

*Pacific
Journal of
Mathematics*

ON THE ADDITIVITY OF THE THURSTON–BENNEQUIN
INVARIANT OF LEGENDRIAN KNOTS

ICHIRO TORISU

ON THE ADDITIVITY OF THE THURSTON–BENNEQUIN INVARIANT OF LEGENDRIAN KNOTS

ICHIRO TORISU

In this article, we consider the maximal value of the Thurston–Bennequin invariant of Legendrian knots which topologically represent a fixed knot type in the standard contact 3-space and we prove a formula of the value under the connected sum operation of knots.

1. Introduction.

The *standard contact structure* ξ_0 on 3-space $\mathbb{R}^3 = \{(x, y, z)\}$ is the plane field on \mathbb{R}^3 given by the kernel of the 1-form $dz - ydx$. A *Legendrian knot* K in the contact manifold (\mathbb{R}^3, ξ_0) is a knot which is everywhere tangent to the contact structure ξ_0 . The *Thurston–Bennequin invariant* $\text{tb}(K)$ of a Legendrian knot K in (\mathbb{R}^3, ξ_0) is the linking number of K and a knot K' which is obtained by moving K slightly along the vector field $\frac{\partial}{\partial z}$. For a topological knot type k in \mathbb{R}^3 , the *maximal Thurston–Bennequin invariant* $\text{mtb}(k)$ is defined to be the maximal value of $\text{tb}(K)$, where K is a Legendrian knot which topologically represents k . For any k , by the *Bennequin’s inequality* in [1], we know that $\text{mtb}(k)$ is an integer (i.e., not ∞). There are several computations of $\text{mtb}(k)$ (for example, see [3], [5], [8], [9], [10], [11]).

In this paper, we prove the following theorem:

Theorem 1.1. *Let $k_1 \# k_2$ be the connected sum of topological knots k_1 and k_2 in \mathbb{R}^3 . Then $\text{mtb}(k_1 \# k_2) = \text{mtb}(k_1) + \text{mtb}(k_2) + 1$.*

Remark 1.2. After writing this paper, the author was informed that J. Etnyre and K. Honda [4] have also obtained a result on connected sum of Legendrian knots which extensively includes Theorem 1.1 and that T. Tanaka [12] have partially proved Theorem 1.1 by using a technique of algebraic knot theory.

2. Fronts.

Let K be a Legendrian knot in $(\mathbb{R}^3, \xi_0 = \ker(dz - ydx))$. Then a diagram (i.e., projection) of K in xz -plane is called *front* as in Figure 1.

A front does not have vertical tangents; generically, its only singularities are transverse double points and semicubical cusps. Note that the number

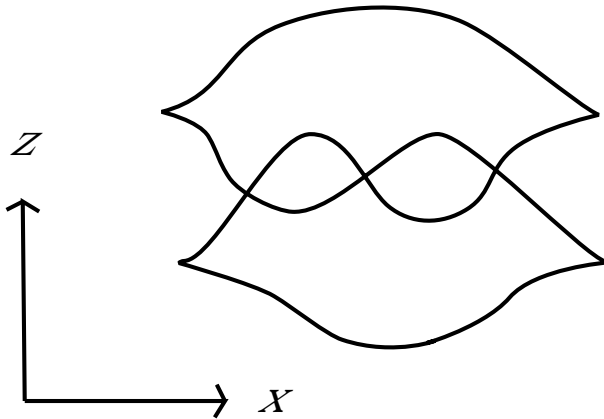


Figure 1.

of the cusps is even. Since $y = \frac{\partial z}{\partial x}$ along K , the missing y coordinate is the slope of the front. Therefore the front of K is free from selftangencies, and, at a double point, the branch with a greater slope is higher along the y axis. Conversely such a diagram uniquely determines K as its front. So, as usual in knot theory, we identify a Legendrian knot K with its front, also denoted by K .

The Thurston-Bennequin invariant $tb(K)$ is computed in terms of the double points and cusps of its front. See Figure 2, where K is oriented and the choice of the orientation is irrelevant for the value of $tb(K)$.

$$tb = \# \begin{array}{c} \nearrow \\ \searrow \\ \nwarrow \\ \nearrow \end{array} + \# \begin{array}{c} \nwarrow \\ \nearrow \\ \swarrow \\ \nwarrow \end{array} - \# \begin{array}{c} \nwarrow \\ \nearrow \\ \swarrow \\ \nwarrow \end{array} - \# \begin{array}{c} \nearrow \\ \searrow \\ \swarrow \\ \nwarrow \end{array} - 1/2 \# \text{ of cusps}$$

Figure 2.

For example, $tb(K) = 1$ for the front in Figure 1.

Proposition 2.1. *For two topological knots k_1 and k_2 , we have $mtb(k_1 \# k_2) \geq mtb(k_1) + mtb(k_2) + 1$.*

Proof. Let K_1 and K_2 be Legendrian knots whose topological types are k_1 and k_2 , respectively and $mtb(k_1) = tb(K_1)$ and $mtb(k_2) = tb(K_2)$. We also regard K_1 and K_2 as fronts. Further we can assume that $K_1 \cap K_2 = \emptyset$ and K_1

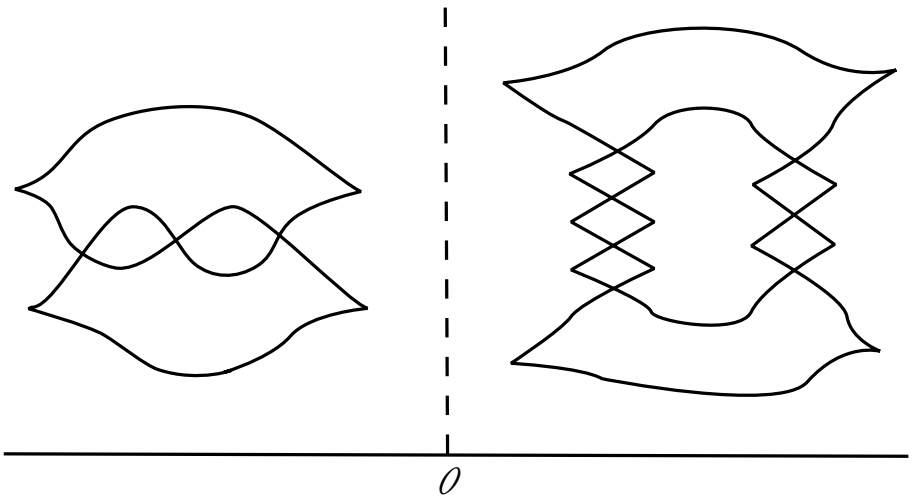


Figure 3.

(resp. K_2) lies in the left (resp. right) region of xz -plane, i.e., $\{(x, z) | x < 0\}$ (resp. $\{(x, z) | x > 0\}$) as in Figure 3.

Then we connect K_1 and K_2 by joining a right cusp of K_1 and a left cusp of K_2 as in Figure 4.

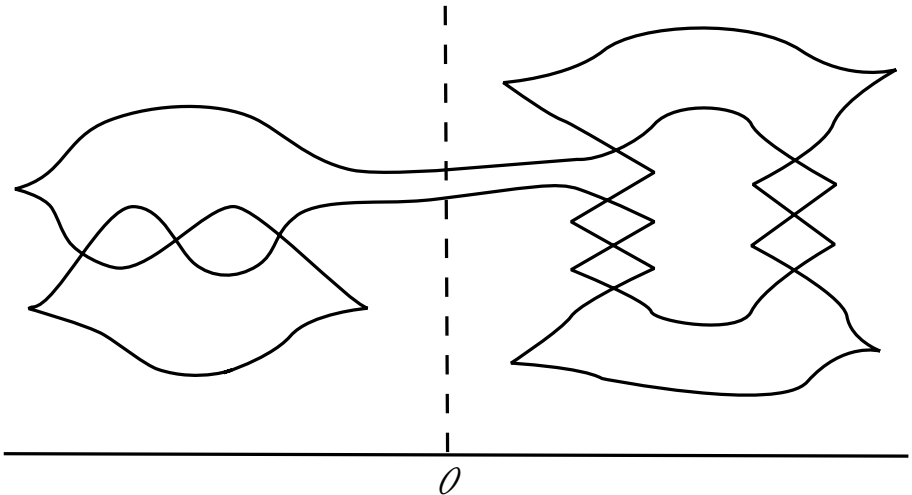


Figure 4.

This procedure produces a Legendrian knot whose topological type is $k_1 \# k_2$ and Thurston-Bennequin invariant is $\text{mtb}(k_1) + \text{mtb}(k_2) + 1$. This completes the proof. \square

3. Preliminaries from contact topology.

In this section, we recall some basic notions and theorems from recent 3-dimensional contact topology. In fact, the proof of Theorem 1.1 essentially relies on the previous foundational work of E. Giroux, Honda and Y. Eliashberg-M. Fraser. In particular, we assume the reader is familiar with convex surface theory started by Giroux in [6]. For details and proofs, see [2], [3], [6], [7], [8]. Let $\xi_n = \ker(\sin(2\pi n z)dx + \cos(2\pi n z)dy)$ be the contact structure on a solid torus $V = \{(x, y, z) \in \mathbb{R}_z^3 \mid x^2 + y^2 \leq \epsilon\}$, where $n \in \mathbb{N}$ and \mathbb{R}_z^3 is \mathbb{R}^3 modulo $z \mapsto z + 1$. The *characteristic foliation* on an embedded surface in a contact 3-manifold is the singular foliation defined by the intersection of the contact structure and the surface. The set of tangents of ξ_n to ∂V forms a disjoint union of two simple closed curves on ∂V , which are called *Legendrian divides*.

The next lemma is proved by a standard Darboux-type argument.

Lemma 3.1. *For any Legendrian knot K in (\mathbb{R}^3, ξ_0) , there exists a sufficiently small neighborhood $N(K)$ such that $(N(K), K, \xi_0)$ is isomorphic to $(V, \{(0, 0, z)\}, \xi_n)$ for some n .*

As ∂V is a *convex surface* (i.e., has a contact vector field transverse to ∂V), the following lemma can be proved by convex surface theory:

Lemma 3.2. *Let T be any embedded torus in (\mathbb{R}^3, ξ_0) and W a solid torus bounded by T . Suppose the characteristic foliation on T is diffeomorphic to that on ∂V and identifying these, the Legendrian divides on T are isotopic to the core curve of W through an isotopy in W . Then (W, ξ_0) is isomorphic to (V, ξ_n) for some n .*

The following theorem on the classification of topologically trivial Legendrian knots due to Eliashberg-Fraser [2] is also needed for the proof of Theorem 1.1:

Theorem 3.3. *Any topologically trivial Legendrian knot is Legendrian isotopic to one of standard forms expressed as fronts in Figure 5.*

4. Proof of Theorem 1.1.

By Proposition 2.1, it is sufficient to show the converse inequality.

Suppose \hat{K} is a Legendrian knot in (\mathbb{R}^3, ξ_0) whose topological type is the connected sum of k_1 and k_2 and its Thurston-Bennequin invariant is maximal. By Lemma 3.1, there exists a neighbourhood $N(\hat{K})$ of \hat{K} such

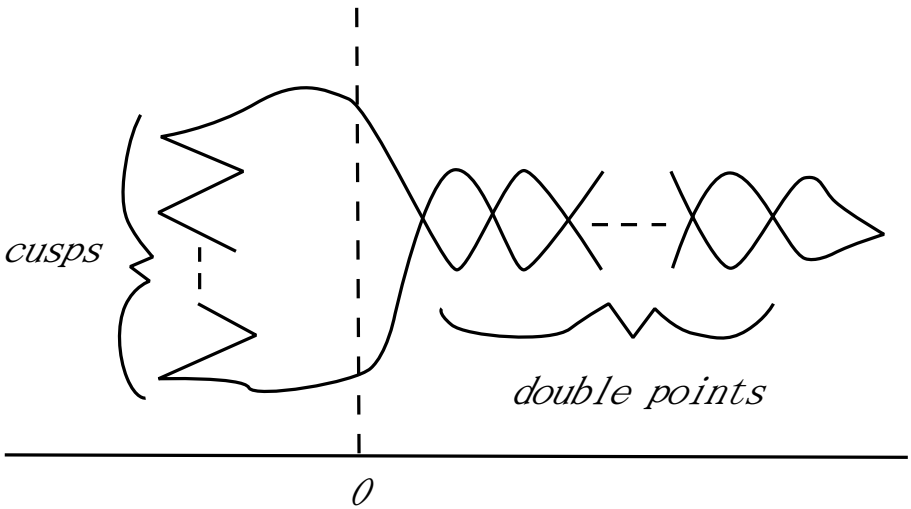


Figure 5.

that $(N(\hat{K}), \xi_0)$ is isomorphic to (V, ξ_n) for some n . Let B_1 and B_2 be 3-balls in \mathbb{R}^3 such that B_1 (resp. B_2) splits \hat{K} into the component corresponding to k_1 (resp. k_2) and $B_1 \cap B_2 = \emptyset$ (Figure 6).

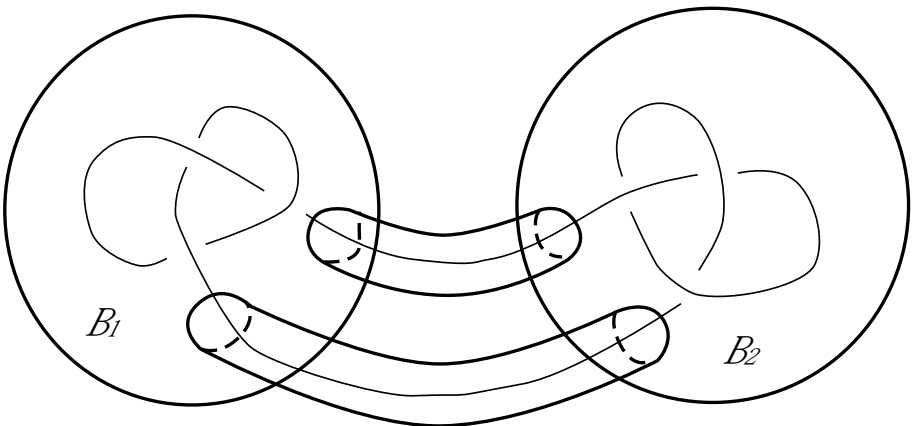


Figure 6.

Further, by convex surface theory, we can assume that (i) ∂B_1 and ∂B_2 are convex and (ii) $\partial B_1 \cap \partial N(\hat{K})$ and $\partial B_2 \cap \partial N(\hat{K})$ are Legendrian knots on ∂B_1 and ∂B_2 , respectively and (iii) each *dividing set* on ∂B_i (i.e., the subset of ∂B_i consisting of tangents of ξ_0 and a contact vector field defining the convex surface) intersects $\partial B_i \cap N(\hat{K})$ as a diameter of the disk.

Then by Edge-Rounding Lemma due to Honda in [7], we have a solid torus W such that (i) W equals $B_1 \cup B_2 \cup N(\hat{K})$ except small neighbourhoods of $\partial B_1 \cap \partial N(\hat{K})$ and $\partial B_2 \cap \partial N(\hat{K})$ and (ii) ∂W is a convex surface whose characteristic foliation is diffeomorphic to that of ∂V . By Lemma 3.2, it follows that (W, ξ_0) is isomorphic to (V, ξ_n) for some n . And notice that W is unknotted in \mathbb{R}^3 and hence the core curve K of W which is Legendrian is also unknotted. Further, by a standard argument, we can assume that K agrees with \hat{K} in the region of $N(\hat{K}) - (B_1 \cup B_2)$. So by Theorem 3.3, K is Legendrian isotopic to one of standard forms in Figure 5. Therefore W is also identified with a small neighbourhood of that of the standard form. Further, by a homogeneous property of V and a parallel translation of W , we can assume that a region of W corresponding to B_1 (resp. B_2) lies in $\{(x, y, z) | x < 0\}$ (resp. $\{(x, y, z) | x > 0\}$). Then, identifying \hat{K} with its front, we can divide \hat{K} along a vertical line into Legendrian knots K_1 and K_2 corresponding to k_1 and k_2 , respectively as the converse procedure in the proof of Proposition 2.1.

Counting the Thurston-Bennequin invariant of K_1 and K_2 , we have $\text{tb}(\hat{K}) = \text{mtb}(k_1 \# k_2) = \text{tb}(K_1) + \text{tb}(K_2) + 1$. Therefore $\text{mtb}(k_1 \# k_2) \leq \text{mtb}(k_1) + \text{mtb}(k_2) + 1$.

This completes the proof of the main theorem.

References

- [1] D. Bennequin, *Entrelacements et équations de Pfaff*, Astérisque, **107-108** (1983), 87-161, [MR 86e:58070](#), [Zbl 0573.58022](#).
- [2] Y. Eliashberg and M. Fraser, *Classification of topologically trivial Legendrian knots*, in ‘Geometry, Topology and Dynamics’ (Montreal, PQ, 1995), CRM Proc. Lecture Notes, **15** (1998), 17-51, [MR 99f:57007](#), [Zbl 0907.53021](#).
- [3] J. Etnyre and K. Honda, *Knots and contact geometry I: Torus knots and the figure eight knot*, preprint.
- [4] ———, *Knots and contact geometry II: Connected sums*, preprint.
- [5] D. Fuchs and S. Tabachnikov, *Invariants of Legendrian and transverse knots in the standard contact space*, Topology, **36** (1997), 1025-1053, [MR 99a:57006](#), [Zbl 0904.57006](#).
- [6] E. Giroux, *Convexité en topologie de contact*, Comment. Math. Helvetici, **66** (1991), 637-677, [MR 93b:57029](#), [Zbl 0766.53028](#).
- [7] K. Honda, *On the classification of tight contact structures I*, Geom. Topol., **4** (2000), 309-368, [MR 2001i:53148](#), [Zbl 0980.57010](#).
- [8] Y. Kanda, *On the Thurston-Bennequin invariant of Legendrian knots and non exactness of Bennequin’s inequality*, Invent. Math., **133** (1998), 227-242, [MR 99g:57030](#), [Zbl 0902.57007](#).
- [9] L. Rudolph, *An obstruction to sliceness via contact geometry and “classical” gauge theory*, Invent. Math., **119** (1995), 155-163, [MR 95k:57013](#), [Zbl 0843.57011](#).

- [10] S. Tabachnikov, *Estimates for the Bennequin number of Legendrian links from state models for knot polynomials*, Math. Research Letters, **4** (1997), 143-156, [MR 98k:57023](#), [Zbl 0877.57001](#).
- [11] T. Tanaka, *Maximal Bennequin numbers and Kauffman polynomials of positive links*, Proc. Amer. Math. Soc., **127** (1999), 3427-3432, [MR 2000b:57014](#), [Zbl 0983.57008](#).
- [12] ———, *On invariants of Legendrian links*, preprint.

Received July 12, 2001 and revised June 27, 2002.

DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING
AKITA UNIVERSITY
1-1 TEGATA GAKUEN-CHO
AKITA 010-8502
JAPAN
E-mail address: torisu@math.akita-u.ac.jp