QUANTUM LENS SPACES AND GRAPH ALGEBRAS

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We construct the $C^*$-algebra $C(L_q(p; m_1, \ldots, m_n))$ of continuous functions on the quantum lens space as the fixed point algebra for a suitable action of $\mathbb{Z}_p$ on the algebra $C(S^{2n-1}_q)$, corresponding to the quantum odd dimensional sphere. We show that $C(L_q(p; m_1, \ldots, m_n))$ is isomorphic to the graph algebra $C^*(F_{2n-1}^{(m_1, \ldots, m_n)})$. This allows us to determine the ideal structure and, at least in principle, calculate the $K$-groups of $C(L_q(p; m_1, \ldots, m_n))$. Passing to the limit with natural imbeddings of the quantum lens spaces we construct the quantum infinite lens space, or the quantum Eilenberg-MacLane space of type $(\mathbb{Z}_p, 1)$.

0. Introduction.

Classical lens spaces $L(p; m_1, \ldots, m_n)$ are defined as the orbit spaces of suitable free actions of finite cyclic groups on odd dimensional spheres (e.g., see [13]). In the present article, we define and investigate their quantum analogues. The $C^*$-algebras of continuous functions on the quantum lens spaces were introduced earlier by Matsumoto and Tomiyama in [18], but our construction leads to different (in general) algebras. (The very special case of the quantum 3-dimensional real projective space was investigated by Podleś [20] and Lance [17], in the context of the quantum $SO(3)$ group.) The starting point for us is the $C^*$-algebra $C(S^{2n-1}_q)$, $q \in (0, 1)$, of continuous functions on the quantum odd dimensional sphere. If $n = 2$ then $C(S^3_q)$ is nothing but $C(SU_q(2))$ of Woronowicz [27]. The construction in higher dimensions is due to Vaksman and Soibelman [26], and from a somewhat different perspective to Nagy [19]. (See also the closely related construction of representations of the twisted canonical commutation relations due to Pusz and Woronowicz [21].) We define the $C^*$-algebra $C(L_q(p; m_1, \ldots, m_n))$ of continuous functions on the quantum lens space as the fixed point algebra for a suitable action of the finite cyclic group $\mathbb{Z}_p$ on $C(S^{2n-1}_q)$. This definition depends on the deformation parameter $q \in (0, 1)$, as well as on positive integers $p \geq 2$ and $m_1, \ldots, m_n$. We normally assume that each of $m_1, \ldots, m_n$ is relatively prime to $p$. On the classical level, this guarantees freeness of the action. In the special case $p = 2$, $m_1 = \cdots = m_n = 1$ we recover
quantum odd dimensional real projective spaces, defined and investigated in our earlier article [11].

The key technical result on which this article depends is Theorem 4.4 of [11], which gives an explicit isomorphism between $C(S^3_\mathbb{Q})$ and the $C^*$-algebra $C^*(L_{2n-1})$ corresponding to the directed graph $L_{2n-1}$. Thus, $C(L_q(p; m_1, \ldots, m_n))$ is isomorphic to the fixed point algebra $C^*(L_{2n-1})^\Lambda$, corresponding to a suitable action $\Lambda : \mathbb{Z}_p \to \text{Aut}(C^*(L_{2n-1}))$. This allows us to employ in our investigations of the quantum lens spaces the huge and comprehensive machinery developed for dealing with Cuntz-Krieger algebras of directed graphs (cf. [6, 5, 16, 12, 15, 14, 2, 9, 24, 10, 22, 25, 1] and references there).

In order to understand the fixed point algebra $C^*(L_{2n-1})^\Lambda$ we first look at the crossed product $C^*(L_{2n-1}) \rtimes_\Lambda \mathbb{Z}_p$. By virtue of the results of [14] this crossed product itself is isomorphic to the $C^*$-algebra of the skew product graph $L_{2n-1} \times_c \mathbb{Z}_p$, corresponding to a suitable labelling $c$ of the edges of $L_{2n-1}$ by elements of $\mathbb{Z}_p$. The action $\Lambda$ is saturated and, hence, $C^*(L_{2n-1})^\Lambda$ is isomorphic to a full corner of $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$. This allows us, at least in principle, to calculate the $K$-groups of $C(L_q(p; m_1, \ldots, m_n))$.

Our main result, Theorem 2.5, shows that $C(L_q(p; m_1, \ldots, m_n))$ itself is isomorphic to the graph algebra $C^*[L_{2n-1}^{(p;m_1,\ldots,m_n)}]$, corresponding to a finite graph $L_{2n-1}^{(p;m_1,\ldots,m_n)}$. As a corollary, we easily deduce the ideal structure of $C(L_q(p; m_1, \ldots, m_n))$. We believe that on the basis of Theorem 2.5 one should be able to determine isomorphisms between the $C^*$-algebras of continuous functions on the quantum lens spaces, but this work is not carried in the present article.

1. Preliminaries.

1.1. Definition. We recall the definition of the $C^*$-algebra corresponding to a directed graph [9]. Let $E = (E^0, E^1, r, s)$ be a directed graph with (at most) countably many vertices $E^0$ and edges $E^1$, and range and source functions $r, s : E^1 \to E^0$, respectively. $C^*(E)$ is, by definition, the universal $C^*$-algebra generated by families of projections $\{P_v \mid v \in E^0\}$ and partial isometries $\{S_e \mid e \in E^1\}$, subject to the following relations:

(G1) $P_vP_w = 0$ for $v, w \in E^0$, $v \neq w$.
(G2) $S_e^*S_f = 0$ for $e, f \in E^1$, $e \neq f$.
(G3) $S_e^*S_e = P_{r(e)}$ for $e \in E^1$.
(G4) $S_eS_e^* \leq P_{s(e)}$ for $e \in E^1$.
(G5) $P_v = \sum_{e \in E^1 : s(e) = v} S_eS_e^*$ for $v \in E^0$, provided $\{e \in E^1 \mid s(e) = v\}$ is finite and nonempty.
Universality in this definition means that if \( \{ Q_v \mid v \in E^0 \} \) and \( \{ T_e \mid e \in E^1 \} \) are families of projections and partial isometries, respectively, satisfying Conditions (G1)–(G5), then there exists a \( \mathcal{C}^* \)-algebra homomorphism from \( \mathcal{C}^*(E) \) to the \( \mathcal{C}^* \)-algebra generated by \( \{ Q_v \mid v \in E^0 \} \) and \( \{ T_e \mid e \in E^1 \} \) such that \( P_v \mapsto Q_v \) and \( S_e \mapsto T_e \) for \( v \in E^0 \), \( e \in E^1 \).

It follows from the universal property that there exists the gauge action \( \gamma : \mathbb{T} \to \text{Aut}(\mathcal{C}^*(E)) \) such that \( \gamma_t(P_v) = P_v \) and \( \gamma_t(S_e) = tS_e \), for all \( v \in E^0 \), \( e \in E^1 \).

1.2. \( K \)-theory. The \( K \)-theory of a graph algebra \( \mathcal{C}^*(E) \) can be calculated as follows: Let \( V \subseteq E \) be the collection of all those vertices which are not sinks and emit finitely many edges, and let \( \mathbb{Z}V \) and \( \mathbb{Z}E^0 \) be free abelian groups on free generators \( V \) and \( E^0 \), respectively. We have

\[
\begin{align*}
K_0(\mathcal{C}^*(E)) &\cong \text{coker}(K_E), \\
K_1(\mathcal{C}^*(E)) &\cong \text{ker}(K_E),
\end{align*}
\]

where \( K_E : \mathbb{Z}V \to \mathbb{Z}E^0 \) is the map defined on generators as

\[
K_E(v) = \left( \sum_{e \in E^1 : s(e) = v} r(e) \right) - v.
\]

(See [5, Proposition 3.1], [16, Corollary 6.12], [22, Theorem 3.2], [24, Proposition 2] and [7, Theorem 3.1].)

1.3. Ideals. We assume that \( E \) is a row-finite graph (i.e., each vertex of \( E \) emits only finitely many edges) without sinks, since this is all we need in the present article. At first we describe closed 2-sided ideals of \( \mathcal{C}^*(E) \) invariant under the gauge action, as well as the corresponding quotients [5, 12, 16, 2, 1, 10]. To this end we consider hereditary and saturated subsets of \( E^0 \). A subset \( X \subseteq E^0 \) is hereditary and saturated if the following two conditions are satisfied:

(HS1) If \( v \in X \), \( w \in E^0 \) and there exists a path from \( v \) to \( w \) then \( w \in X \).

(HS2) If \( v \in E^0 \) and for each \( e \in E^1 \) with \( s(e) = v \) we have \( r(e) \in X \), then \( v \in X \).

We denote by \( \Sigma_E \) the collection of all hereditary and saturated subsets of \( E^0 \). Any hereditary and saturated set \( X \) gives rise to a gauge invariant ideal generated by \( \{ P_v \mid v \in X \} \) and denoted \( J_X \). The quotient \( \mathcal{C}^*(E)/J_X \) is naturally isomorphic to \( \mathcal{C}^*(E/X) \), where \( E/X \) denotes the restriction of the graph \( E \) to \( E^0 \setminus X \). There exists a bijection between \( \Sigma_E \) and the collection of all gauge invariant ideals of \( \mathcal{C}^*(E) \), given by the following two maps:

\[
X \mapsto J_X, \quad J \mapsto \{ v \in E^0 \mid P_v \in J \}.
\]
We now turn to the description of primitive ideals of the graph algebra $C^*(E)$ corresponding to a row-finite graph $E$ with no sinks \[12, 16, 2, 1, 10\]. Key objects used in the classification of primitive ideals of graph algebras are maximal tails, defined as follows: A nonempty subset $M \subseteq E^0$ is called a maximal tail if the following three conditions are satisfied:

(MT1) If $v \in E^0$, $w \in M$ and there is a path in $E$ from $v$ to $w$ then $v \in M$.

(MT2) If $v \in M$ then there exists an edge $e \in E^1$ such that $s(e) = v$ and $r(e) \in M$.

(MT3) For any $v, w \in M$ there is a $y \in M$ such that there exist paths in $E$ from $v$ to $y$ and from $w$ to $y$.

The collection $\mathcal{M}(E)$ of all maximal tails is a disjoint union of its two subcollections $\mathcal{M}_\gamma(E)$ and $\mathcal{M}_\tau(E)$, defined as follows: A maximal tail $M$ belongs to $\mathcal{M}_\gamma(E)$ if and only if every vertex simple loop $(e_1, e_2, \ldots, e_k)$ (where $e_i \in E^1$, $r(e_i) = s(e_{i+1})$, $r(e_k) = s(e_1)$ and $r(e_i) \neq r(e_j)$ for $i \neq j$) whose all vertices $s(e_i)$ belong to $M$ has an exit $e \in E^1$ (that is, $s(e) \in \{s(e_1), \ldots, s(e_k)\}$ but $e \notin \{e_1, \ldots, e_k\}$) with $r(e) \in M$. Otherwise $M$ belongs to $\mathcal{M}_\tau(E)$. It can be shown that each maximal tail from $\mathcal{M}_\gamma(E)$ gives rise to a primitive ideal of $C^*(E)$ invariant under the gauge action, and each maximal tail from $\mathcal{M}_\tau(E)$ gives rise to a circle of primitive ideals none of which is invariant under the gauge action. Let Prim($C^*(E)$) denote the set of all primitive ideals of $C^*(E)$. If $E$ is a finite graph with no sinks then there exists a bijection

$$\mathcal{M}_\gamma(E) \cup (\mathcal{M}_\tau(E) \times \mathbb{T}) \leftrightarrow \text{Prim}(C^*(E)).$$

A complete description of the closure operation in the hull-kernel topology is also available. See \[12, 16, 2, 1, 10\] for the details.

We finish this section with the following lemma, which will be needed in the proof of Theorem 2.5. Recall that a closed 2-sided ideal $J$ of a $C^*$-algebra $A$ is essential if and only if for each nonzero element $a$ of $A$ we have $aJ \neq \{0\}$.

**Lemma 1.1.** If $E$ is a row-finite graph and $X \neq \emptyset$ is a hereditary and saturated subset of $E^0$ then $J_X$ is an essential ideal of $C^*(E)$ if and only if for each vertex $v \in E^0 \setminus X$ there exists a path in $E$ from $v$ to a vertex in $X$.

**Proof.** Suppose that for each vertex $v \in E^0 \setminus X$ there exists a path in $E$ from $v$ to a vertex in $X$. With the gauge action $\gamma : \mathbb{T} \to \text{Aut}(C^*(E))$, the formula $\Gamma(b) = \int_{\mathbb{T}} \gamma_t(b) dt$ (the integration with respect to the normalized Haar measure) defines a faithful conditional expectation from $C^*(E)$ onto the fixed point algebra $C^*(E)^\gamma$. Let $a \neq 0$ be an element of $C^*(E)$ and let $J'$ be the closed 2-sided ideal of $C^*(E)$ generated by $\Gamma(a^*a)$. Since $J'$ is a nonzero gauge invariant ideal there exists a vertex $v \in E^0$ such that $P_v \in J'$ (cf. \[2, Theorem 4.1\]). If $a$ is a path from $v$ to a vertex in $X$ then $S_a \in J_X$
and \( P_vS_\alpha \neq 0 \). Consequently, \( \{0\} \neq \Gamma(a^*a)J_X = (\int_{t \in T} \gamma_t(a^*a)dt) J_X \). Thus, there exists a \( t \in T \) such that \( \gamma_t(a^*a)J_X \neq \{0\} \). Since \( \gamma_t(J_X) = J_X \) this implies \( aJ_X \neq \{0\} \). Therefore, the ideal \( J_X \) is essential, as required. The converse implication is trivial.

**1.4. Quantum odd dimensional spheres.** For \( n = 1, 2, \ldots \) and \( q \in (0,1) \) the \( C^* \)-algebra \( C(S^{2n-1}_q) \) of continuous functions on the quantum sphere \( S^{2n-1} \) is given in [26] as the universal \( C^* \)-algebra generated by elements \( z_1, z_2, \ldots, z_n \), subject to the following relations:

\[
\begin{align*}
(1) & \quad z_j z_i = q z_i z_j \quad \text{for } i < j, \\
(2) & \quad z_j^* z_i = q z_i z_j^* \quad \text{for } i \neq j, \\
(3) & \quad z_i^* z_i = (1 - q^2) \sum_{j > i} z_j z_j^* \quad \text{for } i = 1, \ldots, n, \\
(4) & \quad \sum_{i=1}^n z_i z_i^* = I.
\end{align*}
\]

It is shown in [11, Theorem 4.4] that the \( C^* \)-algebra \( C(S^{2n-1}_q) \) is isomorphic with a graph algebra \( C^*(L_{2n-1}) \). The graph \( L_{2n-1} \) has \( n \) vertices \( \{v_1, \ldots, v_n\} \) and \( n(n+1)/2 \) edges \( \bigcup_{i=1}^n \{e_{i,j} \mid j = i, \ldots, n\} \) with \( s(e_{i,j}) = v_i \) and \( r(e_{i,j}) = v_j \). It is a finite graph without sinks. For example, if \( n = 3 \) then the corresponding graph \( L_5 \) looks as follows:

\[
L_5
\]

The isomorphism \( \phi : C(S^{2n-1}_q) \to C^*(L_{2n-1}) \) is given explicitly on the generators as

\[
\begin{align*}
(5) & \quad \phi : z_n \mapsto \sum_{k_1, \ldots, k_{n-1} \in \mathbb{N}} q^{k_1 + \cdots + k_{n-1}} T(k_1, \ldots, k_{n-1}) S_{e_{n,n}} T(k_1, \ldots, k_{n-1})^*, \\
(6) & \quad \phi : z_i \mapsto \sum_{k_1, \ldots, k_i \in \mathbb{N}} q^{k_1 + \cdots + k_{i-1}} \left( \sqrt{1 - q^2(k_i+1)} - \sqrt{1 - q^2k_i} \right) \times \\
& \quad \times T(k_1, \ldots, k_i) \left( \sum_{j=1}^n S_{e_{i,j}} \right) T(k_1, \ldots, k_i)^*.
\end{align*}
\]
for $i = 1, \ldots, n - 1$. Here for $k_1, \ldots, k_i \in \mathbb{N}$ we denoted

$$ T(k_1, \ldots, k_i) = \left( \sum_{j=1}^{n} S_{e_{1,j}} \right) k_1 \left( \sum_{j=2}^{n} S_{e_{2,j}} \right) \cdots \left( \sum_{j=i}^{n} S_{e_{i,j}} \right) k_i, $$

an element of $C^*(L_{2n-1})$.

### 2. Quantum lens spaces.

We begin by recalling the definition of the classical lens spaces [13]. Namely, for $n = 1, 2, \ldots$ let $S^{2n-1} = \{(y_1, \ldots, y_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} |y_i|^2 = 1 \}$ be the sphere of dimension $2n - 1$. We fix an integer $p \geq 2$ and $n$ integers $m_1, \ldots, m_n$. If $\theta = e^{2\pi i/p}$ then

$$ (y_1, \ldots, y_n) \mapsto (\theta^{m_1}y_1, \ldots, \theta^{m_n}y_n) $$

is a homeomorphism of $S^{2n-1}$ which gives rise to an action of $\mathbb{Z}_p$, the cyclic group of order $p$, on $S^{2n-1}$. The (generalized) lens space $L(p; m_1, \ldots, m_n)$ of dimension $2n - 1$ is defined as the orbit space of this action. It is normally assumed that each of $m_1, m_2, \ldots, m_n$ is relatively prime to $p$. This assumption is equivalent to freeness of the action (8).

We now turn to the quantum case. With the sole exception of Lemma 2.1, we always assume that each of $m_1, m_2, \ldots, m_n$ is relatively prime to $p$. The universal property of $C(S^*_q)$ implies that the assignment

$$ \tilde{\Lambda} : z_i \mapsto \theta^{m_i}z_i, $$

for $i = 1, \ldots, n$, gives rise to an automorphism $\tilde{\Lambda}$ of $C(S^*_q)$ of order $p$. For $q \in (0, 1)$ we define the $C^*$-algebra $C(L_q(p; m_1, \ldots, m_n))$ of continuous functions on the quantum lens space as the fixed point algebra corresponding to this automorphism, i.e.,

$$ C(L_q(p; m_1, \ldots, m_n)) = C(S^*_q)^{\tilde{\Lambda}}. $$

Let $\phi : C(S^*_q) \to C^*(L_{2n-1})$ be the isomorphism given by (5)-(6). Setting $\Lambda = \phi\tilde{\Lambda}\phi^{-1}$ we get

$$ \Lambda : P_{v_i} \mapsto P_{v_i}, $$

$$ \Lambda : S_{e_{i,j}} \mapsto \theta^{m_{i,j}}S_{e_{i,j}}, $$

for $i = 1, \ldots, n$ and $j = i, \ldots, n$. This gives

$$ C(L_q(p; m_1, \ldots, m_n)) = C(S^*_q)^{\tilde{\Lambda}} \cong C^*(L_{2n-1})^\Lambda. $$

Actions of this type have been studied by Kumjian and Pask [14]. Let $c : L^{0}_{2n-1} \to \mathbb{Z}_p$ be a labeling of the edges of $L_{2n-1}$ such that $c(S_{e_{i,j}}) = m_i$. The skew product graph $L_{2n-1} \times_c \mathbb{Z}_p$ is defined so that its vertices are $L^0_{2n-1} \times \mathbb{Z}_p$ and its edges are $L^1_{2n-1} \times \mathbb{Z}_p$ with $s(e_{i,j}, m) = (v_i, m - m_i)$ and
Let \( r(e_{i,j}, m) = (v_j, m) \), for \( m \in \mathbb{Z}_p \), \( i = 1, \ldots, n \) and \( j = i, \ldots, n \). We note that through each vertex of this graph passes precisely one vertex simple loop (composed of \( p \) edges), and for any two vertices \((v_i, m), (v_j, k)\) there exists a path from \((v_i, m)\) to \((v_j, k)\) if and only if \( i \leq j \). For example, if \( n = 2 \), \( p = 3 \), \( m_1 = 1 \) and \( m_2 = 2 \) then \( L_3 \times_c \mathbb{Z}_3 \) looks as follows:

By virtue of [14, Corollary 2.5] there exists a \( C^* \)-algebra isomorphism

\[
C^*(L_{2n-1} \times_c \mathbb{Z}_p) \cong C^*(L_{2n-1}) \times_{\Lambda} \mathbb{Z}_p.
\]

Let \( U \) be a unitary in \( C^*(L_{2n-1}) \times_{\Lambda} \mathbb{Z}_p \) such that \( U^p = I \) and \( U x U^* = \Lambda(x) \) for all \( x \in C^*(L_{2n-1}) \). For \( m = 0, 1, \ldots, p - 1 \) let \( Q_m = \frac{1}{p} \sum_{i=0}^{p-1} \theta^{im} U^i \) be the spectral projection of \( U \). The isomorphism (14) is given explicitly by

\[
P_{(v_i, m)} \mapsto P_{v_i} Q_m, \tag{15}
\]

\[
S_{(e_{i,j}, m)} \mapsto S_{e_{i,j}} Q_m, \tag{16}
\]

for \( i = 1, \ldots, n \), \( j = i, \ldots, n \) and \( m = 0, \ldots, p - 1 \).

We have

\[
Q_0(C^*(L_{2n-1}) \times_{\Lambda} \mathbb{Z}_p) Q_0 = C^*(L_{2n-1})^{\Lambda} Q_0, \tag{17}
\]

and the map \( C^*(L_{2n-1})^{\Lambda} \rightarrow C^*(L_{2n-1})^{\Lambda} Q_0, \ x \mapsto xQ_0 \), is a \( C^* \)-algebra isomorphism. On the other hand, the isomorphism (14) (cf. Formulae (15) and (16)) identifies \( Q_0 = \sum P_{v_i} Q_0 \) with the projection \( \sum P_{(v,0)} \) in \( C^*(L_{2n-1} \times_c \mathbb{Z}_p) \). Consequently, there is a \( C^* \)-algebra isomorphism

\[
C(L_q(p; m_1, \ldots, m_n)) \cong \left( \sum_{i=1}^{n} P_{(v_i, 0)} \right) C^*(L_{2n-1} \times_c \mathbb{Z}_p) \left( \sum_{i=1}^{n} P_{(v_i, 0)} \right). \tag{18}
\]

In the following lemma we only require that \( m_1 \) be relatively prime to \( p \) and no assumptions on the remaining parameters \( m_2, \ldots, m_n \) are made whatever. The lemma says that if \( m_1 \) is relatively prime to \( p \) then the action \( \Lambda \) is saturated, as expected.

**Lemma 2.1.** If \( m_1 \) is relatively prime to \( p \) then for each vertex \((v_k, m)\) there exists a path in \( L_{2n-1} \times_c \mathbb{Z}_p \) from \((v_1, 0)\) to \((v_k, m)\). Thus, Formula
gives an isomorphism between the $C^*$-algebra $C(L_q(p; m_1, \ldots, m_n))$ and a full corner of $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$.

**Proof.** Let $k \in \{1, \ldots, n\}$, $m \in \mathbb{Z}_p$, and let $r$ be a positive integer such that $rm_1 = m$ in $\mathbb{Z}_p$. Then

$$((e_{1,1}, m_1), (e_{1,1}, 2m_1), \ldots, (e_{1,1}, (r-2)m_1), (e_{1,1}, (r-1)m_1), (e_{1,k}, rm_1))$$

is the desired path. Consequently,

$$S_{(e_{1,1}, m_1)}S_{(e_{1,2}, m_1)} \cdots S_{(e_{1,(r-2)}, m_1)}S_{(e_{1,1}, (r-1)m_1)}S_{(e_{1,k}, rm_1)}$$

is a partial isometry in $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$ whose domain projection equals $P_{(v_k, m)}$ and whose range projection is majorized by $P_{(v_1, 0)}$. Thus all projections $P_{(v_k, m)}$, $k = 1, \ldots, n$, $m \in \mathbb{Z}_p$, belong to the ideal of $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$ generated by $\sum_{k=1}^n P_{(v_k, m)}$. Since $I = \sum_{k=1}^n \sum_{m \in \mathbb{Z}_p} P_{(v_k, m)}$ in $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$, Formula (18) implies that the $C^*$-algebra $C(L_q(p; m_1, \ldots, m_n))$ is isomorphic to a full corner of $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$, as claimed.

Lemma 2.1 implies that $C(L_q(p; m_1, \ldots, m_n))$ and $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$ are strongly Morita equivalent [23, Chapter 3]. Consequently, the $K$-groups of these two $C^*$-algebras are isomorphic [4, 8]. In order to calculate the $K$-groups of $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$ we assume that each of $m_1, \ldots, m_n$ is relatively prime to $p$. For short, we write $\Phi$ for the map $K_{f_{2n-1} \times_c \mathbb{Z}_p}$ which determines these $K$-groups (cf. Section 1.2). Thus, the $K_0$ and $K_1$ groups of $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$ are isomorphic to the cokernel and kernel, respectively, of the endomorphism $\Phi$ of the free abelian group with a basis $(L_{2n-1} \times_c \mathbb{Z}_p)^0$, given by

$$\Phi : (v_i, m) \mapsto \left( \sum_{j=1}^n (v_j, m + m_i) \right) - (v_1, m).$$

**Proposition 2.2.** If each of $m_1, \ldots, m_n$ is relatively prime to $p$ then

$$K_1(C(L_q(p; m_1, \ldots, m_n))) \cong \mathbb{Z}.$$

**Proof.** By Lemma 2.1 it suffices to calculate the $K_1$-group of $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$, which is isomorphic to the kernel of the map $\Phi$ from (19). Let $\lambda_i^m \in \mathbb{Z}$, for $i = 1, \ldots, n$ and $m \in \mathbb{Z}_p$, be such that $\Phi(\sum_{i=1}^n \sum_{m \in \mathbb{Z}_p} \lambda_i^m(v_i, m)) = 0$. This can only happen if $\sum_{j=1}^i \lambda_j^{m-m_j} = \lambda_i^m$ for each $i \in \{1, \ldots, n\}$ and $m \in \mathbb{Z}_p$. Setting $i = 1$, we get $\lambda_1^m = \lambda_1^0$ for all $m \in \mathbb{Z}_p$, because $m_1$ is relatively prime to $p$. Then, considering $i = 2$, we get $\lambda_1^m + \lambda_2^{m_2-m_1} = \lambda_2^m$ for all $m \in \mathbb{Z}_p$. Summing this identity over $m$ we see that $\lambda_1^m = 0$. Consequently, $\lambda_i^m = \lambda_i^0$ for all $m \in \mathbb{Z}_p$. Again, we use here the fact that $m_2$ is relatively prime to $p$. Continuing inductively in this manner we get $\lambda_i^m = 0$ for $i = 1, \ldots, n-1$ and $\lambda_n^m = \lambda_n^0$ for $m \in \mathbb{Z}_p$. Thus, the kernel of $\Phi$ is isomorphic to $\mathbb{Z}$, as claimed.

\[\Box\]
It is also possible to calculate the cokernel of the map $\Phi$ and, therefore, the $K_0$ group of $C(L_q(p; m_1, \ldots, m_n))$. This is a simple matter if $n = 2$, and we get

$$K_0(C(L_q(p; m_1, m_2))) \cong \mathbb{Z} \oplus \mathbb{Z}_{p^2},$$

similarly to the result of Matsumoto and Tomiyama [18]. However, if $n \geq 3$ then the calculation becomes a bit more elaborate. We illustrate with a particular case.

**Proposition 2.3.** If $n = 3$, $m_2 = m_3$ and both $m_1$ and $m_2$ are relatively prime to $p$ then

$$K_0(C(L_q(p; m_1, m_2, m_3))) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_{2p} \oplus \mathbb{Z}^2_p & \text{if } p \text{ is even,} \\ \mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p & \text{if } p \text{ is odd.} \end{cases}$$

**Proof.** We must determine the cokernel of $\Phi$. It is easy to see that $\{(v_i, 0) \mid i = 1, 2, 3\}$ together with the range of $\Phi$ generate the entire group $\mathbb{Z}(L_3 \times_c \mathbb{Z}_p)^0$. Now let $d_i, \lambda_i^m \in \mathbb{Z}$ for $i = 1, 2, 3$ and $m \in \mathbb{Z}_p$, be such that $\Phi(\sum_{i=1}^3 \sum_{m \in \mathbb{Z}_p} \lambda_i^m(v_i, m)) = \sum_{i=1}^3 d_i(v_i, 0)$. This is equivalent to

$$d_i = \left( \sum_{j=1}^i \lambda_j^{m-j} \right) - \lambda_i^0, \quad (20)$$

$$0 = \left( \sum_{j=1}^i \lambda_j^{m-j} \right) - \lambda_i^m, \quad \text{for } m \neq 0. \quad (21)$$

If $i = 1$ then (21) gives $\lambda_1^m = \lambda_1^0$ for all $m \in \mathbb{Z}_p$ and then $d_1 = 0$ by (20). If $i = 2$ then substituting $m = km_2$ in (21), with $k = 1, \ldots, p-1$, we get $\lambda_2^{km_2} = k\lambda_1^0 + \lambda_2^0$ for all $k = 0, \ldots, p-1$. Then (20) yields $d_2 = p\lambda_2^0$. If $i = 3$ then substituting $m = km_2$ in (21), with $k = 1, \ldots, p-1$, we get $\lambda_3^{km_2} = k(k+1)\lambda_1^0 + k\lambda_2^0 + \lambda_3^0$ for all $k = 0, \ldots, p-1$. Then (20) yields $d_3 = \frac{p(p+1)}{2} \lambda_1^0 + p\lambda_2^0$. Thus, $(v_1, 0)$ has infinite order in the cokernel. If $p$ is even then $(v_2, 0)$ and $(v_3, 0)$ generate a subgroup of the cokernel isomorphic to $\mathbb{Z}_{2p} \oplus \mathbb{Z}_p^2$. If $p$ is odd then $(v_2, 0)$ and $(v_3, 0)$ generate a subgroup of the cokernel isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$. \qed

We now show that $C(L_q(p; m_1, \ldots, m_n))$ itself is isomorphic to a graph algebra. The following construction of the graph $L^{(p;m_1,\ldots,m_n)}_{2n-1}$ and the argument of Theorem 2.5, below, are similar to [25, Section 4 and Lemma 6]. Again, we assume that each of $m_1, \ldots, m_n$ is relatively prime to $p$.

At first we define the graph $L^{(p;m_1,\ldots,m_n)}_{2n-1}$, as follows: The vertices of the graph $L^{(p;m_1,\ldots,m_n)}_{2n-1}$ are $\{v_1, v_2, \ldots, v_n\}$. The edges of $L^{(p;m_1,\ldots,m_n)}_{2n-1}$ consist of all finite (vertex simple) paths $\alpha = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r))$ in $L_{2n-1} \times_c \mathbb{Z}_p$ such that $k_1 = m_{i_1}$, $k_a \neq 0$ for $a \neq r$, $k_r = 0$ and $(v_{j_a}, k_a) \neq (v_{j_b}, k_b)$ for
a \neq b. The source and range functions are defined as \( s(\alpha) = v_i \) and \( r(\alpha) = v_j \). We note that this is a finite graph without sinks, through each vertex there passes precisely one vertex simple loop (composed of a single edge), and for each pair of vertices \( v_i, v_j \) there exists a path from \( v_i \) to \( v_j \) if and only if \( i \leq j \). For example, if \( n = 2, p = 3, m_1 = 1 \) and \( m_2 = 2 \) then \( L_3^{(3;1,2)} \) looks as follows:

![Diagram of graph](image)

The following Lemma 2.4 essentially follows from [25, Lemma 5]. However, for the sake of completeness and reader’s convenience, we give a self-contained proof.

Lemma 2.4. If each of \( m_1, \ldots, m_n \) is relatively prime to \( p \) then for any \( l \in \{1, \ldots, n\} \) and any \( m \in \mathbb{Z}_p \) we have

\[
P_{(v_l,m)} = \sum_\alpha S_{e_{i_l,j_l,k_1}} \cdots S_{e_{i_r,j_r,k_r}} S_{e_{i_r,j_r,k_r}}^* \cdots S_{e_{i_1,j_1,k_1}}^*
\]

(in \( C^*(L_{2n-1} \times_c \mathbb{Z}_p) \)), where the summation extends over all \( \alpha = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r)) \) in \( L_{2n-1} \times_c \mathbb{Z}_p \) such that the length of \( \alpha \) is not greater than \( \nu \) (and nonzero), \( i_1 = l, k_1 - m_{i_1} = m, k_a \neq 0 \) for \( a \neq r, k_r = 0 \) and \( (v_{j_a}, k_a) \neq (v_{j_b}, k_b) \) for \( a \neq b \).

Proof. For \( \nu = 1, 2, \ldots \) we define \( A_\nu \) to be the collection of all vertex simple paths \( \alpha = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r)) \) in \( L_{2n-1} \times_c \mathbb{Z}_p \) such that the length of \( \alpha \) is not greater than \( \nu \) (and nonzero), \( i_1 = l, k_1 - m_{i_1} = m, k_a \neq 0 \) for \( a \neq r, k_r = 0 \) and \( (v_{j_a}, k_a) \neq (v_{j_b}, k_b) \) for \( a \neq b \), and let \( B_\nu \) be the collection of all paths \( \beta = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r)) \) such that the length of \( \beta \) equals \( \nu, i_1 = l, k_1 - m_{i_1} = m, k_a \neq 0 \) for \( a = 1, \ldots, r \) and \( (v_{j_a}, k_a) \neq (v_{j_b}, k_b) \) for \( a \neq b \). We show, by induction on \( \nu \), that

\[
(22) \quad P_{(v_l,m)} = \sum_{\alpha \in A_\nu} S_\alpha S_\alpha^* + \sum_{\beta \in B_\nu} S_\beta S_\beta^*.
\]

Indeed, the collection of all edges in \( L_{2n-1} \times_c \mathbb{Z}_p \) with source equal to \( (v_l, m) \) is the union of \( A_1 \) and \( B_1 \). Thus, (22) holds with \( \nu = 1 \) by virtue of (G5). Now suppose (22) holds for some \( \nu \). If \( \beta = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r)) \) in \( B_\nu \), then applying Condition (G5) at the range vertex of \( \beta \), equal to \( (v_{j_r}, k_r) \), we get

\[
(23) \quad S_\beta S_\beta^* = S_\beta P_{(v_{j_r}, k_r)} S_\beta^* = \sum_{d=j_r}^n S_\beta S_{e_{j_r,d,k_r+m_{j_r}}} S_{e_{j_r,d,k_r+m_{j_r}}}^*.
\]
Let \( \beta' = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r), (e_{j_r,d}, k_r + m_{j_r})) \). We claim that \((v_d, k_r + m_{j_r}) \neq (v_{j_r}, k_a)\) for \(a = 1, \ldots, r\). This is obvious if \(d \neq j_r\). For \(d = j_r\) let \(b\) be the smallest index such that \(j_b = j_r\). Since \(\beta\) is a path we have \(j_b = j_{b+1} = \cdots = j_r\) and \(k_{b+h} = k_b + hm_{j_b}\) for \(h = 1, \ldots, r - b\). Since \(m_{j_b}\) is relatively prime to \(p\) it follows that \(k_r + m_{j_r} \notin \{k_b, \ldots, k_r\}\), as claimed. Thus \(\beta' \in (A_{p+1} \backslash A_r) \cup B_{p+1}\). Consequently, from the inductive hypothesis, Formula (23) and the above discussion we get

\[
P_{(v_l,m)} = \sum_{\alpha \in A_r} S_{\alpha} S_{\alpha}^* + \sum_{\beta \in B_r} S_{\beta} S_{\beta}^*
\]

and the inductive step follows.

Since \(L_{2n-1} \times c \mathbb{Z}_p\) is a finite graph there exists a \(\nu\) large enough so that \(B_\nu = \emptyset\). With this \(\nu\) Formula (22) gives the lemma. \(\square\)

**Theorem 2.5.** If each of the numbers \(m_1, \ldots, m_n\) is relatively prime to \(p\), then the \(C^*\)-algebra \(C(L_\alpha(p; m_1, \ldots, m_n))\) is isomorphic to \(C^*\left(L_{2n-1}^{(p;m_1,\ldots,m_n)}\right)\).

**Proof.** At first we observe that there exists a \(C^*\)-algebra homomorphism

\[
\psi : C^*\left(L_{2n-1}^{(p;m_1,\ldots,m_n)}\right) \to \left(\sum_{i=1}^{n} P_{(v_i,0)}\right) C^*(L_{2n-1} \times c \mathbb{Z}_p) \left(\sum_{i=1}^{n} P_{(v_i,0)}\right)
\]

such that

\[
\psi : P_{v_l} \mapsto P_{(v_l,0)}, \quad \psi : S_{\alpha} \mapsto S_{(e_{i_1,j_1,k_1})} S_{(e_{i_2,j_2,k_2})} \cdots S_{(e_{i_r,j_r,k_r})},
\]

for all \(l = 1, \ldots, n\) and for all \(\alpha = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r))\), vertex simple paths in \(L_{2n-1} \times c \mathbb{Z}_p\) such that \(k_1 = m_{i_1}, k_a \neq 0\) for \(a \neq r, k_r = 0\) and \(v_{j_r}, k_a \neq (v_{j_r}, k_a)\) for \(a \neq b\). Due to the universal property of \(C^*\left(L_{2n-1}^{(p;m_1,\ldots,m_n)}\right)\), to this end it suffices to verify that the elements \(\{\psi(P_{v_l})\}\), \(\psi(S_{\alpha})\) of \(C^*(L_{2n-1} \times c \mathbb{Z}_p)\) satisfy Conditions (G1)-(G5) for the graph \(L_{2n-1}^{(p;m_1,\ldots,m_n)}\). But it is obvious that Conditions (G1)-(G4) are satisfied, and Condition (G5) is equivalent to Lemma 2.4 with \(m_0 = 0\).

For surjectivity of \(\psi\) it suffices to show that:

(i) If \(\alpha\) is a path in \(L_{2n-1} \times c \mathbb{Z}_p\) such that both \(s(\alpha)\) and \(r(\alpha)\) are in \:\(\{(v_i,0) \mid i = 1, \ldots, n\}\) then \(S_{\alpha}\) belongs to the range of \(\psi\).

(ii) If \(\alpha, \beta\) are two paths such that \(r(\alpha) = r(\beta)\) and both \(s(\alpha)\) and \(s(\beta)\) are in \:\(\{(v_i,0) \mid i = 1, \ldots, n\}\) then \(S_{\alpha} S_{\beta}^*\) belongs to the range of \(\psi\).
To this end we first note that any loop in $L_{2n-1} \times_c \mathbb{Z}_p$ must pass through a vertex of the form $(v_i, 0)$. Now let $\alpha = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_\nu,j_\nu}, k_\nu))$ be a path as in (i). Let $a_1 < a_2 < \cdots < a_\nu = r$ be all the indices for which $k_{a_i} = 0$. We also set $a_0 = 0$. For each $t = 1, \ldots, \nu$ the path $\alpha_t = ((e_{i_{a_t+1},j_{a_t+1}}, k_{1+at-1}), \ldots, (e_{i_{at},j_{at}}, k_{at}))$ is an edge of the graph $L^{((p; m_1, \ldots, m_n)}_{2n-1}$. Hence, $S_\alpha = S_{\alpha_1} \cdots S_{\alpha_\nu}$ belongs to the range of $\psi$, since each $S_{\alpha_t}$ does. Now let $\alpha$ and $\beta$ be two paths as in (ii). Let $\alpha = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_\nu,j_\nu}, k_\nu))$. By virtue of Part (i) it suffices to consider the case $k_r \neq 0$. Let $\mu$ be the greatest index such that $k_\mu = 0$, or if such an index does not exist. We set $\alpha_1 = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_\mu,j_\mu}, k_\mu))$ and $\alpha_2 = ((e_{i_{\mu+1},j_{\mu+1}}, k_{\mu+1}), \ldots, (e_{i_\nu,j_\nu}, k_\nu))$. We have $S_\alpha = S_{\alpha_1} S_{\alpha_2}$ and $S_{\alpha_1}$ is in the range of $\psi$ by Part (i) (if $\mu = 0$ then $\alpha_1 = \emptyset$ and $S_{\alpha_1} = I$). Furthermore, for $\mu + 1 \leq a, b \leq r$ we have $(v_{j_a}, k_b) \neq (v_{j_b}, k_b)$ if $a \neq b$. We have an analogous factorization $S_\beta = S_{\beta_1} S_{\beta_2}$, with $S_{\beta_1}$ in the range of $\psi$. Let $P_{(v_{j_r}, k_r)} = \sum_x S_x S_2^\ast$ be the decomposition as in Lemma 2.4. Then we have

$$S_\alpha S_\beta = S_{\alpha_1} S_{\alpha_2} P_{(v_{j_r}, k_r)} S_{\beta_2} S_{\beta_1} = \sum_x S_{\alpha_1} S_{\alpha_2} S_x S_2^\ast S_x^\ast S_{\beta_2} S_{\beta_1}^\ast.$$ 

Consequently, $S_\alpha S_\beta$ belongs to the range of $\psi$, since $S_{\alpha_1}, S_{\beta_1}$ and all $S_{\alpha_2} S_x$ and $S_{\beta_2} S_x$ do. This completes the proof of surjectivity of $\psi$.

Now we show that the homomorphism $\psi$ is injective. Our argument is essentially the same as in [5, Remark 3]. Since for each $i \in \{1, \ldots, n-1\}$ there exists a path from $v_i$ to $v_n$, the ideal $J$ of $C^* \left( L^{(p; m_1, \ldots, m_n)}_{2n-1} \right)$ generated by $P_{v_n}$ is essential by Lemma 1.1. Thus, it suffices to show that $J \cap \ker(\psi) = \{0\}$. To this end, we notice that in the graph $L^{(p; m_1, \ldots, m_n)}_{2n-1}$ the vertex $v_n$ emits a unique edge, which we call $e$, and the range of this edge is $v_n$. Since there are infinitely many paths from other vertices to $v_n$ it follows (cf. [15] and [5, Remark 3]) that

$$J \cong P_{v_n} J P_{v_n} \otimes K = C^*(S_e) \otimes K \cong C(T) \otimes K.$$ 

Hence, in order to prove injectivity of $\psi$ it suffices to show that $C^*(S_e) \cap \ker(\psi) = \{0\}$. This follows from the fact that

$$\psi(S_e) = S_{(e_{n,n,m_n})} S_{(e_{n,n,2m_n})} \cdots S_{(e_{n,n,p_m n})}$$

is a partial unitary with full spectrum (cf. [15]).

With help of Theorem 2.5 it is easy to determine the ideal structure of $C(L_q(p; m_1, \ldots, m_n))$. For example, we have seen in the proof of Theorem 2.5 that the ideal of $C^* \left( L^{(p; m_1, \ldots, m_n)}_{2n-1} \right)$ generated by $P_{v_n}$ is isomorphic to $C(T) \otimes K$. The corresponding quotient is $C^* \left( L^{(p; m_1, \ldots, m_{n-1})}_{2n-3} \right)$, and this $C^*$-algebra is in turn isomorphic to $C(L_q(p; m_1, \ldots, m_{n-1}))$. Thus, there
exists an exact sequence
\[ 0 \to C(T) \otimes K \to C(L_p(p; m_1, \ldots, m_n)) \to C(L_p(p; m_1, \ldots, m_{n-1})) \to 0. \]

Using the exact sequence (24) or the general results about graph algebras, outlined in Section 1.3, it is easy to understand the primitive spectrum of \( C(L_p(p; m_1, \ldots, m_n)) \). Therefore, we omit the proof of the following proposition:

**Proposition 2.6.** If each of \( m_1, \ldots, m_n \) is relatively prime to \( p \) then the primitive ideal space of \( C(L_p(p; m_1, \ldots, m_n)) \) consists of \( n \) disjoint copies \( C_1, \ldots, C_n \) of the circle. The hull-kernel topology restricted to each of the circles coincides with the natural one. The closure of a point in \( C_k \) contains \( C_1 \cup \cdots \cup C_{k-1} \). Thus, \( \text{Prim}(C(L_p(p; m_1, \ldots, m_n))) \) and \( \text{Prim}(C(S^{2n-1}_p)) \) are homeomorphic (cf. [11, Section 4.1]).

**Concluding remarks.** For a fixed integer \( p \geq 2 \) the infinite lens space \( L(p; \infty) \) is defined as the inductive limit of the lens spaces \( L(p; 1_n) \), corresponding to the natural imbeddings \( L(p; 1_n) \hookrightarrow L(p; 1_{n+1}) \). (If \( m_1 = \cdots = m_n = 1 \) then we simply write \( L(p; 1_n) \) instead of \( L(p; 1, \ldots, 1) \).) It turns out that \( L(p; \infty) \) is identical with the Eilenberg-MacLane space of type \((\mathbb{Z}_p, 1) \) [3].

The results of the previous section lead to quantum versions of this classical topological setting. Namely, the inclusion \( L(p; 1_n) \hookrightarrow L(p; 1_{n+1}) \) corresponds to the surjective homomorphism \( \tilde{\theta}_{n+1} : C(L_p(p; 1_{n+1})) \to C(L_p(p; 1_n)) \) such that the kernel of \( \tilde{\theta}_{n+1} \) is generated by \( z_n z_{n+1}^* \). Consequently, the quantum infinite lens space, or the quantum Eilenberg-MacLane space of type \((\mathbb{Z}_p, 1) \), may be defined as the inverse limit
\[ C(L_p(p; \infty)) = \lim_{\leftarrow} (C(L_p(p; 1_n)), \tilde{\theta}_n). \]

Under the isomorphisms \( C(L_p(p; 1_k)) \cong C^* \left( L^{(p)}_{2k-1} \right) \) (if \( m_1 = \cdots = m_n = 1 \) then we simply write \( L^{(p)}_{2n-1} \) instead of \( L^{(p;1,\ldots,1)}_{2n-1} \)), described in Theorem 2.5, the homomorphism \( \tilde{\theta}_{n+1} \) is carried onto a surjective \( C^* \)-algebra homomorphism \( \theta_{n+1} : C^* \left( L^{(p)}_{2n-1} \right) \to C^* \left( L^{(p)}_{2n-1} \right) \), whose kernel is generated by the projection \( P_{v_{n+1}} \). Thus, we have the \( C^* \)-algebra isomorphism
\[ C(L_p(p; \infty)) \cong \lim_{\leftarrow} \left( C^* \left( L^{(p)}_{2n-1} \right), \theta_n \right). \]

It is not difficult to see, and we omit the details, that this inverse limit itself may be realized as the graph algebra \( C^* \left( L^{(p)}_\infty \right) \). The graph \( L^{(p)}_\infty \) is the increasing limit of the graphs \( L^{(p)}_{2n-1} \), corresponding to the natural imbeddings \( L^{(p)}_{2n-1} \hookrightarrow L^{(p)}_{2n+1} \) such that the \( v_i \) vertex in \( L^{(p)}_{2n-1} \) is identified with
the $v_i$ vertex in $L^{(p)}_{2n+1}$, and the edges from $v_i$ to $v_j$ in $L^{(p)}_{2n-1}$ are bijectively identified with the edges from $v_i$ to $v_j$ in $L^{(p)}_{2n+1}$. The graph $L^{(p)}_{\infty}$ has infinitely many vertices $\{v_1, v_2, \ldots\}$, and for each pair $i \leq j$ there exists at least one edge from $v_i$ to $v_j$. These two properties imply that $C^*\left(L^{(p)}_{\infty}\right)$ is a primitive, purely infinite $C^*$-algebra (but not simple) [1]. Furthermore, $K_0\left(C^*\left(L^{(p)}_{\infty}\right)\right) \cong \bigoplus \mathbb{Z}$ and $K_1\left(C^*\left(L^{(p)}_{\infty}\right)\right) = 0$ [22, 7].

References


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