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Suppose M is a von Neumann algebra with normal, tracial state φ and $\{a_1, \ldots, a_n\}$ is a set of self-adjoint elements in M. We provide an alternative uniform packing description of $\delta_0(a_1, \ldots, a_n)$, the modified free entropy dimension of $\{a_1, \ldots, a_n\}$.

In the attempt to understand the free group factors Voiculescu created a type of noncommutative probability theory. One facet of the theory involves free entropy and free entropy dimension, applications of which have answered some old operator algebra questions ([**1**] and [**4**]). Roughly speaking, given self-adjoint elements $a_1, \ldots, a_n$ in a von Neumann algebra M with normal, tracial state φ a matricial microstate for $\{a_1, \ldots, a_n\}$ is an n-tuple of self-adjoint $k \times k$ matrices which together with the normalized trace, approximate the algebraic behavior of the a_i under φ. Taking a normalization of the logarithmic volume of such microstate sets followed by multiple limiting processes yields a number, $\chi(a_1, \ldots, a_n)$, called the free entropy of $\{a_1, \ldots, a_n\}$. One can think of free entropy as the logarithmic volume of the n-tuple. The (modified) free entropy dimension of $\{a_1, \ldots, a_n\}$ is

$$\delta_0(a_1, \ldots, a_n) = n + \limsup_{\epsilon \to 0} \frac{\chi(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n : s_1, \ldots, s_n)}{|\log \epsilon|}$$

where $\{s_1, \ldots, s_n\}$ is a semicircular family freely independent with respect to $\{a_1, \ldots, a_n\}$ and $\chi(:)$ is a technical modification of χ (see [**4**]).

Free entropy dimension was inspired by Minkowski dimension. Recall that for a subset $A \subset \mathbb{R}^d$ the (upper) Minkowski dimension of A is

$$d + \limsup_{\epsilon \to 0} \frac{\log \lambda(\mathcal{N}_\epsilon(A))}{|\log \epsilon|}$$

where λ above denotes Lebesgue measure and $\mathcal{N}_\epsilon(A)$ is the ϵ-neighborhood of A. Minkowski dimension has an equivalent formulation in terms of uniform packing dimension. The (upper) uniform packing dimension of A is

$$\limsup_{\epsilon \to 0} \frac{\log P_\epsilon(A)}{|\log \epsilon|} = \limsup_{\epsilon \to 0} \frac{\log K_\epsilon(A)}{|\log \epsilon|}$$

where A is endowed with the Euclidean metric, $P_\epsilon(A)$ is the maximum number of elements in a collection of mutually disjoint open ϵ balls of A, and $K_\epsilon(A)$ is the minimum number of open ϵ balls required to cover A (the quantities above make sense in the setting of an arbitrary metric space). It is easy to see that the Minkowski dimension and the uniform packing dimension of A are always equal.

In this paper we present a lemma which formulates a similar metric description of δ_0: Free entropy dimension can be described in terms of packing numbers with balls of equal radius.

The alternative description comes as no surprise in view of both the definition of δ_0 and the techniques in estimations thereof. The proof follows the classical one with the addition of the properties of χ proven in [**3**] and the strengthened asymptotic freeness results of [**5**].

1. Preliminaries.

Throughout M is a von Neumann algebra with normal, tracial state φ and $\{a_1, \ldots, a_n\}$ is a set of self-adjoint elements in M. We use the symbols χ and δ_0 to designate the same quantities introduced in [4]. $M_k^{\mathrm{sa}}(\mathbb{C})$ denotes the set of $k \times k$ self-adjoint complex matrices and $(M_k^{\mathrm{sa}}(\mathbb{C}))^n$ is the set of n-tuples with entries in $M_k^{\mathrm{sa}}(\mathbb{C})$. tr_k is the normalized trace on the $k \times k$ complex matrices. $\|\cdot\|_2$ is the inner product norm on $(M_k^{\mathrm{sa}}(\mathbb{C}))^n$ given by the formula $\|(x_1, \ldots, x_n)\|_2^2 = \sum_{i=1}^n k \cdot \mathrm{tr}_k(x_i^2)$ and vol denotes Lebesgue measure with respect to the $\|\cdot\|_2$ norm. For any $k \in \mathbb{N}$ denote by ρ_k the metric on $(M_k^{\mathrm{sa}}(\mathbb{C}))^n$ induced by the norm $k^{-\frac{1}{2}} \cdot \|\cdot\|_2$. For a metric space (X, d) and $\epsilon > 0$ write $P_\epsilon(X, d)$ for the maximum number of elements in a collection of mutually disjoint open ϵ balls of X and $K_\epsilon(X, d)$ for the minimum number of open ϵ balls required to cover X. Observe that $P_\epsilon(X, d) \geq K_{2\epsilon}(X, d) \geq P_{4\epsilon}(X, d)$. Finally for $S \subset X$ denote by $\mathcal{N}_\epsilon(S)$ the ϵ-neighborhood of S.

2. The lemma.

Definition 2.1. For any $k, m \in \mathbb{N}$, and $R, \gamma, \epsilon > 0$ define successively

$$\mathbb{P}_{\epsilon,R}(a_1, \ldots, a_n; m, k, \gamma) = P_\epsilon(\Gamma_R(a_1, \ldots, a_n; m, k, \gamma), \rho_k),$$

$$\mathbb{P}_{\epsilon,R}(a_1, \ldots, a_n; m, \gamma) = \limsup_{k \to \infty} k^{-2} \cdot \log(\mathbb{P}_{\epsilon,R}(a_1, \ldots, a_n; m, k, \gamma)),$$

$$\mathbb{P}_{\epsilon,R}(a_1, \ldots, a_n) = \inf\{\mathbb{P}_{\epsilon,R}(a_1, \ldots, a_n; m, \gamma) : m \in \mathbb{N}, \gamma > 0\},$$

$$\mathbb{P}_\epsilon(a_1, \ldots, a_n) = \sup_{R>0}\{\mathbb{P}_{\epsilon,R}(a_1, \ldots, a_n)\}.$$

Remark. If $b_1, \ldots, b_p \in M$, then define $\mathbb{P}_\epsilon(a_1, \ldots, a_n : b_1, \ldots, b_p)$ to be the quantity obtained by replacing $\Gamma_R(a_1, \ldots, a_n; m, k, \gamma)$ in the definition with $\Gamma_R(a_1, \ldots, a_n : b_1, \ldots, b_p; m, k, \gamma)$. Similarly, we define $\mathbb{K}_\epsilon(a_1, \ldots, a_n)$ and all its associated quantities by replacing P_ϵ in the first line of Definition 2.1 with K_ϵ. Define $\mathbb{K}_\epsilon(a_1, \ldots, a_n : b_1, \ldots, b_p)$ in the same way $\mathbb{P}_\epsilon(a_1, \ldots, a_n : b_1, \ldots, b_p)$ was defined.

For any self-adjoint elements $h_1, \ldots, h_n \in M$ denote by $\underline{\chi}(h_1, \ldots, h_n)$ the number obtained by replacing the lim sup in the definition of χ with lim inf . $\mathbb{P}_\epsilon(\cdot)$ being a normalized limiting process of the logarithmic microstate space packing numbers we observe just as in the classical case that:

Lemma 2.2. *If $\{h_1, \ldots, h_n\}$ is a set of self-adjoint elements in M which is freely independent with respect to $\{a_1, \ldots, a_n\}$ and $\underline{\chi}(h_1, \ldots, h_n) > -\infty$, then*

$$\begin{aligned} n + \limsup_{\epsilon \to 0} & \frac{\chi(a_1 + \epsilon h_1, \ldots, a_n + \epsilon h_n : h_1, \ldots, h_n)}{|\log \epsilon|} \\ &= \limsup_{\epsilon \to 0} \frac{\mathbb{K}_\epsilon(a_1, \ldots, a_n)}{|\log \epsilon|} \\ &= \limsup_{\epsilon \to 0} \frac{\mathbb{P}_\epsilon(a_1, \ldots, a_n)}{|\log \epsilon|}. \end{aligned}$$

Proof. Clearly it suffices to show equality of the first and last expressions above since $P_\epsilon(\cdot) \geq K_{2\epsilon}(\cdot) \geq P_{4\epsilon}(\cdot)$. Furthermore, we can assume that $\{a_1, \ldots, a_n\}$ has finite dimensional approximants since the equalities hold trivially otherwise. Set $C = \max\{\|h_i\|\}_{1 \leq i \leq n} + 1$. First we show that the free entropy expression is greater than or equal to the $\mathbb{P}_\epsilon$ expression. Suppose $0 < \epsilon < (C\sqrt{n})^{-1}$, $m \in \mathbb{N}$, with $m > n$, $1 > \gamma > 0$, and $R > \max\{\|a_i\|\}_{1 \leq i \leq n}$.

Corollary 2.14 of [**5**] provides an $N \in \mathbb{N}$ such that if $k \geq N$ and σ is a Radon probability measure on $((M_k^{sa}(\mathbb{C}))_{R+1})^{2n}$ (the subset of $(M_k^{sa}(\mathbb{C}))^{2n}$ consisting of $2n$-tuples whose entries have operator norm no greater than $R+1$) invariant under the U_k-action

$$(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) \mapsto (\xi_1, \ldots, \xi_n, u\eta_1 u^*, \ldots, u\eta_n u^*),$$

then $\sigma(\omega_k) > \frac{1}{2}$ where ω_k is

$$\{(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) \in ((M_k^{sa}(\mathbb{C}))_{R+1})^{2n} : \{\xi_1, \ldots, \xi_n\} \text{ and } \{\eta_1, \ldots, \eta_n\} \text{ are } (m, \gamma/4^m)\text{-free}\}.$$

With respect to the ρ_k metric for each k find a collection of mutually disjoint open $C\epsilon\sqrt{n}$ balls of $\Gamma_R(a_1, \ldots, a_n; m, k, \gamma/(8(R+2))^m)$ of maximum cardinality and denote the centers of these balls by $\left\langle \left(x_{1j}^{(k)}, \ldots, x_{nj}^{(k)}\right) \right\rangle_{j \in S_k}$. Let μ_k be the uniform atomic probability measure supported on the centers of these balls and let ν_k be the probability measure obtained by restricting vol to $\Gamma_{C\epsilon}(\epsilon h_1, \ldots, \epsilon h_n; m, k, \gamma/8^m)$ and normalizing appropriately. Then $\mu_k \times \nu_k$

is a Radon probability measure on $((M_k^{\text{sa}}(\mathbb{C}))_{R+1})^{2n}$ invariant under the U_k-action described above. So for $k \geq N$ $(\mu_k \times \nu_k)(\omega_k) > \frac{1}{2}$.

For $k \in \mathbb{N}$ and $j \in S_k$ define F_{jk} to be the set of all $(y_1, \ldots, y_n) \in \Gamma_{C\epsilon}(\epsilon h_1, \ldots, \epsilon h_n; m, k, \gamma/8^m)$ such that $(y_1, \ldots, y_n)$ and $(x_{1j}^{(k)}, \ldots, x_{nj}^{(k)})$ are $\big(m, \frac{\gamma}{4^m}\big)$-free.

$$\begin{aligned}\frac{1}{2} < (\mu_k \times \nu_k)(\omega_k) &= \sum_{j \in S_k} \frac{1}{|S_k|} \cdot \nu_k(F_{jk}) \\ &= \sum_{j \in S_k} \frac{1}{|S_k|} \cdot \frac{\text{vol}\,(F_{jk})}{\text{vol}\,(\Gamma_{C\epsilon}(\epsilon h_1, \ldots, \epsilon h_n; m, k, \gamma/8^m))}.\end{aligned}$$

It follows that for $k \geq N$

$$\frac{1}{2} \cdot |S_k| \cdot \text{vol}\,(\Gamma_{C\epsilon}(\epsilon h_1, \ldots, \epsilon h_n; m, k, \gamma/8^m)) < \sum_{j \in S_k} \text{vol}\,(F_{jk}).$$

Set $E_{jk} = (x_{1j}^{(k)}, \ldots, x_{nj}^{(k)}) + F_{jk}$. F_{jk} is a set contained in the open ball of ρ_k radius $C\epsilon\sqrt{n}$ centered at $(0, \ldots, 0)$. Thus $\langle E_{jk} \rangle_{j \in S_k}$ is a collection of mutually disjoint sets. So

$$\bigsqcup_{j \in S_k} E_{jk} \subset \Gamma_{R+1}(a_1 + \epsilon h_1, \ldots, a_n + \epsilon h_n : \epsilon h_1, \ldots, \epsilon h_n; m, k, \gamma).$$

Thus, for any $(C\sqrt{n})^{-1} > \epsilon > 0$, $m \in \mathbb{N}$ sufficiently large, and $1 > \gamma > 0$

$$\begin{aligned}&\chi_{R+1}(a_1 + \epsilon h_1, \ldots, a_n + \epsilon h_n : \epsilon h_1, \ldots, \epsilon h_n; m, \gamma) \\ &\geq \limsup_{k \to \infty} \left(k^{-2} \cdot \log \left(\text{vol} \left(\bigsqcup_{j \in S_k} E_{jk} \right) \right) + \frac{n}{2} \cdot \log k \right) \\ &\geq \limsup_{k \to \infty} \Bigg[k^{-2} \cdot \log \left(\frac{1}{2} \cdot |S_k| \cdot \text{vol}\,(\Gamma_{C\epsilon}(\epsilon h_1, \ldots, \epsilon h_n; m, k, \gamma/8^m)) \right) \\ &\qquad + \frac{n}{2} \log k \Bigg] \\ &\geq \limsup_{k \to \infty} \left[k^{-2} \cdot \log(|S_k|) \right] \\ &\quad + \liminf_{k \to \infty} \left[k^{-2} \cdot \log(\text{vol}\,(\Gamma_{C\epsilon}(\epsilon h_1, \ldots, \epsilon h_n; m, k, \gamma/8^m))) + \frac{n}{2} \cdot \log k \right] \\ &\geq \mathbb{P}_{C\epsilon\sqrt{n}, R+1}\,(a_1, \ldots, a_n; m, \gamma/(8(R+2))^m) + \underline{\chi_{C\epsilon}}(\epsilon h_1, \ldots, \epsilon h_n) \\ &\geq \mathbb{P}_{C\epsilon\sqrt{n}, R+1}(a_1, \ldots, a_n) + n \log \epsilon + \underline{\chi}(h_1, \ldots, h_n).\end{aligned}$$

By the chain of inequalities of the preceding paragraph it follows that

$$\begin{aligned}&\chi(a_1+\epsilon h_1,\ldots,a_n+\epsilon h_n : h_1,\ldots,h_n)\\&=\chi(a_1+\epsilon h_1,\ldots,a_n+\epsilon h_n : \epsilon h_1,\ldots,\epsilon h_n)\\&\geq \mathbb{P}_{C\epsilon\sqrt{n},R+1}(a_1,\ldots,a_n)+n\log\epsilon+\underline{\chi}(h_1,\ldots,h_n).\end{aligned}$$

This being true for R sufficiently large

$$\begin{aligned}&\chi(a_1+\epsilon h_1,\ldots,a_n+\epsilon h_n : h_1,\ldots,h_n)\\&\geq \mathbb{P}_{C\epsilon\sqrt{n}}(a_1,\ldots,a_n)+n\log\epsilon+\underline{\chi}(h_1,\ldots,h_n).\end{aligned}$$

Dividing by $|\log\epsilon|$ on both sides, taking a lim sup as $\epsilon\to 0$, and adding n to both ends of the inequality above yields

$$\begin{aligned}&n+\limsup_{\epsilon\to 0}\frac{\chi(a_1+\epsilon h_1,\ldots,a_n+\epsilon h_n : h_1,\ldots,h_n)}{|\log\epsilon|}\\&\geq \limsup_{\epsilon\to 0}\frac{\mathbb{P}_{C\epsilon\sqrt{n}}(a_1,\ldots,a_n)}{|\log\epsilon|}\\&=\limsup_{\epsilon\to 0}\frac{\mathbb{P}_{\epsilon}(a_1,\ldots,a_n)}{|\log\epsilon|}.\end{aligned}$$

For the reverse inequality suppose $2\leq m\in\mathbb{N}$ and $\frac{1}{2(C+1)}>\epsilon>\sqrt{\gamma}>0, R>\max_{1\leq j\leq n}\{\|a_j\|\}$. For each $k\in\mathbb{N}$ find an packing by open ρ_k ϵ-balls of $\Gamma_{R+1}(a_1,\ldots,a_n;m,k,\gamma)$ with maximum cardinality. Denote the set of centers of these balls by Ω_k. Clearly

$$\begin{aligned}&\Gamma_{R+\frac{1}{2},\frac{1}{2}}\left(a_1+\epsilon h_1,\ldots,a_n+\epsilon h_n : \epsilon h_1,\ldots,\epsilon h_n;m,k,\frac{\gamma}{2^m}\right)\\&\subset\mathcal{N}_{2C\epsilon\sqrt{n}}(\Gamma_{R+1}(a_1,\ldots,a_n;m,k,\gamma))\\&\subset\mathcal{N}_{4C\epsilon\sqrt{n}}(\Omega_k)\end{aligned}$$

where $\Gamma_{r+\frac{1}{2},\frac{1}{2}}(\cdot)$ denotes the microstate space of $2n$-tuples such that the first n entries have operator norms no larger than $r+\frac{1}{2}$ and the last n entries have operator norms no larger than $\frac{1}{2}$ (see [**4**] for this technical modification). $\mathcal{N}_\epsilon$ is taken with respect to the metric space $(M_k^{\mathrm{sa}}(\mathbb{C}))^n$ with the ρ_k metric. It follows that $\chi_{R+\frac{1}{2},\frac{1}{2}}\left(a_1+\epsilon h_1,\ldots,a_n+\epsilon h_n : \epsilon h_1,\ldots,\epsilon h_n;m,\frac{\gamma}{2^m}\right)$ is dominated by

$$\begin{aligned}&\limsup_{k\to\infty}\left[k^{-2}\cdot\log(\mathrm{vol}\,(\mathcal{N}_{4C\epsilon\sqrt{n}}(\Omega_k)))+\frac{n}{2}\cdot\log k\right]\\&\leq\limsup_{k\to\infty}\left[k^{-2}\cdot\log\left(|\Omega_k|\cdot\frac{\pi^{\frac{nk^2}{2}}\cdot(4C\epsilon\sqrt{nk})^{nk^2}}{\Gamma\left(\frac{nk^2}{2}+1\right)}\right)+\frac{n}{2}\cdot\log k\right]\end{aligned}$$

$$
\begin{aligned}
&\leq \limsup_{k\to\infty} k^{-2}\cdot \log(|\Omega_k|)\\
&\quad + \limsup_{k\to\infty}\left[n\log(4C\epsilon\sqrt{nk\pi}) - k^{-2}\cdot\log\left(\frac{nk^2}{2e}\right)^{\frac{nk^2}{2}} + \frac{n}{2}\cdot\log k\right]\\
&= \limsup_{k\to\infty} k^{-2}\cdot\log(|\Omega_k|)\\
&\quad + \limsup_{k\to\infty}\left(n\log(4C\epsilon\sqrt{n\pi}) - n\log\left(\frac{k\sqrt{n}}{\sqrt{2e}}\right) + n\log k\right)\\
&= \limsup_{k\to\infty} k^{-2}\cdot\log(|\Omega_k|) + n\log(4C\epsilon\sqrt{2\pi e})\\
&= \mathbb{P}_{\epsilon,R+1}(a_1,\ldots,a_n;m,\gamma) + n\log(4C\epsilon\sqrt{2\pi e}).
\end{aligned}
$$

This being true for any $2 \leq m \in \mathbb{N}$, $\frac{1}{2(R+1)} > \epsilon > \sqrt{\gamma} > 0$, and $R > \max_{1\leq j\leq n}\{\|a_j\|\}$ it follows that for sufficiently small $\epsilon > 0$

$$
\begin{aligned}
&\chi(a_1+\epsilon h_1,\ldots,a_n+\epsilon h_n : h_1,\ldots,h_n)\\
&\quad = \chi_{R+\frac{1}{2},\frac{1}{2}}(a_1+\epsilon h_1,\ldots,a_n+\epsilon h_n : \epsilon h_1,\ldots,\epsilon h_n)\\
&\quad \leq \mathbb{P}_\epsilon(a_1,\ldots,a_n) + n\log\epsilon + n\log(4C\sqrt{2\pi e}).
\end{aligned}
$$

Dividing by $|\log\epsilon|$, taking a $\limsup$ as $\epsilon\to 0$, and adding n to both sides of the inequality above yields

$$
n + \limsup_{\epsilon\to 0}\frac{\chi(a_1+\epsilon h_1,\ldots,a_n+\epsilon h_n : h_1,\ldots,h_n)}{|\log\epsilon|} \leq \limsup_{\epsilon\to 0}\frac{\mathbb{P}_\epsilon(a_1,\ldots,a_n)}{|\log\epsilon|}.
$$

□

Remark 2.3. Suppose $b_1,\ldots,b_p$ are contained in the strongly closed algebra generated by the a_i and $R > 0$ is strictly greater than the operator norm of any a_i or b_j. The proof shows that the quanitity

$$
\limsup_{\epsilon\to 0}\frac{\mathbb{K}_{\epsilon,R}(a_1,\ldots,a_n : b_1,\ldots,b_p)}{|\log\epsilon|} = \limsup_{\epsilon\to 0}\frac{\mathbb{P}_{\epsilon,R}(a_1,\ldots,a_n : b_1,\ldots,b_p)}{|\log\epsilon|}
$$

equals

$$
n + \limsup_{\epsilon\to 0}\frac{\chi(a_1+\epsilon h_1,\ldots,a_n+\epsilon h_n : h_1,\ldots,h_n)}{|\log\epsilon|}.
$$

Recall that by [**3**] and [**5**] if $\{s_1,\ldots,s_n\}$ is a free semicircular family, then $\chi(s_1,\ldots,s_n) = \underline{\chi}(s_1,\ldots,s_n) > -\infty$. Thus we have by the lemma:

Corollary 2.4.

$$
\delta_0(a_1,\ldots,a_n) = \limsup_{\epsilon\to 0}\frac{\mathbb{P}_\epsilon(a_1,\ldots,a_n)}{|\log\epsilon|} = \limsup_{\epsilon\to 0}\frac{\mathbb{K}_\epsilon(a_1,\ldots,a_n)}{|\log\epsilon|}.
$$

Both descriptions of δ_0, either in terms of volumes of ϵ-neighborhoods or in terms of packing numbers, can be useful. In the presence of freeness or in the situation with one random variable it is fruitful to use the ϵ-neighborhood description as Voiculescu did ([**3**]). On the other hand when computing δ_0 in some examples it is convenient to use the uniform packing description and this was the implicit attitude taken towards δ_0 in [**2**]. The packing formulation also comes in handy when proving formulas for generators of M when M has a simple algebraic decomposition into a tensor product of a von Neumann algebra N with the $k \times k$ matrices or into a direct sum of algebras.

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