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We give examples showing that Kidwell's inequality for the maximal degree of the Brandt-Lickorish-Millett-Ho polynomial is in general not sharp.

1. Introduction.

The Q (or absolute) polynomial is a polynomial invariant in one variable z of links (and in particular knots) in S^3 without orientation. It can be defined by being 1 on the unknot and the relation

$$A_1 + A_{-1} = z(A_0 + A_\infty). \tag{1}$$

Here A_i denote the Q polynomials of links K_i, such that K_i $(i \in \mathbb{Z} \cup \{\infty\})$ possess diagrams equal except in one tangle site (or "room" in the terminology of [**LM**]), where an i-tangle (in the Conway [**Co**] sense) is inserted. See Figure 1.

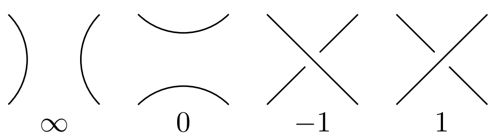


Figure 1. The Conway tangles.

The polynomial was discovered in 1985 independently by Brandt, Lickorish and Millett [**BLM**] and Ho [**Ho**]. Several months after its discovery, Kauffman [**Ka**] found a 2-variable polynomial $F(a, z)$, specializing to Q by setting $a = 1$.

In [**Ki**], Kidwell found a nice inequality for the maximal degree of the Q polynomial:

Theorem 1.1 (Kidwell)**.** *Let D be a diagram of a knot (or link) K. Then*

$$\max\deg Q(K) \le c(D) - d(D), \tag{2}$$

where $c(D)$ is the crossing number of D and $d(D)$ its maximal bridge length, i.e., the maximal number of consecutive crossing over- or underpasses. Moreover, if D is alternating (i.e., $d(D) = 1$) and prime, then equality holds in (2).

In [**Mo**, Problem 4, p. 560] he asked whether (2) always becomes equality when minimizing the r.h.s. over all diagrams D of K. From the theorem it follows that this is true for alternating knots and also for those non-alternating knots K, where $\max\deg Q(K) = c(K) - 2$ (here $c(K)$ denotes the crossing number of K). All non-alternating knots in Rolfsen's tables [**Ro**] have this property except for one – the Perko knot 10_{161} (and its obversed duplication 10_{162}), where $\max\deg Q = 6$. Hence, as quoted by Kidwell, this knot became a promising candidate for strict inequality in (2). To express ourselves more easily, we define:

Definition 1.1. Call a knot K *Q-maximal*, if (2) with the r.h.s. minimized over all diagrams D of K becomes equality.

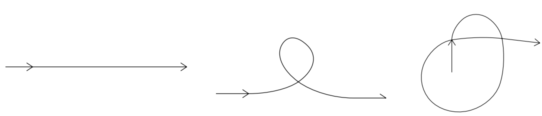
The aim of this note is to show that indeed the Perko knot is not Q-maximal. We give several modifications of our arguments and examples showing how they can be applied to exhibit non-Q-maximality.

2. Plane curves.

We start with some discussion of plane curves.

Definition 2.1. A non-closed *plane curve* is a C^1 map $\gamma : [0,1] \to \mathbb{R}^2$ with $\gamma(0) \ne \gamma(1)$ and only transverse self-intersections. γ carries a natural orientation.

Example 2.1. Here are some plane curves:

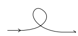
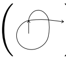


In the following, whenever talking of plane curves we mean non-closed ones with orientation, unless otherwise stated. However, in some cases it is possible to forget about orientation if it is irrelevant. It is convenient to identify γ with $\gamma([0,1])$ wherever this causes no confusion. Whenever we want to emphasize that a line segment in a local picture starts with an endpoint, the endpoint will be depicted as a thickened dot.

Definition 2.2. The *crossing number* $c(\gamma)$ of a curve γ is the number of self-intersections (crossings). A curve γ with $c(\gamma) = 0$ is called *simple* or *trivial*.

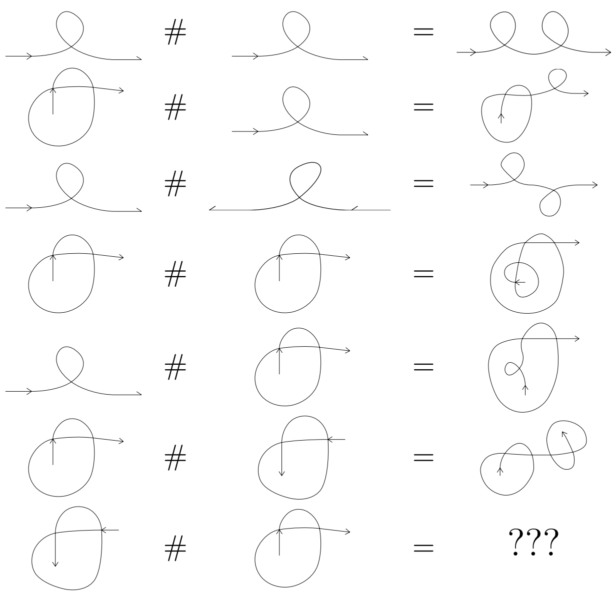
Definition 2.3. We call a non-closed curve $\tilde{\gamma}$ *transversely homotopic* to γ (and denote it by $\tilde{\gamma} \sim \gamma$) if $\tilde{\gamma}(0) = \gamma(0)$, $\tilde{\gamma}(1) = \gamma(1)$ and $\tilde{\gamma}$ intersects γ only transversely. Call the number $d(\gamma) = \min\{\,\#(\gamma \cap \tilde{\gamma}) \,:\, \tilde{\gamma} \sim \gamma\,\} - 2$ the *endpoint distance* of γ. (The term '-2' serves to discount the coincidence of endpoints.)

A curve $\tilde{\gamma}$ realizing this minimum is called *minimal transversely homotopic* to γ. Such $\tilde{\gamma}$ can be chosen to have no self-intersections.

Example 2.2. The curves and  have $d = 0$, while $d\Big($$\Big) = 1$.

Definition 2.4. We call a plane curve γ *composite*, if there is a *closed* plane curve γ' (with no self-intersections) such that γ' intersects γ in exactly one point, transversely, and in both components of $\mathbb{R}^2 \setminus \gamma'$ there are crossings of γ. In this case γ' separates γ into two parts γ_1 and γ_2, which we call *components* of γ. We write $\gamma = \gamma_1 \# \gamma_2$. Conversely, this can be used to define the operation '$\#$' (*connected sum*) of γ_1 and γ_2, wherever $\gamma_1(1)$ or $\gamma_2(0)$ are in the closure of the unbounded component of their complements. We call γ *prime*, if it is not composite.

Example 2.3.



These examples illustrate that the connected sum in general depends on the orientation of the summands and their order.

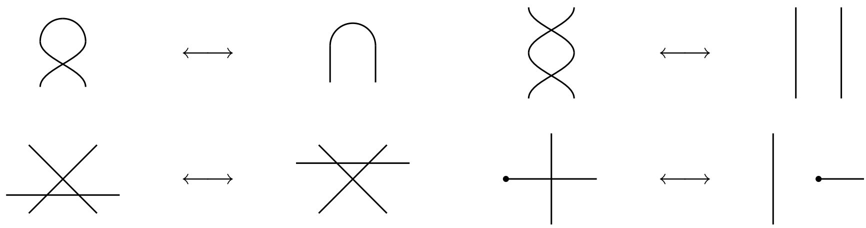
It is clear that the crossing number is additive under connected sum and it's a little exercise to verify that the endpoint distance is as well.

The path we are going to follow starts with the following:

Exercise 2.1. Verify that the complement of a curve γ has $c(\gamma)+1$ connected components, and conclude from that that $d(\gamma) \le c(\gamma)$.

Hint: One way to show that is to observe it when γ is trivial, to prove that you can obtain any γ from the trivial one by the four local moves

(3)



and to trace how the number of components and $c(\gamma)$ change under these moves.

For the Perko knot we need to work a little harder. We need to prove that $d(\gamma) \le \max(3, c(\gamma)-3)$ if γ prime. This was originally achieved by the author by refining the argument in the exercise above. Subsequently, Kidwell further generalized and simplified the argument, showing the following stronger inequality.

Theorem 2.1 (Kidwell)**.** *For any plane curve γ we have $d(\gamma) \le c(\gamma)/2$.*

A proof of this theorem will be given in the Appendix.

3. Non-Q-maximal knots.

Using Kidwell's theorem, we are prepared to exhibit the Perko knot as non-Q-maximal.

Theorem 3.1. *If D is a prime diagram of a knot K of $c(D)$ crossings with a bridge of length $l = c(D) - k$, and D has minimal crossing number among all such diagrams for fixed k, then $l \le k/2$, hence $c(D) \le 3/2 k$.*

From this we have the desired example:

Example 3.1. If 10_{161} were Q-maximal, then we could pose $k = 6$ in the theorem and would obtain a 9 crossing diagram of the knot, which does not exist. Hence 10_{161} is not Q-maximal.

Proof of Theorem 3.1. This is basically Kidwell's theorem. Consider γ' to be the part of D consisting of the maximal (length) bridge and γ consisting of the rest of (the solid line of) D with signs of all crossings ignored. Then the freedom to move the bridge corresponds to the freedom to move γ'. $\square$

Clearly, for many phenomena Rolfsen's tables up to 10 crossings are very limited. Scanning the list of non-alternating knots of at most 15 crossings provided by Thistlethwaite (see [**HTW**]), I found 192 15 crossing knots for which $\max\deg Q \le 8$, and hence for which we would be done showing non-Q-maximality already with Exercise 2.1. The most striking examples are the knots 15_{119574} and 15_{119873}, where $\max\deg Q = 4$. Although for both knots $\max\deg_z F(a,z) = 11$, the coefficients of the 7 highest powers of z cancel when setting $a = 1$. This property exhibits non-Q-minimality in an alternative way, because Theorem 1.1 in fact can be analogously shown to hold with the maximal z-degree of the Kauffman polynomial $F(a,z)$ in place of $\max\deg Q$. Nevertheless, placing emphasis on $\max\deg Q$ rather than $\max\deg_z F(a,z)$ in the above discussion, beside by Kidwell's original work, can be justified at least because for the main example, the Perko knot, $\max\deg Q = \max\deg_z F(a,z)$, and so the geometric argument is needed.

There are several ways that the theorem can be modified.

Theorem 3.2. *If D is a diagram of a knot K of $c(D)$ crossings with a bridge of length $l = c(D) - k$, then $u(K) \le \lfloor k/2 \rfloor$, where $u(K)$ denotes the unknotting number of K.*

Proof. By switching at most half of the crossings in D not involved in the maximal bridge, the remaining part γ of the plane curve (this time *with* signs of the crossings) can be layered, i.e., any crossing is passed the first time as over- and then as under-crossing or vice versa. But reinstalling the bridge to a layered γ gives a layered, and hence unknotted, diagram. $\square$

Corollary 3.1. *If $u(K) > \lfloor \max\deg Q(K)/2 \rfloor$, then K is not Q-maximal.* $\square$

Unfortunately, this corollary does not work to show non-Q-maximality of Perko's knot. Verifying both hand-sides of the inequality (using that the unknotting number of 10_{161} is 3, see [**St**, **Km**, **Ta**]), we find that we just have equality. And that equality does not suffice is seen, e.g., from all 8 closed positive braid knots in Rolfsen's tables (see [**Cr**, **Bu**]) and more generally from the $(2,n)$-torus knots for n odd.

For knots of > 10 crossings unknotting numbers are not tabulated (anywhere I know of) and a general machinery does not exist to compute them, hence when wanting to extend the search space for examples applicable to Corollary 3.1, it makes sense to replace the unknotting number by lower bounds for it, which can be computed straightforwardly. I tried two such bounds. First we have the signature σ:

Corollary 3.2. *If $|\sigma(K)| > \max\deg Q(K)$, then K is not Q-maximal.*

Clearly, replacing $u(K)$ by lower bounds for it makes the condition more and more restrictive. However, when checking the above mentioned list of 192 knots, I found that at least one of them satisfied strict inequality. It is 15_{166028}, where $\sigma = 8$ and $\max\deg Q = 7$.

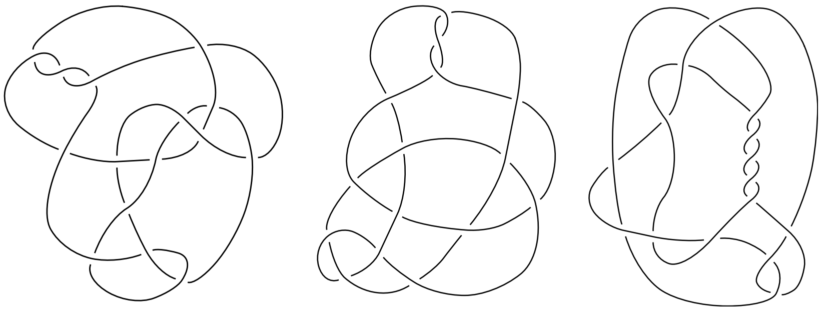


Figure 2. Three non-Q-maximal knots.

Another possibility is to minorate $u(K)$ by the bound coming from the Q polynomial itself.

Corollary 3.3. *If* $2\log_{-3} Q(-1) > \max\deg Q(K)$, *then* K *is not* Q*-maximal.*

Remark 3.1. The negative logarithm base may disturb the reader because such logarithms are usually not defined. But by work of Sakuma, Murakami, Nakanishi (see Theorem 8.4.8 (2) of [**Kw**]) and Lickorish and Millett [**LM**] $Q(-1)$ is always a(n integral) power of -3 and this one it is referred to by this expression.

The inequality in Corollary 3.3 looks rather bizarre. First, the inequality $u(K) \geq \log_{-3} Q(-1)$ is in general much less sharp than the one with the signature and secondly, the inequality in Corollary 3.3 requires the coefficients of Q to be of an average magnitude which grows exponentially with $\max\deg Q$. Thus, non-surprisingly, my quest for applicable examples among the non-alternating 15 and 16 crossing knots ended with no success in this case.

Question 3.1. Is there a knot K with $2\log_{-3} Q(-1) > \max\deg Q(K)$?

I nevertheless gave the above inequality, because it is self-contained w.r.t. Q and would decide about non-Q-maximality from Q itself (without knowing anything else about the knot) and hence is, in some sense, also beautiful.

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Appendix I. A proof of Theorem 2.1 (by M. Kidwell).

By the additivity of c and d under connected sum, it clearly suffices to prove the inequality for γ prime. We can also exclude another degeneracy of γ.

Definition I.1. A *nugatory crossing* of γ is a crossing p such that there is a closed curve γ' with $\gamma \cap \gamma' = \{p\}$ and γ' intersects transversely both strands of γ intersecting at p.

If γ has a nugatory crossing, then one of the components of $\mathbb{R}^2 \setminus \gamma$ has both and the other one has no one of the endpoints of γ. Removing the part of γ in latter component and smoothing γ near p reduces $c(\gamma)$, but not $d(\gamma)$, hence we may (say, by induction on $c(\gamma)$) also assume that γ has no nugatory crossing.

Let $\gamma : [0, 1] \to \mathbb{R}^2$ be a prime curve in the plane with no nugatory crossings. We will assume that γ has only transverse crossings. In fact, we will speak of “right angles” and “straight angles” as if all crossings are orthogonal. Without loss of generality, assume that $\gamma(0)$ lies on the boundary of the unbounded region U of $\mathbb{R}^2 \setminus \gamma$. Define the *index* of any region R of $\mathbb{R}^2 \setminus \gamma$ as the minimum intersection number of a curve with one endpoint in R and the other in U with γ. Thus the unbounded region has index 0. The endpoint distance $d(\gamma)$ of γ is the index of the region with $\gamma(1)$ on its boundary. Call any portion of γ between two crossing points an *edge* of γ. The indices of regions that share an edge along their boundary either differ by one or are equal. Call an edge *neutral* if its two incident regions have the same index. (The edges containing $\gamma(0)$ and $\gamma(1)$ are considered neutral. By the assumption that γ is prime, these are the only edges of γ that border only one region.) We define the index of a neutral edge to be the common index of its two incident regions. We call a curve γ' *efficient* if its endpoints lie in regions of index i and j and γ' intersects γ in exactly $|i-j|$ points.

Lemma I.1. *An efficient curve γ' cannot intersect a neutral edge of γ.*

Proof. Let the endpoints of γ' lie in regions of index i and j with $i \leq j$. Suppose γ' intersects a neutral edge of index k which borders regions R_1 and R_2, and let a and b be points in $[0,1]$ such that $\gamma'(a) \in R_1$ and $\gamma'(b) \in R_2$. Then $\gamma'([0,a])$ intersects γ in at least $|k-i|$ points, $\gamma'([a,b])$ intersects γ in at least one point and $\gamma'([b,1])$ intersects γ in at least $|j-k|$ points. The sum of these three numbers exceeds $j-i$. □

Lemma I.2. *If a crossing point C of γ is incident to a neutral edge, then it is incident to at least two neutral edges.*

Proof. Let the neutral edge be incident to two regions of index i. Then the other two regions incident to C can only have index i, $i-1$, or $i+1$. Since these two regions are incident to a common edge, their indices cannot be $i-1$ and $i+1$. If one of these regions has index i, then two neutral edges form a right angle at C. If these regions have index $i \pm 1$, then two neutral edges form a straight angle at C. Figure 3 illustrates these two cases. □

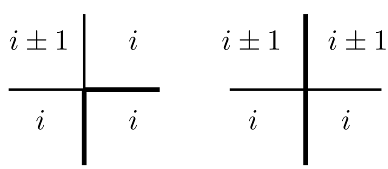


Figure 3. Two types of crossings involving neutral edges (the thickened lines), together with the indices of their neighbored regions.

If three neutral edges are incident to one crossing point, then all four regions incident to that point must have the same index, and thus the fourth edge incident to that point must also be neutral. We next show that this case cannot occur.

Lemma I.3. *Four neutral edges cannot be incident to one crossing point C.*

Proof. One edge incident to C is part of a continuous chain of neutral edges ending at $\gamma(0)$ and another is part of a continuous chain of neutral edges ending at $\gamma(1)$. The other two edges at C must be part of a continuous chain of neutral edges that forms a loop. (All these assertions follow from Lemma I.2.) Let R be any region of $\mathbb{R}^2 \setminus \gamma$ in the interior of this loop. By definition of index, there is an efficient path joining a point in R to a point in U. This path must cross the loop of neutral edges, contradicting Lemma I.1. □

These lemmas taken together show that the union of all neutral edges of γ is a simple curve joining $\gamma(0)$ to $\gamma(1)$. We call this curve the *neutral curve.*

Lemma I.4. *For every number i with $0 \leq i \leq d(\gamma)$, there is at least one neutral edge of index i.*

Proof. The first and last edges along the neutral curve of γ have index 0 and $d(\gamma)$. Figure 3 shows that adjacent neutral edges have equal indices or indices that differ by 1. □

More particularly, adjacent edges that form a straight angle at a crossing point have indices that differ by 1 and edges that form a right angle have equal indices.

Lemma I.5. *Unless γ is a simple curve, there is at least one crossing point C of γ incident to neutral edges which form a right angle.*

Proof. If γ consists entirely of neutral edges then, by Lemma I.3, γ must be a simple curve. The curve that starts at $\gamma(0)$ and goes "straight ahead" at every crossing point traverses all of γ. Thus if the neutral curve is to be a proper subset of γ, it must contain at least one right angle. □

We are now ready to count crossing points, edges and regions in the plane and apply Euler's Theorem.

Theorem I.1. *If γ is prime, without nugatory crossings and has crossing number $c(\gamma)$, then $d(\gamma) \leq \lfloor c(\gamma)/2 \rfloor$.*

Proof. Without changing the Euler characteristic of our configuration, we can eliminate $\gamma(0)$, $\gamma(1)$ and the edges incident to them. We are left with a configuration with $c(\gamma)$ vertices, two with valence 3 and $c(\gamma)-2$ with valence 4.

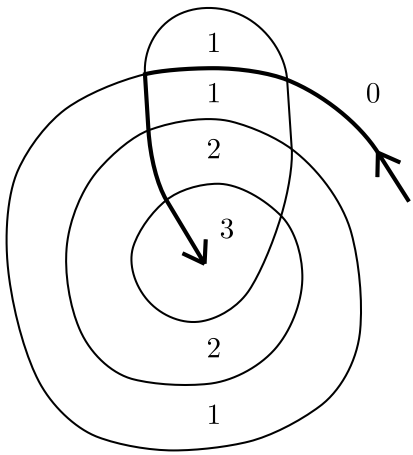


Figure 4. A plane curve and its neutral curve (the thickened part). The numbers in the regions denote their indices.

The edge count is

$$\frac{4\big(c(\gamma)-2\big)+3\cdot 2}{2} = 2c(\gamma)-1.$$

If F is the number of faces (regions of $\mathbb{R}^2 \setminus \gamma$, including the unbounded region), then

$$c(\gamma)-\big(2c(\gamma)-1\big)+F=2, \quad \text{and so} \quad F=c(\gamma)+1.$$

We now count regions that are incident to the (original) neutral curve. (The curve in Figure 4 may aid understanding.) There is the unbounded region of index 0 and the region of index $d(\gamma)$ containing $\gamma(1)$. By Lemma I.4, there are at least two regions of index i for $1 \leq i \leq d(\gamma)-1$, since we are assuming that γ is prime. However, at any crossing point where the neutral curve makes a right angle, there are three regions of the same index, distinct by our assumption that γ has no nugatory crossings. Thus we can compare the total number of regions to the lower bound on the number of regions along the neutral curve:

$$\begin{aligned} c(\gamma)+1 &\geq 2+2\big(d(\gamma)-1\big)+1 = 2d(\gamma)+1, \text{ or} \\ c(\gamma) &\geq 2d(\gamma), \text{ or} \\ d(\gamma) &\leq \frac{c(\gamma)}{2}. \end{aligned}$$

Since the numbers $d(\gamma)$ and $c(\gamma)$ are integers, we have our desired relation. $\square$

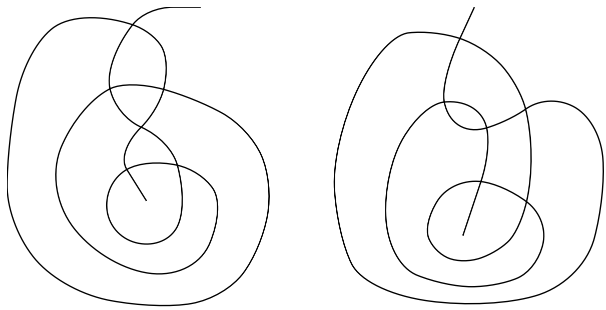
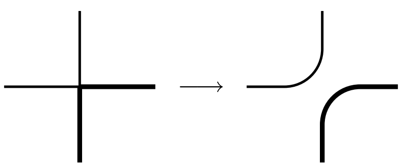


Figure 5. Two more curves with $c = 2d(= 6)$.

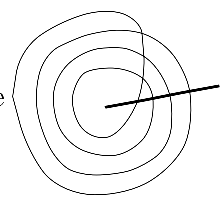
The curve in Figure 4 can be generalized to a family of examples with $c(\gamma) = 2d(\gamma)$ for any even number $c(\gamma)$. Thus the inequality cannot be improved.

These curves are not all curves with $c = 2d$; Figure 5 shows two other ones. However, one can give an explicit description of all such curves. We leave this as a task to the reader.

Exercise I.1. Show that γ satisfies $c(\gamma) = 2d(\gamma)$ if and only if its neutral curve has exactly one angle, and splicing this angle



one obtains a picture like



(with its obvious generalization).

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