MATRICES A_p WEIGHTS VIA MAXIMAL FUNCTIONS

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The matrix A_p condition extends several results in weighted norm theory to functions taking values in a finite-dimensional vector space. Here we show that the matrix A_p condition leads to L^p-boundedness of a Hardy-Littlewood maximal function, then use this estimate to establish a bound for the weighted L^p norm of singular integral operators.

1. Preliminaries.

Weighted Norm theory forms a basic component of the study of singular integrals. Here one attempts to characterize those measure spaces over which a broad class of singular integral operators remain bounded. For the case of singular integral operators on C^d-valued functions in Euclidean space, the answer is given by the Hunt-Muckenhoupt-Wheeden theorem [10]. It states that the necessary and sufficient condition for boundedness in L^p(dμ) is that dμ = W(x) dx and the function W satisfies the A_p condition, namely:

\[ \left( \frac{1}{|B|} \int_B W \, dx \right)^{1/p} \left( \frac{1}{|B|} \int_B W^{-p'/p} \, dx \right)^{1/p'} \leq C \text{ for all balls } B \subset \mathbb{R}^n. \]

The A_p condition requires considerable interpretation in order to apply it to weighted measures of C^d-valued functions. First, the weight W(x) should take values in the space of positive d x d Hermitian forms. This raises concerns about the order in which products are taken, since matrices need not commute, and also what it means for the quantity on the left-hand side to be uniformly bounded. Treil' [21] conjectured that the correct statement of the matrix A_2 condition should be

\[ \sup_B \left\| \left( \frac{1}{|B|} \int_B W \, dx \right)^{1/2} \left( \frac{1}{|B|} \int_B W^{-1} \, dx \right)^{1/2} \right\| < \infty \]

where exponents 1/2 indicate operator powers of a nonnegative matrix. This was subsequently proven in [23] and again in [24].

If p is different from 2, the matrix A_p condition cannot be written in terms of averages of operator powers of weight W. Averages still play a crucial role, however it is more accurate to regard W(x) as a Banach space norm on C^d rather than a matrix. A correct formulation of the matrix A_p condition, which is also the subject of this note, first appeared in [12] and [25]. Because their statements do not appear similar, it is especially...
important to understand what properties matrix $A_p$ weights share with their scalar counterparts. This is discussed further in the next section.

Boundedness estimates on singular integral operators were originally obtained by way of the Hardy-Littlewood maximal function $M$. If a scalar weight $W$ possesses the $A_\infty$ property (several equivalent definitions are given in [18]), then the $L^p$ norm of any singular integral is dominated by the $L^p$ norm of $M$ via a distributional argument commonly known as the good-$\lambda$ inequality. The $A_p$ condition is specifically required to ensure that

$$
\|Mf\|_{L^p(W)} \leq C\|f\|_{L^p(W)}.
$$

Some of these techniques fail to generalize to the case of vector-valued functions with matrix weights. There is no known analogue of the $A_\infty$ property to create simultaneous estimates for every exponent $p$. The weak-$L^p(W)$ spaces used to prove boundedness of the Hardy-Littlewood maximal function are not well defined in this setting. In general, much of the ability to compare objects and dominate one by another is lost when the objects are vectors rather than scalars. The theory of matrix weights has consequently evolved along much different lines. One fundamental technique employed in both [23] and [25] is to choose a good basis (often inspired by Haar functions) in $L^p(W)$ and consider the integral operator as a matrix acting on the coefficient space. Estimates may then be made separately on the matrix and on the coefficient embedding operator. Even in the scalar case these ideas have yielded new results and new ways of approaching weighted norm problems.

In this note we attempt to tackle the difficulties of extending the classical theory, or else circumvent them. Some arguments may be borrowed nearly word for word, some remain intact only if they are presented in a specific manner. Our hope is to discover which properties of scalar $A_p$ weights admit some generalization to the case of vector-valued functions and matrix weights, leading to a more complete understanding of the matrix $A_p$ class.

Let $T$ be a singular integral operator associated to kernel $K(x)$ in the sense that $Tf(x) = \int_{\mathbb{R}^n} K(x - y)f(y)\,dy$ for almost every $x$ outside the support of $f$. The following regularity hypotheses are to be assumed for $K$:

$$
|K(x)| \leq C|x|^{-n} \quad \text{and} \quad |\nabla K(x)| \leq C|x|^{-n-1}
$$

and additionally we suppose that for some choice of $p$, $1 < p < \infty$, the bound $\|Tf\|_{L^p} \leq A\|f\|_{L^p}$ holds for all $f \in L^p$. One may then apply $T$ to functions taking values in $\mathbb{C}^d$ by allowing it to act separately on each coordinate function, that is: $(Tf)_j = Tf_j$. This new operator, also denoted by $T$, is a singular integral operator whose associated convolution kernel is $K$ times the identity matrix.

In a similar manner, define the truncated operators $T_\epsilon$ to be convolution with $K_\epsilon(x) = \chi_{\{|x| > \epsilon\}}K(x)$ for all $\epsilon > 0$. Note that $T$ and the $T_\epsilon$ all commute
with pointwise multiplication by any constant matrix \( \Lambda \), in other words \( \Lambda Tf = T(\Lambda f) \).

A matrix weight \( W \) is a function on \( \mathbb{R}^n \) taking values in \( d \times d \) positive-definite matrices, with weighted norm space \( L^p(W) \) defined by
\[
\| f \|_{L^p(W)}^p = \int_{\mathbb{R}^n} |W^{1/p} f|^p dx.
\]

One is often concerned with the relationship between a weight and its average over arbitrary balls. The most straightforward notion of an average,
\[
W_B = \frac{1}{|B|} \int_B W \, dx,
\]
turns out to be useful only in the study of \( L^2(W) \). With any exponent \( p \neq 2 \), this does not properly respect the structure of the underlying \( L^p \)-space. The following definitions are needed instead:

A metric \( \rho = \rho_x(\cdot) \) denotes a family of Banach space norms on \( C^d \), indexed by \( x \in \mathbb{R}^n \). The weighted norm space \( L^p(\rho) \) is given by
\[
\| f \|_{L^p(\rho)}^p = \int_{\mathbb{R}^n} \left[ \rho_x(f(x)) \right]^p dx.
\]

Note that for any matrix weight \( W \), \( L^p(W) \) is isometrically equivalent to \( L^p(\rho) \) with the metric \( \rho_x(e) = |W^{1/p}(x)e| \). Given a ball \( B \subset \mathbb{R}^n \) and an exponent \( p > 1 \), let \( \rho_{p,B} \) be defined by the formula
\[
\rho_{p,B}(e) = \left( \frac{1}{|B|} \int_B \left[ \rho_x(e) \right]^p dx \right)^{1/p}.
\]

This will be our method for averaging the metric \( \rho \) over a ball \( B \).

The dual metric \( \rho^* \) is defined pointwise in \( x \) to be
\[
\rho_x^*(e) = \sup_{f \in C^d} \frac{|(e,f)|}{\rho_x(f)}.
\]

One immediate consequence is that \( (e,f) \leq \rho_x^*(e) \rho_x(f) \).

**Proposition 1.1.** For any \( e \in C^d \) and any ball \( B \subset \mathbb{R}^n \),
\[
\rho_{p^*,B}(e) \geq (\rho_{p,B})^*(e).
\]

**Proof.** Given two vectors \( e, f \in C^d \),
\[
(e,f) \leq \frac{1}{|B|} \int_B \rho_x^*(e) \rho_x(f) \, dx \\
\leq \left( \frac{1}{|B|} \int_B \left[ \rho_x^*(e) \right]^{p'} dx \right)^{1/p'} \cdot \left( \frac{1}{|B|} \int_B \left[ \rho_x(f) \right]^p dx \right)^{1/p} \\
= \rho_{p',B}^*(e) \rho_{p,B}(f).
\]

In other words, \( \rho_{p',B}^*(e) \geq \frac{(e,f)}{\rho_{p,B}(f)} \). The proof is completed by taking the supremum over all \( f \in C^d \).
A metric $\rho$ is called an $A_p$ metric if there exists some constant $C < \infty$ so that the opposite statement

\begin{equation}
\rho_{\rho', B}(e) \leq C (\rho_{p, B})^*(e) \quad \text{for all balls } B \subset \mathbb{R}^n
\end{equation}

is also true. Since the averages over cubes and balls in $\mathbb{R}^n$ differ by no more that a fixed constant, $A_p$ metrics satisfy an analogous condition for cubes, and vice versa. Stated either way, the $A_p$ condition characterizes an important class of weighted measures.

**Theorem 1** (Nazarov, Treţil' [12], Volberg [25]). Let $d < \infty$. The following statements are equivalent:

1) The Hilbert Transform is bounded on $L^p(\rho)$.

2) $\rho$ is an $A_p$ metric.

We will prove this theorem again for metrics which are induced by some matrix weight $W$. There is no loss of generality because for fixed dimension $d < \infty$ every metric can be uniformly approximated by matrix weights.

**Proposition 1.2.** Let $d < \infty$. Given a Banach space norm $\rho_x$ on $\mathbb{C}^d$, there exists a positive selfadjoint matrix $W_x$ such that

\begin{equation}
\rho_x(e) \leq |W_x(e)| \leq \sqrt{d} \cdot \rho_x(e) \quad \text{for all } e \in \mathbb{C}^d.
\end{equation}

**Proof.** Let $O$ represent the unit ball of $\rho_x$, and $E$ the ellipsoid of maximal volume contained in $O$. There exists a positive selfadjoint matrix $W_x$ such that $W_x(E)$ is the standard unit ball in $\mathbb{C}^d$. The image $W_x(O)$ is a convex balanced set containing the unit ball, and containing no ellipsoid of greater volume.

If there exists a point $v \in W_x(O)$ with $|v| > \sqrt{d}$, then by convexity the boundary of $W_x(O)$ can only be tangent to the unit sphere at points $w$ such that

\[ (w, v) \leq \frac{1}{|v|} < \frac{1}{\sqrt{d}}. \]

For some $\delta > 0$ the ellipsoid with major axis length $e^\delta$ in the direction of $v$ and minor axes length $e^{-\delta/(|v|^2-1)}$ in every direction perpendicular to $v$ is also contained in $W_x(O)$. This has strictly greater volume than the unit ball, contradicting the property of $W_x(O)$ stated above. \qed

It is now possible to state the $A_p$ condition in terms of matrix weights, though some precision is lost in the process. Given a matrix weight $W$ and a ball $B \subset \mathbb{R}^n$, define a Banach space norm $X_B$ on $\mathbb{C}^d$ by considering the $L^p(W)$ norm of characteristic functions on $B$.

\[ \|v\|_{X_B} = |B|^{-1/p} \|\chi_B v\|_{L^p(W)}. \]

By Proposition 1.2 there exists a positive-definite $d \times d$ matrix $V_B$ such that $\|v\|_{X_B} \leq |V_B v| \leq d^{1/2}\|v\|_{X_B}$. From a heuristic standpoint, $V_B$ might
be considered an “$L^p$ average” of $W^{1/p}$ over ball $B$. With $p' = \frac{p}{p-1}$ the dual exponent to $p$, let $V_B'$ be an $L^{p'}$ average of $W^{-1/p}$. In summary, matrices $V_B, V_B'$ enjoy the following properties:

\begin{align}
|V_Bv| &\sim |B|^{-1/p} \| \chi_B W^{1/p} v \|_{L^p} \\
|V_B'v| &\sim |B|^{-1/p'} \| \chi_B W^{-1/p} v \|_{L^{p'}}. 
\end{align}

**Remark.** The definition of $V_B$ and $V_B'$ depends implicitly on the method used to approximate Banach space norms by matrices. For the purposes of our discussion, $V_B$ and $V_B'$ may be any two matrices satisfying (5).

A matrix weight $W$ satisfies the matrix $A_p$ condition if $V_B V_B'$ are uniformly bounded as operators on $C^d$; that is

\begin{align}
\|V_B V_B'\| \leq C < \infty \quad \text{for all balls } B \subset \mathbb{R}^n. 
\end{align}

The exact value of $C$ depends on the choice of $V_B$ and $V_B'$, and is therefore determined here only up to a factor of $d$.

Our approach to Theorem 1 is styled after Coifman and Fefferman’s proof [5] in the scalar ($d = 1$) case. Two technical problems arise immediately: First that general $d \times d$ matrices do not commute with one another, and second the matter of defining a maximal operator for vector-valued functions. To choose pointwise a vector with the largest $\ell^2(C^d)$ magnitude is clearly wrong because the effect of weight $W(x)$ may depend strongly on the direction. In the special case where $W$ is uniformly nonsingular (i.e., $\|W(x)\| \cdot \|W^{-1}(x)\| \leq C$ for all $x$) this can be controlled by a constant factor, but we have no such a priori assumptions about $W$.

For this reason our analysis will take place primarily in unweighted $L^q$ spaces, following [4]. Rather than deal with $T$ directly, we consider the action of $W^{1/p} T W^{-1/p}$ on functions in $L^q(dx)$. With the family of truncated operators $W^{1/p} T_\epsilon W^{-1/p}$ in mind, we define the maximal truncated operator $(W^{1/p} T)_*$ to be

\begin{align}
(W^{1/p} T)_* f(x) = \sup_{\epsilon > 0} |W^{1/p} T_\epsilon f(x)|
\end{align}

with the convention that $f = W^{-1/p} g$ and $g$ is a function in $L^q(dx)$. One estimate from the scalar theory that remains wholly intact is the bound

\begin{align}
|W^{1/p} T W^{-1/p} g(x)| \leq |(W^{1/p} T)_* W^{-1/p} g(x)| + C |g(x)|. 
\end{align}

The constant $C$ depends only on our choice of operator $T$ but not on the function $g$. This will allow us to infer the boundedness of $T$ by controlling the behavior of its truncations. Our primary results are the following four theorems, numbered according to the section in which they appear:
Four Theorems.

(3.2) If \( W \) is a matrix \( A_p \) weight, there exists \( \delta > 0 \) such that the vector Hardy-Littlewood maximal function \( M_w \) (defined in Section 3) is a bounded operator from \( L^q(\mathbb{R}^n; \mathbb{C}^d) \) to \( L^q(\mathbb{R}; \mathbb{R}) \) whenever \( |p - q| < \delta \).

(4.2): Given a singular integral operator \( T \) as above, and a weight \( W \in A_p \), there exists \( \delta > 0 \) such that \( (W^{1/p}T)_*W^{-1/p} \) is a bounded operator from \( L^q(\mathbb{R}^n; \mathbb{C}^d) \) to \( L^q(\mathbb{R}; \mathbb{R}) \) whenever \( |p - q| < \delta \).

(5.1): Consequently \( W^{1/p}TW^{-1/p} \) is bounded on \( L^q(\mathbb{R}^n; \mathbb{C}^d) \) for this range of exponents \( q \).

(5.2): In particular, \( T \) is bounded on \( L^p(W) \) if \( W \in A_p \). With one additional hypothesis on the structure of \( T \), the converse statement is also true.

Remark. The exponent \( W^{1/p} \) is used throughout, even when we are considering functions under an \( L^q \) norm with \( q \neq p \). This places us squarely in the setting of [25], where the \( A_p \) metric \( W^{1/p} \) is the basic object of study. Theorem 5.1 then asserts that any \( A_p \) metric is also an \( A_q \) metric for all \( q \) in some open interval containing \( p \).

2. Properties of \( A_p \) weights.

We would like first to characterize the matrix \( A_p \) class in a more transparent manner by borrowing a lemma from [12]:

**Proposition 2.1.** A metric \( \rho_x \) satisfies the \( A_p \) condition if and only if the operators \( f \rightarrow \chi_B \frac{1}{|B|} \int_B f \, dx \) are uniformly bounded on \( L^p(\rho) \). In fact, the uniform bound is equal to the \( A_p \) constant of \( \rho \).

**Proof.** The \( L^p(\rho) \) norm of \( \chi_B \frac{1}{|B|} \int_B f \, dx \) is given by

\[
\frac{1}{|B|} \left( \int_B \left[ \rho_y \left( \int_B f \, dx \right) \right]^p \, dy \right)^{1/p},
\]

which in turn is equal to \( |B|^{-1/p'} \rho_{p,B}(\int_B f \, dx) \). Therefore

\[
\sup_{\|f\|_{L^p(\rho)} = 1} \|\chi_B \frac{1}{|B|} \int_B f \, dx\|_{L^p(\rho)} = \sup_{f} \sup_{e \in \mathbb{C}^d} |B|^{-1/p'} \int_B (e, f(x)) \, dx / (\rho_{p,B}(e))
\]

\[
= \sup_{e \in \mathbb{C}^d} |B|^{-1/p'} \|\chi_B e\|_{L'^{p'}(\rho^*)} / (\rho_{p,B}(e))
\]

\[
= \sup_{e \in \mathbb{C}^d} \rho_{p',B}(e) / (\rho_{p,B}(e)).
\]

Equality between the first and second lines takes place because \( L^p(\rho) \) is the dual space of \( L'^{p'}(\rho^*) \). \( \square \)
Corollary 2.2. Let \( \rho \) be an \( A_p \) metric. For any vector \( v \in \mathbb{C}^d \), \( \rho_x(v)^p \) is a scalar \( A_p \) weight with constant less than or equal to that of \( \rho \).

Proof. Let \( \phi \) be any scalar function and consider \( f = \phi v \). The weighted norm of \( f \) is \( \|f\|_{L^p(\rho)} = (\int_B \phi^p [\rho_x(v)]^p dx)^{1/p} \). Proposition 2.1 applied to \( f \) states that all maps \( \phi \to \chi_B \frac{1}{|B|} \int_B \phi \, dx \) are uniformly bounded on the \( L^p \) space with measure \( [\rho_x(v)]^p dx \), with norms less than the \( A_p \) constant of \( \rho \). We now apply Proposition 2.1 again, this time in the scalar setting, to conclude that \( [\rho_x(v)]^p \) is a scalar \( A_p \) weight whose constant is also less than the \( A_p \) constant of \( \rho \). \( \square \)

Corollary 2.3. If \( W \) is a matrix \( A_p \) weight, then \( \|W\| \) is a scalar \( A_p \) weight.

Proof. Let \( e_i \) be the standard unit basis for \( \mathbb{C}^d \). Since \( W(x) \) is a nonnegative and selfadjoint operator at each point \( x \),

\[
\|W(x)\| = \|W^{2/p}(x)\|^{p/2} \sim [\text{tr}(W^{2/p}(x))]^{p/2} \\
= \left( \sum_{i=1}^d |W^{1/p}(x)e_i|^2 \right)^{p/2} \\
\sim \sum_{i=1}^d |W^{1/p}(x)e_i|^p
\]

is pointwise in \( x \). By Corollary 2.2, each individual function \( |W^{1/p}(x)e_i|^p \) is a scalar \( A_p \) weight, therefore their sum is as well. \( \square \)

Remarks. Both of these corollaries are proven in [23] for the case \( p = 2 \), and are adapted here with minimal alteration.

From this point forward we will work exclusively in the language of matrix weights. While our primary definition of \( A_p \) weights (6) is decidedly less elegant than that of \( A_p \) metrics (3), the ability to use notation and theorems from linear algebra makes it a worthwhile sacrifice.

One crucial feature in the theory of scalar \( A_p \) weights is the presence of “Reverse Hölder” inequalities estimating the average of \( W^{1+\epsilon} \) in terms of the average of \( W \). We will employ inequalities of a similar character as the centerpiece of our analysis.

Proposition 2.4. Let \( W \) be an \( A_p \) weight. Then there exist \( \delta > 0 \) and constants \( C_q \) such that for all balls \( B \subset \mathbb{R}^n \),

\[
\frac{1}{|B|} \int_B \|W^{1/p}(y)V_B\|q dy \leq C_q, \text{ all } q < p + \delta
\]

(10)

\[
\frac{1}{|B|} \int_B \|V_B W^{-1/p}(y)\|q dy \leq C_q, \text{ all } q < p' + \delta.
\]

(11)
Proof. We will verify only the first of these statements. The second one is proven in an identical manner with the starting point that $W^{-p'/p}$ is an $A_p$ weight.

By Corollary 2.2, all functions of the form $|W^{1/p}(y)V_B'\mathbf{e}|^p$ are scalar $A_p$ weights with $A_p$ norms bounded uniformly in $e$. It is therefore possible to choose $q$ and $C_q$ so that the Reverse Hölder inequality
\[
\frac{1}{|B|} \int_B |W^{1/p}(y)V_B'\mathbf{e}|^q dy \leq C_q \left( \frac{1}{|B|} \int_B |W^{1/p}(y)V_B'\mathbf{e}|^p dy \right)^{q/p}
\]
is satisfied for all $\mathbf{e} \in \mathbb{C}^d$.

Let $\mathbf{e}_i$ once again be the standard unit basis for $\mathbb{C}^d$. It is useful to remember that the norm of any $d \times d$ matrix $M$ (not necessarily Hermitian) is controlled by its action on the vectors $\mathbf{e}_i$ via the formula
\[
\|M\| \leq d^{1/2} \sup_i |M\mathbf{e}_i|.
\]

We may now estimate the desired integral:
\[
\frac{1}{|B|} \int_B \|W^{1/p}(y)V_B'\|^q dy \leq \frac{1}{|B|} \int_B \left( d^{1/2} \sup_i |W^{1/p}(y)V_B'\mathbf{e}_i| \right)^q dy
\]
\[
\leq d^{q/2} \sum_{i=1}^d \frac{1}{|B|} \int_B |W^{1/p}(y)V_B'\mathbf{e}_i|^q dy
\]
\[
\leq C_q \sum_{i=1}^d \left( \frac{1}{|B|} \int_B |W^{1/p}(y)V_B'\mathbf{e}_i|^p dy \right)^{q/p}
\]
\[
\sim C_q \sum_{i=1}^d |V_B V_B'\mathbf{e}_i|^q \leq d \cdot C_q \|V_B V_B'\|^q \leq C_q.
\]

\[\square\]

Note. In later sections we will also use the slightly weaker inequality
\[
|B|^{-1} \int_B \|W^{1/p}(y)V_B^{-1}\|^q dy \leq C_q, \; \text{all } q < p + \delta
\]
whose proof follows the above calculations almost word for word.

3. The Hardy-Littlewood maximal function.

There is a wide variety of possible maximal functions to choose from, each of which has its own advantages and limitations. In [4] we first considered an auxiliary maximal function $M'_w$, given by
\[
M'_w(g)(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |V_B W^{-1/p}(y)g(y)| dy.
\]
Although the intuitive meaning of $M'_w$ is unclear, one may approach it with
the classical tools of weak-type inequalities and interpolation. A direct
application of the second reverse Hölder inequality (11) proves the following
lemma:

**Lemma 3.1.** Let $W$ be an $A_p$ weight. Then there exists $\delta > 0$ such that
\[
\|M'_w g\|_{L^q} \leq C_q \|g\|_{L^q(R^n;C^d)}, \quad \text{all } g \in L^q, \text{ all } q > p - \delta.
\]

*Sketch of Proof.* The operators $g \mapsto \chi_B \frac{1}{|B|} \int_B |V_B W^{-1/p} g| \, dy$ are uniformly bounded in $L^q$ if $p - \delta < q$. This is a consequence of (11) together with the
inequality
\[
\int_B |V_B W^{-1/p}(y)g(y)| \, dy \leq \left( \int_B \|V_B W^{-1/p}(y)\|_{L^q} \, dy \right)^{1/q'} \|g\|_{L^q}
\]
One may use the Vitali Covering Lemma to obtain a weak-type $(q,q)$ es-
timate on the associated maximal function $M'_w$. The lemma then follows
from the Marcinkiewicz Interpolation Theorem. \qed

The vector Hardy-Littlewood maximal function $M_w$ is defined as
\[
(14) \quad M_w g(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |W^{1/p}(x) W^{-1/p}(y)g(y)| \, dy.
\]

The following equivalent definition of $M_w$ is often quite useful:
\[
(15) \quad M_w g(x) = M(|W^{1/p}(x) W^{-1/p}(\cdot)g(\cdot)|)(x).
\]

Here $M$ denotes the classical Hardy-Littlewood maximal operator acting
on scalar-valued functions. The only difference between $M_w$ and $M'_w$ is the
presence of a weight $W^{1/p}(x)$ rather than an average weight $V_B$ over a ball
containing $x$. The reverse Hölder inequalities suggest that $A_p$ weights are
often pointwise comparable to their averages, in which case $\|M_w g\|$ would be
controlled by $\|M'_w g\|$. For a range of exponents near $p$, this line of reasoning
can be made precise.

**Theorem 3.2.** Let $W$ be an $A_p$ weight. Then there exists $\delta > 0$ such that
\[
\|M_w g\|_{L^q} \leq C_q \|g\|_{L^q(R^n;C^d)}, \quad \text{all } g \in L^q, \text{ all } |p - q| < \delta.
\]

*Proof.* Let us suppose for a moment that the suprema defining $M_w g$ and
$M'_w g$ are taken over cubes in some dyadic grid. The entire preceding dis-
}
For each integer $j$, define $\{S_j\}$ to be the collection of dyadic cubes $R = R_x$ that are maximal with respect to the property

$$2^j \leq |R|^{-1} \int_{R} |V_{R}W^{-1/p}(y)g(y)|dy < 2^{j+1}.$$  

Maximality insures that whenever $M_wg(x) \neq 0$ the cube $R_x$ is contained in some $S_j$ with

$$|R_x|^{-1} \int_{R_x} |V_{R_x}W^{-1/p}(y)g(y)|dy \leq 2|S_j|^{-1} \int_{S_j} |V_{S_j}W^{-1/p}(y)g(y)|dy.$$  

When $j$ is fixed, the disjoint union $\bigcup_j S_j$ is contained in the set where $M'_w g(x) \geq 2^j$.

Consider the functions $N_Q(x) = \sup_{x \in R \subset Q} \|W^{1/p}(x)V_R^{-1}\|$, defined on their respective cubes $Q$. By virtue of the preceding two statements, the inequality $M_w g(x) \leq 4 \cdot 2^{j+1} N_{S_j}(x)$ must hold for some number $j$ (this is trivial at the points where $M_w g(x) = 0$). It follows that

$$\|M_w g\|_{L^q}^q \leq C \sum_{j=-\infty}^{\infty} 2^{jq} \sum_{S_j} \int_{S_j} (N_{S_j}(x))^q dx.$$  

By Lemma 3.3 below, we can continue the estimate as follows:

$$\|M_w g\|_{L^q}^q \leq C \sum_{j=-\infty}^{\infty} 2^{jq} \sum_{S_j} |S_j| \leq C \sum_{j=-\infty}^{\infty} 2^{jq} |\{M'_w g \geq 2^j\}| \leq C \|M'_w g\|_{L^q}^q.$$  

The proof is then complete by Lemma 3.1.

**Lemma 3.3.** Let $W$ be a matrix $A_p$ weight and functions $N_Q(x)$ be defined as above. Then there exist $\delta > 0$ and $C_q < \infty$ such that for all dyadic $Q$,

$$\int_Q (N_Q(x))^q dx \leq C_q |Q|$$  

for all $q < p + \delta$.

**Proof.** We present an informal argument here, assuming that $\int_Q N_Q^q \leq B|Q|$ with some finite $B$ then deriving an *a priori* bound for $B$. This may be readily adapted into a rigorous proof.

Let $A < \infty$ be a large constant to be specified later. Denote by $\{R_j\}$ the set of maximal cubes satisfying $\|V_QV_{R_j}^{-1}\| > A$. Outside of $\bigcup_j R_j$, $N_Q(x) \leq A\|W^{1/p}(x)V_R^{-1}\|$. Thus $\int_{Q \cup \bigcup_j R_j} (N_Q(x))^q dx \leq C|Q|$, seen by applying reverse Hölder inequality (12).
We claim that \( \sum_j |R_j| < \frac{1}{2}|Q| \) if \( A \) is sufficiently large. Remember first that \( \|V_Q V^{-1}_{R_j}\| = \|V^{-1}_{R_j} V_Q\| \leq C \|V_{R_j} V_Q\| \), by Proposition 1.1. It follows that

\[
|R_j| \cdot \|V_Q V^{-1}_{R_j}\|^{p'} \leq C \sup_{|e|=1} |R_j| \cdot \|V_{R_j} V_Q e\|^{p'}
\]

\[
\sim C \sup_{|e|=1} \int_{R_j} |W^{-1/p}(y) V_Q e|^{p'} \, dy
\]

\[
\leq C \int_{R_j} \|W^{-1/p}(y) V_Q\|^{p'} \, dy.
\]

The cubes \( R_j \) are disjoint from one another, so

\[
A^{p'} \sum_j |R_j| < C \int_{\cup_j R_j} \|W^{-1/p}(y) V_Q\|^{p'} \, dy
\]

\[
\leq C \int_Q \|W^{-1/p}(y) V_Q\|^{p'} \, dy \leq C |Q|.
\]

This estimate shows that for \( A \) large enough, \( \sum_j |R_j| < \frac{1}{2}|Q| \), and the value of \( A \) may be chosen independently of \( Q \).

Inside each cube \( R_j \), we may assume that \( N_Q(x) = N_{R_j}(x) \), otherwise the bound \( N_Q(x) \leq A \|W^{1/p}(x) V^{-1}_Q\| \) still holds. Then

\[
\int_{\cup_j R_j} (N_Q(x))^q = \sum_j \int_{R_j} (N_{R_j}(x))^q \leq B \sum_j |R_j| < \frac{1}{2} B |Q|.
\]

Putting these pieces together, we would discover that \( B \leq C + \frac{1}{2} B \), where \( C < \infty \) is determined by the constants in the reverse Hölder inequality. \( \square \)

This concludes the proof that matrix \( A_p \) weights enjoy \( L^q \)-boundedness of the dyadic Hardy-Littlewood maximal function for a range of exponents \( |q - p| < \delta \). There is a standard argument employing two incompatible dyadic grids [7] for extending results of this kind to the general setting. Thus the Hardy-Littlewood maximal function as we originally defined it (as a supremum over balls containing \( x \)) is bounded in \( L^q \) for the same range of exponents \( q \).

4. A distributional inequality.

**Proposition 4.1.** Let \( W \) be a matrix \( A_p \) weight and fix \( q < p + \delta \). Then there exist positive constants \( 0 < b < 1, c > 0 \) depending only on \( q \), the \( A_p \)
"norm" of $W$, and the dimensions $d, n$ such that
\begin{equation}
\left\{ x \in \mathbb{R}^n : (W^{1/p}T) * f(x) > \alpha ; \max (M'_w(W^{1/p}f)(x), M_w(W^{1/p}f)(x)) < c\alpha \right\} < \frac{1}{b} |\{ x \in \mathbb{R}^n : (W^{1/p}T) * f(x) > b\alpha \}| \end{equation}
for all $f \in C^\infty_c(\mathbb{R}^n; \mathbb{C}^d)$.

From this point onward we follow as closely as possible in the footsteps of Coifman and Fefferman [5], decomposing the set where $(W^{1/p}T) * f > b\alpha$ into a union of cubes and proving the desired inequality on each cube separately. Our decomposition uses a slightly modified version of the Whitney covering lemma, stated below.

**Covering Lemma.** Given a set $E \subset \mathbb{R}^n$ of finite (Lebesgue) measure, there exists a collection $\{Q_j\}$ of pairwise disjoint cubes such that:

i) $E \subset \cup_j Q_j$ up to sets of measure zero,

ii) $|Q_j \cap E| \geq \frac{1}{2} |Q_j|$, 

iii) $|3Q_j \cap E^c| \geq C_n |3Q_j|$.

A simple consequence of Statements i) and ii) is that $\sum_j |Q_j| \leq 2|E|$.

**Proof.** Let $\{Q_j\}$ be the collection of dyadic cubes maximal under the property that $|Q \cap E| \geq \frac{1}{2} |Q|$. Then Conditions ii) and iii) hold with constant $C_n = \frac{1}{2} \cdot (\frac{2}{3})^n$. The first condition also holds because as $\epsilon \to 0$, the ratio $|B(x, \epsilon) \cap E|/|B(x, \epsilon)| \to 1$ at almost every $x \in E$. $\square$

**Proof of Proposition 4.1.** Write $f = W^{-1/p} g$ and let

$E = \{ x \in \mathbb{R}^n : (W^{1/p}T) * f(x) > b\alpha \}$.

Apply the covering lemma to obtain cubes $\{Q_j\}$ with the specified properties. It suffices to verify that in each cube $Q = Q_j$ there is a distributional inequality
\begin{equation}
\left\{ x \in Q : (W^{1/p}T) * f(x) > \alpha ; \max (M'_w g(x), M_w g(x)) < c\alpha \right\} < \frac{1}{4} b^q |Q|.
\end{equation}

For this we use a construction similar to the one in [5]. Let $O$ be the ball with the same center as $Q_j$ and radius $5 \text{ diam } (Q)$. By the covering lemma and inequality (11), there exists a point $\overline{x} \in 3Q$ such that

$(W^{1/p}T) * f(\overline{x}) < b\alpha$ and $\|V_0 W^{-1/p}(\overline{x})\| < C$.

Let $B = B(\overline{x}, 3 \text{ diam}(Q_j))$. Since $B \subset O$ and is of comparable size, $\|V_B V_0^{-1}\|$ is bounded by a constant and hence $\|V_B W^{-1/p}(\overline{x})\| < C$. 
Assume $|\{x \in Q : M'_w g(x) < c\alpha\}| \geq \frac{1}{4} b^2 |Q|$, otherwise the proposition is trivially satisfied. Then there exists a point $\overline{y} \in Q$ such that 
\[ M_w g(\overline{y}) < c\alpha \text{ and } \|V_B W^{-1/p}(\overline{y})\| \leq C b^{-1}. \]

Write $f_1 = \chi_B f$ and $f_2 = \chi_{B^c} f$. By the sublinearity of $(W^{1/p} T)_*$, the set where $(W^{1/p} T)_* f(x) > \alpha$ is contained in the union of sets $\{(W^{1/p} T)_* f_i(x) > \alpha/2\}$, $i = 1, 2$.

The operator $T_*$ is weak-type $(1, 1)$. This fact is easily obtained from the scalar case when $d$ is finite, but is also true in general [17]. Consequently, 
\[ \left| \left\{ (V_B T)_* f_1(x) > \frac{\alpha}{2R} \right\} \right| \leq \frac{AR}{\alpha} \|V_B f_1\|_{L^1(\mathbb{R}^n, C^d)}. \]

Here we are using the property that operator $T_*$ commutes with multiplication by any constant matrix, in this case $V_B$. Furthermore, 
\[ \|V_B f_1\|_{L^1} = \int_B |V_B f(y)| dy \leq |B| M'_w g(\overline{y}) \leq C c\alpha |Q| \]
with the end result that $\left| \left\{ x \in Q : (V_B T)_* f_1(x) > \frac{\alpha}{2} \right\} \right| \leq C cR |Q|.$

It follows that $\left| \left\{ x \in Q : (W^{1/p} T)_* f_1(x) > \frac{\alpha}{2} \right\} \right| \leq (C cR + C' R^{-p}) |Q|$ for all $R > 0$, because the Reverse Hölder inequality (10) guarantees that $\|W^{1/p}(x)V_B^{-1}\| \leq R$ except on a set of measure less than $C' R^{-p}$. Taking the infimum over $R$,
\[ \left| \left\{ x \in Q : (W^{1/p} T)_* f_1(x) > \alpha/2 \right\} \right| \leq C_0 e^{p/(p+1)} |Q|. \]

For the second estimate, we begin by noting that the point $\overline{x}$ is chosen so that $(W^{1/p} T)_* f(\overline{x}) < b\alpha$ and $\|V_B W^{-1/p}(\overline{x})\| < C$. Then $(V_B T)_* f(\overline{x}) < C b\alpha$. Our estimate for $\left| \left\{ (W^{1/p} T)_* f_2(x) > \alpha/2 \right\} \right|$ relies on the following inequality which holds for all $x \in Q$:
\[ (V_B T)_* f_2(x) \leq (V_B T)_* f(\overline{x}) + C' M\left(\|V_B f\|_{L^1}\right)(\overline{y}) \leq C b\alpha + C' \|V_B W^{-1/p}(\overline{y})\| \cdot M\left(\|W^{1/p} f\|_{L^1}\right)(\overline{y}) \leq C b\alpha + C' \|V_B W^{-1/p}(\overline{y})\| \cdot M_w g(\overline{y}) \leq (C b + C' b^{-1} c) \alpha. \]

In the preceding expressions $M(\cdot)$ denotes the scalar Hardy-Littlewood maximal function.

Imitating the method for the $|(W^{1/p} T)_* f_1|$ estimate, we see that 
\[ \left| \left\{ x \in Q : (W^{1/p} T)_* f_2(x) > R(C b + C' b^{-1} c) \alpha \right\} \right| \leq AR^{-r} |Q| \]
where $r$ may be chosen so that $q < r < p + \delta$. Once again (10) has been invoked, this time to guarantee that $\|W^{1/p} V_B^{-1}\| > R$ only on a set of measure less than $C R^{-r}|B|$. Set $R$ equal to $(4bC)^{-1}$. Then
\[ \left| \left\{ x \in Q : (W^{1/p} T)_* f_2(x) > (1/4 + C_1 b^{-2} c) \alpha \right\} \right| \leq C_2 b^r |Q|. \]
Proof. As in the scalar case, the truncated operators $T_j$ possess a weak limit $T_0$, and $T = T_0 + A$, where $A$ is a bounded pointwise multiplier. In dimensions $d > 1$, $A = A(x)$ is a matrix-valued function, but the hypothesis

Corollary 4.2. With $c$ as in Proposition (4.1),

$$\|(W^{1/p}T)_*f\|_{L^q}^q \leq 2c^{-q}\max\left\{M'_w(W^{1/p}f), M_w(W^{1/p}f)\right\}\|f\|_{L^q}^q$$

for all $f \in C_c^\infty(\mathbb{R}^n; \mathbb{C}^d)$.

Proof. If both sides of (19) are multiplied by $q\alpha^{q-1}$ and integrated over the the interval $0 \leq \alpha < \infty$, the resulting inequality is

$$\int_{\mathbb{R}^n} \left(\left\|\left(W^{1/p}T\right)_*f\right\|_q^q - c^{-q}\max\left\{\left\|M'_w(W^{1/p}f)\right\|_q^q, \left\|M_w(W^{1/p}f)\right\|_q^q\right\}\right) dx$$

$$\leq \frac{1}{2}\int_{\mathbb{R}^n} \left\|\left(W^{1/p}T\right)_*f\right\|_q^q dx$$

from which it follows that

$$\left\|\left(W^{1/p}T\right)_*f\right\|_{L^q}^q - \frac{1}{c^q}\max\left\{\left\|M'_w(W^{1/p}f)\right\|_q^q, \left\|M_w(W^{1/p}f)\right\|_q^q\right\} \leq \frac{1}{2}\left\|\left(W^{1/p}T\right)_*f\right\|_{L^q}^q.$$

The remaining task is to verify that the $L^q$ norm of $(W^{1/p}T)_*f$ is finite. A key estimate is the fact that $T_\ast f(x) \leq C_f(1 + |x|)^{-n}$ for all $f \in C_c^\infty$, where $C_f$ depends on $f$. Then

$$(W^{1/p}T)_*f(x) \leq C\|W\|^{1/p}(1 + |x|)^{-n}.$$

There are many ways to show that the expression on the right-hand side is in $L^q$, all exploiting the fact that $\|W\|$ is a scalar $A_p$ weight. One possibility is to choose any nontrivial (scalar) function $\phi \geq 0 \in C_c^\infty$. We have shown in Theorem 3.2 that $\|W\|^{1/p}M(\|W\|^{1/p}\phi) \in L^q$ whenever $|p - q| < \delta$.

On the other hand, $C(1 + |x|)^{-n} \leq M(\|W\|^{1/p}\phi)$, which completes the proof. □

5. The main theorem.

Theorem 5.1. Let $T$ be a linear operator whose associated convolution kernel $K(x)$ satisfies the hypotheses in (1), and which acts separately on each coordinate function of $f$ (in other words, $(Tf)_j = Tf_j$). Let $W$ be a matrix $A_p$ weight.

There exists $\delta > 0$ such that $W^{1/p}TW^{-1/p}$ is a bounded operator on $L^q(\mathbb{R}^n; \mathbb{C}^d)$ whenever $|q - p| < \delta$.

Proof. As in the scalar case, the truncated operators $T_\epsilon$ possess a weak limit $T_0$, and $T = T_0 + A$, where $A$ is a bounded pointwise multiplier. In dimensions $d > 1$, $A = A(x)$ is a matrix-valued function, but the hypothesis
\((Tf)_j = Tf_j\) requires \(A(x)\) to be a scalar \(L^\infty\) function multiplied by the identity matrix.

The function \(W^{1/p} TW^{-1/p} g\) is dominated pointwise by \(g\) and \((W^{1/p} T)_* (W^{-1/p} g)\), as in Equation (8):

\[
|W^{1/p} TW^{-1/p} g(x)| = |W^{1/p} T_0 W^{-1/p} g(x) + A(x) g(x)| \\
\leq |(W^{1/p} T)_* (W^{-1/p} g)(x)| + C |g(x)|.
\]

The triangle inequality for \(L^q\)-norms immediately yields the result

\[
\|W^{1/p} TW^{-1/p} g\|_{L^q} \leq \|(W^{1/p} T)_* W^{-1/p} g\|_{L^q} + C \|g\|_{L^q}.
\] (24)

For all \(g\) such that \(W^{-1/p} g \in C_c^\infty\), the right-hand side is controlled by \(\|g\|_{L^q}\). Observe that \(W^{q/p}\) is a locally integrable matrix-valued function. Then \(C_c^\infty(\mathbb{R}^n; \mathbb{C}^d)\) is a dense subset of \(L^q(W^{q/p})\). The map \(f \in L^q(W^{q/p}) \rightarrow g = W^{1/p} f \in L^q(dx)\) is an invertible isometry, so its image \(W^{1/p}(C_c^\infty)\) is dense in \(L^q\). Thus the boundedness of \(W^{1/p} TW^{-1/p}\) may then be extended to all functions \(g \in L^q(\mathbb{R}^n; \mathbb{C}^d), \ |p - q| < \delta\).

A converse statement, with some minor modifications, is also true.

**Theorem 5.2.** Suppose that \(T\) is a convolution operator as above, with the additional nondegeneracy hypothesis that there exists some unit vector \(u \in \mathbb{R}^n\) such that \(|K(ru)| \geq a|r|^{-n}\), all \(r \in \mathbb{R} \setminus \{0\}\). If \(T\) is a bounded operator on \(L^p(W)\), then \(W\) is an \(A_p\) weight.

In order to prove this theorem we first need a result about integral operators with bounded and compactly supported kernels:

**Proposition 5.3.** Let \(S\) be an integral operator \(Sf(x) = \int_{\mathbb{R}^n} S(x,y) f(y)\) whose (scalar) kernel \(S(x,y)\) is supported in \(B \times B\) and satisfies the bound \(|S(x,y)| \leq |B|^{-1}\) for all \((x,y) \in B \times B\).

The norm of \(S\) as an operator on \(L^p(W)\) is less than \(C_d \|V_B V_B'\|\), where \(C_d\) is a dimensional constant independent of the particular choice of \(S\). In the special case \(S_0(x,y) = |B|^{-1} \chi_{B \times B}\), the operator norm of \(S_0\) is also greater than \(C_d^{-1} \|V_B V_B'\|\).

**Proof.** This is a straightforward calculation similar to those found in Section 2. Let \(f\) be any function in \(L^p(W)\). We first estimate the size of
$W^{1/p}(x)Sf(x)$ pointwise for each $x$.

$$|W^{1/p}(x)Sf(x)| = \left| W^{1/p}(x) \int_B S(x, y) f(y) \, dy \right|$$

$$= \left| \int_B S(x, y) W^{1/p}(x) f(y) \, dy \right|$$

$$\leq |B|^{-1} \int_B |W^{1/p}(x) f(y)| \, dy$$

$$\leq |B|^{-1} \left( \int_B \|W^{1/p}(x) W^{-1/p}(y)\|_{p'}^p \, dy \right)^{1/p'} \cdot \|f\|_{L^p(W)}.$$  

As in Section 2, we now introduce an orthonormal basis of vectors $e_i$ spanning $C^d$.

$$\left( \int_B \|W^{1/p}(x) W^{-1/p}(y)\|_{p'}^p \, dy \right)^{1/p'}$$

$$\leq \left( \int_B \left( d^{1/2} \sup_i |W^{-1/p}(y) W^{1/p}(x) e_i| \right)^{p'} \, dy \right)^{1/p'}$$

$$\leq d^{1/2} \left( \sum_{i=1}^d \int_B |W^{-1/p}(y) W^{1/p}(x) e_i|_{p'} \, dy \right)^{1/p'}$$

$$\leq C_d \left( \sum_{i=1}^d |B| \cdot |V_B W^{1/p}(x) e_i|_{p'} \right)^{1/p'}$$

$$\leq C_d |B|^{1/p'} \|V_B W^{1/p}(x)\|$$

which leads to the estimate

$$|W^{1/p}(x)Sf(x)| \leq C_d |B|^{-1/p} \|V_B W^{1/p}(x)\| \cdot \|f\|_{L^p(W)}.$$  

Then for all $\|f\|_{L^p(W)} \leq 1$, it follows that

$$\|Sf\|_{L^p(W)} \leq C \left( \int_B \|V_B W^{1/p}(x)\|^p \, dx \right)^{1/p}$$

$$\leq C_d \left( |B|^{-1} \int_B \left( d^{1/2} \sup_i |W^{1/p}(x) V_B e_i| \right)^p \, dx \right)^{1/p}$$

$$\leq C_d \left( \sum_i |B|^{-1} \int_B |W^{1/p}(x) V_B e_i|^p \, dx \right)^{1/p}$$

$$\sim C_d \left( \sum_i |V_B V_B e_i|^p \right)^{1/p} \leq C_d \|V_B V_B\|.$$
The second assertion is a restatement of Proposition 2.1.

Proof of Theorem 5.2. First, let $\epsilon > 0$ be small enough so that $2\epsilon + \epsilon^2 < \frac{1}{2} C_d^{-2}$. There exists a number $t_0 < \infty$ such that

\begin{equation}
|K(v) - K(rt_0 u)| \leq \epsilon |K(rt_0 u)| \quad \text{whenever } v \in B(rt_0 u, 2r), \text{ all } r \in \mathbb{R} \setminus \{0\}.
\end{equation}

This is seen to be true because $|K(rt_0 u)| \geq \frac{a}{r^d |p|}$ but $|\nabla K(x)| \leq \frac{C}{r^{d+1} |p|}$

for all $x \in B(rt_0 u, r)$. It suffices to choose $t_0 > \frac{2C}{\epsilon a}$.

Let $B$ denote the ball $B(y, r)$ in $\mathbb{R}^n$, and $B'$ the translated ball $B' = B(y + rt_0 u, r)$. We wish to consider the operator $S_B$ defined by

$$S_B f = \chi_{B'} (\chi_B T (\chi_B f)).$$

This is an integral operator whose kernel

$$S_B(x, y) = \chi_{B \times B} \int_{B'} K(x - z) K(z - y) \, dz$$

is supported in $B \times B$. If $T$ acts boundedly on $L^p(W)$, so too does $S_B$ with operator norm less than or equal to $\|T\|_2$.

The restrictions $\{x, y \in B, z \in B'\}$ guarantee that $z - y \in B(rt_0 u, 2r)$ and $x - z \in B(-rt_0 u, 2r)$. Thus the values of $K(z - y)$ and $K(x - z)$ do not vary much over the region of integration. Using the bounds established in (26), we rewrite $S_B(x, y)$ as the sum of a characteristic function and a small remainder:

\begin{equation}
S_B(x, y) = |B| K(rt_0 u) K(-rt_0 u) \chi_{B \times B} + S_1(x, y),
\end{equation}

where $|S_1(x, y)| \leq \frac{1}{2} C_d^{-2} |B| \cdot |K(rt_0 u) K(-rt_0 u)|$.

According to Proposition 5.3, the first term corresponds to an operator with norm at least $C \|V_B V_B^t\|$. In terms of other constants, $C$ is proportional to $a^2 t_0^{-2n} C_d^{-1}$. The operator corresponding to the second term has norm no more than half as great. It follows that $\|S_B\| \geq \frac{1}{2} C \|V_B V_B^t\|$. Then

\begin{equation}
\|V_B V_B^t\| \leq 2C^{-1} \|S_B\| \leq 2C^{-1} \|T\|_2^2 < \infty
\end{equation}

for all balls $B \subset \mathbb{R}^n$, and $W$ is an $A_p$ weight.

Corollary 5.4. If $W$ is a matrix $A_p$ weight, there exists $\delta > 0$ such that $W^{q/p}$ is an $A_q$ weight whenever $|q - p| < \delta$. In other words, an $A_p$ metric is also an $A_q$ metric for all $|q - p| < \delta$.

Remarks. We could have proven this statement directly in Section 2, using the reverse Hölder inequality to show that operators $f \rightarrow \chi_B \frac{1}{|B|} \int_B f \, dx$ are uniformly bounded on $L^q(W^{q/p})$. To do so would have added another computation without simplifying the subsequent discussion in any way.
Recall that a matrix weight $W \in A_p$ if and only if the averaging operators $A_B$ defined by

$$A_B f = \chi_B \frac{1}{|B|} \int_B f \, dx$$

are uniformly bounded on $L^p(W)$. An equivalent statement is that the conjugated operators $W^{1/p}A_BW^{-1/p}$ are uniformly bounded on the unweighted space $L^p(\mathbb{C}^d)$. It is trivial to observe that $A_B$ are uniformly bounded on $L^\infty(\mathbb{C}^d)$ with norm 1. By interpolation on the analytic family of operators

$$\{ W^{(1-z)/p}A_BW^{(z-1)/p}, \quad 0 \leq \text{Re}(z) \leq 1 \}$$

we find that $W^{1/r}A_BW^{-1/r}$ are uniformly bounded on $L^r(dx)$ for all $r > p$, leading to another result well-known in the scalar case:

**Proposition 5.5.** If $W$ is a matrix $A_p$ weight, then $W$ is also a matrix $A_r$ weight for all $r > p$.

One crucial difference must be noted. We cannot use the reverse Hölder inequality in this setting to extend the range of exponents to $r > p - \delta$. If we could, then by Corollary 5.4 and Proposition 5.5 for each weight $W \in A_p$ there would exist numbers $r < q < p$ such that $W^{q/r} \in A_q \subset A_p$. Instead, counterexamples are known; in [1] a matrix $A_2$ weight $W$ is constructed for which $W^s \notin A_2$ for any $s > 1$.

On a speculative note, perhaps this (suspected) lack of self-improvement is related to the absence of a unifying matrix $A_\infty$ class whose elements are all contained in some $A_p$ with $p$ finite. We do not claim to have proven anything here, nor have we investigated thoroughly the union of the $A_p$-weight classes in search of a common $A_\infty$ property. It has been suggested [25] that the scalar $A_\infty$ condition generalizes instead to an entire spectrum of $A_p, \infty$ conditions, one for each exponent $p$, in the matrix setting.

### 6. The case $d = \infty$.

Most of the estimates in the preceding discussion fail when the dimension $d$ is infinite. Banach space norms may not be representable by matrices, and traces (when defined at all) are no longer comparable to operator norms. Most importantly, the main theorem is false. Gillespie et al. [9] have constructed operator $A_2$ weights $W$ for which the Hilbert Transform is unbounded on $L^2(W)$.

The test function $f$ in their counterexample is constructed out of Haar functions on different length-scales, with the signs chosen so that each new piece contributes positively to the overall $L^2(W)$ norm of $Tf$. Linearity of $T$ is needed to ensure that the whole of $Tf$ will be equal to the sum of the

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1Following [16], with the slight modification $F_z(\psi) = |\psi|^{-\alpha(z)} - 1 \psi$. 
various parts, and also to bound from below an expectation over choices of signs. When applied to merely sublinear operators such as a maximal function, the argument is less successful. We do not presently know if the Hardy-Littlewood maximal operator $M_w$ is bounded or not.

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DILATION OF MARKOVIAN COCYCLES ON
A VON NEUMANN ALGEBRA

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We consider normal Markovian cocycles on a von Neumann algebra which are adapted to a Fock filtration. Every such cocycle $k$ which is Markov-regular and consists of completely positive contractions is realised as a conditioned $^*$-homomorphic cocycle. This amounts to a stochastic generalisation of a recent dilation result for norm-continuous normal completely positive contraction semigroups. To achieve this stochastic dilation we use the fact that $k$ is governed by a quantum stochastic differential equation whose coefficient matrix has a specific structure, and extend a technique for obtaining stochastic flow generators from Markov semigroup generators, to the context of cocycles. Number/exchange-free dilatability is seen to be related to locality in the case where the cocycle is a Markovian semigroup. In the same spirit unitary dilations of Markov-regular contraction cocycles on a Hilbert space are also described. The paper ends with a discussion of connections with measure-valued diffusion.

0. Introduction.

Let $(P_t)_{t \geq 0}$ be an ultraweakly continuous completely positive contraction semigroup on a von Neumann algebra $\mathcal{M}$. A natural problem is to seek a dilation of $P$. This has various interpretations (see e.g., [EvL], [Vin], [Küm], [Sa1], [Bha]); the following is a minimum: A family $(j_t)_{t \geq 0}$ of normal $^*$-homomorphisms of $\mathcal{M}$ into a larger algebra $\mathcal{N}$, equipped with a conditional expectation $E : \mathcal{N} \to \mathcal{M}$, such that $E \circ j_t = P_t$. One usually further requires $j$ to satisfy a semigroup law. From a physical point of view it is in many situations appropriate to choose the algebra $\mathcal{N}$, to be of the form $\mathcal{M} \otimes \mathcal{B}_k$ where $\mathcal{B}_k = \mathcal{B}(\Gamma_k)$ and $\Gamma_k$ is the symmetric Fock space over $L^2(\mathbb{R}_+; k)$ for some Hilbert space $k$ (see e.g., [AAFL]). In this case the dilation is of Evans-Hudson type ([EH1]) if $E$ is chosen to be the vacuum expectation and if $j$ satisfies a quantum stochastic (QS) differential equation of the form

$$dj_t = j_t \circ \phi_t d\Lambda(t), \quad j_0(a) = a \otimes 1,$$

(0.1)
in which \( \Lambda \) denotes the matrix of fundamental QS integrators (\([HuP],[Par],[Mey]\)) and \( \phi \) is the coefficient matrix of a bounded sesquilinear map \( k \times k \to \mathcal{B}(\mathcal{A}) \), both defined with respect to some basis of the \emph{noise dimension space} \( k \). This kind of stochastic dilation satisfies a cocycle relation (with respect to the shift of the quantum noise) rather than a semigroup law, however combining \( j \) with the shift does result in a semigroup on \( \mathcal{M} \otimes \mathcal{B}_k \). In the recent paper \([GS]\) it is shown that an E-H dilation exists when the semigroup \( P \) has bounded generator. A coordinate-free formulation of QS calculus is given in \([GS]\), using Hilbert \( W^\ast\)-modules, and this copes well with the fact that the noise dimension space \( k \) required for the dilation may in general be nonseparable.

This dilation picture has the following natural generalisation: Let \( B^0_0 = B(\Gamma_{J,\mathcal{k}_0}) \) where \( \Gamma_{J,\mathcal{k}_0} \) is the symmetric Fock space over \( L^2(J;\mathcal{k}_0) \) for a subinterval \( J \) of \( \mathbb{R}_+ \) and a Hilbert space \( \mathcal{k}_0 \), writing simply \( B^0 \) when \( J = \mathbb{R}_+ \). Then \( B^0 \) carries the filtration of subalgebras \( (B^0_0[t,\infty[)_{t \geq 0} \) and the semigroup of shifts \( \sigma = (\sigma_t : B^0 \to B^0[t,\infty[ \subset B^0)_{t \geq 0} \). Suppose that \( k = (k_t : \mathcal{M} \to \mathcal{M} \otimes B^0)_{t \geq 0} \) is an ultraweakly continuous family of completely positive (CP) normal contractions which is \emph{adapted} to the filtration and is a \emph{cocycle} with respect to the semigroup of shifts:

\[
k_{s+t} = \hat{k}_s \circ \sigma_s \circ k_t, \quad k_0(a) = a \otimes 1,
\]

where \( \hat{k}_s \) is the normal extension of \( k_s \) to a map \( \mathcal{M} \otimes B^0_{s,\infty[} \to \mathcal{M} \otimes B^0 \) (\([LW2]\)). Is there a Hilbert space \( \mathcal{k} \supset \mathcal{k}_0 \) and a \(*\)-homomorphic cocycle \( j = (j_t : \mathcal{M} \to \mathcal{M} \otimes B_k)_{t \geq 0} \) satisfying a QS differential equation of the form (0.1), such that \( E_0 \circ j_t = k_t \) where \( E_0 \) is now the conditional expectation which averages out the quantum noise provided by the supplementary Hilbert space \( \mathcal{k} \supset \mathcal{k}_0 \)? In other words, subject to regularity, can every CP contraction cocycle be realised as a conditioned \(*\)-homomorphic cocycle? Note that averaging out \emph{all} of the noise from such a Markovian cocycle yields an ultraweakly continuous CP contraction semigroup \( (P_t = E \circ k_t)_{t \geq 0} \), called the \emph{Markov semigroup} of the cocycle.

The unity of these ideas is further brought out by the recent paper \([LW2]\) where, following \([Bra]\), every such cocycle \( k \) which is Markov-regular (that is, whose Markov semigroup \( P \) is norm-continuous) is seen to satisfy a QS differential equation of the form (0.1). In the deterministic case the cocycle reduces to a semigroup and the equation is \( dP_t = P_t \circ \theta^0_0 dt \), simply expressing the fact that the semigroup has bounded generator \( \theta^0_0 \). We are therefore addressing a \emph{stochastic generalisation} of the dilation problem for CP contraction semigroups.

In the present paper we give an affirmative answer to the dilation problem for normal Markov-regular CP contraction cocycles on a von Neumann algebra which are adapted to the Fock filtration, thus extending the results
Dilation of Markovian cocycles. The construction is based on the infinitesimal structure of CP flows obtained in [LiP] and [LW1], combined with the techniques introduced in [GS] to obtain a stochastic flow generator from the generator of a CP contraction semigroup. We also consider the dilation problem for Markovian contraction cocycles on a Hilbert space. Again every such cocycle that is Markov-regular is governed by a QS differential equation, this time of the Hudson-Parthasarathy type:

\[(0.2) \quad dX_t = X_t F_{\alpha}^\beta \, d\Lambda_{\beta}^\alpha(t), \quad X_0 = 1,\]

for a matrix \(F\) of bounded operators on the Hilbert space ([LW2], [HuL]); moreover the structure of the matrix is once again completely characterised ([LW1], [Fag], [Mo2]).

The plan of the paper is as follows: In Section 1 the basic terminology of quantum stochastic calculus, flows and cocycles is reviewed, and some key results are recalled. Section 2 reviews a characterisation of nonnegative \((2 \times 2)\) operator matrices that is used in the construction of the dilations, and in Section 3 the structure theorems for CP, and CP contraction, flow generators are refined to a form which facilitates stochastic dilation. The \(^*\)-homomorphic dilations of CP contraction cocycles on a von Neumann algebra, and the unitary dilations of contraction cocycles on a Hilbert space, are constructed in Sections 4 and 6 respectively. In Section 5 CP contraction cocycles which have \(^*\)-homomorphic stochastic dilations involving no number/exchange processes are characterised, and the connection with locality for quantum dynamical semigroups is discussed. The final section contains a discussion of how these ideas might be applied to the theory of measure-valued diffusions.

Dilation for CP flows in a rather different sense arise in the work of Belavkin ([Be2]). There the question raised is one of implementing a CP flow \(k\) by conjugation with a solution of a Hudson-Parthasarathy equation of the form (0.2): \(k_t(a) = X_t(a \otimes 1)X_t^*\) — the flow being on the full algebra of operators on a Hilbert space. In this connection the implementation of CP flows on a von Neumann algebra, by inner perturbations of a \(^*\)-homomorphic flow (in the spirit of [EH2]), is treated elsewhere ([GLW]).

**1. Notation, terminology and background results.**

The symmetric Fock space \(\Gamma(H)\) over a Hilbert space \(H\) enjoys the exponential property \(\Gamma(H_1 \oplus H_2) = \Gamma(H_1) \otimes \Gamma(H_2)\), where the natural isomorphism identifying the two spaces is most economically described in terms of exponential vectors: \(\varepsilon(f_1, f_2) \longleftrightarrow \varepsilon(f_1) \otimes \varepsilon(f_2)\) ([Par]). For a Hilbert space \(k\) and subinterval \(J\) of \(\mathbb{R}_+\) we denote \(\Gamma(L^2(J; k))\) by \(\Gamma_{J,k}\), the algebra of all bounded operators on \(\Gamma_{J,k}\) by \(B_{J,k}\) and the vacuum vector \(\varepsilon(0)\) in \(\Gamma_{J,k}\) by \(\Omega_{J,k}\), abbreviating these to \(\Gamma_k, B_k\) and \(\Omega_k\) when \(J = \mathbb{R}_+\). Two instances of the exponential property that will be important for us are when \(k\) is an orthogonal
sum of Hilbert spaces $k_0$ and $k_1$ and when $\mathbb{R}_+$ is a disjoint union of intervals $J_1$ and $J_2$. Thus $\Gamma_k = \Gamma_{k_0} \otimes \Gamma_{k_1}, B_k = B_{k_0} \otimes B_{k_1} \oplus B_{k_0} \otimes 1 \subset B_k$; $\Gamma_k = \Gamma_{J_1,k} \otimes \Gamma_{J_2,k}, B_k = B_{J_1,k} \otimes B_{J_2,k} \oplus B_{J_1,k} \otimes 1 \subset B_k$ (where 1 denotes the relevant identity operator) are all identifications that will be invoked without comment.

Let $A$ be a unital $C^*$-algebra acting on an initial Hilbert space $\mathfrak{h}$. In [LW1] and [LW2] the noise dimension space $k$ is assumed to be separable, the quantum stochastic calculus with infinite degrees of freedom developed there being that initiated by Mohari and Sinha ([MoS]). An orthonormal basis $(e_i)_{i \geq 1}$ of $k$ is fixed, and the matrix of quantum stochastic integrators $[\Lambda_{ij}^0]_{\alpha,\beta \geq 0}$ is defined in terms of these vectors. They fall into four distinct classes: $\Lambda_0^i$ is the time component; $\Lambda_0^0 = A_i$, the $i$th annihilation component $(i \geq 1)$; $\Lambda_j^0 = A_j$, the $j$th creation component $(j \geq 1)$; and $\Lambda_j^i = N_j^i$, the $(i,j)$th number/exchange (also called gauge or preservation) component $(i,j \geq 1)$. The generator of a quantum stochastic flow is then a (possibly infinite) matrix $\theta = [\theta_{ij}^0]$ with entries in $\mathcal{B}(A)$. We write $M_D(\mathcal{B}(A))$, where $D = 1 + \dim k$, for the collection of these mapping matrices, and $\hat{k}$ for the Hilbert space $\mathbb{C} \oplus k$. The Hilbert spaces $\mathfrak{h} \otimes \hat{k}$ and $\bigoplus_{\nu \geq 0} \mathfrak{h}$ may be identified by use of the fixed basis (adding $e_0 = 1 \in \mathbb{C}$ to give a basis of $\hat{k}$). If $\theta$ is the stochastic generator of a CP contraction flow then the $\theta_{ij}^0$ are components, with respect to this basis, of a bounded linear map, also denoted $\theta$, from $A$ into $A'' \otimes \mathcal{B}(k)$ ([LW1], Theorem 5.2). Whenever we use this identification and introduce components we shall use the Einstein summation convention and sum over repeated indices; greek indices running from 0, roman indices from 1.

This global boundedness property of CP contraction flow generators connects with the approach in [GS] where QS calculus is reformulated in a coordinate free manner using Hilbert $W^*$-modules. The modules encountered in [GS] are all of the form $\mathcal{M} \otimes \mathcal{B}(k_0; k_1)$, for Hilbert spaces $k_0$ and $k_1$, and von Neumann algebra $\mathcal{M}$. Such a module may be characterised as the set $\{ T \in \mathcal{B}(\mathfrak{h} \otimes k_0; \mathfrak{h} \otimes k_1) : (a' \otimes 1_1)T = T(a' \otimes 1_0) \forall a' \in \mathcal{M}' \}$, and coincides with the closure of $\mathcal{M} \otimes_{\text{alg}} \mathcal{B}(k_0; k_1)$ in the weak, ultraweak, strong and ultrastrong topologies. We denote the topological dual of $k$ by $k'$ and, since $\mathcal{B}(\mathbb{C}; k)$ is naturally identified with $k$, we write $\mathcal{M} \otimes k$ for $\mathcal{M} \otimes \mathcal{B}(\mathbb{C}; k)$.

Globally bounded mapping matrices on a $C^*$-algebra $A$ will be written in block matrix form:

$$\begin{pmatrix} \tau & \alpha \\ \chi & \nu - \iota \end{pmatrix}$$

where $\tau \in \mathcal{B}(A), \alpha : A \to A'' \otimes k', \chi : A \to A'' \otimes k$ and $\nu, \iota : A \to A'' \otimes \mathcal{B}(k)$ are such that $\theta \in M_D(\mathcal{B}(A))$. Throughout the paper $\iota$ denotes the map $\iota(a) = a \otimes 1_k$ for the relevant $k$, and $\overline{\theta}$ denotes the transformed mapping.
matrix
\[
\hat{\theta} = \begin{bmatrix} \tau & \alpha \\ \chi & \nu \end{bmatrix}.
\]

(1.1)'

For a subspace \( k_0 \) of the noise dimension space \( k \), the \textit{vacuum conditional expectation} \( \mathbb{E}_0 : \mathcal{A}'' \otimes \mathcal{B}_k \to \mathcal{A}'' \otimes \mathcal{B}_{k_0} \) is given by \( \mathbb{E}_0[c] = E^*cE \) where \( E \) is the isometry \( \mathfrak{h} \otimes \Gamma_{k_0} \ni \xi \mapsto \xi \otimes \Omega_{k_0} \in \mathfrak{h} \otimes \Gamma_k \). When \( k_0 = \{0\} \) it is denoted simply by \( \mathbb{E} \).

Processes on a \( C^* \)-algebra \( \mathcal{A} \) with separable noise dimension space \( k \) are defined in generality in [LW1]. Here, apart from in Theorem 3.1, we are exclusively concerned with \textit{contraction processes}, that is pointwise weakly measurable families of contractions \( k = (k_t)_{t \geq 0} \) which are adapted to the Fock filtration: \( k_t : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}_{[0,t],k} \otimes 1 \subset \mathcal{A}'' \otimes \mathcal{B}_k \). When \( \mathcal{A} \) is a von Neumann algebra, \( k \) is called \textit{normal} if each map \( k_t \) is normal. Fock-adapted \textit{Markovian cocycles} \( k \) on a \( C^* \)-algebra \( \mathcal{A} \) are defined in [LW2], following Bradshaw who defined normal \( * \)-homomorphic cocycles on a von Neumann algebra by (1.2) below ([Bra]). They are required to satisfy a Feller property which includes invariance of the algebra under the maps \( \mathcal{P}_t = \mathbb{E} \circ k_t \ (t \geq 0) \), which comprise the \textit{Markov semigroup} of \( k \). A Markovian CP contraction cocycle is called \textit{Markov-regular} when its Markov semigroup is norm continuous. For normal CP contraction processes \( k \) on a von Neumann algebra \( \mathcal{M} \), the cocycle condition reads
\[
k_{s+t} = \widehat{k}_s \circ \sigma_s \circ k_t \quad (s, t \geq 0),
\]
where \( \sigma_s \) is the right shift \( \mathcal{M} \otimes \mathcal{B}_k \to \mathcal{M} \otimes \mathcal{B}_{[s,\infty],k} \subset \mathcal{M} \otimes \mathcal{B}_k \), and \( \widehat{k}_s \) is the normal extension of the map \( \mathcal{M} \otimes_{\text{alg}} \mathcal{B}_{[s,\infty],k} \to \mathcal{M} \otimes \mathcal{B}_k \) given by \( a \otimes b \mapsto k_s(a)(1 \otimes b) \). In the case where \( \mathcal{A} \) is a von Neumann algebra, \( k \) is one-dimensional and \( k \) is normal, unital and \( * \)-homomorphic, the following result was established by Bradshaw ([Bra]). It has been extended to a class of such cocycles whose Markov semigroup is only assumed to be ultraweakly continuous in [AcM].

**Theorem 1.1** ([LW2]). Let \( k \) be a CP contraction process on a unital \( C^* \)-algebra \( \mathcal{A} \), with separable noise dimension space \( k \). Then the following are equivalent:

(i) \( k \) is a Markov-regular cocycle.

(ii) \( k \) weakly satisfies a QS differential equation of the form
\[
dk_t = k_t \circ \theta \alpha \ d\Lambda^\alpha(t), \quad k_0(a) = a \otimes 1 \tag{1.3}
\]

for a mapping matrix \( \theta \in M_{D}(\mathcal{B}(\mathcal{A})) \).

In this case \( k \) satisfies the equation strongly and \( \theta \) defines a (completely) bounded operator \( \mathcal{A} \to \mathcal{B}(\mathfrak{h} \otimes k) \). Moreover, if \( \mathcal{A} \) is a von Neumann algebra then \( k \) is normal if and only if \( \theta \) is.
In view of this result we use the terminology Markovian cocycle and stochastic flow interchangeably. Let $k$ and $j$ be a pair of contraction processes on $\mathcal{A}$ with noise dimension spaces $k_0$ and $k$ respectively, where $k_0$ is a subspace of $k$. We shall call $j$ a stochastic dilation of $k$ if

$$k_t = \mathbb{E}_0 \circ j_t \quad (t \geq 0).$$

Thus, in this terminology, if $k$ is a Markovian cocycle with associated Markov semigroup $\mathcal{P}$ then $k$ is a stochastic dilation of $\mathcal{P}$. When $j$ and $k$ are processes that satisfy QS differential equations of the form (1.3) with noise dimension spaces $k$ and $k_0 \subset k$ respectively, it is easy to determine when $j$ is a stochastic dilation of $k$ by inspecting their generators.

**Lemma 1.2.** Let $\theta : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}((k_0))$ and $\phi : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\hat{k})$ be bounded mapping matrices that weakly generate contraction flows $k$ and $j$, with noise dimension spaces $k_0$ and $k = k_0 \oplus k_1$ respectively. Then $j$ is a stochastic dilation of $k$ if and only if $\phi$ has block matrix form

$$(1.4) \quad \phi = \begin{bmatrix} \theta & * \\ * & * \end{bmatrix}.$$  

**Proof.** If $j$ is a stochastic dilation of $k$ then, for all $u, v \in \mathfrak{h}$ and $f^0, g^0 \in L^2(\mathbb{R}_+; k_0)$,

$$\langle u \varepsilon(f^0), j_t(a) \nu \varepsilon(g^0) \rangle = \langle u \varepsilon(f^0), 0 \rangle, j_t(a) \nu \varepsilon(g^0, 0) \rangle.$$  

Applying the first fundamental formula of quantum stochastic calculus to each side, differentiating the resulting expressions at $t = 0$, and varying the test functions $f^0$ and $g^0$ reveals that $\phi$ has the form (1.4). Conversely, if $\phi$ has the form (1.4) then the process $\mathbb{E}_0 \circ j$ is also a weak solution of the QS differential equation satisfied by $k$. Thus by uniqueness of solutions ([LW1], Theorem 3.1) $k = \mathbb{E}_0 \circ j$. $\square$

For a unital $C^*$-algebra $\mathcal{A}$ acting on the Hilbert space $\mathfrak{h}$, a representation $(\pi, H)$ of $\mathcal{A}$, and operators $R \in \mathcal{B}(\mathfrak{h}; H)$ and $\mathcal{H} \in \mathcal{B}(\mathfrak{h})$ we write $\delta_{R,\pi}$ and $\mathcal{L}_{R,\pi,H}$ for the operators given by

$$\delta_{R,\pi}(a) = Ra - \pi(a)R,$$

$$\mathcal{L}_{R,\pi,H}(a) = R^*\pi(a)R - \frac{1}{2}\{R^*R, a\} + i[H, a].$$

Thus $\delta_{R,\pi} : \mathcal{A} \to \mathcal{B}(\mathfrak{h}; H)$ is a $\pi$-derivation, and $\mathcal{L}_{R,\pi,H} : \mathcal{A} \to \mathcal{B}(\mathfrak{h})$ satisfies

$$\partial \mathcal{L}_{R,\pi,H}(a, b) = \delta_{R,\pi}(a)^* \delta_{R,\pi}(b) + a^* R^* \pi(1)^{1/2} R b$$

where, for a linear map $\tau : \mathcal{A} \to \mathcal{B}(\mathfrak{h})$, $\partial \tau : \mathcal{A} \times \mathcal{A} \to \mathcal{B}(\mathfrak{h})$ is the sesquilinear map defined by

$$\partial \tau(a, b) = \tau(a^*b) - a^* \tau(b) - \tau(a^*)b + a^* \tau(1)b.$$

The map $\tau$ is called real if it satisfies $\tau^\dagger = \tau$, where $\tau^\dagger$ is defined by $\tau^\dagger(a) = \tau(a^*)^*$. The following result is contained in [ChE]:
Theorem 1.3. Let \((\tau, \rho, H, \delta)\) consist of a map \(\tau \in B(A)\), a representation \((\rho, H)\) of \(A\), and a \(\rho\)-derivation \(\delta : A \to B(h; H)\) satisfying \(\partial \tau(a, b) = \delta(a)^* \delta(b) + \delta(1) = 0\). Then there is an operator \(R \in B(h; H)\) which lies in the ultraweak closure of \(\text{Lin}\{\delta(a)b : a, b \in A\}\) and an element \(h \in A''\) such that

\[
\delta(\cdot) = \delta_{R, \rho}(\cdot) \quad \text{and} \quad \tau(\cdot) = L_{R, \rho, h}(\cdot) + \frac{1}{2}\{\tau(1), \cdot\}.
\]

If \(\tau\) is real then \(h\) may be chosen so that \(h = h^*\).


In this section we recall a characterisation of nonnegative operator block matrices that will be exploited in the construction of dilations. Parts (a) and (b) of the lemma below are classical and can be traced back as far as Schur — see [FoF], p. 547 for historical comments. We include a proof for the convenience of the reader. Part (c) is the special case required for the following section.

Lemma 2.1. Let \(T \in B(H_1 \oplus H_2)\) for Hilbert spaces \(H_1\) and \(H_2\).

(a) The following are equivalent:
(i) \(T \geq 0\).
(ii) In block matrix form

\[
T = \begin{bmatrix}
A & A^{1/2}VD^{1/2} \\
D^{1/2}V^*A^{1/2} & D
\end{bmatrix}
\]

where \(A, D \geq 0\) and \(V \in B(H_2; H_1)\) is a contraction.

(b) There is a representation (2.1) in which

\[
\text{Ker } V \supset \text{Ker } D \quad \text{and} \quad \text{Ran } V \subset \overline{\text{Ran } A};
\]

this \(V\) is unique.

(c) If the Hilbert spaces are of the form \(H \otimes h_1\) and \(H \otimes h_2\) respectively, and \(T\) belongs to \(C \otimes B(h_1 \oplus h_2)\) for a von Neumann algebra \(C\) acting on \(H\), then the unique contraction \(V\) satisfying (2.1) and (2.2) belongs to the \(W^*\)-module \(C \otimes B(h_2; h_1)\).

Proof. If (a)(ii) holds then \(T\) may be written

\[
\begin{bmatrix}
A^{1/2} & 0 \\
0 & D^{1/2}
\end{bmatrix}
\begin{bmatrix}
V & 1 - VV^* \\
V^* & 0
\end{bmatrix}
\begin{bmatrix}
A^{1/2} & 0 \\
0 & D^{1/2}
\end{bmatrix}
\]

which is manifestly nonnegative.
Conversely, if $T$ is nonnegative with block matrix form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ then obviously $A, D \geq 0$ and $C = B^*$. By the Cauchy-Schwarz inequality,

$$\langle \xi, B\eta \rangle^2 = \left| \left\langle \begin{pmatrix} \xi \\ 0 \end{pmatrix}, T \begin{pmatrix} 0 \\ \eta \end{pmatrix} \right\rangle \right|^2 \leq \left\langle \begin{pmatrix} \xi \\ 0 \end{pmatrix}, T \begin{pmatrix} \xi \\ 0 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 0 \\ \eta \end{pmatrix}, T \begin{pmatrix} 0 \\ \eta \end{pmatrix} \right\rangle = \| A^{1/2} \xi \|^2 \| D^{1/2} \eta \|^2.$$

First note that this implies that $\| B\eta \| \leq \| A^{1/2} \| \| D^{1/2} \eta \| \ \forall \ \eta \in H_2,$

from which it follows that there is a (unique) operator $J \in \mathcal{B}(H_2; H_1)$ satisfying

$$B = JD^{1/2} \text{ and } \text{Ker } J \supset \text{Ker } D^{1/2}.$$ 

Now substitute this back into the Cauchy-Schwarz estimate:

$$\langle J^*\xi, D^{1/2}\eta \rangle \leq \| A^{1/2} \xi \| \| D^{1/2} \eta \|.$$ 

Since $\overline{\text{Ran } J^*} = (\text{Ker } J)^\perp \subset (\text{Ker } D^{1/2})^\perp = \overline{\text{Ran } D^{1/2}},$ this implies that

$$\| J^*\xi \| \leq \| A^{1/2} \xi \| \ \forall \ \xi \in H_1,$$

from which it follows that there is a (unique) contraction $W \in \mathcal{B}(H_1; H_2)$ satisfying

$$J^* = WA^{1/2} \text{ and } \text{Ker } W \supset \text{Ker } A^{1/2}.$$ 

Since $W$ also satisfies $\overline{\text{Ran } W} = \overline{\text{Ran } J^*},$ if we put $V = W^*$ then we have $B = JD^{1/2} = A^{1/2}VD^{1/2}$ and $\text{Ker } V = (\text{Ran } W)^\perp = \text{Ker } J \supset \text{Ker } D^{1/2} = \text{Ker } D$. Also $\text{Ran } V \subset (\text{Ker } W)^\perp \subset (\text{Ker } A^{1/2})^\perp = \overline{\text{Ran } A^{1/2}} = \overline{\text{Ran } A},$ so (a)(ii) follows along with the first part of (b).

The uniqueness of $V$ subject to (2.2) follows from the identity

$$\langle A^{1/2} \xi, VD^{1/2} \eta \rangle = \left\langle \begin{pmatrix} \xi \\ 0 \end{pmatrix}, T \begin{pmatrix} 0 \\ \eta \end{pmatrix} \right\rangle.$$ 

Under the conditions of (c) obviously $A \in \mathcal{C} \otimes \mathcal{B}(h_1)$ and $D \in \mathcal{C} \otimes \mathcal{B}(h_2)$. Moreover

$$\langle (c' \otimes 1_1)A^{1/2} \xi, V(d' \otimes 1_2)D^{1/2} \eta \rangle = \left\langle (c' \otimes 1) \begin{pmatrix} \xi \\ 0 \end{pmatrix}, T(d' \otimes 1) \begin{pmatrix} 0 \\ \eta \end{pmatrix} \right\rangle$$ 

for $c', d' \in \mathcal{C}'$, where $1_1, 1_2$ and $1$ are the identities on $h_1, h_2$ and $h_1 \oplus h_2$ respectively; it follows that $V \in \mathcal{C} \otimes \mathcal{B}(h_2; h_1)$.

$\square$

In this section we refine the characterisation of CP flow generators on a unital C*-algebra \( A \) found in [LiP] and [LW1] in such a way that, when \( A \) is a von Neumann algebra and the flow is a normal contraction flow, the constituents of the generator may be used to determine a \(^*\)-homomorphic stochastic dilation. The refinement uses the technique introduced in [GS] for dilating CP contraction semigroups.

Fix a separable noise dimension space \( k_0 \), with basis \((e_i)_{i \geq 1}\). To begin with we do not assume that \( k \) is a contraction flow. Therefore (when \( k_0 \) is infinite dimensional), its generator \( \theta \) need not be bounded, and we must work with its components \( \theta_{ij}^a \). Following (1.1)' write 
\[
\begin{bmatrix}
\tau & \chi^\dagger_j \\
\chi^i & \nu^j
\end{bmatrix}
\]
for the \((2 \times 2)\) matrix of maps 
\[
\begin{bmatrix}
\theta_0^0 & \theta_0^j \\
\theta^0_i & \theta_i^j
\end{bmatrix}, \quad i, j \geq 1
\]
when \( \theta \) is real. Let \( h_{00} \) be the subspace of \( \bigoplus_{\gamma \geq 0} h = h \otimes k_0 \) consisting of vectors with only finitely many nonzero components with respect to the chosen basis.

**Theorem 3.1.** Let \( \theta \) be a mapping matrix on the unital C*-algebra \( A \) that weakly generates a stochastic flow \( k \).

(a) The following are equivalent:

(i) \( k \) is completely positive.

(ii) \( \theta \) is real and there is a quintuple \( R = (h, \pi, h, d, \{w_j\}) \) consisting of a Hilbert space \( h \), a \(^*\)-homomorphism \( \pi : A \to A'' \otimes B(h) \) and operators \( h = h^* \in A'', d \in A'' \otimes h \) and \( w_j \in A'' \otimes h, j \geq 1 \), such that

\[
\begin{bmatrix}
\tau(a) & \chi^\dagger_j(a) \\
\chi^i(a) & \nu^j
\end{bmatrix} = \begin{bmatrix}
\mathcal{L}(a) + \frac{1}{2}\{t, a\} & \delta^i(a)w_j + a(c^\dagger)^* \\
\delta^i(a) + c^a & w^i_j \pi(a)w_j
\end{bmatrix}
\]

where \( t = \tau(1), c^i = \chi^i(1), \delta = \delta_{d, \pi}, \mathcal{L} = \mathcal{L}_{d, \pi, h} \) and \( H = \overline{H_0} \) with 
\[
H_0 = \text{Lin} \{ \delta(a)u^0 + \pi(a)w_ju^j : a \in A, (u^a) \in h_{00} \}.
\]

(b) If \( R_1 \) and \( R_2 \) are quintuples satisfying (3.1) then there is a unique partial isometry \( V : h \otimes h_1 \to h \otimes h_2 \) satisfying

\[
V^*V = \pi_1(1); \quad VV^* = \pi_2(1)
\]

Moreover \( V \in A'' \otimes B(h_1; h_2) \).

(c) If \( A \) is a von Neumann algebra and \( R \) is a quintuple satisfying (3.1), then the representation \( (\pi, h \otimes h) \) is normal if and only if each \( \theta_{ij}^a \) is.
Proof. The implication (a)(ii) ⇒ (a)(i) is contained in [LW1], Theorem 4.1.

(a)(i) ⇒ (a)(ii): Suppose that \( k \) is completely positive. Then, by Theorem 4.1 of [LW1], \( \theta \) is real and there is a quadruple \( Q = (\rho, \mathcal{H}, \gamma, \{W_i\}) \) consisting of a representation \( (\rho, \mathcal{H}) \) of \( \mathcal{A} \), a \( \rho \)-derivation \( \gamma : \mathcal{A} \to \mathcal{B}(\mathfrak{h}; \mathcal{H}) \) and a family of operators \( \{W_i : i \geq 1\} \) in \( \mathcal{B}(\mathfrak{h}; \mathcal{H}) \) such that

\[
(3.3i) \quad \begin{bmatrix} \partial \tau(a, b) & \chi^\dagger_j(a) \\ \chi^i(a) & \nu^j(a) \end{bmatrix} = \begin{bmatrix} \gamma(a^* \gamma(b) & \gamma^i(a)W_j + a\chi^j(1) \\ W_i^* \gamma(a) + \chi^i(1)a & W_i^* \rho(a)W_j \end{bmatrix}
\]

(3.3ii) \( \rho \) is unital

(3.3iii) \( \mathcal{H} = \overline{\mathcal{H}_0} \)

where \( \mathcal{H}_0 = \text{Lin}\{\gamma(a)u^0 + \rho(a)W_i u^i : a \in \mathcal{A}, (u^a) \in \mathfrak{h}_{00}\} \). For each unitary \( u' \in \mathcal{A}' \) define bounded linear operators \( \gamma^{u'} : \mathcal{A} \to \mathcal{B}(\mathfrak{h}; \mathcal{H}) \) and \( W^{u'}_j \in \mathcal{B}(\mathfrak{h}; \mathcal{H}) \) by \( \gamma^{u'}(a) = \gamma(a)u' \) and \( W^{u'}_j = W_j u' \), and note the following relations:

\[
\begin{align*}
\gamma^{u'}(a)^* \gamma^{u'}(b) &= u'^* \partial \tau(a, b) u' = \partial \tau(a, b) = \gamma(a)^* \gamma(b); \\
(W^{u'}_i)^* \gamma^{u'}(a) &= u'^* W_i^* \gamma(a) u' = u'^* (\chi^i(a) - \chi^i(1)a) u' = W_i^* \gamma(a); \\
(W^{u'}_i)^* \rho(a) W^{u'}_j &= u'^* W_i^* \rho(a) W_j u' = u'^* \nu^j(a) u' = \nu^j(a);
\end{align*}
\]

\( \mathcal{H}_{0}' = \mathcal{H}_0 \).

In other words the quadruple \( Q^{u'} = (\rho, \mathcal{H}, \gamma^{u'}, \{W^{u'}_i\}) \) also satisfies (3.3). Hence, by the uniqueness part of Theorem 4.1 in [LW1], there is a unique unitary operator \( \rho'(u') \) on \( \mathcal{H} \) such that

\[
(3.4) \quad \rho'(u') W_i = W_i u'; \quad \rho'(u') \gamma(a) = \gamma(a) u'; \quad \rho'(u') \rho(a) = \rho(a) \rho'(u').
\]

The resulting map \( \rho' \) is easily seen to be a unitary representation of the group of unitaries in \( \mathcal{A}' \) by checking matrix elements against vectors from the dense subspace \( \mathcal{H}_0 \). It follows that \( \rho' \) extends linearly to a normal, unital representation of \( \mathcal{A}' \). Hence ([Dix], p. 61) there is a Hilbert space \( \mathfrak{h} \) and an isometry \( V : \mathcal{H} \to \mathfrak{h} \otimes \mathfrak{h} \) such that \( \rho'(x') = V^*(x' \otimes 1)V \) and \( p = VV^* \in \mathcal{A}' \otimes \mathcal{B}(\mathfrak{h}) \). Put \( \mathcal{H} = V \mathcal{H} \) and define \( \pi' : \mathcal{A}' \to \mathcal{B}(\mathfrak{h} \otimes \mathfrak{h}), \pi : \mathcal{A} \to \mathcal{B}(\mathfrak{h} \otimes \mathfrak{h}), \delta : \mathcal{A} \to \mathcal{B}(\mathfrak{h} \otimes \mathfrak{h}) \), and \( w_j \in \mathcal{B}(\mathfrak{h} \otimes \mathfrak{h}) \) for \( j \geq 1 \), by

\[
\pi'(x') = (x' \otimes 1)p, \quad \pi(a) = V \rho(a) V^*, \quad \delta(a) = V \gamma(a), \quad w_j = VW_j.
\]
Since \( p \in \mathcal{A}'' \otimes \mathcal{B}(h) \), algebraic manipulations applied to (3.4) reveal the following identities:

\[
\begin{align*}
\pi(1) = p; & \quad p\delta(a) = \delta(a); \quad pw_j = w_j, \\
\pi(a)(x' \otimes 1) = (x' \otimes 1)\pi(a) = \pi'(x')\pi(a), & \quad \delta(a)x' = (x' \otimes 1)\delta(a), \\
w_j x' = (x' \otimes 1)w_j.
\end{align*}
\]

Thus \( \pi(\mathcal{A}) \subset \mathcal{A}'' \otimes \mathcal{B}(h) \), \( \delta(\mathcal{A}) \subset \mathcal{A}'' \otimes h \) and \( w_j \in \mathcal{A}'' \otimes h \). Moreover \( \pi \) is a representation, \( \delta \) a \( \pi \)-derivation, and

\[
\operatorname{Ran} \pi(1) = \operatorname{Ran} V = \overline{V(\mathcal{H}_0)} = \mathcal{H}_0 = \mathcal{H}.
\]

Now \( \delta(1) = 0 \) and \( \delta(a)^*\delta(b) = \partial\tau(a,b) \), from its definition. Thus by Theorem 1.3 there is some \( d \in \mathcal{L}^{\text{uw}}(\delta(a)b : a, b) \subset \mathcal{A}'' \otimes h \) and \( h = h^* \in \mathcal{A}'' \) such that \( \delta = \delta_{d,\pi} \) and \( \tau(\cdot) = \mathcal{L}_{d,\pi,h}(\cdot) + \frac{1}{2}\{\tau(1), \cdot\} \). Note also that \( \pi(1)d = d \)

(b) Writing \( \mathcal{H}_i \) for \( \operatorname{Ran} \pi_i(1) \subset \mathfrak{h} \otimes h_i, i = 1, 2 \), Theorem 4.1 of [LW1] ensures the existence of a unique unitary operator \( V_0 : \mathcal{H}_1 \to \mathcal{H}_2 \) satisfying

\[
\begin{align*}
V_0w_j^1 = w_j^2; & \quad V_0\delta_1 = \delta_2; \quad V_0\pi_1(a) = \pi_2(a)V_0.
\end{align*}
\]

Let \( V \) be the unique extension of \( V_0 \) to an operator \( V : \mathfrak{h} \otimes h_1 \to \mathfrak{h} \otimes h_2 \) that satisfies (3.2i). Then \( V \) satisfies (3.2ii), and is thus clearly the unique partial isometry satisfying (3.2). That \( V \) belongs to \( \mathcal{A}'' \otimes \mathcal{B}(h_1; h_2) \) follows from the identity

\[
(a' \otimes 1_i)(\delta_i(a)u^0 + \pi_i(a)w_j^i u^i) = \delta_i(a'a^0 + \pi_i(a)w_j^i a^i u^i).
\]

(c) In one direction this is trivial. To obtain normality of \( \pi \) from the normality of all of the \( \theta^a_j \) it is enough to consider the restriction of \( \pi \) to \( \pi(1)h \otimes h \) and apply [LW1], Theorem 4.1(d).

\[\square\]

Remark. If \( \mathcal{A} \) is a von Neumann algebra, the initial space \( \mathfrak{h} \) is separable, and each \( \theta^a_j \) is ultraweakly continuous, then Proposition 4.2 of [LW1] implies that the representation space \( \mathcal{H} \) in the quadruple \( \mathcal{Q} \) is separable too, so that in particular the von Neumann algebra \( \rho'(\mathcal{A}')' \) is \( \sigma \)-finite. In this case we may assume that the Hilbert space \( h \) in Theorem 3.1 is separable too (see [Dix], p. 62).

\textbf{Theorem 3.2.} Let \( \theta \) be a mapping matrix on the unital \( C^* \)-algebra \( \mathcal{A} \) that weakly generates a stochastic flow \( k \). The following are equivalent:

(i) \( k \) is a completely positive contraction cocycle.

(ii) \( \theta \) is real and bounded, and there is a sextuple \( \mathcal{S} = (\mathfrak{h}, \pi, h, d, w, v) \)

consisting of a Hilbert space \( \mathfrak{h} \), a \( * \)-homomorphism \( \pi : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(h) \),
operators $h = h^* \in \mathcal{A}'', d \in \mathcal{A}' \otimes h$ and contractions $w \in \mathcal{A}'' \otimes \mathcal{B}(k_0; h)$ and $v \in \mathcal{A}'' \otimes k_0$ such that, in the notation of (1.1)',

\begin{align}
(3.5i) & \quad \hat{\theta}(a) = \begin{bmatrix} L(a) + \frac{1}{2}\{t, a\} \delta^x(a)w + ac^* \\
w^*\delta(a) + ca \quad w^*\pi(a)w \end{bmatrix} \\
(3.5ii) & \quad \pi(1)d = d \quad \text{and} \quad \pi(1)w = w \\
(3.5iii) & \quad \text{Ran} \pi(1) = H \\
(3.5iv) & \quad c = (1 - w^*w)^{1/2}v(-t)^{1/2} \\
(3.5v) & \quad \text{Ker} \ v \supset \text{Ker} \ t \quad \text{and} \quad \text{Ran} \ v \subset \text{Ran} (1 - w^*w)
\end{align}

where $t = \tau(1) \leq 0, c = \chi(1), \delta = \delta_d, \pi, L = L_d, \pi, h$ and $H = H_0$ with

$$H_0 = \text{Lin} \{ \delta(a)u^0 + \pi(a)w(u^i) : a \in \mathcal{A}, (u^i) \in h_00 \}. $$

Moreover, if $\mathcal{A}$ is a von Neumann algebra then the representation $(\pi, h \otimes h)$ of $\mathcal{A}$ appearing in $\mathcal{S}$ is normal if and only if the process $k$ is normal.

**Proof.** By Lemma 2.1 and [LW1], Proposition 5.1 and Theorem 5.2, (ii) implies (i). Conversely if (i) holds then $\theta$ is bounded and satisfies $\theta(1) \leq 0,$ so that $\nu$ is a completely positive contraction. Letting $(h, \pi, h, d, \{w_i\})$ be the quintuple of Theorem 3.1, define an operator $w : h \otimes k_0 \to h \otimes h$ by $w(v^i) = w_i v^i,$ initially on vectors whose components are eventually zero. Since $\pi(1)w_j = w_j$ for each $j,$ and

$$\|w(v^i)\|^2 = \langle w_i v^i, w_j v^j \rangle = \langle v^i, w_i^* \pi(1)w_j v^j \rangle = \langle v^i, v_j^\delta(1)v^j \rangle = \langle (v^i), \nu(1)(v^i) \rangle \leq \|v^i\|^2,$$

it follows that $w$ is a contraction. Since

\begin{align}
(3.6) & \quad \begin{bmatrix} t & c^* \\
c & w^*w - 1 \end{bmatrix} \leq 0
\end{align}

the existence of $v \in \mathcal{A}'' \otimes k_0$ satisfying (3.5iv) and (3.5v) follows from Lemma 2.1. The remaining properties now follow easily from Theorem 3.1, noting that when $\mathcal{A}$ is a von Neumann algebra, $k$ is normal if and only if each $\theta^a_\beta$ is normal by [LW1], Proposition 3.2. \qed

**Remarks.** (i) The necessary conditions (3.5) for $\theta$ to weakly generate a CP contraction cocycle $k$ have been shown to be sufficient too ([LW3]), since completely bounded mapping matrices are necessarily regular in the sense of [Mey] and [LW1]. This also implies that $\theta$ strongly generates $k.$ Uniqueness of solutions ([LW1], Theorem 3.1) permits us to use the notation $k^0$ for the cocycle generated by $\theta.$

(ii) By [LW1] Proposition 5.1, $k$ is unital if and only if $c$ and $t$ are both zero and $w$ is isometric; this implies that $v$ is zero too.
Since we aim to dilate CP contraction cocycles to \(^*\)-homomorphic cocycles, we need to be able to recognize from the generator when the flow is already \(^*\)-homomorphic. This is addressed in the next result.

**Proposition 3.3.** Let \(\mathcal{M}\) be a von Neumann algebra acting on \(\mathcal{H}\), and let \(k\) be the flow generated by a (completely) bounded mapping matrix \(\theta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{B}(\mathcal{K}_0)\) which is expressible in the form (3.5i) for some normal representation \((\pi, \mathcal{H} \otimes \mathcal{H})\) of \(\mathcal{M}\) and operators \(c \in \mathcal{M} \otimes \mathcal{K}_0\), \(h = h^* \in \mathcal{M}\), \(d \in \mathcal{M} \otimes \mathcal{H}\) and \(w \in \mathcal{M} \otimes \mathcal{B}(\mathcal{K}_0; \mathcal{H})\) satisfying (3.5ii). Then \(k\) is \(^*\)-homomorphic if and only if:

(i) \(w\) is a partial isometry,
(ii) \(wc = 0\),
(iii) \(c^*c = -t\),
(iv) \(ww^* \in \pi(\mathcal{M})'\), and
(v) \(ww^*\delta = \delta\).

In particular, if \(ww^* = \pi(1)\) then \(k\) is \(^*\)-homomorphic if and only if \(wc = 0\) and \(c^*c = -t\).

**Note.** The hypotheses here imply complete positivity of \(k\) (by Theorem 3.1) but not contractivity. If however \(k\) is assumed to be contractive — equivalently \(\theta(1) \leq 0\) — then there is an element \(v\) of \(\mathcal{M} \otimes \mathcal{K}_0\) satisfying (3.5iv), and so (i) implies (ii).

**Proof.** The necessary and sufficient conditions for \(k\) to be \(^*\)-homomorphic ([LW1], Theorem 6.5) may be written

\[
(3.7) \quad w^*\pi(a^*a)w = w^*\pi(a)^*ww^*\pi(a)w \\
(3.8) \quad w^*\delta(a^*a) = w^*\delta(a^*)a + w^*\pi(a)^*w(w^*\delta(a) + ca) \\
\hspace{1cm} (w^*\delta(a) + ca)^*(w^*\delta(a) + ca) = \partial\tau(a, a) - a^*ta \\
\hspace{1cm} = \delta(a)^*\delta(a) - a^*ta
\]

where (1.5) is used in the last line. Suppose (3.7)–(3.9) hold. Then, putting \(a = 1\) and using the fact that \(\pi(1)w = w\), reveals that \((w^*w)^2 = w^*w\), so that \(w\) is a partial isometry. Now \(\delta(1) = 0\), so (3.8) and (3.9) with \(a = 1\) give \(wc = 0\) and \(c^*c = -t\) respectively, which in turn implies that \(\delta(a)^*(1 - ww^*)\delta(a) = 0\), and so \((1 - ww^*)\delta(a) = 0\). Also (3.7) implies that \(w^*\pi(a)^*(1 - ww^*)\pi(a)w = 0\), so \((1 - ww^*)\pi(a)w = 0\), which implies that \(\pi(1)ww^* = ww^*\pi(a)\), and taking adjoints shows that \(ww^*\pi(a) = \pi(a)ww^*\). Thus (i)–(iv) hold.

Conversely if (i)–(iv) hold then (3.7)–(3.9) are easily verified. □

**Remark.** Invoking [GS] Theorem 3.3.6, instead of [LW1] Theorem 6.5, the proposition remains true when the noise dimension space is no longer separable.
4. *-Homomorphic dilation.

In this section we give the main result of the paper, namely necessary and sufficient conditions for a Markovian cocycle to have a normal Markov-regular *-homomorphic dilation. We then consider various special cases.

**Theorem 4.1.** Every normal Markov-regular completely positive contraction cocycle on a von Neumann algebra, with separable noise dimension space, admits a normal *-homomorphic stochastic dilation.

**Proof.** Let \( k \) be a normal Markov-regular CP contraction cocycle on the von Neumann algebra \( \mathcal{M} \) with separable noise dimension space \( k_0 \). By Theorem 3.2 its stochastic generator \( \theta \) satisfies

\[
\hat{\theta}(a) = \begin{bmatrix} \tau(a) & \delta^\dagger(a)w + ac^* \\ w^*\delta(a) + ca & w^*\pi(a)w \end{bmatrix} = mu(a) + \nu(a)m^* + s^*\Psi(a)s
\]

where

\[
m = \begin{bmatrix} 1/2 & t \\ c/2 & 0 \end{bmatrix} \in \mathcal{M} \otimes \mathcal{B}(\hat{k}_0), \quad s = \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix} \in \mathcal{M} \otimes \mathcal{B}(\hat{k}_0; \hat{h}),
\]

\( \Psi : \mathcal{M} \to \mathcal{M} \otimes \mathcal{B}(\hat{h}) \) is the (transformed) mapping matrix

\[
(4.1) \quad \Psi = \begin{bmatrix} \mathcal{L} & \delta^\dagger \\ \delta & \pi \end{bmatrix},
\]

and \((h, \pi, h, d, w, v)\) is the sextuple \( S \) from the theorem satisfying (3.5). Set \( k = k_0 \oplus k_1 \oplus k_2 \) and let \( \phi : \mathcal{M} \to \mathcal{M} \otimes \mathcal{B}(\hat{k}) \) be the mapping matrix defined by

\[
\hat{\phi}(a) = \begin{bmatrix} \tau(a) & \delta^\dagger(a)W + aC^* \\ W^*\delta(a) + Ca & W^*\pi(a)W \end{bmatrix} = M\nu(a) + \nu(a)M^* + S^*\Psi(a)S
\]

where

\[
M = \begin{bmatrix} 1/2 & t \\ c/2 & 0 \end{bmatrix} \in \mathcal{M} \otimes \mathcal{B}(\hat{k}) \quad \text{and} \quad S = \begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix} \in \mathcal{M} \otimes \mathcal{B}(\hat{k}; \hat{h}),
\]

with \( W = [w \quad r \quad 0] \) and \( C = [c \quad g \quad e]^T \), and where the Hilbert spaces \( k_1 \) and \( k_2 \) and operators \( r \in \mathcal{M} \otimes \mathcal{B}(k_1; h), g \in \mathcal{M} \otimes k_1 \) and \( e \in \mathcal{M} \otimes k_2 \) are to be determined.

Since \( \phi \) is unchanged if \( W \) is replaced by \( \pi(1)W \), we may assume that \( \pi(1)r = r \). By Lemma 1.2 \( k^\phi \) is a stochastic dilation of \( k \), and by Proposition 3.3 and the remarks following it, \( k^\phi \) is *-homomorphic if and only if

\[
q^2 = q \in \pi(\mathcal{M})', \quad q\delta = \delta, \quad wc + rg = 0 \quad \text{and} \quad c^*c + g^*g + e^*e = -t,
\]

where \( q := ww^* + rr^* \). The choice \( k_1 = h, k_2 = \mathbb{C} \) and

\[
r = (\pi(1) - ww^*)^{1/2}, \quad g = -wv(-t)^{1/2}, \quad e = (1 - v^*v)^{1/2}(-t)^{1/2},
\]
satisfies these constraints since then $q = \pi(1)$, and by (3.5ii) and (3.5iv)
\[
rg = - (\pi(1) - ww^*)^{1/2}\overline{wv}(-t)^{1/2}
= -w(1 - w^*w)^{1/2}v(-t)^{1/2} = -wc,
\]
and
\[
t + g^*g + e^*e = (-t)^{1/2}[-1 + v^*w^*wv + (1 - v^*v)](-t)^{1/2}
= (-t)^{1/2}v^*(w^*w - 1)v(-t)^{1/2} = -c^*c.
\]

**Remarks.** The particular solution given above serves merely to establish the existence of stochastic dilations. More natural dilations arise in particular cases; moreover one may wish to preserve unitality under dilation. In the remarks below we simplify the constraints (4.2) by considering only dilations for which $q = \pi(1)$.

(i) If $wc = 0$ then it is natural to choose $g = 0$, simplifying the constraints to a question of finding $r \in \mathcal{M} \otimes \mathcal{B}(k_1; h)$ and $e \in \mathcal{M} \otimes k_2$ satisfying
\[
rr^* + ww^* = \pi(1) \quad \text{and} \quad c^*c + e^*e = -t.
\]

(ii) If $w$ is a partial isometry then $r$ is necessarily a partial isometry and (using (3.5iv)) $wc = 0$ so (i) applies.

(iii) If $ww^* = \pi(1)$ then (ii) applies, and since $r$ is necessarily zero we may take $k_1 = \{0\}$. Moreover, by Proposition 3.3, $k$ is already *-homomorphic unless $c^*c \neq -t$, in which case we may take
\[
k_2 = \mathbb{C}, \quad C = \begin{bmatrix} c \\ (-t - c^*c)^{1/2} \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} w & 0 \end{bmatrix}.
\]

(iv) If $k$ has a conservative Markov semigroup then $t = 0$, and so $v$ and $c$ are zero. Also $e$ and $g$ are necessarily zero and we may take $k_2 = \{0\}$. By (iii) $k$ is already *-homomorphic unless $ww^* \neq \pi(1)$. In this case
\[
\hat{\phi}(a) = S^*\Psi(a)S \quad \text{for} \quad S = \begin{bmatrix} 1 & 0 \\ 0 & w^r \end{bmatrix},
\]
with $\Psi$ given by (4.1) and $r \neq 0$ satisfying (4.2)'.

(v) If $k$ is unital then $\theta(1) = 0$ (by [LW1], Proposition 5.1), (iv) applies and also $w$ is necessarily isometric so (ii) applies too. Thus we are left with a choice of Hilbert space $k_1$ and partial isometry $r$ in $\mathcal{M} \otimes \mathcal{B}(k_1; h)$ whose final space is $\text{Ran}(\pi(1) - ww^*)$. The resulting dilation will be unital if and only if $r$ is isometric. In particular, the above scheme can yield a unital *-homomorphic dilation with the choice $k_1 = h$ if and only if the projection $(\pi(1) - ww^*)$ is Murray-von Neumann equivalent to the identity, relative to the von Neumann algebra $\mathcal{M} \otimes \mathcal{B}(h)$. 

\[\Box\]
(vi) If \( k_0 = \{0\} \) then there is no noise and so \( k \) is a semigroup, \( \mathcal{P} \) say. Thus \( c, v \) and \( w \) are all zero, in particular (ii) applies. If \( \mathcal{P} \) is unital then (iv) and (v) apply too, and in this case \( S = \text{diag}[1, r] \). The choice \( k_1 = h \) and \( r = \pi(1) \), so that \( \hat{\phi} = \Psi \), gives the dilation obtained in \([GS]\). If \( \mathcal{P} \) is nonunital then, choosing \( k_2 = C \) and \( k_1 = h \) yields the generator

\[
\phi = \begin{bmatrix}
\tau & \delta & \lambda \dagger \\
\delta & \pi - \iota & 0 \\
\lambda & 0 & -\text{id}
\end{bmatrix},
\]

in which \( \lambda \) denotes left multiplication by \( e = (-t)^{1/2} \).

(vii) If \( k_0 = \{0\} \) and \( \mathcal{M} = \mathcal{B}(\mathfrak{h}) \) where \( \mathfrak{h} \) is infinite dimensional but separable, then (vi) applies and we may choose \( k_1 \) to be (at most) one dimensional. To see this let

\[
\pi(a) = V^*(a \otimes 1_{\mathfrak{h}'})V \quad \text{with} \quad VV^* \in 1_{\mathfrak{h}} \otimes \mathcal{B}(\mathfrak{h}')
\]

be the normal decomposition of the representation \( \pi \). Then, since \( \mathfrak{h}' \) may be assumed to be separable, there is a partial isometry \( U \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathfrak{h}') \) with final space \( \text{Ran}VV^* \). Putting \( r = V^*U \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathfrak{h}) \) yields the transformed dilation generator

\[
\hat{\phi} = \begin{bmatrix}
\tau & \gamma \dagger & \lambda \dagger \\
\gamma & \nu & 0 \\
\lambda & 0 & 0
\end{bmatrix},
\]

where \( \nu(a) = U^*(a \otimes 1_{\mathfrak{h}'})U \) and \( \gamma = \delta_{D, \nu} \) for \( D = U^*Vd \). If \( \partial \tau = 0 \) then \( \delta = 0 \) by (1.5), and it follows from Theorem 1.3 that we may take \( k_1 = \{0\} \) in (4.3) which (unless \( \mathcal{P} \) is already *-homomorphic) gives the generator

\[
\phi = \begin{bmatrix}
\tau & \lambda \dagger \\
\lambda & -\text{id}
\end{bmatrix}.
\]

If \( \partial \tau \neq 0 \) then it follows that \( \pi \) is nonzero and we may choose \( U \) to be isometric, making \( \nu \) unital. When \( \mathcal{P} \) is unital, so that \( k_2 \) may be chosen to be \( \{0\} \), the resulting dilation is that obtained in \([HuS]\).

**Corollary 4.2** ([GS]). Every norm-continuous normal completely positive contraction semigroup on a von Neumann algebra has an Evans-Hudson dilation.

**Corollary 4.3** ([HuS]). Let \( \mathfrak{h} \) be infinite dimensional and separable. Every norm-continuous normal completely positive unital semigroup on \( \mathcal{B}(\mathfrak{h}) \) has a unital Evans-Hudson dilation with one-dimensional quantum noise.
5. Number/exchange-free dilation.

In this section we seek necessary and sufficient conditions for a CP contraction cocycle to have a Markov-regular \(*\)-homomorphic stochastic dilation involving no number/exchange processes. Creation/annihilation-free dilations are also considered. Some interesting connections with locality, for quantum dynamical semigroups, emerge.

**Proposition 5.1.** Let \(k\) be a Markov-regular completely positive contraction cocycle on a von Neumann algebra \(\mathcal{M}\), with separable noise dimension space \(k_0\), and let \(\theta = \begin{bmatrix} \tau & \chi^\dagger \\ \chi & \nu - i_0 \end{bmatrix}\) be its stochastic generator and \(\mathcal{P}\) its Markov semigroup.

(a) Suppose that \(\nu = i_0\). Then the following are equivalent:

(i) \(k\) has a number/exchange-free \(*\)-homomorphic dilation.

(ii) \(k\) is unital, \(\chi\) is an \(i_0\)-derivation and \(\tau\) satisfies

\[
\partial \tau(a, a) - \chi(a)^* \chi(a) = \gamma(a)^* \gamma(a)
\]

where \(\gamma : \mathcal{M} \to \mathcal{M} \otimes k_1\) is an \(i_1\)-derivation for some Hilbert space \(k_1\).

(b) Suppose that \(\chi = 0\) and \(k\) is normal. Then the following are equivalent:

(i) \(k\) has a creation/annihilation-free \(*\)-homomorphic dilation.

(ii) \(\mathcal{P}\) is \(*\)-homomorphic.

**Proof.** (a) Suppose that \(k\) has a number/exchange-free \(*\)-homomorphic dilation. Then, by [LW1] Proposition 6.3 and Lemma 6.4, the generator of this dilation has the form

\[
\begin{bmatrix} l^*(a \otimes 1_k)l - \frac{1}{2} \{l^*l, a\} + i[h, a] & al^* - l^*(a \otimes 1_k)l \\
la - (a \otimes 1_k)l & 0 \end{bmatrix}
\]

where \(l \in \mathcal{M} \otimes k\), \(h = h^* \in \mathcal{M}\) and \(k \supset k_0\). It follows that \(\chi(a) = ma - (a \otimes 1_k)jm\), where \(m = p_0l \in \mathcal{M} \otimes k_0\), \(p_0\) being the orthogonal projection \(h \otimes k \to h \otimes k_0\), and so \(\theta\) satisfies (ii).

Conversely, if \(\theta\) has satisfies (ii) then by Theorem 1.3, \(\chi(a) = ma - (a \otimes 1_k)jm\) and \(\gamma(a) = na - (a \otimes 1_k)n\) for some \(m \in \mathcal{M} \otimes k_0\) and \(n \in \mathcal{M} \otimes k_1\), and \(\tau = \mathcal{L}_{l,t,h}\) where \(\nu(a) = a \otimes 1_{k_0} \oplus k_1, l = [m n]^T\) and \(h = h^* \in \mathcal{M}\), so that \(k\) has the number/exchange-free dilation with generator (5.2).

(b) First note that (ii) is equivalent to \(\tau\) being a derivation. Since the creation component \(\gamma\) of a \(*\)-homomorphic dilation must satisfy \(\tau(a^*a) - a^*\tau(a) - \tau(a)^*a = \tilde{\gamma}(a)^*\tilde{\gamma}(a)\) ([LW1], Section 6), (i) implies (ii).

Conversely, if (ii) holds then \(\mathcal{P}\) is conservative and, since \(k\) is normal, Remark (iv) above applies, with \(\delta = 0\), so \(k\) has a creation/annihilation-free \(*\)-homomorphic dilation. \(\square\)
Remark. From the structure of Markov-regular $^*$-homomorphic generators without number/exchange coefficients (5.2) it is clear that on an abelian von Neumann algebra any nontrivial CP contraction cocycle requires number/exchange processes in its dilation. This is not so when the Markov semigroup is no longer assumed to be norm continuous ($\{G_t\}$).

Proposition 5.2. Let $k$ and $\theta$ be as in Proposition 5.1, and assume further that $\nu = \iota_0$ and $\chi$ is an $\iota_0$-derivation. If $k$ has a number/exchange-free $^*$-homomorphic dilation then $k$ is unital and $\tau$ satisfies

\[(5.3) \quad \partial\tau(az,az) = z^*\partial\tau(a,a)z \quad \forall a \in \mathcal{M}, z \in \mathcal{Z},\]

where $\mathcal{Z}$ denotes the centre of $\mathcal{M}$. The converse holds when $\mathcal{M}'$ is abelian and $k$ is normal.

Proof. The necessity of (5.3) is clear from the form $\tau$ must take according to (5.2). Conversely, suppose that $\mathcal{M}'$ is abelian (and so equals $\mathcal{Z}$), $k$ is unital and normal and (5.3) holds, and let

\[\theta(a) = \begin{bmatrix} \tau(a) & \chi^\dagger(a) \\ \chi(a) & \nu(a) - \iota_0(a) \end{bmatrix} = \begin{bmatrix} \mathcal{L}(a) & \delta^\dagger(a)w \\ w^*\delta(a) & 0 \end{bmatrix}\]

be a representation of $\theta$ according to Theorem 3.2. Thus

\[(5.4) \quad w^*\pi(a)w = a \otimes 1_{k_0} \text{ and } \partial\mathcal{L}(a,a) = \delta(a)^*\delta(a),\]

with $\pi$ normal. Also (5.3) with $a = 1$, and (5.4), imply that $\delta(z) = 0$ for $z \in \mathcal{Z} = \mathcal{M}'$, so that

\[(5.5) \quad \delta(a)a' = \pi(a')\delta(a) \quad \forall a \in \mathcal{M}, a' \in \mathcal{M}'.\]

Now (5.4) also implies that $w$ is isometric and, letting $p$ be the orthogonal projection $ww^* \in \mathcal{M} \otimes \mathcal{B}(h)$,

\[p\pi(a)^*p^\dagger \pi(a)p = p\pi(a^*a)p - p\pi(a^*)p\pi(a)p = w(a^*a)w^* - w(a^* \otimes 1_{k_0})(a \otimes 1_{k_0})w^* = 0,\]

so $p$ commutes with $\pi(\mathcal{M})$ and thus $\pi_1(a) = p^\dagger\pi(a)p^\dagger$ defines a normal representation $\pi_1$ of $\mathcal{M}$. Let

\[\pi_1(a) = V^*(a \otimes 1_{h_1})V \text{ with }VV^* \in \mathcal{M}' \otimes \mathcal{B}(h_1),\]

be its normal decomposition, and define a map $\gamma : \mathcal{M} \to \mathcal{B}(h; h \otimes h_1)$ by $\gamma = V\delta$. Note that $V$ satisfies

\[(5.6) \quad (a \otimes 1_{h_1})V = V\pi(a).\]

By (5.5), (5.6) and (5.4), $\gamma$ satisfies

\[
\begin{align*}
\gamma(a)a' &= V\pi(a')\delta(a) = (a' \otimes 1_{h_1})\gamma(a); \\
\gamma(ab) - \gamma(a)b - (a \otimes 1_{h_1})\gamma(b) &= V\pi(a)\delta(b) - (a \otimes 1_{h_1})V\delta(b) = 0; \\
\gamma(a)^*\gamma(a) + \chi(a)^*\chi(a) &= \delta(a)^*[V^*V + ww^*]\delta(a) = \partial\tau(a,a); \\
\gamma(ab) - \gamma(a)b - (a \otimes 1_{h_1})\gamma(b) &= V\pi(a)\delta(b) - (a \otimes 1_{h_1})V\delta(b) = 0;
\end{align*}
\]
for $a, b \in \mathcal{M}$ and $a' \in \mathcal{M}'$. Thus $\gamma$ is an $\iota_1$-derivation satisfying $\gamma(\mathcal{M}) \subset \mathcal{M} \otimes \mathfrak{h}_1$ and (5.1). The result follows. $\square$

**Corollary 5.3.** A number/exchange-free Markov-regular normal CP unital cocycle on $\mathcal{B}(\mathfrak{h})$ has a number/exchange-free $*$-homomorphic dilation if and only if the creation component of its generator is an $\iota_0$-derivation.

**Remark.** Two consequences of the existence of a number/exchange-free $*$-homomorphic dilation for a Markov-regular CP contraction semigroup $\mathcal{P}$, with generator $\tau$, are

\[(5.7a) \quad \partial \tau(a, za) = \partial \tau(a, z)a + z\partial \tau(a, a)\]

\[(5.7b) \quad \delta(a)z = 0 \text{ whenever } az = 0\]

where $z \in \mathcal{Z}$ and $\delta$ is an $\iota_1$-derivation satisfying $\partial \tau(a, b) = \delta(a)^*\delta(b)$. When $\mathcal{M}$ is abelian (in which case $\mathcal{Z} = \mathcal{M}$) (5.7a) implies that

\[(5.7a') \quad \partial \tau \text{ is a derivation in its second argument.}\]

In his study of locality for quantum dynamical semigroups ([Sa2]) Sauvageot (citing [LeJ], Section 1.5) notes that, in the commutative $C^*$-case, (5.7a)' is equivalent to the locality of $\mathcal{P}$; that is

\[
\lim_{t \downarrow 0} t^{-1} \mathcal{P}_t f(x) = 0 \text{ whenever } f \text{ vanishes in a neighbourhood of } x.
\]

This suggests an interesting link between the analytic property of locality and the quantum probabilistic property of number/exchange-free $*$-homomorphic dilatability, for noncommutative Markov semigroups. Note also that, in the commutative case, (5.7b) is locality for $\delta$.

### 6. Unitary dilation.

In this section we consider contraction cocycles on a Hilbert space, and their stochastic dilation to unitary cocycles. We also apply this to construct an alternative form of dilation for unital CP cocycles on a von Neumann algebra.

A (left) Markovian contraction cocycle on the Hilbert space $\mathfrak{h}$ is a contraction process $(\mathcal{X}_t)_{t \geq 0}$ on $\mathfrak{h}$ adapted to the Fock filtration and satisfying

\[X_{s+t} = X_s \sigma_s(X_t),\]

in which the right shift $\sigma_s$ acts on $\mathcal{B}(\mathfrak{h} \otimes \Gamma_k)$ by

\[\sigma_s(T) = 1_s \otimes S_s TS_s^*,\]

where $S_s$ is the right shift $\mathfrak{h} \otimes \Gamma_k \to \mathfrak{h} \otimes \Gamma_{[s, \infty), k}$, $1_s$ is the identity operator on $\Gamma_{[0, s), k}$, and some natural identifications of tensor products are invoked. The Markov semigroup of $\mathcal{X}$ is defined by

\[P_t = E[X_t] = E^*X_tE \quad (t \geq 0).\]

The cocycle $\mathcal{X}$ is called Markov-regular if $P$ is norm continuous $\mathbb{R}_+ \to \mathcal{B}(\mathfrak{h})$. The counterpart to Theorem 1.1 is:
Theorem 6.1 ([LW2]). Let $X$ be a contraction process on the Hilbert space $\mathfrak{h}$, with separable noise dimension space $k_0$. Then the following are equivalent:

(i) $X$ is a Markov-regular cocycle.

(ii) $X$ weakly satisfies a QS differential equation of the form

$$dX_t = X_tF^\alpha_\beta d\Lambda^\alpha_\beta(t), \quad X_0 = 1,$$

(6.1)

for an operator matrix $F \in M_D(\mathcal{B}(\mathfrak{h}))$.

In this case $X$ satisfies the equation strongly, and $F$ defines a bounded operator on $\mathfrak{h} \otimes \hat{k}_0$. Moreover, for a von Neumann algebra $\mathcal{M}$ acting on $\mathfrak{h}$, $X_t \in \mathcal{M} \otimes \mathcal{B}_{k_0}$ for all $t \geq 0$ if and only if $F \in \mathcal{M} \otimes \mathcal{B}(k_0)$.

Remark. A (unique) strong solution to (6.1) exists whenever $[F^\alpha_\beta]$ defines a bounded operator $F$ on $\mathfrak{h} \otimes \hat{k}$ ([LW1], Theorem 7.1), although the solution may be unbounded. Following (1.1) such $F$ will be written in block matrix form as $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in \mathcal{B}(\mathfrak{h})$, $B,C^* \in \mathcal{B}(\mathfrak{h} \otimes k; \mathfrak{h})$ and $D \in \mathcal{B}(\mathfrak{h} \otimes k)$.

The case where $X$ is unitary was established in [HuL] and [Mo1]. Weakly differentiable Markovian contraction cocycles were shown to satisfy a QS differential equation in [AJL] and [Fag]. An interesting characterisation of strongly continuous unitary Markovian cocycles was obtained in [Jou], which also contains an example of such a cocycle which fails to satisfy a QS differential equation.

Propositions 7.5 and 7.6 of [LW1] may be stated as follows, using the representation of nonnegative operator matrices given in Lemma 2.1 once more:

Theorem 6.2. Let $X$ be a process on the Hilbert space $\mathfrak{h}$ with separable noise dimension space $k_0$, weakly satisfying a QS differential equation of the form (6.1). Then the following equivalences hold:

(ai) $X$ is a contraction process.

(aii) $X$ is isometric.

(aii) $F$ is bounded with block matrix form

$$F = \begin{pmatrix} iH - \frac{1}{2}(M^*M + B^2) & BV - M^*W \\ M & W - 1 \end{pmatrix}$$

where $H = H^*, B \geq 0, \|V\|, \|W\| \leq 1$ and $S = (1 - W^*W)^{1/2}$.

(aiii) $F$ is bounded with block matrix form

$$F = \begin{pmatrix} iH - \frac{1}{2}(LL^* + C^2) & -L \\ WL^* - RV'C & W - 1 \end{pmatrix}$$

where $H = H^*, C \geq 0, \|V'\|, \|W\| \leq 1$ and $R = (1 - WW^*)^{1/2}$.

(bi) $X$ is isometric.

(bii) In (a) $W$ is isometric and $B = 0$. 
(ci) $X$ is coisometric.
(cii) In (a(ii)) $W$ is coisometric and $C = 0$.
(dii) $X$ is unitary
(dii) In (a(ii)) $W$ is unitary and $B = 0$.
(diii) In (a(iii)) $W$ is unitary and $C = 0$.

The representation (a(ii)) is unique provided that $V$ satisfies $\text{Ker} V \supset \text{Ker} S$ and $\text{Ran} V \subset \text{Ran} B$, which may be easily arranged. Uniqueness of the representation (a(iii)) may be similarly arranged.

Remark. Thus associated with a Markov-regular operator contraction cocycle $X$ is a unique octet $(H, W, (M, B, V), (L, C, V'))$, and conversely, $X$ is determined by either one of the two parameterisations $(H, W, M, B, V)$ or $(H, W, L, C, V')$.

Theorem 6.3. Every Markov-regular contraction cocycle on a Hilbert space, with separable noise dimension space, admits a unitary stochastic dilation.

Proof. Let $X$ be a Markov-regular contraction cocycle on $\mathfrak{h}$, with separable noise dimension space $k_0$. By Theorems 6.1 and 6.2, $X$ has a stochastic generator of the form

$$
\begin{bmatrix}
K & -Q \\
N & W-1
\end{bmatrix},
$$

where $K = iH - \frac{1}{2}(N^*N + B^2)$, $Q = N^*W - BV S$, $H = H^*$, $B \geq 0$, $\|W\|, \|V\| \leq 1$ and $S = (1 - W^*W)^{1/2}$.

Set $k = k_0 \oplus k_1 \oplus k_2$, where $k_1$ and $k_2$ are to be determined, and let $G \in \mathcal{B}(\mathfrak{h} \otimes \hat{k})$ with block matrix form

$$
\begin{bmatrix}
K & -L \\
M & U-1
\end{bmatrix}
$$

in which

\begin{equation}
(6.2a)
U^*U = UU^* = 1, \quad M^*M = N^*N + B^2 \quad \text{and} \quad L = M^*U.
\end{equation}

Then $G$ generates a unitary cocycle by Theorem 6.2, and by the operator process analogue of Lemma 1.2 this cocycle is a stochastic dilation of $X$ if and only if

\begin{equation}
(6.2b)
M^0 = N, \quad L_0 = Q \quad \text{and} \quad U_0^0 = W.
\end{equation}

Solutions of the constraints (6.2) are obtained by taking $k_1 = k_0, k_2 = \mathbb{C}$,

$$
M = \begin{bmatrix} N \\ -V^*B \\ M_2 \end{bmatrix}, \quad L = \begin{bmatrix} Q & L_1 & M_2^* \end{bmatrix}, \quad \text{and} \quad U = \begin{bmatrix} W & R & 0 \\ S & -W^* & 0 \\ 0 & 0 & 1 \end{bmatrix},
$$

where $R = (1 - WW^*)^{1/2}$, $L_1 = BVW^* + N^*R$, and $M_2 \in \mathcal{B}(\mathfrak{h})$ is chosen so that $M_2^2M_2 = B(1 - VV^*)B$. \hfill \Box
Remarks. (i) If $X$ is affiliated to a von Neumann algebra $\mathcal{M}$, then it has a unitary dilation which is affiliated to $\mathcal{M}$ also.

(ii) If $B = 0$, in particular if $X$ is isometric (but not unitary), then $M_2 = 0$ and we may take $k_1 = k_0$ and $k_2 = \{0\}$, giving the dilation generator

\[
G = \begin{pmatrix}
iH - \frac{1}{2}N^*N & -N^*W & -N^*R \\
N & W - 1 & R \\
0 & S & -W^* - 1
\end{pmatrix},
\]

with $R$ and $S$ as above.

(iii) If $W$ is unitary (but $B \neq 0$) then $V = 0$ and the simplest solution is $k_1 = \{0\}, k_2 = \mathbb{C}$ and

\[
G = \begin{pmatrix}
iH - \frac{1}{2}(N^*N + B^2) & -N^*W & iB \\
N & W - 1 & 0 \\
iB & 0 & 0
\end{pmatrix}.
\]

(iv) To see what is being stochastically generalised here, let $k_0 = \{0\}$ so that we have a contraction semigroup on $\mathfrak{h}$. Then (iii) applies, giving

\[
G = \begin{pmatrix}
iH - \frac{1}{2}B^2 & iB \\
iB & 0
\end{pmatrix},
\]

and the dilation is effected by the solution $U$ of the stochastic differential equation

\[
dU_t = U_t \left[ iB dQ_t + \left( iH - \frac{1}{2}B^2 \right) dt \right]
\]

where $Q = (A_t + A_t^\dagger)_{t \geq 0}$ is a classical Brownian motion. When the operators $B$ and $H$ commute it is given explicitly by

\[
U_t = e^{i(B \otimes Q_t + tH \otimes I)}.
\]

(v) Using time-reversal techniques, or by taking adjoints, Markov-regular right contraction cocycles are equally seen to have unitary stochastic dilations.

We end this section with an alternative form of dilation for unital CP cocycles.

**Theorem 6.4.** Every normal Markov-regular completely positive unital cocycle $k$, on a von Neumann algebra, with separable noise dimension space $k_0$ and initial space $\mathfrak{h}$, enjoys a factorisation of the following form:

\[
\tilde{k}_t = E_1 \circ \text{ad} U_t \circ \iota
\]

(6.3)

in which $\tilde{k}_t(a) = k_t(a) \otimes 1$ on $\mathfrak{h} \otimes \Gamma_{k_1}$, $\iota(a) = a \otimes 1$ on $\mathfrak{h} \otimes \Gamma_{k_1 \otimes k_1}$, $k_1 \supset k_0$ is separable and $U$ is a Markov-regular unitary cocycle. If $k_0$ is infinite dimensional then we may take $k_1 = k_0$, so that $k = k$. 
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Proof. Theorem 4.5 of [GLW] ensures the existence of a separable Hilbert space $k_1$ containing $k_0$, and a Markov-regular coisometry cocycle $X$ with noise dimension space $k_1$ such that $k_t(a) \otimes 1 = X_t(a \otimes 1)X_t^*$ on $\mathfrak{h} \otimes \Gamma_{k_1}$. Let $k^-$ denote the Markovian cocycle on $\mathcal{B}(\mathfrak{h})$ given by the same formula. The stochastic generators of $X$ and $k^-$ have the respective forms

$$\hat{F} = \begin{bmatrix} iH - \frac{1}{2}M^*M & -M^* \\ W^*M & W^* \end{bmatrix}, \quad \hat{\theta}(a) = \begin{bmatrix} \tau(a) & \delta^\dagger(a)W \\ W^*\delta(a) & W^*\iota_1(a)W \end{bmatrix}$$

where $H = H^*, W^*W = 1, \iota_1(a) = a \otimes 1$ on $\mathfrak{h} \otimes k_1, \delta = \delta_{M,\iota_1}$ and $\tau = \mathcal{L}_{M,\iota_1,H}$. If $X$ is unitary then (6.3) holds with $U_t = X_t \otimes 1$. If $X$ is nonunitary then, by Theorem 6.2, $W^*$ is nonisometric and we let $U$ be the unitary stochastic dilation of $X$ whose generator is defined by

$$\hat{G} = \begin{bmatrix} iH - \frac{1}{2}M^*M & -M^* & 0 \\ W^*M & W^* & 0 \\ RM & R & -W \end{bmatrix}$$

where $R = (1 - WW^*)$, and set $\tilde{j}^- = \text{ad} U_t : a \mapsto U_t(a \otimes 1)U_t^*$. Then the stochastic generator of $\tilde{j}^-$ is given by

$$\phi(a) = \iota(a)G^* + G\iota(a) + G\Delta(a)G^*,$$

where $\Delta(a) = a \otimes \Delta$ and $\Delta$ is the orthogonal projection in $\mathcal{B}(\hat{k})$ with range $k = k_1 \oplus k_1$, so that

$$\hat{\phi}(a) = \begin{bmatrix} \tau(a) & \delta^\dagger(a)W & \delta^\dagger(a)R \\ W^*\delta(a) & W^*\iota_1(a)W & W^*\iota_1(a)R \\ R\delta(a) & R\iota_1(a)W & R\iota_1(a)R + W\iota_1(a)W \end{bmatrix}.$$

Invoking Lemma 1.2 once more, comparison with (6.4) shows that $\tilde{j}^-$ is a stochastic dilation of $k^-$, and the result follows. □

Remark. This need not be a stochastic dilation of $\tilde{k}$ in our sense, since $U$ need not be affiliated to $\mathcal{M}$.

7. Application to classical probability.

Two areas of classical probability where ideas from the theory of CP-valued quantum processes might be applied are filtering theory and measure-valued diffusion. Indeed the Zakai equation (see e.g., [KaK]), governing the conditional distribution of a signal process at time $t$ given the $\sigma$-algebra of an observed process up to time $t$, was a motivation for Belavkin to consider quantum filtering from the point of view of QS differential equations ([Be1]). Below we give a rough outline of how one might view measure-valued diffusions as conditioned classical processes on the underlying state space.

Consider a compact Hausdorff space $X$. Let $\mathcal{P}$ denote the set of regular Borel probability measures on $X$, equipped with the topology of weak
convergence, and let $\mathcal{A}$ denote the unital $C^*$-algebra of continuous complex-valued functions on $X$. Also let $L^2(\mathcal{P})$ be the $L^2$-space of the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ of a family $P_1, \ldots, P_m$ of independent Poisson processes with respective intensities $\lambda_1, \ldots, \lambda_m$. Finally let $\mu := (\mu_t^x : \Omega \to \mathcal{P})_{x \in X, t \geq 0}$ be a family of maps satisfying suitable measurability conditions, the condition that, for each $t \geq 0$ and $f \in \mathcal{A}$, the map $x \mapsto \langle f, \mu_t^x(\omega) \rangle := \int f(y)\mu_t^x(\omega)(dy)$ is continuous for almost all $\omega \in \Omega$, and also the following stochastic differential equation:

\begin{equation}
(7.1) \quad d\langle f, \mu_t^x \rangle = \langle \alpha_0(f), \mu_t^x \rangle \, dt + \sum_{i=1}^m \langle \alpha_i(f), \mu_t^x \rangle \, dP_i^t, \quad \mu_0^x = \delta_x,
\end{equation}

where $\alpha_0, \ldots, \alpha_m$ are bounded linear maps on $\mathcal{A}$ such that $\alpha_i(f) = \overline{\alpha_i(f)}$ and $\alpha_i(1) = 0$. Thus $\mu_t^x$ is a measure-valued diffusion.

In order to cast this in a QS form, first let $\mathcal{M}$ be the universal enveloping algebra of $\mathcal{A}$ and let $\mathfrak{H}$ be the Hilbert space on which it acts. Then each $\alpha_i$ extends to a bounded normal operator on $\mathcal{M}$ — which we denote by $\alpha_i^*$. Under the natural identification of $L^2(\mathcal{P})$ with $\Gamma(L^2(\mathbb{R}_+; \mathbb{C}^m))$, the operator of multiplication by $P_i^t$ corresponds to the following combination of fundamental quantum processes:

\begin{equation}
(7.2) \quad N_i^t(t) + \sqrt{\lambda_i(A_i(t) + A_i^d(t))} + \lambda_i t
\end{equation}

([Mey], p. 74). Let $\overline{k}$ be the Markovian cocycle on $\mathcal{M}$ generated by $\theta$ where

$$
\theta_0^0 = \alpha_0^{**} + \sum_{i=1}^m \lambda_i \alpha_i^{**}, \quad \theta_0^1 = \sqrt{\lambda_i \alpha_i^{**}}, \quad \text{and} \quad \theta_j^1 = \delta_j^i \alpha_i^{**}.
$$

Then $\overline{k}$ is normal, CP and unital. Comparison with (7.1) and (7.2) shows that for $f \in \mathcal{A}$, viewed as a subalgebra of $\mathcal{M}$, $k_i(f)$ corresponds to the operator of multiplication by the function $F_i : \Omega \to \mathcal{A} \subset \mathcal{M}$ given by $F_i(\omega) = \langle f, \mu_t^x(\omega) \rangle$.

Suppose now that $\overline{k}$ has a unital $\ast$-homomorphic dilation $\overline{\mathcal{J}}$. Let $j_i$ be the restriction of $\overline{\mathcal{J}}_i$ to $\mathcal{A}$ then the $C^*$-algebra generated by $\{j_i(f) : t \geq 0, f \in \mathcal{A}\}$ is abelian ([Par], Theorem 28.8). Letting $\Sigma$ be its spectrum, our process may be viewed as consisting of unital $\ast$-homomorphisms $j_i : C(X) \to C(\Sigma)$. By Gelfand theory (specifically the Banach-Stone Theorem) there are continuous maps $\xi_i : \Sigma \to X$ such that $j_i(f) = f \circ \xi_i$. In this way the dilation of $k$ may be viewed as a lifting of the measure-valued diffusion $\mu$ to a “process” $\xi$ taking values in the underlying state space.

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QUANTUM LENS SPACES AND GRAPH ALGEBRAS

Wojciech Szymański

We construct the $C^*$-algebra $C(L_q(p; m_1, \ldots, m_n))$ of continuous functions on the quantum lens space as the fixed point algebra for a suitable action of $\mathbb{Z}_p$ on the algebra $C(S^{2n-1}_q)$, corresponding to the quantum odd dimensional sphere. We show that $C(L_q(p; m_1, \ldots, m_n))$ is isomorphic to the graph algebra $C^*(L_{2n-1}^{p;m_1,\ldots,m_n})$. This allows us to determine the ideal structure and, at least in principle, calculate the $K$-groups of $C(L_q(p; m_1, \ldots, m_n))$. Passing to the limit with natural imbeddings of the quantum lens spaces we construct the quantum infinite lens space, or the quantum Eilenberg-MacLane space of type $(\mathbb{Z}_p, 1)$.

0. Introduction.

Classical lens spaces $L(p; m_1, \ldots, m_n)$ are defined as the orbit spaces of suitable free actions of finite cyclic groups on odd dimensional spheres (e.g., see [13]). In the present article, we define and investigate their quantum analogues. The $C^*$-algebras of continuous functions on the quantum lens spaces were introduced earlier by Matsumoto and Tomiyama in [18], but our construction leads to different (in general) algebras. (The very special case of the quantum 3-dimensional real projective space was investigated by Podleś [20] and Lance [17], in the context of the quantum $SO(3)$ group.) The starting point for us is the $C^*$-algebra $C(S^{2n-1}_q)$, $q \in (0, 1)$, of continuous functions on the quantum odd dimensional sphere. If $n = 2$ then $C(S^3_q)$ is nothing but $C(SU_q(2))$ of Woronowicz [27]. The construction in higher dimensions is due to Vaksman and Soibelman [26], and from a somewhat different perspective to Nagy [19]. (See also the closely related construction of representations of the twisted canonical commutation relations due to Pusz and Woronowicz [21].) We define the $C^*$-algebra $C(L_q(p; m_1, \ldots, m_n))$ of continuous functions on the quantum lens space as the fixed point algebra for a suitable action of the finite cyclic group $\mathbb{Z}_p$ on $C(S^{2n-1}_q)$. This definition depends on the deformation parameter $q \in (0, 1)$, as well as on positive integers $p \geq 2$ and $m_1, \ldots, m_n$. We normally assume that each of $m_1, \ldots, m_n$ is relatively prime to $p$. On the classical level, this guarantees freeness of the action. In the special case $p = 2$, $m_1 = \cdots = m_n = 1$ we recover
quantum odd dimensional real projective spaces, defined and investigated in our earlier article [11].

The key technical result on which this article depends is Theorem 4.4 of [11], which gives an explicit isomorphism between $C(S_{2n}^{2n-1})$ and the $C^*$-algebra $C^*(L_{2n-1})$ corresponding to the directed graph $L_{2n-1}$. Thus, $C(L_{q}(p; m_1, \ldots, m_n))$ is isomorphic to the fixed point algebra $C^*(L_{2n-1})^\Lambda$, corresponding to a suitable action $\Lambda : \mathbb{Z}_p \to \text{Aut}(C^*(L_{2n-1}))$. This allows us to employ in our investigations of the quantum lens spaces the huge and comprehensive machinery developed for dealing with Cuntz-Krieger algebras of directed graphs (cf. [6, 5, 16, 12, 15, 14, 2, 9, 24, 10, 22, 25, 1] and references there).

In order to understand the fixed point algebra $C^*(L_{2n-1})^\Lambda$ we first look at the crossed product $C^*(L_{2n-1}) \rtimes_\Lambda \mathbb{Z}_p$. By virtue of the results of [14] this crossed product itself is isomorphic to the $C^*$-algebra of the skew product graph $L_{2n-1} \rtimes_c \mathbb{Z}_p$, corresponding to a suitable labelling $c$ of the edges of $L_{2n-1}$ by elements of $\mathbb{Z}_p$. The action $\Lambda$ is saturated and, hence, $C^*(L_{2n-1})^\Lambda$ is isomorphic to a full corner of $C^*(L_{2n-1} \rtimes_c \mathbb{Z}_p)$. This allows us, at least in principle, to calculate the $K$-groups of $C(L_{q}(p; m_1, \ldots, m_n))$.

Our main result, Theorem 2.5, shows that $C(L_{q}(p; m_1, \ldots, m_n))$ itself is isomorphic to the graph algebra $C^*(L_{2n-1}(p; m_1, \ldots, m_n))$, corresponding to a finite graph $L_{2n-1}(p; m_1, \ldots, m_n)$. As a corollary, we easily deduce the ideal structure of $C(L_{q}(p; m_1, \ldots, m_n))$. We believe that on the basis of Theorem 2.5 one should be able to determine isomorphisms between the $C^*$-algebras of continuous functions on the quantum lens spaces, but this work is not carried in the present article.

1. Preliminaries.

1.1. Definition. We recall the definition of the $C^*$-algebra corresponding to a directed graph [9]. Let $E = (E^0, E^1, r, s)$ be a directed graph with (at most) countably many vertices $E^0$ and edges $E^1$, and range and source functions $r, s : E^1 \to E^0$, respectively. $C^*(E)$ is, by definition, the universal $C^*$-algebra generated by families of projections $\{P_v \mid v \in E^0\}$ and partial isometries $\{S_e \mid e \in E^1\}$, subject to the following relations:

1. $P_v P_w = 0$ for $v, w \in E^0$, $v \neq w$.
2. $S_e^* S_f = 0$ for $e, f \in E^1$, $e \neq f$.
3. $S_e^* S_e = P_{r(e)}$ for $e \in E^1$.
4. $S_e S_e^* \leq P_{s(e)}$ for $e \in E^1$.
5. $P_v = \sum_{e \in E^1 : s(e) = v} S_e S_e^*$ for $v \in E^0$, provided $\{e \in E^1 \mid s(e) = v\}$ is finite and nonempty.
Universality in this definition means that if \( \{Q_v \mid v \in E^0\} \) and \( \{T_e \mid e \in E^1\} \) are families of projections and partial isometries, respectively, satisfying Conditions (G1)–(G5), then there exists a \( C^* \)-algebra homomorphism from \( C^*(E) \) to the \( C^* \)-algebra generated by \( \{Q_v \mid v \in E^0\} \) and \( \{T_e \mid e \in E^1\} \) such that \( P_v \mapsto Q_v \) and \( S_e \mapsto T_e \) for \( v \in E^0, e \in E^1 \).

It follows from the universal property that there exists the gauge action \( \gamma : \mathbb{T} \to \text{Aut}(C^*(E)) \) such that \( \gamma_t(P_v) = P_v \) and \( \gamma_t(S_e) = tS_e \), for all \( v \in E^0, e \in E^1, t \in \mathbb{T} \).

1.2. \( K \)-theory. The \( K \)-theory of a graph algebra \( C^*(E) \) can be calculated as follows: Let \( V_E \) be the collection of all those vertices which are not sinks and emit finitely many edges, and let \( \mathbb{Z}V_E \) and \( \mathbb{Z}E^0 \) be free abelian groups on free generators \( V_E \) and \( E^0 \), respectively. We have

\[
K_0(C^*(E)) \cong \text{coker}(K_E),
\]

\[
K_1(C^*(E)) \cong \ker(K_E),
\]

where \( K_E : \mathbb{Z}V_E \to \mathbb{Z}E^0 \) is the map defined on generators as

\[
K_E(v) = \left( \sum_{e \in E^1 : s(e) = v} r(e) \right) - v.
\]

(See [5, Proposition 3.1], [16, Corollary 6.12], [22, Theorem 3.2], [24, Proposition 2] and [7, Theorem 3.1].)

1.3. Ideals. We assume that \( E \) is a row-finite graph (i.e., each vertex of \( E \) emits only finitely many edges) without sinks, since this is all we need in the present article. At first we describe closed 2-sided ideals of \( C^*(E) \) invariant under the gauge action, as well as the corresponding quotients \([5, 12, 16, 2, 1, 10]\). To this end we consider hereditary and saturated subsets of \( E^0 \). A subset \( X \subseteq E^0 \) is hereditary and saturated if the following two conditions are satisfied:

(HS1) If \( v \in X, w \in E^0 \) and there exists a path from \( v \) to \( w \) then \( w \in X \).

(HS2) If \( v \in E^0 \) and for each \( e \in E^1 \) with \( s(e) = v \) we have \( r(e) \in X \), then \( v \in X \).

We denote by \( \Sigma_E \) the collection of all hereditary and saturated subsets of \( E^0 \). Any hereditary and saturated set \( X \) gives rise to a gauge invariant ideal generated by \( \{P_v \mid v \in X\} \) and denoted \( J_X \). The quotient \( C^*(E)/J_X \) is naturally isomorphic to \( C^*(E/X) \), where \( E/X \) denotes the restriction of the graph \( E \) to \( E^0 \setminus X \). There exists a bijection between \( \Sigma_E \) and the collection of all gauge invariant ideals of \( C^*(E) \), given by the following two maps:

\[
X \mapsto J_X, \quad J \mapsto \{v \in E^0 \mid P_v \in J\}.
\]

We now turn to the description of primitive ideals of the graph algebra \( C^*(E) \) corresponding to a row-finite graph \( E \) with no sinks \([12, 16, 2, 1, 10]\).
Key objects used in the classification of primitive ideals of graph algebras are maximal tails, defined as follows: A nonempty subset $M \subseteq E^0$ is called a maximal tail if the following three conditions are satisfied:

(MT1) If $v \in E^0$, $w \in M$ and there is a path in $E$ from $v$ to $w$ then $v \in M$.
(MT2) If $v \in M$ then there exists an edge $e \in E^1$ such that $s(e) = v$ and $r(e) \in M$.
(MT3) For any $v, w \in M$ there is a $y \in M$ such that there exist paths in $E$ from $v$ to $y$ and from $w$ to $y$.

The collection $\mathcal{M}(E)$ of all maximal tails is a disjoint union of its two subcollections $\mathcal{M}_\gamma(E)$ and $\mathcal{M}_r(E)$, defined as follows: A maximal tail $M$ belongs to $\mathcal{M}_\gamma(E)$ if and only if every vertex simple loop $(e_1, e_2, \ldots, e_k)$ (where $e_i \in E^1$, $r(e_i) = s(e_{i+1})$, $r(e_k) = s(e_1)$ and $r(e_i) \neq r(e_j)$ for $i \neq j$) whose all vertices $s(e_i)$ belong to $M$ has an exit $e \in E^1$ (that is, $s(e) \in \{s(e_1), \ldots, s(e_k)\}$ but $e \notin \{e_1, \ldots, e_k\}$) with $r(e) \in M$. Otherwise $M$ belongs to $\mathcal{M}_r(E)$. It can be shown that each maximal tail from $\mathcal{M}_\gamma(E)$ gives rise to a primitive ideal of $C^*(E)$ invariant under the gauge action, and each maximal tail from $\mathcal{M}_r(E)$ gives rise to a circle of primitive ideals none of which is invariant under the gauge action. Let $\text{Prim}(C^*(E))$ denote the set of all primitive ideals of $C^*(E)$. If $E$ is a finite graph with no sinks then there exists a bijection

$$\mathcal{M}_\gamma(E) \cup (\mathcal{M}_r(E) \times \mathbb{T}) \leftrightarrow \text{Prim}(C^*(E)).$$

A complete description of the closure operation in the hull-kernel topology is also available. See [12, 16, 2, 1, 10] for the details.

We finish this section with the following lemma, which will be needed in the proof of Theorem 2.5. Recall that a closed 2-sided ideal $J$ of a $C^*$-algebra $A$ is essential if and only if for each nonzero element $a$ of $A$ we have $aJ \neq \{0\}$.

**Lemma 1.1.** If $E$ is a row-finite graph and $X \neq \emptyset$ is a hereditary and saturated subset of $E^0$ then $X^*$ is an essential ideal of $C^*(E)$ if and only if for each vertex $v \in E^0 \setminus X$ there exists a path in $E$ from $v$ to a vertex in $X$.

**Proof.** Suppose that for each vertex $v \in E^0 \setminus X$ there exists a path in $E$ from $v$ to a vertex in $X$. With the gauge action $\gamma : \mathbb{T} \to \text{Aut}(C^*(E))$, the formula $\Gamma(b) = \int_{t \in \mathbb{T}} \gamma_t(b) \, dt$ (the integration with respect to the normalized Haar measure) defines a faithful conditional expectation from $C^*(E)$ onto the fixed point algebra $C^*(E)^\gamma$. Let $a \neq 0$ be an element of $C^*(E)$ and let $J'$ be the closed 2-sided ideal of $C^*(E)$ generated by $\Gamma(a^*a)$. Since $J'$ is a nonzero gauge invariant ideal there exists a vertex $v \in E^0$ such that $P_v \in J'$ (cf. [2, Theorem 4.1]). If $a$ is a path from $v$ to a vertex in $X$ then $S_\alpha \in J_X$ and $P_v S_\alpha \neq 0$. Consequently, $\{0\} \neq \Gamma(a^*a)J_X = \{\int_{t \in \mathbb{T}} \gamma_t(a^*a) \, dt\} J_X$. Thus, there exists a $t \in \mathbb{T}$ such that $\gamma_t(a^*a)J_X \neq \{0\}$. Since $\gamma_t(J_X) = J_X$ this
implies $aJ_X \neq \{0\}$. Therefore, the ideal $J_X$ is essential, as required. The converse implication is trivial. \hfill \Box

1.4. Quantum odd dimensional spheres. For $n = 1, 2, \ldots$ and $q \in (0, 1)$ the $C^*$-algebra $C(S_q^{2n-1})$ of continuous functions on the quantum sphere $S^{2n-1}$ is given in [26] as the universal $C^*$-algebra generated by elements $z_1, z_2, \ldots, z_n$, subject to the following relations:

\begin{align*}
(1) & \quad z_jz_i = qz_iz_j \quad \text{for } i < j, \\
(2) & \quad z_j^*z_i = qz_iz_j^* \quad \text{for } i \neq j, \\
(3) & \quad z_i^*z_i = z_iz_i^* + (1 - q^2)\sum_{j<i} z_jz_j^* \quad \text{for } i = 1, \ldots, n, \\
(4) & \quad \sum_{i=1}^n z_i z_i^* = I. 
\end{align*}

It is shown in [11, Theorem 4.4] that the $C^*$-algebra $C(S_q^{2n-1})$ is isomorphic with a graph algebra $C^*(L_{2n-1})$. The graph $L_{2n-1}$ has $n$ vertices $\{v_1, \ldots, v_n\}$ and $n(n + 1)/2$ edges $\bigcup_{i=1}^n \{e_{i,j} \mid j = i, \ldots, n\}$ with $s(e_{i,j}) = v_i$ and $r(e_{i,j}) = v_j$. It is a finite graph without sinks. For example, if $n = 3$ then the corresponding graph $L_5$ looks as follows:

\[ L_5 \]

The isomorphism $\phi : C(S_q^{2n-1}) \rightarrow C^*(L_{2n-1})$ is given explicitly on the generators as

\begin{align*}
(5) & \quad \phi : z_n \mapsto \sum_{k_1, \ldots, k_{n-1} \in \mathbb{N}} q^{k_1 + \cdots + k_{n-1}} T(k_1, \ldots, k_{n-1}) S_{e_{n,n}} T(k_1, \ldots, k_{n-1})^*, \\
(6) & \quad \phi : z_i \mapsto \sum_{k_1, \ldots, k_{i-1} \in \mathbb{N}} q^{k_1 + \cdots + k_{i-1}} \left( \sqrt{1 - q^2(k_i+1)} - \sqrt{1 - q^2k_i} \right) \times \\
& \quad \times T(k_1, \ldots, k_i) \left( \sum_{j=i}^n S_{e_{i,j}} \right) T(k_1, \ldots, k_i)^*, 
\end{align*}

for $i = 1, \ldots, n - 1$. Here for $k_1, \ldots, k_i \in \mathbb{N}$ we denoted

\begin{align*}
(7) & \quad T(k_1, \ldots, k_i) = \left( \sum_{j=1}^n S_{e_{1,j}} \right)^{k_1} \left( \sum_{j=2}^n S_{e_{2,j}} \right)^{k_2} \cdots \left( \sum_{j=i}^n S_{e_{i,j}} \right)^{k_i},
\end{align*}
an element of $C^*(L_{2n-1})$.

2. Quantum lens spaces.

We begin by recalling the definition of the classical lens spaces [13]. Namely, for $n = 1, 2, \ldots$ let $S^{2n-1} = \{(y_1, \ldots, y_n) \in \mathbb{C}^n \mid \sum_{i=1}^n |y_i|^2 = 1\}$ be the sphere of dimension $2n - 1$. We fix an integer $p \geq 2$ and $n$ integers $m_1, \ldots, m_n$. If $\theta = e^{2\pi i/p}$ then

\begin{equation}
(y_1, \ldots, y_n) \mapsto (\theta^{m_1}y_1, \ldots, \theta^{m_n}y_n)
\end{equation}

is a homeomorphism of $S^{2n-1}$ which gives rise to an action of $\mathbb{Z}_p$, the cyclic group of order $p$, on $S^{2n-1}$. The (generalized) lens space $L(p; m_1, \ldots, m_n)$ of dimension $2n - 1$ is defined as the orbit space of this action. It is normally assumed that each of $m_1, m_2, \ldots, m_n$ is relatively prime to $p$. This assumption is equivalent to freeness of the action (8).

We now turn to the quantum case. With the sole exception of Lemma 2.1, we always assume that each of $m_1, m_2, \ldots, m_n$ is relatively prime to $p$. The universal property of $C(S^{2n-1}_q)$ implies that the assignment

\begin{equation}
\tilde{\Lambda} : z_i \mapsto \theta^{m_i}z_i,
\end{equation}

for $i = 1, \ldots, n$, gives rise to an automorphism $\tilde{\Lambda}$ of $C(S^{2n-1}_q)$ of order $p$. For $q \in (0, 1)$ we define the $C^*$-algebra $C(L_q(p; m_1, \ldots, m_n))$ of continuous functions on the quantum lens space as the fixed point algebra corresponding to this automorphism, i.e.,

\begin{equation}
C(L_q(p; m_1, \ldots, m_n)) = C(S^{2n-1}_q)^{\tilde{\Lambda}}.
\end{equation}

Let $\phi : C(S^{2n-1}_q) \to C^*(L_{2n-1})$ be the isomorphism given by (5)-(6). Setting $\Lambda = \phi \tilde{\Lambda} \phi^{-1}$ we get

\begin{align}
\Lambda : P_{v_i} &\mapsto P_{v_i}, \\
\Lambda : S_{e_{i,j}} &\mapsto \theta^{m_i}S_{e_{i,j}},
\end{align}

for $i = 1, \ldots, n$ and $j = i, \ldots, n$. This gives

\begin{equation}
C(L_q(p; m_1, \ldots, m_n)) = C(S^{2n-1}_q)^\Lambda \cong C^*(L_{2n-1})^\Lambda.
\end{equation}

Actions of this type have been studied by Kumjian and Pask [14]. Let $c : L_{2n-1} \to \mathbb{Z}_p$ be a labeling of the edges of $L_{2n-1}$ such that $c(e_{i,j}) = m_i$. The skew product graph $L_{2n-1} \times_c \mathbb{Z}_p$ is defined so that its vertices are $L^0_{2n-1} \times \mathbb{Z}_p$ and its edges are $L^1_{2n-1} \times \mathbb{Z}_p$ with $s(e_{i,j}, m) = (v_i, m - m_i)$ and $r(e_{i,j}, m) = (v_j, m)$, for $m \in \mathbb{Z}_p$, $i = 1, \ldots, n$ and $j = i, \ldots, n$. We note that through each vertex of this graph passes precisely one vertex simple loop (composed of $p$ edges), and for any two vertices $(v_i, m), (v_j, k)$ there exists a path from $(v_i, m)$ to $(v_j, k)$ if and only if $i \leq j$. For example, if $n = 2$, $p = 3$, $m_1 = 1$ and $m_2 = 2$ then $L_3 \times_c \mathbb{Z}_3$ looks as follows:
By virtue of [14, Corollary 2.5] there exists a $C^*$-algebra isomorphism

$$\text{(14)} \quad C^*(L_{2n-1} \times_c \mathbb{Z}_p) \cong C^*(L_{2n-1}) \times_{\Lambda} \mathbb{Z}_p,$$

Let $U$ be a unitary in $C^*(L_{2n-1}) \times_{\Lambda} \mathbb{Z}_p$ such that $U^p = I$ and $U x U^* = \Lambda(x)$ for all $x \in C^*(L_{2n-1})$. For $m = 0, 1, \ldots, p - 1$ let $Q_m = \frac{1}{p} \sum_{i=0}^{p-1} \theta^{im} U^i$ be the spectral projection of $U$. The isomorphism (14) is given explicitly by

$$\text{(15)} \quad P_{(v_i, m)} \mapsto P_{v_i} Q_m,$$
$$\text{(16)} \quad S_{(e_{i,j}, m)} \mapsto S_{e_{i,j}} Q_m,$$

for $i = 1, \ldots, n$, $j = i, \ldots, n$ and $m = 0, \ldots, p - 1$.

We have

$$\text{(17)} \quad Q_0(C^*(L_{2n-1}) \times_{\Lambda} \mathbb{Z}_p)Q_0 = C^*(L_{2n-1})^\Lambda Q_0,$$
and the map $C^*(L_{2n-1})^\Lambda \to C^*(L_{2n-1})^\Lambda Q_0$, $x \mapsto x Q_0$, is a $C^*$-algebra isomorphism. On the other hand, the isomorphism (14) (cf. Formulae (15) and (16)) identifies $Q_0 = \sum_{i=1}^n P_{v_i} Q_0$ with the projection $\sum_{i=1}^n P_{(v_i, 0)}$ in $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$. Consequently, there is a $C^*$-algebra isomorphism

$$\text{(18)} \quad C(L_q(p; m_1, \ldots, m_n)) \cong \left( \sum_{i=1}^n P_{(v_i, 0)} \right) C^*(L_{2n-1} \times_c \mathbb{Z}_p) \left( \sum_{i=1}^n P_{(v_i, 0)} \right).$$

In the following lemma we only require that $m_1$ be relatively prime to $p$ and no assumptions on the remaining parameters $m_2, \ldots, m_n$ are made whatever. The lemma says that if $m_1$ is relatively prime to $p$ then the action $\Lambda$ is saturated, as expected.

**Lemma 2.1.** If $m_1$ is relatively prime to $p$ then for each vertex $(v_k, m)$ there exists a path in $L_{2n-1} \times_c \mathbb{Z}_p$ from $(v_1, 0)$ to $(v_k, m)$. Thus, Formula (18) gives an isomorphism between the $C^*$-algebra $C(L_q(p; m_1, \ldots, m_n))$ and a full corner of $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$. 
Proof. Let \( k \in \{1, \ldots, n\}, m \in \mathbb{Z}_p \), and let \( r \) be a positive integer such that \( rm_1 = m \) in \( \mathbb{Z}_p \). Then

\[
((e_{1,1}, m_1), (e_{1,1}, 2m_1), \ldots, (e_{1,1}, (r-2)m_1), (e_{1,1}, (r-1)m_1), (e_{1,k}, rm_1))
\]

is the desired path. Consequently,

\[
P_{C} \text{ is a partial isometry in } \mathbb{Z} \text{ isomorphic to a full corner of } C \text{ for } i
\]

(\( i \in \mathbb{Z} \)) generated by \( p \)-groups (cf. Section 1.2). Thus, the \( K \)-groups of these \( C^\ast \)-algebras are isomorphic [4, 8]. In order to calculate the \( K \)-groups of \( C^\ast(L_{2n-1} \times_c \mathbb{Z}_p) \) we assume that each of \( m_1, \ldots, m_n \) is relatively prime to \( p \). For short, we write \( \Phi \) for the map \( K_{L_{2n-1} \times_c \mathbb{Z}_p} \) which determines these \( K \)-groups (cf. Section 1.2). Thus, the \( K_0 \) and \( K_1 \) groups of \( C^\ast(L_{2n-1} \times_c \mathbb{Z}_p) \) are isomorphic to the cokernel and kernel, respectively, of the endomorphism \( \Phi \) of the free abelian group with a basis \((L_{2n-1} \times_c \mathbb{Z}_p)^0\), given by

\[
(19) \quad \Phi : (v_i, m) \mapsto \left( \sum_{j=i}^n (v_j, m + m_i) \right) - (v_i, m).
\]

Proposition 2.2. If each of \( m_1, \ldots, m_n \) is relatively prime to \( p \) then

\[
K_1(C(L_q(p; m_1, \ldots, m_n))) \cong \mathbb{Z}.
\]

Proof. By Lemma 2.1 it suffices to calculate the \( K_1 \)-group of \( C^\ast(L_{2n-1} \times_c \mathbb{Z}_p) \), which is isomorphic to the kernel of the map \( \Phi \) from (19). Let \( \lambda^m_i \in \mathbb{Z} \), for \( i = 1, \ldots, n \) and \( m \in \mathbb{Z}_p \), be such that \( \Phi(\sum_{i=1}^n \sum_{m \in \mathbb{Z}_p} \lambda^m_i(v_i, m)) = 0 \). This can only happen if \( \sum_{j=1}^i \lambda^m_j = \lambda^m_i \) for each \( i \in \{1, \ldots, n\} \) and \( m \in \mathbb{Z}_p \). Setting \( i = 1 \), we get \( \lambda^m_1 = \lambda^m_1 \) for all \( m \in \mathbb{Z}_p \), because \( m_1 \) is relatively prime to \( p \). Then, considering \( i = 2 \), we get \( \lambda^0_1 + \lambda^m_2 = \lambda^m_2 \) for all \( m \in \mathbb{Z}_p \). Summing this identity over \( m \) we see that \( \lambda^0_1 = 0 \). Consequently, \( \lambda^m_2 = \lambda^0_2 \) for all \( m \in \mathbb{Z}_p \). Again, we use here the fact that \( m_2 \) is relatively prime to \( p \). Continuing inductively in this manner we get \( \lambda^m_i = 0 \) for \( i = 1, \ldots, n-1 \) and \( \lambda^m_n = \lambda^0_n \) for \( m \in \mathbb{Z}_p \). Thus, the kernel of \( \Phi \) is isomorphic to \( \mathbb{Z} \), as claimed.
It is also possible to calculate the cokernel of the map $\Phi$ and, therefore, the $K_0$ group of $C(L_q(p; m_1, \ldots, m_n))$. This is a simple matter if $n = 2$, and we get

$$K_0(C(L_q(p; m_1, m_2))) \cong \mathbb{Z} \oplus \mathbb{Z}_p,$$

similarly to the result of Matsumoto and Tomiyama [18]. However, if $n \geq 3$ then the calculation becomes a bit more elaborate. We illustrate with a particular case.

**Proposition 2.3.** If $n = 3$, $m_2 = m_3$ and both $m_1$ and $m_2$ are relatively prime to $p$ then

$$K_0(C(L_q(p; m_1, m_2, m_3))) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_{2p} \oplus \mathbb{Z}_p^2 & \text{if } p \text{ is even,} \\ \mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p^2 & \text{if } p \text{ is odd.} \end{cases}$$

**Proof.** We must determine the cokernel of $\Phi$. It is easy to see that $\{(v_i, 0) \mid i = 1, 2, 3\}$ together with the range of $\Phi$ generate the entire group $\mathbb{Z}(L_5 \times_c \mathbb{Z}_p)^0$. Now let $d_i, \lambda_i^n \in \mathbb{Z}$, for $i = 1, 2, 3$ and $m \in \mathbb{Z}_p$, be such that $\Phi(\sum_{i=1}^3 \sum_{m \in \mathbb{Z}_p} \lambda_i^m (v_i, m)) = \sum_{i=1}^3 d_i(v_i, 0)$. This is equivalent to

$$\sum_{j=1}^i \lambda_j^{-m_j} - \lambda_1^0 = d_i \quad \text{(20)}$$

$$0 = \sum_{j=1}^i \lambda_j^m - \lambda_1^0, \quad \text{for } m \neq 0. \quad \text{(21)}$$

If $i = 1$ then (21) gives $\lambda_1^m = \lambda_1^0$ for all $m \in \mathbb{Z}_p$ and then $d_1 = 0$ by (20). If $i = 2$ then substituting $m = km_2$ in (21), with $k = 1, \ldots, p - 1$, we get $\lambda_2^{km_2} = k\lambda_1^0 + \lambda_2^0$ for all $k = 0, \ldots, p - 1$. Then (20) yields $d_2 = p\lambda_1^0$. If $i = 3$ then substituting $m = km_2$ in (21), with $k = 1, \ldots, p - 1$, we get $\lambda_3^{km_2} = \frac{k(k+1)}{2} \lambda_1^0 + k\lambda_2^0 + \lambda_3^0$ for all $k = 0, \ldots, p - 1$. Then (20) yields $d_3 = \frac{p(p+1)}{2} \lambda_1^0 + p\lambda_2^0$. Thus, $(v_1, 0)$ has infinite order in the cokernel. If $p$ is even then $(v_2, 0)$ and $(v_3, 0)$ generate a subgroup of the cokernel isomorphic to $\mathbb{Z}_{2p} \oplus \mathbb{Z}_p^2$. If $p$ is odd then $(v_2, 0)$ and $(v_3, 0)$ generate a subgroup of the cokernel isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$.

We now show that $C(L_q(p; m_1, \ldots, m_n))$ itself is isomorphic to a graph algebra. The following construction of the graph $L_{2n-1}^{(p;m_1,\ldots,m_n)}$ and the argument of Theorem 2.5, below, are similar to [25, Section 4 and Lemma 6]. Again, we assume that each of $m_1, \ldots, m_n$ is relatively prime to $p$.

At first we define the graph $L_{2n-1}^{(p;m_1,\ldots,m_n)}$, as follows: The vertices of the graph $L_{2n-1}^{(p;m_1,\ldots,m_n)}$ are $\{v_1, v_2, \ldots, v_n\}$. The edges of $L_{2n-1}^{(p;m_1,\ldots,m_n)}$ consist of all finite (vertex simple) paths $a = (e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r)$ in $L_{2n-1} \times_c \mathbb{Z}_p$ such that $k_1 = m_{i_1}$, $k_a \neq 0$ for $a \neq r$, $k_r = 0$ and $(v_{j_a}, k_a) \neq (v_{j_b}, k_b)$ for
$$L^{(3;1,2)}_3$$ looks as follows:

![Diagram](image)

The following Lemma 2.4 essentially follows from [25, Lemma 5]. However, for the sake of completeness and reader’s convenience, we give a self-contained proof.

**Lemma 2.4.** If each of $m_1, \ldots, m_n$ is relatively prime to $p$ then for any $l \in \{1, \ldots, n\}$ and any $m \in \mathbb{Z}_p$ we have

$$P_{(v_l, m)} = \sum_{\alpha} S_{(e_{i_1-j_1,k_1})} \cdots S_{(e_{i_r-j_r,k_r})} S_{(e_{i_r-j_r,k_r})} \cdots S_{(e_{i_1-j_1,k_1})}$$

(in $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$), where the summation extends over all $\alpha = ((e_{i_1-j_1}, k_1), \ldots, (e_{i_r-j_r}, k_r))$, vertex simple paths in $L_{2n-1} \times_c \mathbb{Z}_p$ such that $i_1 = l$, $k_1 - m_{i_1} = m$, $k_a \neq 0$ for $a \neq r$, $k_r = 0$ and $(v_{j_a}, k_a) \neq (v_{j_b}, k_b)$ for $a \neq b$.

**Proof.** For $\nu = 1, 2, \ldots$ we define $A_\nu$ to be the collection of all vertex simple paths $\alpha = ((e_{i_1-j_1}, k_1), \ldots, (e_{i_r-j_r}, k_r))$ in $L_{2n-1} \times_c \mathbb{Z}_p$ such that the length of $\alpha$ is not greater than $\nu$ (and nonzero), $i_1 = l$, $k_1 - m_{i_1} = m$, $k_a \neq 0$ for $a \neq r$, $k_r = 0$ and $(v_{j_a}, k_a) \neq (v_{j_b}, k_b)$ for $a \neq b$, and let $B_\nu$ be the collection of all paths $\beta = ((e_{i_1-j_1}, k_1), \ldots, (e_{i_r-j_r}, k_r))$ such that the length of $\beta$ equals $\nu$, $i_1 = l$, $k_1 - m_{i_1} = m$, $k_a \neq 0$ for $a = 1, \ldots, r$ and $(v_{j_a}, k_a) \neq (v_{j_b}, k_b)$ for $a \neq b$. We show, by induction on $\nu$, that

$$P_{(v_l, m)} = \sum_{\alpha \in A_\nu} S_{(e_{i_1-j_1,k_1})} \cdots S_{(e_{i_r-j_r,k_r})} \cdots S_{(e_{i_1-j_1,k_1})} + \sum_{\beta \in B_\nu} S_{\beta} S_{(e_{i_1-j_1,k_1})} \cdots S_{(e_{i_r-j_r,k_r})} \cdots S_{(e_{i_1-j_1,k_1})}.$$

Indeed, the collection of all edges in $L_{2n-1} \times_c \mathbb{Z}_p$ with source equal to $(v_l, m)$ is the union of $A_1$ and $B_1$. Thus, (22) holds with $\nu = 1$ by virtue of (G5). Now suppose (22) holds for some $\nu$. If $\beta = ((e_{i_1-j_1}, k_1), \ldots, (e_{i_r-j_r}, k_r))$ in $B_\nu$, then applying Condition (G5) at the range vertex of $\beta$, equal to $(v_{j_r}, k_r)$, we get

$$S_{\beta} S_{(e_{i_1-j_1,k_1})} \cdots S_{(e_{i_r-j_r,k_r})} \cdots S_{(e_{i_1-j_1,k_1})} = \sum_{d=j_r}^n S_{\beta} S_{(e_{i_1-j_1,k_1})} \cdots S_{(e_{i_r-j_r,k_r+m_{j_r}})} \cdots S_{(e_{i_1-j_1,k_1})} \cdots S_{(e_{i_r-j_r,k_r+m_{j_r}})} S_{\beta}.$$
Let $\beta' = ((e_{i_1, j_1}, k_1), \ldots, (e_{i_r, j_r}, k_r), (e_{j_r, d}, k_r + m_{j_r}))$. We claim that $(v_a, k_r + m_{j_r}) \not= (v_{j_a}, k_a)$ for $a = 1, \ldots, r$. This is obvious if $d \not= j_r$. For $d = j_r$ let $b$ be the smallest index such that $j_b = j_r$. Since $\beta$ is a path we have $j_b = j_{b+1} = \cdots = j_r$ and $k_{b+h} = k_b + hm_{j_b}$ for $h = 1, \ldots, r - b$. Since $m_{j_b}$ is relatively prime to $p$ it follows that $k_r + m_{j_r} \not\in \{k_b, \ldots, k_r\}$, as claimed.

Thus $\beta' \in (A_{\nu+1} \setminus A_{\nu}) \cup B_{\nu+1}$. Consequently, from the inductive hypothesis, Formula (23) and the above discussion we get

$$P_{(v_l, m)} = \sum_{\alpha \in A_{\nu}} S_{\alpha} S_{\alpha}^* + \sum_{\beta \in B_{\nu}} S_{\beta} S_{\beta}^*$$

$$= \sum_{\alpha \in A_{\nu}} S_{\alpha} S_{\alpha}^* + \sum_{\beta' \in (A_{\nu+1} \setminus A_{\nu})} S_{\beta'} S_{\beta'}^* + \sum_{\beta \in B_{\nu+1}} S_{\beta} S_{\beta}^*$$

and the inductive step follows.

Since $L_{2n-1} \times_c \mathbb{Z}_p$ is a finite graph there exists a $\nu$ large enough so that $B_{\nu} = \emptyset$. With this $\nu$ Formula (22) gives the lemma.

**Theorem 2.5.** If each of the numbers $m_1, \ldots, m_n$ is relatively prime to $p$ then the $C^*$-algebra $C(L_*(p; m_1, \ldots, m_n))$ is isomorphic to $C^*(L_{2n-1}^{(p,m_1,\ldots,m_n)})$.

**Proof.** At first we observe that there exists a $C^*$-algebra homomorphism

$$\psi : C^*\left(L_{2n-1}^{(p,m_1,\ldots,m_n)}\right) \rightarrow \left(\sum_{i=1}^n P_{(v_i,0)}\right) C^*(L_{2n-1} \times_c \mathbb{Z}_p) \left(\sum_{i=1}^n P_{(v_i,0)}\right)$$

such that

$$\psi : P_{v_l} \mapsto P_{(v_l,0)}$$
$$\psi : S_{\alpha} \mapsto S_{(e_{i_1, j_1}, k_1)} S_{(e_{i_2, j_2}, k_2)} \cdots S_{(e_{i_r, j_r}, k_r)}$$

for all $l = 1, \ldots, n$ and for all $\alpha = ((e_{i_1, j_1}, k_1), \ldots, (e_{i_r, j_r}, k_r))$, vertex simple paths in $L_{2n-1} \times_c \mathbb{Z}_p$ such that $k_1 = m_{i_1}$, $k_a \not= 0$ for $a \not= r$, $k_r = 0$ and $(v_{j_a}, k_a) \not= (v_{j_b}, k_b)$ for $a \not= b$. Due to the universal property of $C^*\left(L_{2n-1}^{(p,m_1,\ldots,m_n)}\right)$, to this end it suffices to verify that the elements $\{\psi(P_{v_i}), \psi(S_{\alpha})\}$ of $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$ satisfy Conditions (G1)--(G5) for the graph $L_{2n-1}^{(p,m_1,\ldots,m_n)}$. But it is obvious that Conditions (G1)--(G4) are satisfied, and Condition (G5) is equivalent to Lemma 2.4 with $m = 0$.

For surjectivity of $\psi$ it suffices to show that:

(i) If $\alpha$ is a path in $L_{2n-1} \times_c \mathbb{Z}_p$ such that both $s(\alpha)$ and $r(\alpha)$ are in $\{(v_i,0) \mid i = 1, \ldots, n\}$ then $S_{\alpha}$ belongs to the range of $\psi$.

(ii) If $\alpha, \beta$ are two paths such that $r(\alpha) = r(\beta)$ and both $s(\alpha)$ and $s(\beta)$ are in $\{(v_i,0) \mid i = 1, \ldots, n\}$ then $S_{\alpha} S_{\beta}^*$ belongs to the range of $\psi$. 
To this end we first note that any loop in $L_{2n-1} \times \mathbb{Z}_p$ must pass through a vertex of the form $(v_i, 0)$. Now let $\alpha = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r))$ be a path as in (i). Let $a_1 < a_2 < \cdots < a_\nu = r$ be all the indices for which $k_{a_t} = 0$. We also set $a_0 = 0$. For each $t = 1, \ldots, \nu$ the path $\alpha_t = ((e_{i_1+a_t-1,j_1+a_t-1}, k_1+a_t-1), \ldots, (e_{i_r,j_r}, k_r))$ is an edge of the graph $L_{2n-1}^{(p;m_1,\ldots,m_n)}$. Hence, $S_\alpha = S_{\alpha_1} \cdots S_{\alpha_\nu}$ belongs to the range of $\psi$, since each $S_{\alpha_t}$ does. Now let $\alpha$ and $\beta$ be two paths as in (ii). Let $\alpha = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r))$. By virtue of Part (i) it suffices to consider the case $k_r \ne 0$. Let $\mu$ be the greatest index such that $k_\mu = 0$, or 0 if such an index does not exist. We set $\alpha_1 = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r))$ and $\alpha_2 = ((e_{i_\mu+1,j_{\mu+1}}, k_{\mu+1}), \ldots, (e_{i_r,j_r}, k_r))$. We have $S_\alpha = S_{\alpha_1} S_{\alpha_2}$ and $S_{\alpha_1}$ is in the range of $\psi$ by Part (i) (if $\mu = 0$ then $\alpha_1 = \emptyset$ and $S_{\alpha_1} = I$). Furthermore, for $\mu + 1 \le a, b \le r$ we have $(v_{j_a}, k_a) \ne (v_{j_b}, k_b)$ if $a \ne b$. We have an analogous factorization $S_\beta = S_{\beta_1} S_{\beta_2}$, with $S_{\beta_1}$ in the range of $\psi$. Let $P_{(v_{j_r}, k_r)} = \sum_x S_x S_x^*$ be the decomposition as in Lemma 2.4. Then we have

$$S_\alpha S_\beta = S_{\alpha_1} S_{\alpha_2} P_{(v_{j_r}, k_r)} S_{\beta_2}^* S_{\beta_1} = \sum_x S_{\alpha_1} S_{\alpha_2} S_x S_x^* S_{\beta_2}^* S_{\beta_1}.$$ 

Consequently, $S_\alpha S_\beta$ belongs to the range of $\psi$, since $S_{\alpha_1}$, $S_{\beta_1}$ and all $S_{\alpha_2} S_x$ and $S_{\beta_2} S_x$ do. This completes the proof of surjectivity of $\psi$.

Now we show that the homomorphism $\psi$ is injective. Our argument is essentially the same as in [5, Remark 3]. Since for each $i \in \{1, \ldots, n-1\}$ there exists a path from $v_i$ to $v_n$, the ideal $J$ of $C^* \left( L_{2n-1}^{(p;m_1,\ldots,m_n)} \right)$ generated by $P_{v_n}$ is essential by Lemma 1.1. Thus, it suffices to show that $J \cap \ker(\psi) = \{0\}$. To this end, we notice that in the graph $L_{2n-1}^{(p;m_1,\ldots,m_n)}$ the vertex $v_n$ emits a unique edge, which we call $e$, and the range of this edge is $v_n$. Since there are infinitely many paths from other vertices to $v_n$ it follows (cf. [15] and [5, Remark 3]) that

$$J \cong P_{v_n} J P_{v_n} \otimes K = C^* (S_e) \otimes K \cong C(T) \otimes K.$$

Hence, in order to prove injectivity of $\psi$ it suffices to show that $C^* (S_e) \cap \ker(\psi) = \{0\}$. This follows from the fact that

$$\psi(S_e) = S_{(e_{n,n,m_n})} S_{(e_{n,n,2m_n})} \cdots S_{(e_{n,n,pm_n})}$$

is a partial unitary with full spectrum (cf. [15]).

With help of Theorem 2.5 it is easy to determine the ideal structure of $C(L_q(p; m_1, \ldots, m_n))$. For example, we have seen in the proof of Theorem 2.5 that the ideal of $C^* \left( L_{2n-1}^{(p;m_1,\ldots,m_n)} \right)$ generated by $P_{v_n}$ is isomorphic to $C(T) \otimes K$. The corresponding quotient is $C^* \left( L_{2n-3}^{(p;m_1,\ldots,m_{n-1})} \right)$, and this $C^*$-algebra is in turn isomorphic to $C(L_q(p; m_1, \ldots, m_{n-1}))$. Thus, there
exists an exact sequence

\[ (24) \quad 0 \to C(\mathbb{T}) \otimes K \to C(L_q(p; m_1, \ldots, m_n)) \to C(L_q(p; m_1, \ldots, m_{n-1})) \to 0. \]

Using the exact sequence (24) or the general results about graph algebras, outlined in Section 1.3, it is easy to understand the primitive spectrum of \( C(L_q(p; m_1, \ldots, m_n)) \). Therefore, we omit the proof of the following proposition:

**Proposition 2.6.** If each of \( m_1, \ldots, m_n \) is relatively prime to \( p \) then the primitive ideal space of \( C(L_q(p; m_1, \ldots, m_n)) \) consists of \( n \) disjoint copies \( C_1, \ldots, C_n \) of the circle. The hull-kernel topology restricted to each of the circles coincides with the natural one. The closure of a point in \( C_k \) contains \( C_1 \cup \cdots \cup C_{k-1} \). Thus, \( \text{Prim}(C(L_q(p; m_1, \ldots, m_n))) \) and \( \text{Prim}(C(S^{2n-1}_q)) \) are homeomorphic (cf. [11, Section 4.1]).

**Concluding remarks.** For a fixed integer \( p \geq 2 \) the infinite lens space \( L(p; \infty) \) is defined as the inductive limit of the lens spaces \( L(p; 1_n) \), corresponding to the natural imbeddings \( L(p; 1_n) \hookrightarrow L(p; 1_{n+1}) \). (If \( m_1 = \cdots = m_n = 1 \) then we simply write \( L(p; 1_n) \) instead of \( L(p; 1, \ldots, 1) \).) It turns out that \( L(p; \infty) \) is identical with the Eilenberg-MacLane space of type \( (\mathbb{Z}_p, 1) \) [3].

The results of the previous section lead to quantum versions of this classical topological setting. Namely, the inclusion \( L(p; 1_n) \hookrightarrow L(p; 1_{n+1}) \) corresponds to the surjective homomorphism \( \tilde{\theta}_{n+1} : C(L_q(p; 1_{n+1})) \to C(L_q(p; 1_n)) \) such that the kernel of \( \tilde{\theta}_{n+1} \) is generated by \( z_{n+1}z_{n+1}^* \). Consequently, the quantum infinite lens space, or the quantum Eilenberg-MacLane space of type \( (\mathbb{Z}_p, 1) \), may be defined as the inverse limit

\[ (25) \quad C(L_q(p; \infty)) \cong \lim_{\leftarrow} C(L_q(p; 1_n)), \quad \tilde{\theta}_n. \]

Under the isomorphisms \( C(L_q(p; 1_k)) \cong C^*(L_{2k-1}^{(p)}) \) (if \( m_1 = \cdots = m_n = 1 \) then we simply write \( L_{2n-1}^{(p)} \) instead of \( L_{n;1, \ldots, 1}^{(p)} \), described in Theorem 2.5, the homomorphism \( \tilde{\theta}_{n+1} \) is carried onto a surjective \( C^* \)-algebra homomorphism \( \theta_{n+1} : C^*(L_{2n+1}^{(p)}) \to C^*(L_{2n-1}^{(p)}) \), whose kernel is generated by the projection \( P_{2n+1} \). Thus, we have the \( C^* \)-algebra isomorphism

\[ (26) \quad C(L_q(p; \infty)) \cong \lim_{\leftarrow} C^*(L_{2n-1}^{(p)}, \theta_n). \]

It is not difficult to see, and we omit the details, that this inverse limit itself may be realized as the graph algebra \( C^*(L_{\infty}^{(p)}) \). The graph \( L_{\infty}^{(p)} \) is the increasing limit of the graphs \( L_{2n-1}^{(p)} \), corresponding to the natural imbeddings \( L_{2n-1}^{(p)} \hookrightarrow L_{2n+1}^{(p)} \) such that the \( v_i \) vertex in \( L_{2n-1}^{(p)} \) is identified with
the \(v_i\) vertex in \(L_{2n+1}^{(p)}\), and the edges from \(v_i\) to \(v_j\) in \(L_{2n-1}^{(p)}\) are bijectively identified with the edges from \(v_i\) to \(v_j\) in \(L_{2n+1}^{(p)}\). The graph \(L_{\infty}^{(p)}\) has infinitely many vertices \(\{v_1, v_2\ldots\}\), and for each pair \(i \leq j\) there exists at least one edge from \(v_i\) to \(v_j\). These two properties imply that \(C^*\left(L_{\infty}^{(p)}\right)\) is a primitive, purely infinite \(C^*\)-algebra (but not simple) [1]. Furthermore, \(K_0\left(C^*\left(L_{\infty}^{(p)}\right)\right) \cong \bigoplus \mathbb{Z}\) and \(K_1\left(C^*\left(L_{\infty}^{(p)}\right)\right) = 0\) [22, 7].

References


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A FREE ENTROPY DIMENSION LEMMA

KENLEY JUNG

For Arlan Ramsay

Suppose $M$ is a von Neumann algebra with normal, tracial state $\varphi$ and $\{a_1, \ldots, a_n\}$ is a set of self-adjoint elements in $M$. We provide an alternative uniform packing description of $\delta_0(a_1, \ldots, a_n)$, the modified free entropy dimension of $\{a_1, \ldots, a_n\}$.

In the attempt to understand the free group factors Voiculescu created a type of noncommutative probability theory. One facet of the theory involves free entropy and free entropy dimension, applications of which have answered some old operator algebra questions ([1] and [4]). Roughly speaking, given self-adjoint elements $a_1, \ldots, a_n$ in a von Neumann algebra $M$ with normal, tracial state $\varphi$ a matricial microstate for $\{a_1, \ldots, a_n\}$ is an $n$-tuple of self-adjoint $k \times k$ matrices which together with the normalized trace, approximate the algebraic behavior of the $a_i$ under $\varphi$. Taking a normalization of the logarithmic volume of such microstate sets followed by multiple limiting processes yields a number, $\chi(a_1, \ldots, a_n)$, called the free entropy of $\{a_1, \ldots, a_n\}$. One can think of free entropy as the logarithmic volume of the $n$-tuple. The (modified) free entropy dimension of $\{a_1, \ldots, a_n\}$ is

$$\delta_0(a_1, \ldots, a_n) = n + \limsup_{\epsilon \to 0} \frac{\chi(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n ; s_1, \ldots, s_n)}{|\log \epsilon|}$$

where $\{s_1, \ldots, s_n\}$ is a semicircular family freely independent with respect to $\{a_1, \ldots, a_n\}$ and $\chi(\cdot)$ is a technical modification of $\chi$ (see [4]).

Free entropy dimension was inspired by Minkowski dimension. Recall that for a subset $A \subset \mathbb{R}^d$ the (upper) Minkowski dimension of $A$ is

$$d + \limsup_{\epsilon \to 0} \frac{\log \lambda(\mathcal{N}_\epsilon(A))}{|\log \epsilon|}$$

where $\lambda$ above denotes Lebesgue measure and $\mathcal{N}_\epsilon(A)$ is the $\epsilon$-neighborhood of $A$. Minkowski dimension has an equivalent formulation in terms of uniform packing dimension. The (upper) uniform packing dimension of $A$ is

$$\limsup_{\epsilon \to 0} \frac{\log P_\epsilon(A)}{|\log \epsilon|} = \limsup_{\epsilon \to 0} \frac{\log K_\epsilon(A)}{|\log \epsilon|}$$
where \( A \) is endowed with the Euclidean metric, \( P_{\epsilon}(A) \) is the maximum number of elements in a collection of mutually disjoint open \( \epsilon \) balls of \( A \), and \( K_{\epsilon}(A) \) is the minimum number of open \( \epsilon \) balls required to cover \( A \) (the quantities above make sense in the setting of an arbitrary metric space). It is easy to see that the Minkowski dimension and the uniform packing dimension of \( A \) are always equal.

In this paper we present a lemma which formulates a similar metric description of \( \delta_0 \): Free entropy dimension can be described in terms of packing numbers with balls of equal radius.

The alternative description comes as no surprise in view of both the definition of \( \delta_0 \) and the techniques in estimations thereof. The proof follows the classical one with the addition of the properties of \( \chi \) proven in [3] and the strengthened asymptotic freeness results of [5].

1. Preliminaries.

Throughout \( M \) is a von Neumann algebra with normal, tracial state \( \varphi \) and \{\( a_1, \ldots, a_n \)\} is a set of self-adjoint elements in \( M \). We use the symbols \( \chi \) and \( \delta_0 \) to designate the same quantities introduced in [4]. \( M_{ka}^n(\mathbb{C}) \) denotes the set of \( k \times k \) self-adjoint complex matrices and \((M_{ka}^n(\mathbb{C}))^n\) is the set of \( n \)-tuples with entries in \( M_{ka}^n(\mathbb{C}) \). \( \text{tr}_k \) is the normalized trace on the \( k \times k \) complex matrices. \( \| \cdot \|_2 \) is the inner product norm on \((M_{ka}^n(\mathbb{C}))^n\) given by the formula \( \| (x_1, \ldots, x_n) \|_2^2 = \sum_{i=1}^{n} k \cdot \text{tr}_k(x_i^2) \) and \( \text{vol} \) denotes Lebesgue measure with respect to the \( \| \cdot \|_2 \) norm. For any \( k \in \mathbb{N} \) denote by \( R_k \) the metric on \((M_{ka}^n(\mathbb{C}))^n\) induced by the norm \( k^{-\frac{1}{2}} \cdot \| \cdot \|_2 \). For a metric space \((X, d)\) and \( \epsilon > 0 \) write \( P_\epsilon(X, d) \) for the maximum number of elements in a collection of mutually disjoint open \( \epsilon \) balls of \( X \) and \( K_\epsilon(X, d) \) for the minimum number of open \( \epsilon \) balls required to cover \( X \). Observe that \( P_\epsilon(X, d) \geq K_{2\epsilon}(X, d) \geq P_{4\epsilon}(X, d) \). Finally for \( S \subset X \) denote by \( \mathcal{N}_\epsilon(S) \) the \( \epsilon \)-neighborhood of \( S \).

2. The lemma.

**Definition 2.1.** For any \( k, m \in \mathbb{N} \), and \( R, \gamma, \epsilon > 0 \) define successively

\[
\mathbb{P}_{\epsilon,R}(a_1, \ldots, a_n; m, k, \gamma) = P_{\epsilon}(\Gamma_R(a_1, \ldots, a_n; m, k, \gamma), \rho_k),
\]

\[
\mathbb{P}_{\epsilon,R}(a_1, \ldots, a_n; m, \gamma) = \limsup_{k \to \infty} k^{-2} \cdot \log(\mathbb{P}_{\epsilon,R}(a_1, \ldots, a_n; m, k, \gamma)),
\]

\[
\mathbb{P}_{\epsilon,R}(a_1, \ldots, a_n) = \inf\{\mathbb{P}_{\epsilon,R}(a_1, \ldots, a_n; m, \gamma) : m \in \mathbb{N}, \gamma > 0\},
\]

\[
\mathbb{P}_{\epsilon}(a_1, \ldots, a_n) = \sup_{R > 0}\{\mathbb{P}_{\epsilon,R}(a_1, \ldots, a_n)\}.
\]

**Remark.** If \( b_1, \ldots, b_p \in M \), then define \( \mathbb{P}_{\epsilon}(a_1, \ldots, a_n : b_1, \ldots, b_p) \) to be the quantity obtained by replacing \( \Gamma_R(a_1, \ldots, a_n; m, k, \gamma) \) in the definition with \( \Gamma_R(a_1, \ldots, a_n : b_1, \ldots, b_p; m, k, \gamma) \). Similarly, we define \( \mathbb{K}_{\epsilon}(a_1, \ldots, a_n) \) and
all its associated quantities by replacing \( P_t \) in the first line of Definition 2.1 with \( K_e \). Define \( K_e(a_1, \ldots, a_n : b_1, \ldots, b_p) \) in the same way \( P_e(a_1, \ldots, a_n : b_1, \ldots, b_p) \) was defined.

For any self-adjoint elements \( h_1, \ldots, h_n \in M \) denote by \( \chi(h_1, \ldots, h_n) \) the number obtained by replacing the \( \lim sup \) in the definition of \( \chi \) with \( \lim inf \). \( P_t(\cdot) \) being a normalized limiting process of the logarithmic microstate space packing numbers we observe just as in the classical case that:

**Lemma 2.2.** If \( \{h_1, \ldots, h_n \} \) is a set of self-adjoint elements in \( M \) which is freely independent with respect to \( \{a_1, \ldots, a_n \} \) and \( \chi(h_1, \ldots, h_n) > -\infty \), then

\[
\frac{n + \limsup_{\epsilon \to 0} \frac{\chi(a_1 + \epsilon h_1, \ldots, a_n + \epsilon h_n : h_1, \ldots, h_n)}{|\log \epsilon|}}{\limsup_{\epsilon \to 0} \frac{\chi_e(a_1, \ldots, a_n)}{|\log \epsilon|}} = \frac{\limsup_{\epsilon \to 0} \frac{\chi_e(a_1, \ldots, a_n)}{|\log \epsilon|}}{\limsup_{\epsilon \to 0} \frac{P_e(a_1, \ldots, a_n)}{|\log \epsilon|}}.
\]

**Proof.** Clearly it suffices to show equality of the first and last expressions above since \( P_t(\cdot) \geq K_{2t}(\cdot) \geq P_{4t}(\cdot) \). Furthermore, we can assume that \( \{a_1, \ldots, a_n \} \) has finite dimensional approximants since the equalities hold trivially otherwise. Set \( C = \max\{|h_i|\}_{1 \leq i \leq n} + 1 \). First we show that the free entropy expression is greater than or equal to the \( P_e \) expression. Suppose \( 0 < \epsilon < (C\sqrt{n})^{-1}, m \in \mathbb{N}, \) with \( m > n, 1 > \gamma > 0, \) and \( R > \max\{|a_i|\}_{1 \leq i \leq n} \).

Corollary 2.14 of [5] provides an \( N \in \mathbb{N} \) such that if \( k \geq N \) and \( \sigma \) is a Radon probability measure on \( ((M_k^{sa}(\mathbb{C}))^{R+1})^{2n} \) (the subset of \( M_k^{sa}(\mathbb{C})^{2n} \) consisting of \( 2n \)-tuples whose entries have operator norm no greater than \( R + 1 \)) invariant under the \( U_k \)-action

\[
(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) \mapsto (\xi_1, \ldots, \xi_n, u\eta_1u^*, \ldots, u\eta_nu^*),
\]

then \( \sigma(\omega_k) > \frac{1}{2} \) where \( \omega_k \) is

\[
\{((\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) \in ((M_k^{sa}(\mathbb{C}))^{R+1})^{2n} : \{\xi_1, \ldots, \xi_n\} \quad \text{and} \quad \{\eta_1, \ldots, \eta_n\} \quad \text{are} \quad (m, \gamma/4^m)-\text{free}\}.
\]

With respect to the \( \rho_k \) metric for each \( k \) find a collection of mutually disjoint open \( C\sqrt{n} \) balls of \( \Gamma_R(a_1, \ldots, a_n; m, k, \gamma/(8(R + 2))^m) \) of maximum cardinality and denote the centers of these balls by \( \left\langle \left( x_{ij}^{(k)}, \ldots, x_{nj}^{(k)} \right) \right\rangle_{j \in S_k} \). Let \( \mu_k \) be the uniform atomic probability measure supported on the centers of these balls and let \( \nu_k \) be the probability measure obtained by restricting \( \text{vol} \) to \( \Gamma_{C\sqrt{n}}(\epsilon h_1, \ldots, \epsilon h_n; m, k, \gamma/8^m) \) and normalizing appropriately. Then \( \mu_k \times \nu_k \)
is a Radon probability measure on $((M^m_k(C))_{R+1})^{2n}$ invariant under the $U_k$-action described above. So for $k \geq N (\mu_k \times \nu_k)(\omega_k) > \frac{1}{2}$.

For $k \in \mathbb{N}$ and $j \in S_k$ define $F_{jk}$ to be the set of all $(y_1, \ldots, y_n) \in \Gamma_{C}(\epsilon h_1, \ldots, \epsilon h_n; m, k, \gamma/8^m)$ such that $(y_1, \ldots, y_n)$ and $(x_{1j}^{(k)}, \ldots, x_{nj}^{(k)})$ are $(m, \frac{\gamma}{8^m})$-free.

$$\frac{1}{2} < (\mu_k \times \nu_k)(\omega_k) = \sum_{j \in S_k} \frac{1}{|S_k|} \cdot \nu_k(F_{jk})$$

$$= \sum_{j \in S_k} \frac{1}{|S_k|} \cdot \frac{\text{vol}(F_{jk})}{\text{vol}(\Gamma_{C}(\epsilon h_1, \ldots, \epsilon h_n; m, k, \gamma/8^m))}.$$ 

It follows that for $k \geq N$

$$\frac{1}{2} \cdot |S_k| \cdot \text{vol}(\Gamma_{C}(\epsilon h_1, \ldots, \epsilon h_n; m, k, \gamma/8^m)) < \sum_{j \in S_k} \text{vol}(F_{jk}).$$

Set $E_{jk} = (x_{1j}^{(k)}, \ldots, x_{nj}^{(k)}) + F_{jk}$. $F_{jk}$ is a set contained in the open ball of $\rho_k$ radius $C\epsilon \sqrt{n}$ centered at $(0, \ldots, 0)$. Thus $\langle E_{jk} \rangle_{j \in S_k}$ is a collection of mutually disjoint sets. So

$$\bigcup_{j \in S_k} E_{jk} \subset \Gamma_{R+1}(a_1 + \epsilon h_1, \ldots, a_n + \epsilon h_n; \epsilon h_1, \ldots, \epsilon h_n; m, k, \gamma).$$

Thus, for any $(C\sqrt{n})^{-1} > \epsilon > 0$, $m \in \mathbb{N}$ sufficiently large, and $1 > \gamma > 0$

$$\chi_{R+1}(a_1 + \epsilon h_1, \ldots, a_n + \epsilon h_n; \epsilon h_1, \ldots, \epsilon h_n; m, \gamma)$$

$$\geq \limsup_{k \to \infty} \left( k^{-2} \cdot \log \left( \text{vol} \left( \bigcup_{j \in S_k} E_{jk} \right) \right) + \frac{n}{2} \cdot \log k \right)$$

$$\geq \limsup_{k \to \infty} \left[ k^{-2} \cdot \log \left( \frac{1}{2} \cdot |S_k| \cdot \text{vol}(\Gamma_{C}(\epsilon h_1, \ldots, \epsilon h_n; m, k, \gamma/8^m)) \right) + \frac{n}{2} \cdot \log k \right]$$

$$\geq \limsup_{k \to \infty} \left[ k^{-2} \cdot \log(|S_k|) \right] + \liminf_{k \to \infty} \left[ k^{-2} \cdot \log(\text{vol}(\Gamma_{C}(\epsilon h_1, \ldots, \epsilon h_n; m, k, \gamma/8^m))) + \frac{n}{2} \cdot \log k \right]$$

$$\geq \mathbb{P}_{\Gamma_{C}(\sqrt{n}, R+1)}(a_1, \ldots, a_n; m, \gamma/(8(R + 2))^m) + \chi_{C}(\epsilon h_1, \ldots, \epsilon h_n)$$

$$\geq \mathbb{P}_{\Gamma_{C}(\sqrt{n}, R+1)}(a_1, \ldots, a_n) + n \log \epsilon + \chi(h_1, \ldots, h_n).$$
By the chain of inequalities of the preceding paragraph it follows that
\[
\chi(a_1 + \varepsilon h_1, \ldots, a_n + \varepsilon h_n : h_1, \ldots, h_n) \\
= \chi(a_1 + \varepsilon h_1, \ldots, a_n + \varepsilon h_n : \varepsilon h_1, \ldots, \varepsilon h_n) \\
\geq \mathbb{P}_{C_{\varepsilon \sqrt{n}, R+1}}(a_1, \ldots, a_n) + n \log \varepsilon + \chi(h_1, \ldots, h_n).
\]
This being true for \( R \) sufficiently large
\[
\chi(a_1 + \varepsilon h_1, \ldots, a_n + \varepsilon h_n : h_1, \ldots, h_n) \\
\geq \mathbb{P}_{C_{\varepsilon \sqrt{n}}}(a_1, \ldots, a_n) + n \log \varepsilon + \chi(h_1, \ldots, h_n).
\]
Dividing by \(|\log \varepsilon|\) on both sides, taking a lim sup as \( \varepsilon \to 0 \), and adding \( n \) to both ends of the inequality above yields
\[
n + \limsup_{\varepsilon \to 0} \frac{\chi(a_1 + \varepsilon h_1, \ldots, a_n + \varepsilon h_n : h_1, \ldots, h_n)}{|\log \varepsilon|} \\
\geq \limsup_{\varepsilon \to 0} \frac{\mathbb{P}_{C_{\varepsilon \sqrt{n}}}(a_1, \ldots, a_n)}{|\log \varepsilon|} \\
= \limsup_{\varepsilon \to 0} \frac{\mathbb{P}_\varepsilon(a_1, \ldots, a_n)}{|\log \varepsilon|}.
\]
For the reverse inequality suppose \( 2 \leq m \in \mathbb{N} \) and \( \frac{1}{2(C+1)} > \varepsilon > \sqrt{\gamma} > 0, R > \max_{1 \leq j \leq n} \{\|a_j\|\} \). For each \( k \in \mathbb{N} \) find an packing by open \( \rho_k \varepsilon \)-balls of \( \Gamma_{R+1}(a_1, \ldots, a_n; m, k, \gamma) \) with maximum cardinality. Denote the set of centers of these balls by \( \Omega_k \). Clearly
\[
\Gamma_{R+1, \frac{1}{2}}(a_1 + \varepsilon h_1, \ldots, a_n + \varepsilon h_n : \varepsilon h_1, \ldots, \varepsilon h_n; m, k, \gamma) \\
\subset \mathcal{N}_{2C_{\varepsilon \sqrt{n}}}(\Gamma_{R+1}(a_1, \ldots, a_n; m, k, \gamma)) \\
\subset \mathcal{N}_{4C_{\varepsilon \sqrt{n}}}(\Omega_k)
\]
where \( \Gamma_{r+1, \frac{1}{2}}(\cdot) \) denotes the microstate space of \( 2n \)-tuples such that the first \( n \) entries have operator norms no larger than \( \frac{1}{2} \) and the last \( n \) entries have operator norms no larger than \( \frac{1}{2} \) (see [4] for this technical modification). \( \mathcal{N}_\varepsilon \) is taken with respect to the metric space \( (M_{sa}^n(\mathbb{C}))^n \) with the \( \rho_k \) metric. It follows that \( \chi_{R+1, \frac{1}{2}}(a_1 + \varepsilon h_1, \ldots, a_n + \varepsilon h_n : \varepsilon h_1, \ldots, \varepsilon h_n; m, \frac{\gamma}{2^m}) \) is dominated by
\[
\limsup_{k \to \infty} \left[ k^{-2} \cdot \log(\mathcal{N}_{4C_{\varepsilon \sqrt{n}}}(\Omega_k)) + \frac{n}{2} \cdot \log k \right] \\
\leq \limsup_{k \to \infty} \left[ k^{-2} \cdot \log \left( \frac{\pi \frac{n k^2}{2} \cdot (4C_{\varepsilon \sqrt{n}} k)^n}{\Gamma \left( \frac{n k^2}{2} + 1 \right)} \right) + \frac{n}{2} \cdot \log k \right]
\]
\[ \leq \limsup_{k \to \infty} k^{-2} \cdot \log(|\Omega_k|) \]
\[ + \limsup_{k \to \infty} \left[ n \log(4C\epsilon \sqrt{n\pi}) - k^{-2} \cdot \log\left(\frac{n k^2}{2e}\right) \right] \]
\[ = \limsup_{k \to \infty} k^{-2} \cdot \log(|\Omega_k|) \]
\[ + \limsup_{k \to \infty} \left( n \log(4C\epsilon \sqrt{n\pi}) - n \log\left(\frac{k^2}{2e}\right) + n \log k \right) \]
\[ = \limsup_{k \to \infty} k^{-2} \cdot \log(|\Omega_k|) + n \log(4C\epsilon \sqrt{2}\pi) \]
\[ = \mathbb{P}_{\epsilon,R+1}(a_1, \ldots, a_n; m, \gamma) + n \log(4C\epsilon \sqrt{2}\pi). \]

This being true for any \( 2 \leq m \in \mathbb{N}, \frac{1}{2(R+1)} > \epsilon > \sqrt{\gamma} > 0, \) and \( R > \max_{1 \leq j \leq n} \|a_j\| \) it follows that for sufficiently small \( \epsilon > 0 \)
\[ \chi(a_1 + \epsilon h_1, \ldots, a_n + \epsilon h_n : h_1, \ldots, h_n) \]
\[ = \chi_{R+\frac{1}{2} \frac{1}{2}}(a_1 + \epsilon h_1, \ldots, a_n + \epsilon h_n : h_1, \ldots, h_n) \]
\[ \leq \mathbb{P}_{\epsilon}(a_1, \ldots, a_n) + n \log \epsilon + n \log(4C\epsilon \sqrt{2}\pi). \]

Dividing by \(|\log \epsilon|\), taking a lim sup as \( \epsilon \to 0 \), and adding \( n \) to both sides of the inequality above yields
\[ n + \limsup_{\epsilon \to 0} \frac{\chi(a_1 + \epsilon h_1, \ldots, a_n + \epsilon h_n : h_1, \ldots, h_n)}{|\log \epsilon|} \leq \limsup_{\epsilon \to 0} \frac{\mathbb{P}_{\epsilon}(a_1, \ldots, a_n)}{|\log \epsilon|}. \]

\textbf{Remark 2.3.} Suppose \( b_1, \ldots, b_p \) are contained in the strongly closed algebra generated by the \( a_i \) and \( R > 0 \) is strictly greater than the operator norm of any \( a_i \) or \( b_j \). The proof shows that the quantity
\[ \limsup_{\epsilon \to 0} \frac{\mathbb{K}_{\epsilon,R}(a_1, \ldots, a_n ; b_1, \ldots, b_p)}{|\log \epsilon|} = \limsup_{\epsilon \to 0} \frac{\mathbb{P}_{\epsilon,R}(a_1, \ldots, a_n ; b_1, \ldots, b_p)}{|\log \epsilon|} \]
equals
\[ n + \limsup_{\epsilon \to 0} \frac{\chi(a_1 + \epsilon h_1, \ldots, a_n + \epsilon h_n : h_1, \ldots, h_n)}{|\log \epsilon|}. \]

Recall that by \([3]\) and \([5]\) if \( \{s_1, \ldots, s_n\} \) is a free semicircular family, then \( \chi(s_1, \ldots, s_n) = \chi(s_1, \ldots, s_n) > -\infty \). Thus we have by the lemma:

\textbf{Corollary 2.4.}
\[ \delta_0(a_1, \ldots, a_n) = \limsup_{\epsilon \to 0} \frac{\mathbb{P}_{\epsilon}(a_1, \ldots, a_n)}{|\log \epsilon|} = \limsup_{\epsilon \to 0} \frac{\mathbb{K}_{\epsilon}(a_1, \ldots, a_n)}{|\log \epsilon|}. \]
Both descriptions of $\delta_0$, either in terms of volumes of $\epsilon$-neighborhoods or in terms of packing numbers, can be useful. In the presence of freeness or in the situation with one random variable it is fruitful to use the $\epsilon$-neighborhood description as Voiculescu did ([3]). On the other hand when computing $\delta_0$ in some examples it is convenient to use the uniform packing description and this was the implicit attitude taken towards $\delta_0$ in [2]. The packing formulation also comes in handy when proving formulas for generators of $M$ when $M$ has a simple algebraic decomposition into a tensor product of a von Neumann algebra $N$ with the $k \times k$ matrices or into a direct sum of algebras.

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ON CERTAIN MAXIMAL CYCLIC MODULES FOR THE QUANTIZED SPECIAL LINEAR ALGEBRA AT A ROOT OF UNITY

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By properly specializing the parameters irreducible modules of maximal dimension for the De Concini-Kac version of the Drinfeld-Jimbo quantum algebra in type $A$ may be transformed into modules over Lusztig’s infinitesimal quantum algebra. Thus obtained modules have a simple socle and a simple head, and share the same dimension as the infinitesimal Verma modules. Despite these common features we find that they are never isomorphic to infinitesimal Verma modules unless they are irreducible. The same carry over to the modular setup for the special linear groups in positive characteristic.

The finite dimensional irreducible representations of the De Concini-Kac version of the Drinfeld-Jimbo quantized enveloping algebra at a complex $\ell$-th root of 1 have their dimensions bounded above, generically attaining a maximal dimension $[DK]$. In type A Date, Jimbo, Miki and Miwa [DJMM] have given a concrete realization of most of those of maximal dimension. By properly specializing their parameters the second named author of the present paper found in $[N]$ that they afford modules $\mathcal{V}$, rarely irreducible, for Lusztig’s “infinitesimal” quantum algebra $u$ [L1] and that each $\mathcal{V}$ has a unique, up to scalar, invariant vector $u_{\hat{0}}$ relative to a Borel subalgebra $u^\sharp$ of $u$, and hence that $\mathcal{V}$ has a simple socle generated by $u_{\hat{0}}$. The dimension being right, it is tempting to compare $\mathcal{V}$ with Humphreys’ “infinitesimal” Verma modules $[H]$ quantized by Andersen, Polo and Wen [APW], which are the standard objects of study in the representation theory of $u$.

It is easy to see that $\mathcal{V}$ is isomorphic to an infinitesimal Verma module as $u^\sharp$-module, which in turn shows that $\mathcal{V}$ has the same simple head as the infinitesimal Verma module. The explicit description of the actions of the standard generators of $u$ on $\mathcal{V}$ allows us, however, to find that $\mathcal{V}$ has also a unique, up to scalar, invariant vector $u_-$ with respect to the opposite infinitesimal Borel subalgebra. It follows that $\mathcal{V}$ does not lift to an integrable $uU_0^\mathbb{C}$-module, and hence $\mathcal{V}$ can not be isomorphic to any infinitesimal Verma module as $u$-module unless $\mathcal{V}$ is simple, where $U_0^\mathbb{C}$ is the Cartan part of Lusztig’s quantum algebra $U_\mathbb{C}$ at the $\ell$-th root of 1.
By construction $\mathcal{V}$ may be defined over $\mathcal{B} = \mathbb{Z}[v, v^{-1}]/(\phi_\ell)$ with $\phi_\ell$ the $\ell$-th cyclotomic polynomial in indeterminate $v$. If $\ell$ is an odd prime $p$, $\mathcal{B}/(v - 1)$ is a finite field $\mathbb{F}_p$ of $p$-elements. Let $G$ be the special linear group scheme over $\mathbb{F}_p$ with opposite Borel subgroups $B$ and $B^+$, and let $G_1$, $B_1$, $B_1^+$ be the Frobenius kernel of $G$, $B$, and $B^+$, respectively. If $\mathcal{V}_B$ is the $B$-form of $\mathcal{V}$, then $\mathcal{V}_p = \mathcal{V}_B \otimes_B \mathbb{F}_p$ is naturally a $G_1$-module. We find that $u_\bar{0} \otimes 1$ (resp. $u_\bar{0} \otimes 1$) remains a unique, up to $\mathbb{F}_p^\times$, $B_1^+$- (resp. $B_1$-) invariant vector in $\mathcal{V}_p$. Hence $\mathcal{V}_p$ is isomorphic to an infinitesimal Verma module as $B_1^+$-module, but not as $G_1$-module unless $\mathcal{V}_p$ is simple.

If the simple $u$-module generated by $u_\bar{0}$ in $\mathcal{V}$ and the simple $G_1$-module generated by $u_\bar{0} \otimes 1$ in $\mathcal{V}_p$ have the same dimension, Lusztig's conjecture for the irreducible characters of $G$-modules will follow from the celebrated theorems of Kazhdan and Lusztig [KL] and Kashiwara and Tanisaki [KT].

If $C$ is a category, $C(A, B)$ will denote the set of morphisms of $C$ from object $X$ of $C$ to object $Y$ of $C$. If $A$ is a ring, $A \text{Mod}$ will denote the category of left $A$-modules.

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### 1. Infinitesimal Verma modules.

In this section we recollect some facts about infinitesimal Verma modules over an arbitrary quantum algebra of finite type.

**1.1.** Let $\mathbb{Q}(v)$ be the fractional field of the Laurent polynomial ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ in indeterminate $v$, $\mathcal{A} = \langle A_{ij} \rangle$ an indecomposable Cartan matrix of finite type and let $\mathcal{U}$ be the associated Drinfeld-Jimbo quantum algebra over $\mathbb{Q}(v)$ with generators $E_i$, $F_i$, and $K_i^{\pm 1}$, $i \in [1, n]$. Let $\mathcal{U}$ be Lusztig’s $\mathcal{A}$-subalgebra of $\mathcal{U}$ generated by $E_i^{(r)} = E_i^{r}[v, v^{-1}]$, $F_i^{(r)} = F_i^{r}[v, v^{-1}]$, $K_i^{\pm 1}$, $i \in [1, n]$, $r \in \mathbb{N}$, where $[r]_i! = \prod_{s=1}^{r}[s]_i$ with $[s]_i = \frac{v_i^{s} - v_i^{-s}}{v_i - v_i^{-1}}$, $v_i = v^{d_i}$, $d_i \in \{1, 2, 3\}$ minimal such that the matrix $\{d_i A_{ij}\}$ is symmetric.

Let $R$ (resp. $\Lambda$) be the root system (resp. the weight lattice) associated to $\mathcal{A}$ and $R^+$ a positive subsystem of $R$ with the simple roots $\alpha_i$, $i \in [1, n]$. We equip $\Lambda$ with a partial order defined by $R^+$ as usual. Let $\Lambda^\vee$ be the colattice of $\Lambda$ and denote by $\langle , \rangle : \Lambda \times \Lambda^\vee \to \mathbb{Z}$ the perfect pairing. If $\alpha \in R$, let $\alpha^\vee$ be its coroot. We set $\lambda_i = \langle \lambda, \alpha_i^\vee \rangle$ for $\lambda \in \Lambda$. Let $U^0$ be the $\mathcal{A}$-subalgebra of $U$ generated by $K_i^{\pm 1}$ and $[K_i; c]_r = \prod_{s=1}^{r} \frac{K_i^{\epsilon_{c+1} - \epsilon_{c+s-1}} - K_i^{-\epsilon_{c+1} + \epsilon_{c+s-1}}}{v_i^{\epsilon_{c+1} - \epsilon_{c+s-1}}}$, $i \in [1, n]$, $c \in \mathbb{Z}$, $r \in \mathbb{N}$. Each $\lambda \in \Lambda$ defines an $\mathcal{A}$-algebra homomorphism $\chi_\lambda : U^0 \to \mathcal{A}$ such that

$$K_i \mapsto v_i^{\lambda_i}, \quad [K_i; c]_r \mapsto \left[\frac{\lambda_i + c}{r}_i\right]_i = \prod_{s=0}^{r-1} \frac{\lambda_i + c - s}{[r]_i!}, \quad \forall i \in [1, n], c \in \mathbb{Z}, r \in \mathbb{N}.$$
1.2. Let $\ell$ be a positive integer greater than 2 prime to all entries $A_{ij}$ of the Cartan matrix $A$, $K = \mathbb{Q}[v]/(\phi_\ell)$, and set $U_K = U \otimes_A K$. Let $u$ (resp. $u^+$; $u^-$; $u^0$) be the $K$-subalgebra of $U_K$ generated by $E_i \otimes 1$, $F_i \otimes 1$, $K_i \otimes 1$ (resp. $E_i \otimes 1$; $F_i \otimes 1$; $K_i \otimes 1$), $i \in [1, n]$. Let also $u^x = u^+u^0$ and $u^y = u^0u^-$. We will abbreviate $x \otimes 1$ of $U_K$ as $x$, and $\chi_\lambda \otimes_A K$ as $\chi_\lambda$.

Let $\tilde{u}$ be the $K$-subalgebra of $U_K$ generated by $u$ and $U_0^K = U^0 \otimes_A K$, and let $\tilde{u}^x = u^xU^K_0$, $\tilde{u}^y = U^K_0u^y$. Each $\lambda \in \Lambda$ defines a 1-dimensional $\tilde{u}^x$-module by $\chi_\lambda$ annihilating all $F_i$, which we will still denote by $\lambda$. Let $\tilde{\nabla}(\lambda) = \tilde{u}^x \text{Mod}(\tilde{u}, \lambda)$. We make $\tilde{\nabla}(\lambda)$ into a $\tilde{u}$-module by setting $xf = f(\Psi(x))$ for each $x \in \tilde{u}$ and $f \in \tilde{\nabla}(\lambda)$.

Let $\Lambda^{\text{res}} = \{\nu \in \Lambda \mid \nu_i \in [0, \ell - 1] \ \forall i\}$. If we write $\lambda = \lambda^0 + \ell \lambda^1$ with $\lambda^0 \in \Lambda^{\text{res}}$ and $\lambda^1 \in \Lambda$, one has from [APW, 1.9] an isomorphism of $\tilde{u}$-modules

\[
\tilde{\nabla}(\lambda) \simeq \tilde{\nabla}(\lambda^0) \otimes_K \ell \lambda^1,
\]

where $\ell \lambda^1$ is a 1-dimensional $\tilde{u}$-module defined by $\chi_{\ell \lambda^1}$ annihilating all $E_i$, $F_i$ and $K_i - 1$.

On the other hand, the natural gradation on $u^+$ assigning each $E_i$ grade $\alpha_i$ equips $u^+$ with a structure of $\tilde{u}$-module such that $u^+$ act by the left multiplication and $U^K_\nu$ by $\chi_{-\lambda + \nu}$ on the $\nu$-th homogeneous part of $u^+$, $\nu \in \sum_i N \alpha_i$. Recall antiautomorphism $\Psi$ on $U$ such that $E_i \mapsto E_i$, $F_i \mapsto F_i$ and $K_i \mapsto K_i^{-1}$, $i \in [1, n]$. If $M$ is a $\tilde{u}$-module of finite type, we will denote by $M^{\Psi}$ the $K$-linear dual of $M$ made into $\tilde{u}$-module by setting $xf = f(\Psi(x))$ for each $x \in \tilde{u}$, $f \in \text{Mod}_K(M, K)$. Then we have an isomorphism of $\tilde{u}$-modules

\[
\tilde{\nabla}(\lambda) \simeq (u^+)^{\Psi} \text{ via } f \mapsto f \circ (\Psi \otimes_A K).
\]

One can likewise define a $u$-module $\nabla(\lambda) = u^x \text{Mod}(u, \lambda)$. By restricting the $\tilde{u}$-action to $u$, $\tilde{\nabla}(\lambda)$ yields $\nabla(\lambda)$. Then the isomorphism (2) restricts to an isomorphism of $\tilde{u}$-modules

\[
\nabla(\lambda) \simeq (u^+)^{\Psi}.
\]

1.3. Recall from Xi [X, 2.5] that

\[
(1) \quad u^+ \text{ has a simple socle } K \prod_{\alpha \in R^+} E^{\ell - 1}_\alpha \text{ as } u^+-\text{module},
\]

where $E_\alpha$ is a root vector of $u^+$ associated to $\alpha \in R^+$ [L2] and the product is taken in a certain specific order. It follows that $u^+$ is indecomposable as $u^+$-module, and hence that

\[
(2) \quad u^+ \text{ is a projective cover of trivial module } K \text{ as } u^+-\text{module}.
\]
By an integrable \( \tilde{u} \)- (resp. \( u \))-module \( M \) we will mean a \( \tilde{u} \)- (resp. \( u \))-module \( M \) such that

\[
M = \prod_{\mu \in \Lambda} M_\mu, \quad \text{with} \quad M_\mu = \{ m \in M \mid tm = \chi_\mu(t)m \ \forall t \in U^0_\mathcal{K} \ \text{(resp. } u^0) \}.
\]

Each \( \tilde{\nabla}(\lambda) \) (resp. \( \nabla(\lambda) \)) is an integrable \( \tilde{u} \)- (resp. \( u \))-module. Define integrable \( \tilde{u}^2 \)- and \( u^2 \)-modules likewise. One obtains from (2), (1.2.2) and (1.2.3):

**Proposition.** For each \( \lambda \in \Lambda \) the \( u^2 \)- (resp. \( \tilde{u}^2 \))-module \( \nabla(\lambda) \) (resp. \( \tilde{\nabla}(\lambda) \)) is an injective hull of \( \lambda \) in the category of integrable \( u^2 \)- (resp. \( \tilde{u}^2 \))-modules.

**1.4.** Let \( \zeta \) be the image of \( v \) in \( \mathcal{K} \), and let \( U_\zeta \) be the De Concini-Kac algebra [DK] over \( \mathcal{K} \) associated to the Cartan matrix \( A \) with the generators \( E_i, F_i, K_i^{\pm 1}, i \in [1,n] \), and the same relations for \( U \) with \( v \) replaced by \( \zeta \). For each \( \alpha \in R^+ \) let \( E_\alpha \) (resp. \( F_\alpha \)) be the root vector of \( U_\zeta \) associated with \( \alpha \) (resp. \( -\alpha \)), and let \( \mathcal{U}_\zeta = U_\zeta/(E_\alpha, F^{\ell}_\alpha \mid \alpha \in R^+) \). If \( \mathcal{U}_\zeta^{\ell} \) is the \( \mathcal{K} \)-subalgebra of \( U_\zeta \) generated by all \( F_\alpha, \alpha \in R^+ \), and \( K_i, i \in [1,n] \), each \( \lambda \in \Lambda \) defines a 1-dimensional \( \mathcal{U}_\zeta \)-module by annihilating all \( F_\alpha \) and letting \( K_i \) act by \( \zeta^{d_i \lambda} \). Then \( \mathcal{U}_\zeta \otimes \mathcal{U}_\zeta^{\ell} \lambda \) comes equipped with a structure of \( \tilde{u} \)-module [AJS, 2.10] such that each \( x \in U^0_\mathcal{K} \) acts on \( \prod_{\alpha \in R^+} E^{c_\alpha}_\alpha \otimes 1 \), \( c_\alpha \in \mathbb{N} \), by the scalar \( \chi_{\lambda+\sum_{\alpha} c_\alpha \alpha}(x) \). Put \( \rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \).

**Proposition.** For each \( \lambda \in \Lambda \) we have an isomorphism of \( \tilde{u} \)-modules

\[
\tilde{\nabla}(\lambda) \simeq \mathcal{U}_\zeta \otimes \mathcal{U}_\zeta^{\ell} (\lambda - 2(\ell - 1)\rho).
\]

**Proof.** Let \( \varepsilon_\lambda \in \tilde{\nabla}(\lambda) \) be the element induced by the counit of \( u^+ \). By the universality of \( \tilde{\nabla}(\lambda) \) [APW, 0.8.1] there is a homomorphism of \( \tilde{u} \)-modules

\[
(1) \quad \mathcal{U}_\zeta \otimes \mathcal{U}_\zeta^{\ell} (\lambda - 2(\ell - 1)\rho) \rightarrow \tilde{\nabla}(\lambda) \quad \text{such that} \quad E^+ \otimes 1 \mapsto \varepsilon_\lambda,
\]

where \( E^+ = \prod_{\alpha \in R^+} E^{\ell-1}_\alpha \). On the other hand, by [AJS, 4.9]

\[
(2) \quad \text{the } \tilde{u} \text{-socle of } \mathcal{U}_\zeta \otimes \mathcal{U}_\zeta^{\ell} (\lambda - 2(\ell - 1)\rho) \text{ is simple of highest weight } \lambda.
\]

It follows that the map (1) is injective, and hence bijective by dimension.

**1.5.**

**Corollary.** Each \( \tilde{\nabla}(\lambda) \) (resp. \( \nabla(\lambda) \)), \( \lambda \in \Lambda \), is the projective cover of \( \lambda - 2(\ell - 1)\rho \) as integrable \( \tilde{u}^2 \)- (resp. \( u^2 \))-module.
1.6. Because of the isomorphism (1.4) we call \( \tilde{\nabla}(\lambda) \) and also by abuse of language \( \nabla(\lambda) \) the infinitesimal Verma module of highest weight \( \lambda \). By [AJS, 6.3 and 4.10.1]

(1) \( \nabla(\lambda) \) (resp. \( \nabla(\lambda) \)) is simple as \( \tilde{u} \)- (resp. \( u \)-) module

\[ \text{iff } \lambda \equiv (\ell - 1)\rho \mod \ell \Lambda. \]

Let \( u^{++} \) (resp. \( u^{--} \)) be the augmentation ideal of \( u^+ \) (resp. \( u^- \)). If \( M \) is a \( u^{\pm} \)-module, let \( M^{u^{\pm\pm}} \) denote the annihilator of \( u^{\pm\pm} \) in \( M \). By (1.3)

(2)

\[ \nabla(\lambda) u^{++} = \nabla(\lambda) u^{++} = \lambda. \]

If \( \lambda \equiv (\ell - 1)\rho \mod \ell \Lambda \), then \( \nabla((\ell - 1)\rho + \ell \nu) = \nabla((\ell - 1)\rho), \nu \in \Lambda \), is simple, called the Steinberg module, and hence

(3)

\[ \nabla((\ell - 1)\rho + \ell \nu) u^{--} = -(\ell - 1)\rho + \ell \nu. \]

In general, the lowest weight of \( \nabla(\lambda) \) (resp. the socle of \( \nabla(\lambda) \)) is \( \lambda - 2(\ell - 1)\rho \) (resp. \( w_0\lambda^0 + \ell \lambda^1 \)) if \( w_0 \) is an element of the Weyl group of \( R \) such that \( w_0 R^+ = -R^+ \) and if one writes \( \lambda = \lambda^0 + \ell \lambda^1 \) with \( \lambda^0 \in \Lambda_{\text{res}} \) and \( \lambda^1 \in \Lambda \) [AJS, 4.2.5]). It follows that

(4)

\[ \dim \nabla(\lambda) u^{--} \geq 2 \text{ unless } \nabla(\lambda) \text{ is } \tilde{u}\text{-simple}. \]

1.7. Let \( B = A/(\phi \ell) \) and let \( \tilde{u}_B \) be the \( B \)-subalgebra of \( U \otimes_A B \) generated by \( E_i \otimes 1, F_i \otimes 1, K_i \otimes 1, \left[K_i; c\right] \otimes 1, i \in [1, n], c \in \mathbb{Z}, r \in \mathbb{N} \). Define its \( B \)-subalgebras \( \tilde{u}_B, u^\flat_B, \tilde{u}^\flat_B, \) and \( u^\flat_B \) as for \( \tilde{u} \). An infinitesimal Verma module may be defined over \( B \); \( \nabla_B(\lambda) = \tilde{u}^\flat_B \text{Mod}(\tilde{u}_B, \lambda), \lambda \in \Lambda, \) admits a structure of \( \tilde{u}_B \)-module like \( \nabla(\lambda) \), and we have an isomorphism of \( \tilde{u} \)-modules

\[ \nabla_B(\lambda) \otimes_B \mathcal{K} \simeq \tilde{\nabla}(\lambda). \]

Restricting the \( \tilde{u}_B \)-action to \( u_B \), one obtains \( u_B \)-module

\[ \nabla_B(\lambda) = u^\flat_B \text{Mod}(u_B, \lambda). \]

Assume now that \( \ell \) is a prime \( p \). Then \( B/(\nu - 1) \) is a finite filed \( F_p \) of \( p \)-elements. Let \( G \) be a simply connected simple algebraic group over \( F_p \) associated to the Cartan matrix \( A \) with a Borel subgroup \( B \) and a maximal torus \( T \) of \( B \) both split over \( F_p \) such that the roots of \( B \) are \( -R^+ \).

Let \( G_1 \) (resp. \( B_1 \)) be the Frobenius kernel of \( G \) (resp. \( B \)). If \( \text{Dist}(G_1) \) (resp. \( \text{Dist}(B_1) \)) is the algebra of distributions of \( G_1 \) (resp. \( B_1 \)), there are isomorphisms of \( F_p \)-algebras [L2]

\[
\begin{align*}
\text{u}_B/(K_i - 1 \mid i \in [1, n]) \otimes_B F_p & \simeq \text{Dist}(G_1), \\
\text{u}^\flat_B/(K_i - 1 \mid i \in [1, n]) \otimes_B F_p & \simeq \text{Dist}(B_1),
\end{align*}
\]
and each \( \nabla_p(\lambda) := \nabla_B(\lambda) \otimes_B \mathbb{F}_p, \lambda \in \Lambda \), admits a structure of \( G_1T \)-module (cf. [J, II.9]):

\[
\nabla_p(\lambda) \simeq \text{Dist}(G_1) \otimes_{\text{Dist}(B_1)} (\lambda - 2(p-1)\rho).
\]

Likewise \( \nabla_B(\lambda) \otimes_B \mathbb{F}_p \) yields a \( G_1 \)-module, which we will denote by \( \nabla_p(\lambda) \).

2. Maximal cyclic modules.

In this section we assume that our Cartan matrix is of type \( A_n \). Then all \( d_i = 1 \), and we will suppress \( i \) from \([\ ]_i \). By properly specializing the parameters a maximal cyclic module \( \mathcal{V} \) for \( U_\xi \) of [DJMM] factors through \( \mathfrak{u} \), having a unique, up to scalar, \( \mathfrak{u}^+ \)-primitive vector. Thus \( \mathcal{V} \) is of dimension \( \ell^{[R_+]} \) and has a simple \( \mathfrak{u} \)-socle, inviting us to compare \( \mathcal{V} \) with infinitesimal Verma modules.

2.1. Fix \( \lambda \in \Lambda \). We define as we may \( \mathcal{V} \) to be a \( \mathcal{K} \)-linear space of basis \( \mathfrak{u}_m, m \in (\mathbb{Z}/\ell\mathbb{Z})^{[R_+]} \). After \([N]\) we reindex \( R_+ \) by the pairs \((i,j)\), \( 1 \leq i \leq j \leq n \), and we will denote the \((i,j)\)-component of \( m \) by \( m_{ij} \). Then \( \mathcal{V} \) admits a structure of integrable \( \mathfrak{u} \)-module as proved in \([N, 5.2]\) such that for each \( i \in [1,n] \) and \( m \in (\mathbb{Z}/\ell\mathbb{Z})^{[R_+]} \)

\[
(1) \quad E_i u_m = \sum_{k=1}^n [m_{ik} + m_{i,k-1} - m_{i-1,k} - m_{i+1,k}] u_{m+\epsilon(i,k)+\cdots+\epsilon(i,n)},
\]

\[
(2) \quad F_i u_m = \sum_{k=1}^i [-\lambda_i + m_{i+1,k} + m_{i-1,k} - m_{i,k} + m_{i-1,k} + m_{i,k-1}] u_{m-\epsilon(i+1,k,n+1-k) - \epsilon(i,k,n+2-k) - \cdots - \epsilon(i,n)},
\]

\[
(3) \quad K_i u_m = \xi^{\lambda_i + 2m_{i0} - m_{i-1,0} - 2m_{i+1,0}} u_m,
\]

where \( [r] = \xi^r - \xi^{-r} \), \( \epsilon(i,j) \in (\mathbb{Z}/\ell\mathbb{Z})^{[R_+]} \) such that \( \epsilon(i,j)_{st} = \delta_{is}\delta_{jt} \) for each \( s \) and \( t \), and any meaningless terms in the sums should be read as 0. As the structure of \( \mathfrak{u} \)-module on \( \mathcal{V} \) depends on \( \lambda \), to be precise, we will denote the \( \mathfrak{u} \)-module \( \mathcal{V} \) by \( \mathcal{V}(\lambda) \).

A main theorem of \([N]\) is that \( \mathcal{V}(\lambda) \) has a unique, up to \( \mathcal{K} \times, \mathfrak{u}^+ \)-primitive element, i.e.,

\[
(4) \quad \mathcal{V}(\lambda)^{++} = \mathcal{K} u_\bar{0} \quad \text{with} \quad \bar{0} = (0, \ldots, 0),
\]

and hence by Engel’s theorem

\[
(5) \quad \mathcal{V}(\lambda) \text{ has a simple } \mathfrak{u} \text{-socle generated by } u_\bar{0}.
\]

It also follows from (1.3) by dimension that:

**Proposition.** There is an isomorphism of \( \mathfrak{u}^\ast \)-modules \( \mathcal{V}(\lambda) \simeq \nabla(\lambda) \).
2.2. Recall antiautomorphism $\tau$ of $u$ such that

$$E_i \mapsto F_i, \quad F_i \mapsto E_i, \quad K_i \mapsto K_i, \quad \forall i \in [1, n].$$

If $M$ is a $u$-module, let $M^\tau$ be the $K$-dual space of $M$ with a $u$-action given by

$$xf = f(\tau(x)), \quad f \in M^\ast, x \in u.$$

Then the isomorphism of $u^\sharp$-modules $\mathcal{V}(\lambda) \simeq \nabla(\lambda)$ from (2.1) yields an isomorphism of $u^-$-modules

$$\mathcal{V}(\lambda)^\tau \simeq \nabla(\lambda)^\tau$$

$$\simeq \{ \overline{U} \otimes \overline{U} \lambda \}^\tau \quad \text{by (1.4)}$$

$$\simeq \overline{U} \lambda \quad \text{by [AJS, 4.10]}$$

$$\simeq u^-,$$

where $\overline{U}$ is the $K$-subalgebra of $\overline{U}$ generated by all $E_{\alpha}$, $\alpha \in R^+$, and $K_i$, $i \in [1, n]$. It follows from [X, 2.5] again that $\mathcal{V}(\lambda)^\tau$ has a unique, up to $K^\times$, $u^-$-primitive vector, and hence:

**Corollary.** $\mathcal{V}(\lambda)$ has the same simple $u$-socle and the same simple $u$-head head as $\nabla(\lambda)$.

2.3. We find, moreover, that:

**Theorem.** The $u$-module $\mathcal{V}(\lambda)$ has a unique, up to $K^\times$, $u^-$-primitive vector $u_m$ with

$$m_{ij} \equiv - \sum_{s=1}^{i} \sum_{t=1}^{j} \lambda_{n+s-t} \mod \ell \quad \forall i \leq j,$$

which has $u^0$-weight $w_0 \lambda$.

**Proof.** The argument is the same as for (2.1.4) from [N, 4.2]; let

$$\sum_{m \in (\mathbb{Z}/\ell \mathbb{Z})^{(R^+)}} c_m u_m \in \mathcal{V}(\lambda)^{u^--}, \quad c_m \in K.$$

As $F_i \sum c_m u_m = 0$ for all $F_i$, we obtain successively $c_m = 0$ unless

$$(1) \quad -\lambda_i + m_{i+1-k,n-k} - m_{i+1-k,n+1-k} + m_{i-k,n+1-k} - m_{i-k,n-k} \equiv 0 \mod \ell$$

for each $i$ and $k \in [1, i]$. If $\sum c_m u_m \neq 0$, the system (1) of equations determines $m$ with $c_m \neq 0$ uniquely as asserted.
2.4.

**Corollary.** Let \( \lambda \in X \).

(i) There is an isomorphism of \( u^0 \)-modules

\[
\mathcal{V}(\lambda) \simeq \overline{U}_\zeta \otimes_{\overline{U}_0} (w_0 \lambda + 2(\ell - 1)\rho).
\]

(ii) The \( u \)-module \( \mathcal{V}(\lambda) \) lifts to an integrable \( \tilde{u} \)-module iff \( \lambda \equiv (\ell - 1)\rho \mod \ell \Lambda \). In particular, unless \( \lambda \equiv (\ell - 1)\rho \mod \ell \Lambda \), \( \mathcal{V}(\lambda) \) is not isomorphic as \( u \)-module to any infinitesimal Verma module.

**Proof.** For (i) argue as in (2.1) and (2.2).

(ii) If \( \lambda \equiv (\ell - 1)\rho \mod \ell \Lambda \), the simple \( u \)-socle of \( \mathcal{V}(\lambda) \) has dimension \( \ell |R^+| = \dim \mathcal{V}(\lambda) \) and hence the assertion follows. We may therefore assume that \( \lambda \not\equiv (\ell - 1)\rho \mod \ell \Lambda \).

Just suppose \( \mathcal{V}(\lambda) \) lift to an integrable \( \tilde{u} \)-module. Then we would have from (1.3) an isomorphism of \( \tilde{u} \)-modules

\[
\mathcal{V}(\lambda) \simeq \overline{V}(\lambda + \ell \nu) \quad \text{for some } \nu \in \Lambda.
\]

Then the unique \( u^- \)-primitive in \( \mathcal{V}(\lambda) \) should have by (1.4) weight \( \lambda + \ell \nu - 2(\ell - 1)\rho \). That would yield, arguing as in (2.3), an isomorphism of \( \tilde{u} \)-modules

\[
\mathcal{V}(\lambda) \simeq \overline{U}_\zeta \otimes_{\overline{U}_0} (\lambda + \ell \nu).
\]

Then \( \mathcal{V}(\lambda) = u^- u_0^- \) as \( u_0^- \) is a highest weight vector of \( \mathcal{V}(\lambda) \). But \( u^- u_0^- = uu_0^- \) is the simple socle of \( \mathcal{V}(\lambda) \) and of dimension \( < \dim \mathcal{V}(\lambda) \) by (1.6.1), absurd.

2.5. Assume now that \( \ell \) is a prime \( p \) and let \( \mathcal{B} = \mathcal{A}/(\phi_p) = \mathbb{Z}[v]/(\phi_p) \) as in (1.7). By construction \( \mathcal{V}(\lambda) \) may be defined over \( \mathcal{B} \): Let \( \mathcal{V}_B(\lambda) \) be the free \( \mathcal{B} \)-module of basis \( v_m \), \( m \in (\mathbb{Z}/p\mathbb{Z})|R^+| \), with the \( u \mathcal{B} \)-action given by (2.1.1)-(2.1.3). Regarding \( \mathbb{F}_p \) as the quotient \( \mathcal{B}/(v - 1) \), let \( \mathcal{V}_p(\lambda) = \mathcal{V}_B(\lambda) \otimes_{\mathcal{B}} \mathbb{F}_p \). Then \( \mathcal{V}_p(\lambda) \) is naturally a \( G_1 \)-module for \( G = \text{SL}_{n+1} \otimes \mathbb{F}_p \) in the setup of (1.7). The proof of (2.1.4) from \([N]\) and the argument of (2.3) carry over to obtain:

**Theorem.** Let \( \lambda \in \Lambda \).

(i) There is an isomorphism of \( B_1^+ \)-modules \( \mathcal{V}_p(\lambda) \simeq \nabla_p(\lambda) \).

(ii) \( \mathcal{V}_p(\lambda) \) has a unique, up to \( \mathbb{F}_p^\times \), \( B_1^+ \)- and \( B_1 \)-invariant vector, respectively, and hence has a simple \( G_1 \)-socle and a simple \( G_1 \)-head, where \( B_1^+ \) is the Frobenius kernel of the Borel subgroup \( B^+ \) of \( G \) opposite to \( B \).

(iii) If \( \lambda \not\equiv (p - 1)\rho \mod p\Lambda \), the structure of \( G_1 \)-module on \( \mathcal{V}_p(\lambda) \) does not lift to \( G_1T \)-module, and hence \( \mathcal{V}_p(\lambda) \) is not isomorphic to any infinitesimal Verma module as \( G_1 \)-module.
2.6.

Remark. If the simple $u$-module generated by $u_0$ in $\mathcal{V}(\lambda)$ and the simple $G_1$-module generated by $u_0 \otimes 1$ in $\mathcal{V}(\lambda) \otimes_{\mathbb{Z}} \mathbb{F}_p$ have the same dimension, then Lusztig’s conjecture for the irreducible $\text{SL}_{n+1} \otimes_{\mathbb{Z}} \mathbb{F}_p$-modules will follow from [KL] and [KT]; for $p \geq 2n - 1$ the converse is also expected to hold.

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EFFICIENT FUNDAMENTAL CYCLES OF CUSPED HYPERBOLIC MANIFOLDS

THILO KUESSNER

Let $M$ be a manifold (with boundary) of dimension $\geq 3$, such that its interior admits a hyperbolic metric of finite volume. We discuss the possible limits arising from sequences of relative fundamental cycles approximating the simplicial volume $\| M, \partial M \|$, using ergodic theory of unipotent actions. As applications, we extend results of Jungreis and Calegari from closed hyperbolic to finite-volume hyperbolic manifolds:

a) Strict subadditivity of simplicial volume with respect to isometric glueing along geodesic surfaces, and

b) nontriviality of the foliated Gromov norm for “most” foliations with two-sided branching.

1. Introduction.

Gromov defined the simplicial volume $\| M, \partial M \|$ of a manifold $M$ as the “minimal cardinality of a triangulation with real coefficients”. That means, for an $n$-dimensional compact, connected, orientable manifold $M$ with (possibly empty) boundary $\partial M$, define

$$\| M, \partial M \| := \inf \left\{ \sum_{i=1}^{r} a_i : \sum_{i=1}^{r} a_i \sigma_i \text{ represents } [M, \partial M] \right\} .$$

Here, $[M, \partial M] \in H_n(M, \partial M; \mathbb{R})$ is the image of a generator of $H_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$ under the canonical homomorphism $H_n(M, \partial M; \mathbb{Z}) \rightarrow H_n(M, \partial M; \mathbb{R})$.

The simplicial volume quantifies the topological complexity of a manifold. It is nontrivial if $\text{int}(M)$ admits a complete metric of sectional curvature $\leq -a^2$ and finite volume. In particular the Gromov-Thurston theorem ([9], [19]) states for finite-volume hyperbolic manifolds $\| M, \partial M \| = \frac{1}{V_n} \text{Vol}(M)$, where $V_n$ is the volume of a regular ideal simplex in $\mathbb{H}^n$. This exhibits hyperbolic volume as a homotopy invariant, complementing the Chern-Gauß-Bonnet theorem, which implies homotopy invariance of hyperbolic volume for even-dimensional manifolds. Homotopy invariance of hyperbolic volume was used by Gromov to give a more topological proof of Mostow’s rigidity theorem. In the meantime, various more general rigidity theorems have been proved, again using the simplicial volume.
On a finite-volume hyperbolic manifold, there does not exist a fundamental cycle actually having $l^1$-norm $\frac{1}{V_n} \text{Vol}(M)$. However, there is a measure cycle, supported on the set of regular ideal simplices, the so-called smearing of a regular ideal simplex, having this norm: After identifying the set of (ordered) regular ideal simplices with $\text{Isom}(\mathbb{H}^n) = \text{Isom}^+(\mathbb{H}^n) \cup \text{Isom}^-(\mathbb{H}^n)$, it is the signed measure $\frac{1}{2V_n} (\text{Haar} - r^*\text{Haar})$, where $\text{Haar}$ is the Haar measure on $\text{Isom}^+(\mathbb{H}^n)$ and $r$ is an orientation-reversing isometry. This measure cycle can be approximated by authentic singular chains, i.e., finite linear combinations of (nonideal) simplices (and this proves the Gromov-Thurston theorem, cf. [2] for details of the proof in the case of closed manifolds).

Technically, the main part of this paper is devoted to the question of to which extent this construction is unique, i.e., whether there exist sequences of fundamental cycles with $l^1$-norms converging to $\frac{1}{V_n} \text{Vol}(M)$ which do not approximate Gromov’s smearing construction.

For closed manifolds of dimension $\geq 3$, it was shown in [11] by Jungreis that any such sequence must converge to Gromov’s smearing cycle. In this paper we extend this rigidity results to hyperbolic manifolds of finite volume which are either of dimension $\geq 4$ or which are of dimension 3 and not Gieseking-like (see Definition 4.4).

Moreover, we obtain restrictions on sequences of fundamental cycles with $l^1$-norms converging to $\frac{1}{V_n} \text{Vol}(M)$ on (possibly Gieseking-like) finite-volume hyperbolic manifolds of dimension $\geq 3$, which allow to conclude: If $F$ is a closed geodesic hypersurface, then the limits of such sequences do invoke simplices intersecting $F$ ‘transversally’ (see Definition 5.1). This property can actually be restated as $\|M_F, \partial M_F\| > \|M, \partial M\|$, where $M_F$ is obtained by cutting $M$ along $F$.

As applications, we extend results of Jungreis and Calegari to hyperbolic manifolds with cusps.

**Glueing along boundaries.** Consider manifolds $M_1, M_2$, a homeomorphism $f : A_1 \to A_2$ between subsets $A_i \subset \partial M_i$, and let $M = M_1 \cup_f M_2$ be the glued manifold. In general, it is hard to compare $\|M\|$ to $\|M_1\| + \|M_2\|$. One can prove “$\geq$” if the $A_i$ are incompressible and amenable, and even “$=$” if, in addition, they are connected components of $\partial M_i$, cf. [9] and [12].

**Theorem 6.3.** Let $n \geq 3$ and let $M_1, M_2$ be compact $n$-manifolds with boundaries $\partial M_i = \partial_0 M_i \cup \partial_1 M_i$, such that $M_i - \partial_0 M_i$ admit incomplete hyperbolic metrics of finite volume with totally geodesic boundaries $\partial_1 M_i$. If $\partial_1 M_i$ are not empty, $f : \partial_1 M_1 \to \partial_1 M_2$ is an isometry and $M = M_1 \cup_f M_2$, then

$$\|M, \partial M\| < \|M_1, \partial M_1\| + \|M_2, \partial M_2\|.$$ 

The same statement holds if one glues only along some connected components of $\partial_1 M_i$. One also has an analogous statement if two totally geodesic
boundary components of the same hyperbolic manifold are glued by an isometry.

One point of interest in Theorem 6.3 is that it serves, in the case of 3-manifolds, as a main step for a general glueing inequality. In [12], we prove:

**Theorem.** For a compact 3-manifold $M$, $\|DM\| < 2\|M, \partial M\|$ holds if and only if $\|\partial M\| > 0$, i.e., if $\partial M$ consists not only of spheres and tori.

(Here, $DM$ is the manifold obtained by glueing two differently oriented copies of $M$ via the identity of $\partial M$. Note that $\|DM\| \leq 2\|M, \partial M\|$ trivially holds.) This theorem may be seen as a generalisation of Theorem 6.3, saying that any efficient fundamental cycle on a 3-manifold with $\mathbb{Z}_2$-symmetry has to intersect the fixed point set ‘transversally’. It is maybe worth pointing out that for the proof of this theorem in [12] we need to have Theorem 6.3 also for the case of cusps.

Another (direct) corollary from Theorem 6.3 and Mostow rigidity is that (under the assumptions of Theorem 6.3), in dimensions $\geq 4$, we get the same inequality for any homeomorphisms $f$. This theorem seems to be hardly available by topological methods. The analogous statement in dimension 3 was recently shown to be wrong by Soma ([18]). He proved: If $M_1, M_2$ are hyperbolic 3-manifolds of totally geodesic boundary and $f: \partial M_1 \to \partial M_2$ is pseudo-Anosov, then $\lim_{n \to \infty} \|M_1 \cup f^n M_2\| = \infty$.

**Foliated Gromov norm.** The Gromov norm of a foliation/lamination $\mathcal{F}$ on a manifold $M$, as introduced in [4], is

$$\|M, \partial M\|_{\mathcal{F}} := \inf \left\{ \sum_{i=1}^r |a_i| : \sum_{i=1}^r a_i \sigma_i \text{ represents } [M, \partial M], \sigma_i \text{ transverse to } \mathcal{F} \right\}.$$  

The difference $\|M, \partial M\|_{\mathcal{F}} - \|M, \partial M\|$ seems to quantify the amount of branching of the leaf space. Calegari proved:

- $\|M\|_{\mathcal{F}} = \|M\|$, when the leaf space is branched in at most one direction, and
- $\|M\|_{\mathcal{F}} > \|M\|$ for asymptotically separated laminations of closed hyperbolic manifolds of dimension $\geq 3$.

The first statement generalizes easily to manifolds with boundary. We extend the second statement as follows:

**Theorem 7.5.** Assume that the interior of $M$ is a hyperbolic $n$-manifold of finite volume. If $n \geq 3$ and $M$ is not Gieseking-like (Definition 4.4), and if $\mathcal{F}$ is an asymptotically separated lamination, then

$$\|M, \partial M\| < \|M, \partial M\|_{\mathcal{F}}.$$
We want to outline the content of this paper. In Chapter 3 we give a definition of “efficient fundamental chains”, exhibit them as signed measures $\mu$ on the space of regular ideal simplices, show that they are absolute cycles (the boundary “escapes to infinity” for $\epsilon \to 0$), and derive ergodic decompositions of $\mu$ with respect to certain groups of reflections. Such (different) decompositions exist associated to all vertices of a fixed simplex $\Delta_0$. We show that the ergodic decomposition corresponding to the $i$-th vertex of $\Delta_0$ uses only the Haar measure and measures determined on the set of simplices having $i$-th vertex in a parabolic fixed point of $\Gamma$.

This is used in Chapter 5 to prove Theorem 5.3: If $F$ is a closed, totally geodesic hypersurface in a finite-volume hyperbolic manifold of dimension $\geq 3$ and $\mu$ is an efficient fundamental cycle, then $\mu(S_F) > 0$, where $S_F$ is the set of simplices intersecting $F$ transversally. To give a rough explanation of the proof: The Haar measure does not vanish on $S_F$, hence $\mu(S_F) = 0$ would imply that $\mu$ is determined on the set of simplices with all vertices in parabolic points, contradicting the fact that it must invoke simplices with faces in the cuspless hypersurface $F$.

In Chapter 6, Theorem 6.3 is derived from Theorem 5.3. Chapter 7 is devoted to the foliated Gromov norm and the proof of Theorem 7.5.

The simplicial volume of a nonorientable, disconnected manifold is the sum over the connected components of half of the simplicial volumina of the orientation coverings. We will give all proofs for connected, oriented manifolds, since all statements generalise directly. This includes that the orientations of glued manifolds are understood to fit together.

2. Preliminaries.

2.1. Volume of straight simplices. A simplex in hyperbolic space $\mathbb{H}^n$, with vertices $p_0, \ldots, p_i$, is called straight if it is the barycentric parametrization of the geodesic simplex with vertices $p_0, \ldots, p_i$.

Given two regular ideal (straight) $n$-simplices $\Delta_0$ and $\Delta$ in $\mathbb{H}^n$, with fixed orderings of their vertices, there is a unique $g \in \text{Isom}(\mathbb{H}^n)$ mapping $\Delta_0$ to $\Delta$.

Hence, fixing a reference simplex $\Delta_0$, we have an $\text{Isom}(\mathbb{H}^n)$-equivariant bijection between the set of ordered regular ideal $n$-simplices and $\text{Isom}(\mathbb{H}^n)$, this bijection being unique up to the choice of $\Delta_0$, i.e., up to multiplication with a fixed element of $\text{Isom}(\mathbb{H}^n)$.

As another consequence, all regular ideal $n$-simplices in $\mathbb{H}^n$ have the same volume, to be denoted $V_n$.

By [10], any straight $n$-simplex $\sigma$ in $\mathbb{H}^n$ satisfies $\text{Vol}(\sigma) \leq V_n$ and equality is achieved only for regular ideal simplices.

2.2. Ergodic decomposition. For a topological space $X$, we consider Radon measures $\mu$ on $X$. This are, by definition, elements of $C_b^*(X)$, the
dual of the space of compactly supported continuous functions. They have a decomposition \( \mu = \mu^+ - \mu^- \) with \( \mu^+, \mu^- \) nonnegative Radon measures. (We will refer to \( \mu \) as signed measure and to \( \mu^\pm \) as measures.) A probability measure on \( X \) is a measure \( \mu \) with \( \mu(X) = 1 \).

Let a group \( G \) act on a topological space \( X \). A probability measure \( \mu \) is called ergodic if any \( G \)-invariant set has measure 0 or 1. Denote by \( \mathcal{E} \) the set of ergodic \( G \)-invariant probability measures on \( X \). Let \( \mathcal{A} \) be the weak measure class induced by the measure class on \( X \), i.e., the smallest \( \sigma \)-algebra \( \mathcal{A} \) on \( \mathcal{E} \) such that for all Borel sets \( A \subset X \) the application \( f_A : \mathcal{E} \to \mathbb{R} \) defined by \( f_A(\mu) := \mu(A) \) is measurable.

**Lemma 2.1.** Let a group \( G \) act on a complete separable metric space \( X \). If there exists a \( G \)-invariant probability measure on \( X \), then the set \( \mathcal{E} \) of ergodic \( G \)-invariant measures on \( X \) is not empty and there is a decomposition map \( \beta : X \to \mathcal{E} \).

Here, a decomposition map is a \( G \)-invariant map \( \beta : X \to \mathcal{E} \), which is:

- Measurable with respect to \( \mathcal{A} \),
- satisfies \( e(\{ x \in X : \beta(x) = e \}) = 1 \) for all \( e \in \mathcal{E} \), and
- for all \( G \)-invariant probability measures \( \mu \) and Borel sets \( A \subset X \) the following equality holds:

\[
\mu(A) = \int_X \beta(x)(A) \, d\mu(x) .
\]

For a proof of Lemma 2.1, see Theorem 4.2 in [21].

For later reference we state the following lemma, Part (i) of which is known as Alaoglu’s theorem, whereas a proof of Part (ii) can be found in Lemma 3.2 of [6].

**Lemma 2.2.**

(i) Any weak-\( * \)-bounded sequence of signed Radon measures on a locally compact metric space has an accumulation point in the weak-\( * \)-topology.

(ii) If \( \mu \) is the weak-\( * \)-limit of a sequence \( \mu_n \) of measures on a space \( X \), and \( U \subset X \) is an open subset, then \( \mu(U) \leq \liminf \mu_n(U) \).

Moreover, we recall that the support of a measure \( \mu \) on \( X \) is defined as the complement of the largest open set \( U \subset X \) with the property \( \mu(U) = 0 \).

2.3. Measure homology. The following explanations are not necessary (from a logical point of view) for our arguments, but may be helpful to understand the framework. For a topological space \( X \), let \( C^0(\Delta^k, X) \) be the space of singular \( k \)-simplices in \( X \), topologized by the compact-open-topology. For a signed measure \( \mu \) on \( C^0(\Delta^k, X) \), one has its decomposition
\( \mu = \mu^+ - \mu^- \) as difference of two (nonnegative) Borel measures, and one defines its total variation as \( \| \mu \| = \int d\mu^+ + \int d\mu^- \).

Let \( \mathcal{C}_k (X) \) be the vector space of all signed measures \( \mu \) on \( C^0 (\Delta^k, X) \) which have compact support and finite total variation. (We assume finite total variation because we want \( \| . \| \) to define a norm on \( \mathcal{C}_k (X) \). The condition ‘compact support’ is imposed because otherwise the map \( j : H_*(X, \mathbb{R}) \to \mathcal{H}_*(X) \) defined below would, in general, not be surjective, see [19], 6.1. for examples of this phenomenon.) Let \( \eta_i : \Delta^k \to \Delta^{k-1} \) be the \( i \)-th face map. It induces a map \( \partial_i = (\eta_i^*)_* : \mathcal{C}_k (X) \to \mathcal{C}_{k-1} (X) \). We define the boundary operator \( \partial := \sum_{i=0}^k \partial_i \), to make \( \mathcal{C}_* (X) \) a chain complex. We denote the homology groups of this chain complex by \( \mathcal{H}_* (X) \).

We have an obvious inclusion \( j : \mathcal{C}_* (X) \to \mathcal{C}_* (\overline{X}) \), where \( \mathcal{C}_* (\overline{X}) \) are the singular chains, considered as finite linear combination of atomic measures. Clearly, \( j \) is a chain map. Zastrow’s Theorem 3.4. in [22] says that we get an isomorphism \( j_* : H_* (M) \to \mathcal{H}_* (M) \) if \( M \) is a smooth manifold (but not for arbitrary topological spaces \( X \)).

The \( l^1 \)-norm on \( \mathcal{C}_* (M) \) extends to the total variation \( || . || \) on \( \mathcal{C}_* (M) \), and we get an induced pseudonorm on \( \mathcal{H}_* (M) \). Thurston conjectured in [20] that the isomorphism \( j_* \) should be an isometry for this pseudonorm. There seems not to exist a proof of this general conjecture so far, but if \( M \) is a closed hyperbolic \( n \)-manifold, it follows easily from the identity \( || M || = \frac{1}{1} \text{Vol} (M) \) ([5], [9]) that \( j_n : H_n (M) \to \mathcal{H}_n (M) \) is an isometry.

2.4. Intersection numbers.

**Definition 2.3.** Let \( M \) be an oriented differentiable \( n \)-manifold. For an immersed differentiable \( n \)-simplex \( \sigma : \Delta^n \to M \), and \( x \in M \), define

\[
\Phi_x (\sigma) = \sum_{y \in \sigma^{-1} (x)} \text{sign} \ \det \sigma (y).
\]

For a singular chain \( c = \sum_{j=1}^r a_j \sigma_j \), let \( \Phi_x (c) = \sum_{j=1}^r a_j \Phi_x (\sigma_j) \).

**Lemma 2.4.** Let \( M \) be a connected, oriented, smooth, noncompact \( n \)-manifold, \( M' \) an \( n \)-submanifold with boundary, such that \( M - M' \) is compact. Let \( c = \sum_{j=0}^r a_j \sigma_j \) be a smooth singular \( n \)-chain representing the relative fundamental class \([M, M']\). Assume that all \( \sigma_j \) are immersed smooth \( n \)-simplices. Then \( \Phi_x (c) = 1 \) holds for almost all \( x \in M - M' \).

**Proof.** Let \( K = \cup_{j=0}^r \text{Im} (\partial \sigma_j) \). \( K \) is of measure zero, by Sard’s lemma.

We want to show that \( \Phi_x (c) \), as a function of \( x \), is constant on \( M - (M' \cup K) \). It is obvious that it is locally constant on \( M - (M' \cup K) \), since all \( \sigma_i \) are immersed. It remains to prove: For all \( x \in K \cap \text{int} (M - M') \), there is a neighborhood \( U \) of \( x \) in \( M \) such that \( \Phi_x (c) \) is constant on \( U \cap (M - K) \). The point \( x \) is contained in the image of finitely many \((n-1)\)-simplices \( \kappa_1, \ldots, \kappa_k \), which are boundary faces of some \( \sigma_{i_1}, \ldots, \sigma_{i_k} \). (Note that the
σ_i, j’s need not be distinct and that there might be further σ_i’s containing
x in the interior of their image.) Since \( \partial \sum_{j=1}^{r} a_j \sigma_j \) invokes only simplices
whose image is contained in \( M - M' \), we necessarily have that all \( \sigma_{i_1}, \ldots, \sigma_{i_k} \)
cancel each other, i.e., there is a partition of \( \{i_1, \ldots, i_k\} \) in some subsets,
such that for each of these subsets of indices the sum of the corresponding
coefficients \( a_{i_j} \), multiplied with a sign according to orientation of \( \sigma_{i_j} \), adds
up to zero. This implies that \( \Phi \) is constant in the intersection of a small
neighborhood of \( x \) with the complement of \( K \) and, hence, also constant on
all of \( M - (M' \cup K) \).

We now prove that this constant does not depend on the representative
of the relative fundamental class. This implies that the constant must be 1,
since one can choose a triangulation as representative of the relative funda-
mental class.

If \( c \) and \( c' \) are different representatives of \([M, M']\), we have that \( c - c' = \partial w + t \) for some \( w \in C_{n+1} (M) \) and \( t \in C_n (M') \). Because \( \partial w \) is a cycle, the
same argument as above gives that \( \Phi_\cdot (\partial w) \) is a.e. constant on all of \( M \). This
constant must be zero, since \( \partial w \) has compact support in the noncompact
manifold \( M \). That means that \( \Phi_x (c) - \Phi_x (c') = \Phi_x (t) \) for almost all \( x \in M \).
But \( \Phi_x (t) = 0 \) for all \( x \in \text{int} (M - M') \). □

2.5. Convergence of fundamental cycles. A major point of the next
chapter will be to consider limiting objects of sequences of relative funda-
mental cycles of a finite-volume hyperbolic manifold \( M \) with \( l^1 \)-norms
converging to the simplicial volume. Since straight simplices have volume
smaller than \( V_n \), there do not exist relative fundamental cycles actually
having \( l^1 \)-norm equal to \( \frac{1}{V_n} \text{Vol} (M) \). Hence, the limits of such sequences
can not be just singular chains. What we are going to do is to embed the
singular chain complex into a larger space, where any bounded sequence has
accumulation points. A straightforward idea would be to use the inclusion
\( j : C_n (M) \to C_n (M) \) and to consider weak-\( * \) accumulation points in \( C_n (M) \).
This works perfectly well, however it is easy to see that the weak-\( * \) limits are
just trivial measures. The reason is roughly the following: A singular chain
with \( l^1 \)-norm close to \( \frac{1}{V_n} \text{Vol} (M) \) has to have a very large part of its mass on
simplices \( \sigma \) with \( \text{vol} (\text{str} (\sigma)) \) quite close to \( V_n \). If we consider a compact set
of simplices, it will have some upper bound (better than \( V_n \)) on \( \text{vol} (\text{str} (\cdot)) \).
Hence, it will contribute very little to an almost efficient fundamental cycle,
and the limiting measure will actually vanish on this set of simplices.

Therefore, to get nontrivial accumulation points, we are obliged to con-
consider the larger space of simplices which might be ideal, i.e., whose lifts to \( \mathbb{H}^n \) might have vertices in \( \partial_\infty \mathbb{H}^n \). This, however, raises another problem:
The space of ideal simplices in \( M = \Gamma \backslash \mathbb{H}^n \) is not Hausdorff, and there is
no theorem guaranteeing existence of weak-\( * \) accumulation points for signed
measures on non-Hausdorff spaces.
Let \( M = \Gamma \backslash \mathbb{H}^n \) has finite volume, then the action of \( \Gamma \) on \( \partial_\infty \mathbb{H}^n \) has dense orbits. Thus the quotient \( \Gamma \backslash \mathbb{H}_n^1 (M) \) can not be Hausdorff as long as \( \mathbb{H}_n^1 (M) \) contains degenerate simplices. We will show in Section 2.5.3, however, that the action of \( \Gamma \) on the subspace of nondegenerate \( n \)-simplices is properly discontinuous, i.e., after throwing away the degenerate simplices we get a Hausdorff quotient. A similar idea seems to have been exploited in the proof of Lemma 2.2 in [11] where the author restricted to a compact subset of \( \mathbb{H}_n^1 (M) \), i.e., to simplices with a lower volume bound.

### 2.5.1. Straightening alternating chains.

The symmetric group \( S_{n+1} \) acts on the standard \( n \)-simplex \( \Delta^n \): Any permutation \( \pi \) of vertices can be realised by a unique affine map \( f_\pi : \Delta^n \to \Delta^n \). For a singular simplex \( \sigma : \Delta^n \to M \) let \( \text{alt}(\sigma) := \sum_{\pi \in S_{n+1}} \text{sgn}(\pi) f_\pi \), and for a singular chain \( c = \sum_{j=1}^r a_j \sigma_j \) define \( \text{alt}(c) := \frac{1}{(n+1)!} \sum_{j=1}^r a_j \text{alt}(\sigma_j) \). Clearly, \( \| \text{alt}(c) \| \leq \| c \| \).

For a simplex \( \sigma \) in \( \mathbb{H}^n \), we denote by \( \text{Str}(\sigma) \) the straight simplex with the same vertices as \( \sigma \) (as in Section 2.1). A straight simplex in a hyperbolic manifold \( M = \Gamma \backslash \mathbb{H}^n \) is the image of a straight simplex in \( \mathbb{H}^n \) under the projection \( p : \mathbb{H}^n \to \Gamma \backslash \mathbb{H}^n = M \). For a simplex \( \sigma \) in \( M \), its straightening \( \text{Str}(\sigma) \) is defined as \( p(\text{Str}(\sigma)) \), where \( \sigma \) is a simplex in \( \mathbb{H}^n \) projecting to \( \sigma \). Since straightening in \( \mathbb{H}^n \) commutes with isometries, the definition of \( \text{Str}(\sigma) \) does not depend on the choice of \( \tilde{\sigma} \).

Finally, the straightening of a singular chain \( c = \sum_{j=1}^r a_j \sigma_j \) is defined as \( \text{Str}(c) = \sum_{j=1}^r a_j \text{Str}(\sigma_j) \). \( \text{Str}(c) \) is homologous to \( c \), and clearly \( \| \text{Str}(c) \| \leq \| c \| \) for any \( c \in C_* (M) \). (\( \text{Str}(c) \) may possibly have smaller norm than \( c \), since different simplices can have the same straightenings.)

If \( M' \subset M \) is a convex subset (meaning that \( \sigma \in M' \) implies \( \text{str}(\sigma) \subset M' \), then \( \text{Str} : C_*(M, M') \to C_*(M, M') \) is well-defined.

### 2.5.2. Nondegenerate chains.

Let \( M \) be a hyperbolic manifold. We call a straight \( i \)-simplex \( \sigma : \Delta^i \to N \) degenerate if two of its vertices are mapped to the same point, nondegenerate otherwise.

For \( M' \subset M \) a convex subset of \( M \), we consider \( \text{algvol} : C_n (M, M') \to R \) which maps \( \sigma \in C_n (M) \) to the algebraic volume (see [2], p. 107) of \( \text{str}(\sigma) \cap (M - M') \). (Since \( M' \) is convex, \( \text{algvol} \) is well-defined on the relative chain complex.) It follows from Stokes theorem that we get an induced map \( \text{algvol}_* : H_n (M, M'; \mathbb{R}) \to \mathbb{R} \).

**Lemma 2.5.** Let \( M \) be a hyperbolic \( n \)-manifold, \( M' \) a convex subset such that \( \text{algvol} : H_n (M, M'; \mathbb{R}) \to \mathbb{R} \) is an isomorphism. Let \( \sum_{i \in I} a_i \sigma_i \in C_n (M, M'; \mathbb{R}) \) be a straight relative \( n \)-cycle. Then there is a subset of indices \( J \subset I \) such that all \( \sigma_j \) with \( j \in J \) are nondegenerate and \( \sum_{j \in J} a_j \sigma_j \) is relatively homologous to \( \sum_{i \in I} a_i \sigma_i \).

**Proof.** Let \( K := \{ k \in I : \sigma_k \text{ degenerate} \} \) be the set of indices of degenerate simplices occurring in \( \sum_{i \in I} a_i \sigma_i \). We claim that \( \sum_{k \in K} a_k \sigma_k \) is a relative
cycle. Indeed, the degenerate faces of \( \sum_{k \in K} a_k \sigma_k \) cancel each other (relatively), since they cancel in \( \partial (\sum_{i \in I} a_i \sigma_i) \) and they can not cancel against faces of nondegenerate simplices. Moreover, the nondegenerate faces of degenerate simplices cancel anyway: If \((a, v_1, \ldots, v_n)\) and \((b, v_1, \ldots, v_n)\) are nondegenerate faces of a degenerate simplex, then necessarily \(a = b\). Thus this face contributes twice to the boundary, with opposite signs.

We have obtained that \( \sum_{k \in K} a_k \sigma_k \) is a relative cycle. But, since all \( \sigma_k \) are degenerate, they have vanishing volume, and we have that the relative homology class \([\sum_{k \in K} a_k \sigma_k]\) \(\in\ker (\text{algvol}^*) = 0\) (since \(\text{algvol}^*\) is an isomorphism, by assumption), i.e., \(\sum_{k \in K} a_k \sigma_k \in \ker (\text{algvol}) = 0\) is a relative boundary. Then choose \(J = I - K\).

In conclusion, if \(M' \subset M\) convex and \(n = \dim (M)\), then to any relative \(n\)-cycle \(c \in C_n (M, M'; \mathbb{R})\) we find \(c' \in C_n (M, M'; \mathbb{R})\) homologous to \(c\) in \(C_* (M, M'; \mathbb{R})\), such that \(\|c'\| \leq \|c\|\) and \(c'\) is an alternating linear combination of nondegenerate straight simplices.

### 2.5.3. Straight chains as measures.

We explained in 2.3 that singular chains may be considered as measures on the space of singular simplices, thus getting a homomorphism \(C_* (M; \mathbb{R}) \rightarrow C_* (M; \mathbb{R})\). As we said, to get nontrivial results, we should consider not only \(C_* (M; \mathbb{R})\), but measures on the space of possibly ideal simplices. Since it is hard to prove existence of accumulation points in this measure space, we will consider measures on smaller sets of simplices.

Let \(M\) be a hyperbolic manifold. The set of nondegenerate, possibly ideal, straight \(i\)-simplices in \(M = \Gamma \setminus \mathbb{H}^n\) is

\[
SS_i (M) := \Gamma \setminus \{(p_0, \ldots, p_i) : p_0, \ldots, p_i \in \mathbb{H}^n, p_j \neq p_k \text{ if } j \neq k\},
\]

where \(g \in \Gamma\) acts by \(g (p_0, \ldots, p_n) = (gp_0, \ldots, gp_n)\).

Denote \(\mathcal{M} (SS_i (M))\) the space of signed regular measures on \(SS_i (M)\). Straight singular chains \(c = \sum_{j=1}^{r} a_j \sigma_j \in C_i (M; \mathbb{R})\), with all \(\sigma_j\) nondegenerate, can be considered as discrete signed measures on \(SS_i (M)\) defined by

\[
c (B) = \sum_{\{j : \sigma_j \subset B\}} |a_j|
\]

for any Borel set \(B \subset SS_i (M)\).

Let \(n = \dim (M)\). To apply Alaoglu’s theorem to \(\mathcal{M} (SS_n (M))\), we need to know that \(SS_n (M)\) is locally compact (which is obvious) and metrizable.

**Lemma 2.6.** Let \(M\) be a hyperbolic manifold of dimension \(n \geq 3\). Then \(SS_n (M)\) is metrizable.

**Proof.** We have to show that \(\Gamma\)-orbits on \(\Pi_{j=0}^n \mathbb{H}^n - D\) are closed, \(D\) being the set of degenerate straight simplices. On the complement of \(\Pi_{j=0}^n \partial \infty \mathbb{H}^n\) this follows from proper discontinuity of the \(\Gamma\)-action on \(\mathbb{H}^n\).
Now we assume \( n \geq 3 \). To any \( n \)-tuple \( (v_0, \ldots, v_{n-1}) \in \Pi_{j=0}^{n-1} \partial_\infty \mathbb{H}^n \) of distinct points corresponds a unique \( v_n \in \partial_\infty \mathbb{H}^n \) such that \( (v_0, \ldots, v_n) \) is a positively oriented regular ideal \( n \)-simplex. (If \( n = 2 \), then \( v_n \) is not uniquely determined.) Together with 2.1, we get a \( \Gamma \)-equivariant homeomorphism \( \Pi_{j=0}^{n-1} \partial_\infty \mathbb{H}^n \to \text{Isom}^+ (\mathbb{H}^n) \). Since \( \Gamma \backslash \mathbb{H}^n \) is a manifold, we know that \( \Gamma \) acts properly discontinuously on \( \text{Isom}^+ (\mathbb{H}^n) \), thus also on \( \Pi_{j=0}^{n-1} \partial_\infty \mathbb{H}^n - D \). This implies of course that it acts properly discontinuously on \( \Pi_{j=0}^{n} \partial_\infty \mathbb{H}^n - D \). Thus, \( \Gamma \)-orbits are closed. □

3. Degeneration.

3.1. Efficient fundamental cycles. For a closed hyperbolic manifold \( M \), we know that \( \| M \| = \frac{1}{n} \text{Vol} (M) \). This means that, for any \( \epsilon > 0 \), there is some fundamental cycle \( c_\epsilon \) satisfying \( \| c_\epsilon \| \leq \| M \| + \frac{\epsilon}{n} \). By 2.5.1 and 2.5.2, we can choose \( c_\epsilon \) to be an alternating chain consisting of nondegenerate straight simplices, without increasing the \( l^1 \)-norm. To speak about limits of sequences of \( c_\epsilon \), one has to regard them as elements of some locally compact space, namely the space of signed Radon measures on \( SS_n (M) = \Gamma \backslash \left( \Pi_{j=0}^{n} \partial_\infty \mathbb{H}^n - D \right) \) with the weak-\( * \)-topology, as in 2.5.3.

Jungreis results from [11], for closed hyperbolic manifolds of dimension \( \geq 3 \), can be rephrased as follows:

- Any sequence of \( c_\epsilon \) as above, with \( \epsilon \to 0 \), converges,
- the limit is a signed measure \( \mu \), which is supported on the set of regular ideal simplices (to be identified with \( \text{Isom} (\mathbb{H}^n) \)), and
- up to a multiplicative factor one has \( \mu = \mu^+ - \mu^- \) with \( \mu^+ \) the Haar measure on \( \text{Isom}^+ (\mathbb{H}^n) \) and \( \mu^- = r^* \mu^+ \) for an arbitrary orientation reversing \( r \in \text{Isom} (\mathbb{H}^n) \).

The aim of this chapter is to generalize these results to finite-volume hyperbolic manifolds. For these cusped hyperbolic manifolds, there arises a technical problem: We wish to consider chains representing the relative fundamental class of a manifold with boundary, but we have a hyperbolic metric (and a notion of straightening) only on the interior. In the following, we will get around this problem and analyse the possible limits:

Let \( M \) be a compact \( n \)-manifold with boundary \( \partial M \) such that \( \text{int} (M) \) carries a hyperbolic metric of finite volume. With respect to this hyperbolic metric, denote \( M_{[a,b]} := \{ x \in \text{int} (M) : a \leq \text{inj} (x) \leq b \} \). It is a well-known consequence of the Margulis lemma ([2], D.3.12.) that, for sufficiently small \( \epsilon > 0 \), the ‘\( \epsilon \)-thin part’ \( M_{[\epsilon,\infty]} \) is a product neighborhood of \( \partial M \), i.e., homeomorphic to \( \partial M \times [0, \infty) \). Thus, one has a retraction \( r_\epsilon \) from \( M \) to the ‘\( \epsilon \)-thick part’ \( M_{[\epsilon,\infty]} \) which induces a homeomorphism of pairs.
\( r_\epsilon : (M, \partial M) \to (M_{[\epsilon, \infty]}, \partial M_{[\epsilon, \infty]}) \) and, thus, an isomorphism

\[ r_{\epsilon*} : H_* (M, \partial M) \to H_* (M_{[\epsilon, \infty]}, \partial M_{[\epsilon, \infty]}) \].

(This applies to all \( \epsilon < \epsilon_0 \), where \( \epsilon_0 \) depends on \( M \).)

It should be noted that \( M_{[0, \epsilon]} \) is convex and that one has the isomorphism

\[ \text{algvol} : H_n (M, M_{[0, \epsilon]}; \mathbb{R}) \to \mathbb{R}. \]

Convexity of \( M_{[0, \epsilon]} \) implies that the straightening homomorphism

\[ \text{Str} : C_* (\text{int} (M), M_{[0, \epsilon]}) \to C_* (\text{int} (M), M_{[0, \epsilon]}), \]

is well-defined and induces an isomorphism in relative homology. Moreover, there is the inclusion

\[ \text{exc} : C_* (M_{[\epsilon, \infty]}, \partial M_{[\epsilon, \infty]}) \to C_* (\text{int} (M), M_{[0, \epsilon]}), \]

which induces an isomorphism in homology by the excision theorem. In conclusion,

\[ \text{Str} (\text{exc} (r_{\epsilon*} \cdot)) : C_n (M, \partial M; \mathbb{R}) \to C_n (\text{int} (M), M_{[0, \epsilon]}; \mathbb{R}) \]

induces an isomorphism in homology and does not increase \( l^1 \)-norms.

Let, for \( \epsilon < \epsilon_0 \), \( c_\epsilon \in C_n (M, \partial M; \mathbb{R}) \) be some relative fundamental cycle satisfying

\[ \|c_\epsilon\| \leq \|M, \partial M\| + \frac{\epsilon}{V_n}. \]

By the above arguments, we may replace \( c_\epsilon \) by

\[ \text{Str} (\text{exc} (r_{\epsilon*} c_\epsilon)) \in C_n (\text{int} (M), M_{[0, \epsilon]}; \mathbb{R}) \]

without increasing the \( l^1 \)-norm. Abusing notation, we will continue to denote this new relative cycle by \( c_\epsilon \).

**Definition 3.1.** A signed measure \( \mu \) on \( SS_n (M) \) is called an efficient fundamental chain if there exists a sequence of \( \epsilon \) with \( \epsilon \to 0 \) and a sequence of \( c_\epsilon \in C_n (M, \partial M; \mathbb{R}) \) representing the relative fundamental class \([M, \partial M]\), which are alternating chains invoking only nondegenerate simplices and which satisfy \( \|c_\epsilon\| \leq \|M, \partial M\| + \frac{\epsilon}{V_n} \), such that the sequence \( \text{Str} (\text{exc} (r_{\epsilon*} c_\epsilon)) \in C_n (\text{int} (M), M_{[0, \epsilon]}; \mathbb{R}) \) converges to \( \mu \) in the weak-\( * \)-topology of \( \mathcal{M} (SS_n (M)) \), the space of signed measures on the space of straight nondegenerate simplices.

**Lemma 3.2.** Assume that \( M \) is a manifold of dimension \( n \geq 3 \), such that \( \text{int} (M) \) admits a hyperbolic metric of finite volume. Then there is at least one efficient fundamental chain.
Proof. Considering some sequence of \( c_\epsilon \) with \( \epsilon \to 0 \), we may by Lemma 2.5 assume that the support of the \( c_\epsilon \) consists of only straight nondegenerate simplices. \( \text{Str} \ (\text{exc} (r_\epsilon c_\epsilon)) \) may be regarded as a sequence of signed measures on the locally compact metric space \( SS_n (M) \), see 2.5.3. The sequence \( c_\epsilon \) is bounded by its definition and, hence, Lemmas 2.2 and 2.6 guarantee the existence of a weak-*-accumulation point \( \mu \). (The condition \( n \geq 3 \) is needed to apply Lemma 2.6.)

We recall that excision and straightening, as well as the homeomorphism \( r_\epsilon \) induce isomorphisms in relative homology. Hence, any new \( c_\epsilon \) represents the relative fundamental class in \( H_n (\text{int} (M), M_{[0, \epsilon]} ; \mathbb{R}) \). As a special case of Lemma 2.4 we have:

**Lemma 3.3.** Let \( c_\epsilon \) be a representative of the relative fundamental class \( [\text{int} (M), M_{[0, \epsilon]}] \). Then \( \Phi_x (c_\epsilon) = 1 \) holds for almost all \( x \in M_{[\epsilon, \infty]} \).

**Definition 3.4.** For a hyperbolic manifold \( M \), let \( S_\delta \subset SS_n (M) \) be the set of nondegenerate straight simplices \( \sigma \in M \) with \( \text{vol} (\sigma) < V_n - \delta \).

**Lemma 3.5.** An efficient fundamental chain \( \mu \) is supported on \( SS_n (M) - S_0 \), i.e., on the set of straight simplices of volume \( V_n \).

**Proof.** It suffices to show that \( \mu (S_\delta) = 0 \) holds for any \( \delta > 0 \). By Lemma 2.2, (ii), and openness of \( S_\delta \), this follows if we can prove \( \lim_{\epsilon \to 0} c_\epsilon (S_\delta) = 0 \) for any \( \delta > 0 \). Here, \( c_\epsilon = \sum_{j=1}^r a_j \sigma_j \) is the sequence from Definition 3.1.

From Lemma 3.3, we conclude \( \int_M \Phi_x (c_\epsilon) \ d\text{vol} (x) \geq \text{Vol} (M_{[\epsilon, \infty]}) \). But \( \int_M \Phi_x (c_\epsilon) \ d\text{vol} (x) = \sum_{j=1}^r a_j \int_M \Phi_x (\sigma_j) \ d\text{vol} (x) = \sum_{j=1}^r a_j \text{algvol} (\sigma_j) \), where \( \text{algvol}(\sigma_j) \) is \( \text{Vol}(\sigma_j) \) with a sign according to orientation. As a consequence:

\[
\sum | a_j | \text{Vol} (\sigma_j) \geq \text{Vol} (M_{[\epsilon, \infty]}) .
\]

On the other hand, we want \( c_\epsilon = \sum_{j=1}^r a_j \sigma_j \) to satisfy \( V_n \sum | a_j | \leq \text{Vol} (M) + \epsilon \). Subtracting the two inequalities yields

\[
\sum | a_j | (V_n - \text{Vol} (\sigma_j)) \leq \epsilon + \text{Vol} (M_{[0, \epsilon]}) .
\]

We get

\[
\epsilon + \text{Vol} (M_{[0, \epsilon]}) \\
\geq \sum | a_j | (V_n - \text{Vol} (\sigma_j)) \\
= \sum_{j: \text{Vol}(\sigma_j) \geq V_n - \delta} | a_j | (V_n - \text{Vol} (\sigma_j)) \\
+ \sum_{j: \text{Vol}(\sigma_j) < V_n - \delta} | a_j | (V_n - \text{Vol} (\sigma_j))
\]
Let \( \mathbf{E} \text{fficient fundamental chains} \) denote by \( \mathbf{D} \text{enote by} \) \( \mathbf{I} \text{n the case of closed manifolds, this lemma is, of course, an im-
mediate consequence of the fact that} \). Choose a continuous \( f \) satisfying \( \|c_\epsilon^+\| \leq \|c_\epsilon\| \) and is 1 on the complement of some \( \delta \). But clearly, \( \|c_\epsilon\| \geq ||M, \partial M|| \) implies that one of \( \lim_{\epsilon \to 0} c_\epsilon^\pm (SS_n (M)) \) must be at least \( \frac{1}{2} ||M, \partial M|| \), thus positive.

**Lemma 3.6.** Let \( \mu \) be an efficient fundamental chain. Then \( \mu \neq 0 \).

**Proof.** Choose a continuous \( f : SS_n (M) \to [0,1] \), which vanishes on some \( S_\delta \) and is 1 on the complement of some \( S_\delta \). As \( f \) is compactly supported, we have \( \mu (f) = \lim_{\epsilon \to 0} c_\epsilon (f) \). (Here we use that we are admitting ideal simplices: Otherwise the support of \( f \) would not be compact.)

Now using \( \lim_{\epsilon \to 0} c_\epsilon (S_\delta) = 0 \) we have

\[
\mu^\pm (f) = \lim_{\epsilon \to 0} c_\epsilon^\pm (f) \geq \lim_{\epsilon \to 0} c_\epsilon^\pm (SS_n (M)) - c_\epsilon^\pm (S_\delta) = \lim_{\epsilon \to 0} c_\epsilon^\pm (SS_n (M)).
\]

But \( c_\epsilon^+ (SS_n (M)) + c_\epsilon^- (SS_n (M)) = ||c_\epsilon^+|| + ||c_\epsilon^-|| = ||c_\epsilon|| \geq ||M, \partial M|| \) implies that one of \( \lim_{\epsilon \to 0} c_\epsilon^\pm (SS_n (M)) \) must be at least \( \frac{1}{2} ||M, \partial M|| \), thus positive. \( \square \)

**Lemma 3.7.** Efficient fundamental chains \( \mu \) are cycles, i.e., \( (\partial \mu)^+ = (\partial \mu)^- = 0 \).

**Proof.** Denote by \( T_i^\epsilon (M) \) the set of \( (\text{possibly ideal}) i \)-simplices intersecting \( M_{[\epsilon, \infty)} \). For all \( \delta < \epsilon \), we get by the convexity of \( M_{[0, \delta]} \subset M_{[0, \epsilon]} \):

\[
B \subset T_{\epsilon}^{n-1} (M) \text{ measurable} \Rightarrow \partial c_\delta^\pm (B) = 0.
\]

When \( \partial \mu^\pm \) is a weak-* accumulation point of a sequence \( \partial c_\delta^\pm \), we conclude \( \partial \mu^\pm (B) = 0 \) for all measurable sets \( B \) contained in some \( T_{\epsilon}^{n-1} (M) \) by Part (ii) of Lemma 2.2, since we may consider them as subsets of an open set still contained in some slightly larger \( T_{\epsilon}^{n-1} (M) \).

But clearly, \( \cup_{k=1}^\infty T_{\frac{1}{k}}^{n-1} (M) \) is the set of all \( (\text{even ideal}) (n-1) \)-simplices, hence the claim of the lemma. \( \square \)

**Remark.** In the case of closed manifolds, this lemma is, of course, an immediate consequence of the fact that \( ||\partial|| \leq n + 1 \).
3.2. Invariance under ideal reflection group. Since we have an ordering of the vertices of a simplex $\Delta$, we can speak of the $i$-th face of $\Delta$, the codimension 1-face not containing the $i$-th vertex.

Definition 3.8. Fix a regular ideal simplex $\Delta_0 \subset \mathbb{H}^n$ and, for $i = 0, \ldots, n$, let $r_i$ be the reflection in the $i$-th face of $\Delta_0$. Let $R \subset \text{Isom}(\mathbb{H}^n)$ be the subgroup generated by $r_0, \ldots, r_n$ and let $R^+ = R \cap \text{Isom}^+(\mathbb{H}^n)$.

We know that $\mu^\pm$ are measure cycles supported on the set of regular ideal simplices. By 2.1, we may consider $\mu^\pm$ as measures on $\Gamma \setminus \text{Isom}(\mathbb{H}^n)$, after fixing some regular ideal simplex $\Delta_0$ in $\mathbb{H}^n$.

We will use the convention that $\gamma \in \text{Isom}(\mathbb{H}^n)$ corresponds to the simplex $\gamma \Delta_0$, i.e., we let $\text{Isom}(\mathbb{H}^n)$, and in particular $\Gamma$, act from the left. It will be important to note that, after this identification, the right-hand action of $R$ corresponds to the following operation on the set of simplices: $r_i$ maps a simplex to the simplex obtained by reflection in the $i$-th face. This is clear from the picture above.

Lemma 3.9. For $n \geq 3$, efficient fundamental chains are invariant under the right-hand action of $R^+$ on $\Gamma \setminus \text{Isom}(\mathbb{H}^n)$.

Note. If $\Delta = g\Delta_0$ for some $g \in \Gamma \setminus \text{Isom}(\mathbb{H}^n)$, then the reflection $s_i$ in the $i$-th face of $\Delta$ maps $\Delta = g\Delta_0$ to $gr_i(\Delta_0)$. In other words, the choice of another reference simplex changes the identification with Isom($\mathbb{H}^n$) by left multiplication with $g \in \text{Isom}(\mathbb{H}^n)$, but does not alter the right-hand action of $R^+$ on Isom($\mathbb{H}^n$). This implies that the truth of Lemma 3.9 is independent of the choice of $\Delta_0$.

Lemma 3.9 follows from:

Lemma 3.10. In dimensions $n \geq 3$, a signed alternating measure $\mu$ on the set of maximal volume simplices is a cycle iff $r_i^*(\mu) = -\mu$ for all $i = 0, \ldots, n$. 
Proof. If \( n \geq 3 \), then for any ordered regular ideal \((n-1)\)-simplex \( \tau \), there
are exactly two ordered regular ideal \( n \)-simplices, \( \tau^+_i \) and \( \tau^-_i \), having \( \tau \) as \( i \)-th
face. (By the way, this is besides Lemma 2.6 and its ‘corollary’ Lemma 3.2
the only point entering the proofs of our theorems which uses \( n \geq 3 \).) We
fix them such that \( \tau^+_i \) is positively oriented. For a measurable set \( B \subset \{ \text{ordered regular ideal } (n-1)\text{-simplices} \} \)
define
\[
B^+_i = \{ \tau^+_i : \tau \in B \} \quad \text{and} \quad B^-_i = \{ \tau^-_i : \tau \in B \}.
\]
Since \( \mu \) is supported on the set of regular ideal \( n \)-simplices, we have that
\[
\partial \mu^+ (B) = \sum_{k=0}^{n} (-1)^k \mu^+ (\partial_k^{-1} (B))
= \sum_{k=0}^{n} (-1)^k (\mu^+ (B^+_k) + \mu^+ (B^-_k)).
\]
We may assume that \( \mu \) is alternating, in particular \( \pi^*_{ik} \mu = (-1)^{i-k} \mu \), where
\( \pi_{ik} \) is induced by the affine map realizing the transposition of the \( i \)-th and
\( k \)-th vertex. \( \pi_{ik} \) maps \( B_i^+ \) to \( B_k^- \) and \( B_i^- \) to \( B_k^- \). Therefore, for any \( i \in \{0, \ldots, n\}, \) we get
\[
\partial \mu^+ (B) = \sum_{k=0}^{n} (-1)^k (-1)^{i-k} (\pi^*_{ik} \mu^+ (B_k^+) + \pi^*_{ik} \mu^+ (B_k^-))
= \sum_{k=0}^{n} (-1)^i (\mu^+ (B_i^+) + \mu (B_i^-))
= (-1)^i (n + 1) (\mu^+ (B_i^+) + \mu^+ (B_i^-)).
\]
In particular \( \partial \mu (B) = 0 \) holds if and only if \( \mu (B_i^+) = -\mu (B_i^-) \) for \( i = 0, \ldots, n \).

The action of \( r_i \) maps \( B_i^+ \) bijectively to \( B_i^- \) and vice versa. This implies
the ‘if’-part of Lemma 3.10.

To get the ‘only if’-part, we use that \( \partial \mu = 0 \) implies that \( \tau^+_i \mu (B_i^+) = -\mu (B_i^+) \)
holds, for any set \( B \subset \{ \text{ordered regular ideal } (n-1)\text{-simplices} \} \).
Now let \( C \subset \{ \text{ordered regular ideal } n\text{-simplices} \} \) be an arbitrary set. We
divide \( C = C^+ \cup C^- \), where \( C^+ = \{ \sigma \in C : \sigma \text{ positively oriented } \} \). Consider
\( B := \{ \partial_i \sigma : \sigma \in C^+ \}. \) (\( i \) is arbitrary, e.g., \( i = 0 \).) Then we have \( B_i^+ = C^+ \),
because for any ordered regular ideal \( (n-1)\)-simplex \( \partial_i \sigma \in B, \sigma \) is the unique
positively oriented ordered regular ideal \( n \)-simplex having \( \partial_i \sigma \) as its \( i \)-th face.
Thus \( \mu (C^+) = -\tau^+_i (C^+) \). The same way one gets \( \mu (C^-) = -\tau^+_i \mu (C^-) \), thus
\( \mu (C) = -\tau^+_i \mu (C) \). Since \( C \) was arbitrary, this proves the ‘only if’-part. \( \Box \)

Remark. A different proof of the same fact is given in Lemma 2.2 of [11].
4. Decomposition of efficient fundamental cycles.

If \( n \geq 4 \), then the group generated by reflections in the faces of a regular ideal \( n \)-simplex in \( \mathbb{H}^n \) is dense in \( \text{Isom}(\mathbb{H}^n) \). We get therefore from Lemma 3.9 that efficient fundamental cycles are invariant under the right-hand action of \( \text{Isom}^+(\mathbb{H}^n) \). This implies that they are a multiple of \( \text{Haar} - r^*\text{Haar} \), where \( \text{Haar} \) is the Haar measure on \( \text{Isom}^+(\mathbb{H}^n) \). (It is well-known that all invariant measures on a Lie group are multiples of the Haar measure.)

In the following we will discuss the case \( n = 3 \).

We wish to recall some facts from the ergodic theory of unipotent actions.

The Iwasawa decomposition \( G = KAN \) of \( G = \text{Isom}^+(\mathbb{H}^n) \) is as follows: Fix some \( v_\infty \in \partial_\infty \mathbb{H}^n \) and some \( p \in \mathbb{H}^n \). Then we may take \( K \) to be the group of isometries fixing \( p \), \( A \) the group of translations along the geodesic through \( p \) and \( v_\infty \), and \( N \) the group of translations along the horosphere through \( p \) and \( v_\infty \).

We will consider the natural right-hand action of \( N \) on \( G = KAN \).

The next lemma follows from [5]. It is nowadays a special case of the Ragnanathan conjecture, which was proved by Ratner.

**Lemma 4.1.** Let \( G = KAN \) be the Iwasawa decomposition of a simple Lie group of \( \mathbb{R} \)-rank 1, and \( \Gamma \subset G \) a discrete subgroup of finite covolume. If \( \mu \) is a finite \( N \)-invariant ergodic measure on \( \Gamma \backslash G \), then \( \mu \) is either a multiple of the Haar measure or it is supported on a compact \( N \)-orbit.

The following lemma is a straightforward generalisation of Theorem 4.4. in [6]:

**Lemma 4.2.** Let \( G = KAN \) be the Iwasawa decomposition of a simple Lie group of \( \mathbb{R} \)-rank 1, and \( \Gamma \subset G \) a discrete subgroup of finite covolume. Let \( N' \subset N \) be a closed subgroup such that \( N/N' \) is compact. Then any \( N' \)-invariant ergodic measure on \( \Gamma \backslash G \) is either a multiple of the Haar measure or is supported on a compact \( N \)-orbit.

**Proof.** We will use several times the following basic fact: If \( G_1 \) and \( G_2 \) are subgroups of a group \( G \) endowed with a measure \( \mu \), then the left action of \( G_1 \) on \( G/G_2 \) is ergodic if and only if the right action of \( G_2 \) on \( G_1 \backslash G \) is ergodic. (This is known as Moore-equivalence.)

By Moore-equivalence, ergodic measures for the \( N' \)-action on \( \Gamma \backslash G \) correspond to ergodic measures for the action of \( \Gamma \) on \( G/N' \). Consider, therefore, \( \mu \) as a measure on \( G/N' \), ergodic with respect to the \( \Gamma \)-action. Let \( \text{pr} : G/N' \to G/N \) be the projection. Since \( N/N' \) is compact, we have a locally finite measure \( \text{pr}_*\mu \) on \( G/N \) which is easily seen to be ergodic with respect to the \( \Gamma \)-action. By Lemma 4.1 and Moore-equivalence, \( \text{pr}_*\mu \) must either be the Haar measure or correspond to an \( N \)-invariant measure on \( \Gamma \backslash G \) which is determined on a compact orbit \( \Gamma \backslash gN \subset \Gamma \backslash G \).
If $\text{pr}_*\mu = \text{Haar measure}$, it follows easily that $\mu$ is absolutely continuous with respect to the Haar measure and then one gets, from ergodicity of the $\Gamma$-action (Theorem 7 in [14]), that $\mu$ is a multiple of the Haar measure.

In the second case, $\text{pr}_*\mu$ must be supported on the $\Gamma$-orbit of some $gN \in G/N$. Therefore, $\mu$ is supported on the $\Gamma \times N$-orbit of $gN' \in G/N'$. By Moore-equivalence we get a measure supported on the compact $N$-orbit. □

Lemmas 4.1 and 4.2 apply in particular to $G = \text{Isom}^+ (\mathbb{H}^n)$ with the Iwasawa decomposition described above.

Back to the situation of Section 3.2. Let $v$ be an ideal vertex of the reference simplex $\Delta_0$. Let $N_v \subset \text{Isom}^+ (\mathbb{H}^3)$ be the subgroup of parabolic isometries fixing $v$. We may consider $N_v$ as the $N_v$-factor in the Iwasawa decomposition $\text{Isom}^+ (\mathbb{H}^3) = K_v A_v N_v$. (That means we use $v$ and some arbitrary $p \in \mathbb{H}^3$ to construct the Iwasawa decomposition. In the following, we will fix some arbitrary $p \in \mathbb{H}^3$ but consider various $v \in \partial_{\infty} \mathbb{H}^3$, therefore the labelling of the Iwasawa decompositions.)

Instead of $R^+$ defined in Section 3.2, we consider only the subgroup $T_v' \subset R^+ \subset \text{Isom}^+ (\mathbb{H}^3)$ generated by products of even numbers of reflections in those faces of $\Delta_0$ which contain $v$. $\mu$ is, of course, also invariant under the smaller group $T_v'$. In [11] it is shown that $T_v'$ contains a subgroup $T_v$ which is a cocompact subgroup of $N_v$ (if $n = 3$).

The signed measure $\mu$ decomposes as a difference of two (nonnegative) measures $\mu^+$ and $\mu^-$. Both are invariant under the right-hand action of $T_v$. From Lemma 2.1, we get that the probability measures $\mu^\pm$, obtained by rescaling the restrictions of $\mu^\pm$ to $\Gamma \backslash \text{Isom}^+ (\mathbb{H}^3)$, have decomposition maps with respect to the action of $T_v$,

$$\beta^\pm_v : \Gamma \backslash \text{Isom}^+ (\mathbb{H}^3) \rightarrow \mathcal{E}.$$ 

Here, $\mathcal{E}$ is the set of ergodic $T_v$-invariant measures on $\Gamma \backslash \text{Isom}^+ (\mathbb{H}^3)$. From Lemma 4.2, we get that $\mathcal{E}$ consists of Haar (the Haar measure, rescaled to a probability measure) and measures determined on compact $N_v$-orbits. The following lemma is well-known:

**Lemma 4.3.** An orbit $gN_v$ is compact in $\Gamma \backslash \text{Isom} (\mathbb{H}^n)$ iff all simplices $gh\Delta_0$ with $h \in N_v$ have its ideal vertex $g(v)$ in a parabolic fixed point of $\Gamma$.

**Proof.** Parametrise elements of $N_v$ as $u(s), s \in \mathbb{R}^{n-1}$ (identifying a stabilized horosphere with euclidean $(n - 1)$-space). The $N_v$-orbit of $g$ on $\Gamma \backslash \text{Isom} (\mathbb{H}^n)$ is compact if and only if, for all $s \in \mathbb{R}^{n-1}$, one finds $\gamma \in \Gamma$ and $t \in \mathbb{R}$ such that $gu(ts) = \gamma g$. This $\gamma$ is then conjugated to $u(ts)$ and, in particular, is parabolic, i.e., has only one fixed point. The fixed point of $\gamma$ must be $g(v)$, since $\gamma g(v) = gu(ts)(v) = g(v)$.

The other implication is straightforward. □
To summarize, we have the following statement: For any vertex $v$ of the reference simplex $\Delta_0$, the ergodic decomposition of the rescaled $\mu^\pm$ with respect to the right-hand action of $T_v$ uses the Haar measure and measures determined on the set of those simplices $g\Delta_0$ which have the vertex $g(v)$ in a parabolic fixed point of $\Gamma$.

4.1. Manifolds which are not Gieseking-like.

Definition 4.4. A 3-manifold is Gieseking-like if it has a hyperbolic structure $M = \Gamma \backslash \mathbb{H}^3$ of finite volume such that $Q(\omega) \cup \{\infty\} \subset \partial_\infty \mathbb{H}^3$ are parabolic fixed points of $\Gamma$.

Here, we have used the upper half space model of $\mathbb{H}^3$, and identified the ideal boundary with $\mathbb{C} \cup \{\infty\}$. $\omega = \frac{1}{2} + \frac{\sqrt{-3}}{2}$ is the 4th vertex of a regular ideal simplex with vertices $0, 1, \infty$. The condition is, of course, equivalent to the condition that $\Gamma$ is conjugate to a discrete subgroup of $\text{PSL}_2 \mathbb{Q}(\omega)$ after the identification of $\text{Isom}^+ (\mathbb{H}^3)$ with $\text{PSL}_2 \mathbb{C}$. One does not seem to know any example of a Gieseking-like manifold which is not a finite cover of the Gieseking manifold (communicated to the author by Alan Reid, see also [13]).

Theorem 4.5. Let $M$ be a compact manifold of dimension $n \geq 3$ such that $\text{int}(M)$ admits a hyperbolic metric of finite volume. Assume that $M$ is either of dimension $\geq 4$ or that $M$ is of dimension $3$ and is not Gieseking-like.

If $\mu$ is an efficient fundamental cycle on $M$, then $\mu = K (\text{Haar} - r^* \text{Haar})$ for some real number $K$.

Proof. By the first remark of Chapter 4, we may restrict to dimension 3. We have to exclude the existence of a signed measure $\nu$ which is supported on the set of regular ideal simplices with vertices in cusps and which satisfies $r^* \nu = \pm \nu$ for all $r \in R$. However, the existence of such a nontrivial signed measure would imply the existence of an $R$-invariant family $\{\Delta r : r \in R\}$ of simplices with vertices in the cusps of $M = \Gamma \backslash \mathbb{H}^3$. By 2.1, there is $g \in \text{Isom} (\mathbb{H}^3)$ with $\Delta = g\Delta_0$, where $\Delta_0$ is the ideal simplex with vertices $0, 1, \infty, \omega$ in the upper half-space model. We get that all vertices of the form $gv_\infty r$ with $r \in R$ and $v_\infty$ one of $0, 1, \infty, \omega$ must be parabolic fixed points of $\Gamma$. Note that $\{v_\infty r : v_\infty \in \{0, 1, \omega, \infty\}, r \in R\} = Q(\omega) \cup \{\infty\}$. Thus, conjugating $\Gamma$ with $g$ we get a hyperbolic structure with all of $Q(\omega) \cup \{\infty\}$ as parabolic fixed points. \qed

5. Cycles not transversal to geodesic surfaces.

In the last section, we classified efficient fundamental cycles on finite-volume hyperbolic manifolds which are not Gieseking-like. In this chapter, we will see that for arbitrary (possibly Gieseking-like) finite-volume hyperbolic manifolds we can still obtain information which in Chapter 6 will be used to derive glueing inequalities.
Definition 5.1. For a hyperbolic manifold $M$ and a two-sided totally geodesic codimension-1 submanifold $F \subset M$ call:

- $S^i_{\text{cusp}}$ the set of positively oriented ideal $i$-simplices with all vertices in parabolic fixed points of $M$, and
- $S^i_F$, the set of (possibly ideal) positively oriented $i$-simplices that intersect $F$ transversally.

Here, a simplex $\sigma$ is said to intersect $F$ transversally if it intersects both components of any regular neighborhood of $F$.

Lemma 5.2. If $M$ is a hyperbolic manifold and $F$ is a two-sided totally geodesic codimension-1-submanifold, then $S^n_F \cap \{\text{regular ideal simplices}\} \subset \{\text{regular ideal simplices}\}$ has positive Haar measure.

Proof. It is easy to see that $S^n_F \cap \{\text{regular ideal simplices}\}$ is an open, nonempty subset of $\{\text{regular ideal simplices}\}$. □

Theorem 5.3. Let $M$ be a compact manifold of dimension $n \geq 3$ such that int $(M)$ admits a hyperbolic metric of finite volume, and let $F \subset M$ be a closed totally geodesic codimension-1-submanifold.

If $\mu$ is an efficient fundamental cycle (with $\mu^+ \mid _{\Gamma \backslash \text{Isom}^+(\mathbb{H}^n)} \neq 0$), then $\mu^+ (S^n_F) \neq 0$.

Proof. Very roughly, the idea is the following: If $\mu^+ (S^n_F)$ vanishes, then the Haar measure can only give a zero contribution to the ergodic decomposition of $\mu^+$, hence, $\mu^+$ is supported on $S^i_{\text{cusp}}$. In particular, $\mu^+$ vanishes on the set of simplices with boundary faces in $F$, and this will give a contradiction.

Rescale $\mu^+ \mid _{\Gamma \backslash \text{Isom}^+(\mathbb{H}^n)}$ to a probability measure $\overline{\mu}^+$.

Assume for some totally geodesic surface $F$ we had $\overline{\mu}^+ (S^n_F) = 0$.

Let $v$ be a vertex of the reference simplex $\Delta_0$. Using the ergodic decomposition with respect to the $T_v$-action on $\Gamma \backslash G = \Gamma \backslash \text{Isom}^+ (\mathbb{H}^n)$ yields

$$0 = \overline{\mu}^+ (S^n_F) = \int_{\Gamma \backslash G} \beta_v (g) (S^n_F) d\overline{\mu}^+ (g)$$
$$\geq \int_{\gamma \in \Gamma \backslash G; \beta_v (g) = \text{Haar}} \beta_v (g) (S^n_F) d\overline{\mu}^+ (g)$$
$$= \int_{\gamma \in \Gamma \backslash G; \beta_v (g) = \text{Haar}} \text{Haar} (S^n_F) d\overline{\mu}^+ (g)$$
$$= \text{Haar} (S^n_F) \int_{\gamma \in \Gamma \backslash G; \beta_v (g) = \text{Haar}} d\overline{\mu}^+ (g).$$
By Lemma 5.2, \( \text{Haar} (S_F^n) \neq 0 \) and, thus,

\[
\int_{g \in \Gamma \setminus G : \beta_h (g) = \text{Haar}} d\mu^+ (g) = 0.
\]

We will conclude that \( \mu^+ \) is supported on \( S^n_{\text{cusp}} \) by means of Lemma 5.5, which we state separately because it will be of independent use in Chapter 7.

**Definition 5.4.** Let \( \Gamma \subset G = \text{Isom}^+ (\mathbb{H}^n) \) be a cocompact discrete subgroup, \( v \in \partial_\infty \mathbb{H}^n, T_v \subset \text{Isom}^+ (\mathbb{H}^n) \) the subgroup defined in Chapter 4 and \( \beta \) a decomposition map for the right-hand action of \( T_v \), as defined in Chapter 4. Define

\[
H_v = \{ g \in \Gamma \setminus G : \beta_v (g) = \text{Haar} \}.
\]

**Lemma 5.5.** Let \( v_0, \ldots, v_n \) be the vertices of a regular ideal simplex in \( \mathbb{H}^n \) and \( \mu^+ \) a probability measure on \( \Gamma \setminus G := \Gamma \setminus \text{Isom}^+ (\mathbb{H}^n) \), invariant with respect to the right-hand action of \( R^+ \). If \( \mu^+ (H_{v_i}) = 0 \) for \( i = 0, \ldots, n \), then \( \mu^+ \) is supported on \( S^n_{\text{cusp}} \).

**Proof.** Let

\[
A_i = \{ g \in \Gamma \setminus G : gv_i \text{ is cusp of } \Gamma \}
\]

and

\[
B_i = \{ g \in \Gamma \setminus G : \Gamma g N_{v_i} \text{ is compact} \}.
\]

We have

\[
\Gamma \setminus G - S^n_{\text{cusp}} = \Gamma \setminus G - \bigcup_{j=0}^n A_j = \Gamma \setminus G - \bigcup_{j=0}^n B_j = \bigcap_{j=0}^n \Gamma \setminus G - B_j,
\]

where the second equality holds by Lemma 4.3.

If \( e \) is a \( T_{v_i} \)-ergodic measure supported on a compact \( N_{v_i} \)-orbit, then

\[
e (\Gamma \setminus G - B_i) = 0.
\]

Thus (abbreviating \( \beta_g := \beta_{v_i} (g) \)),

\[
\mu^+ (\Gamma \setminus G - B_i)
\]

\[
= \int_{\Gamma \setminus G} \beta_g (\Gamma \setminus G - B_i) \, d\mu^+ (g)
\]

\[
= \int_{H_{v_i}} \beta_g (\Gamma \setminus G - B_i) \, d\mu^+ (g) + \int_{\Gamma \setminus G - H_{v_i}} \beta_g (\Gamma \setminus G - B_i) \, d\mu^+ (g)
\]

\[
= \text{Haar} (\Gamma \setminus G - B_i) \, \mu^+ (H_{v_i}) + \int_{\Gamma \setminus G - H_{v_i}} \beta_g (\Gamma \setminus G - B_i) \, d\mu^+ (g)
\]

\[
= \text{Haar} (\Gamma \setminus G - B_i) \, 0 + \int_{\Gamma \setminus G - H_{v_i}} 0 \, d\mu^+ (g)
\]

\[
= 0
\]
and, therefore,
\[
\pi^+ (\Gamma \setminus G - S_{\text{cusp}}^n) = \pi^+ \left( \bigcup_{i=0}^n \Gamma \setminus G - B_i \right) \\
\leq \sum_{i=0}^n \pi^+ (\Gamma \setminus G - B_i) = 0.
\]

\[\square\]

We are now going to finish the proof of Theorem 5.3:

We know (from the proof of Lemma 3.3) that \(\Phi_x (c_i^+) \geq \Phi_x (c_i) \geq 1\) for all \(x \in M_{[\epsilon, \infty]}\). \(F\) is a closed totally geodesic hypersurface. Therefore \(F \subset M_{[\epsilon, \infty]}\) for sufficiently small \(\epsilon\). We conclude \(\Phi_x (c_i^+) \geq 1\) for all \(x \in F\).

For \(x \in M\) let \(S^n_x\) be the set of straight \(n\)-simplices \(\Delta\) containing \(x\) in their image. If \(x \in F\) is contained in the totally geodesic submanifold \(F\), then \(S^n_x\) is the union of the following two sets of simplices:

– Simplices in \(S^n_x\) which intersect \(F\) transversally, and
– Simplices in \(S^n_x\) which have a vertex in \(F\).

\(\mu^+\) vanishes on the second set, since it is determined on \(S_{\text{cusp}}^n\) and the closed totally geodesic hypersurface \(F\) can not have cusps. Thus, we obtain
\[
\mu^+ (S^n_x) = \mu^+ (S^n_F \cap S^n_x) \leq \mu^+ (S^n_F),
\]
i.e., it suffices to show that \(\mu^+ (S^n_F) > 0\).

For a measure \(\mu^+\) on \(SS_n (M)\), let \(\Phi_x (\sigma) = \int_{SS_n(M)} \Phi_x (\sigma) \, d\mu^+ (\sigma)\), where \(\Phi_x (\sigma)\) is given in Definition 2.3. Weak-*-convergence implies \(\Phi_x (\mu^+) = \lim_{\epsilon \to 0} \Phi_x (c_i^+) \geq 1\).

On the other hand, \(\Phi_x (\sigma) = 0\) if \(\sigma \notin S^n_x\), hence
\[
\Phi_x (\mu^+) = \int_{S^n_x} \Phi_x (\sigma) \, d\mu^+ (\sigma).
\]
If \(\mu^+ (S^n_F) = 0\), then \(\Phi_x (\mu^+) = \int_{S^n_x} \Phi_x (\sigma) \, d\mu^+ (\sigma) = 0\) (regardless whether \(\Phi_x\) is bounded or not), giving a contradiction. Thus \(\mu^+ (S^n_F) > 0\), implying \(\mu^+ (S^n_F) > 0\).

\[\square\]

Remark. If \(\mu^+ \big|_{\text{isom}^+ (H^n)} = 0\), then \(\mu^- \big|_{\text{isom}^+ (H^n)} \neq 0\) because of Lemma 3.6 and Lemma 3.10, and we get with an analogous proof \(\mu^- (S^n_F) \neq 0\).

6. Acylindrical hyperbolic manifolds.

In this chapter we extend Corollary 1 from [11] to manifolds with cusps.

If \(M\) is a hyperbolic manifold, define its convex core to be the smallest closed convex subset \(C_M\) of \(M\) such that the embedding \(C_M \to M\) is a homotopy equivalence. \(C_M\) is either contained in a geodesic codimension 1 submanifold, or it is a codimension 0 submanifold with boundary \(\partial C_M\). (If \(\dim (M) = 3\), then \(\partial C_M\) is, in general, a pleated surface (see [19]), i.e.,
is almost everywhere totally geodesic and is bent along a family of disjoint geodesics.) We say that $M$ has totally geodesic boundary if $C_M$ is homeomorphic to $M$ and $\partial C_M$ is a nonempty totally geodesic submanifold of $M$. Note that we admit that $C_M$ may have cusps. If $M$ is an orientable geometrically finite hyperbolic 3-manifold (with $C_M$ not contained in a geodesic codimension 1 submanifold), then $\partial C_M$ is homeomorphic to the union of all non-torus components of the topological boundary $\partial M$. This applies in particular to any hyperbolic 3-manifold $M$ with totally geodesic boundary $\partial C_M$.

Although hyperbolic structures of infinite volume are not necessarily rigid, it follows easily from Mostow’s rigidity theorem that on a manifold of dimension $\geq 3$, there can be at most one hyperbolic metric $g_0$ admitting totally geodesic boundary, up to isometry. In particular, the volume of the convex core with respect to the metric $g_0$ is a topological invariant. Actually, it was shown in [3] that $g_0$ minimizes the volume of the convex core among all hyperbolic metrics on $M$.

**Lemma 6.1.** Let $M$ be a compact 2-manifold with boundary $\partial M = \partial_0 M \cup \partial_1 M$, such that $M - \partial_0 M$ admits an incomplete hyperbolic metric of finite volume with $\partial_1 M$ totally geodesic and the ends corresponding to $\partial_0 M$ complete. Then

$$\|M, \partial M\| = \frac{1}{V_2} \text{Vol} (M).$$

**Proof.** It is well-known that any (possibly bounded) surface of nonpositive Euler characteristic satisfies $\|M, \partial M\| = -2\chi(M)$. By the Gauss-Bonnet-formula, this is the same as $\frac{1}{2} \text{Vol} (M) = \frac{1}{V_2} \text{Vol} (M)$. □

**Corollary 6.2.** Let $n \geq 3$ and let $M$ be a compact $n$-manifold with boundary $\partial M = \partial_0 M \cup \partial_1 M$, such that $M - \partial_0 M$ admits an incomplete hyperbolic metric of finite volume with $\partial_1 M$ totally geodesic and the ends corresponding to $\partial_0 M$ complete. Then,

$$\|M, \partial M\| > \frac{1}{V_n} \text{Vol} (M).$$

**Proof.** $\|M, \partial M\| \geq \frac{1}{V_n} \text{Vol} (M)$ follows from the familiar argument that fundamental cycles can be straightened to invoke only simplices of volume smaller than $V_n$ or, equivalently, from the trivial inequality $\|DM\| \leq 2\|M, \partial M\|$.

Suppose we had equality $\|M, \partial M\| = \frac{1}{V_n} \text{Vol} (M)$. Glue two differently oriented copies of $M$ via $\text{id}_{\partial M}$ to get $N = DM$. The incomplete metrics can be glued along the totally geodesic boundary and, hence, we have that $N$ is a complete hyperbolic manifold of finite volume $\text{Vol} (N) = 2\text{Vol} (M)$. A relative fundamental cycle for $M$ of norm smaller than $\frac{1}{V_n} \text{Vol} (M) + \frac{1}{2}$ fits together with its reflection to give a relative fundamental cycle $c_\epsilon$ on $N$. 


of $l^1$-norm smaller than $2\frac{1}{\ell_n} \text{Vol}(M) + \epsilon = \frac{1}{\ell_n} \text{Vol}(N) + \epsilon$, but consisting of simplices which do not intersect transversally the totally geodesic surface $\partial M \subset N$, i.e., $c^\pm(S^\epsilon_\partial M) = 0$.

Recall that a representative $c_\epsilon$ of $[N, \partial N]$ was used in Chapter 3 to get a representative $\text{Str}(\text{exc}(r_\epsilon c^\pm_\epsilon))$ of $[\text{int}(N), N_{[\epsilon, \infty]}]$ with (at most) the same $l^1$-norm. Here $r_\epsilon : (N, \partial N) \to (N_{[\epsilon, \infty]}, \partial N_{[\epsilon, \infty]})$ was a homeomorphism, $\text{exc}$ was the canonical inclusion, and $\text{Str}$ means straightening. $c^\pm_\epsilon(S^\epsilon_\partial M) = 0$ implies $\text{Str}(\text{exc}(r_\epsilon c^\pm_\epsilon))(S^\epsilon_\partial M) = 0$, because $r_\epsilon$ can be chosen to be the identity in a neighborhood of the totally geodesic hypersurface $\partial M \subset N$ (which belongs to the thick part if $\epsilon$ is small enough), and because straightening in $N = DM$ preserves the set of simplices not intersecting transversally the totally geodesic surface $\partial M$.

By Lemma 3.2, we have some accumulation point $\mu$ of $\text{Str}(\text{exc}(r_\epsilon c_\epsilon))$ for a sequence of $\epsilon$ tending to zero. Similarly to Lemma 5.2, it is easy to see that $S^\epsilon_\partial M$ is open in $SS_n(N)$. Hence, we can apply Part (ii) of Lemma 2.2 to get $\mu^+(S^\epsilon_\partial M) = 0$. But this contradicts Theorem 5.3. \hfill $\square$

**Theorem 6.3.**

(a) Let $n \geq 3$ and let $M_i, i = 1, 2$ be compact $n$-manifolds with boundaries $\partial M_i = \partial_0 M_i \cup \partial_1 M_i$, such that $M_i - \partial_0 M_i$ admit incomplete hyperbolic metrics of finite volume with $\partial_1 M_i$ totally geodesic and the ends corresponding to $\partial_0 M_i$ complete. If $\partial_1^i M_i \subset \partial_1 M_i$ are nonempty sets of connected components of $\partial_1 M_i$, $f : \partial_1^i M_i \to \partial_1^j M_j$ is an orientation-reversing isometry, and $M = M_1 \cup_f M_2$, then

$$\|M, \partial M\| < \|M_1, \partial M_1\| + \|M_2, \partial M_2\|.$$  

(b) Let $n \geq 3$ and let $M_0$ be a compact $n$-manifold with boundary $\partial M_0 = \partial_0 M_0 \cup \partial_1 M_0$, such that $M_0 - \partial_0 M_0$ admits an incomplete hyperbolic metric of finite volume with $\partial_1 M_0$ totally geodesic and the ends corresponding to $\partial_0 M_0$ complete. If $\partial_1^0 M_0 \subset \partial_1 M_0$ is a nonempty set of connected components of $\partial_1 M_0$, and $f : \partial_1^0 M_0 \to \partial_1^0 M_0$ is an orientation-reversing isometry of $\partial_1^0 M_0$ exchanging the connected components by pairs, then, letting $M = M_0 / f$,

$$\|M, \partial M\| < \|M_0, \partial M_0\|.$$  

**Proof.** (a) The incomplete hyperbolic metrics on $M_1$ and $M_2$ glue together to give a complete hyperbolic metric on $M$ of volume $\text{Vol}(M) = \text{Vol}(M_1) + \text{Vol}(M_2)$. By the Gromov-Thurston theorem, we know that $\|M, \partial M\| = \frac{1}{\ell_n} \text{Vol}(M)$ and, by Corollary 6.2, we have $\|M_i, \partial M_i\| > \frac{1}{\ell_n} \text{Vol}(M_i)$. The claim follows.

The proof of (b) is similar. \hfill $\square$

If $\partial M_i$ has dimension $\geq 3$, any homeomorphism is homotopic to an isometry, by Mostow rigidity. We conclude:
Corollary 6.4. Let \( n \geq 4 \), and let \( M_1, M_2, M_0 \) and \( \partial'_i M_i \) satisfy all assumptions of Theorem 6.3.

If \( f : \partial'_1 M_1 \to \partial'_1 M_2 \) is a homeomorphism, then
\[
\|M_1 \cup_f M_2, \partial (M_1 \cup_f M_2)\| < \|M_1, \partial M_1\| + \|M_2, \partial M_2\|.
\]

If \( f : \partial'_1 M_0 \to \partial'_1 M_0 \) is an orientation-reversing homeomorphism of \( \partial'_1 M_0 \) exchanging the boundary components by pairs, then
\[
\|M_0/f, \partial (M_0/f)\| < \|M_0, \partial M_0\|.
\]


We are going to extend results of [4] to manifolds with cusps (which are not Gieseking-like).

In this chapter, we always consider foliations/laminations of codimension 1.

For more background on the Gromov norm of foliations (and foliations in general), we refer to [4].

Definition 7.1. Let \( M \) be a manifold, possibly with boundary, and \( \mathcal{F} \) a lamination of \( M \). Define
\[
\|M, \partial M\|_{\mathcal{F}} := \inf \left\{ \sum_{i=1}^{r} |a_i| : \sum_{i=1}^{r} a_i \sigma_i \in [M, \partial M], \sigma_i \text{ transverse to } \mathcal{F} \right\}.
\]

Here, a simplex \( \sigma \) is said to be transverse to the lamination \( \mathcal{F} \), if the induced lamination \( \mathcal{F}|_{\sigma} \) is topologically conjugate to the subset of a foliation of \( \sigma \) by level sets of an affine map \( f : \sigma \to \mathbb{R} \).

A typical example for non-transversality of a tetrahedron \( \Delta \) to a lamination \( \mathcal{F} \) is the following: Let \( e_1, e_2, e_3 \) be the three edges of a face \( \tau \subset \Delta \). If \( \mathcal{F}|_{\tau} \) contains three lines which connect respectively \( e_1 \) to \( e_2 \), \( e_2 \) to \( e_3 \) and \( e_3 \) to \( e_1 \), then \( \Delta \) can not be transverse to \( \mathcal{F} \).

Remark. If \( \mathcal{F} \) is not transverse to \( \partial M \) nor contains \( \partial M \) as a leaf, then \( \|M, \partial M\|_{\mathcal{F}} = \infty \). Otherwise the foliated Gromov norm is finite. In the following, we will always assume that either \( \mathcal{F} \) is transverse to \( \partial M \) or that \( \partial M \) is a leaf of \( \mathcal{F} \).

7.1. Laminations with no or one-sided branching. Recall that a codimension 1 lamination \( \mathcal{F} \) of an \( n \)-manifold \( M \) is a decomposition of a closed subset \( \lambda \subset M \) into codimension 1 submanifolds (leaves) so that \( M \) is covered by charts of the form \( I^{n-1} \times I \), the intersection of a leaf with a chart being of the form \( I^{n-1} \times \{*\} \). A lamination \( \mathcal{F} \) of a 3-manifold \( M \) with image \( \lambda \subset M \) is called essential if no leaf is a sphere or a torus bounding a solid torus, \( M - \lambda \) is irreducible, and \( \partial M - \lambda \) is incompressible and end-incompressible in \( M - \lambda \), where the closure of \( M - \lambda \) is taken w.r.t. any
metric (see [8], Ch.1). E.g., if $M$ is a 3-manifold and $\mathcal{F}$ a foliation without Reeb components, this is an essential lamination.

To motivate the following results about the relation between foliated Gromov norm and branching of laminations, we recall a notion from [8]. An order tree is a set $T$ together with a collection $S$ of linearly ordered segments $\sigma$, each having distinct least and greatest elements, $e(\sigma)$ and $i(\sigma)$, respectively, satisfying the following conditions:

- If $\sigma \in S$, then $-\sigma \in S$ (the set $\sigma$ with reversed order),
- any closed subset of $\sigma \in S$ belongs to $S$,
- any $v, w \in T$ can be joined by some $\sigma_1, \ldots, \sigma_k \in S$, i.e., $v = i(\sigma_1)$, $e(\sigma_j) = i(\sigma_{j+1})$ and $w = e(\sigma_k)$,
- any cyclic word $\sigma_0, \ldots, \sigma_{k-1}$ with $e(\sigma_j) = i(\sigma_{j+1})$ for all $j$ and $e(\sigma_{k-1}) = i(\sigma_0)$ becomes trivial after subdividing the $\sigma_i$'s and performing trivial cancellations,
- if $\sigma_1 \cap \sigma_2 = \{i(\sigma_2)\} = \{e(\sigma_1)\}$, then $\sigma_1 \cup \sigma_2 \in S$.

To a codimension 1 lamination $\mathcal{F}$ of $M$, one considers the pull-back lamination $\tilde{\mathcal{F}}$ of $\tilde{M}$ with image $\lambda \subset \tilde{M}$, and one constructs $(T, S)$ as follows: Elements of $T$ are either leaves of $\tilde{\mathcal{F}}$ not contained in $\tilde{M} - \lambda$ or components of $\tilde{M} - \lambda$. For each directed arc $\alpha$ intersecting leaves of $\tilde{\mathcal{F}}$ transversally, having nonempty intersection with at least two leaves and being effective (no two points on the intersection with a leaf can be cancelled by an obvious homotopy), let $\sigma$ be the set of elements (of $T$) intersected by $\alpha$, with the inherited linear order.

According to [8], Prop. 6.10, $(T, S)$ is an order tree if $\mathcal{F}$ is an essential lamination. $\mathcal{F}$ is then called $\mathbb{R}$-covered, one-sided branched, or two-sided branched according to whether the leaf space of $\tilde{\mathcal{F}}$, considered as an order tree, is $\mathbb{R}$, branched in one direction, or branched in both directions.

**Lemma 7.2.** If $\mathcal{F}$ is an $\mathbb{R}$-covered or one-sided branched essential lamination on a 3-manifold $M$ such that $\mathcal{F}|_{\partial M}$ is $\mathbb{R}$-covered, then $\|M, \partial M\| = \|M, \partial M\|_{\mathcal{F}}$.

**Proof.** This is shown in Theorems 2.2.10 and 2.5.9 of [4], assuming that $M$ is closed. However, the proof works also for manifolds with boundary.

Indeed, since $\partial M$ is either transverse to $\mathcal{F}$ or is a leaf of $\mathcal{F}$, the straightening defined in Lemma 2.2.8 of [4], for chains with vertices on comparable leaves, preserves $C_*(\partial M)$. This implies, in particular, the claim for $\mathbb{R}$-covered foliations. In the case of one-sided branching (say in positive direction), the argument in 2.5.9 of [4] was then to isotope a chosen lift of the finite singular chain in $\tilde{M}$ in the negative direction until its vertices are on comparable leaves. (This has to be done $\pi_1 M$-equivariantly in the sense that the projection to $M$ stays a relative cycle.) If $\partial M$ is a leaf of $\mathcal{F}$, then
one can leave all vertices on $\partial M$ fixed and only isotope the other vertices. If $\partial M$ is transversal to $\mathcal{F}$, the isotopy can clearly be performed in such a way that vertices on $\partial M$ (which already are on comparable leaves since $\mathcal{F}|_{\partial M}$ is $\mathbb{R}$-covered) are isotoped inside $\partial M$.

Hence, in any case, the straightening maps $C_*(\partial M)$ to $C_*(\partial M)$ and, by the five lemma, it induces the identity map in relative homology. Thus, it maps relative fundamental cycles to relative fundamental cycles transversal to $\mathcal{F}$, not increasing the $l^1$-norm. □

7.2. Asymptotically separated laminations.

Definition 7.3. Let $\text{int}(M^n)$ be hyperbolic and let $\mathcal{F}$ be a lamination of $M$. Let $\tilde{\mathcal{F}}|_{\text{int}(\tilde{M})}$ be the pull-back lamination of $\mathbb{H}^n$. $\mathcal{F}$ is called asymptotically separated if, for some leaf $F \in \tilde{\mathcal{F}}$, there are two geodesic $(n - 1)$-planes on distinct sides of $F$.

We include a proof of the following lemma, implicit in [4], for lack of an explicit reference and because it might help to understand the idea behind Theorem 7.5.

Lemma 7.4. If $\mathcal{F}$ is an asymptotically separated lamination of a finite-volume hyperbolic manifold $M = \Gamma \backslash \mathbb{H}^n$, then $\mathcal{F}$ is two-sided branched.

Proof. Let $F$ be a leaf of $\tilde{\mathcal{F}}$ such that there exist geodesic $(n - 1)$-planes on distinct sides of $F$. These two planes cut out two half-spaces $U_1$ and $U_2$ on distinct sides of $F \subset \mathbb{H}^n$. Let $H$ be the complement of $U_1$ and let $H_1$ and $H_2$ be disjoint half-spaces in $U_2$. Note that $F \subset H$.

If $\Gamma \backslash \mathbb{H}^n$ has finite volume, then it is well-known that the $\Gamma$-orbits on the space of pairs of distinct points in $\partial_\infty \mathbb{H}^n$ are dense.
In particular, fixing some arbitrary $\gamma \in \Gamma$ with fixed points $p_1, p_2$, one finds conjugates of $\gamma$ in $\Gamma$, such that their fixed points come arbitrarily close to two given points $q_1 \neq q_2$ in $\partial_\infty \mathbb{H}^n$. (Namely, conjugate with elements of $\Gamma$ which map $p_1$ close to $q_1$ and $p_2$ close to $q_2$.)

It follows that, in a finite-covolume subgroup $\Gamma \subset \text{Isom}^+(\mathbb{H}^n)$, to any given disk $D \subset \partial_\infty \mathbb{H}^n$, one finds loxodromic isometries with both fixed points in $\partial_\infty \mathbb{H}$, after sufficiently many iterations, inside any neighborhood of the attracting fixed point. Hence, replacing $\alpha_1$ and $\alpha_2$ by sufficiently large powers, we get that $\alpha_1(F) \subset H_1$ and $\alpha_2(F) \subset H_2$.

Since $\tilde{F}$ is $\Gamma$-invariant, we have found incomparable leaves $\alpha_1(F)$ and $\alpha_2(F)$ above $F$ and, by analogous arguments, we also get incomparable leaves below $F$. □

**Remark.** A conjecture of Fenley would imply that a foliation of a finite-volume hyperbolic 3-manifold $(M)$ is two-sided branched if and only if it is asymptotically separated, see the discussion in Chapter 2.5. of [4]. Namely, Calegari proves that a two-sided branched foliation (on an arbitrary hyperbolic manifold) either is asymptotically separated or the leaves have as limit sets all of $\partial_\infty \mathbb{H}^3$. On the other hand, Fenley conjectures that for foliations of finite-volume hyperbolic manifolds (which are transversal to the boundary $\partial M$), the limit set of a leaf can be all of $\partial_\infty \mathbb{H}^3$ only if $\mathcal{F}$ is $\mathbb{R}$-covered.

The following Theorem 7.5 extends Theorem 2.4.5 in [4] to the cusped case.

**Theorem 7.5.** If the interior of $M$ is a hyperbolic $n$-manifold of finite volume which is not Gieseking-like, $n \geq 3$, and if $\mathcal{F}$ is an asymptotically separated lamination, then

$$\|M, \partial M\| < \|M, \partial M\|_\mathcal{F}.$$  

**Proof.** We want to give an outline of the proof. We will show that there exist three half-spaces $D_0, D_1, D_2$ such that the following holds: Whenever a straight simplex has at least one vertex in each of $D_0, D_1, D_2$, it can not be transverse to $\mathcal{F}$. Assuming $\|M, \partial M\|_\mathcal{F} = \|M, \partial M\|$, we would have an efficient fundamental cycle $\mu$ which actually comes from a sequence of fundamental cycles transverse to $\mathcal{F}$. If $M$ is closed, one gets easily that $\mu^\pm$ have to vanish on the set of those ideal simplices with at least one vertex in each of $\partial_\infty D_0, \partial_\infty D_1, \partial_\infty D_2$. If $M$ has cusps, we still get the slightly weaker statement that $\mu^\pm$ have to vanish on the set of those ideal simplices with at least one vertex in each of $\partial_\infty D_0 - P, \partial_\infty D_1 - P, \partial_\infty D_2 - P$, where $P$ is the
set of parabolic fixed points of \( \Gamma \). We can then use our knowledge of \( \mu \) to derive a contradiction.

Let \( F \) be a leaf which has the property in the definition of “asymptotically separated”, i.e., there are planes, and hence half-spaces \( U_1 \) and \( U_2 \), in disjoint components of \( \mathbb{H}^n - F \). We choose in \( U_2 \) two smaller disjoint half-spaces \( H_1 \) and \( H_2 \). Like in the proof of Lemma 7.4, one finds loxodromic isometries \( \alpha_1 \in \Gamma \) with both fixed points in \( H_1 \) and \( \alpha_2 \in \Gamma \) with both fixed points in \( H_2 \). Replacing, if necessary, \( \alpha_1 \) and \( \alpha_2 \) by sufficiently large powers, we arrange that \( \alpha_1(U_1) \subset H_1 \) and \( \alpha_2(U_1) \subset H_2 \), and that \( F, \alpha_1(F), \alpha_2(F) \) are disjoint. Letting \( D_0 = U_2, D_1 = \alpha_1(U_1) \), and \( D_2 = \alpha_2(U_1) \), the remark after Definition 7.1 tells us that there is no tetrahedron transverse to \( \tilde{F} \) with one vertex in each of \( D_0, D_1 \) and \( D_2 \).

For the convenience of the reader, we first explain the proof for closed manifolds. Assume that we have straight fundamental cycles \( c_\epsilon \), transverse to \( F \), with \( \|c_\epsilon\| < \|M\| + \frac{\epsilon}{n} \), and that \( \mu \) is the weak-\( \ast \)-limit of \( c_\epsilon \).

Denoting by \( S_{D_0,D_1,D_2} \) the open set of straight (possibly ideal) simplices with one vertex in each of \( D_0, D_1 \) and \( D_2 \), we have just seen that transversality to \( F \) implies \( c_\pm^+(S_{D_0,D_1,D_2}) = 0 \). This implies \( \mu^\pm(S_{D_0,D_1,D_2}) = 0 \), contradicting the fact that \( \mu^+ \) is the Haar measure. (A similar argument was given by Calegari in 2.4.5 of [4].)

Now we are going to consider hyperbolic manifolds of finite volume. Let \( P \subset \partial_\infty \mathbb{H}^n \) be the parabolic fixed points of \( \Gamma \), and \( H_\epsilon = p^{-1}(M_{[0,\epsilon]} \subset \mathbb{H}^n \) the preimage of the \( \epsilon \)-thin part. It is the union of horoballs centered at the points of \( P \). For \( \delta \) sufficiently small, \( D_0 - \overline{\partial_\delta D_0}, D_1 - \overline{\partial_\delta D_1} \) and \( D_2 - \overline{\partial_\delta D_2} \) are nonempty. Fix such a \( \delta \). Let

\[
S_{D_0,D_1,D_2} = \left\{ \text{simplices having vertices } v_0 \in D_0 - \overline{\partial_\delta D_0}, \quad \begin{array}{c} v_1 \in D_1 - \overline{\partial_\delta D_1}, \quad v_2 \in D_2 - \overline{\partial_\delta D_2} \end{array} \right\},
\]

where we admit ideal simplices.

We have seen that simplices in \( S_{D_0,D_1,D_2} \) are not transversal to \( \tilde{F} \). Moreover, we define

\[
\text{Str} \left( S_{D_0,D_1,D_2} \right) := \{ \text{Str} \left( \sigma \right) : \sigma \in S_{D_0,D_1,D_2} \}
\]

and

\[
U := \left\{ \text{pos. or. regular ideal simplices } (v_0, \ldots, v_n) : v_i \in \partial_\infty D_i - P \quad \text{for } i = 0, 1, 2 \right\}.
\]

Now suppose we had the equality \( \|M, \partial M\| = \|M, \partial M\|_F \). We will stick to the notations of Chapter 3. Take some transverse relative fundamental
cycle $c_{\epsilon}$ of norm smaller than $\|M, \partial M\| + \frac{1}{V_{\epsilon}}$ and make it, via the homeomorphism $r_{\epsilon} : (M, \partial M) \to (M_{[\epsilon, \infty]}, \partial M_{[\epsilon, \infty]})$, to a relative fundamental cycle $r_{\epsilon*}(c_{\epsilon})$ of the $\epsilon$-thick part, which is transverse to the foliation $r_{\epsilon} (\mathcal{F})$. We may arrange $r_{\epsilon}$ to be the identity on the $\epsilon'$-thick part for $\epsilon'$ slightly larger than $\epsilon$. Then, the lift of $r_{\epsilon*}(c_{\epsilon})$ to $\mathbb{H}^n$ is transverse to $\tilde{\mathcal{F}}$ outside $H_{\epsilon'}$. By choosing $\epsilon$ and $\epsilon'$ sufficiently small, one may make this exceptional set $H_{\epsilon'}$ as small as one wishes.

Decompose $S_{D_0, D_1, D_2}$ as a countable union $S_{D_0, D_1, D_2} = \bigcup_{i=1}^{\infty} V_i$, where $V_i \subset S_{D_0, D_1, D_2}$ is the open subset of (possibly ideal) positively oriented simplices $\sigma \in S_{D_0, D_1, D_2}$ satisfying $\sigma \cap H_{\frac{1}{2}} = \emptyset$. (The union is all of $S_{D_0, D_1, D_2}$ because any ideal or non-ideal simplex with vertices outside $H_{\delta}$ must remain outside some $H_{\frac{1}{2}}$ for sufficiently large $i$.) Let $W_i = \text{Str} (V_i) = \{ \text{str} (\sigma) : \sigma \in V_i \}$. For $\epsilon$ sufficiently small (such that we can choose $\epsilon' < \frac{1}{7}$), we have $r_{\epsilon*}(c_{\epsilon}^\pm)(V_i) = 0$, since $r_{\epsilon*}(c_{\epsilon})$ is transverse to $\mathcal{F}$ outside $H_{\frac{1}{2}}$ and $V_i$ consists of simplices which do not intersect $H_{\frac{1}{2}}$ and which are not transverse to $\mathcal{F}$. As a consequence, $\text{Str} (\text{exc} (r_{\epsilon*}(c_{\epsilon})))(W_i) = 0$. If $\mu$ is the weak-* limit of the sequence $\text{Str} (\text{exc} (r_{\epsilon*}(c_{\epsilon})))$ with $\epsilon \to 0$, we get $\mu^\pm(W_i) = 0$ by openness of $W_i$ and Part (ii) of Lemma 2.2.

$W = \text{Str} (S_{D_0, D_1, D_2}) = \{ \text{Str} (\sigma) : \sigma \in S_{D_0, D_1, D_2} \}$ is a countable increasing union $W = \bigcup_{i=1}^{\infty} W_i$. Hence $\mu^\pm(W) = 0$. $U \subset W$ implies $\mu^\pm(U) = 0$.

On the other hand, $U$ has nontrivial Haar measure. Indeed, $\text{Isom}^+(\mathbb{H}^n)$ corresponds to ordered $n$-tuples of points in $\partial_\infty \mathbb{H}^n$, because any such ordered $n$-tuple is the set of first $n$ vertices for some unique positively oriented regular ideal simplex. Hence, the set of positive regular ideal simplices, with $v_i \in \partial_\infty D_i$ for $i = 0, 1, 2$, corresponds to an open set of positive Haar measure in $\text{Isom}^+(\mathbb{H}^n)$. Clearly, a discrete subgroup of $\text{Isom}^+(\mathbb{H}^n)$ has a countable number of parabolic fixed points. Thus, $U$ has positive Haar measure.

Recall the notation from Chapter 4: $v \in \partial_\infty \mathbb{H}^n$ is an arbitrary vertex of the reference simplex $\Delta_0$ and $\beta_v(g)$ is the ergodic component of $g \in \Gamma \setminus \text{Isom}^+(\mathbb{H}^n)$ with respect to the $T_v$-action. We define

$$H_v = \{ g \in \Gamma \setminus G : \beta_v(g) = \text{Haar} \}.$$  

$\text{Haar}(U) \neq 0$ implies $\mu^\pm(H_v) = 0$. Indeed, from Lemma 4.2 and Lemma 4.3 we know that the complement of $H_v$ in the set of regular ideal simplices is the set of simplices $g \Delta_0$ with the vertex $g(v)$ in a parabolic fixed point of $\Gamma$. $\Gamma$ has a countable number of parabolic fixed points and, therefore, this complement is a set of trivial Haar measure. Thus,

$$\text{Haar}(U \cap H_v) = \text{Haar}(U) > 0.$$
and we apply the ergodic decomposition from 2.2 to get
\[ 0 = \mu^\pm (U \cap H_v) = \text{Haar} (U \cap H_v) \mu^\pm (H_v) \]
which implies
\[ \mu^\pm (H_v) = 0. \]

This discussion applies to all vertices \( v_i \) of the reference simplex \( \Delta_0 \). By Lemma 5.5, we can conclude that \( \mu^\pm \) are determined on \( S_{\text{cusp}}^n \).

In particular, since \( \mu \neq 0 \), there necessarily are regular simplices with all vertices in parabolic fixed points. By Lemma 3.10, \( \mu \) is invariant up to sign under the right-hand action of the regular ideal reflection group \( R \) defined in Section 3.2. Hence, there must even be an \( R \)-invariant family of regular ideal simplices with vertices in parabolic fixed points. This is only possible in dimension 3 and, after conjugating with an isometry, \( Q(\omega) \cup \{\infty\} \) must be parabolic fixed points of \( \Gamma \).

A surface \( F \) in a 3-manifold \( M \) is called a virtual fiber if there is some finite cover \( p : \overline{M} \to M \) and some fibration \( \overline{F} \to \overline{M} \to S^1 \) with \( \overline{F} \) isotopic to \( p^{-1}(F) \).

A theorem of Thurston and Bonahon asserts that a properly embedded compact \( \pi_1 \)-injective surface in a finite-volume hyperbolic 3-manifold is either quasigeodesic or a virtual fiber. Since quasigeodesic surfaces are asymptotically separated, one gets analogously to [4], Theorem 4.1.4:

**Corollary 7.6.** If the interior of \( M \) is a hyperbolic 3-manifold of finite volume which is not Gieseking-like and \( F \subset M \) is a properly embedded compact \( \pi_1 \)-injective surface, then \( F \) is a virtual fiber if and only if \( \|M, \partial M\|_F = \|M, \partial M\| \).

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**References**


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THE VECTOR BUNDLE DECOMPOSITION

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In this paper, we studied the real vector bundle decomposition problem. We first give a general decomposition result which relates a given vector bundle to some cohomology classes with local coefficients in the homotopy group of a Grassmann manifold; it is those classes that obstruct the decomposition. Those classes are natural with respect to the induced vector bundle by a map. For some special decompositions, we gave a relationship between those classes and the well-known characteristic classes such as Stiefel-Whitney classes and Chern classes. We determined the local coefficients in the cohomology group which contain the decomposition obstruction classes. We find applications in the study of subbundles of low codimension. In particular, codimension 1 decomposition classes are investigated in which we find that one of the two decomposition classes for the universal bundle over $BO(2n + 1)$ is in $H^{2n+1}(BO(2n + 1), Z)$. This result gives rise to a new geometric interpretation for the order two elements in the integral cohomology group in odd dimension. We further make use of the cellular structure of the classifying space $BO(n)$ to see the ‘local’ structure for the restriction of the universal bundle to each cell. In this way, we can construct the obstruction classes for the codimension 1 vector bundle decomposition. We gave an example to calculate the decomposition obstruction for the tangent bundle of $RP^{2n}$, which turns out to be the generator in the cohomology of $RP^{2n}$ with twisted integer coefficients. On the other hand, we exhibit a trivial summand in the tangent bundle for any odd dimensional cobordism classes.

1. Introduction.

The classification for vector bundles is a classical problem which has been studied by many mathematicians. Grothendieck proved that every algebraic vector bundle over $CP^1$ can be decomposed as a direct sum of complex line bundles which gives rise to a complete classification for algebraic vector bundle over $CP^1$. Hirzebruch [6] applied the Riemann-Rock theorem to the vector bundles over $CP^n$ and obtained an integral condition on the Chern classes of the vector bundle. Using the Hirzebruch’s results, Schwarzenberger
[12] gave a partial classification for complex algebraic 2-vector bundles over $CP^2$ and formulated the following conditions for the Chern classes $c_1$ and $c_2$ of a 2-vector bundle over $CP^n$:

$$S : \binom{\delta_1}{k} + \binom{\delta_2}{k} \in Z, \quad 2 \leq k \leq n,$$

where $\delta_1 + \delta_2 = c_1$, $\delta_1\delta_2 = c_2$.


In the real case the literature is sketchy at best. In this paper we shall begin the study of the decomposition of real vector bundles. Unlike complex vector bundles, the decomposition problem for real vector bundles will involve cohomology groups with local coefficients. We give a general decomposition result (Theorem 2.1.5) which relates a given vector bundle to some cohomology classes with local coefficients in the homotopy group of a Grassmann manifold; it is those classes that obstruct the decomposition. Those classes are natural with respect to the induced vector bundle by a map (see 2.1.7). For some special decompositions, we gave a relationship between those classes and the well-known characteristic classes such as Stiefel-Whitney classes and Chern classes (see 2.2.8, 2.2.9 and 2.2.10). We find applications in the study of subbundles of low codimension. In particular, codimension 1 decomposition classes are investigated in 2.2.6 in which we find that one of the two decomposition classes for the universal bundle over $BO(2n+1)$ is in $H^{2n+1}(BO(2n+1), \mathbb{Z})$. This result gives rise to a new geometric interpretation for the order two elements in the integral cohomology group in odd dimension. We further make use of the cellular structure of the classifying space $BO(n)$ to see the ‘local’ structure (see 2.2.11) for the restriction of the universal bundle to each cell. In this way, we can construct the obstruction classes for the codimension 1 vector bundle decomposition. In example 3.1, we calculated the decomposition obstruction for the tangent bundle of $RP^{2n}$, which turns out to be the generator in the cohomology of $RP^{2n}$ with twisted integer coefficients. On the other hand, in Example 3.4, we exhibit a trivial summand in the tangent bundle for any odd dimensional cobordism classes.

Our approach is based on the following considerations. Let $G$ be a compact Lie group and $H$ be a closed subgroup of $G$. Designated by $BG$ and $BH$ the classifying spaces of $G$ and $H$ respectively, classical results on compact Lie groups and their classifying spaces give us an important fibration $BH \to BG$ with fiber $G/H$ [15]. The lifting problem for certain fibrations
between the classifying spaces of classical Lie groups and their closed sub-
groups is an extensively studied problem during the past twenty years [7],
[8], [11] and [9]. We will discuss the following fibration in some detail:

\[ B(O(n) \times O(m)) \longrightarrow BO(n+m). \]

The fibre of the fibration is \( O(n+m)/O(n) \times O(m) = G_{n,m} \), the Grassmann
manifold. Let \( X \) be a CW-complex and \( \xi^n \) be an \( n \)-dimensional vector
bundle over \( X \), then by classification theorem there is a continuous map,
the classifying map of \( \xi^n, f : X \longrightarrow BO(n) \) such that \( \xi^n \approx f^*(\eta_n) \) where \( \eta_n \)
is the canonical \( n \)-dimensional vector bundle over \( BO(n) \). A vector bundle
\( \xi^n \) can be decomposed into a Whitney sum \( \xi^n = \xi^k \oplus \xi^{n-k} \) of two bundles
if and only if the structure group of \( \xi^n \) can be reduced into the subgroup
\( O(k) \times O(n-k) \) which means that the classifying map \( f \) of \( \xi^n \) can be lifted
to the classifying space \( B(O(k) \times O(n-k)) \) up to a homotopy. So the problem
of decomposing a vector bundle is equivalent to the lifting problem of its
classifying map for the fibration \( B(O(k) \times O(n-k)) \longrightarrow BO(n) \). We apply
the obstruction theory to study the corresponding lifting problem for this
fibration.

2. Main results.

2.1. The general decomposition results. In this section we will use
the obstruction theory to consider some problems about vector bundles, in
particular the decomposition problem of vector bundles over a CW-complex.

**Lemma 2.1.1.** Let \( \xi^n \) be an \( m \)-dimensional vector bundle over a paracom-
pact space \( X \), then \( \xi^n \) has a Whitney sum decomposition \( \xi^k \oplus \xi^{n-k} \) if and
only if there exists a commutative diagram up to homotopy:

\[
\begin{array}{ccc}
B(O(k) \times O(m-k)) & \longrightarrow & BO(m) \\
p \downarrow & & \\
X & \longrightarrow & BO(m)
\end{array}
\]

where \( f \) is the classifying map of \( \xi^n \) and \( p \) is the map between the classifying
spaces induced by the inclusion \( O(k) \times O(m-k) \subset O(m) \).

**Proof.** This result can be proved by considering the structure group of a
vector bundle. \( \Box \)

There are different ways to construct the classifying space \( BG \) for a comp-
act Lie-group \( G \). Here I give a geometric way to construct \( B(O(k) \times O(m-k)) \) so that there is a natural fibration
\( p : B(O(k) \times O(m-k)) \longrightarrow BO(m) \)
with fiber \( G_k(R^m) \), the Grassmann. Let \( \eta_m \) be the universal \( m \)-dimensional
section for the fibration

Proof.

Let a pull-back diagram:

\[ G \]

Lemma 2.1.3.

with \( \xi \) case, \( G \) homeomorphism \( h \) \( k \) the space of all \( Y \) where \( Y \) is any \( m \)-dimensional subspace of \( X \), such that if \( f: G \rightarrow X \) is a fibration with fiber \( G_k(R^m) \). The universal bundle over \( G_k(\eta_m) \) is \( p^* (\eta_m) \) which is isomorphic to \( \omega_k \oplus \nu_{m-k} \), whose total spaces are as follows:

\[ E(\omega_k) = \{ ((X,Y),\nu) | \nu \in Y \} \subset G_k(\eta_m) \times R^\infty; \]
\[ E(\nu_{m-k}) = \{ ((X,Y),\nu) | \nu \in X, \nu \bot Y \} \subset G_k(\eta_m) \times R^\infty. \]

Proof. The proof is based on the uniqueness of the classifying space for compact Lie group and the fact that the canonical inclusion \( i: G_k(R^\infty) \times G_{m-k}(R^\infty) \rightarrow G_m(R^\infty) \) factors through \( G_k(\eta_m) \).

By the local triviality of the universal bundle \( \eta_m \), \( G_k(\eta_m) \) is also a local trivial fiber space with fiber \( G_k(R^\infty) \), therefore, the projection \( p \) is a fibration with fiber \( G_k(R^m) \).

Let \( \xi^m \) be any \( m \)-dimensional vector bundle over a CW-complex \( X \) with classifying map \( f: X \rightarrow G_m(R^\infty) \). We define \( G_k(\xi^m) \rightarrow X \), called the Grassmann bundle associated with \( \xi^m \), to be the pull-back of the fibration \( p: G_k(\eta_m) \rightarrow G_m(R^\infty) \) by \( f \). To justify the definition, we need to prove that if \( f \) is homotopic to \( g \), then their pull-backs must be homeomorphism. By the classification theorem, \( f^* (\eta_m) \approx g^* (\eta_m) \approx \xi^m \). Now we can define a homeomorphism \( h^* \) \( f^* (G_k(\eta_m)) \approx g^* (G_k(\eta_m)) \) by

\[ h^*(x,(f(x),Y)) = (x,(h(f(x)),h(Y))) = (x,(g(x),h(Y))) \]

where \( Y \) is any \( k \)-dimensional subspace of \( f(x) \). Geometrically, \( G_k(\xi^m) \) is the space of all \( k \)-dimensional subspaces in the fibers of \( \eta_m \). As a special case, \( G_1(\xi^m) \) is the well-known projective bundle space \( RP(\xi^m) \) associated with \( \xi^m \).

Lemma 2.1.3. The projection \( p_\xi : G_k(\xi^m) \rightarrow X \) is a fibration with fiber \( G_k(R^m) \). \( \xi^m \) has a decomposition \( \xi^m \approx \xi_1^k \oplus \xi_2^{m-k} \) if and only if there is a section for the fibration \( p_\xi \).

Proof. Let \( f: X \rightarrow G_m(R^\infty) \) be the classifying map for \( \xi^m \), then we have a pull-back diagram:

\[
\begin{array}{ccc}
G_k(\xi^m) & \xrightarrow{f^*} & G_k(\eta_m) \\
p_\xi \downarrow & & \downarrow p \\
X & \xrightarrow{f} & G_m(R^\infty).
\end{array}
\]
By 2.1.1 and 2.1.2 $\xi^m \approx \xi_1^k \oplus \xi_2^{m-k}$ if and only if there is a lifting for the fibration $p : G_k(\eta_m) \to G_m(R^\infty)$ which is equivalent to the existence of a section of the fibration $p_\xi : G_k(\xi^m) \to X$. □

Let $p : E \to B$ be a fibration with fiber $F$, and let $E, B$ and $F$ be path-connected CW-spaces. Then there exist fibrations $q_n$ and maps $h_n$ making the diagram

\[ \begin{array}{ccc}
E & \xrightarrow{p} & B \\
\downarrow & & \downarrow q_1 \\
E_1 & \xleftarrow{q_2} & \cdots \xleftarrow{q_{n-1}} & E_n & \xleftarrow{q_n} & \cdots
\end{array} \]

commute, and such that for $n > 1$ (if $n = 1$, $\pi_1(F)$ needs to be abelian):
1. $q_n$ is a fibration with fiber $K(\pi_n(F), n)$, the Eilenberg-Maclane space.
2. $h_n$ is $(n + 1)$-connected.

In the following, we will use the above so-called Postnikov decomposition for a fibration to study $p : G_k(\eta_m) \to G_m(R^\infty)$ or $p_\xi : G_k(\xi^m) \to X$. We start with a typical fibration in the Postnikov decomposition.

**Lemma 2.1.4.** Let $p : E \to B$ be a fibration with fiber $K(\Pi, n)$, where $\Pi$ is an abelian group, and $f : X \to B$ be a map between connected CW-spaces. Let $\hat{\Pi}_p$ be the local coefficients and $ob(p) \in H^{n+1}(B; \hat{\Pi}_p)$ be the primary cohomology obstruction with respect to the trivial section on the base point. Then $f^*(ob(p)) = 0 \in H^{n+1}(X; f^*\hat{\Pi}_p)$ if and only if $f$ can be lifted to $\tilde{f} : X \to E$.

**Proof.** See [3].

Now we can apply the above results to prove our general decomposition theorem.

**Theorem 2.1.5.** Let $\xi^m$ be an $m$-dimensional vector bundle over a connected CW-complex $X$, $p_\xi : G_k(\xi^m) \to X$ be the Grassmann bundle of $\xi^m$ with Postnikov decomposition $\{\tilde{X}_n, \tilde{h}_n, \tilde{\eta}_n\}$. Let

\[ ob^k_n(\xi^m) \in H^{n+1}(\tilde{X}_{n-1}, \pi_n(G_{m-k,k})), \quad n = 1, 2, \ldots \]

be the $n$-th Postnikov invariance for $p_\xi$. For $n \geq 1$ and a given homomorphism $\theta : \pi_1(X) \to \pi_1(\tilde{X}_1)$, set

\[ OB^k_{n,\theta}(\xi^m) = \left\{ s_{n-1}^*(ob^k_n(\xi^m)) \mid \text{for all sections} \right. \]

\[ \left. s_{n-1} : X \to \tilde{X}_{n-1} \text{ s.t. } (\tilde{q}_2 \cdots \tilde{q}_n s_{n-1})_* = \theta \right\} \subset H^{n+1}(X, \theta_*\pi_n(G_{m-k,k})) \]
where \( s_{n-1}^* : H^{n+1}(\tilde{X}_{n-1}; \tilde{\pi}_n(G_{m-k,k})) \rightarrow H^{n+1}(X; \theta_\ast \tilde{\pi}_n(G_{m-k,k})) \) is the induced homomorphism in the cohomology groups with local coefficients. Then
\[
\{ \text{ob}_n^k(\xi^m) \mid n = 1, 2, \ldots \} \quad \text{and} \quad \{ \text{OB}_{n,\theta}^k(\xi^m) \mid n = 1, 2, \ldots \}
\]
have the following properties:

1. If \( \text{ob}_n^k(\xi^m) = 0 \) for every \( n < \dim X \), then \( \xi^m \) can be decomposed as a Whitney sum \( \xi^m = \xi^k \oplus \xi^{m-k} \).
2. If \( \xi^m \) can be decomposed as a Whitney sum \( \xi^m = \xi^k \oplus \xi^{m-k} \), then there exists a homomorphism \( \theta : \pi_1(X) \rightarrow \pi_1(\tilde{X}_1) \) such that \( 0 \in \text{OB}_{n,\theta}^k(\xi^m) \) for \( n = 1, 2, \ldots \).
3. If \( N = \dim X < \infty \) and \( 0 \in \text{OB}_{N-1,\theta}^k(\xi^m) \), then \( \xi^m \) can be decomposed as a Whitney sum \( \xi^m = \xi^k \oplus \xi^{m-k} \).
4. \( \xi^m \) can be decomposed as a Whitney sum \( \xi^m = \xi^k \oplus \xi^{m-k} \) if and only if there exists a section \( s : X \rightarrow \lim_n \tilde{X}_n \) for the fibration \( p : X \rightarrow \lim_n \tilde{X}_n \).

**Proof.** By Lemma 2.1.3, the problem of decomposition \( \xi^m = \xi^k \oplus \xi^{m-k} \) is equivalent to the existence of a section for the fibration \( p_\xi : G_k(\xi^m) \rightarrow X \) which has fiber \( G_{m-k,k} \).

If \( k = 1 \) and \( m = 2 \), then \( G_{m-k,k} = G_{1,1} = S^1 \) and \( \pi_1(G_{1,1}) = Z \). If \( m > 2 \), then it is well-known that
\[
\pi_1(G_{m-k,k}) \approx \pi_0(O(k)) = Z_2.
\]
As a result, \( \pi_1(G_{m-k,k}) \) is always an abelian group so that we can apply the obstruction theory for the fibration \( p_\xi : G_k(\xi^m) \rightarrow X \).

**Proof of (1).** By Lemma 2.1.4, \( \text{ob}_n^k(\xi^m) = 0 \) if and only if the fibration \( \tilde{q}_n : \tilde{X}_n \rightarrow \tilde{X}_{n-1} \) has a section. So if \( \text{ob}_n^k(\xi^m) = 0 \) for each \( n \), then there exists a section \( s : X \rightarrow \lim_n \tilde{X}_n \) for the fibration \( p : \lim_n \tilde{X}_n \rightarrow X \), which is the composition of all sections \( \tilde{q}_n : \tilde{X}_n \rightarrow \tilde{X}_{n-1} \). From [3], \( \lim_n \tilde{h}_n : G_k(\xi^m) \rightarrow \lim_n \tilde{X}_n \) is a weak homotopy equivalence. Since \( X \) is a CW-complex, one can apply J.H.C. Whitehead theorem, which in our situation says that \( h^* : [X, G_k(\xi^m)] \approx [X, \lim_n \tilde{X}_n] \), to get a section for the fibration \( p_\xi : G_k(\xi^m) \rightarrow X \). By Lemma 2.1.3, \( \xi^m \) can be decomposed as a Whitney sum: \( \xi^m = \xi^k \oplus \xi^{m-k} \).

**Proof of (2).** By Lemma 2.1.3, \( \xi^m = \xi^k \oplus \xi^{m-k} \) implies \( p_\xi : G_k(\xi^m) \rightarrow X \) has a sections \( s : X \rightarrow G_k(\xi^m) \) which gives rise to a sequence of sections \( \{ s_n = h_n s : X \rightarrow \tilde{X}_n \mid s.t. \tilde{q}_n s_n = s_{n-1}, \ n = 1, 2, \ldots \} \). Take \( \theta = s_1 : \pi_1(X) \rightarrow \pi_1(\tilde{X}_1) \). Then for every \( n \), consider the pull-back diagram:
Now we look at the local coefficients. From the long exact sequence of the fibration $\tilde{q}_{n} : \tilde{X}_{n} \to \tilde{X}_{n-1}$. Since $s_{n-1}$ has a lifting $s_n$, by Lemma 2.1.4,

$$s^*_{n-1}(ob^k_n(\xi^m)) = 0 \in H^{n+1}(X; s_{n-1} \pi_n (G_{m-k,k})).$$

Now we look at the local coefficients. From the long exact sequence of the fibration $\tilde{q}_{n} : \tilde{X}_{n} \to \tilde{X}_{n-1}$:

$$\cdots \to \partial_2 \pi_1(K(\pi_n(F), n)) \to i_* \pi_1(\tilde{X}_{n}) \to \tilde{q}_n^* \pi_1(\tilde{X}_{n-1}) \to 0$$

we see that $\tilde{q}_n^* : \pi_1(\tilde{X}_{n}) \to \pi_1(\tilde{X}_{n-1})$ is an isomorphism for $n \geq 2$. Noticing that

$$\tilde{q}_2 \cdots \tilde{q}_{n-1}s_{n-1} = s_1,$$

we thus prove that $s_{n-1} \pi_n(G_{k,m-k}) = \theta_{s} \pi_n(G_{k,m-k})$, and

$$s^*_{n-1}(ob^k_n(\xi^m)) = 0 \text{ in } H^{n+1}(X; \theta_{s} \pi_n(G_{m-k,k})).$$

By definition, $0 \in OB^k_{n,0}(\xi^m)$, for $n = 1, 2, \ldots$.

**Proof of (3).** By definition, $0 \in OB^k_{N-1,0}(\xi^m)$ implies that there exists a section $s_{N-2}$ such that $s^*_{N-2}(ob_{N-1}(\xi^m)) = 0$ in $H^N(X, \theta_{s} \pi_{N-1}(G_{m-k,k}))$, by Lemma 2.1.4, $s_{N-2}$ has a lifting $s_{N-1}$ such that $p_{N-1}s_{N-1} = s_{N-2}$. Since $\dim X = N$, for any local coefficients $\tilde{G}$ on $X$

$$H^i(X, \tilde{G}) = 0 \quad \text{for } i > N.$$

But $s^*_{N-1}(ob_N(\xi^m)) \in H^{N+1}(X, s_{N-1} \pi_N (G_{m-k,k})) = 0$, so $s^*_{N-1}(ob_N(\xi^m)) = 0$, and by Lemma 2.1.4, there exists a section $s_{N}$ such that $p_NS = s_{N-1}$. By repeating this procedure, we can obtain a sequence of sections $\{s_n \mid \text{s.t. } p_n s_n = s_{n-1}\}$ which gives a section for the fibration $p_G : G_k(\xi^m) \to X$ and by Lemma 2.1.3, $\xi^m = \xi^k \oplus \xi^{m-k}$.

**Proof of (4).** Since $\lim_n \tilde{h}_n : G_k(\xi^m) \to \lim_n \tilde{X}_n$ is a weak homotopy equivalence and $X$ is a CW-complex, from Lemma 2.1.1 and J.H.C. Whitehead theorem as in proof of (1) the result follows.

**Definition 2.1.6.** The classes $\{ob^k_n(\xi^m) \in H^{n+1}(\tilde{X}_{n-1}, \pi_n(G_{m-k,k})) \}$, $n = 1, 2, \ldots$ in the above theorem are called the decomposition obstructions of $\xi^m = \xi^k \oplus \xi^{m-k}$. 
Corollary 2.1.7. \( \{ \text{ob}_n^k(\xi^m) \in H^{n+1}(\tilde{\pi}^{-1}(\xi^m), G_{m-k,k}) \}, n = 1, 2, \ldots \} \) are natural in the following sense: If \( g : Y \rightarrow X \) is a map, then \( g \) pulls the tower
\[
\begin{array}{c}
X \\
\vdots \\
\tilde{X}_1 \\
\vdots \\
\tilde{X}_{n-1} \\
\tilde{X}_n \\
\end{array}
\]
back over \( Y \) such that
\[
g_{n-1}^* (\text{ob}_n^k(\xi^m)) = \text{ob}_n^k(g^*(\xi^m)) \quad \text{and} \quad g^*(\text{OB}_n^k(\xi^m)) \subset \text{OB}_n^k(g_\ast^*(\xi^m))
\]
where \( g_{n-1} : g^*(\tilde{X})_{n-1} \rightarrow \tilde{X}_{n-1} \) is the induced map at \((n-1)\) stage, in particular,
\[
\text{ob}_n^k(\xi^m) = \text{ob}_n^k(f^*(\eta_m)) = f_{n-1}^* (\text{ob}_n^k(\eta_m)) \quad \text{and} \quad f^*(\text{OB}_n^k(\eta_m)) \subset \text{OB}_n^k(f_\ast^*(\xi^m))
\]
where \( \eta_m \) is the universal \( m \)-vector bundle, and \( f \) is the classifying map of \( \xi^m \).

Proof. The proof is essentially based on the naturality of the primary cohomology obstruction and that of the Postnikov decomposition. By definition \( \text{ob}_n^k(\xi^m) \) is the \( n \)-th Postnikov invariant in the induced Postnikov decomposition. But the Postnikov invariants are natural since they are defined to be the primary cohomology obstructions.

To prove that \( g^*(\text{OB}_n^k(\xi^m)) \subset \text{OB}_n^k(g_\ast^*(\xi^m)) \), let \( s_{n-1} : X \rightarrow \tilde{X}_{n-1} \) be a section such that \((\tilde{q}_2 \cdots \tilde{q}_{n-1} s_{n-1})_* = \theta\), consider the commutative diagram:
\[
\begin{array}{ccc}
g^*(X_{n-1}) & \xrightarrow{g_{n-1}} & \tilde{X}_{n-1} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g} & X.
\end{array}
\]
Since \( s_{n-1} \) induces a section \( g^*(s_{n-1}) \) for the induced fibration by \( g \) such that \( s_{n-1} g = g_{n-1} g^*(s_{n-1}) \), by the naturality of cohomology with local coefficients, we have
\[
g^*(s_{n-1}^* (\text{ob}_n^k(\xi^m))) = (s_{n-1} g)^* (\text{ob}_n^k(\xi^m))
\]
\[
= (g_{n-1} g^*(s_{n-1}))^* (\text{ob}_n^k(\xi^m))
\]
\[
= (g^*(s_{n-1}))^* g_{n-1}^* (\text{ob}_n^k(\xi^m))
\]
\[
= (g^*(s_{n-1}))^* (\text{ob}_n^k(g^*(\xi^m)))
\]
which means that \( g^*(\text{OB}_n^k(\xi^m)) \subset \text{OB}_n^k(g_\ast^*(\xi^m)) \). \[\square\]
In the following we study the decomposition obstructions in some details. Consider the inclusions:

\[ O(m - k) \subset O(m - k) \times O(k) \subset O(m) \]

which induce fibrations in their classifying spaces:

\[ BO(m - k) \to B(O(m - k) \times O(k)) \to BO(m). \]

We look at the following two fibrations:

\[ p : BO(m - k) \to BO(m) \quad \text{and} \quad p' : B(O(m - k) \times O(k)) \to BO(m) \]

which have fibers \( O(m)/O(m - k) = V_{m,k} \) and \( O(m)/(O(m - k) \times O(k)) = G_{m-k,k} \) respectively, and view the third fibration \( BO(m - k) \to B(O(m - k) \times O(k)) \) as a map between the two fibrations. By the naturality of Postnikov decomposition, there exists a commutative diagram:

\[
\begin{array}{ccc}
E & \stackrel{j}{\longrightarrow} & E' \\
\downarrow{h_n} & & \downarrow{h'_n} \\
E_n & \stackrel{f_n}{\longrightarrow} & E'_n \\
\downarrow{q_{n-1}} & & \downarrow{q'_{n-1}} \\
E_{n-1} & \stackrel{f_{n-1}}{\longrightarrow} & E'_{n-1} \\
\downarrow{p_{n-1}} & & \downarrow{p'_{n-1}} \\
BO(m) = & & \text{BO}(m)
\end{array}
\]

where \( \{E_n, q_n, h_n\} \) and \( \{E'_n, q'_n, h'_n\} \) are the Postnikov decompositions of \( p : BO(m - k) \to BO(m) \) and \( p' : B(O(m - k) \times O(k)) \to BO(m) \) respectively.

Now we study the Postnikov decomposition \( \{E_n, q_n, h_n\} \) of \( p : BO(m - k) \to BO(m) \). Since \( q_n \) is a fibration with fiber \( K(\pi_n(V_{m,k}), n) \) and the Stiefel manifold \( V_{m,k} \) is \( (m - k - 1) \)-connected, the Postnikov invariant

\[ k_i(p) \in H^{i+1}(E_{i-1}; \tilde{\pi}_i(V_{m,k})) \]

vanishes for \( 0 < i < m - k \). Applying Lemma 2.1.4 repeatedly, we have:

**Proposition 2.1.8.** There exists a section: \( BO(m) \to E_{m-k-1} \) in the Postnikov decomposition \( \{E_n, q_n, h_n\} \) of the fibration \( p : BO(m - k) \to BO(m) \).

**Corollary 2.1.9.** There exists a section: \( BO(m) \to E'_{m-k-1} \) in the Postnikov decomposition \( \{E'_n, q'_n, h'_n\} \) of the fibration \( p' : B(O(m - k) \times O(k)) \to BO(m) \).
Proof. This is a direct result of the Proposition 2.1.8 and the above commutative diagram. \(\square\)

**Corollary 2.1.10.** Let \(\xi^m\) be an \(m\)-dimensional vector bundle over a connected \(N\)-dimensional CW-complex \(X\). If \(N \leq m - k\), then \(\xi^m\) can be decomposed as a Whitney sum \(\xi^m = \xi^k \oplus \xi^{m-k}\).

**Proof.** By Corollary 2.1.9, we have a section: \(s : BO(m) \to E'_{m-k-1}\) in the Postnikov decomposition \(\{E'_n, q'_n, h'_n\}\) of the fibration \(p' : BO((m-k) \times O(k)) \to BO(m)\). So there exists a section: \(\tilde{s} : X \to \tilde{X}_{m-k-1}\) in the induced Postnikov decomposition. By definition, \(0 \in \text{OB}^{k}_{m-k-1, \theta}(\xi^m)\), using Theorem 2.1.5 (3), one concludes that \(\xi^m\) can be decomposed as a Whitney sum \(\xi^m = \xi^k \oplus \xi^{m-k}\). \(\square\)

In Theorem 2.1.5, the obstruction set \(\text{OB}^{k}_{n, \theta}(\xi^m)\) depends on \(\theta : \pi_1(X) \to \pi_1(\tilde{X}_1)\). In the following, we will see that there are exactly two different \(\theta\)'s for the universal \(m\)-vector bundle \(\eta^m\).

**Theorem 2.1.11.** In the Postnikov decomposition \(\{E'_n, q'_n, h'_n\}\) of the fibration \(p' : BO((m-k) \times O(k)) \to BO(m)\), if \(m > 2\), then there are exactly two homomorphisms

\[\theta_1, \theta_2 : \pi_1(BO(m)) \to \pi_1(E'_1)\]

which are induced by some sections from \(BO(m)\) to \(E'_1\).

**Proof.** Recall that in the Postnikov decomposition \(\{E'_n, q'_n, h'_n\}\), \(q'_1 : E'_1 \to BO(m)\) is a fibration with fiber \(K(\pi_1(G_{m-k}), 1)\). Consider the homotopy sequence of the fibration

\[\cdots \to \pi_2(BO(m)) \overset{\partial}{\to} \pi_1(K(\pi_1(G_{m-k}), 1)) \overset{i_*}{\to} \pi_1(E'_1) \overset{q'_{1*}}{\to} \pi_1(BO(m)) \to 0\]

and the fact that \(\pi_1(G_{m-k}) = Z_2\) for \(m > 2\) and \(\pi_1(BO(m)) = Z_2\). Since \(B(O(m-k) \times O(k)) \simeq BO(m-k) \times BO(k)\), we have

\[\pi_1(B(O(m-k) \times O(k))) \approx \pi_1(BO(m-k)) \oplus \pi_1(BO(m-k)) \approx Z_2 \oplus Z_2\]

But \(h' : B(O(m-k) \times O(k)) \to E'_1\) is 2-connected, so

\[h'_{1*} : \pi_1(B(O(m-k) \times O(k))) \to \pi_1(E'_1)\]

is an isomorphism and hence \(\pi_1(E'_1) \approx Z_2 \oplus Z_2\). Therefore, the above exact sequence actually is the following sequence:

\[\pi_2(BO(m)) \overset{\partial}{\to} Z_2 \overset{i_*}{\to} Z_2 \oplus Z_2 \overset{q'_{1*}}{\to} Z_2 \to 0.\]

By the exactness, the image of \(\partial\) must be 0, and we get a short exact sequence:

\[0 \to Z_2 \overset{i_*}{\to} Z_2 \oplus Z_2 \overset{p'_{1*}}{\to} Z_2 \to 0.\]
It is easy to see that there are exactly two homomorphisms
\[ \theta_1, \theta_2 : \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \]
which satisfy the condition: \( p'_s \circ \theta_1 = p'_s \circ \theta_2 = 1 \). By the 2-extendability theorem [3], we get two sections: \( s_1, s_2 : BO(m) \to E'_1 \) such that
\[ s_i^* = \theta_i : \pi_1(BO(m)) = \mathbb{Z}_2 \to \pi_1(E'_1) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad \text{for } i = 1, 2. \]
This is what we need to prove.

2.2. Codimension 1 decomposition. Now we consider two special cases in which there is only one decomposition obstruction. The first one is that the dimension of the vector bundle is 2. The other one is the case in which the codimension is 1.

**Proposition 2.2.1.** For any two dimensional vector bundle \( \xi^2 \) over a connected CW-complex \( X \), there is only one decomposition obstruction \( \text{ob}_1(\xi^2) \in H^2(X; f \ast \tilde{Z}) \) such that \( \text{ob}_1(\xi^2) = 0 \) if and only if \( \xi^2 \) can be decomposed as \( \xi^2 = \xi^1 \oplus \eta^1 \), where \( f : X \to BO(2) \) is the classifying map of \( \xi^2 \), and \( \tilde{Z} \) is the twisted integer.

**Proof.** In the fibration \( p : B(O(1) \times O(1)) \to BO(2) \), the fiber is
\[ O(2)/(O(1) \times O(1)) \approx G_{1,1} \approx S^1 \approx K(\mathbb{Z}, 1), \]
so the fibration itself is the Postnikov decomposition, and
\[ \text{ob}_n(\xi^2) = 0 \quad \text{for } n > 1. \]
Since \( \text{ob}_1(\xi^2) \) is the only non trivial decomposition obstruction, by Theorem 2.1.5, \( \text{ob}_1(\xi^2) = 0 \) if and only if \( \xi^2 \) can be decomposed as \( \xi^2 = \xi^1 \oplus \eta^1 \). \( \square \)

**Corollary 2.2.2.** Let \( \eta_2 \) be the universal principal \( O(2) \)-bundle, then for any two dimensional vector bundle \( \xi^2 \) over a connected CW-complex \( X \) with classifying map \( f \),
\[ f^*(\text{ob}_1(\eta_2)) = 0 \quad \text{if and only if } \xi^2 \text{ can be decomposed as } \xi^2 = \xi^1 \oplus \eta^1. \]

**Proof.** By the naturality of the decomposition obstruction
\[ f^*(\text{ob}_1(\eta_2)) = \text{ob}_1(f^*(\eta_2)) = \text{ob}_1(\xi^2). \]
From Proposition 2.2.1, we have this corollary. \( \square \)

Now we turn to consider the second special case where the dimension of the vector bundle is same as that of the base space. The decomposition is such that one of the bundles in the sum is a line bundle.

In our definition of decomposition obstructions \( \text{ob}_n(\xi^m) \), we use the Postnikov decomposition induced by the classifying map of the vector bundle \( \xi^m \).
from the fibration \( p : B(O(k) \times O(m-k)) \to BO(m) \). The advantage of this approach is that those decomposition obstruction classes \( \text{ob}_n^k(\xi^m) \) only depend on the vector bundle \( \xi^m \) and are natural in the sense as indicated in Corollary 2.1.7. If one has a section \( s \) to the \((n-1)\)-level in the induced Postnikov decomposition, then the vanishing of \( s^*(\text{ob}_n^k(\xi^m)) \) is equivalent to the existence of the \( n \)-level lifting for \( s \). So the class \( s^*(\text{ob}_n^k(\xi^m)) \) has the similar property as that of the obstruction cohomology class for a section.

On the other hand, as stated in [3], the construction for the Postnikov decomposition is much more difficult than that of CW-decomposition. In some special cases, one may give a detail description of the Postnikov decomposition [14]. However, Eckmann and Hilton [5] showed that the cohomology obstructions for Postnikov and CW-decompositions are equivalent. To say precisely, let \((X,A)\) be a relative CW-complex with CW-decomposition \( X = \lim_n X_n \), and let \( p : E \to X \) be a fibration with Postnikov decomposition \( \{E_n, q_n, h_n\} \), then there is a bijection:

\[
\lambda : \langle X_n, E \rangle_{u,\theta}^{u,\theta} | X_{n-1} \approx \langle X, E_{n-1} \rangle_{\theta}^{\theta'}
\]

where \( u : A \to p^{-1}(A) = E_{-1} \) is a partial section, \( \theta : \pi_1(X) \to \pi_1(E) \) is a splitting of \( p \) (see [3]) and \( \theta' = h_1 \circ \theta : \pi_1(X) \to \pi_1(E_1) \), and \( \langle X_n, E \rangle_{u,\theta}^{u,\theta} \) denoted the section homotopy classes relative to \( u \) and compatible with \( \theta \), and \( \langle X_n, E \rangle_{u,\theta}^{u,\theta} | X_{n-1} \) is the set of all the restrictions on \( X_{n-1} \). The bijection mapping \( \lambda \) is given in the following way:

Let \( \phi_n \in \langle X_n, E \rangle_{u,\theta}^{u,\theta} \), then \( h_{n-1} \phi_n | X_{n-1} : X_{n-1} \to E_{n-1} \) has an extension \( h_{n-1} \phi_n : X_n \to E_{n-1} \). But the fiber of the fibration: \( E_{n-1} \to X \) has no non-vanishing homotopy groups in dimension greater than \( n-1 \), so the section \( h_{n-1} \phi_n : X_n \to E_{n-1} \) can be extended to a section over \( X \) which is defined to be \( \lambda(\phi_n | X_{n-1}) \). From [3], this is well-defined bijection. Under this bijection, Eckmann and Hilton’s result says that

\[
[\text{ob}_n(\phi_n)] = \phi_{n-1}^* (k_n(q_n)) \in H^{n+1}(X,A; \theta \ast \pi_n(F))
\]

where \( \phi_{n-1} = \lambda(\phi_n | X_{n-1}) \), and \( k_n(q_n) \in H^{n+1}(E_{n-1}; \pi_n(F)q_n) \) is the \( n \)-th Postnikov invariant for the fibration \( q_n : E_n \to E_{n-1} \).

Using the above result, we see that from the Postnikov invariants, one can recover all the obstructions classes defined by using the CW-decomposition of the base space. It is because of this, we can compute the obstruction \( \text{OB}_{n\theta}^k(\xi^m) \) without knowing the Postnikov decomposition. This makes it possible to compute \( \text{OB}_{n\theta}^k(\xi^m) \) by using only the CW-decomposition of the base space.

In order to actually compute \( \text{OB}_{n\theta}^k(\xi^m) \subset H^{n+1}(X,\theta \ast \pi_n(G_{m-k,k})) \), one still needs to know the local coefficients \( \pi_n(G_{m-k,k}) \). By the naturality 2.1.7, we know that \( f^*(\text{OB}_{n\theta}^k(\eta_m)) \subset \text{OB}_{n\theta}^k(\xi^m) \), where \( f \) is the classifying map for the vector bundle \( \xi^m \), and \( \eta_m \) is the universal \( m \)-plane bundle over \( BO(m) \). So it will be enough to determine the local coefficients for the
fibration $p : B(O(k) \times O(m - k)) \to BO(m)$ which is difficult in general because one does not know $\pi_n(G_{k,m-k})$ for each $n$. In order to relate this coefficients to some known coefficients, we need the following lemma:

**Lemma 2.2.3.** Let $E_2 \xrightarrow{q_2} E_1 \xrightarrow{q_1} X$ be a tower of fibrations, and let $F_2$ and $F_1$ be the fibers of $q_1 q_2$ and $q_1$ respectively, then for each $n$, the induced homomorphism $q_{2*} : \pi_n(F_2) \to \pi_n(F_1)$ is a homomorphism $q_{2*} : \pi_n(F_2)^{q_1 q_2} \to \pi_n(F_1)^{q_1}$ between the two systems of local coefficients determined by the fibrations $q_1 q_2$ and $q_1$.

**Proof.** Let $\hat{E}_2, \hat{E}_2$, be the mapping cylinders of $q_1 q_2$ and $q_1$ respectively, then $q_2$ induces an obvious map $\hat{q}_2 : (\hat{E}_2, E_2) \to (\hat{E}_1, E_1)$ which induces operator homomorphisms between the two homotopy exact sequences:

$$\begin{array}{cccccccc}
\cdots & \to & \pi_n(E_2) & \xrightarrow{i_*} & \pi_n(\hat{E}_2) & \xrightarrow{j_*} & \pi_n(\hat{E}_1, E_2) & \xrightarrow{\partial} & \pi_{n-1}(E_2) & \to & \cdots \\
& & \downarrow{q_{2*}} & & \downarrow{\hat{q}_{2*}} & & \downarrow{\hat{q}_{2*}} & & \downarrow{q_{2*}} & & \\
\cdots & \to & \pi_n(E_1) & \xrightarrow{i_*} & \pi_n(\hat{E}_1) & \xrightarrow{j_*} & \pi_n(\hat{E}_1, E_1) & \xrightarrow{\partial} & \pi_{n-1}(E_1) & \to & \cdots.
\end{array}$$

In particular, $\hat{q}_{2*} : \pi_q(\hat{E}_2, E_2) \to \pi_q(\hat{E}_1, E_1)$ is an operator homomorphism. Recall that $\pi_n(F_1)^{q_1}$ is defined via an isomorphism $\Delta' : \pi_q(\hat{E}_1, E_1) \to \pi_{q-1}(F_1)$. In fact, the isomorphism is given by the following composition:

$$\Delta' : \pi_q(\hat{E}_1, E_1) \xrightarrow{k_1^{-1}} \pi_q(\hat{F}_1, F_1) \xrightarrow{\partial} \pi_{q-1}(F_1)$$

where $k_1$ is the inclusion which induces an isomorphism [15]. Thus we have a commutative diagram:

$$\begin{array}{cccccccc}
\pi_q(\hat{E}_2, E_2) & \xrightarrow{k_1^{-1}} & \pi_q(\hat{F}_2, F_2) & \xrightarrow{\partial} & \pi_{q-1}(F_2) \\
\downarrow{\hat{q}_{2*}} & & \downarrow{\hat{q}_{2*}} & & \downarrow{q_{2*}} & & \\
\pi_q(\hat{E}_1, E_1) & \xrightarrow{k_1^{-1}} & \pi_q(\hat{F}_1, F_1) & \xrightarrow{\partial} & \pi_{q-1}(F_1).
\end{array}$$

Therefore $q_{2*} : \pi_n(F_2)^{q_1 q_2} \to \pi_n(F_1)^{q_1}$ is a homomorphism between the two local systems. \hfill \Box

**Corollary 2.2.4.** With the above notations, for any $n$-skeleton section $s : X^n \to E_2$ ($n \geq 2$), there is an induced homomorphism:

$$q_{2*} : H^{n+1}(X; s_* \pi_n(F_2)) \to H^{n+1}(X; q_2 \circ s_* \pi_n(F_1))$$

such that

$$q_{2*}(ob_n(s)) = ob_n(q_2 \circ s).$$
Proof. By Lemma 2.2.3, \(q_2^*: \pi_n(F_2) \rightarrow \pi_n(F_1)\) is a homomorphism \(q_2^*: \pi_n(F_2)^{q_1,q_2} \rightarrow \pi_n(F_1)^{q_1}\) between the two systems of local coefficients determined by the fibrations \(q_1\) and \(q_2\). The sections \(s: X^n \rightarrow E_2\) and \(q_2\) \(s: X^n \rightarrow E_1\) pull the two systems of local coefficients back on \(X^n\) so that \(q_2^*: \pi_n(F_2)^{q_1,q_2} \rightarrow \pi_n(F_1)^{q_1}\) induces a homomorphism of the systems of local coefficients:

\[
q_2^*: s_\ast \pi_n(F_2) \rightarrow (q_2s)_\ast \pi_n(F_1).
\]

From [15], \(q_2^*\) induces a homomorphism for the cohomology groups with local coefficients:

\[
q_2^*: H^{n+1}(X; s_\ast \pi_n(F_2)) \rightarrow H^{n+1}(X; q_2 \circ s_\ast \pi_n(F_1)).
\]

Now we consider the diagram:

\[
\begin{array}{ccc}
\pi_{n+1}(E_2^{n+1} , E_2^n) & \xrightarrow{\partial} & \pi_n(E_2^n) \\
q_2^* \downarrow & & \downarrow q_2^* \\
\pi_{n+1}(E_1^{n+1} , E_1^n) & \xrightarrow{\partial} & \pi_n(E_1^n)
\end{array}
\]

where all the spaces and homomorphisms are the same as in the definition of obstruction cocycle (see [3]). We only need to check the commutativity of the right square.

Let \(\alpha \in \pi_n(E_2^n)\), by definition, we have

\[
i_\ast q_2^*s_\ast \#(\alpha) = q_2^*i_\ast s_\ast \#(\alpha) = q_2^*(\alpha - s_\ast(q_1q_2)_\ast(\alpha))
\]

\[
= q_2^*(\alpha) - q_2^*s_\ast(q_1q_2)_\ast(\alpha)
\]

\[
= q_2^*(\alpha) - (q_2s)_\ast(q_1q_2)_\ast(\alpha)) = i_\ast(q_2 \circ s)_\ast \#q_2^*(\alpha)
\]

Since \(i_\ast\) is injective, we obtain that \(q_2^*s_\ast \# = (q_2 \circ s)_\ast \#q_2^*\). Therefore, we have

\[
q_2^*s_\ast \# \partial q_2^* = (q_2 \circ s)_\ast \# \partial.
\]

By the definition of cocycle, and that of the induced homomorphism,

\[
q_2^*(\text{ob}_n(s)) = \text{ob}_n(q_2 \circ s).
\]

This completes the proof. \(\square\)

Corollary 2.2.5. The inclusions \(O(m) \subset O(m) \times O(n) \subset O(m+n)\) induce a tower of fibrations \(BO(m) \xrightarrow{q_1} B(O(m) \times O(n)) \xrightarrow{q_2} BO(m+n)\). Let \(m \geq n\) and \(m+n > 2\). Then \(q_2\) induces a homomorphism:

\[
q_2^*: H^{q+1}(BO(m+n); \pi_q(V_{m+n,n})_{q_1q_2}) \rightarrow H^{q+1}(BO(m+n); \theta_1, \pi_q(G_{m,n}))
\]

where \(\theta_1: \pi_1(BO(m+n)) = \pi_1(BO(m)) = Z_2 \xrightarrow{q_2^*} \pi_1(B(O(m) \times O(n))) = Z_2 \oplus Z_2\). If \(s\) is a \(q\)-skeleton section with \(q \geq 2\), then

\[
q_2^*(\text{ob}_q(s)) = \text{ob}_q(q_2 \circ s).
\]
Proof. The fibers of $q_1$ and $q_1 q_2$ are $G_{m,n}$ and $V_{m+n,n}$ respectively. By our assumption, $m \geq 3$, so $V_{m+n,n}$ is at least 1-connected and the fibration $q_1 q_2$ has a unique (up to homotopy) 2-skeleton section $s$ and

$$s_* : \pi_1(BO(m+n)) = Z_2 \rightarrow \pi_1(BO(m)) = Z_2$$

is an isomorphism. The result follows by Corollary 2.2.4 if we simply think $s_*$ as an identity. □

The following theorem is the main result about the codimension 1 decomposition which reveals an obstruction class in $H^{2n+1}(BO(2n+1); Z)$, the cohomology with ordinary integer coefficients.

**Theorem 2.2.6.** Let $\eta_m$ be the $m$-dimensional universal bundle over $BO(m)$ with $m \geq 3$, then there are exactly two obstruction classes in dimension $m$ for the decomposition $\eta_m \approx \xi^{m-1} \oplus \lambda$. Furthermore, one of the two classes is the primary obstruction for the decomposition $\eta_m \approx \xi^{m-1} \oplus R$, which is in $H^m(BO(m); \tilde{Z})$, where $\tilde{Z}$ is the twisted integers; the other is in $H^m(BO(m); Z)$ if $m$ is even and is in $H^m(BO(m); Z)$ if $m$ is odd.

Proof. Consider the tower of fibrations:

$$BO(m-1) \xrightarrow{q_2} B(O(m-1) \times O(1)) \xrightarrow{q_1} BO(m)$$

by Corollary 2.2.5, $q_2$ induces a homomorphism:

$$q_{2*} : H^{q+1}(BO(m); \pi_q(V_{m,1}) \times q_1) \rightarrow H^{q+1}(BO(m); \theta_1, \pi_q(G_{m-1,1}))$$

where

$$\theta_1 : \pi_1(BO(m)) = \pi_1(BO(m-1)) = Z_2 \xrightarrow{q_{2*}} \pi_1(B(O(m-1) \times O(1))).$$

Noticing that $V_{m,1} = S^{m-1}$ and $G_{m-1,1} = RP^{m-1}$, we know that $q_2$ induces an isomorphism $q_{2*} \pi_{m-1}(V_{m,1}) \approx Z \rightarrow \pi_{m-1}(G_{m-1,1}) \approx Z$ between the two systems of local coefficients. As is well-known, the coefficients in the first cohomology is the twisted integer $\tilde{Z}$ when $q = m-1$, so is the coefficients in the second one. Thus $q_2$ induces an isomorphism

$$q_{2*} : H^m(BO(m); \tilde{Z}) \rightarrow H^m(BO(m); \tilde{Z}).$$

Let $s$ be any $(m-1)$-skeleton section for the fibration $q_1 q_2 : BO(m-1) \rightarrow BO(m)$, then $q_2 s$ is a $(m-1)$-skeleton section for the fibration $q_1 : B(O(m-1) \times O(1)) \rightarrow BO(m)$. From Corollary 2.2.5, we obtain that

$$q_{2*}(\text{ob}_{m-1}(s)) = \text{ob}_{m-1}(q_2 \circ s).$$

From [15], $\text{ob}_{m-1}(s)$ is the primary obstruction for the vector bundle decomposition $\eta_m \approx \xi^{m-1} \oplus R$. From the primary obstruction theorem, the primary obstruction is independent with the choice of sections. Hence $\text{ob}_{m-1}(q_2 \circ s)$ is one of the primary obstruction for the decomposition $\eta_m \approx$
The universal covering space of $\theta$ which are given by $\eta$ and there are exactly two splitting homomorphisms: $q_1, q_2$.

But the action of $(0, \beta)$ on $\pi_{m-1}(G_{m-1,1})$ is the same as that of $\beta$, and the action of $(\alpha, 0)$ on $\pi_{m-1}(G_{m-1,1})$ is the same as that of $\alpha$ which is the twisted action, i.e.,

$$\tau_\alpha(\xi) = -\xi, \text{ for any } \xi \text{ in } \pi_{m-1}(G_{m-1,1}) = Z.$$ 

Now the action induced by $\theta_2(\alpha) = (\alpha, \beta)$ is simply the product of the actions of $\alpha$ and $\beta$. So we have $\theta_2(\alpha)(\xi) = (-1)^{m+1}\xi$, and thus the action is either the twisted integer $\tilde{Z}$ if $m$ is even or the ordinary integer $Z$ if $m$ is odd.

The most interesting part in the theorem is the following corollary:
Corollary 2.2.7. There is an element \( o_{2n+1} \) in \( H^{2n+1}(BO(2n+1); \mathbb{Z}) \) such that for any map \( f : X^{2n+1} \to BO(2n+1) \), if \( f^*(o_{2n+1}) = 0 \), then \( f^*(\eta_{2n+1}) \approx \xi^{2n} \oplus \lambda \), where \( X^{2n+1} \) is a \((2n+1)\)-dimensional CW-complex and \( \eta_{2n+1} \) is the universal vector bundle over \( BO(2n+1) \).

Proof. Let \( o_{2n+1} \) be the obstruction corresponding to the decomposition \( \xi^{2n} \oplus \lambda \) as in Theorem 2.2.6. By the primary obstruction theorem, there is a \((2n)\)-skeleton section \( s \) for the fibration \( p : B(O(2n) \times O(1)) \to BO(2n+1) \) such that \( o_{2n+1} = \text{ob}(s) \). By cellular approximation theorem, we may assume that \( f \) is a cellular map. By the naturality of the primary obstruction class, we have

\[
f^*(o_{2n+1}) = f^*(\text{ob}(s)) = \text{ob}(f^*s)
\]

where \( f^*(s) \) is the induced section by \( f \) over the \((2n)\)-skeleton of \( X^{2n+1} \). So if \( f^*(o_{2n+1}) = 0 \), then \( f^*(s) \) can be extended to a section over \( X^{2n+1} \), hence \( f : X^{2n+1} \to BO(2n+1) \) has a lifting for the fibration \( p : B(O(2n) \times O(1)) \to BO(2n+1) \). By Lemma 2.1.1, \( f^*(\eta_{2n+1}) \approx \xi^{2n} \oplus \lambda \). □

The following corollary gives us some relationship between the decomposition obstruction and the well-known characteristic classes such as Stiefel-Whitney classes, Euler classes and Chern classes.

Corollary 2.2.8. If the coefficient is reduced to \( \mathbb{Z}_2 \), then one of the two obstruction classes in Theorem 2.2.6 will be the top Stiefel-Whitney class of the universal bundle.

Proof. By definition, the top universal Stiefel-Whitney class can be defined as the primary obstruction class for the vector bundle decomposition \( \eta_m \approx \xi^{m-1} \oplus R \) reduced the coefficient to \( \mathbb{Z}_2 \). The corollary follows by Theorem 2.2.6. □

Recall that the classifying space for oriented \( m \)-dimensional vector bundles is \( BSO(m) \), and the inclusion \( SO(m) \to O(m) \) induced a universal covering map

\[
\pi : BSO(m) \to BO(m)
\]

which is the classifying map for the universal oriented \( m \)-dimensional vector bundle \( \zeta_m \) over \( BSO(m) \).

Corollary 2.2.9. Let \( OB^{1}_{m-1, \theta_1}(\eta_m) \) and \( OB^{1}_{m-1, \theta_2}(\eta_m) \) be the two decomposition obstruction classes as in Theorem 2.2.6, and \( \pi : BSO(m) \to BO(m) \) be the projection, then

\[
\pi^*(OB^{1}_{m-1, \theta_1}(\eta_m)) = \pi^*(OB^{1}_{m-1, \theta_2}(\eta_m)) \in H^m(BSO(m), \mathbb{Z})
\]

which is the Euler class of the universal oriented \( m \)-dimensional vector bundle \( \zeta_m \) over \( BSO(m) \).
Proof. By the naturality of the obstruction classes, both \( \pi^*(\text{OB}_1^m(\eta_m)) \) and \( \pi^*(\text{OB}_1^m(\eta_m)) \) are the primary obstruction classes corresponding to the splitting homomorphism \( \theta_1 \pi_*$ and \( \theta_2 \pi_* \) for the codimension one decomposition of \( \zeta_m \). But \( \text{BSO}(m) \) is 1-connected, hence \( \theta_1 \pi_* = \theta_2 \pi_* = 0 \). By the uniqueness of the primary obstruction class, we have \( \pi^*(\text{OB}_1^m(\eta_m)) = \pi^*(\text{OB}_1^m(\eta_m)) \in H^m(\text{BSO}(m), Z) \). Noticing that any line bundle over \( \text{BSO}(m) \) is trivial, we see that \( \pi^*(\text{OB}_1^m(\eta_m)) \) is the obstruction for the decomposition \( \zeta_m = \zeta^{m-1} \oplus R \). But Euler class can be defined to be the primary obstruction of the decomposition \( \zeta_m = \zeta^{m-1} \oplus R \). \( \square \)

Recall that any \( n \)-dimensional complex vector bundle \( \zeta^n \) can be regarded as a \( 2n \)-dimensional real vector bundle Re \( (\zeta^n) \). In terms of the structure groups, one has an inclusion \( U(n) \subset O(2n) \) which induces a fibration \( p : BU(n) \longrightarrow BO(2n) \). If \( \gamma_n \) is the universal \( n \)-dimensional complex vector bundle over \( BU(n) \), then Re \( (\gamma_n) = p^*(\eta_{2n}) \). Let \( c_n(\zeta^n) \) be the \( n \)-th Chern class of a complex vector bundle \( \zeta^n \), then we have the following corollary:

**Corollary 2.2.10.** With the above notations,

\[
c_m(\gamma_m) = p^*(\text{OB}_1^{2m-1,\theta_1}(\eta_{2m})) \\
= p^*(\text{OB}_1^{2m-1,\theta_2}(\eta_{2m})) \in H^{2m}(BU(m), Z).
\]

Proof. Since \( BU(m) \) is 1-connected, Re \( (\gamma_n) = p^*(\eta_{2n}) \) is oriented vector bundle, as in Corollary 2.2.7, we know that

\[
p^*(\text{OB}_1^{2m-1,\theta_1}(\eta_{2m})) = p^*(\text{OB}_1^{2m-1,\theta_2}(\eta_{2m})),
\]

which is the Euler class of Re \( (\gamma_n) \). But the top Chern class of \( \gamma_n \) is just the Euler class of Re \( (\gamma_n) \) (see [10]). \( \square \)

In the following, we further consider the codimension 1 decomposition, that is the decomposition \( \xi^m \approx \xi^{m-1} \oplus \lambda \). We try to begin from CW-structure of the classify space to see the restriction of the universal bundle to each cell.

Recall that a partition of \( r \geq 0 \) is an unordered sequence \( (i_1, i_2, \ldots, i_s) \) of positive numbers such that the sum of the numbers is equal to \( r \). For our purpose, we define a partition of \( r \geq 0 \) with length \( n \) is an unordered sequence \( (i_1, i_2, \ldots, i_n) \) of non-negative numbers such that the sum of the numbers is equal to \( r \). We can always assume that \( 0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \). Then there is an one to one correspondence \( (i_1, i_2, \ldots, i_n) \leftrightarrow (\sigma_1, \sigma_2, \ldots, \sigma_n) \) given by \( \sigma_j = i_j + j \), for \( j = 1, 2, \ldots, n \). From [10], the Schubert symbol \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \) determines an unique \( r \)-cell \( e(\sigma) \), which is the set of \( n \)-planes in \( R^m \) such that:

\[
e(\sigma) = \left\{ X \mid \dim(X \cap R^{n-i}) = i, \dim(X \cap R^{n-i+1}) = i-1; i = 1, 2, \ldots, n \right\}.
\]
From [10], we know that the Grassmann manifold $G_n(R^m)$ has the following CW-structure:

$$G_n(R^m) = \{ \bigcup e(\sigma) \mid \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \\
\text{is such that } 0 < \sigma_1 < \sigma_2 < \cdots < \sigma_n \leq m \}.$$ 

Taking the direct limit as $m \to \infty$, one gets the infinite Grassmann manifold $G_n(R^\infty)$, which is the classifying space $BO(n)$.

**Proposition 2.2.11.** Let $e(\sigma)$ be a $(2k+1)$-dimensional cell in $G_n(R^\infty)$, then

$$\eta_{2k+1} \mid \overline{e}(\sigma) \approx \xi^{2k} \oplus \lambda$$

where $\eta_{2k+1}$ is the universal vector bundle over $G_n(R^\infty)$.

**Proof.** Let $\sigma_j = i_j + j$, for $j = 1, 2, \ldots, 2k+1$, then

$$\dim e(\sigma) = \sum_{j=1}^{2k+1} i_j = 2k + 1, \quad \text{and } 0 \leq i_1 \leq i_2 \leq \ldots \leq i_{2k+1}.$$ 

It is easy to see that the only $(2k+1)$-cell such that $i_1 > 0$ is the cell with partition $(1,1,\ldots,1)$. The corresponding Schubert symbol is $\sigma = (2,3,\ldots,2k+2)$. By definition, $e(\sigma)$ consists of all the $(2k+1)$-subspace $X$ in $R^{2k+2}$ such that $\dim (X \cap R^1) = 0$. It is not difficult to count all the faces of this cell, in fact, all the faces in the partition form are $(0,0,\ldots,0,1,\ldots,1)$. So the closure $\overline{e}(\sigma)$ for this cell is just $G_{2k+1}(R^{2k+2}) = RP^{2k+1}$. And $\eta_{2k+1} | G_{2k+1}(R^{2k+2})$ is the canonical $(2k+1)$-plane bundle of $G_{2k+1}(R^{2k+2})$ which is the tangent bundle for $k > 0$. It is well-known that there exists an nowhere 0 vector field for $\tau(S^{2k+1})$, in fact,

$$(x_0, x_1, \ldots, x_{2k}, x_{2k+1}) \mapsto ((x_0, x_1, \ldots, x_{2k}, x_{2k+1}), (x_1, -x_0, \ldots, x_{2k+1}, -x_{2k}))$$

is such a vector field. It is easy to see that this vector field induces a section for the fibration $RP(\eta_{2k+1} | \overline{e}(\sigma)) \to \overline{e}(\sigma)$, the projective space bundle. By Lemma 2.1.3

$$\eta_{2k+1} | \overline{e}(\sigma) \approx \xi^{2k} \oplus \lambda$$

where $\sigma = (2,3,\ldots,2k+2)$.

For any other $(2k+1)$-cell $e(\sigma)$, $\sigma_1$ must be equal to 1 and each face of $e(\sigma)$ must also have the first entry 1 which means that $X \cap R^1 = R^1$ for any $X \in \overline{e}(\sigma)$. So $RP(\eta_{2k+1} | \overline{e}(\sigma)) \to \overline{e}(\sigma)$ has a section given by $X \mapsto (X, R^1)$. Again by Lemma 2.1.3,

$$\eta_{2k+1} | \overline{e}(\sigma) \approx \xi^{2k} \oplus \lambda$$

where $\lambda$ can be even chosen to be trivial line bundle. Thus we complete the proof of the proposition. \qed
3. Two examples.

In this section, we will give two examples. In the first example, we demonstrate a method to calculate the obstruction classes. In the second one, we try to find as many as possible the trivial lines in the tangent bundle of a manifold up to cobordism.

**Example 3.1.** $H^{2n}(RP^{2n}, \bar{Z}) \approx Z$ and the generator is the obstruction class for the tangent bundle decomposition $\tau(RP^{2n}) \approx \xi^{2n-1} \oplus \lambda$.

Let $p : S^{2n} \to RP^{2n}$ be the covering map and $\tau : S^{2n} \to S^{2n}$ be the antipodal map. By Eilenberg Theorem $H^{2n}(RP^{2n}, \bar{Z}) \approx E^{2n}(S^{2n}, \bar{Z})$, where $E^{2n}(S^{2n}, \bar{Z})$ is the equivariant cohomology group which can be determined by the complex

$$\left\{ H_{p}(S^{p}, S^{p-1}, Z) \xrightarrow{\partial} H_{p-1}(S^{p-1}, S^{p-2}, Z) \mid p = 1, 2, \ldots, 2n \right\}.$$

From [15], one can choose the orientation $e^{p}$ of the cell $E_{\tau}^{p}$, the upper hemisphere so that $H_{p}(S^{p}, S^{p-1}, Z) = \langle e^{p}, \tau e^{p} \rangle$ and the boundary operator is given by

$$\partial(e^{p}) = e^{p-1} + (-1)^{p+1} \tau e^{p-1}$$

where $\tau e^{p}$ is the induced orientation on the cell of lower hemisphere by the antipodal map. As an equivariant group $H_{p}(S^{p}, S^{p-1}, Z)$ has one generator $e^{p}$, so the set of equivariant homomorphism $\text{Hom}^{Z_{2}}(H_{p}(S^{p}, S^{p-1}, Z), \bar{Z})$ has one generator $c_{e^{p}}$ which maps $e^{p}$ to 1, where the antipodal map generates $Z_{2}$ and acts on integers by multiplying $(-1)$. Now we can calculate the coboundary operator:

$$\delta(c_{e^{p}})(e^{p+1}) = c_{e^{p}}(\partial e^{p+1}) = c_{e^{p}}(e^{p} + (-1)^{p+1} \tau e^{p}) = 1 + (-1)^{p}$$

where we use the fact that $c_{e^{p}}(\tau e^{p}) = \tau(c_{e^{p}}(e^{p})) = -1$. Now it is easy to see that $E^{2n}(S^{2n}, \bar{Z}) = Z$ with generator $c_{e^{p}}$. So $H^{2n}(RP^{2n}, \bar{Z}) \approx Z$.

In the following obstruction class for the decomposition $\tau(RP^{2n}) \approx \xi^{2n-1} \lambda$ is generator of $H^{2n}(RP^{2n}, \bar{Z}) \approx Z$. Consider the fibration

$$p : RP(\tau(RP^{2n})) \longrightarrow RP^{2n}$$

whose fiber is $RP^{2n-1}$. The obstruction for the existence of a section for $p$ is the same as that of for the decomposition $\tau(RP^{2n}) \approx \xi^{2n-1} \oplus \lambda$. If we have a section $s_{2n-1}$ for $p : RP(\tau(RP^{2n})) \longrightarrow RP^{2n}$ on the $(2n-1)$-skeleton of $RP^{2n}$. Let

$$h : (B^{2n}, S^{2n-1}) \longrightarrow (RP^{2n}, RP^{2n-1})$$

be the characteristic map of the only $2n$-cell of $RP^{2n}$. In the pull-back diagram
The trivial bundle
the pull-back of the canonical
\( O \)
be the principal
pull-back of fibration
specific characteristic map
is in
The obstruction cocycle \( \text{ob}(s) \)
which in turn determines an element
which determines a map
which maps the decomposition obstructions of
momorphism:
By the naturality of the decomposition obstruction classes, \( \text{induces a ho-} \)
To see this, we consider the natural inclusion:
From Theorem 2.2.6, the obstruction for the decomposition \( \eta \) is
But the inclusion \( i \) induces an isomorphism on the fundamental groups:
By the naturality of the decomposition obstruction classes, \( i \) induces a homomorphism:
which maps the decomposition obstructions of \( \eta \) to that of \( i * (\eta) = \tau(\text{RP}(2n)) \).
Thus
In order to calculate \( [h] \) we need to choose a specific characteristic map \( h \) for the 2n-cell, and find the trivilization of the pull-back of fibration \( \text{RP}(2n) \rightarrow \text{RP}(2n) \). Let
be the principal \( O(k) \)-bundle, where \( V_k(R^m) \) is the Stiefel manifold, then the pull-back of the canonical \( k \)-bundle over \( G_k(R^m) \) by \( q \) is isomorphic to the trivial bundle
\( V_k(R^m) \times R^k \rightarrow V_k(R^m) \).
The isomorphism is given by

\[(\nu_1, \nu_2, \ldots, \nu_k, (t_1, t_2, \ldots, t_k)) \mapsto (\nu_1, \nu_2, \ldots, \nu_k, (V, \sum_{i=1}^{k} t_i \nu_i))\]

where \(\nu_1, \nu_2, \ldots, \nu_k\) are \(k\) unit orthogonal vectors in \(\mathbb{R}^m\), and \((V)\) is the \(k\)-dimensional subspace in \(\mathbb{R}^m\) generated by \(\nu_1, \nu_2, \ldots, \nu_k\). The inverse is given by

\[(\nu_1, \nu_2, \ldots, \nu_k, (V, \nu)) \mapsto ((\nu_1, \nu_2, \ldots, \nu_k), (\nu \cdot \nu_1, \nu \cdot \nu_2, \ldots, \nu \cdot \nu_k)).\]

So any vector bundle factors out though \(q\):

\[V^k(\mathbb{R}^m) \to G^k(\mathbb{R}^m)\]

is trivial and the trivialization is given by the above isomorphisms. In particular, we know the trivialization on any subspace of \(V^k(\mathbb{R}^m)\). The following theorem \cite{10} states that the characteristic map for any cell in \(G^k(R^m)\) can be chosen to be the restriction of \(q: V^k(\mathbb{R}^m) \to G^k(R^m)\) to some subspace of \(V^k(R^m)\).

**Theorem 3.2.** In the CW-structure of \(G_n(R^m)\),

\[G_n(R^m) = \{ \cup e(\sigma) \mid \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \]

is such that \(0 < \sigma_1 < \sigma_2 \cdots < \sigma_n \leq m\}

the characteristic map of the cell \(e(\sigma)\) can be given by

\[q|_{\overline{H}^1 \times \cdots \times \overline{H}^n} : V_n(R^m) \cap \overline{H}^1 \times \cdots \times \overline{H}^n \to G_n(R^m)\]

where \(\overline{H}^i = \{(x_1, x_2, \ldots, x_{\sigma_i}, 0, \ldots, 0) \in \mathbb{R}^m \mid x_{\sigma_i} \geq 0\}\).

Before further considering the characteristic map for the top cell in \(G_{2n}(R^{2n+1})\), we need the following lemma:

**Lemma 3.3.** Let \((\nu_1, \nu_2, \ldots, \nu_k) \in V_k(R^{k+1})\) be written in matrix form:

\[
\begin{bmatrix}
\nu_1 \\
\nu_2 \\
\vdots \\
\nu_k
\end{bmatrix}
= \begin{bmatrix}
\nu_{1,1} & \nu_{1,2} & \cdots & \nu_{1,k+1} \\
\nu_{2,1} & \nu_{2,2} & \cdots & \nu_{2,k+1} \\
\vdots & \vdots & \ddots & \vdots \\
\nu_{k,1} & \nu_{k,2} & \cdots & \nu_{k,k+1}
\end{bmatrix}
\]

then the vector \((\nu_1, \ldots, \nu_k)^\perp = (A_1, A_2, A_{k+1})\), where

\[A_i = (-1)^{i+1} \begin{bmatrix}
\nu_{1,1} & \cdots & \hat{\nu}_{1,i} & \cdots & \nu_{1,k+1} \\
\cdots & \cdots & \cdots & \cdots \\
\nu_{k,1} & \cdots & \hat{\nu}_{k,i} & \cdots & \nu_{k,k+1}
\end{bmatrix}\]

is an unit vector and orthogonal to each \(\nu_i\) for \(1, 2, \ldots, k\).
Proof. Let \( w = (w_1, w_2, \ldots, w_{k+1}) \) be the unique unit vector that is orthogonal to each \( \nu_i \) for \( i = 1, 2, \ldots, k \) and such that

\[
\begin{pmatrix}
\nu_1, & \cdots & \nu_i, & \cdots & \nu_{1,k+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\nu_{k,1} & \cdots & \nu_{k,i} & \cdots & \nu_{k,k+1} \\
w_1 & \cdots & w_{k,i} & \cdots & w_{k+1}
\end{pmatrix}
= (-1)^k.
\]

From the determinant properties we know that \((\nu_1, \ldots, \nu_k) \perp = (A_1, A_2, \ldots, A_{k+1})\) is orthogonal to each \( \nu_i \) for \( i = 1, 2, \ldots, k \) and so

\[
(\nu_1, \ldots, \nu_k) \perp = tw
\]

hence \( \langle (\nu_1, \ldots, \nu_k) \perp, w \rangle = t \langle w, w \rangle = t \). Expanding the determinant \((*)\) on the last row, we see that 

\[
(-1)^k \sum_{i=1}^{k+1} w_i A_i = (-1)^k,
\]

that is

\[
\langle (\nu_1, \ldots, \nu_k) \perp, w \rangle = 1.
\]

Thus we proved that \((\nu_1, \ldots, \nu_k) \perp = w\) and hence \((\nu_1, \ldots, \nu_k) \perp \) is an unit vector and orthogonal to each \( \nu_i \) for \( i = 1, 2, \ldots, k \).

Now we consider the \( 2n \)-cell in \( G_{2n}(R^{2n+1}) \) which corresponds to the Schubert symbol \((2, 3, \ldots, 2n+1)\). In this special case, we can further give a specific homeomorphism:

\[
h_{2n} : I^{2n} \longrightarrow V_{2n}(R^{2n+1}) \cap H^2 \times \cdots \times H^{2n+1}
\]

by

\[
h_{2n}(x_1, x_2, \ldots, x_{2n}) = (\nu_1, \nu_2, \ldots, \nu_{2n})
\]

where

\[
\nu_1 = x_1 e_1 + \sqrt{1 - x_1^2} e_2
\]

\[
\nu_2 = x_2 \left( \sqrt{1 - x_1^2} e_1 - x_1 e_2 \right) + \sqrt{1 - x_2^2} e_3
\]

\[
\cdots\cdots\cdots
\]

\[
\nu_k = x_k (\nu_1, \ldots, \nu_{k-1}) \perp + \sqrt{1 - x_k^2} e_{k+1}
\]

with inverse given by

\[
h_{2n}^{-1}(\nu_1, \nu_2, \ldots, \nu_{2n}) = ((\nu_1, e_1), (\nu_2, (\nu_1) \perp), \ldots, (\nu_{2n}, (\nu_1, \ldots, \nu_{2n-1}) \perp)).
\]

Now we consider the \((2n - 1)\)-skeleton section \( s_{2n-1} \) for the fibration

\[
p : RP(\tau(RP^{2n})) \longrightarrow RP^{2n} = G_{2n}(R^{2n+1})
\]

given by

\[
s_{2n-1}(X) = (X, [e_1]).
\]

In fact, for any cell \( e(\sigma) \) in \( BO(2n) \) and \( X \in e(\sigma) \), \( \dim(X \cap R^1) = 1 \) if \( \dim e(\sigma) < 2n \). So \( s_{2n-1} \) is a section over the \((2n - 1)\)-skeleton for the fibration

\[
RP(\eta_{2n}) \longrightarrow BO(2n).
\]
Now we can calculate the obstruction cocycle for the section $s_{2n-1}$ as follows: In the pull-back diagram

$$h^* (R\!P(\tau(RP^{2n}))) \xrightarrow{h^*} R\!P(\tau(RP^{2n}))$$

$$\downarrow \quad \quad \quad \quad \downarrow p$$

$$(B^{2n}, S^{2n-1}) \xrightarrow{h} (RP^{2n}, RP^{2n-1})$$

if we choose the characteristic map to be

$$qh_{2n} : I^{2n} \longrightarrow V_{2n}(R^{2n+1}) \cap \overline{H}^{2n+1} \times \cdots \times \overline{H}^{2n+1} \longrightarrow G_{2n}(R^{2n+1})$$

then the pull-back diagram will be equivalent to the following diagram

$$I^{2n} \times RP^{2n-1} \xrightarrow{qh_{2n}^*} R\!P(\tau(RP^{2n}))$$

$$\downarrow \quad \quad \quad \downarrow p$$

$I^{2n} \xrightarrow{qh_{2n}} G_{2n}(R^{2n+1})$

where $I^{2n} \times RP^{2n-1} \xrightarrow{qh_{2n}^*} R\!P(\tau(RP^{2n}))$ is given by

$$qh_{2n}^*((x_1, \ldots, x_{2n}), (t_1, \ldots, t_{2n})) = \left(qh_{2n}(x_1, \ldots, x_{2n}), \sum_i t_i \nu_i\right)$$

where $h_{2n}(x_1, x_2, \ldots, x_{2n}) = (\nu_1, \nu_2, \ldots, \nu_{2n})$. The induced section $s_{2n-1}^*$ over $\partial(I^{2n}) \approx S^{2n-1}$ is given by

$$s_{2n-1}^*(x_1, \ldots, x_{2n}) = ((x_1, \ldots, x_{2n}), [(e_1, \nu_1), \ldots, (e_1, \nu_{2n})])$$

which induces a map $h : \partial(I^{2n}) \approx S^{2n-1} \longrightarrow RP^{2n-1}$ given by

$$h(x_1, \ldots, x_{2n}) = [(e_1, \nu_1), \ldots, (e_1, \nu_{2n})]$$

Taking a close look at the formula for $h_{2n}(x_1, x_2, \ldots, x_{2n}) = (\nu_1, \nu_2, \ldots, \nu_{2n})$, we find that

$$[(e_1, \nu_1), \ldots, (e_1, \nu_{2n})] = \left[x_1, x_2 \sqrt{1 - x_1^2}, \ldots, x_{2n} \sqrt{1 - x_1^2} \cdots \sqrt{1 - x_{2n-1}^2}\right].$$

Thus $h(x_1, \ldots, x_{2n}) = \left[x_1, x_2 \sqrt{1 - x_1^2}, \ldots, x_{2n} \sqrt{1 - x_1^2} \cdots \sqrt{1 - x_{2n-1}^2}\right]$ which is homotopic to the map $(x_1, \ldots, x_{2n}) \longrightarrow [x_1, \ldots, x_{2n}]$ via the following homotopy

$$H_t(x_1, \ldots, x_{2n}) = \left[x_1, x_2 \sqrt{1 - tx_1^2}, \ldots, x_{2n} \sqrt{1 - tx_1^2} \cdots \sqrt{1 - tx_{2n-1}^2}\right].$$

Therefore $[h]$ represents the generator in $\pi_{2n-1}(RP^{2n-1})$, hence the obstruction class $[\text{ob}(s_{2n-1})]$ for the decomposition $\tau(RP^{2n}) \approx \xi^{2n-1} \oplus \lambda$ is the generator in $H^{2n}(RP^{2n}Z) \approx Z$. Thus we complete our first example.
In the following example, we will see that the decomposition $\xi^{2n+1} \approx \xi^{2n} \oplus \lambda$ is often possible. Let $MO_*$ denote the Thom cobordism ring. It is well-known that $MO_* = \sum_{n \ge 0} MO_n = \mathbb{Z}_2[X_n | n \neq 2^k - 1]$ is a graded polynomial algebra over $\mathbb{Z}_2$ with one generator in each dimension $n$ not in the form $2^k - 1$ for all $k > 0$.

**Example 3.4.** Let $[M_{2k+1}] \in MO_{2k+1}$ be a $(2k+1)$-dimensional cobordism class, then we can choose $M_{2k+1}$ such that $\tau(M_{2k+1}) \approx \xi^{2k} \oplus R$.

**Proof.** For any odd dimensional generator, we will choose such a representative. Consider the vector bundle $\lambda_1 \oplus \cdots \oplus \lambda_m$ over $RP(n_1) \times \cdots \times RP(n_m)$, where $\lambda_1$ is the pull-back of the canonical line bundle over $i$-th factor. Let $RP(n_1, \ldots, n_m)$ be the projective space bundle of $\lambda_1 \oplus \cdots \oplus \lambda_m$, then it is an $n$-dimensional smooth manifold, where $n = \sum n_i + m - 1$. From [13], $RP(n_1, \ldots, n_m)$ is indecomposable in $MO_n$ if and only if $\left( n - 1 \right) + \cdots + \left( n - 1 \right) \equiv 1 \mod 2$.

Let $n$ be a positive odd number that is not in the form $2^k - 1$, then $n$ can be uniquely written as $n = 2^p q + 1$, where $p, q$ are positive integers. Let $X_n$ be the manifold $RP(2^p, 1, \ldots, 1, 0)$ where the number of 1’s is $2^p q - 1$ which is greater than 0. Noticing that $2^p$ and 1 do not appear in the binary expression of $n - 1$, one can check that $RP(2^p, 1, \ldots, 1, 0)$ is indecomposable. From Borel-Hirzebruch [4], the tangent bundle of the projective space bundle $RP(\xi)$ associated with a vector bundle $\xi$ over a smooth manifold $M$ always splits:

$$\tau(RP(\xi)) \approx p^*(\tau(M)) \oplus \tau_2$$

where $p : RP(\xi) \to M$ is the projection, and $\tau_2$ is the bundle along the fiber. From this result, noticing that the tangent bundle of $RP^1$ is trivial, we see that the tangent bundle of $RP(2^p, 1, \ldots, 1, 0)$ has a $(2^p q - 1)$-dimensional trivial summand.

For even number $n$, we may just choose $RP^n$ to be the generator. From the polynomial structure of $MO_*$, any $[M_{2k+1}] \in MO_{2k+1}$ has the form:

$$[M_{2k+1}] = \sum_{I} \varepsilon_I X_{i_1} \cdots X_{i_r}$$

where $\varepsilon \in \mathbb{Z}_2$, and $\sum_{j=1}^{r} i_j = 2k + 1$. So at least one of the generator has odd dimension. Therefore the tangent bundle of $\sum_{I} \varepsilon_I X_{i_1} \cdots X_{i_r}$ has a trivial summand. □

Since the characteristic numbers are cobordism invariants, noticing that $w_{2n+1}(\xi^{2n} \oplus R) = 0$, and using the above example, we get a different proof of the fact that $\langle w_{2n+1}[M^{2n+1}] \rangle = 0$ for any odd dimensional closed manifold $M^{2n+1}$.  

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QUATERNIONIC REPRESENTATIONS OF EXCEPTIONAL LIE GROUPS

Hung Yean Loke

Let $G$ be a quaternionic real form of an exceptional group of real rank 4. Gross and Wallach show that three representations in the continuation of the quaternionic discrete series are unitarizable (see Gross and Wallach, 1996). In this paper we will determine the restrictions of these representations to certain subgroups of $G$ by computing explicitly the intersections of orbits. In particular we will determine certain compact dual pair correspondences of the minimal representation of $G$.

1. Introduction.

1.1. We refer to §3 of [GW2] or §2 of [L3] for the definition of the double cover $G$ of a quaternionic real form $G_0$ of a complex Lie group $G(\mathbb{C})$. $G$ has maximal compact subgroup $K$ of the form $M_1 \times M$ where $M_1$ is isomorphic to $SU_2$. Its Lie algebra has complexified Cartan decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$. Here $\mathfrak{p} = \mathbb{C}^2 \otimes V_M$ where $V_M$ is a self dual representation of $M(\mathbb{C})$.

1.2. Let $G(\mathbb{C})$ be one of the following simply connected complex exceptional groups:

$$F_4(\mathbb{C}), \quad E_6(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}, \quad E_7(\mathbb{C}), \quad E_8(\mathbb{C}).$$

We will index these four cases by $s = 1, 2, 4, 8$ respectively. Let $G(\mathbb{C})^0$ denote the connected component of $G(\mathbb{C})$. Then there exists a unique connected quaternionic real form $G_0^0$ of $G(\mathbb{C})^0$ (cf. §1.1). It has a real root system of type $F_4$. We will denote $G_0^0$ by $F_{4,4}, E_{6,4}, E_{7,4}$ and $E_{8,4}$ respectively. Set $G_0 = F_{4,4}, E_{6,4} \rtimes \mathbb{Z}/2\mathbb{Z}, E_{7,4}$ and $E_{8,4}$ respectively and let $G$ denote the corresponding double cover of $G_0$.

It is known that $\mathbb{P}V_M$ is a union of four $M(\mathbb{C})$-orbits. One of the orbits is Zariski dense and we will follow [GW2] and denote the Zariski closure of the remaining three orbits by $X, Y$ and $Z$. Here $Z$ is the unique closed orbit in $\mathbb{P}V_M$ and

$$\mathbb{P}V_M \supset X \supset Y \supset Z.$$  

Let $\mathcal{O}$ be either $X, Y$ or $Z$ and let $\bigoplus_n A^n(\mathcal{O})$ denote the coordinate ring of $\mathcal{O}$ in $\mathbb{P}V_M$. Note that $A^n(\mathcal{O})$ is a representation of $M$.  

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In [GW1], [GW2] Gross and Wallach construct a unitary representation $\sigma_O$ in the continuations of the quaternionic discrete series which is associated to $O$ in the sense that it has $K$-types ($K = SU_2 \times M$)

$$\sum_{n=0}^{\infty} \text{Sym}^{n+k}(\mathbb{C}^2) \otimes A^n(O)$$

(2)

where $k$ is the integer $3s + 3, 2s + 2$ and $s + 2$ if $O$ is $X, Y$ and $Z$ respectively. Again we will follow the notations of [GW2] and denote the three representations by $\sigma_X, \sigma_Y$ and $\sigma_Z$.

1.3. Let $G$ be one of the exceptional groups of real rank 4 in §1.2. A quaternionic Lie subgroup $G'$ of $G$ is defined as a semisimple Lie subgroup of $G$ containing $M_1$. Let $M' = G' \cap M$.

We refer to §2 and its references for the definition of a quaternionic representation of $G$ and $G'$. In [L3] we show that the restriction of $\sigma_O$ from $G$ to $G'$ decomposes discretely into a direct sum of quaternionic representations of $G'$. There we derived a formula to compute the spectrum of decompositions (See Thm. 3.4.1 [L3]) and we deduced that most of the irreducible components of the restriction to $G'$ are determined by the coordinate ring of the intersection of $O \cap \mathbb{P}V_0$. Here $\mathbb{P}V_0$ is a $M'(\mathbb{C})$-invariant subspace of $\mathbb{P}V_M$ and hence $O \cap \mathbb{P}V_0$ is a $M'(\mathbb{C})$-invariant projective variety. We will briefly recall these results in §3. Unfortunately the formula cannot be applied immediately to some interesting situations. In particular the coordinate ring of $O \cap \mathbb{P}V_0$ is in general difficult to compute.

This paper is a continuation of [L3]. The main objective of this paper is to determine the restriction of $\sigma_O$ in the following five situations (cf. (8)):

$$\text{G} = \tilde{E}_{6,4} \ni G' = \tilde{E}_{6,4} \times (\mathbb{Z}/2\mathbb{Z})$$

(3)

$$\text{G} = \tilde{E}_{7,4} \ni G' = \tilde{E}_{7,4} \times \mu_2 \times U_1$$

(6)

Here the tilde above the group denotes its double cover. We will compute the decompositions of the restrictions of the representations $\sigma_O$ by explicitly computing the intersections $O \cap \mathbb{P}V_0$. We have mentioned that one of the difficulty is to determine the intersections and their coordinate rings. Fortunately in each of the above $G'$ it is known that $\mathbb{P}V_0$ is a union of finitely many $M'(\mathbb{C})$ orbits and the intersection $O \cap \mathbb{P}V_0$ can therefore be effectively determined. We remark that the above subgroups $G'$ are just a few such examples. Our method presented in this paper should be applicable to other subgroups $G'$ where $M'(\mathbb{C})$ exhibit a dense orbit in $\mathbb{P}V_0$. 
The restrictions provide an efficient method for finding many quaternionic representations of the exceptional quaternionic Lie groups which are unitarizable. Since $\sigma_O$ has small Gelfand-Kirillov dimensions, the quaternionic representations obtained in the restriction would have small Gelfand-Kirillov dimensions too.

1.4. The representation $\sigma_Z$ is a ladder representation and it is annihilated precisely by the Joseph ideal and it is thus called the *minimal* representation of $G$. A pair of reductive subgroups $H_1 \times_C H_2$ in $G$ is called a (reductive) *dual pair* if the centralizers of $H_1$ and $H_2$ in $G$ are $H_2$ and $H_1$ respectively. The dual pair is called compact if either $H_1$ or $H_2$ is compact. Note that $G'$ in (3) to (7) are examples of compact dual pairs. In the appendix of [L3], we showed that certain correspondences of dual pairs in §1.3 exist and the second objective of this paper is to find the rest (if any) of the correspondences. We will also describe the dual pairs correspondences of $\text{Spin}(4,4) \times \mu_2^2 U_2$ in $E_{6,4}$, $E_{6,4} \times SU_3$ in $E_{8,4}$ in Corollary 4.10.1.

We remark that the restrictions of the minimal representations to dual pairs are of great interest. Exceptional dual pairs correspondences have been investigated by [HPS], [Li1], [Li2], [GS] and [L2]. In [GS] and [MS], the authors computed certain dual pair correspondences of $p$-adic exceptional groups by determining the intersection of orbits.

1.5. The organization of the paper is as follows: In §2 we will recall the definition of quaternionic groups and quaternionic representations. In §3 we will briefly state the restriction formula of [L3]. The main results of this paper are stated in §4. In §5 we describe the closures of the orbits $X, Y, Z$ in detail. The rest of the paper is devoted to the proofs of the main results in §4.

1.6. We define some notations. $\pi_G(a_1 \varpi_1 + \cdots + a_n \varpi_n)$ will denote the irreducible finite dimensional complex representation of a semisimple Lie group $G$ with highest weight $a_1 \varpi_1 + \cdots + a_n \varpi_n$ where $\varpi_i$ are the fundamental weights given in Planches [Bou]. If $V$ is a representation of $G$, then $S^n(V) = \text{Sym}^n V$ will denote its $n$-th symmetric product and $V^*$ its dual representation. $S^n$ will denote the representation $\text{Sym}^n(C^2)$ of $SU_2$. $\chi_1$ will denote a fundamental character of the compact torus $U_1$. $\mu_n$ will denote the cyclic group of order $n$. Suppose $H_1$ and $H_2$ are subgroups of $G$ and $C$ lies in the centers of both $H_1$ and $H_2$, then we denote

$$H_1 \times_C H_2 := (H_1 \times H_2)/\{(z,z) : z \in C\}.$$  

2. Quaternionic groups and representations.

2.1. In this section we define some notations and briefly recall the definitions of the quaternionic real form of an algebraic group and quaternionic representations.
2.2. We refer to §3 of [GW2] and §2 of [L3] for the definition of the quaternionic real form \( g_0 \) of a complex simple Lie algebra \( g \). It has Cartan decomposition \( g = \mathfrak{k} \oplus \mathfrak{p} \) where

\[
\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{m}.
\]

Here \( \mathfrak{m} \) is a reductive Lie subgroup of \( g \) and \( \mathfrak{p} = \mathbb{C}^2 \otimes V_M \) where \( V_M \) is a self dual representation of \( \mathfrak{m} \). \( \mathfrak{k} \) contains a Cartan subalgebra \( \mathfrak{h} \) of \( g \). We choose a positive root system \( \Phi^+ \) with respect to \( \mathfrak{h} \) such that the \( \mathfrak{su}_2 \) in \( \mathfrak{k} \) corresponds to the highest root \( \tilde{\alpha} \). Denote this Lie algebra by \( \mathfrak{su}_2(\tilde{\alpha}) \). \( t_1 = \mathfrak{h} \cap \mathfrak{su}_2(\tilde{\alpha}) \) is a Cartan subalgebra of \( \mathfrak{su}_2(\tilde{\alpha}) \). Then \( \mathfrak{l} = t_1 \oplus \mathfrak{m} \) is a Levi subalgebra of a maximal parabolic subalgebra \( g \) whose nilpotent radical \( \mathfrak{u} \) is a Heisenberg Lie algebra. The one dimensional center of \( \mathfrak{u} \) is spanned by the highest root space \( \mathfrak{g}_\alpha \) and let \( \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}] = V_M \) denote the representation of \( \mathfrak{m} \). \( \tilde{\mathfrak{g}} = \mathfrak{l} \oplus \mathfrak{u} \) will denote the opposite parabolic subalgebra.

Let \( G(\mathbb{C}) \) be a complex simply connected simple Lie group with Lie algebra \( g \) and let \( G_0 \) be a real form of \( G(\mathbb{C}) \) having Lie algebra \( g_0 \). We denote the real Lie subgroups in \( G(\mathbb{C}) \) corresponding to the various real forms of the Lie subalgebras \( \mathfrak{m}_0, \mathfrak{t}_0 \) and \( \mathfrak{e}_0 \) by \( M, L \) and \( K_0 = \text{SU}_2(\tilde{\alpha}) \times \mathbb{R}_2 M \) respectively. Here \( K_0 \) is the maximal compact subgroup of \( G_0 \) and \( \text{SU}_2(\tilde{\alpha}) = M_1 \) in §1.1. Let \( G \) denote the double cover of \( G_0 \) with maximal compact subgroup \( K = \text{SU}_2(\tilde{\alpha}) \times M \). We will call \( G \times H \) a quaternionic Lie group if \( G \) is a quaternionic simple Lie group and \( H \) is a compact Lie group.

Set \( 2d = \dim V_M \). If \( g \) is of type \( D_4, F_4, E_6, E_7, E_8 \), then \( d = 3s + 4 \) where \( s = 0, 1, 2, 4, 8 \) respectively. We tabulate some of the \( M(\mathbb{C}) \) and \( V_M \) in Table 1 below.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( G )</th>
<th>( M(\mathbb{C}) )</th>
<th>( V_M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>( \text{Spin}(d,4), d \geq 4 )</td>
<td>( SL_2 \times \text{Spin}(d) )</td>
<td>( \mathbb{C}^2 \otimes \mathbb{C}^d )</td>
</tr>
<tr>
<td>( e_1 )</td>
<td>( F_4,4 )</td>
<td>( Sp_6 )</td>
<td>( \pi(\varpi_3) )</td>
</tr>
<tr>
<td>( e_2 )</td>
<td>( E_6,4 \times \mathbb{Z}/2\mathbb{Z} )</td>
<td>( SL_6 \times \mathbb{Z}/2\mathbb{Z} )</td>
<td>( \pi(\varpi_3) )</td>
</tr>
<tr>
<td>( e_3 )</td>
<td>( E_7,4 )</td>
<td>( \text{Spin}(12) )</td>
<td>( \pi(\varpi_6) )</td>
</tr>
<tr>
<td>( e_8 )</td>
<td>( E_8,4 )</td>
<td>( \text{simply connected} \ E_7 )</td>
<td>( \pi(\varpi_1) )</td>
</tr>
</tbody>
</table>

Table 1.

2.3. We will define and review the properties of quaternionic representations. We refer to [S1], [W1], [W2], [GW1], [GW2, §5] and Theorem 3.3.1 of [L2] for proofs and details.

Let \( W[k] = e^{-\frac{k\tilde{\alpha}}{2}} \otimes W \) be an irreducible finite dimensional representation of \( L = U_1 \times M \). We extend \( W[k] \) trivially to a representation of \( \tilde{\mathfrak{g}} \) and denote

\[
H(G, W[k]) = H(W[k]) := \Gamma^1_{K/L} (\text{Hom} \mathcal{U}(\tilde{\mathfrak{g}}), (U(\mathfrak{g}), W[k])_L)
\]

as the Harish-Chandra module of \( G \) where \( \Gamma^1 \) is the first Zuckerman derived functor. It has infinitesimal character \( \mu + \rho(G) - \frac{k\tilde{\alpha}}{2} \). If \( k \geq 2 \), then \( H(W[k]) \)
has $K$-types ($K = SU_2(\tilde{\alpha}) \times M$)

\[ \sum_{n=0}^{\infty} S_{\tilde{\alpha}}^{k-2+n}(\mathbb{C}^2) \otimes (\text{Sym}^n(V_M) \otimes W). \]  

(9)

It contains a unique irreducible $(\mathfrak{g}, K)$-submodule denoted by $\sigma(G, W[k])$ which is generated by the translates of the lowest $K$-types

\[ S_{\tilde{\alpha}}^{k-2}(\mathbb{C}^2) \otimes W. \]

(10)

We will call $H(G, W[k])$ and $\sigma(G, W[k])$ quaternionic representations.


3.1. Let $G_0$ be one of the four exceptional groups given in Table 1 (es) indexed by $s = 1, 2, 4, 8$. Let $G$ be its double cover.

3.2. Suppose $G'$ is a quaternionic real Lie subgroup of $G$ containing $SU_2(\tilde{\alpha})$. We have correspondingly $K' = G' \cap K$, $M' = G' \cap M$ and the Lie algebras $\mathfrak{g}', \mathfrak{m}', \mathfrak{u}' = \mathfrak{u} \cap \mathfrak{g}'$. Write $\mathfrak{u} = \mathfrak{u}' + V_0$. We have $V_{M'} \subset V_M$ and we define $V_0 = V_M/V_{M'}$ as a representation of $M'$.

We will use $\text{Res}_{G}^{G'} \sigma$ to denote the restriction of a Harish-Chandra module $\sigma$ of $G$ to that of $G'$.

3.3. First we review the work of [GW2]. Denote $H_k := H(G, \mathbb{C}[k])$ and its unique submodule $\sigma(G, \mathbb{C}[k])$ by $\sigma_k$. $H_k$ is irreducible and unitarizable if $k \geq 3s + 4$. If $k > 6s + 8$, then it belongs to the discrete series. We recall Corollary 4.2.2 of [L2].

**Proposition 3.3.1.** There exists a filtration $H_n$ of $H_k$ such that $H_{n+1}/H_n = H(G', S^n(V_0)[n + k])$. If $H_k$ is unitarizable, then

\[ \text{Res}_{G}^{G'} H_k = \sum_{n=0}^{\infty} H(G', S^n(V_0)[n + k]) \]

and each summand on the right is irreducible and unitarizable.

3.4. Since $V_M$ is a self dual representation of $M$, we identify $V_M$ with its dual. In §1.2 we remarked that $\mathbb{P}V_M$ is a union of four $M(\mathbb{C})$-orbits. One of the orbit is Zariski dense and we denote the Zariski closure of the other three non-dense orbits by $X$, $Y$ and $Z$ satisfying (1). We will describe these three orbits in greater detail in §5.

Let $(k, m, \mathcal{O})$ be one of the three following sets of data:

\[ (3s + 3, 4, X), \ (2s + 2, 3, Y), \ (s + 2, 2, Z). \]

Let $I^s(\mathcal{O}) = \bigoplus_{n \geq m} I^n(\mathcal{O})$, ($I^n(\mathcal{O}) \neq 0$) be the homogeneous ideal of $\mathcal{O}$ in $\mathbb{P}V_M$ and $A^s(\mathcal{O}) = \bigoplus A^n(\mathcal{O})$ be its coordinate ring. Then $I^s$ is generated as a $S^s(V_m)$ module by $I^n(\mathcal{O})$ and each graded piece $A^n(\mathcal{O})$ is a representation of $M$. 
Gross and Wallach showed that $\sigma_k$ is a unitarizable proper submodule of $H_k$ with $K$ types given in (2). Furthermore it satisfies an exact sequence

$$0 \rightarrow \sigma_k \rightarrow H(G, \mathbb{C}[k]) \overset{\phi}{\rightarrow} H(G, I^m(O)[k + m]).$$

As a representation of $M(\mathbb{C})$, $I^m(O) = \mathbb{C}, V_M$ and $m$ respectively for the three values of $k$. When $k = 3s + 3$, $\phi$ in (11) is a surjection (cf. §8 of [GW2]). We will follow the notation of [GW1] and denote the three representations $\sigma_{3s+3}$, $\sigma_{2s+2}$ and $\sigma_{s+2}$ by $\sigma_X$, $\sigma_Y$ and $\sigma_Z$ respectively.

The annihilator ideal of $\sigma_Z$ is the Joseph ideal in $U(g)$ so $\sigma_Z$ is called the minimal representation of $G$. It has $K$-types

$$\sum_{n=0}^{\infty} S_{\alpha}^{s+n}(\mathbb{C}^2) \otimes \pi_M(n\lambda)$$

where $\lambda$ is the highest weight of $V_M$. Note that all the representations descend to representations of $G_0$ except $\sigma_X$ for groups of type $E_n$ and $\sigma_Z$ of $\tilde{F}_{4,4}$.

3.5. The inclusion $I^m(O) \subset S^m(V_M)$ gives rise to the following natural maps of $M$-modules:

$$\text{Sym}^{n-m}(V_M) \otimes I^m(O) \rightarrow \text{Sym}^{n-m}(V_M) \otimes \text{Sym}^m(V_M) \rightarrow \text{Sym}^n(V_M).$$

Let $r'_n$ denote the composite of the above maps. The direct sum $V_M = U' \oplus V_0$ (cf. §3.2) induces a natural map of $M'$-modules

$$r''_n : \text{Sym}^n(V_M) \rightarrow \text{Sym}^n(V_0).$$

We define $r_n = r''_n \circ r'_n$ for $n \geq m$. For $0 \leq n < m$, we set $r_n$ to be the zero map into $\text{Sym}^n(V_0)$. Let $R_n$ denote the cokernel of $r_n$ and let $R_* := \bigoplus_{n=0}^{\infty} R_n$. Note that $R_n$ is a representation of $M'$ and we write

$$R_n = \sum_j W_{n,j}$$

where $W_{n,j}$ are the irreducible subrepresentations of $M'$.

Let $O' = O \cap \mathbb{P} V_0$ and we denote its coordinate ring in $\mathbb{P} V_0$ by $A^* (O') = \bigoplus A^n(O')$. Then $O'$ is cut out by $r_m(I^m(O))$ and $R_*/\text{Nil}(R_*) = A^*(O')$.

We can now state Theorem 3.3.1 and Corollary 2.8.1 of [L3].
Theorem 3.5.1.

(a) \( \text{Res}^G_{G'} \sigma_k = \sum_{n=0}^{\infty} \sigma(G', R_n[k + n]) = \sum_{n=0}^{\infty} \sum_j \sigma(G', W_{n,j}[k + n]). \)

(b) \( \text{Res}^G_{G'} \sigma_k \supseteq \sum_{n=0}^{\infty} \sigma(G', A^n(O')|k + n|). \)

Equality holds if and only if \( r_m(I^m(O')) \) generates the ideal of \( O' \).

(c) If \( r_m \) is surjective, then \( r_n \) is surjective for \( n \geq m \) and

\( \text{Res}^G_{G'} \sigma_k = \sum_{n=0}^{m-1} \sigma(G', S^n V_0|k + n|). \)

4. The main results.

4.1. In this section we will state the main results on the restrictions of the quaternionic representations \( \sigma_O \) of the exceptional group \( G \) to its subgroup \( G' \). The proofs will be given in the later sections. We will replace \( G \) by \( G_0 \) if the representation descends to \( G_0 \). If \( H(G', W[n]) \) appears in the restriction formula, it means that \( H(G', W[n]) \) is irreducible and its \( K \)-types are given by (9).

4.2. Let \( G = \tilde{F}_{4,4} \supseteq G' = \tilde{\text{Spin}}(5,4) \) and \( M' = SU_2 \times \text{Spin}(5) \).

Let \( V_{m,n} = S^m(C^2) \otimes \pi_{\text{Spin}(5)}(n\pi_2) \) be the representation of \( M' \).

Theorem 4.2.1.

(a) \( \text{Res}^G_{G'} \sigma_Z = \sigma(G', \mathbb{C}[3]) + \sigma(G', V_{0,1}[4]). \)

(b) \( \text{Res}^{F_{4,4}}_{\text{Spin}(5,4)} \sigma_Y = \sum_{n=0}^{\infty} \sigma(\text{Spin}(5,4), V_{0,n}[4 + n]). \)

(c) \( \text{Res}^{F_{4,4}}_{\text{Spin}(5,4)} \sigma_X = H(\text{Spin}(5,4), \mathbb{C}[6]) + \sum_{n=1}^{\infty} \sigma(\text{Spin}(5,4), V_{0,n}[6 + n]). \)

(d) \( \text{Res}^G_{G'} H(G, \mathbb{C}[k]) = \sum_{n=0}^{\infty} H(G', V_{0,n}[k + n]) \text{ if } k \geq 7. \)

If \( n \geq 1 \) then the summands in (c) satisfy the following exact sequence:

\( 0 \rightarrow \sigma(F_{4,4}, V_{0,n}[6 + n]) \rightarrow H(F_{4,4}, V_{0,n}[6 + n]) \rightarrow H(F_{4,4}, V_{0,n}[9 + n]) \rightarrow 0. \)
4.3. Suppose \( G = \widetilde{F}_{4,4} \) and \( G' = \widetilde{\text{Spin}}(4,4) \). The maximal compact subgroup of \( G' \) is \( SU_2(\alpha) \times M' \) where \( M' = SU_2(A) \times SU_2(B) \times SU_2(C) \). Here \( \alpha, A, B, C \) are the (orthogonal) compact roots of \( \widetilde{\text{Spin}}(4,4) \). Let

\[
S(a, b, c) := S^a_A(C^2) \otimes S^b_B(C^2) \otimes S^c_C(C^2)
\]
denote the irreducible representation of \( M' \).

**Theorem 4.3.1.**

(a) \( \text{Res}_{G'}^{G} \sigma_Z = \sigma(G', C[3]) + \sigma(G', S(1, 0, 0)[4]) + \sigma(G', S(0, 1, 0)[4]) + \sigma(G', S(0, 0, 1)[4]) \).

(b) \( \text{Res}_{\text{Spin}(4,4)}^{F_{4,4}} \sigma_Y = \sum_{abc=0} \sigma(\text{Spin}(4,4), S(a, b, c)[4 + a + b + c]) \).

(c) \( \text{Res}_{\text{Spin}(4,4)}^{F_{4,4}} \sigma_X = \sum_{abc=0} H(\text{Spin}(4,4), S(a, b, c)[6 + a + b + c]) + \sum_{a,b,c>0} \sigma(\text{Spin}(4,4), S(a, b, c)[6 + a + b + c]) \).

(d) \( \text{Res}_{G}^{G'} H(G, C[k]) = \sum_{a,b,c \geq 0} H(G', S(a, b, c)[k + a + b + c]) \) if \( k \geq 7 \).

When \( a, b, c \) are strictly positive, the summands in (c) satisfy the following exact sequence:

\[
0 \to \sigma(\text{Spin}(4,4), S(a, b, c)[6 + a + b + c]) \to H(\text{Spin}(4,4), S(a, b, c)[6 + a + b + c]) \to H(\text{Spin}(4,4), S(a - 1, b - 1, c - 1)[7 + a + b + c]) \to 0.
\]

4.4. Note that \( \widetilde{\text{Spin}}(4,4) \) and \( \widetilde{\text{Spin}}(5,4) \) are components of dual pairs in (3) and (4). In §6 we will describe the action of \( (\mathbb{Z}/2\mathbb{Z})^2 \) and \( (\mathbb{Z}/2\mathbb{Z})^3 \) on the summands. In particular Theorems 4.2.1(a) and 4.3.1(a) give the dual pairs correspondences of the minimal representation \( \sigma_Z \) of \( \widetilde{F}_{4,4} \). The \( K \)-types of summands are given in Theorem 6.8.1 and 6.9.1.

4.5. Let \( G = \widetilde{E}_{6,4} \times \mathbb{Z}/2\mathbb{Z} \) and \( G' = \widetilde{F}_{4,4} \times \mathbb{Z}/2\mathbb{Z} \). Let \( \chi \) be the nontrivial character of \( \mathbb{Z}/2\mathbb{Z} \).
Theorem 4.5.1.

(a) \( \text{Res}_{E_{6,4} \times Z/2Z}^{E_{6,4} \times Z/2Z} \sigma_Z = \sigma(F_{4,4}, \mathbb{C}[4]) \otimes \chi^0 + \sigma(F_{4,4}, \mathbb{C}^6[5]) \otimes \chi. \)

(b) \( \text{Res}_{E_{6,4} \times Z/2Z}^{E_{6,4} \times Z/2Z} \sigma_Y = \sum_{n=0}^{\infty} \sigma(F_{4,4}, S^n(\mathbb{C}^6)[6 + n]) \otimes \chi^n. \)

(c) \( \text{Res}_{G}^{C} \sigma_X = H(\mathcal{F}_{4,4}, \mathbb{C}[9]) \otimes \chi^0 + H(\mathcal{F}_{4,4}, \mathbb{C}^6[10]) \otimes \chi + \sum_{n=2}^{\infty} \sigma(\mathcal{F}_{4,4}, S^n(\mathbb{C}^6)[9 + n]) \otimes \chi^n. \)

(d) \( \text{Res}_{G}^{C} H(G, \mathbb{C}[k]) = \sum_{n=0}^{\infty} H(\mathcal{F}_{4,4}, S^n(\mathbb{C}^6)[10 + n]) \otimes \chi^n \text{ if } k \geq 10. \)

If \( n \geq 2 \) then the summands in (c) satisfy the following exact sequence:

\[
(15) \quad 0 \rightarrow \sigma(\mathcal{F}_{4,4}, S^n(\mathbb{C}^6)[9 + n]) \rightarrow H(\mathcal{F}_{4,4}, S^n(\mathbb{C}^6)[9 + n])
\rightarrow H(\mathcal{F}_{4,4}, S^n-2(\mathbb{C}^6)[11 + n]) \rightarrow 0.
\]

4.6. Each of the summands of \( \sigma_X \) in Theorems 4.2.1(c), 4.3.1(c) and 4.5.1(c) satisfies a short exact sequence. The \( K' \)-types of the middle and the last term of the exact sequence is given by (9). Hence it is possible to determine the \( K' \)-types, the Gelfand-Kirillov dimensions and the Bernstein degrees of the summands.

4.7. Let \( G = \bar{E}_{7,4}, G' = \bar{E}_{6,4} \times_{\mu_3} U_1 \) and \( M' = SU_6 \times_{\mu_3} U_1 \).
Let \( V_{a,b} = \pi_{SU_6}(a\varpi_1 + b\varpi_6). \)

Theorem 4.7.1.

(a) \( \text{Res}_{E_{6,4} \times U_1}^{E_{7,4}} \sigma_Z = \sum_{a,b \geq 0, a+b=0} \sigma(E_{6,4}, V_{a,b}[6 + a + b]) \otimes \chi_1^{a-b}. \)

(b) \( \text{Res}_{E_{6,4} \times U_1}^{E_{7,4}} \sigma_Y = H(E_{6,4}, \mathbb{C}[12]) \otimes \chi_1^0
+ \sum_{a,b \geq 0} \sigma(E_{6,4}, V_{a,b}[10 + a + b]) \otimes \chi_1^{a-b}. \)

(c) \( \text{Res}_{G}^{C} \sigma_X = \sum_{n=0, a,b \geq 0} H(\mathcal{E}_{6,4}, V_{a,b}[15 + 2n + a + b]) \otimes \chi_1^{a-b}. \)

(d) \( \text{Res}_{G}^{C} H(G, \mathbb{C}[k]) = \sum_{a,b,n \geq 0} H(\mathcal{E}_{6,4}, V_{a,b}[k + 2n + a + b]) \otimes \chi_1^{a-b}
\text{ if } k \geq 16. \)

4.8. Let \( G = \bar{E}_{8,4}, G' = \bar{E}_{7,4} \times_{\mu_2} SU_2 \) and \( M' = \text{Spin}(12) \times_{\mu_2} SU_2 \). Let \( V_{a,b} = \pi_{\text{Spin}(12)}(a\varpi_1 + b\varpi_2). \)
Theorem 4.8.1.

(a) \( \text{Res}_{E_7,4 \times SU_2} \sigma_Z = \sum_{n=0}^{\infty} \sigma(E_7,4, V_{n,0}[10 + n]) \otimes S^n(C^2). \)

(b) \( \text{Res}_{E_7,4 \times SU_2} \sigma_Y = \sum_{a+2b+2c=n, bc=0} \sigma(E_7,4, V_{a,c}[18 + n]) \otimes S^{a+2b}(C^2). \)

(c) \( \text{Res}_{G \times G} \sigma_X = \sum \ast \vartheta(E_7,4, V_{a,c}[2n+27]) \otimes S^{a+2b}(C^2). \)

(d) \( \text{Res}_{G \times G} \vartheta(G, \mathbb{C}[k]) = \sum_{m=0}^{\infty} \sum \ast \vartheta(E_7,4, V_{a,c}[k+n+4m]) \otimes S^{a+2b}(C^2) \) if \( k \geq 28 \)

where the summation \( \sum^* \) is taken over all nonnegative integers \( a, b, c, d, n \) satisfying the relations

\( n - 2a \leq a + 2b + 2c + 4d \leq n, \quad cd = 0, \quad a \equiv n \mod (2). \)

4.9. Some of the restrictions stated in this section are known. We have included them for the sake of completeness. Moreover they follow with little additional effort from the proofs of the rest of the statements. The following results are known:

(i) Theorems 4.5.1(a) and 4.7.1(a) are unpublished results of B. Gross [G]. See §6 [GW1] for Theorem 4.8.1(a). The method of proof is by considering the decompositions of the \( K \)-types. Also see Theorem 12.1.1 [L2].

(ii) See §4.6 of [L2] for Theorems 4.3.1(c) and (d).

4.10. Using the above theorems, we will deduce the following dual pair correspondence in §9.4:

Corollary 4.10.1.

\( \text{Res}_{E_6,4 \times SU_2} \sigma_Z = \sum_{a,b \geq 0} \Theta(a,b) \otimes \pi_{SU_3}(a\varpi_1 + b\varpi_2) \)

where

\( \Theta(a,b) = \begin{cases} \sigma(E_6,4, \pi_{SU_6}(a\varpi_1 + b\varpi_5)[a + b + 10]) & \text{if } (a,b) \neq (0,0) \\ \vartheta(E_6,4, \mathbb{C}[10]) \oplus \vartheta(E_6,4, \mathbb{C}[12]) & \text{if } (a,b) = (0,0). \end{cases} \)

The dual pair correspondence of \( \text{Spin}(4,4) \times SU_2^2 \in E_6,4 \) will be given in Theorem 7.3.1.
4.11. The proofs of part (c) of Theorems 4.2.1 to 4.8.1 are similar. By (11) we have an exact sequence
\[ 0 \to \sigma_X \to \mathbf{H}(G, \mathbb{C}[d - 1]) \to \mathbf{H}(G, \mathbb{C}[d + 3]) \to 0. \]
The term on the right is irreducible and unitarizable. By Proposition 3.3.1 the restriction of the last term to $G'$ decomposes into
\[ \sum_{n=0}^{\infty} \mathbf{H}(G', S^n V_0[d + 3 + n]). \]
Applying the filtration in Proposition 3.3.1 to the middle and the last term of (18) gives a homomorphism of graded modules
\[ 0 \to \text{Res}_{G}^{G'} \sigma_X \to \sum_{n=0}^{\infty} \mathbf{H}(G', S^n V_0[d - 1 + n]) \to \sum_{n=0}^{\infty} \mathbf{H}(G', S^n V_0[d + 3 + n]) \to 0. \]
One can show that the above sequence is an exact sequence. We shall use (20) to prove part (c) of the above theorems.

5. Orbits computations.

5.1. In this section we will give a realization of the $M(\mathbb{C})$ action on $\mathbb{P}V_M$ and describe its subvarieties $X$, $Y$ and $Z$ (cf. §1.2). Please refer to [Ba], [Kim], [J1], [GW2] and [GL] for more details.

5.2. Let $s = 1, 2, 4, \text{ or } 8$ and let $\mathcal{K} = \mathcal{K}_s$ denote the composition algebra over $\mathbb{C}$ of dimension $s$. Then up to isomorphism, $\mathcal{K}_1 = \mathbb{C}$, $\mathcal{K}_4$ is the set of $2 \times 2$ complex matrices $M_2(\mathbb{C})$, $\mathcal{K}_2$ is the subset of diagonal elements in $M_2(\mathbb{C})$ and $\mathcal{K}_8$ is the set of Cayley numbers [J2]. Each algebra has an anti-automorphism $z \mapsto \bar{z}$ called conjugation such that $N(z) = z\bar{z} = \bar{z}z$ is a nondegenerate bilinear form on $\mathcal{K}$. Moreover $N(zz') = N(z)N(z')$. Define $\text{tr}(z) = z + \bar{z}$ and $\langle z, z' \rangle = \text{tr}(z\bar{z'})$. There are obvious embeddings $\mathcal{K}_1 \subset \mathcal{K}_2 \subset \mathcal{K}_4$.

5.3. Let $(U_0, \langle , \rangle)$ be the 3 dimensional complex inner product space with orthonormal basis $\{e_1, e_2, e_3\}$. Then $\mathcal{K}_8$ can be realized as elements of the form
\[ x = (a, d; v_1, v_2) := \begin{pmatrix} a & v_1 \\ v_2 & d \end{pmatrix} \]
where $a, d \in \mathbb{C}$ and $v_1, v_2 \in U_0$. The multiplication is given in [J2, p. 142] and $N(x) = ad - \langle v_1, v_2 \rangle$. We define an action of $g \in SL(U_0)$ on $\mathcal{K}_8$ by
\[ g : (a, d; v_1, v_2) \mapsto (a, d; hv_1, t_h^{-1}v_2). \]
We embed $\mathcal{K}_4 \subset \mathcal{K}_8$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & be_1 \\ ce_1 & d \end{pmatrix}$.
5.4. Let $\mathcal{J} = \mathcal{J}(\mathcal{K})$ be the Jordan algebra consisting of 3 by 3 Hermitian symmetric matrices of the form

$$J = (\gamma_1, \gamma_2, \gamma_3; c_1, c_2, c_3) := \left( \begin{array}{ccc} \gamma_1 & c_3 & \overline{c_2} \\ \overline{c_3} & \gamma_2 & c_1 \\ c_2 & \overline{c_1} & \gamma_3 \end{array} \right)$$

where $\gamma_i \in \mathbb{C}$ and $c_i \in \mathcal{K}$. The composition in $\mathcal{J}$ is given by $J_1 \circ J_2 = \frac{1}{2} (J_1 J_2 + J_1 J_2)$. Define an inner product on $\mathcal{J}$ given by $\langle X, Y \rangle = \text{Tr}(X \circ Y)$ where $\text{Tr}$ denotes the usual trace of matrices. There is a cubic form

$$\det(J) = \gamma_1 \gamma_2 \gamma_3 - \gamma_1 N(c_1) - \gamma_2 N(c_2) - \gamma_3 N(c_3) + \text{tr}(c_1(c_2 c_3))$$

on $\mathcal{J}$ which induces a trilinear form on $\mathcal{J}$ such that $(J, J, J) = \det J$. For $\mathcal{J}(\mathcal{K}_1)$ and $\mathcal{J}(\mathcal{K}_2)$, $\det$ is the usual determinant of 3 by 3 matrices. Finally we define a bilinear map $\mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J}$, $(J, J') \mapsto J \times J'$ such that in the notation of (23)

$$J \times J = (\gamma_2 \gamma_3 - N(c_1), \gamma_3 \gamma_1 - N(c_2), \gamma_1 \gamma_2 - N(c_3); \overline{c_2 c_3} - \gamma_1 c_1, \overline{c_3 c_1} - \gamma_2 c_2, \overline{c_1 c_2} - \gamma_3 c_3).$$

$J \neq 0$ is said to have rank 1 if $J \times J = 0$. $J$ has rank 0 if $J = 0$. $J \neq 0$ has rank 2 if $\det(J) = 0$ and it is not of rank 1.

Define

$$V_M := \mathbb{C} \oplus \mathcal{J} \oplus \mathcal{J} \oplus \mathbb{C}$$

and we denote a vector in $V_M$ by $(\xi, J, J', \xi')$. There is a realization of the $M(\mathbb{C})$ action on $V_M$ (see [Ba]). Let $p : V_M \setminus \{0\} \rightarrow \mathbb{P} V_M$ be the canonical projection.

We refer to (23) and define

$$\mathcal{J}_1 := \{(0, 0, 0; c_1, 0, 0) \in \mathcal{J} : c_1 \in \mathcal{K}\}.$$

Similarly we define $\mathcal{J}_2$ and $\mathcal{J}_3$. Define $W_i = \{(0, J, J', 0) \in V_M : J, J' \in \mathcal{J}_i\}$ for $i = 1, 2, 3$. We will need these definitions in §6 and §7.

5.5. The smallest orbit $Z$ is generated by the highest weight vector spanned by $p(1, 0, 0, 0)$. The stabilizer of $p(1, 0, 0, 0)$ is a maximal abelian parabolic $Q = L' \rtimes N'$ in $M(\mathbb{C})$. We denote $\overline{Q}' = L \rtimes \overline{N}'$ to be the opposite parabolic subgroup. Then $\overline{Q}'$ stabilizes the flag

$$\mathbb{C} \subset \mathcal{J} \oplus \mathbb{C} \subset \mathcal{J} \oplus \mathcal{J} \oplus \mathbb{C} \subset V_M.$$

There is a bijection $\mathcal{J}(\mathcal{K}) \rightarrow \overline{N}'$ given by $B \mapsto p_B$ where $p_B$ acts on $V_M$ by (see [Kim])

$$p_B : (0, J, J', 0) \mapsto (0, J, J' + 2B \times J, (B, B, J) + (B, J')).$$

We recall a version of Lemma 7.5 of [MS].
Lemma 5.5.1. Representatives of the $\mathcal{N}$-orbits on $Z$ are

$$v_1 = p(1, 0, 0, 0) \quad v_2 = p(0, J, 0, 0)$$
$$v_3 = p(0, 0, J', 0) \quad v_4 = p(0, 0, 0, 1)$$

where $J$ and $J'$ are rank 1 elements in $\mathcal{J}$.

5.6. The variety $X$ is the hypersurface given by the degree 4 polynomial (cf. [J1])

$$f_4(\xi, J, J', \xi') = (J \times J, J' \times J') - \xi \det(J) - \xi' \det(J') - \frac{1}{4}(\langle J, J' \rangle - \langle \xi, \xi' \rangle)^2.$$  

If $J_1 = (0, 0, 0; x, y, z)$ and $J_2 = (0, 0, 0; x', y', z')$, then

$$f_4(0, J_1, J_2, 0) = N(x)N(x') + N(y)N(y') + N(z)N(z') + \langle yz, y'z' \rangle$$
$$+ \langle zx, z'x' \rangle + \langle xy, x'y' \rangle - \frac{1}{4}(\langle x, x' \rangle + \langle y, y' \rangle + \langle z, z' \rangle)^2.$$  

Clearly $(0, J, 0, 0)$ and $(0, J_1, J_1, 0) \in X$.

5.7. $Y$ is the algebraic set cut out by the set of degree 3 polynomials $\{\partial f_4/\partial v : v \in V_M\}$. It contains the point $p(0, J, 0, 0)$ (resp. $p(0, 0, J', 0)$) if and only if $J$ (resp. $J'$) has rank at most 2. Similarly $Z$ contains the point iff $J$ (resp. $J'$) has rank 1.

5.8. Let $K = \mathbb{C}$, then the nontrivial outer automorphism of $M = SL_6$ in Table 1 (e2) acts on $V_M$ by sending $(\xi, J, J', \xi') \mapsto (\xi, J, J', \xi')$ where $J$ and $J'$ denotes taking conjugation of the entries.

5.9. Let $G_0 = \widetilde{Spin}(4, 4)$ and by setting $J$ to be the set of diagonal 3 by 3 complex matrices in (25), $V_M$ is the representation $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ of $M(\mathbb{C}) = SL_3(\mathbb{C})$. We index this case by $s = 0$.

6. Dual pairs in $\widetilde{F}_{4,4}$.

6.1. In this section let $G = \widetilde{F}_{4,4}$. It has maximal compact subgroup $K' = SU_2(\alpha) \times Sp_6$. Let $G' = \widetilde{Spin}(5, 4)$ and $G'' = \widetilde{Spin}(4, 4)$. In this section we will prove Theorems 4.2.1 and 4.3.1. We will retain the notations of §4.2 and §4.3.

6.2. The center $C$ of $\widetilde{Spin}(4, 4)$ is

$$C := \{(\epsilon_1 \epsilon_2 \epsilon_3, \epsilon_1, \epsilon_2, \epsilon_3) \in SU_2(\alpha) \times SU_2 : \epsilon_i = \pm 1 \in SU_2 \} \simeq \mu_2^3.$$  

$(-1, -1, -1, -1) \in C$ is the nontrivial center of $\widetilde{F}_{4,4}$. We would like to interpret the subgroup $\widetilde{Spin}(4, 4)$ in $\widetilde{F}_{4,4}$ as the dual pair

$$\widetilde{Spin}(4, 4) \times_C C.$$
We denote a character of $C$ by $\chi(s_1, s_2, s_3)$, $s_i \in (\mathbb{Z}/2\mathbb{Z})$ such that

$$(\epsilon_1, \epsilon_2, \epsilon_3) \mapsto \epsilon_1^{s_1} \epsilon_2^{s_2} \epsilon_3^{s_3}.$$ 

Let $C^\wedge$ denote the character group of $C$.

The element $\epsilon_1 \in C$ acts on $F_{4,4}$ by conjugation and it fixes the subgroup $\widetilde{\text{Spin}}(5, 4)$ in $\widetilde{F}_{4,4}$. It has maximal compact subgroup

$$K = SU_2(\alpha) \times (SU_2(A) \times \text{Spin}(5)).$$

The center $C_1 \subset C$ of $\widetilde{\text{Spin}}(5, 4)$ is the Klein 4 group

$$C_1 := \{ (\epsilon_1, \epsilon_1, \epsilon_2, \epsilon_2) \in C : \epsilon_i = \pm 1 \}.$$ 

We denote the character group of $C_1$ by $C_1^\wedge$. It consists of characters

$$\chi(s_1, s_2) : (\epsilon_1, \epsilon_2) \mapsto \epsilon_1^{s_1} \epsilon_2^{s_2},$$

where $s_i \in \mathbb{Z}/2\mathbb{Z}$.

We have the following see-saw pairs in $\widetilde{F}_{4,4}$:

$$\widetilde{\text{Spin}}(5, 4) \triangleleft C$$

(30)

$$\widetilde{\text{Spin}}(4, 4) \triangleleft C_1.$$

6.3. The $S_3$ outer automorphism group on $\widetilde{\text{Spin}}(4, 4)$ permutes the 3 factor subgroups of $SU_2^3 \subset M''$ and $G$ contains

(31) 

$$(\widetilde{\text{Spin}}(4, 4) \times C) \rtimes S_3.$$ 

Hence if $\text{Res}_{\widetilde{\text{Spin}}(4, 4)} \sigma(\mathbb{C}[k])$ contains $\sigma(\text{Spin}(4, 4), S(a, b, c)[k])$, then it will also contain

$$^s \sigma = \sigma \left( \text{Spin}(4, 4), S_{s(a)}^a \otimes S_{s(B)}^b \otimes S_{s(C)}^c | k \right),$$

where $s \in S_3$.

6.4. By (31) $\text{Spin}(4, 4) \rtimes S_3 \subset F_{4,4}$ and $M''(\mathbb{C}) = SL_3^2(\mathbb{C}) \rtimes S_3$. Set $K = \mathbb{C}$ and we recall the definition of $J_i$ and $W_i$ ($i = 1, 2, 3$) in (26). The $W_i$'s give the standard representations of each of the 3 factor groups of $SU_2^3 \subset M''$. $V_0 = W_1 \oplus W_2 \oplus W_3$. The outer automorphism group $S_3$ acts on $V_0$ by permuting the $W_i$'s. Then $\mathbb{P}V_0$ has a dense $M''(\mathbb{C})$-orbit. There are two more orbits and their closures are

$$X_2 := \mathbb{P}(W_1 \oplus W_2) \cup \mathbb{P}(W_2 \oplus W_3) \cup \mathbb{P}(W_3 \oplus W_1)$$

$$X_1 := \mathbb{P}W_1 \cup \mathbb{P}W_2 \cup \mathbb{P}W_3 \subset X_2.$$ 

The homogeneous ideal of $X_2$ in $\mathbb{P}V_0$ is generated by

(32) 

$$W_0 = W_1 \otimes W_2 \otimes W_3 \subset S^3(V_0)$$.
and it has coordinate ring
\[ A^n(X_2) = \sum_{a,b,c} S^a(W_1) \otimes S^b(W_2) \otimes S^c(W_3) \]
where the sum is taken over all nonnegative integers \(a, b, c\) such that \(a + b + c = n\) and \(abc = 0\).

Lemma 6.4.1.
(a) \(\mathbb{P}V_0 \cap X = \mathbb{P}V_0\).
(b) \(\mathbb{P}V_0 \cap Y = X_2\).
(c) \(\mathbb{P}V_0 \cap Z\) is the empty set.

Proof. Let \(J_1 = (0, 0, 0; 0, 0, 1)\), \(J_2 = (0, 0, 0; 0, 1, 1)\), \(J_3 = (0, 0, 0; 1, 1, 1)\) and \(v_i = (0, J_i, 0) \in X\) for \(i = 1, 2, 3\). \(p(v_3) \in (X \cap \mathbb{P}V_0) \setminus X_2\) so \(\mathbb{P}V_0 \cap X\) strictly contains \(X_2\) and thus equals \(\mathbb{P}V_0\). This proves (a). Note that \(p(v_3) \notin Y\) so \(\mathbb{P}V_0 \cap Y \subset X_2\). On the other hand \(p(v_2) \in Y \cap X_2\). This proves (b). Finally \(p(v_3) \in X_1\) but it does not lie in \(Z\). This proves (c).

6.5. Consider \(\text{Spin}(5, 4) \subset F_{4,4}\) and \(M' = SU_2 \times \text{Spin}(5) = SU_2 \times Sp_4\). Then \(V'_0 = W_2 \oplus W_3\) gives the standard representation of \(Sp_4\). Note that \(\mathbb{P}V'_0\) is a single \(Sp_4\) orbit.

Lemma 6.5.1.
(a) \(\mathbb{P}V'_0 \cap X = \mathbb{P}V'_0 \cap Y = \mathbb{P}V'_0\).
(b) \(\mathbb{P}V'_0 \cap Z\) is the empty set.

Proof. Since \(V'_0 \subset V_0\), the lemma follows from Lemma 6.4.1.

6.6. Proof of Theorems 4.2.1(c)(d) and 4.3.1(c)(d). Part (d) follows easily from Proposition 3.3.1 since \(S^nV'_0 = V_{0,n}\) and \(S^nV_0 = \sum_{a+b+c=n} S(a, b, c)\) respectively.

(c) By (20) we have
\[
0 \rightarrow \text{Res} \sigma_X \rightarrow \sum_{n=0}^{\infty} \text{H}(\text{Spin}(5, 4), V_{0,n}[d - 1 + n]) \rightarrow \sum_{n=0}^{\infty} \text{H}(\text{Spin}(5, 4), V_{0,n}[d + 3 + n]) \rightarrow 0.
\]
Considering the infinitesimal characters of the summands gives (13).

The restriction of \(\sigma_X\) to \(\text{Spin}(4, 4)\) and (14) have been proven in Prop. 4.6.1 of [L2]. This proves (c).

Proof of Theorem 4.2.1(b). \(r_3 : V_M \rightarrow S^3(V'_0)\) and since the codomain is irreducible, \(r_3\) is either the zero map or a surjection. However the image of \(r_3\) has to cut out the empty set so \(r_3 = 0\). This implies \(r_n = 0\) and \(R_n = S^n(V_0)\) for all \(n\).
6.7. Proof of Theorem 4.3.1(b). $r_3 : V_M \rightarrow S^3(V_0)$ and

$$V_M = S^1_A + S^1_B + S^1_C + (S^1_A \otimes S^1_B \otimes S^1_C)$$

$$S^3(V_0) = \sum_{a+b+c=3} S(a, b, c)$$

as representations of $M''$. Thus image of $r_3$ is either 0 or $W_0$ (cf. (32)). By Lemma 6.4.1 the image has to cut out $X_2$ so it is $W_0$. Moreover $W_0$ generates the ideal of $X_2$ and thus $R_n = A^n(X_2)$. □

6.8. Recall that the minimal representation $\sigma_Z$ of $\tilde{F}_{4,4}$ has $K$-types

$$\sum_{n=0}^{\infty} S^m_{\alpha}((\mathbb{C}^2) \otimes \pi_{Sp_6}(n\varpi_3)).$$

Suppose

$$\text{Res}_{Spin(5,4) \times C_4} \sigma_Z = \sum_{\chi \in C_4^+} \Theta'(\chi) \otimes \chi.$$ 

The center of $\tilde{F}_{4,4}$ acts nontrivially on $\sigma_Z$ and hence $C_1$ only acts by the characters $\chi(1, 0)$ and $\chi(1, 1)$. Theorem 4.2.1(a) is a consequence of the following theorem:

**Theorem 6.8.1.** Let $\epsilon = 0, 1$. Then $\Theta'(\chi(1, \epsilon)) = \sigma(\tilde{\text{Spin}}(5, 4), V_0[4])$ and it has $K$-types $(K = SU_2(\widetilde{\alpha}) \times \mu_2 (SU_2 \times \text{Spin}(5)))$

$$\sum_{a, b \geq 0} S_{\alpha}^{a+2b+1+\epsilon}((\mathbb{C}^2) \otimes S^a(\mathbb{C}) \otimes V_{a, 2b+\epsilon}).$$

**Proof.** By Lemma 6.5.1, $O'$ is empty so $r'_2 : S^2V_0' \rightarrow S^2V_0'$ is not the zero map. Hence $r'_2$ is a surjection and Theorem 3.5.1(a) applies. The following lemma proves the claim about the $K$-types and completes the proof of the theorem:

**Lemma 6.8.2.**

$$\text{Res}_{SU_2 \times \text{Spin}(5) \times C_4} \pi(n\varpi_3) = \sum_{a+b=n} S^a(\mathbb{C}) \otimes \pi_{\text{Spin}}(5)(a\varpi_1 + b\varpi_2) \otimes \chi(a, b).$$

We omit the proof of the lemma since branching law of $Sp_6$ is known (see Equation (25.27) [FH]). □

6.9. Suppose

$$\text{Res}_{Spin(4,4) \times C} \sigma_Z = \sum_{\chi \in C^\wedge} \Theta(\chi) \otimes \chi.$$ 

Since the center of $\tilde{F}_{4,4}$ acts nontrivially on $\sigma_Z$, $C$ can only act by the characters $\chi(s_1, s_2, s_3)$ where $s_1 + s_2 + s_3$ is an odd integer.

Theorem 4.5.1(a) is a consequence of the following theorem:
Theorem 6.9.1.
(a) $\Theta(\chi(1, 1, 1)) = \sigma(\widetilde{\text{Spin}}(4, 4), \mathbb{C}[3])$.
(b) $\Theta(\chi(1, 0, 0)) = \sigma(\widetilde{\text{Spin}}(4, 4), S(1, 0, 0)[4])$.
(c) $\Theta(\chi(0, 1, 0)) = \sigma(\widetilde{\text{Spin}}(4, 4), S(0, 1, 0)[4])$.
(d) $\Theta(\chi(0, 0, 1)) = \sigma(\widetilde{\text{Spin}}(4, 4), S(0, 0, 1)[4])$.

$\Theta(\chi(1, 1, 1))$ (resp. $\Theta(\chi(1, 0, 0))$) has $K$-types ($K = SU_2 \times SU_2^2$)

\[
\sum_{n,a,b,c \geq 0} S_{n,a}^{1+n}(\mathbb{C}) \otimes S(a, b, c)
\]

where the sum is taken over all nonnegative integers $n, a, b, c$ such that $a + b + c \geq n$, $a + b - c \leq n$, $a - b + c \leq n$, $b + c - a \leq n$ and $n \equiv a \equiv b \equiv c \pmod{2}$ (resp. $n \equiv a \not\equiv b \equiv c \pmod{2}$).

The $K$-types of $\Theta(\chi(0, 1, 0))$ and $\Theta(\chi(0, 0, 1))$ differ from that of $\Theta(\chi(1, 0, 0))$ by permuting $a, b, c$ in (34) accordingly under the action of $S_3$ (cf. §6.3).

Proof. By the action of $S_3$, it suffices to prove (a) and (b). By the see-saw pair (30), we note that

\[
\Theta'(\chi(1, 0)) = \Theta(\chi(1, 1, 1)) + \Theta(\chi(1, 0, 0)).
\]

The branching rule from Spin (5) to Spin (4) = $SU_2^2$ is well-known (see eqn (25.34) [FH]). Applying this to Theorem 6.8.1 shows that the sum of the $K$-types of (a) and (b) agrees with the $K$-types in (35). A $K$-type

\[ S^n(\mathbb{C}^2) \otimes S(a, b, c) \]

in (35) will belong to $\Theta(\chi(1, 1, 1))$ (resp. $\Theta(\chi(1, 0, 0))$) if and only if $a - b$ is an even (resp. odd) integer. This proves (34).

By naturality, the composition of the map

\[ I^2(Z) \to S^2 V_0 \to S^2 V_0' = S^2_B + S^2_C + \mathbb{C}_B \otimes \mathbb{C}_C \]

is the map $r_2^*$ in the proof of Theorem 6.8.1 and it is surjective. This implies that the image contains $S^2_B + S^2_C + \mathbb{C}_B \otimes \mathbb{C}_C$. Since $r_2$ commutes with the action of $S_3$ (cf. §6.3), $r_2$ is surjective and the theorem follows from Theorem 3.5.1(c). \hfill \Box

7. Dual pairs in $E_{6,4}$.

7.1. Let $G' \subset G$ be one of the dual pairs (5) to (7). We set $s = 2, 4, 8$ and we define

\[ J^p_s = \{(0, 0, 0; x_1, x_2, x_3) \in J(K_s) : x_i \in K_s, \langle x_i, z \rangle = 0 \text{ for all } z \in K_s/2 \} \]

then

\[ V_0 = \{(0, J, J', 0) \in V_M : J, J' \in J^p_s \}. \]
It is known that $M'(\mathbb{C})$ has a dense orbit in $\mathbb{P}V_0$ [SK]. The orbits and their coordinate rings have been extensively studied and they are documented in §7 [GW2]. We will make use of these results to determine $R_\bullet$.

7.2. Consider $G = E_{6,4} \ltimes \mathbb{Z}/2\mathbb{Z} \supset G' = \tilde{F}_{4,4} \ltimes \mathbb{Z}/2\mathbb{Z}$. Set $s = 2$. Then $V_0 = \mathbb{C}^6$ in (36) is the standard representation of $M'(\mathbb{C}) = Sp_6(\mathbb{C}) \times \mathbb{Z}/2\mathbb{Z}$. It is well-known that $\mathbb{P}V_0$ is a single orbit of $M'(\mathbb{C})$.

Lemma 7.2.1.

(a) $\mathbb{P}V_0 \cap X = \mathbb{P}V_0 \cap Y = \mathbb{P}V_0$.

(b) $\mathbb{P}V_0 \cap Z$ is an empty set.

Proof. (a) It suffices to show that $\mathbb{P}V_0 \cap Y$ is nonempty since $X \supset Y$ and $\mathbb{P}V_0$ is a single orbit of $M'(\mathbb{C})$. Let $J = (0, 0, 0; s, 0, 0) \in J(K_2)$ where $s = \text{diag}(1, -1) \in K_2$. By (36) $(0, J, 0, 0) \in V_0$. Since $\det(J) = 0$, $p(J) \in Y$ by §5.7.

(b) We will prove this by contradiction. Suppose on the contrary $\mathbb{P}V_0 \cap Z$ is nonempty and it equals $\mathbb{P}V_0$. Let $v = (0, J, J', 0) \in V_0$ as given in (36) and assume that $J \neq 0$. Since $p(v) \in Z$, by Lemma 5.5.1 and (27) $J$ has rank at most 1. By (24), $J = 0$. This yields the contradiction. □

Proof of Theorem 4.5.1. (a) By Lemma 7.2.1(b), $r_2 : \mathfrak{s}_6 \rightarrow S^2(\mathbb{C}^6)$ is nonzero. Since the image is irreducible $r_2$ is a surjection. (a) follows from Theorem 3.5.1(c).

(b) and (c) follow from Lemma 7.2.1(a) and Theorem 3.5.1(a) since $R_n = S^n(\mathbb{C}^6)$.

By (20) we have

$$0 \rightarrow \text{Res} \sigma_X \rightarrow \sum_{n=0}^{\infty} \mathbf{H}(\tilde{F}_{4,4}, S^n(\mathbb{C}^6)[9 + n]) \otimes \chi^n$$

$$\rightarrow \sum_{n=0}^{\infty} \mathbf{H}(\tilde{F}_{4,4}, S^n(\mathbb{C}^6)[13 + n]) \otimes \chi^n \rightarrow 0.$$ 

Finally considering the infinitesimal characters of the above summands gives (15). □

7.3. Consider $G_0 = E_{6,4} \ltimes \mathbb{Z}/2\mathbb{Z} \supset G''_0 = \text{Spin}(4, 4) \times \mu_2^2 (U_1^2 \times \mathbb{Z}/2\mathbb{Z})$ and $M'' = SU_2^3 \times \mu_2^2 (U_1^2 \times \mathbb{Z}/2\mathbb{Z})$. Here we identify $U_1^2 = \{(t_1, t_2, t_3) : t_i \in U_1, t_1t_2t_3 = 1\}$.

Let $\chi_0(a_1, a_2, a_3) : (t_1, t_2, t_3) \mapsto t_1 a_1 t_2 a_2 t_3 a_3$ be a character of $U_1^2$ where $a_i \in \mathbb{Z}$. Let $\chi(a_1, a_2, a_3)$ be the unique irreducible representation of $U_1^2 \times \mathbb{Z}/2\mathbb{Z}$ containing $\chi_0(a_1, a_2, a_3)$. Note that $\chi(a, a, a) = \mathbb{C}$ and if not all the $a_i$’s are the same, then $\chi(a_1, a_2, a_3) = \chi_0(a_1, a_2, a_3) + \chi_0(-a_1, -a_2, -a_3)$. Clearly
\(\chi(a_1, a_2, a_3) = \chi(a_1 - a, a_2 - a, a_3 - a) = \chi(-a_1, -a_2, -a_3)\). Therefore we may assume that \((a_1, a_2, a_3)\) takes values from the set

\[
T := \{(a_1, a_2, a_3) \in \mathbb{Z}^3 : a_i > a_{i+1} = 0 \geq a_{i+2}
\text{ for some } i = 1, 2, 3\} \cup \{(0, 0, 0)\}.
\]

Set \(\mathcal{K} = \mathcal{K}_2\) and we define \(\mathcal{J}_i\) and \(W_i\) \((i = 1, 2, 3)\) using (26). Then \(V_0 = W_1 \oplus W_2 \oplus W_3\). \(W_i = W_{i1} \oplus W_{i2}\) where \(W_{i1}\) and \(W_{i2}\) are the standard representations of \(SU_2\). \((t_1, t_2, t_3) \in U_\mathbb{C}^2\) acts on \(W_{i1}\) (resp. \(W_{i2}\)) by multiplication by \(t_i\) (resp. \(t_i^{-1}\)).

Write \(V_0 = \sum_{i,j} W_{ij}\) and we denote an element of \(V_0\) by \((w_{ij})\) where \(w_{ij} \in W_{ij}\). Let \(\mathcal{V}\) denote the subset of \(V_0\) consisting of \((w_{ij})\) satisfying the following:

1. For each \(j = 1, 2\), at least 2 of \(w_{1j}, w_{2j}, w_{3j}\) is zero.
2. For each \(i = 1, 2, 3\), either \(w_{i1} = 0\) or \(w_{i2} = 0\).

Then one can show that \(Z \cap \mathbb{P}V_0 = \mathbb{P}\mathcal{V}\) and the ideal of \(\mathbb{P}\mathcal{V}\) is generated by degree 2 polynomials. Its coordinate ring is

\[
A^n(\mathbb{P}\mathcal{V}) = \sum S(|a_1|, |a_2|, |a_3|) \otimes \chi(a_1, a_2, a_3)
\]

where the sum is taken over all \((a_1, a_2, a_3) \in T\) satisfying \(|a_1| + |a_2| + |a_3| = n\).

By Theorem 3.5.1(c) we get the dual pair correspondence of \(G''\) in \(G\).

**Theorem 7.3.1.**

\[
\operatorname{Res}_{\operatorname{Spin}(4, 4) \times (U_\mathbb{C}^2 \times \mathbb{Z}/2\mathbb{Z})}^{E_6, 4 \times \mathbb{Z}/2\mathbb{Z}} \sigma_Z
\]

\[
= \sum_{(a_1, a_2, a_3) \in T} \sigma(\operatorname{Spin}(4, 4), S(|a_1|, |a_2|, |a_3|)[|a_1| + |a_2| + |a_3| + 4]) \otimes \chi(a_1, a_2, a_3).
\]

The \(K\)-types of the summands could be calculated using (12) by applying the branching rule from \(M = SU_6\) to \(SU_3^2 \times U_\mathbb{C}^2 \subset M''\). This in turn could be computed using the Littlewood-Richardson rule (see [FH, p. 456]).

Note that the summands of the above correspondence also appear in Theorem 4.3.1(b). This follows from the fact that the dual pairs \(G_0''\) and \(E_{4, 4} \times \mathbb{Z}/2\mathbb{Z}\) form a see-saw pair in \(G_0\). Similarly restrictions of \(\sigma_Y\) and \(\sigma_X\) to \(G_0''\) will yield representations of \(\operatorname{Spin}(4, 4)\) appearing in Theorem 4.3.1(c) and (d).

**8. Dual pair in \(E_{7, 4}\).**

**8.1.** Consider \(G_0 = E_{7, 4} \supset G_0' = (E_{6, 4} \times U_1) \times \mathbb{Z}/2\mathbb{Z}\) and \(M'(\mathbb{C}) = GL_6 \rtimes \mathbb{Z}/2\mathbb{Z}\). The nontrivial element in \(\mathbb{Z}/2\mathbb{Z}\) acts on \(E_{6, 4}\) as the outer automorphism. It also acts on \(U_1\) by sending \(z \mapsto z^{-1}\).

Set \(s = 4\) and define \(V_0\) as in (36). \(z \in U_1(\mathbb{C}) = \mathbb{C}^*\) will act on \(V_0\) by

\[
(0, J, J', 0) \rightarrow (0, zJ, z^{-1}J', 0).
\]
Recall $K_4 = M_2(\mathbb{C})$ and let $K_u$ (resp. $K_l$) denote the subspace of strictly upper (resp. lower) triangular matrices. Define

$$J_a = \{(0, 0, 0; c_1, c_2, c_3) \in J(K_4) : c_i \in K_u\}$$
$$V_a = \{(0, J, J', 0) \in V_0 : J, J' \in J_a\}.$$

Similarly we define $J_l$ and $V_l$ by replacing $K_u$ with $K_l$. $V_u$ (resp. $V_l$) gives the standard (resp. dual) representation of $GL_6$. Thus $V_0 = \mathbb{C}^6 \oplus (\mathbb{C}^6)^*$ as a representation of $M'(\mathbb{C}) = GL_6 \times \mathbb{Z}/2\mathbb{Z}$.

There are only two nontrivial proper orbits of $M'$ in $\mathbb{P}V_0$ (cf. §6 [GW2]). Their closures are

$$X_1 = \mathbb{P}C^6 \cup \mathbb{P}(\mathbb{C}^6)^*$$
$$\mathcal{F} = \{(v, v^*) \in \mathbb{P}V_0 : f_0 := \langle v, v^* \rangle = 0\}.$$

Note that $X_1 \subset \mathcal{F}$. The inner product $f_0$ defining $\mathcal{F}$ is an $M'$-invariant quadratic form in $S^2V_0$ and this in turn induces an inclusion

$$S^nV_0 : f_0 \mapsto S^{n+2}V_0.$$ 

The coordinate ring $A^n(\mathcal{F})$ is the quotient of the above inclusion. Recall §4.7 where we define $V_{a,b} = \pi_{SL_6}(a\varpi_1 + b\varpi_6)$. Then

$$A^n(\mathcal{F}) = \sum_{a+b=n} V_{a,b} \otimes \chi_1^{a-b}$$
$$A^n(\mathbb{P}V_0) = A^n(\mathcal{F})[f_0] = \sum_{a+b+2m=n} V_{a,b} \otimes \chi_1^{a-b}$$

as a representation of $GL_6 = SL_6 \times \mu_6 U_1$.

Lemma 8.1.1.

(a) $\mathbb{P}V_0 \cap Z = \mathbb{P}C^6 \cup \mathbb{P}(\mathbb{C}^6)^*$.
(b) $\mathcal{F} = \mathbb{P}V_0 \cap X = \mathbb{P}V_0 \cap Y$.

Proof. Define

$$s = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in K_4 = M_2(\mathbb{C}),$$
$$x_1 = (0, 0, 0; s, 0, 0), \quad x_2 = (0, 0, 0; t, 0, 0) \in J(K_4),$$
$$T_1 = (0, x_1, 0, 0), \quad T_2 = (0, x_1 + x_2, 0, 0), \quad T_3 = (0, x_1, x_2, 0) \in V_0.$$

By Lemma 5.5.1 $p(T_1) \in Z \cap \mathbb{P}V_0$. Hence $Z \cap \mathbb{P}V_0$ is nontrivial and it must contain $X_1$. $x_1 + x_2$ has rank 2 so $p(T_2) \in Y \setminus Z$ (cf. §5.7) and $p(T_2) \notin X_1$. This proves (a) and implies that $\mathbb{P}V_0 \cap Y \supseteq \mathcal{F}$.

Next $p(T_3) \in \mathbb{P}V_0 \setminus \mathcal{F}$. $f_4(T_3) \neq 0$ so $T_3 \notin X$. Since $\mathcal{F}$ is maximal proper, $\mathcal{F} \supseteq X \cap \mathbb{P}V_0 \supseteq Y \cap \mathbb{P}V_0$. This proves (b).
Proof of Theorem 4.7.1.  (a) The image of \( r_2 \) must be cut out \( X_1 \) so it is either \( V_{1,1} \) or \( \mathbb{C}f_0 + V_{1,1} \) in \( S^2 V_0 \).

We claim that the image cannot be \( V_{1,1} \). Indeed if otherwise by Theorem 3.5.1(b) the restriction of \( \sigma_Z \) will contain \( \sigma' = \sigma(E_{6,4}, \mathbb{C}[6]) \). \( \sigma' \) contains the lowest \( K' \)-type \( \tau = S^4_0(\mathbb{C}^2) \otimes \mathbb{C} (K' = SU_2(\tilde{\alpha}) \times SU_6) \). Therefore \( \tau \) is a subrepresentation of the \( K \)-type

\[
S^4_\alpha(\mathbb{C}^2) \otimes \pi_{\text{Spin}(12)}(2\varpi_6)
\]

in \( \sigma_Z \). However the tables of [KP] show that the above does not contain \( \tau \). This proves the claim.

Finally since \( V_{1,1} + \mathbb{C}f_0 \) generates the homogeneous ideal of \( X_1 \) so \( R_* = A^*(\mathcal{F}) \) (cf. (37)). This proves (a).

(b) Similar to (a) the image of \( r_3 \) cuts out \( \mathcal{F} \) and it has to be \( V_0 : f_0 \subset S^3 V_0 \). The ideal generated by \( V_0 f_0 \) contains all homogeneous polynomials vanishing on \( \mathcal{F} \) except \( \mathbb{C}f_0 \). Thus \( R_* = A^*(\mathcal{F}) + \mathbb{C}f_0 \). Finally we note that \( \mathbf{H}(E_{6,4}, \mathbb{C}[12]) \) is irreducible (cf. §3.3).

(c) By (20) we get

\[
0 \rightarrow \text{Res} \sigma_X \rightarrow \sum_{a,b} \mathbf{H}(\tilde{E}_{6,4}, V_{a,b}[15 + 2n + a + b]) \otimes \chi_1^{a-b} \\
\rightarrow \sum_{a,b,n} \mathbf{H}(\tilde{E}_{6,4}, V_{a,b}[19 + 2n + a + b]) \otimes \chi_1^{a-b} \rightarrow 0.
\]

The summands on the right are all irreducible and unitary, and they also appear as summands in the middle term. Therefore removing these representations from (38) gives

\[
0 \rightarrow \text{Res} \sigma_X \rightarrow \sum_{n=0} \sum_{a,b} \mathbf{H}(\tilde{E}_{6,4}, V_{a,b}[15 + 2n + a + b]) \otimes \chi_1^{a-b} \rightarrow 0.
\]

This completes the proof of (c). \( \square \)

Note that in (c) the image of \( r_4 \) is one dimensional and it cuts out \( \mathcal{F} \) and it has to be \( (f_0)^2 \subset S^4 V_0 \). Therefore \( R_* \) is not reduced. This example shows that the containment in Theorem 3.5.1(b) may be proper.
9. Dual pairs in $E_{8,4}$.

9.1. In this section we consider $\tilde{E}_{8,4} \supset \tilde{E}_{7,4} \times_{\mu_3} SU_2$ and $M' = \text{Spin}(12) \times SU_2$. Set $s = 8$ and define $V_0$ by (36).

9.2. Recall $U_0$ in §5.3. Define $U_e := \mathbb{C}e_2 \oplus \mathbb{C}e_3 \subset U_0$. By (22) $SL(U_e)$ acts on $\mathcal{K}_8$. This induces an action of $SL(U_e)$ on $(0, J, J', 0) \in V_0$ where $SL(U_e)$ acts uniformly on each of the nonzero entries of $J$ and $J'$.

For $i = 1, 2$, define $J_i = (0, 0, 0; x_{i1}, x_{i2}, x_{i3}) \in \mathcal{J}$ where $x_{ij} = (0, 0; w_{ij1}e_2, -w_{ij2}e_3)$ in $\mathcal{K}_8$ (cf. (21)) and $w_{ijk} \in \mathbb{C}$. Let $v = (0, J_1, J_2, 0) \in V_0$. Denote the set of such vectors $v$ in $V_0$ by $V_1$.

Similarly for $i = 1, 2$, define $J'_i = (0, 0, 0; x'_{i1}, x'_{i2}, x'_{i3}) \in \mathcal{J}$ where $x'_{ij} = (0, 0; w'_{ij1}e_3, w'_{ij2}e_2) \in \mathcal{K}_8$ and $w'_{ijk} \in \mathbb{C}$. Let $v' = (0, J'_1, J'_2, 0) \in V_0$. Denote the set of such vectors $v'$ in $V_0$ by $V_2$.

Note that $V_0 = V_1 \oplus V_2$. Both $V_1$ and $V_2$ give the standard representations of $\text{Spin}(12)$ in $M(\mathbb{C})$. The invariant quadratic forms on $V_1$ and $V_2$ are given respectively by

$$q_1(w_{ijk}) = \sum_{j=1}^{3} (w_{1j1}w_{2j2} - w_{1j2}w_{2j1})$$

$$q_2(w'_{ijk}) = \sum_{j=1}^{3} (w'_{1j1}w'_{2j2} - w'_{1j2}w'_{2j1}).$$

Let $\omega = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) \in SL(U_e)$, $(w_{ijk}) \in V_1$ and $(w'_{ijk}) \in V_2$. The action of $\omega$ on $V_0$ commutes with that of $\text{Spin}(12)$ and $\omega V_2 = V_1$ by sending $(w'_{ijk}) \mapsto (w_{ijk})$ where $w_{ijk} = w'_{ijk}$. Let $w_1, w_2 \in V_1$, define $\phi : V_1 \otimes U_e \to V_0$ by

$$\phi : w_1 \otimes e_2 + w_2 \otimes e_3 \to w_1 + \omega^{-1}w_2.$$ 

Then $\phi$ is an isomorphism of representations of $M'(\mathbb{C}) = \text{Spin}(12) \times SL(U_e)$.

9.3. We will describe the orbits of $M'(\mathbb{C})$ on $\mathbb{P}V_0$. $\mathbb{P}V_0$ has a dense $M'(\mathbb{C})$ orbit. It contains four additional orbits and we denote their closures by $X_1$, $Y_1$, $Y_2$, $Z_1$. Let $\langle , \rangle$ denote the inner product induced by quadratic form $q_1$ on $V_1$ and let $v = w_1 \otimes e_2 + w_2 \otimes e_2$. Then $X_1$ is a hypersurface whose ideal is generated by a degree 4 polynomial $f'_4$

$$f'_4(v) = \det \left( \begin{array}{cc} \langle w_1, w_1 \rangle & \langle w_1, w_2 \rangle \\ \langle w_2, w_1 \rangle & \langle w_2, w_2 \rangle \end{array} \right).$$

$Y_1$ is the complete intersection of the 3 quadrics

$$\langle w_1, w_1 \rangle = \langle w_1, w_2 \rangle = \langle w_2, w_2 \rangle = 0.$$
\( Y_2 \subset X_1 \) is the subvariety \( \mathbb{P}W \times \mathbb{P}U \). Let \( Q \subset \mathbb{P}W \) be defined by \( \langle w_1, w_1 \rangle = 0 \). Then \( Z_1 = Q \times \mathbb{P}U = Y_1 \cap Y_2 \) is the unique minimal closed orbit in \( \mathbb{P}V_0 \). \( Y_1 \cup Y_2 \) is cut out by cubics

\[
\begin{align*}
f_a(v) &:= \det \begin{pmatrix}
\langle w_1, w_1 \rangle & \langle w_1, a \rangle \\
\langle w_2, w_1 \rangle & \langle w_2, a \rangle
\end{pmatrix} = 0, \\
f_b'(v) &:= \det \begin{pmatrix}
\langle b, w_1 \rangle & \langle b, w_2 \rangle \\
\langle w_2, w_1 \rangle & \langle w_2, w_2 \rangle
\end{pmatrix} = 0
\end{align*}
\]

for all \( a, b \in W \). Let \( I_0 \) be the homogeneous ideal generated by \( \{ f_a, f_b' : a, b \in W \} \). We claim that \( I_0 \) is the homogeneous ideal of \( Y_1 \cup Y_2 \). Indeed suppose \( f \) vanishes on \( Y_1 \cup Y_2 \), by (39) we may assume that modulo \( I_0 \)

\[
f(w_1, w_2) = h_1(w_1)\langle w_1, w_1 \rangle + h_2(w_1, w_2)\langle w_1, w_2 \rangle + h_3(w_2)\langle w_2, w_2 \rangle
\]

where \( h_1, h_2, h_3 \) are polynomials on \( V_1, W \) and \( V_2 \) respectively. Since \( f(w_1, w_2) = 0 \) whenever \( w_1 \) is parallel to \( w_2 \), we get \( h_1 = h_3 = 0 \) and \( h_2 \) vanishes on \( Y_2 \). Thus \( f \in I_0 \) and this proves the claim.

Recall §4.8 where \( V_{a,b} = \pi_{Spin(12)}(a\overline{w}_1 + b\overline{w}_2) \). Then the coordinate rings are:

\[
\begin{align*}
(41) & \quad A^\bullet(\mathbb{P}V_0) = A^\bullet(X_1)[f_4'] \\
& \quad A^n(X_1) = \sum a V_{a+2d,c} \otimes S^{a+2b}(U) \\
& \quad \text{where the sum } \sum \text{ is taken over } a, b, c, d \text{ satisfying (16).} \\
(42) & \quad A^n(Y_1) = \sum_{a+2c=n} V_{a,c} \otimes S^n(U). \\
(43) & \quad A^n(Y_2) = S^n(W) \otimes S^n(U) = \sum_{a+2b=n} V_{a,0} \otimes S^n(U). \\
(44) & \quad A^n(Y_1 \cup Y_2) = \sum_{a+2b+2c=n, bc=0} V_{a,c} \otimes S^{a+2b}(U). \\
(45) & \quad A^n(Z_1) = V_{n,0} \otimes S^n(U).
\end{align*}
\]

The coordinate rings except (44) are given in §5, §6 [GW2]. Since \( A^n(Y_1 \cup Y_2) \) is a quotient of \( A^n(X_1) \) which is multiplicity free, (44) follows from (42) and (43).

**Lemma 9.3.1.**

(a) \( X \cap \mathbb{P}V_0 = X_1 \).

(b) \( Y \cap \mathbb{P}V_0 = Y_1 \cup Y_2 \).

(c) \( Z \cap \mathbb{P}V_0 = Z_1 \).

**Proof.** For the ease of notations, suppose \( J_i = (0, 0, 0; c_{i1}, c_{i2}, c_{i3}) \in J \) \( (i = 1, 2) \), then we denote \( (0, J_1, J_2, 0) \in V_0 \) by \( (c_{11}, c_{12}, c_{13}|c_{11}, c_{12}, c_{13}) \). For \( i = 2, 3 \), let \( x_i = (0, 0; e_i, e_i), y_i = (0, 0; e_i, 0) \in K_8 \) and (cf. (21)).
(a) Set \( v_1 := (x_2, x_2, x_2)(0, 0, 0) \in V_0 \). Then \( f_4(v_1) \neq 0 \). Hence \( X \cap PV_0 \) is a hypersurface in \( PV_0 \) of degree 4 and it has to be \( X_1 \).

(b) \( J_4 = (0, 0, 0; x_2, x_2, x_2) \) has rank 3 so \( p(0, J_4, 0, 0) \in (X \setminus Y) \cap PV_0 \). Hence \( Y \cap PV_0 \subseteq Y_1 \cup Y_2 \). \( J_5 = (0, 0, 0; x_2, 0, 0) \) has rank 2 so \( p(0, J_5, 0, 0) \in (Y \setminus Z) \cap (Y_1 \setminus Z_1) \). Hence \( Z \cap PV_0 \subseteq Y_2 \) and \( Y_1 \subseteq Y \cap PV_0 \).

Let \( v_3 = (y_2, 0, 0|y_3, 0, 0) \) so that \( pv_3 \in Y_2 \setminus Z_1 \). It is easy to check that \( v_3 \) satisfies \( \partial f_4 / \partial v = 0 \) in \( \S 5.7 \) so \( pv_2 \in Y \). This implies that \( Y \supseteq Y_2 \).

(c) We have seen in (b) that \( Z \cap PV_0 \subseteq Y_2 \). \( J_6 = (0, 0, 0; y_2, 0, 0) \) has rank 1 so \( p(0, J_6, 0, 0) \in Z \cap PV_0 \). Hence \( Z \cap PV_0 \supseteq Z_1 \). To complete the proof it suffices to show that \( pv_3 \not\in Z \). Indeed otherwise, by Lemma 5.5.1, \( v_3 = p_B(0, J_6, 0, 0) \) for some \( B = (\beta_i; d_i) \in \mathcal{J} \) (cf. (27)). Computations show that \( p_B(J_6) = (0, J_6, 2B \times J_6, 0) \). However \( 2B \times J_6 = (*, *, *; -\beta_1 y_2, *, *) \neq (0, 0, 0; y_3, 0, 0) \). Hence \( pv_3 \not\in Z \). \( \square \)

**Proof of Theorem 4.8.1.** (a) This is determined by the map

\[
\begin{align*}
\tau_2 : & \mathfrak{e}_7 = \mathbb C \otimes S^2U + V_{0,1} + U \otimes \pi_{\text{Spin}_{12}}(w_6) \rightarrow S^2(W \otimes U) = V_{2,0} \otimes S^2U + S^2(U) + V_{0,1}.
\end{align*}
\]

The image of \( \tau_2 \) is nonzero so it is either \( S^2U, V_{0,1} \) or the sum. By (42) (resp. (43)) \( S^2U \) (resp. \( V_{0,1} \)) vanishes on \( Y_1 \) (resp. \( Y_2 \)). By Lemma 9.3.1(c) the image cuts out \( Z_1 \) and hence it is must be the sum. Since \( I^\ast(Z_1) \) is generated by degree 2 polynomials, \( R_n = A^n(Z_1) \).

(b) \[
\begin{align*}
\tau_3 : & V_M = W \otimes U + \pi_{\text{Spin}_{12}}(w_5) \rightarrow S^3(V_0) = V_{3,0} \otimes S^3(U) + W \otimes U + W \otimes S^3(U) + V_{1,2} \otimes U.
\end{align*}
\]

Since \( \tau_3 \) is nontrivial, the image has to be \( W \otimes U \) and they are the set of cubics in (40). We have shown that the set of cubics generates the ideal of \( Y_1 \cup Y_2 \) and (b) follows from Theorem 3.5.1(b) and (44).

(c) By (20) and (41) we get

\[
0 \rightarrow \text{Res} \sigma_X \rightarrow \sum_{n,m} H(G', A^m(X_1)[27 + m + 4n]) \rightarrow \sum_{n,m} H(G', A^m(X_1)[31 + m + 4n]) \rightarrow 0.
\]

The summands on the right also appear in the middle term. Therefore by removing these representations from (46) we get

\[
0 \rightarrow \text{Res} \sigma_X \rightarrow \sum_{m=0}^{\infty} H(G', A^m(X_1)[27 + m + n]) \rightarrow 0.
\]

This completes the Proof of Theorem 4.8.1(c).
(d) This follows from Proposition 3.3.1 and (41). □

9.4. Proof of Corollary 4.10.1. First we recall a well-known fact [FH].

Lemma 9.4.1. The 1 dimensional character $\det^{a-b}$ of $U_2$ is the only $SU_2$ fixed vector in $\pi_{SU_3}(a\varpi_1 + b\varpi_2)$.

Consider the see-saw pair

$$E_{7,4} \quad SU_3$$

$$\downarrow \updownarrow$$

$$E_{6,4} \quad SU_2$$

By Theorem 4.8.1(a), the trivial representation of $SU_2$ corresponds to the representation $\sigma_Y$ of $E_{7,4}$. Applying Lemma 9.4.1 to the see-saw pair (47) gives

$$\sum_{a,b \geq 0} \Theta(a,b) \otimes \chi_1^{a-b}$$

$$= \text{Res}_{E_{6,4} \times U_1} \sigma_Y$$

$$= H(E_{6,4}, \mathbb{C}[12]) \otimes \chi_1^0 + \sum_{a,b \geq 0} \sigma(E_{6,4}, \pi_{SU_6}(a\varpi_1 + b\varpi_5)(a + b + 10)) \otimes \chi_1^{a-b}.$$

The second equality is Theorem 4.7.1(b). By Table 1B of the Appendix to [L3], $\Theta(a,b)$ contains the right-hand side of (17). Alternatively, one can deduce this by considering the correspondence of the infinitesimal characters [Li3]. By (48) the containment is an equality. This proves Corollary 4.10.1. □

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A WEAK MULTIPLICITY-ONE THEOREM FOR SIEGEL MODULAR FORMS

 Rudolf Scharlau and Lynne H. Walling

Using the explicit action of the Hecke operators $T(p)$ acting on the Fourier coefficients of Siegel modular forms of arbitrary degree and level, a short and elementary proof and a generalization of a result by Breulmann and Kohnen is obtained, which says that cuspidal eigenforms are determined by their coefficients on matrices of square-free content.

1. Introduction.

In a recent paper by Breulmann and Kohnen [BK99], the authors obtain a weak multiplicity-one result on (integral weight) Siegel-Hecke cuspidal eigenforms of degree 2, showing that such forms are completely determined by their coefficients on matrices of the form $mS$, where $S$ is primitive and $m$ is square-free. To show this, they twist Andrianov’s identity relating the Maaß-Koecher series and the spinor zeta function of an eigenform [An74] by a Grössencharacter. This allows them to then use Imai’s converse theorem for degree 2 forms [Im80] and thereby obtain their result.

In this note, we use an elementary algebraic argument to reprove and extend their result to Siegel modular forms of arbitrary degree $n$ and arbitrary level which are only assumed to be eigenforms for the operators $T(p)$ (but not necessarily for the full Hecke algebra). We first show that such an eigenform must have primitive matrices in the support of its Fourier development. Then it is immediate from the explicit action of the Hecke operators on Fourier coefficients that if two such forms have the same eigenvalues for all $T(p)$ and the same coefficients on primitive matrices then their difference must be zero. Since, moreover, the assumption of coinciding eigenvalues can be derived from the above stated assumption of Breulmann and Kohnen, we recover their result for $n = 2$.

Note that Andrianov’s identity and Imai’s converse theorem are currently only known for $n = 2$ and level 1, so the analytic approach used in [BK99] cannot at this time be extended to general $n$. 

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2. Preliminaries.

Let $F$ be a degree $n$ Siegel modular form with Fourier expansion

$$F(\tau) = \sum_S c(S)e\{S\tau\},$$

where $S$ runs over all symmetric positive semidefinite even integral $n \times n$ matrices $S$ and $e\{\tau\} = \exp(\pi i \text{trace } \tau)$. We consider each $S$ to be a quadratic form on a $\mathbb{Z}$-lattice $\Lambda$ of rank $n$ relative to some basis for $\Lambda$. As $S$ varies, the pair $(\Lambda, S)$ varies over all isometry classes of rank $n$ lattices with even integral positive semi-definite quadratic forms. Also, the isometry class of $(\Lambda, S)$ is that of $(\Lambda, S')$ if and only if $S' = S[G] = G^tSG$ for some $G \in \text{GL}_n(\mathbb{Z})$.

When $k$ is even, $F(\tau[G]) = F(\tau)$ for all $G \in \text{GL}_n(\mathbb{Z})$, so it follows that $c(S[G]) = c(S)$. Hence, (with $k$ even) we can rewrite the Fourier expansion of $F$ in the form

$$F(\tau) = \sum_{\text{class } \Lambda} c(\Lambda)e^*\{\Lambda\tau\},$$

where $c(\Lambda) = c(S)$ for any matrix $S$ representing the quadratic form on $\Lambda$, and with $O(\Lambda)$ the orthogonal group of $\Lambda$ we set

$$e^*\{\Lambda\tau\} = \sum_{G \in O(\Lambda) \setminus \text{GL}_n(\mathbb{Z})} e\{S[G]\tau\}.$$

When $k$ is odd, we have $F(\tau[G]) = \det G \cdot F(\tau)$, so $c(S[G]) = \det G \cdot c(S)$, and a completely analogous formula holds with $\Lambda$ considered as an oriented lattice (i.e., a pair consisting of a lattice and one of the two orientation classes of its bases), and the sum in the definition of $e^*$ being over $\text{SO}(\Lambda) \setminus \text{SL}_n(\mathbb{Z})$.

In what follows we make use of the ‘content’ and the ‘discriminant’ of a lattice. When $\Lambda$ is a lattice with quadratic form $q$, the content $\text{cont } \Lambda$ of $\Lambda$ is defined as

$$\text{cont } \Lambda := \gcd\{q(x,x)/2 \mid x \in \Lambda\}.$$

If $q$ on $\Lambda$ has the Gram matrix $S$, with respect to some basis, then $\text{cont } \Lambda$ is just the gcd of the entries $s_{ij}, i \neq j, s_{ii}/2$ of $S$. (The term ‘content’ is standard for symmetric matrices, but not for lattices; 2$\text{cont } \Lambda$ is equal to what is usually called the ‘norm’ of the lattice $\Lambda$; see [O’M71] for further information.) The determinant of $S$ does not depend on the choice of the basis and is called the discriminant $\text{disc } \Lambda$ of $\Lambda$. It is zero if the lattice (i.e., its quadratic form) is degenerate. In this case, $q$ induces a nondegenerate form on $\overline{\Lambda} := \Lambda/\text{rad } \Lambda$, where $\text{rad } \Lambda \subseteq \Lambda$ denotes the radical of $q$. The dimension of $\overline{\Lambda}$ is also called the rank of $\Lambda$, or rather of $(\Lambda, q)$. We write

$$\text{rank } \Lambda := \dim \overline{\Lambda} = \dim \Lambda - \dim \text{rad } \Lambda.$$

For a positive rational number $\alpha$, the notation $\Lambda^\alpha$ means we “scale” $\Lambda$, or rather the pair $(\Lambda, q)$, by $\alpha$, i.e., $\Lambda^\alpha$ is equipped with the quadratic form $\alpha q$. 

We summarize here the results on content, scaling and discriminant used in the proofs below; $\Lambda$ is a lattice of rank $r$ and $\Omega$ a sublattice of finite index.

- $[\Lambda : \Omega] = m \implies \text{disc } \Omega = m^2 \text{disc } \Lambda$.
- $\text{disc } \Lambda^\alpha = \alpha^r \text{disc } \Lambda$.
- $\text{cont } \Lambda^\alpha = \alpha \text{cont } \Lambda$.

The first formula is well-known and is verified e.g., by taking a pair of elementary divisor bases of $\Omega \subseteq \Lambda$ and their corresponding Gram matrices; the other two formulas are obvious.

We now recall from e.g., [Fr83], Kapitel IV, the notion of the Hecke operators $T(p)$, for all primes $p$, acting on Siegel modular forms degree $n$, level $\ell$ and character $\chi$ (and any fixed weight $k$). The Siegel modular form $T(p)F$ or $F|T(p)$, with $F$ as above, is defined by averaging $F$ over the double coset $\Gamma_n(\ell)g\Gamma_n(\ell)$ of the rational symplectic similitude $g = \begin{pmatrix} \alpha I_n & 0 \\ 0 & I_n \end{pmatrix}$, where $\Gamma_n(\ell) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_n(\mathbb{Z}) \mid C \equiv 0 \pmod{\ell} \right\}$.

See e.g., [Fr83] for the precise definition (and for the definition of the other Hecke operators $T_j(p^2)$, $j = 1, \ldots, n-1$, which apparently cannot be used to improve the result below).

We denote by $T(p) c(\Lambda)$ the $\Lambda$'th Fourier coefficient of $T(p)F$. In §4 and §6 of [HaWa], it is shown that, for $p$ prime,

$$T(p) c(\Lambda) = \sum_{p\Lambda \subseteq \Omega \subseteq \Lambda} \gamma(\Omega)c(\Omega^{1/p}) + c(\Lambda^p),$$

with

$$\gamma(\Omega) = \chi(p^{n-j}) p^{k(n-j)+j(j+1)/2-n(n+1)/2}$$

where $[\Lambda : \Omega] = p^j$. For level $\ell = 1$, this description of $T(p)$ is essentially already contained in the classical work [Ma51] by Maaß; it is also readily derived from the well-known coset representatives for the above double coset, as described e.g., in [Fr83], Kapitel 4. The generalization to arbitrary $\ell$ is easy since we are only dealing with the $T(p)$; notice that the sum over $\Omega$ in the above formula disappears for $p|\ell$.

When $F$ is an eigenform and $T(p)F = \lambda_F(p)F$, we shall refer to this formula as the ‘Hecke eigenform equation’.

3. The result.

We immediately proceed to our main result.

**Theorem 3.1.** Suppose $F, G$ are degree $n$ eigenforms of arbitrary level and character, for all $T(p)$, with the same eigenvalues (i.e., $\lambda_F(p) = \lambda_G(p)$ for all $p$), and that their Fourier coefficients agree on primitive lattices and on $0$. Then $F = G$. 
Proof. By the support \( \text{supp} F \) of a Siegel modular form as above, we mean the support of its Fourier coefficients, i.e., the set of lattices \( \Lambda \) with \( c(\Lambda) \neq 0 \).

Suppose \( F \neq G \). Then \( F - G \) is an eigenform for all \( T(p) \) with no primitive lattice in its support. But this is impossible by the following lemma:

**Lemma 3.2.** Let \( F \) be a degree \( n \) eigenform for \( T(p) \) for all primes \( p \). Then there is at least one primitive lattice in the support of \( F \).

**Proof.** Suppose the contrary. For \( 1 \leq r \leq n \) let

\[
\text{supp}_r F = \{ \Lambda \in \text{supp} F \mid \text{rank} \Lambda = n - \text{dim} \text{rad} \Lambda = r \}.
\]

We first consider the case \( \text{supp}_n F \neq \emptyset \). Let \( N \) be the minimal content of lattices in \( \text{supp}_n F \) (so \( N > 1 \)). Take a prime \( p | N \). Then among the lattices in \( \text{supp}_n F \) with content \( N \), choose \( \Lambda \) s.t. the \( p \)-part of disc \( \Lambda \) is minimal.

Since \( \Lambda^{1/p} \) is integral, the Hecke eigenform equation says

\[
\lambda_F(p)c_F(\Lambda^{1/p}) = \sum_{p\Lambda \subseteq \Omega \subseteq \Lambda} \gamma(\Omega)c_F(\frac{1}{p^2} \Omega) + c_F(\Lambda).
\]

For \( \Omega \) s.t. \( p\Lambda \subseteq \Omega \subseteq \Lambda \), we have \( \Lambda \subseteq \frac{1}{p} \Omega \) and hence the \( p \)-part of disc \( \frac{1}{p} \Omega \) is strictly smaller than that of disc \( \Lambda \). Similarly, disc \( \Lambda^{1/p} = p^{-n} \text{disc} \Lambda \). Hence \( \Lambda^{1/p}, \frac{1}{p} \Omega \notin \text{supp}_n F \), for \( p\Lambda \subseteq \Omega \subseteq \Lambda \); so the Hecke eigenform equation says \( 0 = c_F(\Lambda) \), contradicting that \( \Lambda \) was chosen in \( \text{supp}_n F \).

Now we consider the case \( \text{supp}_n F = \emptyset \) and let \( r \) be maximal with \( \text{supp}_r F \neq \emptyset \). Note that \( r > 0 \). We use Siegel’s \( \Phi \)-operator (see [Fr83], Kapitel IV, §4) to reduce this case to the previous one. Indeed, this operator lowers the degree by 1 and commutes up to a constant with \( T(p) \), thus \( G := F|\Phi^{n-r} \) is a degree \( r \) eigenform for all \( T(p) \). Moreover

\[
\text{supp}_r F = \{ \Lambda \perp 0_{n-r} \mid \Lambda \in \text{supp}_r G \}.
\]

Since, by the preceding case, \( \text{supp}_r G \) contains a primitive lattice, so does \( \text{supp}_r F \).

The next lemma shows, for cusp forms, the equivalence between our assumption of coinciding eigenvalues and the assumption used in [BK99].

**Lemma 3.3.** Let \( F, G \) be degree \( n \) cuspidal eigenforms for each \( T(p) \), \( p \) prime, s.t. the coefficients of \( F \) and \( G \) agree on primitive lattices. Then

\[
\lambda_F(p) = \lambda_G(p)
\]

for all \( p \) if and only if the coefficients of \( F, G \) agree on all primitive lattices scaled by non-squares.

**Proof.** Suppose \( \lambda_F(p) = \lambda_G(p) \) for all \( p \). Let \( Q \in \mathbb{N} \) be square-free, and let \( p \) be a prime not dividing \( Q \). Suppose we know that the coefficients of \( F, G \) agree on all primitive lattices scaled by divisors of \( Q \). (Note that we are assuming this for \( Q = 1 \).) We show that the coefficients of \( F, G \) must then agree on all primitive lattices scaled by divisors of \( pQ \).
Let $\Lambda$ be a primitive lattice scaled by some divisor of $Q$ s.t. $p \nmid \text{disc } \Lambda$. (Note that $\text{disc } \Lambda \neq 0$ since $F$ is cuspidal.) Then for $p\Lambda \subset \Omega \subset \Lambda$, we have $[\Lambda : \Omega] = p^s$ with $s < n$. Hence $\text{disc } \Omega = p^{2s} \cdot \text{disc } \Lambda$, so $p^{2s} \| \text{disc } \Omega$. Thus $p^2 \nmid \text{cont } \Omega$ (else $p^{2n} \mid \text{disc } \Omega$), so $\Omega$ is a primitive lattice scaled by some divisor of $pQ$. This means either $\Omega^{1/p}$ is not integral or is a primitive lattice scaled by a divisor of $Q$; in either case $c_F(\Omega^{1/p}) = c_G(\Omega^{1/p})$. This together with the Hecke eigenform equation then gives us

$$c_F(\Lambda^p) = \lambda_F(p)c_F(\Lambda) - \sum_{p\Lambda \subset \Omega \subset \Lambda} \gamma(\Omega)c_F(\Omega^{1/p})$$

$$= \lambda_G(p)c_G(\Lambda) - \sum_{p\Lambda \subset \Omega \subset \Lambda} \gamma(\Omega)c_G(\Omega^{1/p})$$

$$= c_G(\Lambda^p).$$

Now suppose that for some $t \geq 1$ we know the coefficients of $F, G$ agree on primitive lattices $\Delta$ scaled by a divisor of $pQ$ provided $p^t \nmid \text{disc } \Delta$. Let $\Lambda$ be a primitive lattice scaled by a divisor of $Q$ s.t. $p^t \| \text{disc } \Lambda$. Take $\Omega$ s.t. $p\Lambda \subset \Omega \subset \Lambda$. Since $p^2 \| \text{cont } (p\Lambda)$ and $\text{cont } \Omega \mid \text{cont } (p\Lambda)$, we have $p^3 \nmid \text{cont } \Omega$. Thus $\frac{1}{p}\Omega$ is either non-integral, or primitive scaled by some divisor of $pQ$. Also since $p\Lambda \subset \Omega$, we know $[\Lambda : \Omega] = p^r$ for some $r < n$, so $p^{2(r-n)+t} \| \text{disc } \frac{1}{p}\Omega$. Hence by hypothesis, $c_F(\frac{1}{p}\Omega) = c_G(\frac{1}{p}\Omega)$. Consequently, the Hecke eigenform equation, with all terms rescaled by $\frac{1}{p}$, gives us $c_F(\Lambda) = c_G(\Lambda)$.

Induction on $t$ shows that $c_F(\Lambda) = c_G(\Lambda)$ for all $\Lambda$ that are primitive lattices scaled by a divisor of $pQ$. Induction on the number of primes dividing $Q$ shows $c_F(\Lambda) = c_G(\Lambda)$ for all $\Lambda$ that are primitive lattices scaled by non-squares.

Conversely, suppose the coefficients of $F, G$ agree on all primitive lattices scaled by non-squares. Fix a prime $p$. Choose a primitive lattice $\Lambda \in \text{supp } F$. Then as shown above, for $\Omega$ s.t. $p\Lambda \subset \Omega \subset \Lambda$, $\Omega^{1/p}$ is either non-integral or a primitive lattice scaled by a non-square. Thus

$$\sum_{\Omega} \gamma(\Omega)c_F(\Omega^{1/p}) + c_F(\Lambda^p) = \sum_{\Omega} \gamma(\Omega)c_G(\Omega^{1/p}) + c_G(\Lambda^p),$$

so the Hecke eigenform equation implies

$$\lambda_F(p)c_F(\Lambda) = \lambda_G(p)c_G(\Lambda).$$

Also $c_F(\Lambda) = c_G(\Lambda)$ by hypothesis, and $c_F(\Lambda) \neq 0$. Hence $\lambda_F(p) = \lambda_G(p)$.

**Remark 3.4.** Using the $\Phi$-operator as before, one can easily remove the restriction to cusp forms in the previous lemma.
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FREE FISHER INFORMATION FOR NON-TRACIAL STATES

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We extend Voiculescu’s microstates-free definitions of free Fisher information and free entropy to the non-tracial framework. We explain the connection between these quantities and free entropy with respect to certain completely positive maps acting on the core of the non-tracial non-commutative probability space. We give a condition on free Fisher information of an infinite family of variables, which guarantees factoriality of the von Neumann algebra they generate.

1. Introduction.

Free entropy and free Fisher information were introduced by Voiculescu [13], [14] and [16] in the context of his free probability theory [17] as analogs of the corresponding classical quantities. These quantities are usually considered in the framework of tracial non-commutative probability spaces; not surprisingly, the most striking applications of free entropy theory were to tracial von Neumann algebras (see e.g., [15], [2] and [10]). Recently, however, it turned out that some type III factors associated with free probability theory [5] have certain properties in common with their type II1 cousins [9]. This gives rise to a speculation that there is room for free entropy to exist outside of the context of tracial non-commutative probability spaces.

The goal of this paper is to initiate the development of free Fisher information, based on Voiculescu’s microstates-free approach [16], in the non-tracial framework. The key idea is that all of the ingredients going into the definition of free Fisher information in this case must behave covariantly with respect to the modular group [11] of the non-tracial state. The principal example of a family of variables for which free Fisher information is nontrivial, and which belong to an algebra not having any traces, are semicircular generators of free Araki-Woods factors, taken with free quasi-free states [5].

We describe another route towards free Fisher information, which is based on first converting the non-tracial von Neumann algebra into a larger algebra, the core (having an infinite trace), and then considering free Fisher information relative to a certain completely positive map (in the spirit of [7]). We should point out that it is this approach that is most likely to connect with the microstates free entropy (as suggested by [3]; see also [8]).
since it is at present unclear what a microstates approach to free entropy in the non-tracial framework should be.

We finish the paper with a look at free Fisher information on von Neumann algebras that have traces. Our first result is that once the algebra has a trace, the free Fisher information is automatically infinite when computed with respect to a non-tracial state. It is likely that on any von Neumann algebra, free Fisher information can be finite for only very special states (however, we do not have any results in this direction in the non-tracial category). Another result of the present paper is a statement guaranteeing factoriality of a tracial von Neumann algebra, once we know that it has an infinite generating family whose free Fisher information is bounded in a certain way.

2. Free Fisher information for arbitrary KMS states.

2.1. Free Brownian motion in the presence of a modular group. Let $M$ be a von Neumann algebra, $\phi : M \to \mathbb{C}$ be a normal faithful state on $M$. Denote by $\sigma^\phi_t$ the modular group of $\phi$. Denote by $L^2(M^{sa}, \phi) \subset L^2(M, \phi)$ the closure of the real subspace of self-adjoint elements $M^{sa} \subset M$.

Let $X = X^* \in M$ and let $B \subset M$ be a subalgebra. Assume that $\sigma^\phi_t(B) \subset B$ for all $t \in \mathbb{R}$.

Consider the von Neumann algebra $N = \Gamma(L^2(M^{sa}, \phi)) \subset L^2(M, \phi)$, taken with the free quasi-free state $\phi_M$. Consider the element $Y = \sigma^\phi_t(Y) \in M$ (see [5] for definitions and notation). Then $\phi_M(Y \sigma^\phi_M(Y)) = \phi(X \sigma^\phi_t(X))$, for all $t \in \mathbb{R}$.

Consider the algebra $\mathcal{N} = (M, \phi) \ast (M, \phi_M)$, and denote by $\hat{\phi}$ the free product state on $\mathcal{N}$. Note that $\sigma^\hat{\phi}_t = \sigma^\phi_t \ast \sigma^\phi_M$. The elements $X_\varepsilon = X + \sqrt{\varepsilon}Y$, $\varepsilon \geq 0$ form a natural free Brownian motion, which behaves nicely under the action of the modular group. In particular, note that for all $\varepsilon \geq 0$ and $t \in \mathbb{R}$,

$$\hat{\phi}(X, \sigma^\hat{\phi}_t(X_\varepsilon)) = \phi(X \sigma^\phi_t(X)) \cdot (1 + \varepsilon).$$

Furthermore, for each $\varepsilon > 0$, the distribution of $X_\varepsilon$ is that of a free Brownian motion at time $\varepsilon$, starting at $X$; this is because $Y$ is a semicircular variable, free from $X$.

2.2. Conjugate variables. Let $B[X]$ denote that algebra generated by $B$ and all translates $\sigma^\phi_t(X)$, $t \in \mathbb{R}$. Assume that $\{\sigma^\phi_t(X)\}$ are algebraically free over $B$, i.e., satisfy no algebraic relations modulo $B$. Denote by $\partial_X : B[X] \to \mathcal{N}$ the derivation given by:

1. $\partial_X(\sigma^\phi_t(X)) = \sigma^\hat{\phi}_t(Y)$
2. $\partial_X(b) = 0$, $b \in B$.

Notice that the range of $\partial_X$ actually lies in the subspace $B[X] \cdot \text{span}\{\sigma^\hat{\phi}_t(Y) : t \in \mathbb{R}\} \cdot B[X] \subset \mathcal{N}$. Note also that since $\partial_X(\sigma^\phi_t(X))$ is self-adjoint, we have
that for $P \in B[X]$, $\partial_X(P^*) = \partial_X(P)^*$, i.e., $\partial_X$ is a $*$-derivation. Observe finally that $\partial_X$ is covariant with respect to the modular groups $\sigma_t^\phi$ and $\dot{\sigma}_t^\phi$:

$$\partial_X(\sigma_t^\phi(P)) = \sigma_t^\phi(\partial_X(P)), \quad P \in B[X].$$

Define the conjugate variable $J_\phi(X : B) \in L^2(B[X], \phi)$ to be such a vector $\xi$ that

$$\langle \xi, P \rangle_{L^2(B[X], \phi)} = \langle Y, \partial_X(P) \rangle_{L^2(N, \hat{\phi})}, \quad \forall P \in B[X],$$

if a vector $\xi$ satisfying such properties exists. Formally, this means that $\xi = \partial_X^*(Y)$, where $\partial_X : L^2(B[X], \phi) \to L^2(N, \hat{\phi})$ is viewed as a densely defined operator.

It is clear, because of the density of $B[X]$ in $L^2(B[X], \phi)$, that $\xi$ is unique, if it exists.

It is convenient to talk about $J_\phi(X : B)$ even in the case that $\{\sigma_t(X)\}_{t \in \mathbb{R}}$ are not algebraically free over $B$ (such is the case, for example, when $\phi$ is a trace, and hence $\sigma_t^\phi(X) = X$ for all $t$). In this case, one can view $\partial_X$ as a multi-valued map, the set of values given by the results of application of the definition of $\partial_X$ in all possible ways; the definition of $J_\phi$ is then that (2.1) is valid for all values of $\partial_X$.

Note that $J_\phi(X : B)$ depends on more than just the joint distribution of $X$ and $B$ with respect to the state $\phi$; it depends on the joint distribution of the family $B \cup \{\sigma_t^\phi(X) : t \in \mathbb{R}\}$.

We continue to denote by $\sigma_t^\phi$ the extension of $\sigma_t^\phi$ to the Hilbert space $L^2(M, \phi)$ (this is precisely the one-parameter group of unitaries $\Delta_t^\phi$, where $\Delta_\phi$ is the modular operator). In particular, if $\phi$ is a trace, then the definition of $J_\phi$ is (up to a multiple) precisely that of the conjugate variable of Voiculescu [16].

**Lemma 2.1.** Assume that $\xi = J_\phi(X : B)$ exists. Then $\xi \in L^2(M^{sa}, \phi)$, and $\sigma_t^\phi J_\phi(X : B)) = J_\phi(\sigma_t^\phi(X) : B)$.

**Proof.** We note that, because $\partial_X$ is a $*$-derivation,

$$\langle P^*, \xi \rangle = \langle Y, \partial_X(P^*) \rangle = \langle \partial_X(P), Y \rangle = \langle \xi, P \rangle.$$

From this it follows that $\xi$ is in the domain of the $S$ operator of Tomita theory, and moreover that $S\xi = \xi$. Hence $\xi \in L^2(M^{sa})$.

One also has

$$\langle \sigma_t^\phi(\xi), P \rangle = \langle Y, \partial_X(P) \rangle = \langle \sigma_t^\phi(Y), \partial_{\sigma_t^\phi(X)}(P) \rangle,$$

since the joint distributions of $B[X]$ and $\{\sigma_t^\phi(Y)\}_{t \in \mathbb{R}}$ is the same as $B[X]$ and $\{\sigma_{s+t}^\phi(Y)\}_{t \in \mathbb{R}}$, for any $s$. It follows that $\sigma_t^\phi(J_\phi(X : B)) = J_\phi(\sigma_t^\phi(X) : B)$. \qed
Lemma 2.2. Let $P,Q \in B[X]$, and assume that $\xi = J_\phi(X)$ exists and is in $M$. Then
\begin{align*}
\phi(P\xi Q) &= \hat{\phi}(PY\partial_X(Q)) + \hat{\phi}(\partial_X(P)YQ).
\end{align*}

Proof. Recall that $\phi$ (and $\hat{\phi}$) satisfy the KMS condition: For all $a,b \in M$ (or $\in \mathcal{N}$), there exists a (unique) function $f(z)$, analytic on the strip \{ $z : 0 < \Im z < 1$ \}, and so that (writing $\sigma_t$ for either $\sigma_t^\phi$ or $\sigma_t^\hat{\phi}$)
\begin{align*}
\phi(a\sigma_t(b)) &= f(t), \\
\phi(\sigma_t(b)a) &= f(t + i), \quad t \in \mathbb{R}.
\end{align*}

Fix $P,Q \in B[X]$ and let $f$ be as above, so that
\begin{align*}
\phi(\sigma_t^\phi(P)\xi Q) &= f(t + i), \\
\phi(\xi Q\sigma_t^\phi(P)) &= f(t).
\end{align*}

Then
\begin{align*}
f(t) &= \langle \xi, Q\sigma_t^\phi(P) \rangle \\
&= \langle Y, \partial_X(Q\sigma_t^\phi(P)) \rangle \\
&= \langle Y, \partial_X(Q)\sigma_t^\phi(P) \rangle + \langle Y, Q\sigma_t^\phi(\partial_X(P)) \rangle,
\end{align*}

where in the last step we used the fact that $\partial_X$ intertwines $\sigma_t^\phi$ and $\sigma_t^\hat{\phi}$. Using the KMS-condition for $\hat{\phi}$, we then get
\begin{align*}
f(t + i) &= \hat{\phi}(\sigma_t^\hat{\phi}(P)Y\partial_X(Q)) + \hat{\phi}(\sigma_t^\hat{\phi}(\partial_X(P))YQ) \\
&= \hat{\phi}(\sigma_t^\hat{\phi}(P)Y\partial_X(Q)) + \hat{\phi}(\partial_X(\sigma_t^\hat{\phi}(P))YQ).
\end{align*}

Since $f(t + i) = \phi(\sigma_t^\hat{\phi}(P)\xi Q)$, we get, setting $t = 0$, that
\begin{align*}
\phi(P\xi Q) &= \hat{\phi}(PY\partial_X(Q)) + \hat{\phi}(\partial_X(P)YQ),
\end{align*}

as claimed. \hfill \Box

2.3. Conjugate variables as free Brownian gradients. As pointed out above, $X + \sqrt{\varepsilon}Y$ is a natural free Brownian motion, which is covariant with respect to the appropriate modular groups. The following proposition shows that $J_\phi(X : B)$ plays the role of the free Brownian gradient of $X$:

Proposition 2.3. Assume that $\xi = J_\phi(X : B)$ exists and belongs to $M \subset L^2(M, \phi)$. Let $P(Z_1, \ldots, Z_n)$ be any non-commutative polynomial in $n$ variables $Z_1, \ldots, Z_n$, with coefficients from $B$. Write $X_t = \sigma_t^\phi(X)$, $Y_t = \sigma_t^\hat{\phi}(Y)$, $\xi_t = \sigma_t^\phi(\xi)$, $t \in \mathbb{R}$.

Then for all $t_1, \ldots, t_n \in \mathbb{R}$, we have
\begin{align*}
\hat{\phi}(P(X_{t_1} + \sqrt{\varepsilon}Y_{t_1}, \ldots, X_{t_n} + \sqrt{\varepsilon}Y_{t_n})) \\
&= \frac{1}{2} \phi(P(X_{t_1} + \varepsilon\xi_{t_1}, \ldots, X_{t_n} + \varepsilon\xi_{t_n})) + O(\varepsilon^2).
\end{align*}
Proof. We may assume, by linearity, that $P$ is a monomial, i.e., $P(Z_1, \ldots, Z_n) = b_0 Z_1 \ldots b_{n-1} Z_n b_n$, for $b_j \in B$. In this case, we have

$$\hat{\phi}(b_0 (X_{t_1} + \sqrt{\epsilon} Y_{t_1}) b_1 \ldots b_n)$$

$$= \phi(P(X_{t_1}, \ldots, X_{t_n})) + O(\epsilon^2)$$

$$+ \epsilon \sum_{k<l} \hat{\phi}(b_0 X_{t_1} \ldots b_k Y_{t_{l+1}} b_k Y_{t_{l+1}} b_{k+1} X_{t_{k+2}} \ldots X_{t_n} b_n)$$

$$= \phi(P(X_{t_1}, \ldots, X_{t_n})) + O(\epsilon^2)$$

$$+ \frac{1}{2} \epsilon \sum_{l} \hat{\phi}(b_0 X_{t_1} \ldots X_{t_l} b_l Y_{t_{l+1}} \partial X_{t_{l+1}} (b_{l+1} X_{t_{l+2}} \ldots X_{t_n} b_n))$$

$$+ \frac{1}{2} \epsilon \sum_{k} \hat{\phi}((\partial X_{t_{k+1}} (b_0 X_{t_1} \ldots X_{t_k} b_k) Y_{t_k} b_{k+1} X_{t_{k+2}} \ldots X_{t_n} b_n))$$

$$= \phi(P(X_{t_1}, \ldots, X_{t_n})) + O(\epsilon^2)$$

$$+ \frac{1}{2} \sum_{k} \phi(b_0 X_{t_1} \ldots X_{t_k} b_k \epsilon \xi_{t_k} b_{k+1} X_{t_{k+2}} \ldots X_{t_n} b_n),$$

the last equality by Lemma 2.2. \qed

2.4. Examples of conjugate variables.

2.4.1. Tracial case. We have seen before that if $\phi$ is a trace, then the definition of $J_{\phi}(X : B)$ coincides with the definition of conjugate variables given by Voiculescu, up to a constant (which has to do with the fact that we choose $Y$ so that $\|Y\|_{L^2(\hat{\phi})} = \|X\|_{L^2(\phi)}$, and not 1). In particular,

$$J_{\phi}(X : B) = J(X : B) \cdot \frac{1}{\|X\|_{L^2(\phi)}^2}, \quad \text{if } \phi \text{ is a trace.}$$

2.4.2. Free quasi-free states. Let $\mu$ be a positive finite Borel measure on $\mathbb{R}$, so that $\mu$ is symmetric, $\mu(-X) = \mu(X)$ for all Borel subsets $X \subset \mathbb{R}$. Let $\mathcal{H}_R$ be the real Hilbert space of $\mu$-square-integrable functions, satisfying $f(-x) = \overline{f(x)}$ for all $x \in \mathbb{R}$. Denote by $U_t$ the representation of $\mathbb{R}$ on $\mathcal{H}_R$, given by

$$(U_t f)(x) = e^{2\pi i t x} f(x), \quad x, t \in \mathbb{R}.$$ 

Let $h$ denote the vector $1 \in \mathcal{H}_R$, and consider

$$M = \Gamma(\mathcal{H}_R, U_t)'', \quad \phi = \phi_U, \quad X = s(h) \in M$$

(see [5] for definitions and notation).
Then $X = J_\phi(X : \mathbb{C})$. Indeed, set $X_t = \sigma_t^\phi(X) = s(U_t h)$; then we have
\[
\phi(X \cdot X_{t_1} \ldots X_{t_n}) = \sum_k \phi(X X_{t_k}) \phi(X_{t_1} \ldots X_{t_{k-1}}) \cdot \phi(X_{t_{k+1}} \ldots X_{t_n})
\]
\[
= \sum_k \hat{\phi}(Y \sigma_{t_k}^\phi(Y)) \phi(X_{t_1} \ldots X_{t_{k-1}}) \cdot \phi(X_{t_{k+1}} \ldots X_{t_n})
\]
\[
= \sum_k \hat{\phi}(Y X_{t_1} \ldots X_{t_{k-1}} Y X_{t_{k+1}} \ldots X_{t_n})
\]
\[
= \hat{\phi}(Y \partial_X(X_{t_1} \ldots X_{t_n})),
\]
so that $X$ satisfies the defining property of $J_\phi(X : \mathbb{C})$, and hence $J_\phi(X : \mathbb{C})$ exists and equals $X$.

2.5. Free Fisher information. Following [16], we define the free Fisher information $\Phi^*_\phi(X : B)$ to be
\[
\Phi^*_\phi(X : B) = \|J_\phi(X : B)\|_2^2 \cdot \|X\|^{-2}
\]
(the extra factor $\|X\|^{-2}$ comes from the fact that $\partial_X(X)$ does not have unit norm in our definition). For several variables, we set
\[
\Phi^*_\phi(X_1, \ldots, X_n)
\]
\[
= \sum \Phi^*_\phi(X_i : W^*(\sigma_{t_1}(X_1), \ldots, \hat{X}_i, \ldots, \sigma_{t_n}(X_n) : t_1, \ldots, t_n \in \mathbb{R}))
\]
(here $\hat{X}_i$ means that $X_i$ is omitted).

3. Free Fisher information relative to the core.
Recall [11] that if $(M, \phi)$ is as above, its core is defined to be the von Neumann algebra crossed product $P = M \rtimes_{\sigma^\phi} \mathbb{R}$. There is a canonical inclusion $M \subset P$, and $P$ is densely spanned by elements of the form
\[
mU_t, \ t \in \mathbb{R},
\]
where $m \in M$, and $U_t$ satisfy $U_t m U_t^* = \sigma_t^\phi(m)$. The elements $U_t : t \in \mathbb{R}$ generate a copy of the group von Neumann algebra $L(\mathbb{R}) \subset P$; the map
\[
E^\phi : mU_t \mapsto \phi(m) U_t, \ m \in M, t \in \mathbb{R}
\]
extends to a normal conditional expectation from $P$ onto $L(\mathbb{R})$.

For $X \in M$ self-adjoint, define the completely positive map $\eta_X : L(\mathbb{R}) \to L(\mathbb{R})$ by
\[
\eta_X(g) = E^\phi(X g X), \ g \in L(\mathbb{R}).
\]
Identify $L(\mathbb{R})$ with $L^\infty(\mathbb{R})$ via Fourier transform. For each $t \in \mathbb{R}$, set
\[
\eta(t) = \langle X, \sigma_t^\phi(X) \rangle = E^\phi(X U_t X).
\]
Then $\eta_X(f) = \hat{\eta} \ast f$, if $f \in L^\infty(\mathbb{R}) \cong L(\mathbb{R})$; here $\hat{\eta}$ denotes Fourier transform.
Define on \( P \) an \( L(\mathbb{R}) \)-valued inner product
\[
\langle a, b \rangle_{L(\mathbb{R})} = E^\phi(a^*b), \quad a, b \in P.
\]
Denote by \( L^2(P, E^\phi) \) the \( L(\mathbb{R}) \)-Hilbert bimodule arising from the completion of \( P \) with respect to the norm induced by this inner product. Note that the restriction of \( \langle \cdot, \cdot \rangle_{L(\mathbb{R})} \) to \( M \subset P \) is valued in the complex field, and coincides with the inner product \( \langle a, b \rangle = \phi(a^*b) \) on \( L^2(M) \).

Denote by \( \langle \cdot, \cdot \rangle_\eta \) the \( L(\mathbb{R}) \)-valued inner product on \( P \otimes P \) (algebraic tensor product) given by
\[
\langle a \otimes b, a' \otimes b' \rangle_\eta = E^\phi(b^*\eta(E^\phi(a^*a')b')), \quad a, a', b, b' \in P.
\]
Denote by 1 \( \otimes 1 \) the vector \( 1 \otimes 1 \in P \otimes P \).

Let \( \delta_X : B[X] \cdot L(\mathbb{R}) \to P \otimes P \) be determined by
\[
\delta_X(X) = 1 \otimes 1, \quad \delta_X(B \cdot L(\mathbb{R})) = 0
\]
and the fact that \( \delta_X \) is a derivation.

**Theorem 3.1.** Let \( (M, \phi) \) be as above, and let \( P \) be its core. Let \( i : L^2(M, \phi) \to L^2(P, E^\phi) \) be the extension of the inclusion of \( M \subset P \). Then \( \zeta = i(J_\phi(X : B)) \) satisfies
\[
(3.1) \quad \langle \zeta, Q \rangle_{L(\mathbb{R})} = (1 \otimes 1, \delta_X(Q))_{\eta_X}
\]
for all \( Q \in B[X] \vee L(\mathbb{R}) \). Conversely, if there exists a vector \( \zeta \in L^2(P, E^\phi) \), so that (3.1) is satisfied, then \( J_\phi(X : B) \) exists and \( \zeta = i(J_\phi(X : B)) \).

**Proof.** Assume first that \( J_\phi(X : B) \) exists. Set \( \zeta = i(J_\phi(X : B)) \). We must verify that (3.1) holds. By linearity, and the fact that \( L(\mathbb{R})BL(\mathbb{R}) \subset BL(\mathbb{R}) \), it is sufficient to consider the case when \( Q = b_0U^{s_1}X_{t_1}b_1U^{s_2} \cdots X_{t_n}b_nU^{s_n} \), with \( b_j \in B \) and \( X_t = \sigma^\phi_t(X) \). Then \( Q = P \cdot U^r \), where \( r = \sum s_j \), and \( P = b_0X_{t_1}b_1' \cdots b_n' \), with \( b_j' = \sigma_{s_j \cdots s_1}^\phi(b_j) \). Let \( t_j' = s_j - 1 + \cdots + s_1 + t_j \). Note that for \( x, y, x', y' \in P \), \( \langle x \otimes y, x' \otimes Uy' \rangle_{\eta} = \langle x \otimes y, x'U \otimes y \rangle_{\eta} \), and \( \langle x \otimes y, U^r(x' \otimes y')U^s \rangle_{\eta} = \langle U^{-r}x \otimes yU^{-s}, x' \otimes y' \rangle_{\eta} \). Using this, we get
\[
\langle \zeta, Q \rangle_{L(\mathbb{R})}
= \langle \zeta, P \rangle_{L(\mathbb{R})}g
= \langle \zeta, P \rangle_{L^2(M, \phi)} \cdot U^r
= \phi(Y \partial_X(P)) \cdot U^r
= \sum_j \phi(b_0'X_{t_1} \cdots X_{t_j}b_j') \phi(b_{j+1}'X_{t_{j+2}} \cdots X_{t_n}b_n') \cdot \phi(YY_{t_j})U^r
= \sum_j \phi(b_0'X_{t_1} \cdots X_{t_j}b_j') \phi(b_{j+1}'X_{t_{j+2}} \cdots X_{t_n}b_n') \cdot \phi(XU^{t_j}XU^{-t_j})U^r
= \sum_j \phi(b_0'X_{t_1} \cdots X_{t_j}b_j') \phi(b_{j+1}'X_{t_{j+2}} \cdots X_{t_n}b_n') \cdot E^\phi(XU^{t_j}XU^{-t_j})U^r
\]
\[
\begin{align*}
\sum_j \phi(b'_0 X_{t'_1} \ldots X_{t'_{j+2}} b'_j) \phi(b'_{j+1} X_{t'_{j+2}} \ldots X_{t'_n} b'_{n}) \cdot \eta X(U_{t'_j}) U^{r-t'_j} \\
= \sum_j \eta X \circ E^\phi(b'_0 X_{t'_1} \ldots X_{t'_{j+2}} b'_j) \cdot E^\phi(U^{r-t'_j} b'_{j+1} X_{t'_{j+2}} \ldots X_{t'_n} b'_{n}) \\
= \sum_j \langle 1 \otimes \eta 1, b'_0 X_{t'_1} \ldots X_{t'_{j+2}} b'_j U_{t'_j} \otimes U^{r-t'_j} b'_{j+1} X_{t'_{j+2}} \ldots X_{t'_n} b'_{n} \rangle \eta \\
= \sum_j \langle 1 \otimes \eta 1, b_0 U^{s_1} X_{t_1} \ldots X_{t_j} b_j U^{s_j} U_{t'_j} \otimes U^{r-t_j} U^{s_{j+1}} b_{j+1} X_{t_{j+2}} \ldots X_{t_n} b_n U^{s_n} U^{-r} \rangle \eta \\
= \langle 1 \otimes \eta 1, \delta X(Q) \rangle \eta.
\end{align*}
\]

Conversely, assume that \( \zeta \) satisfying (3.1) exists. Since the argument above is reversible, it is sufficient to prove that \( \zeta \) is in the image of \( i : L^2(M) \to L^2(P, E^\phi) \). Let \( \theta_t \) be the dual action of \( \mathbb{R} \) on \( P \), given by \( \theta_t(U_s) = \exp(2\pi i s t) \), \( \theta_t(m) = m \), \( m \in M \). It is sufficient to prove that \( \theta_t(\zeta) = \zeta \), since \( i(L^2(M)) \) consists precisely of those vectors, which are left fixed by \( \theta \).

It is sufficient to prove that \( \theta_t(E^\phi(\zeta m U_t)) = \exp(2\pi i s t) \) if \( m \in M \). Since \( \zeta \) is assumed to be in the closure of \( B[X] \vee L(\mathbb{R}) \), it is sufficient to check this for \( m \in B[X] \). But then by (3.1),

\[
E^\phi(\zeta m U_t) = \langle \zeta, m U_t \rangle_{L(\mathbb{R})} = \langle 1 \otimes \eta 1, \delta_X(m U) \rangle \eta = \langle 1 \otimes \eta 1, \delta_X(m) \rangle \eta U \in M \cdot U_t,
\]

which gives the desired result, since \( \theta_s \) acts trivially on \( M \). \qed

Note that (3.1) means that \( \zeta \) is equal to \( J(X : B \vee L(\mathbb{R}), \eta) \) in the notation of [7]. (This is strictly speaking incorrect, since the setting of [7] presumes the existence of a finite trace on \( B[X] \vee L(\mathbb{R}) \); however, it is not hard to check that the arguments in [7] go through also in the case of a semifinite trace, which exists in our case.)

This fact has many consequences for the conjugate variables \( J_\phi(X : B) \), coming from the properties of \( J(X : B \vee L(\mathbb{R}), \eta) \). Note in particular that if \( W^*(D, \{ \sigma_i^\phi(X) : t \in \mathbb{R} \}) \) is free from \( B \) with amalgamation over \( D \subset B \) with respect to some conditional expectation \( E : B \to D \), and \( E \) is \( \phi \)-preserving, then \( X \) is free from \( B \vee L(\mathbb{R}) \) with amalgamation over \( D \vee L(\mathbb{R}) \) (see [12], [6]). We record this as:

\[
\begin{align*}
&= \sum_j \phi(b'_0 X_{t'_1} \ldots X_{t'_{j+2}} b'_j) \phi(b'_{j+1} X_{t'_{j+2}} \ldots X_{t'_n} b'_{n}) \cdot \eta X(U_{t'_j}) U^{r-t'_j} \\
&= \sum_j \eta X \circ E^\phi(b'_0 X_{t'_1} \ldots X_{t'_{j+2}} b'_j) \cdot E^\phi(U^{r-t'_j} b'_{j+1} X_{t'_{j+2}} \ldots X_{t'_n} b'_{n}) \\
&= \sum_j \langle 1 \otimes \eta 1, b'_0 X_{t'_1} \ldots X_{t'_{j+2}} b'_j U_{t'_j} \otimes U^{r-t'_j} b'_{j+1} X_{t'_{j+2}} \ldots X_{t'_n} b'_{n} \rangle \eta \\
&= \sum_j \langle 1 \otimes \eta 1, b_0 U^{s_1} X_{t_1} \ldots X_{t_j} b_j U^{s_j} U_{t'_j} \otimes U^{r-t_j} U^{s_{j+1}} b_{j+1} X_{t_{j+2}} \ldots X_{t_n} b_n U^{s_n} U^{-r} \rangle \eta \\
&= \langle 1 \otimes \eta 1, \delta X(Q) \rangle \eta.
\end{align*}
\]
Theorem 3.2. Assume that $E : B \to D$ is a $\phi$-preserving conditional expectation. If $W^*(D, \{\sigma_t^\phi(X) : t \in \mathbb{R}\})$ is free from $B$ over $D$, then

$$J_{\phi}(X : B) = J_{\phi}(X : D).$$

In a similar way, one can generalize to $J_{\phi}(X : B)$ all the properties of the conjugate variable $J(X : B, \eta)$ proved in [7]. Reformulating gives the following properties of $\Phi_{\phi}$, which we list for reader's convenience, since they are needed in the rest of the paper:

Theorem 3.3. Let $\phi$ be a normal faithful state on $M$, $B \subset M$ be globally fixed by the modular group (i.e., $\sigma_t^\phi(B) = B$ for all $t$), and $X_i \in M$. Then:

(a) $\Phi_{\phi}^\ast(\lambda X_1, \ldots, \lambda X_n : B) = \lambda^{-2} \Phi_{\phi}^\ast(X_1, \ldots, X_n : B)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$.

(b) If $B \subset A \subset M$ and $A$ is globally fixed by $\sigma^\phi$, then $\Phi_{\phi}^\ast(X_1, \ldots, X_n : A) \geq \Phi^\ast(X_1, \ldots, X_n : B)$.

(c) If $C \subset M$ is globally fixed by $\sigma^\phi$, and $W^*(X_1, \ldots, X_n)$ and $B$ are free with amalgamation over $C$ (with respect to the unique $\phi$-preserving conditional expectation from $M$ onto $C$), then $\Phi_{\phi}^\ast(X_1, \ldots, X_n : B \vee C) = \Phi_{\phi}^\ast(X_1, \ldots, X_n : C)$.

(d) If $Y_i \in M$ are self-adjoint, $D \subset B$, $D \subset C$ subalgebras of $M$, which are globally fixed by $\sigma^\phi$, and $B[X_1, \ldots, X_n]$ is free from $C[X_1, \ldots, X_n]$ over $D$ (with respect to the unique $\phi$-preserving conditional expectation from $M$ onto $D$), then $\Phi_{\phi}^\ast(X_1, \ldots, X_n, Y_1, \ldots, Y_m : B \vee C) = \Phi_{\phi}^\ast(X_1, \ldots, X_n : B) + \Phi_{\phi}^\ast(Y_1, \ldots, Y_m : C)$.

(e) $\Phi_{\phi}^\ast(X_1, \ldots, X_n, Y_1, \ldots, Y_m : B) \geq \Phi_{\phi}^\ast(X_1, \ldots, X_n : B) + \Phi_{\phi}^\ast(Y_1, \ldots, Y_m : B)$.

(f) $\Phi_{\phi}^\ast(X_1, \ldots, X_n : B) \cdot \phi(\sum X_i^2) \geq n^2$. Equality holds iff $\{\sigma_t(X_1), \ldots, \sigma_t(X_n) : t_1, \ldots, t_n \in \mathbb{R}\}$ have the same distribution as the semi-circular family $\{\kappa \sigma_t(X_1), \ldots, \kappa \sigma_t(X_n) : t_1, \ldots, t_n \in \mathbb{R}\}$ with respect to the free quasi-free state, $\kappa > 0$.

We mention that all of the statements in Sections 3 and 4 of [7] remain valid for $\Phi_{\phi}^\ast$; we leave details to the reader.

One can also define and study free entropy $\chi_{\phi}^\ast(X_1, \ldots, X_n)$ by setting $X_i^\varepsilon = X_i + \sqrt{\varepsilon} Y_i$ to be the free Brownian motion described in the beginning of the paper, and letting

$$\chi_{\phi}^\ast(X_1, \ldots, X_n) = \frac{1}{2} \int_0^\infty \left( \frac{n}{1 + t} - \Phi_{\phi}^\ast(X_1^t, \ldots, X_n^t) \right) dt.$$

The properties of $\chi^\ast(\ldots, \eta)$ once again generalize to $\chi_{\phi}^\ast$ (compare Section 8 of [7]).
4. States on a II$_1$ factor.

4.1. $\Phi^*_\phi$ vs. $\Phi^*_\tau$. The following theorem is somewhat surprising, since it shows that $\Phi^*_\phi$ is identically infinite for most states $\phi$ on a II$_1$ factor (the analogy with classical Fisher information would instead suggest that $\phi \mapsto \Phi^*_\phi$ would have some nice convexity properties). This, on the other hand, goes well with the “degenerate convexity” property of the microstates free entropy $\chi$ [15] (which is reflected in that it is identically $-\infty$ on generators of any von Neumann algebra with more than one unital trace).

**Theorem 4.1.** Let $M$ be a tracial von Neumann algebra, $\phi$ a faithful normal state on $M$, $B \subset M$ a subalgebra so that $\sigma^\phi_t(B) = B$ for all $t$, and $X = X^* \in M$. Then if $J_\phi(X : B)$ exists, the modular group of $\phi$ must fix $X$.

**Proof.** Let $d \in M$ be a positive element, so that $\phi(x) = \tau(dx)$, where $\tau$ is a normal faithful trace on $M$, and $d$ is an unbounded operator on $L^2(M, \tau)$, affiliated to $M$. The modular group of $\phi$ is then given by $\sigma^\phi_t(x) = d^{it}xd^{-it}$, $x \in M$. Denoting by $X_t$ the element $\sigma^\phi_t(X)$, we then get

$$X = X_0 = d^{-it}Xtd^{it}, \quad t \in \mathbb{R}. $$

Consider

$$\phi(X_0^2) = \phi(J_\phi(X : B) \cdot X_0) = \phi(J_\phi(X : B)d^{-it}Xtd^{it}).$$

Let $b_1$ and $b_2$ be two elements in the domain of $\partial_X$, so that $b_1 = b_2^*$. Then we get, writing $Y_t = \sigma^\phi_t(Y)$:

$$\phi(J_\phi(X : B)b_1X_tb_2) = \hat{\phi}(Y_0b_1Y_t b_2)$$

$$\quad + \hat{\phi}(Y_0\partial_X(b_1)X_tb_2) + \hat{\phi}(Y_0b_1X_t\partial_X(b_2))$$

$$\quad = \phi(b_1)\phi(b_2)\hat{\phi}(Y_0Y_t)$$

$$\quad + \hat{\phi}(Y_0\partial_X(b_1)X_tb_2) + \hat{\phi}(\partial_X(b_2^*)X_tb_1Y_0^*)$$

$$\quad = \phi(b_1)\phi(b_2^*)\hat{\phi}(Y_0Y_t)$$

$$\quad + \hat{\phi}(Y_0\partial_X(b_1)X_tb_1^*) + \hat{\phi}(\partial_X(b_1)X_tb_1^*Y_0^*).$$

Now, for all $m, n \in M$, we have

$$\hat{\phi}(Y_0mY_0n) = \phi(m)\phi(n) = \hat{\phi}(mY_0nY_0),$$
so that
\[
\hat{\phi}(Y_0 \partial_X (b_1) X_t b_1^*) + \hat{\phi}([\partial_X (b_1) X_t b_1^*] Y_0^*) \\
= \hat{\phi}(Y_0 \partial_X (b_1) X_t b_1^*) + \hat{\phi}([Y_0 \partial_X (b_1) X_t b_1^*] Y_0^*) \\
= \hat{\phi}(Y_0 \partial_X (b_1) X_t b_1^*) + \hat{\phi}(Y_0 [\partial_X (b_1) X_t b_1^*]) \\
\in \mathbb{R}
\]

It follows that
\[
\Im \hat{\phi}(J_\phi(X : B) b_1 X_t b_1^*) = \Im \hat{\phi}(Y_0 b_1 Y_t b_1^*).
\]

Now fix \( t \in \mathbb{R} \) and choose \( a_n \) in the domain of \( \partial_X \), \( \|a_n\| \leq 1 \), so that \( a_n \to d^it, \ a_n^* \to d^{-it} \) strongly.

One can choose \( a_n \), for example, to be elements of the algebra \( B[X] \). Then
\[
0 = \Im \phi(X_0^2) = \Im \phi(J_\phi(X : B) \cdot X_0) \\
= \Im \phi(J_\phi(X : B) d^{-it} X_t d^{it}) \\
= \lim_{n \to \infty} \Im \phi(J_\phi(X : B) a_n X_t a_n^*) \\
= \lim_{n \to \infty} \Im \hat{\phi}(Y_0 a_n Y_t a_n^*) \\
= \lim_{n \to \infty} \Im (\phi(a_n) \phi(a_n^*) \hat{\phi}(Y_0 Y_t)) \\
= \lim_{n \to \infty} \phi(a_n) \phi(a_n^*) \Im \hat{\phi}(Y_0 Y_t) \\
= \phi(d^{it}) \phi(d^{-it}) \Im \hat{\phi}(Y_0 Y_t).
\]

Since \( \hat{\phi}(Y_0 Y_t) = \phi(X_0 X_t) \), for \( t \) sufficiently close to zero (so that \( \phi(d^{it}) \neq 0 \)), we get that
\[
\phi(XX_t) \in \mathbb{R}.
\]

Thus
\[
0 = \tau(dXX_t) - \tau(dXX_t)^* \\
= \tau(dXd^{it} Xd^{-it} - d^{it}Xd^{-it} Xd) \\
= \tau((dX - Xd)d^{it}Xd^{-it}) \\
= \tau([d, X]d^{it}Xd^{-it}).
\]

Differentiating this in \( t \), and noting that \( (d/dt)_{t=0}(d^{it}Xd^{-it}) = i[d, X] \) gives
\[
i\tau([d, X]^2) = 0.
\]

Since \([d, X]\) is anti-self-adjoint, this implies that \( \tau([d, X]^2) = 0 \), so that \([d, X] = 0\), because \( \tau \) is faithful. This means that \( \sigma^\phi_t(X) = X \) for all \( t \). \( \square \)
Corollary 4.2. Suppose that $X_1, \ldots, X_n$ are self-adjoint generators of a $\text{II}_1$ factor $M$. Let $\phi$ be a normal faithful state on $M$, and denote by $\tau$ the unique faithful normal trace on $M$. Then $\Phi^*_\phi(X_1, \ldots, X_n) < +\infty$ implies that:

1. $\Phi^*_\phi(X_1, \ldots, X_n) < \infty$, and
2. $\phi$ is a multiple of the trace $\tau$ on $M$.

Proof. Clearly, the second statement implies the first. To get the second statement, write $\phi(\cdot) = \tau(d\cdot)$ and apply the theorem to conclude that $[d, X_i] = 0$. Since $X_1, \ldots, X_n$ generate $M$, $d$ must be in the center of $M$, which must consist of multiples of identity, since $M$ is a factor. But then $d$ is a scalar multiple of identity, so that $\phi$ and $\tau$ are proportional. □

4.2. Factoriality. Voiculescu showed [15] that for his microstates entropy $\chi$ the following implication holds:

$$\chi(X_1, \ldots, X_n) > -\infty \Rightarrow W^*(X_1, \ldots, X_n) \text{ is a factor}.$$ 

In fact, the conclusion is stronger: Not only is the center of $W^*(X_1, \ldots, X_n)$ trivial, but so is its asymptotic center. Unfortunately, we don’t know if the same implication holds for the non-microstates free entropy $\chi^*$ introduced by Voiculescu in [16], or even under the stronger assumption that $\Phi^*(X_1, \ldots, X_n)$ is finite. We prove a weaker version of the assertion above for $\Phi^* = \Phi^*_\tau$. We first need a technical lemma:

Lemma 4.3. Let $\phi$ be a normal faithful state on $M$. Let $X \in M$ be self-adjoint and $B \subset M$ be a subalgebra, so that $\sigma^\phi_t(B) = B$ for all $t$. Assume that $p \in B$ is a self-adjoint projection, $\phi(p) = \alpha$, and so that $\sigma^\phi_t(p) = p$ for all $t$. Assume that $\|X, p\|_2 < \delta$. Then

$$\Phi^*_\phi(X : B) > 4 \frac{\alpha^2(1 - \alpha)^2}{\delta^2}.$$ 

Proof. Let $(A, \tau)$ be a copy of $L(\mathbb{F}_2)$, free from $B[X]$. Since $\Phi^*_\phi(X : B) = \Phi^*_\phi(X : B \vee A)$, and since the centralizer of $\{\sigma^\phi_t(B)\}_{t \in A}$ is a factor $[1]$, we can find a projection $q \in B \vee A$, which is fixed by the modular group, and so that $\|X, q\|_2 < \delta$, and $\tau(q) = \beta = m/n$ is rational and close to $\alpha$. We may moreover find a family of matrix units $e_{ij} \in B \vee A$, $1 \leq i, j \leq n$, fixed by the modular group, and so that

$$e^*_{ij} = e_{ji}, \quad e_{ij}e_{kl} = \delta_{jk}e_{il} \quad \tau(e_{ii}) = \frac{1}{n}, \quad q = \sum_{i=1}^n e_{ii}.$$ 

Denote by $C$ the algebra generated in $B \vee A$ by $\{e_{ij}\}_{1 \leq i, j \leq n}$. Note that $C \cong M_{n \times n}$, the algebra of $n \times n$ matrices. The restriction of $\phi \ast \tau$ to $C$ is the usual matrix trace. Then

$$\Phi^*_\phi(X : B) = \Phi^*_\phi(X : B \vee A) \geq \Phi^*_\phi(X : C).$$
Write $X_{ij} = e_{11} X e_{j1}$. Then the inequality $||[X, q]||_2 < \delta$ implies that

$$
\delta > ||qx - qx||_2 \\
= ||qx + (1 - q)xq - qx - q(1 - q)||_2 \\
= ||(1 - q)xq - q(1 - q)||_2 \\
= \sqrt{2} \cdot ||xq(1 - q)||_2,
$$

since $(1 - q)xq$ and $qx(1 - q)$ are orthogonal. Hence

$$
\|qX(1 - q)\|_2 < \delta / \sqrt{2}.
$$

It follows that

$$
\sum_{1 \leq i \leq m, m < j \leq n} \phi(X_{ij}^* X_{ij}) + \sum_{m < i \leq n, 1 \leq i \leq n} \phi(X_{ij}^* X_{ij}) < \delta^2.
$$

Denote by $\phi'$ the state $n(\phi * \tau)(e_{11} \cdot e_{11})$ on $e_{11}W^*(X, C)e_{11}$. Then

$$
\Phi^*_{\phi'}(\{X_{ij}\}) \geq \sum_{i,j} \Phi^*_{\phi'}(\{X_{ij}\}) \\
> 2m(n - m) \frac{1}{n(\delta^2 / 2m(n - m))} \\
= \frac{(2m(n - m))^2}{n \delta^2} \\
= n^3 \frac{\beta^2(1 - \beta)^2}{\delta^2}.
$$

Arguing exactly as in [4, Proposition 4.1], we get that

$$
\Phi^*_{\phi*\tau}(X : C) = \frac{1}{n^2} \Phi^*_{\phi'}(\{X_{ij}\}) > \beta^2 \frac{(1 - \beta)^2}{\delta^2}.
$$

Since $\beta$ was a rational number, arbitrarily close to $\alpha$, we get the desired estimate for $\Phi^*_\phi(X : B)$.

**Theorem 4.4.** Assume that $M$ is a von Neumann algebra with a faithful normal trace $\tau$, and $X_i$ are a family of self-adjoint elements in $M$, $\|X_i\| = 1$. Assume that $B_i$ form an increasing sequence of subalgebras of $M$, so that $M = \bigcup_{i=1}^\infty B_i$. Assume further that for some normal faithful state $\phi$ on $M$,

$$
\sup_i \liminf_j \Phi^*_\phi(X_i : B_j) < +\infty.
$$

Then $M$ is a factor.

**Proof.** In view of Theorem 4.1, we may assume that $\phi$ is a trace, $\tau$. Assume that $M$ is not a factor. Then there exists a central projection $p \in M$ of some trace $\alpha = \tau(p)$, $\alpha(1 - \alpha) \neq 0$. Moreover, $[p, X_i] = 0$ for all $i$. Since $B_i$
increase to all of $M$, given $\delta > 0$, there is a large enough $j$ and a projection $q \in B_j$, so that $\|q - p\|_2 < \delta/2$. Then for any $k$,
\[
\|[q, X_k]\|_2 = \|qX_k - X_kq\|_2 \\
= \|(q - p)X_k - X_k(q - p) + pX_k - X_kp\|_2 \\
\leq \|(q - p)X_k\|_2 + \|X_k(q - p)\|_2 + 0 \\
\leq 2\|(q - p)\|_2\|X_k\| \\
< 2(\delta/2) = \delta.
\]
Now applying Lemma 4.3, we deduce that for any $i > j$,
\[
\Phi^*_i(X_i : B_j) > 4\frac{\alpha^2(1 - \alpha)^2}{\delta^2}.
\]
Hence $\liminf_i \Phi^*_i(X_i : B_j) > 4\alpha^2(1 - \alpha)^2/\delta^2$, which is a contradiction, since $\delta$ was arbitrary. \qed

The hypothesis of the theorem is satisfied for some von Neumann algebras. For example, let $M = L(F_\infty)$ generated by an infinite semicircular family $X_i$, $i = 1, 2, 3, \ldots$. Then if $B_i = W^*(X_j : j < i)$, the assumptions of the theorem are satisfied. In fact if $Y_i$ is any family of elements of a tracial von Neumann algebra, so that $\|Y_i\|_1 = 1$, and $X_i$ are a free semicircular family, then letting $Z_i(\epsilon) = Y_i + \epsilon X_i$, $M_\epsilon = W^*(Z_1(\epsilon), Z_2(\epsilon), \ldots)$ and $B_j = W^*(Z_1(\epsilon), \ldots, Z_j(\epsilon))$, we see that $M_\epsilon$ is a factor. In other words, generators of an arbitrary tracial von Neumann algebra can be perturbed (in a certain representation of this algebra) by an arbitrarily small amount $\epsilon$ in uniform norm, to produce a II_1 factor. Another way of putting it is to note that the free Brownian motion $\epsilon \mapsto Z_j(\epsilon)$ started at $\{Y_1, Y_2, \ldots\}$ generates a factor at any time $\epsilon > 0$.

4.3. Factoriality in the non-tracial case. In a similar way, we get the following:

**Theorem 4.5.** Assume that $M$ is a von Neumann algebra with a faithful normal state $\phi$, and $X_i$ are a family of self-adjoint elements in $M$, $\|X_i\| = 1$. Assume that $B_i$ form an increasing sequence of subalgebras of $M$, $\sigma^\phi_t(B_i) = B_i$ for all $t$ and $i$, and assume that $M^\phi = \bigcup(B_i \cap M^\phi)$. Let $R_i$ be the operator of right multiplication by $X_i$ densely defined on $L^2(M, \phi)$. Assume that $\sup_i \|R_i\| = C < +\infty$. Assume further that
\[
\sup_i \liminf_j \Phi^*_i(X_i : B_j) < +\infty.
\]
Then $M$ is a factor.

**Proof.** Assume that $M$ is not a factor. Then there exists a central projection $p \in M$, $\alpha = \phi(p)$, $\alpha(1 - \alpha) \neq 0$. Moreover, $[p, X_i] = 0$ for all $i$. Since automatically $p \in M^\phi$ and $B_i \cap M^\phi$ increase to all of $M^\phi$, given $\delta > 0$, there
is a large enough $j$ and a projection $q \in B_j \cap M^\phi$, so that $\|q-p\|_2 < \delta/(1+C)$. Then for any $k$,

$$
\| [q, X_k] \|_2 = \| q X_k - X_k q \|_2
= \| (q-p) X_k - X_k (q-p) + p X_k - X_k p \|_2
\leq \| (q-p) X_k \|_2 + \| X_k (q-p) \|_2 + 0
\leq \| R_k \| \| (q-p) \|_2 + \| (q-p) \|_2 \| X_k \|
< (1+C)(\delta/(1+C)) = \delta.
$$

The rest of the argument proceeds just like in the tracial case. $\square$

Note that the assumption on the norms of $R_i$ is satisfied if each $X_i$ is analytic for $\sigma_0$ and satisfies $\sup_k \| \sigma_0(X_k) \| = C < +\infty$.

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NONWANDERING, NONRECURRENT FATOU COMPONENTS IN $P^2$

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Let $\Omega$ be a nonwandering, nonrecurrent Fatou component for a holomorphic self-map $f$ of $P^2$ of degree $d \geq 2$, and let $h$ be a normal limit of the family of iterates of $f$. We prove that $\Sigma := h(\Omega)$ is either a fixed point of $f$ or its normalization is a hyperbolic Riemann surface, so that the dynamics of $f|\Sigma$ may be lifted to the unit disk. We also show that basins of attraction for holomorphic self-maps of $P^k$ of degree $d \geq 2$ are taut.

1. Introduction.

Let $f : P^k \to P^k$ be holomorphic. By definition, therefore, there exists a homogeneous polynomial mapping $\tilde{f} : C^{k+1} \setminus \{0\} \to C^{k+1} \setminus \{0\}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
C^{k+1} \setminus \{0\} & \xrightarrow{\tilde{f}} & C^{k+1} \setminus \{0\} \\
\downarrow p & & \downarrow p \\
P^k & \xrightarrow{f} & P^k.
\end{array}
$$

Here $p$ denotes the standard projection from $C^{k+1} \setminus \{0\}$ onto $P^k$. The degree $d$ of $f$ is by definition the degree of $\tilde{f}$. Throughout this paper we assume that $d > 1$.

The Fatou set $\mathcal{F}(f)$ is the largest open subset of $P^k$ on which the family $\{f^n\}_{n \in \mathbb{N}}$ is normal. In [7], Ueda shows that $\tilde{f}$ has a bounded basin of attraction $A$ to the origin. Let $\Omega$ be any connected component of $\mathcal{F}(f)$. Ueda shows that there exists a set $\tilde{\Omega} \subset \partial A$ such that the restriction of $p$ to $\tilde{\Omega}$ is a holomorphic covering map onto $\Omega$. A corollary of this construction is the Kobayashi hyperbolicity of $\Omega$. Fornaess and Sibony have exploited this fact in their classification of recurrent Fatou components for holomorphic maps on $P^2$ ([4]).

Suppose now that $\Omega$ is a fixed, nonrecurrent Fatou component; that is, $\Omega$ satisfies $f(\Omega) = \Omega$ and $f^n(z) \to \partial \Omega$ for all $z \in \Omega$. Let $h$ be a normal limit of some subsequence of $\{f^n\}$, so that $f^{n_i} \to h$ locally uniformly on $\Omega$ as
$i \to \infty$. Then $\Sigma := h(\Omega) \subset \partial \Omega$. The principal aim of this paper is to prove the following result:

**Theorem 1.** Suppose that $f : \mathbb{P}^2 \to \mathbb{P}^2$ is holomorphic, and $\Omega$ a fixed, nonrecurrent Fatou component for $f$. Let $\Sigma$ be as described above. Then either $\Sigma$ is a fixed point of $f$, or there exists a locally injective holomorphic mapping $\sigma : \Delta \to \Sigma$, where $\Delta \subset \mathbb{C}$ is the unit disk, and a holomorphic function $F : \Delta \to \Delta$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\Delta & \xrightarrow{F} & \Delta \\
\sigma \downarrow & & \downarrow \sigma \\
\Sigma & \xrightarrow{f} & \Sigma.
\end{array}
$$

In the latter case, $F$ must either be conjugate to an irrational rotation, or $F^n(z) \to \partial \Delta$ for all $z \in \Delta$.

The proof is given in Section 2.

**Remark 1.** A more general theorem was stated by Fornaess and Sibony in [3], but the proof seems incomplete.

A complex manifold $M$ is called **taut** if the family of maps from the unit disk $\Delta$ to $M$ is normal. Abate has asked ([1]) whether Fatou components for holomorphic self-maps of $\mathbb{P}^k$ are taut. In Section 3, we prove the following:

**Theorem 2.** Let $\Omega$ be a Fatou component for $f : \mathbb{P}^k \to \mathbb{P}^k$ which is preperiodic to a basin of attraction. Then $\Omega$ is taut.

### 2. Proof of Theorem 1.

Let $f$ be a holomorphic self-map of $\mathbb{P}^k$, and $\Omega$ a fixed, nonrecurrent Fatou component. Choose and fix some subsequence $f_{n_i}$ which converges locally uniformly on $\Omega$. Let $h = \lim_{i \to \infty} f_{n_i}$, and let $\Sigma = h(\Omega)$. Then $\Sigma \subset \partial \Omega$.

**Lemma 1.** Let $\Sigma$ be as above. Then $f(\Sigma) = \Sigma$.

**Proof.** Since $h = \lim_{i \to \infty} f_{n_i}$, $h$ commutes with $f$ on $\Omega$. Let $z \in \Sigma$, $x \in h^{-1}(z)$. Let $y \in f^{-1}(x) \cap \Omega$. Then $f(z) = f(h(x)) = h(f(x)) \in \Sigma$, so $f(\Sigma) \subset \Sigma$. And $h(y) \in \Sigma$ with $f(h(y)) = h(f(y)) = h(x) = z$. Thus $f(\Sigma) = \Sigma$. $\square$

Let $p$ be the natural projection from $\mathbb{C}^{k+1} \setminus \{0\}$ to $\mathbb{P}^k$, and $\bar{f} : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{C}^{k+1} \setminus \{0\}$ the homogeneous polynomial lift of $f$ by $p$. It was shown by Ueda ([7]) that any homogeneous polynomial self-map of $\mathbb{C}^k \setminus \{0\}$ has a bounded basin of attraction to the origin. Let $A$ be the bounded basin of attraction to the origin for $\bar{f}$. Ueda showed further the existence of a set
\[ \tilde{\Omega} \subset \partial A \] such that the restriction of \( p \) to \( \tilde{\Omega} \) is a holomorphic covering map onto \( \Omega \).

**Lemma 2.** Let \( U \) be an open subset of \( \Omega \) sufficiently small that a local inverse \( q : U \to \tilde{\Omega} \) of \( p|_{\tilde{\Omega}} \) may be defined. Then there exists \( \hat{h} : U \to \partial A \) holomorphic (as a mapping into \( \mathbb{C}^{k+1} \)) such that \( p \circ \hat{h} = h \). Furthermore, if \( \hat{h}_1 \) is one such lift, then \( \hat{h}_2 \) is another if and only if \( \hat{h}_2 = e^{i\theta} \hat{h}_1 \) for some real \( \theta \).

**Proof.** Write \( h = \lim f^{n_i} \). On \( U \), we have \( p \circ (\tilde{f}^{n_i} \circ q) = f^{n_i} \). Since \( \{\tilde{f}^{n_i} \circ q\} \) is uniformly bounded as a family of mappings into \( \mathbb{C}^{k+1} \), by passing to a subsequence, if necessary, we may assume that it has a holomorphic limit \( \hat{h} \) on \( U \). Taking limits of both sides of

\[ p \circ (\tilde{f}^{n_i} \circ q) = f^{n_i} \]

gives

\[ p \circ \hat{h} = h. \]

To prove the second statement, note that \( \hat{h}_1(z) \) and \( \hat{h}_2(z) \) are in the same fiber of \( p \) for all \( z \in U \); i.e., in the same complex line in \( \mathbb{C}^{k+1} \). Thus

\[ h_1(z) = \lambda(z) \hat{h}_2(z) \]

for \( z \in U \), \( \lambda : U \to \mathbb{C} \) holomorphic. Recall also that \( h_1(z) \), \( h_2(z) \) are contained in \( \partial A \). If \( G \) is the Green’s function for \( A \), we have \( \partial A = \{G = 0\} \). It is shown in [7] that for \( \lambda \in \mathbb{C} \), \( G \) satisfies

\[ G(\lambda z) = G(z) + \log |\lambda|. \]

Thus

\[ 0 = G(\hat{h}_1(z)) = G(\hat{h}_2(z)) = G(\lambda(z) \hat{h}_1(z)) = G(\hat{h}_1(z)) + \log |\lambda(z)|. \]

Thus \( |\lambda(z)| = 1 \) for all \( z \in U \). Since \( \lambda \) is holomorphic, this gives \( \lambda \equiv e^{i\theta} \) for some \( \theta \in \mathbb{R} \).

This shows that any two lifts \( \hat{h} \) of \( h \) differ by a multiplicative constant of absolute value one. Conversely, it is easy to check that if \( \hat{h} : U \to \partial A \) is a lift of \( h \), then so is \( e^{i\theta} \hat{h} : U \to \partial A \).

The next lemma is part of the classical construction of the desingularization of a Riemann surface; see [5]. We omit the proof.

**Lemma 3.** Let \( f \) be a germ at 0 of a nonconstant holomorphic mapping from \( \mathbb{C} \) to \( \mathbb{C}^n \). Then there exists another germ \( g \) at 0 of a holomorphic mapping from \( \mathbb{C} \) to \( \mathbb{C}^n \) such that \( g \) is injective in a neighborhood of 0, and such that the images of \( f \) and \( g \) agree as germs.
Lemma 4. Given $z \in \Sigma$, let $x \in h^{-1}(z)$, and let $L$ be a complex line through $x$ such that $h|_L$ is not constant. Then there exists a ball $U$ centered at $x$ such that the restriction of $p$ to $\hat{h}(L \cap U)$ is injective.

Proof. Let $U$ be sufficiently small that we may define $\hat{h}$ on $U$, as in Lemma 2. By shrinking $U$, if necessary, we may assume that $x$ is the only critical point of both $\hat{h}$ and of $p \circ \hat{h}$ in $L \cap U$. Let $D = L \cap U$, and $D^* = D \setminus \{x\}$. By Lemma 3, shrinking $U$ further, we may assume that both $\hat{h}(D^*)$ and $p \circ \hat{h}(D^*)$ are biholomorphic to punctured disks. Thus if $p|_{h(D)}$ is not injective, we may assume, making the Böttcher coordinate change, that it is of the form $w \mapsto w^s$ for some $s \geq 2$.

But then we can replace $\hat{h}$ by another lift $g \circ \hat{h}$, where $g$, in the appropriate coordinates, is a nontrivial rotation of $\hat{h}(D)$ about $\hat{h}(x)$. In particular, $g \circ \hat{h}(x) = \hat{h}(x)$. But by Lemma 2, $g \circ \hat{h}$ must be of the form $e^{i\theta} \hat{h}$. Furthermore, $\hat{h}(x) \neq 0$, since it is in $\partial A$. Thus $e^{i\theta} = 1$, and $g$ is the trivial rotation. This contradiction establishes the lemma.

For the remainder of this section, we will assume that $k = 2$, and that $h : \Omega \to \partial \Omega$ is nonconstant. In this case, for $x \in \Omega$, there is an irreducible piece $\Sigma_x$ of a Riemann surface, possibly with singularities, and a neighborhood $U(x)$ such that $h(U(x)) = \Sigma_x$. We define $R$ to be the abstract union $\bigcup \Sigma_x$ for a covering $\{U(x)\}$ of $\Omega$, with identifications of $z_i \in \Sigma_{x_i}$ to $z_j \in \Sigma_{x_j}$ if the images under $h$ agree there as germs. $R$ is Hausdorff, by the identity theorem. It is a one-dimensional Riemann surface, possibly with singularities. Let $S$ be its smooth normalization. The map $h$ factors naturally as $\pi_1 \circ h_1$, where $h_1 : \Omega \to S$ and $\pi_1 : S \to \Sigma$.

Near a regular value of $h_1$, $h_1$ has an inverse $q$ onto some linear disk in $\Omega$. Define $f_1$ locally by $f_1 = h_1 \circ f \circ q$. It is straightforward to check that $f_1$ is thereby well-defined and holomorphic away from critical values of $h_1$, and may be extended continuously to $\Sigma$. Thus $f : \Sigma \to \Sigma$ lifts naturally by $\pi_1$ to $f_1 : S \to S$.

Lemma 5. The Riemann surface $S$ described above is hyperbolic.

Proof. Given $z_0 \in S$, let $U$ be a neighborhood of $z_0$ sufficiently small that $\pi_2(U) \subset R$ contains at most one singular point, $w_0 := \pi_2(z_0)$. Assume also that $U$ is small enough that there exists a linear disk $L \subset \Omega$ such that $p$ maps $\hat{h}(L)$ injectively onto some set containing $\pi_1 \circ \pi_2(U)$, as in Lemma 4.

Let $z_1 \in U \setminus \{z_0\}$. Then there exists a neighborhood $V$ of $z_1$ and an open subset $W \subset L$ such that $g := h|_W$ is a biholomorphism onto $\pi_1 \circ \pi_2(V)$. Consider

$$
\phi : V \to \partial A
$$

$$
z \mapsto \hat{h} \circ g^{-1} \circ \pi_1 \circ \pi_2(z).
$$
Then $\phi$ is holomorphic, and $p \circ \phi = \pi_1 \circ \pi_2$. Any other choice of $\phi$ (obtained by choosing a different subset $W \subset L$) must therefore differ from the first by a multiplicative constant of absolute value one. Since $z_1$ was arbitrary, $\phi$ may therefore be extended along any path in $U \setminus \{z_0\}$. Since $p|_{h(L)}$ is injective, this extension gives rise to a single-valued holomorphic mapping, of which $z_0$ is a removable singularity. Thus $\phi$ is holomorphic on $U$, with $p \circ \phi = \pi_1 \circ \pi_2$. Again, any other choice of $\phi$ must differ from this one by a multiplicative constant of absolute value one; and since $z_0$ was arbitrary, $\phi$ may therefore be extended along any path in $S$. But this defines a covering surface $\tilde{S} \subset \partial A$ of $S$. Since $S$ is covered by a bounded subset of $C^3$, it is hyperbolic. □

There are four a priori possibilities for $f_2 : S \to S$ (see [6]):

1. Some iterate of $f_2$ is the identity.
2. There exists $a \in \mathbb{R}$ such that $f_2^n(z) \to a$ for all $z \in S$.
3. $f_2^n(z)$ diverges to infinity with respect to the Poincaré metric on $S$ for all $z \in S$.
4. $S$ is conformally a disk, punctured disk, or annulus, and the action of $f_2$ on $S$ is conjugate to irrational rotation.

In our case, (1) is impossible, since then some iterate of $f$ would fix $\Sigma$. But by Bezout’s theorem the number of fixed points of a holomorphic self-map of complex projective space is finite. In Case (2), the point $a$ would be an attractive or semi-attractive fixed point of $f$. But then the topological dynamics in a neighborhood $U$ of $a$ are well understood. In both cases, if $U$ is sufficiently small, points in $\mathcal{F}(f) \cap U$ cannot converge to $\Sigma \setminus \{a\}$. But this contradicts our assumption that $h$ is nonconstant. Thus (2) is also impossible.

Now, we note that $f_2$ can in turn be lifted to a holomorphic self-map $F$ of the unit disk, $\Delta$. Cases (3) and (4) above give the following possibilities for $F$:

1. $F^n(z) \to \partial \Delta$ locally uniformly on $\Delta$.
2. $F$ is an irrational rotation of $\Delta$.

Collecting the preceding lemmas gives us the following theorem:

**Theorem 1.** If $h$ is a limit of some subsequence $f^{n_i}$ on $\Omega$ and $\Sigma := h(\Omega)$, then either $\Sigma$ is a fixed point of $f$ or there exists a surjective, locally injective holomorphic mapping $\sigma : \Delta \to \Sigma := h(\Omega)$, and a holomorphic self-map $F$ of $\Delta$ satisfying (1) or (2) above, such that the following diagram commutes:

$$
\begin{array}{ccc}
\Delta & \xrightarrow{F} & \Delta \\
\sigma \downarrow & & \sigma \\
\Sigma & \xrightarrow{f} & \Sigma.
\end{array}
$$
Since $\Sigma$ does not contain an entire curve of singularities, Case (2) gives that $\Sigma$ is a disk, punctured disk, or annulus, with at most one singularity, at the fixed point. An example of this type of behavior is the following: Take $f : \mathbb{P}^2 \to \mathbb{P}^2$ 

$$
\left[ z : w : t \right] \mapsto \left[ z t + z^2 : \lambda w t + w^2 : t^2 \right],
$$

where $\lambda = e^{2\pi i \theta}$ and $\theta$ satisfies a Diophantine condition. Let $S$ be the Siegel disk centered at 0 for the mapping $w \mapsto \lambda w + w^2$. Then $\{f^n\}$ is compactly divergent on the Fatou component containing the point $[-1 : 0 : 1]$, any uniform limit $h$ satisfies

$$
\Sigma = h(\Omega) = \{[0, w, 1] : w \in S\}
$$

(note that $\Sigma$ is conformally a disk), and $f|_{\Sigma}$ is conjugate to multiplication by $\lambda$.

In Case (1) above, the mapping $\sigma$ may be very complicated. I have no example of this type of behavior, nor a proof that it cannot occur.

3. Proof of Theorem 2.

**Theorem 2.** Let $\Omega$ be a Fatou component for $f : \mathbb{P}^k \to \mathbb{P}^k$ which is preperiodic to a basin of attraction. Then $\Omega$ is taut.

**Proof.** Replacing $f$ by an iterate, we may assume that $\Omega$ is an invariant basin of attraction to $q \in \Omega$. Assume, to get a contradiction, that there exists a sequence of holomorphic mappings $\{g_i : \Delta \to \Omega\}$ with no convergent subsequence. Since $\Omega$ is covered by a bounded set in $\mathbb{C}^{k+1}$, the family $\{g_i\}$ is normal as a family of maps from $\Delta$ into $\mathbb{P}^k$. Thus, passing to a subsequence if necessary, we may assume that

$$
g_i \to g : \Delta \to \overline{\Omega}.
$$

But, by assumption, $g(\Delta) \not\subset \Omega$ and $g(\Delta) \not\subset \partial \Omega$.

For each $i$, let $\tilde{g}_i : \Delta \to \partial A$ be a lift of $g_i$. Then $\{\tilde{g}_i\}$ is uniformly bounded as a family of maps into $\mathbb{C}^{k+1}$, so it is normal. By passing to a subsequence if necessary, we may assume that

$$
\tilde{g}_i \to \tilde{g} : \Delta \to \partial A.
$$

Taking limits of both sides of

$$
p \circ \tilde{g}_i = g_i
$$

gives

$$
p \circ \tilde{g} = g.
$$

Now,

$$
p \circ \tilde{f^n} \circ \tilde{g}_i = f^n \circ p \circ \tilde{g}_i = f^n \circ g_i.
$$
Thus, for each \( n \) and each \( i \), \( \tilde{f}^n \circ \tilde{g}_i \) is a lift of \( f^n \circ g_i \). Taking limits with respect to \( i \) gives
\[
p \circ \tilde{f}^n \circ \tilde{g} = f^n \circ g.
\]
But \( \{ \tilde{f}^n \circ \tilde{g} \} \) is uniformly bounded as a family of mappings into \( \mathbb{C}^{k+1} \). Thus it is normal, and so therefore is \( \{ f^n \circ g \} \). Let \( h \) be a normal limit of \( \{ f^n \circ g \} \). Then \( h \equiv q \) on \( g^{-1}(g(\Delta) \cap \Omega) \), so \( h \equiv q \) on \( \Delta \). But this is impossible, since \( f^n \circ g(z) \in \partial \Omega \) for all \( z \in g^{-1}(g(\Delta) \cap \partial \Omega) \). \( \square \)

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