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A PROPERTY OF FREE ENTROPY

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**We show that the restriction on the uniform norms of approximating matricial microstates can be removed when defining free entropy.**

### 1. Introduction.

Denote by  $\mathfrak{M}_k$  the algebra of complex  $k \times k$  matrices, and by  $\tau_k$  the normalized trace on  $\mathfrak{M}_k$ , i.e.,  $\tau_k(A) = \frac{1}{k} \text{Tr}(A)$  for  $A \in \mathfrak{M}_k$ . Consider for each  $k$  a standard Gaussian Hermitian random matrix  $X_k$ . Thus, if  $E$  denotes expected value,  $E\tau_k(X_k) = 0$  and  $E\tau_k(X_k^2) = 1$ . It was shown by E. Wigner [9] that, as  $k \rightarrow \infty$ ,  $X_k$  tends in distribution to a semicircular law, i.e., the limits

$$\mu_p = \lim_{k \rightarrow \infty} E\tau_k(X_k^p)$$

exist, and they can be calculated as

$$\mu_p = \frac{1}{2\pi} \int_{-2}^2 t^p \sqrt{4 - t^2} dt$$

for  $p = 1, 2, \dots$ . If we have several independent standard Gaussian Hermitian random matrices  $(X_k(i))_{i=1}^n$ , D. Voiculescu [4] proved that, as  $k \rightarrow \infty$ , these sets of variables converge in distribution to a free semicircular family. Briefly, this means that given indices  $i_j \in \{1, 2, \dots, n\}$  such that  $i_j \neq i_{j+1}$  for  $j = 1, 2, \dots, m - 1$ , and given positive integers  $p_1, p_2, \dots, p_m$ , the limit

$$\lim_{k \rightarrow \infty} E\tau_k(X_k(i_1)^{p_1} X_k(i_2)^{p_2} \dots X_k(i_m)^{p_m})$$

exists, and

$$\lim_{k \rightarrow \infty} E\tau_k[(X_k(i_1)^{p_1} - \mu_{p_1})(X_k(i_2)^{p_2} - \mu_{p_2}) \dots (X_k(i_m)^{p_m} - \mu_{p_m})] = 0.$$

It is natural to look for large deviation principles associated with these limit laws. For this purpose (and also with motivation from information theory and statistical physics) Voiculescu introduced in [6] (cf. also [5]) the notion of free entropy. The original definition of free entropy, which will be reviewed below, involves a bound  $R > 0$  on the operator norm of approximating matricial microstates, and this may perhaps obscure its significance for large deviations. It is our purpose here to show that this bound can be removed — roughly speaking, one can set  $R = \infty$  in the

definition of free entropy. This result applies to other notions of free entropy which appeared subsequently (see for instance [7] for free entropy in the presence of additional variables, [8] for free entropy using an ultrafilter, and [3] for free entropy of a nonselfadjoint variable). We will only provide the proof for the original quantity  $\chi$  defined in [6], but it should be obvious how the argument applies in the other situations.

It should be noted that a large deviation theorem for Wigner's result has been proved by G. Ben Arous and A. Guionnet [1], where the natural topology of weak convergence of probability measures on the real line is used. The rate function is closely related with free entropy. For several variables, a thorough study of large deviations was undertaken by T. Cabanal Duvallard and A. Guionnet [2]. The rate function they determine is related with another version of free entropy (microstate free).

## 2. The main result.

For the remainder of this note we fix a positive integer  $n$ . We will denote by  $I$  the collection of all multiindices  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$  with  $m \geq 1$  and  $\alpha_j \in \{1, 2, \dots, n\}$  for all  $j = 1, 2, \dots, m$ . In other words,  $I = \bigcup_{m=1}^{\infty} \{1, 2, \dots, n\}^m$ . A multiindex of the form  $(\alpha, \alpha, \dots, \alpha)$  will also be denoted  $\alpha^m$ . We consider the space  $\mathbb{S}$  consisting of all families  $(\mu(\alpha))_{\alpha \in I}$  of complex numbers indexed by  $I$ . The space  $\mathbb{S}$  will be endowed with the topology of componentwise convergence.

Consider now a tracial  $W^*$ -probability space  $(\mathfrak{A}, \tau)$ . That is,  $\mathfrak{A}$  is a von Neumann algebra, and  $\tau$  is a normal trace state on  $\mathfrak{A}$ . We will write  $\mathfrak{A}^{\text{sa}}$  for the space of selfadjoint elements of  $\mathfrak{A}$ . Given an  $n$ -tuple  $X = (X_1, X_2, \dots, X_n) \in (\mathfrak{A}^{\text{sa}})^n$ , its distribution  $\mu_X \in \mathbb{S}$  is defined by

$$\mu_X(\alpha) = \tau(X_\alpha),$$

where  $X_\alpha = X_{\alpha_1} X_{\alpha_2} \dots X_{\alpha_m}$  for  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \in I$ . This notation applies in particular to  $n$ -tuples of selfadjoint matrices in  $\mathfrak{M}_k$ . Voiculescu's entropy measures the extent to which the distribution of  $X$  can be approximated by distributions of the form  $\mu_A$  with  $A \in (\mathfrak{M}_k^{\text{sa}})^n$ . Note first that  $\mathfrak{M}_k^{\text{sa}}$  is a real Hilbert space with the Hilbert-Schmidt norm  $\|A\|_2 = \text{Tr}(A^2)$ , and  $\lambda_k$  will denote the corresponding Lebesgue measure (i.e., a cube whose sides form an orthonormal basis has measure equal to one). On the space  $(\mathfrak{M}_k^{\text{sa}})^n$  we have the product measure  $\lambda_k^{\otimes n}$ .

Given  $X \in (\mathfrak{A}^{\text{sa}})^n$ , and a neighborhood  $U$  of  $\mu_X$  in  $\mathbb{S}$ , we set

$$\Gamma(X; k, U) = \{A \in (\mathfrak{M}_k^{\text{sa}})^n : \mu_A \in U\}.$$

Given in addition a positive number  $R$ ,

$$\Gamma_R(X; k, U) = \{A \in \Gamma(X; k, U) : \|A_j\| < R \text{ for all } j\}.$$

We can then define the quantities

$$\chi_R(X; U) = \liminf_{k \rightarrow \infty} \left[ \frac{1}{k^2} \log \lambda_k^{\otimes n}(\Gamma_R(X; k, U)) + \frac{n}{2} \log k \right],$$

and

$$\chi_R(X) = \inf_U \chi_R(X; U),$$

where  $U$  runs over a neighborhood base of  $\mu_X$  in  $\mathbb{S}$ . Finally, the free entropy is defined as

$$\chi(X) = \sup_{R > 0} \chi_R(X).$$

We also set

$$\chi_\infty(X; U) = \liminf_{k \rightarrow \infty} \left[ \frac{1}{k^2} \log \lambda_k^{\otimes n}(\Gamma(X; k, U)) + \frac{n}{2} \log k \right],$$

and  $\chi_\infty(X) = \inf_U \chi_\infty(X; U)$ . This quantity was introduced in the concluding remarks of [6], where other possible definitions of free entropy are discussed briefly. The inequalities

$$\chi_R(X) \leq \chi(X) \leq \chi_\infty(X)$$

are obvious for  $R > 0$ , and Proposition 2.4 of [6] states that  $\chi_R(X) = \chi(X)$  if  $R$  is sufficiently large;  $R > \max_j \|X_j\|$  will suffice. Our main result is as follows:

**Proposition 2.1.** *For every  $X \in (\mathfrak{A}^{\text{sa}})^n$  we have  $\chi(X) = \chi_\infty(X)$ .*

The proof of this result is a refinement of the proof of Proposition 2.4 in [6]. We begin by considering the diffeomorphism  $f$  of the real line onto  $(-2, 2)$  defined by  $f(t) = t$  for  $t \in [-1, 1]$ ,  $f(t) = 2 - \frac{1}{t}$  for  $t > 1$ , and  $f(t) = -2 - \frac{1}{t}$  for  $t < -1$ . Observe that  $f'$  does not have any local minimum, and therefore

$$\frac{f(s) - f(t)}{s - t} \geq \min\{f'(s), f'(t)\} \geq f'(s)f'(t)$$

for all  $s$  and  $t$ . The function  $F_n : (\mathfrak{M}_k^{\text{sa}})^n \rightarrow (\mathfrak{M}_k^{\text{sa}})^n$  defined by

$$F_n(A_1, A_2, \dots, A_n) = (f(A_1), f(A_2), \dots, f(A_n))$$

is also differentiable, and we need to estimate the Jacobian determinant  $(JF_n)(A)$ . Since

$$(JF_n)(A) = (JF_1)(A_1)(JF_1)(A_2) \dots (JF_1)(A_n),$$

it suffices to do this in one variable. As pointed out in [6], if  $A$  is a  $k \times k$  matrix with eigenvalues  $\mu_1, \mu_2, \dots, \mu_k$ , we have

$$(JF_1)(A) = \left( \prod_{i \neq j} \frac{f(\mu_i) - f(\mu_j)}{\mu_i - \mu_j} \right) \cdot \prod_{i=1}^k f'(\mu_i).$$

By the estimate for difference quotients shown above,

$$\begin{aligned} (JF_1)(A) &\geq \left( \prod_{i \neq j} f'(\mu_i) f'(\mu_j) \right) \cdot \prod_{i=1}^k f'(\mu_i) \\ &= \prod_{i=1}^k f'(\mu_i)^{2k-1} = \prod_{|\mu_i| > 1} \mu_i^{-2(2k-1)}. \end{aligned}$$

Denoting  $\log^+(t) = \max\{\log t, 0\}$ , we obtain

$$\begin{aligned} \log(JF_1)(A) &\geq -2(2k-1) \sum_{i=1}^k \log^+ \mu_i \\ &= -2k(2k-1) \frac{1}{k} \sum_{i=1}^k \log^+ \mu_i \\ &= -2k(2k-1) \tau_k(\log^+ |A|). \end{aligned}$$

We have therefore proved the following estimate:

**Lemma 2.2.** *Given  $A = (A_1, A_2, \dots, A_n) \in (\mathfrak{M}_k^{\text{sa}})^n$ , we have*

$$(JF_n)(A) \geq \exp \left[ -2k(2k-1) \sum_{j=1}^n \tau_k(\log^+ |A_j|) \right].$$

Note for further use that, for a selfadjoint  $k \times k$  matrix  $A$ ,  $\tau_k(\log^+ |A|)$  can be estimated in terms of the moments  $\tau_k(A^{2p})$ ,  $p \geq 1$ . In fact,  $\log^+ t = \frac{1}{2p} \log^+ t^{2p} \leq \frac{1}{2p} t^{2p}$ , and therefore

$$\tau_k(\log^+ |A|) \leq \frac{1}{2p} \tau_k(A^{2p}).$$

We need one more ingredient.

**Lemma 2.3.** *Let  $X \in (\mathfrak{A}^{\text{sa}})^n$  satisfy  $\max_j \|X_j\| < 1$ , and let  $U$  be a neighborhood of  $\mu_X$  in  $\mathbb{S}$ . There exists a neighborhood  $V$  of  $\mu_X$  in  $\mathbb{S}$  such that*

$$F_n(\Gamma(X; k, V)) \subset \Gamma_2(X; k, U) \text{ for all } k.$$

*Proof.* Clearly it suffices to prove the lemma for neighborhoods of the form

$$U = \{\mu \in \mathbb{S} : |\mu(\alpha) - \tau(X_\alpha)| < \varepsilon\},$$

where  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \in I$  and  $\varepsilon > 0$  are fixed. Using the Hölder inequality

$$\|A_\alpha\|_1 \leq \|A_{\alpha_1}\|_m \|A_{\alpha_2}\|_m \cdots \|A_{\alpha_m}\|_m,$$

we see that it is sufficient to choose  $V$  so that, for all  $A \in \Gamma(X; k, V)$ , we have  $|\tau_k(A_\alpha) - \tau(X_\alpha)| < \varepsilon/2$ ,  $\|A_j\|_m \leq 1$ , and  $\|A_j - f(A_j)\|_m < \varepsilon/2m$  for

$j = 1, 2, \dots, n$ . Choose a number  $r < 1$  so that  $r > \|X_j\|$  for all  $j$ , and choose an even integer  $q > m$  such that  $r^{q/m} < \varepsilon/2m$ . Define next

$$V = \left\{ \mu \in \mathbb{S} : |\mu(\alpha) - \tau(X_\alpha)| < \frac{\varepsilon}{2} \text{ and } |\mu(j^q)| < r^q \text{ for } j = 1, 2, \dots, n \right\};$$

recall that  $j^q$  denotes the  $q$ -multiindex with all entries equal to  $j$ . Consider now  $A \in \Gamma(X; k, V)$ , and note that the inequalities  $|\tau_k(A_\alpha) - \tau(X_\alpha)| < \varepsilon/2$  are obviously satisfied. Also,

$$\|A_j\|_m \leq \|A_j\|_q = \tau(A_j^q)^{1/q} \leq r < 1.$$

Finally, if  $\mu_1, \mu_2, \dots, \mu_k$  are the eigenvalues of  $A_j$ ,

$$\begin{aligned} \|A_j - f(A_j)\|_m &\leq \left( \frac{1}{k} \sum_{|\mu_j| \geq 1} |\mu_j|^m \right)^{1/m} \leq \left( \frac{1}{k} \sum_{|\mu_j| \geq 1} |\mu_j|^q \right)^{1/m} \\ &\leq (\tau_k(A_j^q))^{1/m} \leq r^{q/m}, \end{aligned}$$

and this quantity is less than  $\varepsilon/2m$ .  $\square$

*Proof of Proposition 2.1.* It suffices to prove the proposition in case  $\|X_j\| < 1$  for all  $j$ . From the results of [6] we know that  $\chi_2(X) = \chi(X)$ , and clearly  $\chi_2(X) \leq \chi_\infty(X)$ . To prove the opposite inequality  $\chi_2(X) \geq \chi_\infty(X)$ , let  $U$  be a neighborhood of  $\mu_X$  in  $\mathbb{S}$ , and let  $V$  be the neighborhood of  $\mu_X$  furnished by Lemma 2.3, i.e.,  $F_n(\Gamma(X; k, V)) \subset \Gamma_2(X; k, U)$  for all  $k \geq 1$ . Given a positive integer  $p$ , we may also assume that  $\tau_k(A_j^{2p}) \leq 1$  whenever  $A = (A_1, A_2, \dots, A_n) \in \Gamma(X; k, V)$ . It follows then from Lemma 2.2 (and the remark following its statement) that

$$(JF_n)(A) \geq \exp \left[ -2k(2k-1) \frac{n}{2p} \right]$$

for all  $A \in \Gamma(X; k, V)$ . Since the function  $F_n$  is one-to-one, we deduce that

$$\begin{aligned} \lambda_k^{\otimes n}(\Gamma_2(X; k, U)) &\geq \lambda_k^{\otimes n}(F_n(\Gamma(X; k, V))) \\ &\geq \exp \left[ -2k(2k-1) \frac{n}{2p} \right] \lambda_k^{\otimes n}(\Gamma(X; k, V)). \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{1}{k^2} \log \lambda_k^{\otimes n}(\Gamma_2(X; k, U)) + \frac{n}{2} \log k \\ &\geq \frac{1}{k^2} \log \lambda_k^{\otimes n}(\Gamma(X; k, V)) + \frac{n}{2} \log k - \left( 2 - \frac{1}{k} \right) \frac{n}{p}, \end{aligned}$$

and as  $k \rightarrow \infty$  this yields

$$\chi_2(X; U) \geq \chi_\infty(X; V) - \frac{2n}{p}.$$

Since  $p$  is arbitrary, we deduce that  $\chi_2(X;U) \geq \chi_\infty(X;V) \geq \chi_\infty(X)$ , and the proof is concluded by taking the infimum over  $U$ .  $\square$

We remark that a suitable modification of the above proof yields directly that  $\chi_\infty(X) = \chi_R(X)$  if  $\|X_j\| < R$ . One needs an appropriate version of the function  $f$ , and that is easily constructed.

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