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SIXTEEN-DIMENSIONAL LOCALLY COMPACT TRANSLATION PLANES ADMITTING $\mathbf{SL_2}\mathbb{H}$ AS A GROUP OF COLLINEATIONS

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We give an explicit description of all 16-dimensional locally compact translation planes admitting the unimodular quaternion group $\mathbf{SL_2}\mathbb{H}$ as a group of collineations. Moreover, we shall also determine the full collineation groups of these planes.

1. Introduction.

In this paper, all 16-dimensional locally compact translation planes admitting the unimodular quaternion group $\mathrm{SL}_2\,\mathbb{H}$ as a group of collineations will be determined explicitly. Besides the classical plane over the octonions there are a vast number of planes having this property, cf. the Classification Theorem (2.8). Indeed, the class of these planes covers an interesting borderline case: Among all 16-dimensional locally compact translation plane, only the classical plane admits the action of a noncompact almost simple Lie group of dimension larger than $\dim \mathrm{SL}_2\,\mathbb{H} = 15$, cf. [**7**, Theorem A].

The connected component $\mathbb{G}^e$ of the automorphism group $\mathbb{G}$ of a nonclassical example is composed of the translation group, the group of homotheties, the group $\mathrm{SL}_2\,\mathbb{H}$, and a compact group Δ isomorphic to $\{e\}, \mathrm{SO}_2\,\mathbb{R}$, $\mathrm{SO}_2\,\mathbb{R} \times \mathrm{SO}_2\,\mathbb{R}$, or $\mathrm{SU}_2\,\mathbb{C}$, cf. Theorem 3.8. Thus, $\dim \mathbb{G}$ is at most 35.

It is worth mentioning that $\Gamma = \mathbb{G}^e$ leaves precisely one projective line (namely the translation axis) invariant, but does not fix any projective points. In general, a 16-dimensional compact projective plane whose automorphism group contains a closed connected subgroup Γ having this property and satisfying $\dim \Gamma \geq 35$ is necessarily a translation plane, thanks to a theorem of H. Salzmann [**10**]. Recently, H. Hähl has shown in [**4**] that there are precisely three families of such planes: A subfamily of the planes considered here[1], and the planes admitting $\mathrm{SU}_4\,\mathbb{C}\cdot\mathrm{SU}_2\,\mathbb{C}$ or $\mathrm{SU}_4\,\mathbb{C}\cdot\mathrm{SL}_2\,\mathbb{R}$ as a group of collineations, determined in Hähl [**5**]. In particular, $\dim \Gamma \geq 36$ implies that the plane is isomorphic to the octonion plane.

[1] More precisely: The planes for which the group Δ mentioned above equals $\mathrm{SU}_2\,\mathbb{C}$; see 3.8(2) for further details.

Organization. The second section is devoted to the proof of the Classification Theorem (2.8) which is based on the general theory of noncompact semisimple groups acting on locally compact translation planes. (See [**7**] and [**8**], and compare 2.2, 2.3 and 2.5 for the particular applications.)

In 2.8, we shall assign to each continuous function $\sigma : \mathrm{Spin}(3) \to \mathbb{R}_{\mathrm{pos}}$ a 16-dimensional translation plane $\mathcal{P}^\sigma$ admitting the action of $\mathrm{SL}_2\,\mathbb{H}$. One of the quasifields belonging to such a plane $\mathcal{P}^\sigma$ will be obtained in 2.11.

In 3.7 we shall give a necessary and sufficient condition for two functions to define isomorphic planes. Finally, we determine the automorphism groups in 3.8 by computing the reduced stabilizer $\mathrm{S}\mathbb{G}_0$ of each plane $\mathcal{P}^\sigma$. With the exception of the octonion plane, the automorphism groups of the planes under consideration have dimension at most 35.

1.1. Notation. Let Spin(3) denote the group of quaternions of length 1.

For $\vec{x}, \vec{y} \in \mathbb{H}^2$, we put $\langle \vec{x}, \vec{y} \rangle := x_1\overline{y_1} + x_2\overline{y_2}$. For the orthogonal complement of a subspace X with respect to this scalar product we shall write $X^\perp$.

If A is an element of $\mathrm{SL}_2\,\mathbb{H}$, then A^* denotes the inverse of the adjoint map of A with respect to $\langle \cdot, \cdot \rangle$, i.e., $A^* := (\overline{A}^t)^{-1}$. We emphasize that we have

$$\langle \vec{x} \cdot A^*, \vec{y} \cdot A \rangle = \langle \vec{x}, \vec{y} \rangle \text{ for all } \vec{x}, \vec{y} \in \mathbb{H}^2, A \in \mathrm{SL}_2\,\mathbb{H}. \tag{1}$$

Let $\mathrm{SH}_2^+\mathbb{H}$ be the set of positive definite Hermitian (2×2)-matrices over $\mathbb{H}$ with determinant 1. Notice that $\mathrm{SH}_2^+\mathbb{H}$ coincides with the set of all $(A^*)^{-1}A = \overline{A}^t A$, $A \in \mathrm{SL}_2\,\mathbb{H}$. (Recall the polar decomposition of unimodular matrices.)

Finally, $\mathrm{diag}(x_1, \ldots, x_n)$ denotes a diagonal matrix with the given entries.

2. The classification.

2.1. The general situation. We consider a 16-dimensional locally compact affine translation plane $(P, \mathcal{L})$ which is represented in the usual[2] way: The point space P is a 16-dimensional real vector space, the line pencil $\mathcal{L}_0$ through the origin consists of 8-dimensional vector subspaces of P, and the other lines are the affine cosets of the elements of $\mathcal{L}_0$. Moreover, the spread $\mathcal{L}_0$ is a compact subset of the Grassmannian manifold of all 8-dimensional vector subspaces of P. In fact, $\mathcal{L}_0$ is homeomorphic to the 8-sphere.

The group $\mathbb{G}$ of all automorphisms (i.e., continuous collineations) is a semidirect product $\mathbb{G} = \mathbb{G}_0 \ltimes T$ of the translation group T (which coincides with the group of all vector translations of P) and the stabilizer $\mathbb{G}_0$ of the origin. The latter group is a closed subgroup of $\mathrm{GL}(P)$ and, hence, is a Lie group.

[2]Basic facts concerning 16-dimensional locally compact translation planes are collected in Chapter 8 of [**11**]; results used without a reference can be found there.

2.2. The group action of $\mathrm{SL}_2\,\mathbb{H}$ on $(P, \mathcal{L})$. Throughout this paper we suppose that $\mathrm{SL}_2\,\mathbb{H}$ acts on the translation plane $(P, \mathcal{L})$ as a group of collineations, i.e., we have a Lie homomorphism $\Phi : \mathrm{SL}_2\,\mathbb{H} \to \mathbb{G}$ with discrete kernel. Since $\mathbb{G}$ is an almost direct product of $\mathbb{G}_0$ and the abelian translation group, we may assume that the image

$$\Lambda = \Phi(\mathrm{SL}_2\,\mathbb{H})$$

is contained in $\mathbb{G}_0$. In fact, Φ is a representation of $\mathrm{SL}_2\,\mathbb{H}$ on P.

According to [**7**, 6.8], Φ is a direct product of the obvious representation of $\mathrm{SL}_2\,\mathbb{H}$ on $\mathbb{H}^2$ and the contragredient representation on $\mathbb{H}^2$. Therefore, P and the left quaternion vector space $\mathbb{H}^4$ can be identified in such a way that the representation Φ of $\mathrm{SL}_2\,\mathbb{H}$ on $P = \mathbb{H}^4$ is given by right multiplication with the matrices

$$\Phi(A) = \begin{pmatrix} A^* & 0 \\ 0 & A \end{pmatrix} \text{ for } A \in \mathrm{SL}_2\,\mathbb{H}.$$

We emphasize that the Λ-invariant subspaces $\mathbb{H}^2 \times \{\vec{0}\}$ and $\{\vec{0}\} \times \mathbb{H}^2$ are *not* elements of $\mathcal{L}_0$, since the noncompact almost simple group Λ does not fix any affine line, cf. [**7**, Theorem B].

2.3. The weight sphere. We apply the general theory of noncompact almost simple subgroups of $\mathbb{G}_0$ (for which [**7**] contains the details) to our particular case: Being the image of $\mathrm{diag}(-1, 1) \in \mathfrak{sl}_2\mathbb{H}$ under the derivative of Φ, the real diagonal matrix $d := \mathrm{diag}(1, -1, -1, 1)$ is an element of the Lie algebra $\mathfrak{L}\Lambda$.

Since d has precisely two eigenvalues, [**7**, 5.3] implies that both eigenspaces of d are elements of the spread $\mathcal{L}_0$. Collecting all the eigenspaces of all real diagonalizable elements of $\mathfrak{L}\Lambda$ yields the so-called *weight sphere* $\mathcal{S} \subseteq \mathcal{L}_0$ of Λ, see [**7**] for details. The main result [**7**, Theorem B] concerning the weight sphere asserts that Λ acts transitively on it. Therefore, $\mathcal{S}$ is the Λ-orbit of the eigenspace $E := \mathbb{H} \cdot (1, 0) \times \mathbb{H} \cdot (0, 1)$ of d with respect to 1.

Lemma 2.4.

(a) *The weight sphere $\mathcal{S}$ of Λ consists precisely of the subspaces $X \times X^\perp$, where X is a 1-dimensional $\mathbb{H}$-linear subspace of $\mathbb{H}^2$.*

(b) *A vector $(\vec{x}, \vec{y}) \in \mathbb{H}^2 \times \mathbb{H}^2$ is contained in some element of $\mathcal{S}$ if and only if $\vec{x}$ is perpendicular to $\vec{y}$.*

Proof. (a) Let $G_1\mathbb{H}^2$ be the set of 1-dimensional $\mathbb{H}$-linear subspaces of $\mathbb{H}^2$. We have to show that the sets $\mathcal{S}$ and $\mathcal{S}' := \{X \times X^\perp \mid X \in G_1\mathbb{H}^2\}$ coincide. For this, let $X \in G_1\mathbb{H}^2$ and $A \in \mathrm{SL}_2\,\mathbb{H}$. Derive $(XA^*)^\perp = X^\perp A$ from Equation (1) in 1.1. This shows that

$$(X \times X^\perp)\Phi(A) = XA^* \times X^\perp A = XA^* \times (XA^*)^\perp,$$

whence $\mathcal{S}'$ is Λ-invariant. In fact, $\mathcal{S}'$ is a Λ-orbit, because $\mathrm{SL}_2\,\mathbb{H}$ acts transitively on $G_1\mathbb{H}^2$. We already know that the weight sphere $\mathcal{S}$ is a Λ-orbit, too. Since $E = \mathbb{H}\cdot(1,0)\times\mathbb{H}\cdot(0,1)$ is an element of $\mathcal{S}\cap\mathcal{S}'$, we conclude that $\mathcal{S} = \mathcal{S}'$. Part (b) is an easy consequence of (a). □

2.5. Stabilizers. Let $\mathcal{O}$ be an orbit of Λ in $\mathcal{L}_0$ different from $\mathcal{S}$. From [**8**, 6.2.b] we infer that the stabilizer of an element of $\mathcal{O}$ is a compact group. Thus, there exists a line $L \in \mathcal{O}$ such that Λ_L is contained in the particular maximal compact subgroup $\Delta := \Phi(\mathrm{U}_2\mathbb{H})$ of Λ.

Being a subset of the 8-sphere $\mathcal{L}_0$, the orbit $\mathcal{O}$ has dimension at most 8. Applying Halder's dimension formula yields

$$\dim \Lambda_L = \dim \Lambda - \dim L^{\Lambda} \geq 15 - 8 = 7.$$

This implies that Λ_L equals Δ, because $\mathrm{U}_2\mathbb{H}$ does not contain proper subgroups of dimension at least 7: The rank of a subgroup Σ of $\mathrm{U}_2\mathbb{H}$ is at most 2. Moreover, Σ is not isomorphic to the group $\mathrm{SU}_3\,\mathbb{C}$ which does not have a representation of the quaternion vector space $\mathbb{H}^2$, cf. [**11**, 95.10]. Checking all compact groups of rank at most 2 yields the assertion.

2.6. Δ-invariant subspaces and their orbits. Note that a matrix $B \in \mathrm{SL}_2\,\mathbb{H}$ is an element of $\mathrm{U}_2\mathbb{H}$ if and only if $B^* = B$, whence we obtain

$$\Phi(B) = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \text{ for all } B \in \mathrm{U}_2\mathbb{H}.$$

Thus, the restriction of Φ to $\mathrm{U}_2\mathbb{H}$ is a direct sum of two copies of the irreducible representation of $\mathrm{U}_2\mathbb{H}$ on $\mathbb{H}^2$. By [**2**, p. 43, Prop. 6], precisely the following proper $\mathbb{R}$-linear subspaces of P are invariant under $\Delta = \Phi(\mathrm{U}_2\mathbb{H})$:

$$U_h := \{(\vec{x}, h\vec{x}) \mid \vec{x} \in \mathbb{H}^2\} \text{ for } h \in \mathbb{H} \quad \text{and} \quad U_\infty := \{\vec{0}\} \times \mathbb{H}^2.$$

We compute the image of U_h, $h \in \mathbb{H}$, under $\Phi(A)$, $A \in \mathrm{SL}_2\,\mathbb{H}$:

$$U_h^{\Phi(A)} = \{(\vec{x}A^*, h\vec{x}A) \mid \vec{x} \in \mathbb{H}^2\} = \{(\vec{y}, h\vec{y}\,\overline{A}^t A) \mid \vec{y} \in \mathbb{H}^2\}.$$

Recall that $\overline{A}^t A$ is an element of $\mathrm{SH}_2^+\mathbb{H}$ and that every element of $\mathrm{SH}_2^+\mathbb{H}$ has this form. Therefore, the Λ-orbit U_h^Λ consists precisely of the subspaces

$$\{(\vec{x}, h\vec{x}S) \mid \vec{x} \in \mathbb{H}^2\} \text{ where } S \in \mathrm{SH}_2^+\mathbb{H}.$$

Lemma 2.7.

(a) *A nonzero vector $(\vec{x}, \vec{y}) \in \mathbb{H}^2 \times \mathbb{H}^2$ is contained in an element of the Λ-orbit U_h^Λ if and only if $\langle \vec{x}, \vec{y}\rangle = r\bar{h}$ holds for some $r \in \mathbb{R}_{\mathrm{pos}}$.*

(b) *For every $h \in \mathbb{H}^\times$ the set $\mathcal{S} \cup U_h^\Lambda$ is a partial spread.*

(c) *If h and l are distinct nonzero quaternions, then $U_h^\Lambda \cup U_l^\Lambda$ is a partial spread if and only if $h/|h| \neq l/|l|$.*

Proof. (a) Let $A \in \mathrm{SL}_2\,\mathbb{H}$. Then every nonzero vector belonging to the element $\{(\vec{x}A^*, h\vec{x}A) \mid \vec{x} \in \mathbb{H}^2\}$ of U_h^Λ has the desired property, because $\langle \vec{x}A^*, h\vec{x}A\rangle = \langle \vec{x}, h\vec{x}\rangle = \|\vec{x}\|^2 \cdot \bar{h}$. Conversely, let $(\vec{x}, \vec{y})$ be an element of $\mathbb{H}^2 \times \mathbb{H}^2$ such that $\langle \vec{x}, \vec{y}\rangle = r\bar{h}$ holds for some $r \in \mathbb{R}_{\mathrm{pos}}$. Without loss of generality we may assume $\vec{x} = (1, 0)$ (otherwise, replace $(\vec{x}, \vec{y})$ by $(\vec{x}B^*, \vec{y}B)$, where $B \in \mathrm{SL}_2\,\mathbb{H}$ satisfies $\vec{x}B^* = (1, 0)$). Then $\langle \vec{x}, \vec{y}\rangle = r\bar{h}$ implies that $\vec{y} = (rh, l)$ holds for some $l \in \mathbb{H}$. Put

$$A := \begin{pmatrix} \sqrt{r^{-1}} & -\sqrt{r^{-1}}h^{-1}l \\ 0 & \sqrt{r} \end{pmatrix}$$

and observe that $\Phi(A)$ maps $(\vec{x}, \vec{y}) = (1, 0, rh, l)$ to the element $(\sqrt{r}, 0, \sqrt{r}h, 0)$ of U_h. This proves the claim.

(b) It is easy to see that the weight sphere $\mathcal{S}$ is a partial spread. If U_h and $U_h^{\Phi(A)}$, $A \in \mathrm{SL}_2\,\mathbb{H}$, have some nonzero vector $(\vec{x}, \vec{y})$ in common, then $\vec{y} = h\vec{x} = h\vec{x}\overline{A}^t A$ implies that 1 is an eigenvalue of the positive definite Hermitian (2×2)-matrix $\overline{A}^t A$. Since $\det \overline{A}^t A = 1$, we derive that $A^* = A$ is an element of $\mathrm{U}_2\mathbb{H}$, whence U_h and $U_h^{\Phi(A)}$ coincide. Consequently, U_h^Λ is a partial spread. Moreover, the sets of nonzero vectors covered by $\mathcal{S}$ and U_h^Λ, respectively, are disjoint, cf. (a) and 2.4(b). This proves the assertion.

(c) If $h/|h| \neq l/|l|$, then (a) shows that the sets of nonzero vectors covered by the partial spreads U_h^Λ and U_l^Λ, respectively, are disjoint. Thus, $U_h^\Lambda \cup U_l^\Lambda$ is a partial spread, too. For the converse direction we suppose that $h/|h| = l/|l|$. Then U_h^Λ and U_l^Λ are partial spreads covering the same set of vectors, thanks to (a), hence their union fails to be a partial spread unless $U_h^\Lambda = U_l^\Lambda$. The latter condition implies that U_l equals $U_h^{\Phi(A)}$ for some $A \in \mathrm{SL}_2\,\mathbb{H}$, and consequently $l\vec{x} = h\vec{x}\overline{A}^t A$ holds for all $\vec{x} \in \mathbb{H}^2$. This implies that every vector is an eigenvector of $\overline{A}^t A$ with respect to the eigenvalue $h^{-1}l$. Since A is unimodular, we derive that $l = h$. □

Classification Theorem 2.8. *Let $\sigma : \mathrm{Spin}(3) \to \mathbb{R}_{\mathrm{pos}}$ be a continuous function. Then the set*

$$\mathcal{L}_0^\sigma := \mathcal{S} \cup \left\{ U_{\sigma(p)p}^\lambda \,\middle|\, p \in \mathrm{Spin}(3), \lambda \in \Lambda \right\}$$

is a Λ-invariant compact spread on $P = \mathbb{H}^2 \times \mathbb{H}^2$ and, hence, defines a 16-dimensional locally compact translation plane $\mathcal{P}^\sigma$ whose automorphism group contains the group $\Lambda \cong \mathrm{SL}_2\,\mathbb{H}$. (Recall the definition of Λ, $\mathcal{S}$ and U_h in 2.2, 2.4 and 2.6, respectively.)

Conversely, if $\mathcal{P}$ is a 16-dimensional locally compact translation plane admitting the group $\mathrm{SL}_2\,\mathbb{H}$ as a group of collineations, then $\mathcal{P}$ is isomorphic to $\mathcal{P}^\sigma$ for some continuous function $\sigma : \mathrm{Spin}(3) \to \mathbb{R}_{\mathrm{pos}}$.

Proof. (1) The set $\mathcal{L}_0^\sigma$ is a partial spread, thanks to 2.7(b) and 2.7(c). Let $(\vec{x},\vec{y})$ be an element of P with $\langle\vec{x},\vec{y}\rangle = h$. If h vanishes, then $(\vec{x},\vec{y})$ is covered by $\mathcal{S}$, cf. 2.4. For $h\neq 0$ put $l=\sigma(\bar{h}/|h|)\cdot\bar{h}/|h|$ and use 2.7(a) to infer that $(\vec{x},\vec{y})$ is contained in an element of U_l^Λ. Thus, $\mathcal{L}_0^\sigma$ covers P and we have shown that $\mathcal{L}_0^\sigma$ is a spread.

(2) If we can prove that $\mathcal{L}_0$ is closed in the Grassmannian manifold of all 8-dimensional subspaces of P, then $\mathcal{L}_0$ is a compact spread and, hence, $\mathcal{P}^\sigma$ is a locally compact translation plane. Therefore, we consider a sequence $(L_i)_i$, $L_i\in\mathcal{L}_0$, which is convergent to some 8-dimensional vector subspace $L\leq P$.

If L_i is an element of the compact weight sphere $\mathcal{S}$ for infinitely many i, then also L is an element of $\mathcal{S}$. Thus, we may assume that $L_i\in\mathcal{L}_0^\sigma\setminus\mathcal{S}$ holds for all i. By the definition of $\mathcal{L}_0^\sigma$, we have that $L_i=U_{h_i}^{\lambda_i}$, where $\lambda_i\in\Lambda$ and where $h_i=\sigma(p_i)p_i$ for some $p_i\in\mathrm{Spin}(3)$.

For $r\in\mathbb{R}_{\mathrm{pos}}$ we put $\rho(r):=\Phi(\mathrm{diag}(r,r^{-1}))=\mathrm{diag}(r^{-1},r,r,r^{-1})$. By the KAK-decomposition [**6**, 7.39] of $\mathrm{SL}_2\,\mathbb{H}$, there are $\gamma_i,\delta_i\in\Delta$, $r_i\in\mathbb{R}_{\mathrm{pos}}$ such that $\lambda_i=\gamma_i\rho(r_i)\delta_i$. Note that Δ and $\mathrm{Spin}(3)$ are compact and that σ is continuous. By passing to a subsequence we may achieve the following:

(a) p_i is convergent to $p\in\mathrm{Spin}(3)$, whence $\sigma(p_i)p_i$ is convergent to $h:=\sigma(p)p$,
(b) δ_i is convergent to $\delta\in\Delta$,
(c) r_i is convergent to $r\in\mathbb{R}_{\mathrm{pos}}\cup\{0,\infty\}$, and
(d) $U_{h_i}^{\rho(r_i)}$ is convergent to some 8-dimensional vector subspace K of P.

We claim that K is an element of $\mathcal{L}_0^\sigma$ — then $L=\lim\limits_{i\to\infty}U_{h_i}^{\lambda_i}=\lim\limits_{i\to\infty}(U_{h_i}^{\rho(r_i)})^{\delta_i}=K^\delta$ is an element of $\mathcal{L}_0^\sigma$ as well and we are done. If $r\notin\{0,\infty\}$, then it is easy to see that $K=U_h^{\rho(r)}$. If $r=0$, then we have that

$$\begin{aligned}
K &= \lim_{i\to\infty}U_{h_i}^{\rho(r_i)}=\lim_{i\to\infty}\{(r_i^{-1}x,r_iy,r_ih_ix,r_i^{-1}h_iy)\mid x,y\in\mathbb{H}\}\\
&=\lim_{i\to\infty}\{(u,r_i^2v,r_i^2h_iu,h_iv)\mid u,v\in\mathbb{H}\}\\
&=\{(u,0,0,hv)\mid u,v\in\mathbb{H}\}\\
&=(\mathbb{H}\cdot(1,0))\times(\mathbb{H}\cdot(1,0))^\perp,
\end{aligned}$$

whence K is an element of $\mathcal{S}$. The case $r=\infty$ can be treated analogously.

(3) Let $\mathcal{P}$ be a 16-dimensional locally compact translation plane admitting the group $\mathrm{SL}_2\,\mathbb{H}$ as a group of automorphisms. By 2.2 we can identify P and $\mathbb{H}^4$ such that the defining spread $\mathcal{L}_0$ of $\mathcal{P}$ is Λ-invariant. Every element of $\mathcal{L}_0$ is either an element of the weight sphere $\mathcal{S}$ or is contained in an orbit U_h^Λ for some $h\in\mathbb{H}^\times$, thanks to 2.3 and 2.5. Combine 2.7(a) and 2.7(c) to infer the following: For every $p\in\mathrm{Spin}(3)$ there exists precisely one $r\in\mathbb{R}_{\mathrm{pos}}$ such that U_{rp}^Λ is a subset of $\mathcal{L}_0$. Putting $\sigma(p):=r$ we obtain a

function σ : Spin(3) $\to \mathbb{R}_{\mathrm{pos}}$ and observe $\mathcal{L}_0 = \mathcal{L}_0^\sigma$. It remains to show the continuity of σ. For this, consider a sequence $(p_i)_i$ in Spin(3) which converges to p. In order to check $\lim_{i\to\infty} \sigma(p_i) = \sigma(p)$ we prove that $\sigma(p)$ is the only accumulation point of $(\sigma(p_i))_i$ in the interval $[0, \infty]$: Let r be such an accumulation point. It is easy to see that $(U_{\sigma(p_i)p_i})$ is convergent to U_{rp} in the Grassmannian topology. Since $\mathcal{L}_0$ is compact, we infer that U_{rp} is an element of $\mathcal{L}_0$ and, hence, that $r = \sigma(p)$. □

Remark 2.9. The projective closures of the planes specified in 2.8 yield all 16-dimensional compact projective translation planes admitting the group $\mathrm{SL}_2\,\mathbb{H}$ as a group of collineations: According to [**11**, 64.4.c], such a plane either is classical, or the translation axis is invariant under all automorphisms, whence the group $\mathrm{SL}_2\,\mathbb{H}$ acts on the affine part as well.

Remark 2.10. The classification of 4-dimensional translation planes admitting the group $\mathrm{SL}_2\,\mathbb{R}$ as a group of collineations is due to D. Betten, see [**11**, 73.13 and 73.19] for the results. Note that there is an example with an irreducible $\mathrm{SL}_2\,\mathbb{R}$-action. The 8-dimensional translation planes admitting an $\mathrm{SL}_2\,\mathbb{C}$-action were completely determined by H. Hähl, see [**3**].

2.11. Coordinatizing quasifields of $\mathcal{P}^\sigma$. We consider a function σ: Spin(3) $\to \mathbb{R}_{\mathrm{pos}}$. Our aim is to introduce coordinates[3] for the affine translation plane $\mathcal{P}^\sigma$ with respect to the triangle $o = (0,0,0,0)$, $w = (1,0,0,0)$, $s = (0,1,0,0)$. We claim that the resulting quasifield Q^σ is obtained as follows: For $h \in \mathbb{H}$, we put

$$\zeta(h) := \begin{cases} 0 & \text{if } h = 0, \\ \sigma(-h/|h|)^{-2}h & \text{if } h \neq 0. \end{cases}$$

Then the quasifield in question is $Q^\sigma = \mathbb{H}^2$ with its natural addition, while the multiplication is given by

$$(h,l) \circ_\sigma (x,y) := (xh - \overline{\zeta(l)}y, lx + y\bar{h}) \text{ for } (h,l),(x,y) \in \mathbb{H}^2.$$

The line $G_{(h,l)} \in \mathcal{L}_0$ with slope $(h,l) \in \mathbb{H}^2$ is given by

$$G_{(h,l)} = \{(x, xh - \overline{\zeta(l)}y, -lx - y\bar{h}, y) \mid x, y \in \mathbb{H}^2\};$$

notice that $o \vee (w+s) = G_{(1,0)} = \{(x,y,-x,y) \mid x,y \in \mathbb{H}\}$. Moreover, the vertical axis equals

$$G_\infty = o \vee s = \{0\} \times \mathbb{H} \times \mathbb{H} \times \{0\}.$$

We have to show that $\mathcal{L}_0^\sigma = \{G_z \mid z \in \mathbb{H}^2 \cup \{\infty\}\}$. To this end it suffices to prove that:

(1) $\mathcal{S} = \{G_{(h,0)} \mid h \in \mathbb{H}\} \cup \{G_\infty\}$, and
(2) $U^\Lambda_{-\sigma(-p)p} = \{G_{(h,rp)} \mid r \in \mathbb{R}_{\mathrm{pos}}, h \in \mathbb{H}\}$ for all $p \in$ Spin(3).

[3]For details on how to coordinatize translation planes by quasifields we refer to [**1**].

Property (1) can be easily derived from the following equations:

$$G_\infty = (\mathbb{H}\cdot(1,0)) \times (\mathbb{H}\cdot(1,0))^\perp$$
$$G_{(h,0)} = \{(x, xh, -y\bar{h}, y) \mid x, y \in \mathbb{H}\} = (\mathbb{H}\cdot(1,h)) \times (\mathbb{H}\cdot(1,h))^\perp,$$

recall the description of the elements of $\mathcal{S}$ in 2.3. We turn to (2): Directly from the definition of $U_{-\sigma(p)p}$ we see that

$$G_{(0,\sigma(-p)p)} = \{(x, -\sigma(-p)^{-1}\bar{p}y, -\sigma(-p)px, y) \mid x, y \in \mathbb{H}^2\} = U_{-\sigma(-p)p}.$$

Consider an element λ in Λ. By the Iwasawa decomposition [**6**, 6.46] of $\mathrm{SL}_2\,\mathbb{H}$, there are elements $B \in \mathrm{U}_2\mathbb{H}$, $s \in \mathbb{R}_{\mathrm{pos}}$, and $h \in \mathbb{H}$ such that

$$\Phi^{-1}(\lambda) = B \cdot \mathrm{diag}(s, s^{-1}) \cdot \begin{pmatrix} 1 & 0 \\ -\bar{h} & 1 \end{pmatrix}.$$

A short computation shows that $U^\lambda_{-\sigma(p)p} = G^\lambda_{(0,\sigma(-p)p)} = G_{(h,s^2\sigma(-p)p)}$, and Property (2) follows easily. This finishes the proof.

Corollary 2.12. *We consider the constant map $\sigma : \mathrm{Spin}(3) \to \mathbb{R}_{\mathrm{pos}}$; $p \mapsto 1$. Then $\mathcal{P}^\sigma$ is isomorphic to the affine plane over the octonions.*

Proof. The multiplication of the quasifield of $\mathcal{P}^\sigma$ determined in 2.11 is $(h,l)\circ(x,y) = (xh - \bar{l}y, lx + y\bar{h})$. Indeed, this is the multiplication of the division algebra $\mathbb{O}$. □

3. Isomorphisms and automorphisms.

3.1. General remarks. Let $\mathcal{P}$ be a 16-dimensional locally compact translation plane whose group $\mathbb{G}$ contains a subgroup Λ which is locally isomorphic to $\mathrm{SL}_2\,\mathbb{H}$. Following the previous section, we identify P and $\mathbb{H}^4$ such that Λ is the group specified in 2.2.

Moreover, let T be the group of vector translations of P and let Y be the group of homotheties of P with a positive real scalar. Putting $\mathrm{S}\mathbb{G}_0 := \mathbb{G}_0^e \cap \mathrm{SL}(P)$, we infer that $\mathbb{G}^e = (Y \times \mathrm{S}\mathbb{G}_0) \ltimes T$. (The exponent e refers to the connected component of a Lie group.) The group $\mathrm{S}\mathbb{G}_0$ is called the "reduced stabilizer" of $\mathcal{P}$, see [**11**, 81.0] for details. In particular, we have that

$$\dim \mathbb{G} = \dim \mathrm{S}\mathbb{G}_0 + \dim Y + \dim T = \dim \mathrm{S}\mathbb{G}_0 + 17.$$

Proposition 3.2. *Retain the notation above. If Λ is not normal in the reduced stabilizer $\mathrm{S}\mathbb{G}_0$, then $\mathcal{P}$ is isomorphic to the affine plane over the octonions. In every other case, $\mathrm{S}\mathbb{G}_0$ is an almost direct product $\mathrm{S}\mathbb{G}_0 = \Lambda \cdot \Psi$ of Λ and a compact connected subgroup Ψ of the centralizer of Λ in $\mathrm{GL}(P)$.*

Proof. Observe that $\mathrm{S}\mathbb{G}_0$ is a noncompact group which fixes no affine lines of $\mathcal{P}$, since its subgroup Λ has this property. According to [**8**, 1.1], $\mathrm{S}\mathbb{G}_0$ is an almost direct product of a almost simple Lie group S of real rank 1 and a compact group Ψ. From [**7**, Theorem B] we conclude that $\mathcal{P}$ is isomorphic to the octonion plane, or that $S = \Lambda$ is a normal subgroup of $\mathrm{S}\mathbb{G}_0$. In the latter case, Ψ indeed is a subgroup of the centralizer Ξ of Λ. $\square$

Remark 3.3. The reduced stabilizer of the octonion plane is isomorphic to the almost simple Lie group $\mathrm{Spin}_{10}(\mathbb{R}, 1)$. Thus, the group of affine automorphisms of the octonion plane has dimension $45 + 17 = 62$.

3.4. The normalizer of $\boldsymbol{\Lambda}$. We shall determine the normalizer Γ of Λ in $\mathrm{GL}(P)$. To this end we consider the automorphism group A of the Lie algebra $\mathfrak{L}\Lambda \cong \mathfrak{sl}_2\mathbb{H}$. Notice that the adjoint representation Ad is a Lie homomorphism from Γ to A whose kernel coincides with the centralizer Ξ of Λ in $\mathrm{GL}(P)$. Observe that the map

$$\iota : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}^2 \times \mathbb{H}^2; (\vec{x}, \vec{y}) \mapsto (\vec{y}, \vec{x})$$

is an element of Γ and that $\mathrm{Ad}\,\iota$ equals the automorphism $X \mapsto -\overline{X}^t$ of $\mathfrak{sl}_2\mathbb{H}$. From [**9**, §4(c)] we infer that $A = \mathrm{Ad}(\Lambda \cdot \langle\iota\rangle)$. (The group of inner automorphism has index 2 in A and $\mathrm{Ad}\,\iota$ is an outer automorphism.) Indeed, we have that

$$\Gamma = \langle\iota\rangle \cdot \Lambda \cdot \Xi. \tag{2}$$

The subrepresentations of Φ on $\mathbb{H}^2 \times \{\vec{0}\}$ and on $\{\vec{0}\} \times \mathbb{H}^2$ are inequivalent, irreducible quaternion representations. Thus, the centralizer Ξ of Λ consists precisely of the following maps:

$$\xi_{a,b} : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}^2 \times \mathbb{H}^2;\ (\vec{x}, \vec{y}) \mapsto (a \cdot \vec{x}, b \cdot \vec{y}) \text{ with } a, b \in \mathbb{H}^\times.$$

Proposition 3.5. *Let $\sigma, \tau : \mathrm{Spin}(3) \to \mathbb{R}_{\mathrm{pos}}$ be continuous maps and let $a, b \in \mathrm{Spin}(3)$, $r, s \in \mathbb{R}_{\mathrm{pos}}$. Then the following holds:*

- (a) *The map $\xi_{ra,sb} : (\vec{x}, \vec{y}) \mapsto (ra\vec{x}, sb\vec{y})$ is an isomorphism from $\mathcal{P}^\sigma$ onto $\mathcal{P}^\tau$ if and only if $\tau(h) = r^{-1}s\sigma(b^{-1}ha)$ holds for every $h \in \mathrm{Spin}(3)$.*
- (b) *The map $\iota\xi_{ra,sb} : (\vec{x}, \vec{y}) \mapsto (sb\vec{y}, ra\vec{x})$ is an isomorphism from $\mathcal{P}^\sigma$ onto $\mathcal{P}^\tau$ if and only if $\tau(h) = r^{-1}s[\sigma(a^{-1}h^{-1}b)]^{-1}$ holds for all $h \in \mathrm{Spin}(3)$.*

Proof. Let f be an element of $\langle\iota\rangle \cdot \Xi$. Observe that f leaves the weight sphere $\mathcal{S}$ invariant. Moreover, notice that f centralizes the maximal compact subgroup Δ of Λ. This implies that f is an isomorphism from $\mathcal{P}^\sigma$ onto $\mathcal{P}^\tau$ if and only if f maps every Δ-invariant line $U_{\sigma(h)h}$, $h \in \mathrm{Spin}(3)$, of $\mathcal{L}_0^\sigma$ to a Δ-invariant line $U_{\tau(l)l}$ of $\mathcal{L}_0^\tau$. A short computation shows:

$$\{(\vec{x}, \sigma(h)h\vec{x}) \mid \vec{x} \in \mathbb{H}^2\}^{\xi_{ra,sb}} = \{(\vec{y}, r^{-1}s\sigma(h)bha^{-1}\vec{y}) \mid \vec{y} \in \mathbb{H}^2\}$$
$$\{(\vec{x}, \sigma(h)h\vec{x}) \mid \vec{x} \in \mathbb{H}^2\}^{\iota\xi_{ra,sb}} = \{(\vec{y}, r^{-1}s\sigma(h)^{-1}bh^{-1}a^{-1}\vec{y}) \mid \vec{y} \in \mathbb{H}^2\}.$$

From these equations we infer easily the assertions of the proposition. $\square$

Proposition 3.6. *We consider a continuous function* $\sigma : \mathrm{Spin}(3) \to \mathbb{R}_{\mathrm{pos}}$. *Let* $\mathbb{G}_0$ *the stabilizer of the connected component of the automorphism group of* $\mathcal{P}^\sigma$. *Then the following statements are equivalent:*

(1) $\mathcal{P}^\sigma$ *is isomorphic to the affine plane over the octonions.*
(2) σ *is a constant map.*
(3) $\mathbb{G}_0$ *contains the group* $\{\xi_{a,1} \mid a \in \mathrm{Spin}(3)\}$.
(4) $\mathbb{G}_0$ *contains the group* $\{\xi_{1,b} \mid b \in \mathrm{Spin}(3)\}$.

Proof. Use 3.5(a) to derive $(2 \Leftrightarrow 3 \Leftrightarrow 4)$.

$(1 \Rightarrow 3)$: If $\mathcal{P}^\sigma$ is isomorphic to the affine plane over the octonions, then $\mathbb{G}_0 \cap \mathrm{SL}(P)$ is isomorphic to $\mathrm{Spin}_{10}(\mathbb{R}, 1)$. Moreover, the centralizer of $\Lambda \cong \mathrm{SL}_2\,\mathbb{H}$ in $\mathrm{Spin}_{10}(\mathbb{R}, 1)$ is locally isomorphic to $\mathrm{SU}_2\,\mathbb{C} \cdot \mathrm{SU}_2\,\mathbb{C}$. (Up to conjugation, the Lie algebra $\mathfrak{so}_{10}(\mathbb{R}, 1)$ contains only one subalgebra which is isomorphic to $\mathfrak{so}_6(\mathbb{R}, 1) \cong \mathfrak{sl}_2\mathbb{H}$, see [**7**, 6.9].) Therefore, the maximal compact subgroup $\{\xi_{a,b} \mid a, b \in \mathrm{Spin}(3)\}$ of the centralizer of Λ in $\mathrm{GL}(P)$ consists of automorphisms of $\mathcal{P}^\sigma$.

$(2 \Rightarrow 1)$: Since the automorphism group of the octonion plane $\mathcal{P}$ contains a subgroup isomorphic to $\mathrm{SL}_2\,\mathbb{H}$ (see above), we infer that $\mathcal{P}$ is isomorphic to $\mathcal{P}^\sigma$ for some σ by the Classification Theorem (2.8). By "$1 \Rightarrow 2$", σ is a constant map, i.e., $\sigma \equiv r$ holds for some $r \in \mathbb{R}_{\mathrm{pos}}$. If $\tau \equiv s$, $s \in \mathbb{R}_{\mathrm{pos}}$, is an arbitrary constant map, then $\xi_{1,r/s}$ is an isomorphism between $\mathcal{P}^\tau$ and $\mathcal{P}^\sigma$, whence $\mathcal{P}^\tau$ is isomorphic to the octonion plane. □

Theorem 3.7. *Let* $\sigma, \tau : \mathrm{Spin}(3) \to \mathbb{R}_{\mathrm{pos}}$ *be continuous functions. Then* $\mathcal{P}^\sigma$ *and* $\mathcal{P}^\tau$ *are isomorphic if and only if there exists* $a, b \in \mathrm{Spin}(3)$ *and* $r \in \mathbb{R}_{\mathrm{pos}}$ *such that one of the following two properties is satisfied:*

$$\tau(h) = r\sigma(ahb) \textit{ for all } h \in \mathrm{Spin}(3) \textit{ or}$$
$$\tau(h) = r[\sigma(a\bar{h}b)]^{-1} \textit{ for all } h \in \mathrm{Spin}(3).$$

Proof. If one of the two properties above holds, then use 3.5 to obtain an isomorphism between $\mathcal{P}^\sigma$ and $\mathcal{P}^\tau$.

Conversely, suppose that $\mathcal{P}^\sigma$ and $\mathcal{P}^\tau$ are isomorphic. Then there exists an $\mathbb{R}$-linear map $f : \mathbb{H}^4 \to \mathbb{H}^4$ which maps $\mathcal{L}_0^\sigma$ onto $\mathcal{L}_0^\tau$.

If $\sigma \equiv t$, $t \in \mathbb{R}_{\mathrm{pos}}$, is a constant map, then $\mathcal{P}^\sigma$ and, hence, $\mathcal{P}^\tau$ are isomorphic to the octonion plane (3.6). This implies that $\tau \equiv t'$, $t' \in \mathbb{R}_{\mathrm{pos}}$ is a constant map (3.6). Thus, $\tau(h) = t'/t \cdot \sigma(h)$ holds for all $h \in \mathrm{Spin}(3)$.

Finally, suppose that neither σ nor τ is a constant map. Then the reduced stabilizers of $\mathcal{P}^\sigma$ is an almost direct product of Λ and some compact group, see 3.2. Since this assertion holds for $\mathcal{P}^\tau$ as well, f is an element of the normalizer of Λ. Modifying f with elements of Λ, we may achieve that f is an element of $\langle\iota\rangle \cdot \Xi$, and the desired property follows from 3.5. □

Theorem 3.8. *Let $\mathcal{P}$ be a 16-dimensional translation plane with automorphism group $\mathbb{G}$ and reduced stabilizer $\mathrm{S}\mathbb{G}_0$. If $\mathbb{G}$ contains a subgroup locally isomorphic to $\mathrm{SL}_2\,\mathbb{H}$, then only the following (mutually exclusive) possibilities can occur:*

(1) *$\mathrm{S}\mathbb{G}_0$ is isomorphic to $\mathrm{Spin}_{10}(\mathbb{R},1)$ and $\mathcal{P}$ is isomorphic to the octonion plane.*

(2) *$\mathcal{P}$ is isomorphic to $\mathcal{P}^\sigma$, where $\sigma : \mathrm{Spin}(3) \to \mathbb{R}_{\mathrm{pos}}$ is a continuous function which is not constant and depends only on the real part of its argument. The reduced stabilizer of $\mathcal{P}^\sigma$ is the almost direct product of $\Lambda \cong \mathrm{SL}_2\,\mathbb{H}$ and the group*

$$\Psi = \{(\vec{x},\vec{y}) \mapsto (a\vec{x}, a\vec{y}) \mid a \in \mathrm{Spin}(3)\} \cong \mathrm{SU}_2\,\mathbb{C}.$$

In particular, the dimension of $\mathbb{G}$ equals 35.

(3) *$\mathcal{P}$ is isomorphic to $\mathcal{P}^\sigma$, where σ is derived from a continuous, not constant function $\rho : [0;1] \to \mathbb{R}_{\mathrm{pos}}$, as follows:*

$$\sigma : \mathrm{Spin}(3) = \{u + jv \mid u, v \in \mathbb{C}, |u|^2 + |v|^2 = 1\} \to \mathbb{R}_{\mathrm{pos}};\ u + jv \mapsto \rho(|u|).$$

In this case, the reduced stabilizer of $\mathcal{P}^\sigma$ is the almost direct product of $\Lambda \cong \mathrm{SL}_2\,\mathbb{H}$ and the group

$$\Psi = \{(\vec{x},\vec{y}) \mapsto (a\vec{x}, b\vec{y}) \mid a, b \in \mathrm{Spin}(3) \cap \mathbb{C}\} \cong \mathrm{SO}_2\,\mathbb{R} \times \mathrm{SO}_2\,\mathbb{R}.$$

In particular, the dimension of $\mathbb{G}$ equals 34.

(4) *The reduced stabilizer of $\mathcal{P}^\sigma$ an almost direct product of Λ and an at most 1-dimensional compact group, and $\dim \mathbb{G} \in \{32, 33\}$.*

Proof. By the Classification Theorem (2.8), there exists a continuous function $\sigma : \mathrm{Spin}(3) \to \mathbb{R}_{\mathrm{pos}}$ such that $\mathcal{P}$ is isomorphic to $\mathcal{P}^\sigma$.

We suppose that $\mathcal{P}^\sigma$ is not isomorphic to the octonion plane. Then σ is not constant (3.6) and the reduced stabilizer of $\mathcal{P}^\sigma$ is an almost direct product of Λ and a connected compact group Ψ, cf. 3.2. Indeed, Ψ is a subgroup of the centralizer Ξ of Λ in $\mathrm{GL}(P)$ and, hence, Ψ is contained in the maximal compact subgroup $\Xi' = \{\xi_{a,b} \mid a, b \in \mathrm{Spin}(3)\}$ of Ξ. By 3.5(a), we infer that

$$(3) \qquad \Psi = \{\xi_{a,b} \mid a, b \in \mathrm{Spin}(3),\ \sigma(b^{-1}ha) = \sigma(h) \text{ for all } h \in \mathrm{Spin}(3)\}.$$

We emphasize that we are allowed to replace Ψ by $\xi_{a,b}^{-1}\Psi\xi_{a,b}$ for arbitrary $a, b \in \mathrm{Spin}(3)$: This corresponds to the replacement of $\mathcal{P}^\sigma$ by the isomorphic plane $\mathcal{P}^\tau$, $\tau(h) = \sigma(b^{-1}ha)$, see 3.7. Checking the connected subgroups of $\Xi' \cong \mathrm{SU}_2\,\mathbb{C} \times \mathrm{SU}_2\,\mathbb{C}$ yields the following fact: Up to conjugation, there are precisely the following possibilities for Ψ:

(i) Ψ has dimension 0 or 1.

(ii) $\Psi = \{\xi_{a,a} \mid a \in \mathrm{Spin}(3)\}$.

(iii) $\Psi = \{\xi_{a,b} \mid a, b \in \mathrm{Spin}(3) \cap \mathbb{C}\}$.

(iv) Ψ contains the group $\{\xi_{a,1} | a \in \mathrm{Spin}(3)\}$ or the group $\{\xi_{1,b} | b \in \mathrm{Spin}(3)\}$.

Since $\mathcal{P}^\sigma$ is not isomorphic to the octonion plane, case (iv) can not occur, see 3.6. Using Equation (3), it is not hard to see that Ψ equals $\{\xi_{a,a} \mid a \in \mathrm{Spin}(3)\}$ if and only if $\sigma(h)$ depends only on the real part of h.

If σ is one of the functions specified in Part (3) of the theorem, then we derive that $\Psi = \{\xi_{a,b} \mid a, b \in \mathrm{Spin}(3) \cap \mathbb{C}\}$ from Equation (3). Conversely, suppose that $\xi_{a,b}$ is an automorphism of $\mathcal{P}^\sigma$ for every $a, b \in \mathrm{Spin}(3)$. Let $u + jv$ be an arbitrary element of Spin(3) with $u, v \in \mathbb{C}$, $|u|^2 + |v|^2 = 1$. If $u = |u|e^{ir}$ and $v = |v|e^{is}$ are the polar decompositions, then we put $a = e^{-i(r+s)/2}$ and $b = e^{i(r-s)/2}$. Then $\xi_{a,b}$ is an automorphism of $\mathcal{P}^\sigma$ and we infer from 3.5(a) that

$$\begin{aligned}\sigma(u + jv) &= \sigma(e^{-i(r-s)/2}(|u|e^{ir} + j|v|e^{is})e^{-i(r+s)/2}) = \sigma(|u| + j|v|) \\ &= \sigma(|u| + j\sqrt{1 - |u|^2}),\end{aligned}$$

whence σ depends only on $|u|$, as asserted in Part (3). This finishes the proof. $\square$

Corollary 3.9. *Let $\mathcal{P}$ be a 16-dimensional locally compact translation plane admitting $\mathrm{SL}_2\,\mathbb{H}$ as a group of collineations. If the dimension of the automorphism group of $\mathcal{P}$ strictly exceeds 35, then $\mathcal{P}$ is isomorphic to the octonion plane.* $\square$

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