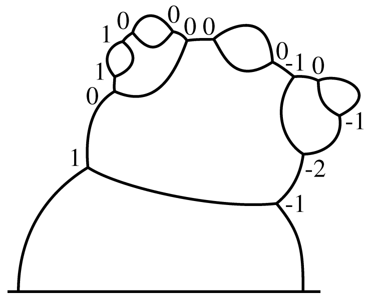

Figure 9. This nonstandard double bubble has the same associated tree structure as those in Figures 6 and 7, but some vertices have been rotated a notch or two left or right, as indicated.

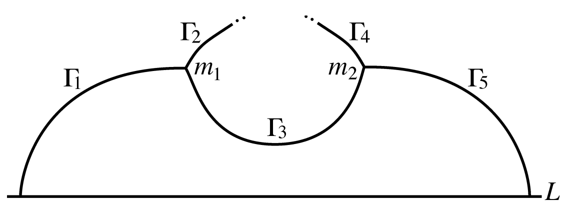

Figure 10. A root involves five arcs: Γ_2, Γ_3, Γ_4, and the two circular caps Γ_1 and Γ_5. In general, each vertex can be rotated m_i notches counterclockwise from the pictured configuration, in which all arcs are graphs and $m_1 = m_2 = 0$.

6. Root stability.

The "root" of a nonstandard minimizing double bubble corresponds to the root of its associated tree of Theorem 3.2 and Figure 4. The root involves five arcs including circular caps to either side, as in Figure 10.

Proposition 6.1. *In a minimizer, consider a root with the notation of Figure* 10. *Then* $(m_1, m_2) \in \{(-1,-1), (0,-1), (0,0), (1,-1), (1,0), (1,1)\}$.

Proof. Γ_1 and Γ_5 are parts of semi-circles, turning inward by positive pressure of the regions. It follows that $m_1, m_2 \in \{-1, 0, 1\}$.

If $(m_1, m_2) = (-1, 0)$ as in Figure 11, then $[-\infty, 3B) \subset f(\Gamma_3)$, where $3B$ denotes the image under f of the right-hand endpoint of Γ_3. Consideration of vertex $v_{345} \equiv \overline{\Gamma}_3 \cap \overline{\Gamma}_4 \cap \overline{\Gamma}_5$ gives $4B < 3B$. Hence $4B \in f(\Gamma_3)$, giving a separating set through Γ_3 and Γ_4. Since this contradicts Corollary 4.4, $(m_1, m_2) \neq (-1, 0)$. Similarly, $(m_1, m_2) \neq (0, 1)$.

If $(m_1, m_2) = (-1, 1)$, then Γ_3 goes vertical twice in its interior, violating Corollary 4.3. Hence $(m_1, m_2) \neq (-1, 1)$, as asserted. $\square$

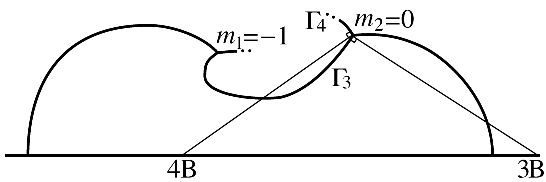

Figure 11. If $(m_1, m_2) = (-1, 0)$, then $[-\infty, 3B) \subset f(\Gamma_3)$ and $4B < 3B$, so $4B \in f(\Gamma_3)$.

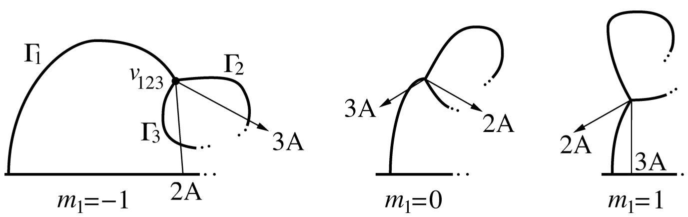

Figure 12. If Γ_2 turns vertical downward after leaving the left circular cap, then the rotation number m_1 of vertex v_{123} is either -1, 0 or 1.

Proposition 6.2. *In a minimizer, consider a root with the notation of Figure* 10*. Then neither* Γ_2 *nor* Γ_4 *can turn vertical downward after leaving a circular cap.*

Proof. Suppose Γ_2 turns vertical downward after leaving vertex v_{123}. By Theorem 3.3 and positive pressure of the regions, Γ_2 is a concave rightward nodoid. By Proposition 6.1, $m_1 \in \{-1, 0, 1\}$, as shown in Figure 12.

First, consider $m_1 = -1$. Then $(-\infty, 2A) \subset f(\Gamma_2)$. By Proposition 6.1, $m_2 \in \{-1, 0, 1\}$, so Γ_3 goes vertical before reaching v_{345}. Hence $[-\infty, 3B) \subset f(\Gamma_3)$.

Second, consider $m_1 = 0$. Then $(-\infty, 2A) \subset f(\Gamma_2)$ again, and $3A < 2A$.

Third, consider $m_1 = 1$. Then $f(\Gamma_2) = L \cup \{\infty\}$.

In each case, $f(\Gamma_2) \cap f(\Gamma_3) \neq \emptyset$, giving a separating set contrary to Corollary 4.4. Therefore Γ_2 cannot turn vertical downward after leaving v_{123}. Symmetrical considerations show that Γ_4 cannot turn vertical downward after leaving v_{345}. □

7. Leaf stability.

A "leaf" of a nonstandard minimizing double bubble corresponds to a leaf of its associated tree of Theorem 3.2 and Figure 4. A leaf involves four arcs, with a standard notation as in Figure 13. We say that one case "models" another if they are symmetrical under horizontal reflection and/or relabelling.

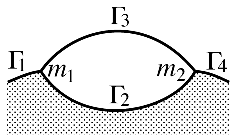

Figure 13. A leaf involves four arcs: Γ_1, Γ_2, Γ_3, Γ_4. In general each vertex can be rotated m_i notches counterclockwise from the pictured configuration, in which all arcs are graphs and $m_1 = m_2 = 0$.

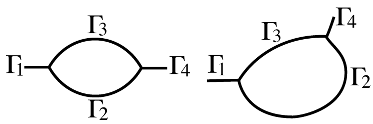

Figure 14. The two near graph cases $(0, 0)$ and $(0, 1)$, shown here, and the cases $(0, 2)$ and $(2, 1)$ model all leaves belonging to the smaller region. Case $(0, 0)$ models case $(3, 3)$, and case $(0, 1)$ models cases $(-2, -3)$, $(-1, 0)$ and $(3, 2)$, modulo $(6, 6)$.

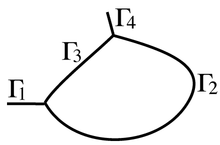

Figure 15. Case $(0, 2)$ models cases $(-2, 0)$, $(-1, -3)$ and $(3, 1)$.

Case $(m_1, m_2) = (a, b)$ models cases (a, b), $(-b, -a)$, $(b + 3, a + 3)$, and $(3 - a, 3 - b)$, up to rotation by $(6, 6)$.

Proposition 7.1. *In a minimizer for unequal volumes, consider a leaf belonging to the smaller region, with the notation of Figure* 13*. Then the only possible* (m_1, m_2) *cases, up to rotation by* $(6, 6)$*, are:*

$$(0, 0), (3, 3); (-2, -3), (-1, 0), (0, 1), (3, 2);$$
$$(-2, 0), (-1, -3), (0, 2), (3, 1); \quad \textit{and} \quad (-2, -1), (-1, -2), (1, 2), (2, 1).$$

In particular, every leaf is modeled on one of the four of Figures 14*,* 15 *and* 16*.*

Proof. First we need Lemma 7.2, in which we will use verticality arguments (refer to Corollary 4.3) to obtain general bounds on (m_1, m_2) cases. We will then eliminate as unstable individual cases.

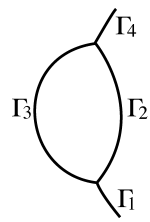

Figure 16. Case $(2,1)$ models cases $(-2,-1)$, $(-1,-2)$ and $(1,2)$.

Lemma 7.2. *With the assumptions as above, the only possibly stable rotation cases for* (m_1, m_2)*, up to rotation by* $(6,6)$*, are:*

$$\begin{aligned} m_1 \in \{-2,-1\} &\implies -3 \le m_2 \le 0 \\ m_1 = 0 &\implies -3 \le m_2 \le 3 \\ m_1 \in \{1,2\} &\implies 0 \le m_2 \le 3 \\ m_1 = 3 &\implies 0 \le m_2 \le 6 \end{aligned}$$

Proof. First, consider $m_1 \in \{-2,-1\}$. Γ_3 leaves v_{123} to the right, while Γ_2 leaves v_{123} to the left. If $m_2 = -4$, Γ_3 enters v_{234} from the left, thus goes vertical twice (in its interior) if $m_2 \le -4$. If $m_2 = 1$, Γ_2 enters v_{234} from the right, thus goes vertical twice if $m_2 \ge 1$. By Corollary 4.3, $-3 \le m_2 \le 0$, as asserted.

Second, consider $m_1 = 0$. Γ_3 leaves v_{123} going upward to the right, while Γ_2 leaves v_{123} going downward to the right. If $m_2 = -4$, Γ_3 enters v_{234} from the left, thus goes vertical twice if $m_2 \le -4$. If $m_2 = 4$, Γ_2 enters v_{234} from the left, thus goes vertical twice if $m_2 \ge 4$. By Corollary 4.3, $-3 \le m_2 \le 3$, as asserted.

Third, consider $m_1 \in \{1,2\}$. Considerations symmetrical to those of the cases $m_1 \in \{-2,-1\}$ give $0 \le m_2 \le 3$, as asserted.

Fourth, consider $m_1 = 3$. Considerations symmetrical to those of the case $m_1 = 0$ give $0 \le m_2 \le 6$, as asserted. □

To finish the proof of Proposition 7.1, we will use slightly more involved arguments to show that leaves with the following (m_1, m_2) rotation pairs cannot belong to the smaller region of a minimizer:

$$(-2,-2),(-1,-1),(0,-3),(0,-2),(0,-1),(0,3),$$
$$(1,0),(1,1),(1,3),(2,0),(2,2),(2,3),(3,0),(3,4),(3,5),(3,6).$$

Note that by Lemma 3.1, the leaf component has positive pressure larger than that of the adjacent component; Γ_2 and Γ_3 rotate about L to form hypersurfaces of positive mean curvature into the leaf.

- Case $(1,1)$ models cases $(-2,-2)$, $(-1,-1)$, $(2,2)$; see Figure 17.
  $4A \in f(\Gamma_3)$, since $3B < 4A$ and $(3B, +\infty] \subset f(\Gamma_3)$.

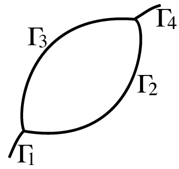

Figure 17. Case $(1,1)$ models cases $(-2,-2)$, $(-1,-1)$, $(2,2)$.

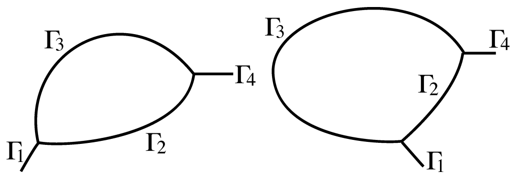

Figure 18. Case $(1,0)$ models cases $(0,-1)$, $(2,3)$, $(3,4)$. Case $(2,0)$ models cases $(0,-2)$, $(1,3)$, $(3,5)$.

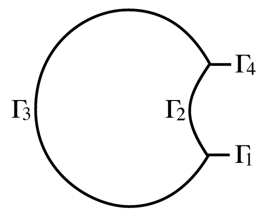

Figure 19. Case $(3,0)$ models cases $(0,-3)$, $(0,3)$, $(3,6)$.

Γ_2 goes vertical, so by Theorem 3.3 must be part of a concave leftward nodoid. Hence the normal to Γ_4 at v_{234} stays right of Γ_2 and in particular of v_{123}, implying $2A < 4A$. Since $(2A, +\infty] \subset f(\Gamma_2)$, $4A \in f(\Gamma_2)$.

Corollary 4.4 for Γ_2, Γ_3, Γ_4 implies instability.

- Case $(1,0)$ models cases $(0,-1)$, $(2,3)$, $(3,4)$, and case $(2,0)$ models cases $(0,-2)$, $(1,3)$, $(3,5)$; see Figure 18.

  For both cases, again $3B < 4A$ and $(3B, +\infty] \subset f(\Gamma_3)$ imply $4A \in f(\Gamma_3)$. Since $4A < 2B$, Corollary 4.4 for Γ_2, Γ_3, Γ_4 yields $4A \leq 2A$.

  Therefore, the net angle θ_3 through which Γ_3 turns satisfies $\theta_3 > 180$ degrees, since v_{123} is clearly left of v_{234}. Also, Γ_3 leaves v_{234} above the horizontal, and Corollary 4.2 for Γ_3 gives $3A \leq 3B$. Since Γ_2 rotates about L to form a hypersurface of positive mean curvature upwards into the leaf, by Theorem 3.3 Γ_2 is a (strictly convex) nodoid. We can now apply Lemma 7.3 to obtain $2A < 4A$, a contradiction.

- Case $(3,0)$ models cases $(0,-3)$, $(0,3)$, $(3,6)$; see Figure 19.

  Γ_2 goes vertical, so by Theorem 3.3 must be part of a concave rightward nodoid or a hyperplane, contradicting Lemma 3.1. $\square$

Lemma 7.3 ([**HMRR**, Corollary 5.10])**.** *Consider the* $(1,0)$*,* $(2,0)$ *and* $(2,1)$ *cases of Figures* 18 *and* 16*. Assume that the net angle* θ_3 *through which* Γ_3 *turns exceeds* 180 *degrees, that* Γ_3 *leaves* v_{234} *at or above the horizontal, that* $3A \leq 3B$*, and that* Γ_2 *is strictly convex. Then* $2A < 4A$*.*

Corollary 7.4. *For a* $(2,1)$ *leaf in a minimizer, as in Figure* 16*,* Γ_3 *leaves* v_{234} *below the horizontal.*

Proof. $4A \in f(\Gamma_3)$ since $3B < 4A$ and $(3B, +\infty] \subset f(\Gamma_3)$. Since also $(2A, +\infty] \subset f(\Gamma_2)$, Corollary 4.4 for Γ_2, Γ_3, Γ_4 yields $4A \leq 2A$.

Γ_2 goes vertical, so by Theorem 3.3 must be part of a concave leftward nodoid. Hence the normal to Γ_4 at v_{234} stays right of Γ_2 and in particular of v_{123}, implying $\theta_3 > 180$ degrees. By Corollary 4.2 for Γ_3, $3A \leq 3B$.

Now if Γ_3 leaves v_{234} at or above the horizontal, then Lemma 7.3 gives $2A < 4A$, a contradiction. □

Lemma 7.5. *For a* $(0,0)$ *or* $(0,1)$ *leaf in a minimizer, as in Figure* 14*,* $1B \leq f(\Gamma_3) \leq 4A$ *and* $1B < 4A$*.*

Proof. First, consider case $(0,0)$. Then $2A < 1B$ and $4A < 2B$. Since v_{123} is left of v_{234}, $1B < 2B$ and $2A < 4A$. Hence $1B, 4A \in f(\Gamma_2)$.

Second, consider case $(0,1)$. Then $2A < 1B$. Since v_{123} is left of v_{234}, $2A < 4A$. Since $(2A, +\infty] \subset f(\Gamma_2)$, again $1B, 4A \in f(\Gamma_2)$.

In both cases, $1B < 3A$ and $3B < 4A$. Corollary 4.4 for Γ_1, Γ_2, Γ_3, and for Γ_2, Γ_3, Γ_4 gives $1B \leq f(\Gamma_3)$ and $f(\Gamma_3) \leq 4A$, respectively. Since $1B < 3A$, $1B < 4A$, as claimed. □

Proposition 7.6. *An arc of outer boundary cannot turn vertical downward after leaving a leaf of the smaller region of a minimizer.*

Proof. First, consider cases $(0,0)$ and $(0,1)$ of Figure 14. By Lemma 7.5, $1B < 4A$. If Γ_1 turns vertical downward, $(1B, +\infty) \subset f(\Gamma_1)$. Since by positive pressure Γ_4 is not a hyperplane, $f(\Gamma_1) \cap f(\Gamma_4) \neq \emptyset$. Corollary 4.4 implies instability, so Γ_1 cannot turn vertical downward. Similarly, Γ_4 cannot turn vertical downward.

Second, consider case $(0,2)$ of Figure 15.

If Γ_1 turns vertical downward, then $(1B, +\infty) \subset f(\Gamma_1)$. Also, $(2A, +\infty] \subset f(\Gamma_2)$, and consideration of v_{123} gives $2A < 1B < 3A$. Therefore, $f(\Gamma_1) \cap f(\Gamma_2) \cap f(\Gamma_3) \neq \emptyset$ (by Lemma 3.1, Γ_3 is not a vertical line), contrary to Corollary 4.4. Hence Γ_1 cannot turn vertical downward.

If Γ_4 turns vertical downward and to the right, then $f(\Gamma_4) = L \cup \{\infty\}$. If Γ_4 turns vertical downward and to the left, then $(4A, +\infty) \subset f(\Gamma_4)$. Either way, $(2A, +\infty] \subset f(\Gamma_2)$. Consideration of v_{234} gives $4A < 3B$, while $2A < 3B$ since v_{123} is left of v_{234}. Also, $3B \neq +\infty$; otherwise Γ_3 is a concave leftward nodoid or a hyperplane, both disallowed by Lemma 3.1. Thus, $3B \in f(\Gamma_2) \cap f(\Gamma_4)$. Corollary 4.4 for Γ_2, Γ_3, Γ_4 implies instability. Hence Γ_4 cannot turn vertical downward.

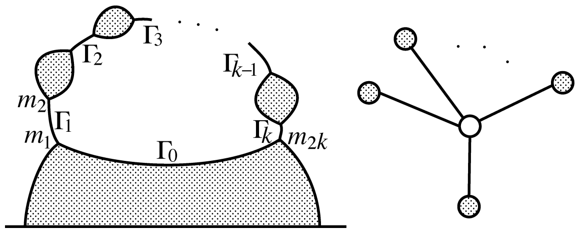

Figure 20. The associated tree T for a $1+k$ double bubble has just one branch from the root and $k-1$ leaves above the connected middle region. (The pictured notation is for Proposition 8.2.)

Third, consider case $(2,1)$ of Figure 16. Since $\infty \in f(\Gamma_2) \cap f(\Gamma_3)$, if either Γ_1 or Γ_4 goes vertical at all, then there is a separating set involving that arc, Γ_2 and Γ_3, violating Corollary 4.4.

By Proposition 7.1, every locally stable leaf of larger pressure can be modeled by one of the cases $(0,0)$, $(0,1)$, $(0,2)$ or $(2,1)$. We conclude that an arc of outer boundary cannot turn vertical downward after leaving any such leaf. □

Corollary 7.7. *In a minimizer, any arc of outer boundary between two leaves of the smaller region, from a circular cap to such a leaf, or between the two circular caps is graph.*

Proof. If such an arc goes vertical in its interior, then it is a nodoid or vertical line, by Theorem 3.3. By positive pressure, it must be a nodoid. Therefore it turns vertical downward after leaving either a circular cap or a leaf of the smaller region, contradicting Proposition 6.2 or Proposition 7.6, respectively. □

8. $1+k$ double bubbles.

For any minimal $1+k$ double bubble, i.e., an area-minimizing nonstandard double bubble in which one region is connected and the other region has k components, by Theorem 3.2 the associated tree T has just one branch from the root and $k-1$ leaves above the connected middle region, as in Figure 20.

Proposition 8.1. *In a $1+k$ minimizer, $k>1$, in which the larger region is connected, there can be no leaf with left rotation number $m_1 \geq 3$ or with right rotation number $m_2 \leq -3$. In particular, with the standard notation of Figure 13, each leaf has rotation pair*

$$(m_1, m_2) \in \{(0,0), (-1,0), (0,1), (-2,0), (0,2), \pm(1,2), \pm(2,1)\}.$$

Proof. Assume that there is a leaf with $m_2 \leq -3$. Let m_3 measure the rotation of the right endpoint of Γ_4.

Suppose Γ_4 connects to another leaf. If $m_2 = -3$ or -4, then by positive pressure Γ_4 is a convex nodoid and $m_3 \leq -3$. If $m_2 \leq -5$, then by Corollary 7.7 $m_3 \leq -5$. In either case, by Proposition 7.1, m_4 the right rotation number of this adjacent leaf satisfies $m_4 \leq -3$ also. By induction, every leaf combinatorially to the right of the original leaf has right rotation number at most -3.

Hence we may assume that Γ_4 connects to the right circular cap. By positive pressure, $m_3 \leq -2$, beyond the range of $\{-1, 0, 1\}$ allowed by Proposition 6.1.

From this contradiction, it follows that $m_2 \geq -2$ for each leaf. Symmetrical considerations give $m_1 \leq 2$ for each leaf. The second assertion follows from applying these inequalities to the possible (m_1, m_2) pairs of Proposition 7.1. □

Define a leaf to be *near graph* if it has rotation pair $(m_1, m_2) \in \{(-1, 0), (0, 0), (0, 1)\}$, as in Figure 14.

Proposition 8.2. *Consider a* $1 + k$ *minimizer, with the notation of Figure* 20*, in which the first* j*,* $0 \leq j \leq k - 1$*, leaves on the left are near graph. If* $m_1 \in \{-1, 0\}$ *— necessarily true if* $j > 0$*, or if* $j = 0$ *and* $m_2 \leq 1$ *— then* $m_{2k} \in \{-1, 0\}$ *and* $0B \leq (j + 1)B$*.*

Proof. By Corollary 7.7, $\Gamma_1, \ldots, \Gamma_k$, the outer boundaries of the middle component indexed from left to right, are graph.

If $m_2 \leq 1$ and $m_1 = 1$, then Γ_1 turns vertical in its interior, a contradiction. Hence $m_2 \leq 1$ — trivially true if $j > 0$ — implies $m_1 \in \{-1, 0\}$, the only remaining possibilities of Proposition 6.1.

Suppose $m_1 \in \{-1, 0\}$; by Proposition 6.1, $m_{2k} \in \{-1, 0\}$. If $m_1 = -1$, then $[-\infty, 0B) \subset f(\Gamma_0)$. If $m_1 = 0$, then consideration of v_{01} gives $0A < 1A$, implying by Corollary 4.4 for Γ_0, Γ_1 that $0A \leq 1B$. In either case, Corollary 4.4 for Γ_0, Γ_1 gives $0B \leq 1B$, the statement for $j = 0$.

Now assume $j > 0$. Suppose $0B \leq iB$, where $1 \leq i \leq j$. By Lemma 7.5 with relabelling, $iB < (i + 1)A$. Hence $0B < (i + 1)A$, implying by Corollary 4.4 for Γ_0, Γ_{i+1} that $0B \leq (i+1)B$. The statement follows by induction in i. □

Corollary 8.3. *A* $1 + k$ *minimizer must have at least one leaf above the middle component which is not near graph. In particular, a nonstandard* $1 + 1$ *double bubble cannot be minimizing.*

Proof. Suppose all the leaves are near graph (true if $k = 1$, when there are no leaves). By Proposition 8.2 applied once to each side, $m_1 = m_2 = 0$, and $0B \leq kB$. Consideration of v_{0k} (the right vertex, if $k = 1$) gives $kB < 0B$, a contradiction. □

Lemma 8.4. *A* $1+k$ *minimizer includes at most one* $(2,1)$*-modeled leaf: Cases* $\pm(1,2)$ *and* $\pm(2,1)$*.*

Proof. In a $(2,1)$-modeled leaf as in Figure 16, $\infty \in f(\Gamma_2) \cap f(\Gamma_3)$. If the minimizer includes more than one such leaf, $\{f^{-1}(\infty)\}$ separates Γ, violating Corollary 4.4. □

Lemma 8.5. *In a* $1+k$ *minimizer in which the larger region is connected, only a* $(0,2)$ *leaf may be directly to the left of a* $(2,1)$ *leaf. Also, every* $(0,2)$ *leaf in the minimizer must be directly to the left of a* $(2,1)$ *leaf. The minimizer does not include any* $(1,2)$ *or* $(-2,-1)$ *leaves.*

Proof. Consider a $(2,1)$ leaf as in Figure 16, and assume that Γ_1 connects to another leaf. Let m_0 measure the rotation of the (combinatorially) left endpoint of Γ_1. By positive pressure, Γ_1 is a concave rightward nodoid, so $m_0 \geq 2$. By Proposition 8.1, the adjacent leaf has rotation pair $(0,2)$ or $(1,2)$, and Lemma 8.4 disallows the latter possibility. Hence indeed, if the minimizer contains a $(2,1)$ leaf, then either that leaf is the leftmost leaf in the minimizer or it is just to the right of a $(0,2)$ leaf.

Now assume there is a leaf with rotation pair $(m_1, m_2) \in \{(0,2),(1,2)\}$, as in Figure 15 or Figure 16 with reflection and relabelling. Let m_3 measure the rotation of the right endpoint of Γ_4.

If Γ_4 connects to the right circular cap, by Proposition 6.1 $m_3 \in \{-1,0,1\}$. Hence Γ_4 turns vertical downward after leaving the root, violating Proposition 6.2.

Therefore Γ_4 connects to another leaf. By positive pressure, Γ_4 is a concave rightward nodoid, implying $m_3 \leq 2$. For $m_3 \leq 1$, Γ_4 turns vertical downward after leaving the adjacent leaf, violating Proposition 7.6. Thus $m_3 = 2$, whence by Proposition 8.1 the adjacent leaf has rotation pair $(2,1)$.

Therefore, if $(m_1, m_2) = (0,2)$, then the leaf is directly to the left of a $(2,1)$ leaf, as asserted.

If on the other hand $(m_1, m_2) = (1,2)$, then again the leaf is directly to the left of a $(2,1)$ leaf. But now the minimizer includes both a $(1,2)$ and a $(2,1)$ leaf, contradicting Lemma 8.4. Hence the minimizer includes no $(1,2)$ leaves. By symmetry, nor does it include any $(-2,-1)$ leaves. □

Proposition 8.6. *A nonstandard double bubble in* $\mathbf{R^n}$, $n \geq 3$, *in which the larger region is connected and the smaller region has* $k \geq 1$ *components is not minimizing.*

Proof. Suppose otherwise and consider the generating curves of the minimizer.

By Corollary 8.3, the minimizer includes at least one leaf which is not near graph. By Proposition 8.1, the possibilities for this leaf, up to horizontal reflection, are $(0,2)$, $(1,2)$ or $(2,1)$. Lemma 8.5 rules out case $(1,2)$ (and,

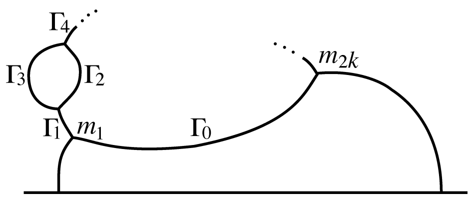

Figure 21. If the $(2, 1)$ leaf is the leftmost leaf, then all other leaves are near graph.

symmetrically, case $(-2, -1)$). If the minimizer includes a $(0, 2)$ leaf, then there is a $(2, 1)$ leaf directly to its right, also by Lemma 8.5.

Hence, after horizontal reflection if necessary, the minimizer includes at least one $(2, 1)$ leaf. By Lemma 8.4, the minimizer includes exactly one $(2, 1)$ leaf, and no $(-1, -2)$ leaves. Therefore, since by Lemma 8.5 every $(0, 2)$ leaf is directly to the left of a $(2, 1)$ leaf, there is at most one $(0, 2)$ leaf, and no $(-2, 0)$ leaves. By Lemma 8.5, if there is a leaf to the left of the $(2, 1)$ leaf, then it is the $(0, 2)$ leaf. If there is no leaf to the left of the $(2, 1)$ leaf, then there are no $(0, 2)$ leaves.

All leaves not directly to the left of the $(2, 1)$ leaf must be near graph, the only possibilities allowed by Proposition 8.1 which still remain.

First, assume the $(2, 1)$ leaf is the leftmost leaf, as in Figure 21. By Corollary 7.7, Γ_1 is graph, implying by Proposition 6.1 that $m_1 = 1$. Consider the downward normal n to Γ_4 at v_{234}. By Lemma 3.1 and Theorem 3.3, Γ_2 is a concave leftward nodoid, so n stays above and to the right of Γ_2. By Corollary 7.4, Γ_3 leaves v_{234} below the horizontal, implying that n is counterclockwise from the downward tangent to Γ_1 at v_{01}. Since by positive pressure Γ_1 is a concave rightward nodoid, n is counterclockwise from every downward tangent to Γ_1. Therefore, n stays above Γ_1 and $0A < 4A$. But by Proposition 8.2 applied from the right, $4A \leq 0A$, a contradiction. Hence the $(2, 1)$ leaf is not the leftmost leaf.

Second, assume there is a $(0, 2)$ leaf to the left of the $(2, 1)$ leaf, with the notation of Figure 22. Again, similar arguments using Corollary 7.4 show that the downward normal n to Γ_7 at v_{567} stays to the right of Γ_4. Since v_{123} is left of v_{234}, n is counterclockwise and to the right of the downward normal to Γ_1 at v_{123}, whence $1B < 7A$. But by Proposition 8.2 applied once to each side, $m_1 = m_{2k} = 0$, $0B \leq 1B$ and $7A \leq 0A$. Combining the inequalities yields $0B < 0A$, a clear impossibility when $m_1 = m_{2k} = 0$. Hence there can be no leaf to the left of the $(2, 1)$ leaf.

Therefore, a $1 + k$ minimizer cannot include a $(2, 1)$ leaf, a contradiction. Thus indeed, a $1 + k$ bubble in which the larger region is connected cannot be minimizing. $\square$

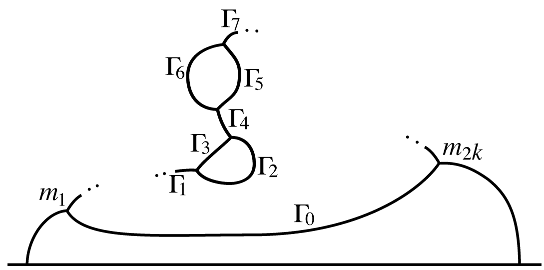

Figure 22. If the $(2,1)$ leaf is just to the right of a $(0,2)$ leaf, then all other leaves are near graph.

9. Proofs of the Double Bubble Conjecture.

Theorem 9.1 (Double Bubble Conjecture in $\mathbf{R^4}$)**.** *In* $\mathbf{R^4}$*, the standard double bubble is the unique area-minimizing double bubble.*

Proof. For equal volumes, both regions are connected by Proposition 2.5. By Corollary 8.3, the area-minimizing double bubble is the standard double bubble.

For unequal volumes, the larger region is connected by Proposition 2.5, and the smaller region has a finite number of components by Corollary 2.4. By Proposition 8.6, the area-minimizing double bubble is the standard double bubble. □

Theorem 9.2 (Double Bubble Conjecture for disparate volumes)**.** *In* $\mathbf{R^n}$, $n \geq 3$*, the standard double bubble is the unique area-minimizing double bubble for prescribed volumes* v, w, *with* $v > 2w$.

Proof. The larger region is connected by Corollary 2.2, and the smaller region has a finite number of components by Corollary 2.4. By Proposition 8.6, an area-minimizing double bubble must be the standard double bubble. □

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