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Let G be a quaternionic real form of an exceptional group of real rank 4. Gross and Wallach show that three representations in the continuation of the quaternionic discrete series are unitarizable (see Gross and Wallach, 1996). In this paper we will determine the restrictions of these representations to certain subgroups of G by computing explicitly the intersections of orbits. In particular we will determine certain compact dual pair correspondences of the minimal representation of G .

1. Introduction.

1.1. We refer to §3 of [GW2] or §2 of [L3] for the definition of the double cover G of a quaternionic real form G_0 of a complex Lie group $G(\mathbb{C})$. G has maximal compact subgroup K of the form $M_1 \times M$ where M_1 is isomorphic to SU_2 . Its Lie algebra has complexified Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Here $\mathfrak{p} = \mathbb{C}^2 \otimes V_M$ where V_M is a self dual representation of $M(\mathbb{C})$.

1.2. Let $G(\mathbb{C})$ be one of the following simply connected complex exceptional groups:

$$F_4(\mathbb{C}), E_6(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}, E_7(\mathbb{C}), E_8(\mathbb{C}).$$

We will index these four cases by $s = 1, 2, 4, 8$ respectively. Let $G(\mathbb{C})^0$ denote the connected component of $G(\mathbb{C})$. Then there exists a unique connected quaternionic real form G_0^0 of $G(\mathbb{C})^0$ (cf. §1.1). It has a real root system of type F_4 . We will denote G_0^0 by $F_{4,4}$, $E_{6,4}$, $E_{7,4}$ and $E_{8,4}$ respectively. Set $G_0 = F_{4,4}$, $E_{6,4} \rtimes \mathbb{Z}/2\mathbb{Z}$, $E_{7,4}$ and $E_{8,4}$ respectively and let G denote the corresponding double cover of G_0 .

It is known that $\mathbb{P}V_M$ is a union of four $M(\mathbb{C})$ -orbits. One of the orbits is Zariski dense and we will follow [GW2] and denote the Zariski closure of the remaining three orbits by X, Y and Z . Here Z is the unique closed orbit in $\mathbb{P}V_M$ and

$$(1) \quad \mathbb{P}V_M \supset X \supset Y \supset Z.$$

Let \mathcal{O} be either X, Y or Z and let $\bigoplus_n A^n(\mathcal{O})$ denote the coordinate ring of \mathcal{O} in $\mathbb{P}V_M$. Note that $A^n(\mathcal{O})$ is a representation of M .

In [GW1], [GW2] Gross and Wallach construct a unitary representation $\sigma_{\mathcal{O}}$ in the continuations of the quaternionic discrete series which is associated to \mathcal{O} in the sense that it has K -types ($K = SU_2 \times M$)

$$(2) \quad \sum_{n=0}^{\infty} \text{Sym}^{n+k}(\mathbb{C}^2) \otimes A^n(\mathcal{O})$$

where k is the integer $3s + 3$, $2s + 2$ and $s + 2$ if \mathcal{O} is X , Y and Z respectively. Again we will follow the notations of [GW2] and denote the three representations by σ_X, σ_Y and σ_Z .

1.3. Let G be one of the exceptional groups of real rank 4 in §1.2. A quaternionic Lie subgroup G' of G is defined as a semisimple Lie subgroup of G containing M_1 . Let $M' = G' \cap M$.

We refer to §2 and its references for the definition of a quaternionic representation of G and G' . In [L3] we show that the restriction of $\sigma_{\mathcal{O}}$ from G to G' decomposes discretely into a direct sum of quaternionic representations of G' . There we derived a formula to compute the spectrum of decompositions (See Thm. 3.4.1 [L3]) and we deduced that most of the irreducible components of the restriction to G' are determined by the coordinate ring of the intersection of $\mathcal{O} \cap \mathbb{P}V_0$. Here $\mathbb{P}V_0$ is a $M'(\mathbb{C})$ -invariant subspace of $\mathbb{P}V_M$ and hence $\mathcal{O} \cap \mathbb{P}V_0$ is a $M'(\mathbb{C})$ -invariant projective variety. We will briefly recall these results in §3. Unfortunately the formula cannot be applied immediately to some interesting situations. In particular the coordinate ring of $\mathcal{O} \cap \mathbb{P}V_0$ is in general difficult to compute.

This paper is a continuation of [L3]. The main objective of this paper is to determine the restriction of $\sigma_{\mathcal{O}}$ in the following five situations (cf. (8)):

- (3) $G = \widetilde{F}_{4,4} \supset G' = \widetilde{\text{Spin}}(5, 4) \times_{(\mathbb{Z}/2\mathbb{Z})^2} (\mathbb{Z}/2\mathbb{Z})^2$
- (4) $G = \widetilde{F}_{4,4} \supset G' = \widetilde{\text{Spin}}(4, 4) \times_{(\mathbb{Z}/2\mathbb{Z})^3} (\mathbb{Z}/2\mathbb{Z})^3$
- (5) $G = \widetilde{E}_{6,4} \times (\mathbb{Z}/2\mathbb{Z}) \supset G' = \widetilde{F}_{4,4} \times (\mathbb{Z}/2\mathbb{Z})$
- (6) $G = \widetilde{E}_{7,4} \supset G' = \widetilde{E}_{6,4} \times_{\mu_3} U_1$
- (7) $G = \widetilde{E}_{8,4} \supset G' = \widetilde{E}_{7,4} \times_{\mu_2} SU_2$.

Here the tilde above the group denotes its double cover. We will compute the decompositions of the restrictions of the representations $\sigma_{\mathcal{O}}$ by explicitly computing the intersections $\mathcal{O} \cap \mathbb{P}V_0$. We have mentioned that one of the difficulty is to determine the intersections and their coordinate rings. Fortunately in each of the above G' it is known that $\mathbb{P}V_0$ is a union of finitely many $M'(\mathbb{C})$ orbits and the intersection $\mathcal{O} \cap \mathbb{P}V_0$ can therefore be effectively determined. We remark that the above subgroups G' are just a few such examples. Our method presented in this paper should be applicable to other subgroups G' where $M'(\mathbb{C})$ exhibit a dense orbit in $\mathbb{P}V_0$.

The restrictions provide an efficient method for finding many quaternionic representations of the exceptional quaternionic Lie groups which are unitarizable. Since $\sigma_{\mathcal{O}}$ has small Gelfand-Kirillov dimensions, the quaternionic representations obtained in the restriction would have small Gelfand-Kirillov dimensions too.

1.4. The representation σ_Z is a ladder representation and it is annihilated precisely by the Joseph ideal and it is thus called the *minimal* representation of G . A pair of reductive subgroups $H_1 \times_C H_2$ in G is called a (reductive) *dual pair* if the centralizers of H_1 and H_2 in G are H_2 and H_1 respectively. The dual pair is called compact if either H_1 or H_2 is compact. Note that G' in (3) to (7) are examples of compact dual pairs. In the appendix of [L3], we showed that certain correspondences of dual pairs in §1.3 exist and the second objective of this paper is to find the rest (if any) of the correspondences. We will also describe the dual pairs correspondences of $\text{Spin}(4, 4) \times_{\mu_2^2} U_1^2$ in $E_{6,4} \times \mathbb{Z}/2\mathbb{Z}$ in Theorem 7.3.1 and $E_{6,4} \times SU_3$ in $E_{8,4}$ in Corollary 4.10.1.

We remark that the restrictions of the minimal representations to dual pairs are of great interest. Exceptional dual pairs correspondences have been investigated by [HPS], [Li1], [Li2], [GS] and [L2]. In [GS] and [MS], the authors computed certain dual pair correspondences of p -adic exceptional groups by determining the intersection of orbits.

1.5. The organization of the paper is as follows: In §2 we will recall the definition of quaternionic groups and quaternionic representations. In §3 we will briefly state the restriction formula of [L3]. The main results of this paper are stated in §4. In §5 we describe the closures of the orbits X, Y, Z in detail. The rest of the paper is devoted to the proofs of the main results in §4.

1.6. We define some notations. $\pi_G(a_1\varpi_1 + \cdots + a_n\varpi_n)$ will denote the irreducible finite dimensional complex representation of a semisimple Lie group G with highest weight $a_1\varpi_1 + \cdots + a_n\varpi_n$ where ϖ_i are the fundamental weights given in Planches [Bou]. If V is a representation of G , then $S^n(V) = \text{Sym}^n V$ will denote its n -th symmetric product and V^* its dual representation. S^n will denote the representation $\text{Sym}^n(\mathbb{C}^2)$ of SU_2 . χ_1 will denote a fundamental character of the compact torus U_1 . μ_n will denote the cyclic group of order n . Suppose H_1 and H_2 are subgroups of G and C lies in the centers of both H_1 and H_2 , then we denote

$$(8) \quad H_1 \times_C H_2 := (H_1 \times H_2) / \{(z, z) : z \in C\}.$$

2. Quaternionic groups and representations.

2.1. In this section we define some notations and briefly recall the definitions of the quaternionic real form of an algebraic group and quaternionic representations.

2.2. We refer to §3 of [GW2] and §2 of [L3] for the definition of the quaternionic real form \mathfrak{g}_0 of a complex simple Lie algebra \mathfrak{g} . It has Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where

$$\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{m}.$$

Here \mathfrak{m} is a reductive Lie subgroup of \mathfrak{g} and $\mathfrak{p} = \mathbb{C}^2 \otimes V_M$ where V_M is a self dual representation of \mathfrak{m} . \mathfrak{k} contains a Cartan subalgebra \mathfrak{h} of \mathfrak{g} . We choose a positive root system Φ^+ with respect to \mathfrak{h} such that the \mathfrak{su}_2 in \mathfrak{k} corresponds to the highest root $\tilde{\alpha}$. Denote this Lie algebra by $\mathfrak{su}_2(\tilde{\alpha})$. $\mathfrak{t}_1 = \mathfrak{h} \cap \mathfrak{su}_2(\tilde{\alpha})$ is a Cartan subalgebra of $\mathfrak{su}_2(\tilde{\alpha})$. Then $\mathfrak{l} = \mathfrak{t}_1 \oplus \mathfrak{m}$ is a Levi subalgebra of a maximal parabolic subalgebra \mathfrak{q} whose nilpotent radical \mathfrak{u} is a Heisenberg Lie algebra. The one dimensional center of \mathfrak{u} is spanned by the highest root space $\mathfrak{g}_{\tilde{\alpha}}$ and let $\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}] = V_M$ denote the representation of \mathfrak{m} . $\bar{\mathfrak{q}} = \mathfrak{l} \oplus \bar{\mathfrak{u}}$ will denote the opposite parabolic subalgebra.

Let $G(\mathbb{C})$ be a complex simply connected simple Lie group with Lie algebra \mathfrak{g} and let G_0 be a real form of $G(\mathbb{C})$ having Lie algebra \mathfrak{g}_0 . We denote the real Lie subgroups in $G(\mathbb{C})$ corresponding to the various real forms of the Lie subalgebras $\mathfrak{m}_0, \mathfrak{l}_0$ and \mathfrak{k}_0 by M, L and $K_0 = SU_2(\tilde{\alpha}) \times_{\mu_2} M$ respectively. Here K_0 is the maximal compact subgroup of G_0 and $SU_2(\tilde{\alpha}) = M_1$ in §1.1. Let G denote the double cover of G_0 with maximal compact subgroup $K = SU_2(\tilde{\alpha}) \times M$. We will call $G \times H$ a *quaternionic* Lie group if G is a quaternionic simple Lie group and H is a compact Lie group.

Set $2d = \dim V_M$. If \mathfrak{g} is of type D_4, F_4, E_6, E_7, E_8 , then $d = 3s + 4$ where $s = 0, 1, 2, 4, 8$ respectively. We tabulate some of the $M(\mathbb{C})$ and V_M in Table 1 below.

	G_0	$M(\mathbb{C})$	V_M
	$\text{Spin}(d, 4), d \geq 4$	$SL_2 \times \text{Spin}(d)$	$\mathbb{C}^2 \otimes \mathbb{C}^d$
(e ₁)	$F_{4,4}$	Sp_6	$\pi(\varpi_3)$
(e ₂)	$E_{6,4} \rtimes \mathbb{Z}/2\mathbb{Z}$	$SL_6 \rtimes \mathbb{Z}/2\mathbb{Z}$	$\pi(\varpi_3)$
(e ₄)	$E_{7,4}$	$\text{Spin}(12)$	$\pi(\varpi_6)$
(e ₈)	$E_{8,4}$	simply connected E_7	$\pi(\varpi_1)$

Table 1.

2.3. We will define and review the properties of quaternionic representations. We refer to [S1], [W1], [W2], [GW1], [GW2, §5] and Theorem 3.3.1 of [L2] for proofs and details.

Let $W[k] = e^{-k\tilde{\alpha}/2} \otimes W$ be an irreducible finite dimensional representation of $L = U_1 \times M$. We extend $W[k]$ trivially to a representation of $\bar{\mathfrak{q}}$ and denote

$$\mathbf{H}(G, W[k]) = \mathbf{H}(W[k]) := \Gamma_{K/L}^1(\text{Hom}_{\mathcal{U}(\bar{\mathfrak{q}})}(\mathcal{U}(\mathfrak{g}), W[k])_L)$$

as the Harish-Chandra module of G where Γ^1 is the first Zuckerman derived functor. It has infinitesimal character $\mu + \rho(G) - k\frac{\tilde{\alpha}}{2}$. If $k \geq 2$, then $\mathbf{H}(W[k])$

has K -types ($K = SU_2(\tilde{\alpha}) \times M$)

$$(9) \quad \sum_{n=0}^{\infty} S_{\tilde{\alpha}}^{k-2+n}(\mathbb{C}^2) \otimes (\text{Sym}^n(V_M) \otimes W).$$

It contains a unique irreducible (\mathfrak{g}, K) -submodule denoted by $\sigma(G, W[k])$ which is generated by the translates of the lowest K -types

$$(10) \quad S_{\tilde{\alpha}}^{k-2}(\mathbb{C}^2) \otimes W.$$

We will call $\mathbf{H}(G, W[k])$ and $\sigma(G, W[k])$ *quaternionic* representations.

3. Restrictions.

3.1. Let G_0 be one of the four exceptional groups given in Table 1 (e_s) indexed by $s = 1, 2, 4, 8$. Let G be its double cover.

3.2. Suppose G' is a quaternionic real Lie subgroup of G containing $SU_2(\tilde{\alpha})$. We have correspondingly $K' = G' \cap K$, $M' = G' \cap M$ and the Lie algebras $\mathfrak{g}', \mathfrak{m}', \mathfrak{u}' = \mathfrak{u} \cap \mathfrak{g}'$. Write $\mathfrak{u} = \mathfrak{u}' + V_0$. We have $V_{M'} \subset V_M$ and we define $V_0 = V_M/V_{M'}$ as a representation of M' .

We will use $\text{Res}_{G'}^G \sigma$ to denote the restriction of a Harish-Chandra module σ of G to that of G' .

3.3. First we review the work of [GW2]. Denote $\mathbf{H}_k := \mathbf{H}(G, \mathbb{C}[k])$ and its unique submodule $\sigma(G, \mathbb{C}[k])$ by σ_k . \mathbf{H}_k is irreducible and unitarizable if $k \geq 3s + 4$. If $k > 6s + 8$, then it belongs to the discrete series. We recall Corollary 4.2.2 of [L2].

Proposition 3.3.1. *There exists a filtration H_n of \mathbf{H}_k such that $H_{n+1}/H_n = \mathbf{H}(G', S^n(V_0)[n+k])$. If \mathbf{H}_k is unitarizable, then*

$$\text{Res}_{G'}^G \mathbf{H}_k = \sum_{n=0}^{\infty} \mathbf{H}(G', S^n(V_0)[n+k])$$

and each summand on the right is irreducible and unitarizable.

3.4. Since V_M is a self dual representation of M , we identify V_M with its dual. In §1.2 we remarked that $\mathbb{P}V_M$ is a union of four $M(\mathbb{C})$ -orbits. One of the orbit is Zariski dense and we denote the Zariski closure of the other three non-dense orbits by X, Y and Z satisfying (1). We will describe these three orbits in greater detail in §5.

Let (k, m, \mathcal{O}) be one of the three following sets of data:

$$(3s + 3, 4, X), \quad (2s + 2, 3, Y), \quad (s + 2, 2, Z).$$

Let $I^\bullet(\mathcal{O}) = \bigoplus_{n \geq m} I^n(\mathcal{O})$, ($I^m \neq 0$) be the homogeneous ideal of \mathcal{O} in $\mathbb{P}V_M$ and $A^\bullet(\mathcal{O}) = \bigoplus A^n(\mathcal{O})$ be its coordinate ring. Then I^\bullet is generated as a $S^\bullet(V_m)$ module by $I^m(\mathcal{O})$ and each graded piece $A^n(\mathcal{O})$ is a representation of M .

Gross and Wallach showed that σ_k is a unitarizable proper submodule of \mathbf{H}_k with K types given in (2). Furthermore it satisfies an exact sequence

$$(11) \quad 0 \rightarrow \sigma_k \rightarrow \mathbf{H}(G, \mathbb{C}[k]) \xrightarrow{\phi} \mathbf{H}(G, I^m(\mathcal{O})[k+m]).$$

As a representation of $M(\mathbb{C})$, $I^m(\mathcal{O}) = \mathbb{C}$, V_M and \mathfrak{m} respectively for the three values of k . When $k = 3s+3$, ϕ in (11) is a surjection (cf. §8 of [GW2]). We will follow the notation of [GW1] and denote the three representations σ_{3s+3} , σ_{2s+2} and σ_{s+2} by σ_X , σ_Y and σ_Z respectively.

The annihilator ideal of σ_Z is the Joseph ideal in $\mathcal{U}(\mathfrak{g})$ so σ_Z is called the *minimal* representation of G . It has K -types

$$(12) \quad \sum_{n=0}^{\infty} S_{\alpha}^{s+n}(\mathbb{C}^2) \otimes \pi_M(n\lambda)$$

where λ is the highest weight of V_M . Note that all the representations descend to representations of G_0 except σ_X for groups of type E_n and σ_Z of $\tilde{F}_{4,4}$.

3.5. The inclusion $I^m(\mathcal{O}) \subset S^m(V_M)$ gives rise to the following natural maps of M -modules:

$$\text{Sym}^{n-m}(V_M) \otimes I^m(\mathcal{O}) \rightarrow \text{Sym}^{n-m}(V_M) \otimes \text{Sym}^m(V_M) \rightarrow \text{Sym}^n(V_M).$$

Let r'_n denote the composite of the above maps. The direct sum $V_M = \mathfrak{u}' \oplus V_0$ (cf. §3.2) induces a natural map of M' -modules

$$r''_n : \text{Sym}^n(V_M) \rightarrow \text{Sym}^n(V_0).$$

We define $r_n = r''_n \circ r'_n$ for $n \geq m$. For $0 \leq n < m$, we set r_n to be the zero map into $\text{Sym}^n(V_0)$. Let R_n denote the cokernel of r_n and let $R_{\bullet} := \bigoplus_{n=0}^{\infty} R_n$. Note that R_n is a representation of M' and we write

$$R_n = \sum_j W_{n,j}$$

where $W_{n,j}$ are the irreducible subrepresentations of M' .

Let $\mathcal{O}' = \mathcal{O} \cap \mathbb{P}V_0$ and we denote its coordinate ring in $\mathbb{P}V_0$ by $A^{\bullet}(\mathcal{O}') = \bigoplus A^n(\mathcal{O}')$. Then \mathcal{O}' is cut out by $r_m(I^m(\mathcal{O}))$ and $R_{\bullet}/\text{Nil}(R_{\bullet}) = A^{\bullet}(\mathcal{O}')$.

We can now state Theorem 3.3.1 and Corollary 2.8.1 of [L3].

Theorem 3.5.1.

(a)

$$\text{Res}_{G'}^G \sigma_k = \sum_{n=0}^{\infty} \sigma(G', R_n[k+n]) = \sum_{n=0}^{\infty} \sum_j \sigma(G', W_{n,j}[k+n]).$$

(b)

$$\text{Res}_{G'}^G \sigma_k \supseteq \sum_{n=0}^{\infty} \sigma(G', A^n(\mathcal{O}')[k+n]).$$

Equality holds if and only if $r_m(I^m(\mathcal{O}))$ generates the ideal of \mathcal{O}' .

(c) *If r_m is surjective, then r_n is surjective for $n \geq m$ and*

$$\text{Res}_{G'}^G \sigma_k = \sum_{n=0}^{m-1} \sigma(G', S^n V_0[k+n]).$$

4. The main results.

4.1. In this section we will state the main results on the restrictions of the quaternionic representations $\sigma_{\mathcal{O}}$ of the exceptional group G to its subgroup G' . The proofs will be given in the later sections. We will replace G by G_0 if the representation descends to G_0 . If $\mathbf{H}(G', W[n])$ appears in the restriction formula, it means that $\mathbf{H}(G', W[n])$ is irreducible and its K -types are given by (9).

4.2. Let $G = \widetilde{F}_{4,4} \supset G' = \widetilde{\text{Spin}}(5, 4)$ and $M' = SU_2 \times \text{Spin}(5)$. Let $V_{m,n} = S^m(\mathbb{C}^2) \otimes \pi_{\text{Spin}(5)}(n\varpi_2)$ be the representation of M' .

Theorem 4.2.1.

(a) $\text{Res}_{G'}^G \sigma_Z = \sigma(G', \mathbb{C}[3]) + \sigma(G', V_{0,1}[4]).$

(b) $\text{Res}_{\text{Spin}(5,4)}^{F_{4,4}} \sigma_Y = \sum_{n=0}^{\infty} \sigma(\text{Spin}(5, 4), V_{0,n}[4+n]).$

(c) $\text{Res}_{\text{Spin}(5,4)}^{F_{4,4}} \sigma_X = \mathbf{H}(\text{Spin}(5, 4), \mathbb{C}[6]) + \sum_{n=1}^{\infty} \sigma(\text{Spin}(5, 4), V_{0,n}[6+n]).$

(d) $\text{Res}_{G'}^G \mathbf{H}(G, \mathbb{C}[k]) = \sum_{n=0}^{\infty} \mathbf{H}(G', V_{0,n}[k+n])$ if $k \geq 7$.

If $n \geq 1$ then the summands in (c) satisfy the following exact sequence:

(13)

$$0 \rightarrow \sigma(F_{4,4}, V_{0,n}[6+n]) \rightarrow \mathbf{H}(F_{4,4}, V_{0,n}[6+n]) \rightarrow \mathbf{H}(F_{4,4}, V_{0,n}[9+n]) \rightarrow 0.$$

4.3. Suppose $G = \widetilde{F}_{4,4}$ and $G' = \widetilde{\text{Spin}}(4, 4)$. The maximal compact subgroup of G' is $SU_2(\widetilde{\alpha}) \times M'$ where $M' = SU_2(A) \times SU_2(B) \times SU_2(C)$. Here $\widetilde{\alpha}, A, B, C$ are the (orthogonal) compact roots of $\widetilde{\text{Spin}}(4, 4)$. Let

$$S(a, b, c) := S_A^a(\mathbb{C}^2) \otimes S_B^b(\mathbb{C}^2) \otimes S_C^c(\mathbb{C}^2)$$

denote the irreducible representation of M' .

Theorem 4.3.1.

- (a) $\text{Res}_{G'}^G \sigma_Z = \sigma(G', \mathbb{C}[3]) + \sigma(G', S(1, 0, 0)[4])$
 $\quad + \sigma(G', S(0, 1, 0)[4]) + \sigma(G', S(0, 0, 1)[4]).$
- (b) $\text{Res}_{\text{Spin}(4,4)}^{F_{4,4}} \sigma_Y = \sum_{abc=0} \sigma(\text{Spin}(4, 4), S(a, b, c)[4 + a + b + c]).$
- (c) $\text{Res}_{\text{Spin}(4,4)}^{F_{4,4}} \sigma_X = \sum_{abc=0} \mathbf{H}(\text{Spin}(4, 4), S(a, b, c)[6 + a + b + c])$
 $\quad + \sum_{a,b,c>0} \sigma(\text{Spin}(4, 4), S(a, b, c)[6 + a + b + c]).$
- (d) $\text{Res}_{G'}^G \mathbf{H}(G, \mathbb{C}[k]) = \sum_{a,b,c \geq 0} \mathbf{H}(G', S(a, b, c)[k + a + b + c])$ if $k \geq 7$.

When a, b, c are strictly positive, the summands in (c) satisfy the following exact sequence:

$$(14) \quad 0 \rightarrow \sigma(\text{Spin}(4, 4), S(a, b, c)[6 + a + b + c])$$

$$\quad \rightarrow \mathbf{H}(\text{Spin}(4, 4), S(a, b, c)[6 + a + b + c])$$

$$\quad \rightarrow \mathbf{H}(\text{Spin}(4, 4), S(a - 1, b - 1, c - 1)[7 + a + b + c]) \rightarrow 0.$$

4.4. Note that $\widetilde{\text{Spin}}(4, 4)$ and $\widetilde{\text{Spin}}(5, 4)$ are components of dual pairs in (3) and (4). In §6 we will describe the action of $(\mathbb{Z}/2\mathbb{Z})^2$ and $(\mathbb{Z}/2\mathbb{Z})^3$ on the summands. In particular Theorems 4.2.1(a) and 4.3.1(a) give the dual pairs correspondences of the minimal representation σ_Z of $\widetilde{F}_{4,4}$. The K -types of summands are given in Theorem 6.8.1 and 6.9.1.

4.5. Let $G = \widetilde{E}_{6,4} \rtimes \mathbb{Z}/2\mathbb{Z}$ and $G' = \widetilde{F}_{4,4} \times \mathbb{Z}/2\mathbb{Z}$. Let χ be the nontrivial character of $\mathbb{Z}/2\mathbb{Z}$.

Theorem 4.5.1.

- (a) $\text{Res}_{F_{4,4} \times \mathbb{Z}/2\mathbb{Z}}^{E_{6,4} \times \mathbb{Z}/2\mathbb{Z}} \sigma_Z = \sigma(F_{4,4}, \mathbb{C}[4]) \otimes \chi^0 + \sigma(F_{4,4}, \mathbb{C}^6[5]) \otimes \chi.$
- (b) $\text{Res}_{F_{4,4} \times \mathbb{Z}/2\mathbb{Z}}^{E_{6,4} \times \mathbb{Z}/2\mathbb{Z}} \sigma_Y = \sum_{n=0}^{\infty} \sigma(F_{4,4}, S^n(\mathbb{C}^6)[6+n]) \otimes \chi^n.$
- (c) $\text{Res}_{G'}^G \sigma_X = \mathbf{H}(\tilde{F}_{4,4}, \mathbb{C}[9]) \otimes \chi^0 + \mathbf{H}(\tilde{F}_{4,4}, \mathbb{C}^6[10]) \otimes \chi$
 $+ \sum_{n=2}^{\infty} \sigma(\tilde{F}_{4,4}, S^n(\mathbb{C}^6)[9+n]) \otimes \chi^n.$
- (d) $\text{Res}_{G'}^G \mathbf{H}(G, \mathbb{C}[k]) = \sum_{n=0}^{\infty} \mathbf{H}(\tilde{F}_{4,4}, S^n(\mathbb{C}^6)[10+n]) \otimes \chi^n$ if $k \geq 10.$

If $n \geq 2$ then the summands in (c) satisfy the following exact sequence:

$$(15) \quad 0 \rightarrow \sigma(\tilde{F}_{4,4}, S^n(\mathbb{C}^6)[9+n]) \rightarrow \mathbf{H}(\tilde{F}_{4,4}, S^n(\mathbb{C}^6)[9+n])$$

$$\rightarrow \mathbf{H}(\tilde{F}_{4,4}, S^{n-2}(\mathbb{C}^6)[11+n]) \rightarrow 0.$$

4.6. Each of the summands of σ_X in Theorems 4.2.1(c), 4.3.1(c) and 4.5.1(c) satisfies a short exact sequence. The K' -types of the middle and the last term of the exact sequence is given by (9). Hence it is possible to determine the K' -types, the Gelfand-Kirillov dimensions and the Bernstein degrees of the summands.

4.7. Let $G = \tilde{E}_{7,4}$, $G' = \tilde{E}_{6,4} \times_{\mu_3} U_1$ and $M' = SU_6 \times_{\mu_3} U_1$. Let $V_{a,b} = \pi_{SU_6}(a\varpi_1 + b\varpi_6)$.

Theorem 4.7.1.

- (a) $\text{Res}_{E_{6,4} \times U_1}^{E_{7,4}} \sigma_Z = \sum_{a,b \geq 0, ab=0} \sigma(E_{6,4}, V_{a,b}[6+a+b]) \otimes \chi_1^{a-b}.$
- (b) $\text{Res}_{E_{6,4} \times U_1}^{E_{7,4}} \sigma_Y = \mathbf{H}(E_{6,4}, \mathbb{C}[12]) \otimes \chi_1^0$
 $+ \sum_{a,b \geq 0} \sigma(E_{6,4}, V_{a,b}[10+a+b]) \otimes \chi_1^{a-b}.$
- (c) $\text{Res}_{G'}^G \sigma_X = \sum_{n=0,1} \sum_{a,b \geq 0} \mathbf{H}(\tilde{E}_{6,4}, V_{a,b}[15+2n+a+b]) \otimes \chi_1^{a-b}.$
- (d) $\text{Res}_{G'}^G \mathbf{H}(G, \mathbb{C}[k]) = \sum_{a,b,n \geq 0} \mathbf{H}(\tilde{E}_{6,4}, V_{a,b}[k+2n+a+b]) \otimes \chi_1^{a-b}$
if $k \geq 16.$

4.8. Let $G = \tilde{E}_{8,4}$, $G' = \tilde{E}_{7,4} \times_{\mu_2} SU_2$ and $M' = \text{Spin}(12) \times_{\mu_2} SU_2$. Let $V_{a,b} = \pi_{\text{Spin}(12)}(a\varpi_1 + b\varpi_2)$.

Theorem 4.8.1.

- (a) $\text{Res}_{E_{7,4} \times SU_2}^{E_{8,4}} \sigma_Z = \sum_{n=0}^{\infty} \sigma(E_{7,4}, V_{n,0}[10+n]) \otimes S^n(\mathbb{C}^2).$
- (b) $\text{Res}_{E_{7,4} \times SU_2}^{E_{8,4}} \sigma_Y = \sum_{a+2b+2c=n, bc=0} \sigma(E_{7,4}, V_{a,c}[18+n]) \otimes S^{a+2b}(\mathbb{C}^2).$
- (c) $\text{Res}_G^G \sigma_X = \sum * \mathbf{H}(\tilde{E}_{7,4}, V_{a+2d,c}[27+n]) \otimes S^{a+2b}(\mathbb{C}^2).$
- (d) $\text{Res}_G^G \mathbf{H}(G, \mathbb{C}[k]) = \sum_{m=0}^{\infty} \sum * \mathbf{H}(\tilde{E}_{7,4}, V_{a+2d,c}[k+n+4m]) \otimes S^{a+2b}(\mathbb{C}^2)$ if $k \geq 28$

where the summation \sum^* is taken over all nonnegative integers a, b, c, d, n satisfying the relations

$$(16) \quad n - 2a \leq a + 2b + 2c + 4d \leq n, \quad cd = 0, \quad a \equiv n \pmod{2}.$$

4.9. Some of the restrictions stated in this section are known. We have included them for the sake of completeness. Moreover they follow with little additional effort from the proofs of the rest of the statements. The following results are known:

(i) Theorems 4.5.1(a) and 4.7.1(a) are unpublished results of B. Gross [G]. See §6 [GW1] for Theorem 4.8.1(a). The method of proof is by considering the decompositions of the K -types. Also see Theorem 12.1.1 [L2].

(ii) See §4.6 of [L2] for Theorems 4.3.1(c) and (d).

4.10. Using the above theorems, we will deduce the following dual pair correspondence in §9.4:

Corollary 4.10.1.

$$\text{Res}_{E_{6,4} \times_{\mu_3} SU_3}^{E_{8,4}} \sigma_Z = \sum_{a,b \geq 0} \Theta(a, b) \otimes \pi_{SU_3}(a\varpi_1 + b\varpi_2)$$

where

$$(17) \quad \Theta(a, b) = \begin{cases} \sigma(E_{6,4}, \pi_{SU_6}(a\varpi_1 + b\varpi_5)[a+b+10]) & \text{if } (a, b) \neq (0, 0) \\ \mathbf{H}(E_{6,4}, \mathbb{C}[10]) \oplus \mathbf{H}(E_{6,4}, \mathbb{C}[12]) & \text{if } (a, b) = (0, 0). \end{cases}$$

The dual pair correspondence of $\text{Spin}(4, 4) \times_{\mu_2} U_1^2 \in E_{6,4}$ will be given in Theorem 7.3.1.

4.11. The proofs of part (c) of Theorems 4.2.1 to 4.8.1 are similar. By (11) we have an exact sequence

$$(18) \quad 0 \rightarrow \sigma_X \rightarrow \mathbf{H}(G, \mathbb{C}[d-1]) \rightarrow \mathbf{H}(G, \mathbb{C}[d+3]) \rightarrow 0.$$

The term on the right is irreducible and unitarizable. By Proposition 3.3.1 the restriction of the last term to G' decomposes into

$$(19) \quad \sum_{n=0}^{\infty} \mathbf{H}(G', S^n V_0[d+3+n]).$$

Applying the filtration in Proposition 3.3.1 to the middle and the last term of (18) gives a homomorphism of graded modules

$$(20) \quad 0 \rightarrow \text{Res}_{G'}^G \sigma_X \rightarrow \sum_{n=0}^{\infty} \mathbf{H}(G', S^n V_0[d-1+n]) \rightarrow \sum_{n=0}^{\infty} \mathbf{H}(G', S^n V_0[d+3+n]) \rightarrow 0.$$

One can show that the above sequence is an exact sequence. We shall use (20) to prove part (c) of the above theorems.

5. Orbits computations.

5.1. In this section we will give a realization of the $M(\mathbb{C})$ action on $\mathbb{P}V_M$ and describe its subvarieties X, Y and Z (cf. §1.2). Please refer to [Ba], [Kim], [J1], [GW2] and [GL] for more details.

5.2. Let $s = 1, 2, 4,$ or 8 and let $\mathcal{K} = \mathcal{K}_s$ denote the composition algebra over \mathbb{C} of dimension s . Then up to isomorphism, $\mathcal{K}_1 = \mathbb{C}$, \mathcal{K}_4 is the set of 2 by 2 complex matrices $M_2(\mathbb{C})$, \mathcal{K}_2 is the subset of diagonal elements in $M_2(\mathbb{C})$ and \mathcal{K}_8 is the set of Cayley numbers [J2]. Each algebra has an anti-automorphism $z \mapsto \bar{z}$ called conjugation such that $N(z) := z\bar{z} = \bar{z}z$ is a nondegenerate bilinear form on \mathcal{K} . Moreover $N(zz') = N(z)N(z')$. Define $\text{tr}(z) = z + \bar{z}$ and $\langle z, z' \rangle = \text{tr}(z\bar{z}')$. There are obvious embeddings $\mathcal{K}_1 \subset \mathcal{K}_2 \subset \mathcal{K}_4$.

5.3. Let (U_0, \langle, \rangle) be the 3 dimensional complex inner product space with orthonormal basis $\{e_1, e_2, e_3\}$. Then \mathcal{K}_8 can be realized as elements of the form

$$(21) \quad x = (a, d; v_1, v_2) := \begin{pmatrix} a & v_1 \\ v_2 & d \end{pmatrix}$$

where $a, d \in \mathbb{C}$ and $v_1, v_2 \in U_0$. The multiplication is given in [J2, p. 142] and $N(x) = ad - \langle v_1, v_2 \rangle$. We define an action of $g \in SL(U_0)$ on \mathcal{K}_8 by

$$(22) \quad g : (a, d; v_1, v_2) \mapsto (a, d; hv_1, {}^t h^{-1}v_2).$$

We embed $\mathcal{K}_4 \subset \mathcal{K}_8$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & be_1 \\ ce_1 & d \end{pmatrix}$.

5.4. Let $\mathcal{J} = \mathcal{J}(\mathcal{K})$ be the Jordan algebra consisting of 3 by 3 Hermitian symmetric matrices of the form

$$(23) \quad J = (\gamma_1, \gamma_2, \gamma_3; c_1, c_2, c_3) := \begin{pmatrix} \gamma_1 & c_3 & \overline{c_2} \\ \overline{c_3} & \gamma_2 & c_1 \\ c_2 & \overline{c_1} & \gamma_3 \end{pmatrix}$$

where $\gamma_i \in \mathbb{C}$ and $c_i \in \mathcal{K}$. The composition in \mathcal{J} is given by $J_1 \circ J_2 = \frac{1}{2}(J_1 J_2 + J_2 J_1)$. Define an inner product on \mathcal{J} given by $\langle X, Y \rangle = \text{Tr}(X \circ Y)$ where Tr denotes the usual trace of matrices. There is a cubic form

$$\det(J) = \gamma_1 \gamma_2 \gamma_3 - \gamma_1 N(c_1) - \gamma_2 N(c_2) - \gamma_3 N(c_3) + \text{tr}(c_1(c_2 c_3))$$

on \mathcal{J} which induces a trilinear form on \mathcal{J} such that $(J, J, J) = \det J$. For $\mathcal{J}(\mathcal{K}_1)$ and $\mathcal{J}(\mathcal{K}_2)$, \det is the usual determinant of 3 by 3 matrices. Finally we define a bilinear map $\mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J}$, $(J, J') \mapsto J \times J'$ such that in the notation of (23)

$$(24) \quad J \times J = (\gamma_2 \gamma_3 - N(c_1), \gamma_3 \gamma_1 - N(c_2), \gamma_1 \gamma_2 - N(c_3); \overline{c_2 c_3} - \gamma_1 c_1, \overline{c_3 c_1} - \gamma_2 c_2, \overline{c_1 c_2} - \gamma_3 c_3).$$

$J \neq 0$ is said to have *rank* 1 if $J \times J = 0$. J has rank 0 if $J = 0$. $J \neq 0$ has rank 2 if $\det(J) = 0$ and it is not of rank 1.

Define

$$(25) \quad V_M := \mathbb{C} \oplus \mathcal{J} \oplus \mathcal{J} \oplus \mathbb{C}$$

and we denote a vector in V_M by (ξ, J, J', ξ') . There is a realization of the $M(\mathbb{C})$ action on V_M (see [Ba]). Let $p : V_M \setminus \{0\} \rightarrow \mathbb{P}V_M$ be the canonical projection.

We refer to (23) and define

$$(26) \quad \mathcal{J}_1 := \{(0, 0, 0; c_1, 0, 0) \in \mathcal{J} : c_1 \in \mathcal{K}\}.$$

Similarly we define \mathcal{J}_2 and \mathcal{J}_3 . Define $W_i = \{(0, J, J', 0) \in V_M : J, J' \in \mathcal{J}_i\}$ for $i = 1, 2, 3$. We will need these definitions in §6 and §7.

5.5. The smallest orbit Z is generated by the highest weight vector spanned by $p(1, 0, 0, 0)$. The stabilizer of $p(1, 0, 0, 0)$ is a maximal abelian parabolic $Q = L' \rtimes N'$ in $M(\mathbb{C})$. We denote $\overline{Q}' = L \rtimes \overline{N}'$ to be the opposite parabolic subgroup. Then \overline{Q}' stabilizes the flag

$$\mathbb{C} \subset \mathcal{J} \oplus \mathbb{C} \subset \mathcal{J} \oplus \mathcal{J} \oplus \mathbb{C} \subset V_M.$$

There is a bijection $\mathcal{J}(\mathcal{K}) \rightarrow \overline{N}'$ given by $B \mapsto p_B$ where p_B acts on V_M by (see [Kim])

$$(27) \quad p_B : (0, J, J', 0) \mapsto (0, J, J' + 2B \times J, (B, B, J) + (B, J')).$$

We recall a version of Lemma 7.5 of [MS].

Lemma 5.5.1. *Representatives of the \overline{N}' -orbits on Z are*

$$\begin{aligned} v_1 &= p(1, 0, 0, 0) & v_2 &= p(0, J, 0, 0) \\ v_3 &= p(0, 0, J', 0) & v_4 &= p(0, 0, 0, 1) \end{aligned}$$

where J and J' are rank 1 elements in \mathcal{J} .

5.6. The variety X is the hypersurface given by the degree 4 polynomial (cf. [J1])

$$f_4(\xi, J, J', \xi') = \langle J \times J, J' \times J' \rangle - \xi \det(J) - \xi' \det(J') - \frac{1}{4}(\langle J, J' \rangle - \xi \xi')^2.$$

If $J_1 = (0, 0, 0; x, y, z)$ and $J_2 = (0, 0, 0; x', y', z')$, then (28)

$$\begin{aligned} f_4(0, J_1, J_2, 0) &= N(x)N(x') + N(y)N(y') + N(z)N(z') + \langle yz, y'z' \rangle \\ &\quad + \langle zx, z'x' \rangle + \langle xy, x'y' \rangle - \frac{1}{4}(\langle x, x' \rangle + \langle y, y' \rangle + \langle z, z' \rangle)^2. \end{aligned}$$

Clearly $(0, J, 0, 0)$ and $(0, J_1, J_1, 0) \in X$.

5.7. Y is the algebraic set cut out by the set of degree 3 polynomials $\{\partial f_4/\partial v : v \in V_M\}$. It contains the point $p(0, J, 0, 0)$ (resp. $p(0, 0, J', 0)$) if and only if J (resp. J') has rank at most 2. Similarly Z contains the point iff J (resp. J') has rank 1.

5.8. Let $\mathcal{K} = \mathbb{C}$, then the nontrivial outer automorphism of $M = SL_6$ in Table 1 (e₂) acts on V_M by sending $(\xi, J, J', \xi') \mapsto (\xi, \overline{J}, \overline{J'}, \xi')$ where \overline{J} and \overline{J}' denotes taking conjugation of the entries.

5.9. Let $G_0 = \text{Spin}(4, 4)$ and by setting \mathcal{J} to be the set of diagonal 3 by 3 complex matrices in (25), V_M is the representation $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ of $M(\mathbb{C}) = SL_2^3(\mathbb{C})$. We index this case by $s = 0$.

6. Dual pairs in $\widetilde{F}_{4,4}$.

6.1. In this section let $G = \widetilde{F}_{4,4}$. It has maximal compact subgroup $K' = SU_2(\tilde{\alpha}) \times Sp_6$. Let $G' = \widetilde{\text{Spin}}(5, 4)$ and $G'' = \widetilde{\text{Spin}}(4, 4)$. In this section we will prove Theorems 4.2.1 and 4.3.1. We will retain the notations of §4.2 and §4.3.

6.2. The center C of $\widetilde{\text{Spin}}(4, 4)$ is

$$(29) \quad C := \{(\epsilon_1 \epsilon_2 \epsilon_3, \epsilon_1, \epsilon_2, \epsilon_3) \in SU_2(\tilde{\alpha}) \times SU_2^3 : \epsilon_i = \pm 1 \in SU_2\} \simeq \mu_2^3.$$

$(-1, -1, -1, -1) \in C$ is the nontrivial center of $\widetilde{F}_{4,4}$. We would like to interpret the subgroup $\widetilde{\text{Spin}}(4, 4)$ in $\widetilde{F}_{4,4}$ as the dual pair

$$\widetilde{\text{Spin}}(4, 4) \times_C C.$$

We denote a character of C by $\chi(s_1, s_2, s_3)$, $s_i \in (\mathbb{Z}/2\mathbb{Z})$ such that

$$(\epsilon_1, \epsilon_2, \epsilon_3) \mapsto \epsilon_1^{s_1} \epsilon_2^{s_2} \epsilon_3^{s_3}.$$

Let C^\wedge denote the character group of C .

The element $\epsilon_1 \in C$ acts on $F_{4,4}$ by conjugation and it fixes the subgroup $\widetilde{\text{Spin}}(5, 4)$ in $\widetilde{F}_{4,4}$. It has maximal compact subgroup

$$K = SU_2(\tilde{\alpha}) \times (SU_2(A) \times \text{Spin}(5)).$$

The center $C_1 \subset C$ of $\widetilde{\text{Spin}}(5, 4)$ is the Klein 4 group

$$C_1 := \{(\epsilon_1, \epsilon_1, \epsilon_2, \epsilon_2) \in C : \epsilon_i = \pm 1\}.$$

We denote the character group of C_1 by C_1^\wedge . It consists of characters

$$\chi(s_1, s_2) : (\epsilon_1, \epsilon_2) \mapsto \epsilon_1^{s_1} \epsilon_2^{s_2}$$

where $s_i \in \mathbb{Z}/2\mathbb{Z}$.

We have the following see-saw pairs in $\widetilde{F}_{4,4}$:

$$(30) \quad \begin{array}{ccc} \widetilde{\text{Spin}}(5, 4) & & C \\ & \bowtie & \\ \widetilde{\text{Spin}}(4, 4) & & C_1. \end{array}$$

6.3. The S_3 outer automorphism group on $\widetilde{\text{Spin}}(4, 4)$ permutes the 3 factor subgroups of $SU_2^3 \subset M''$ and G contains

$$(31) \quad (\widetilde{\text{Spin}}(4, 4) \times_C C) \rtimes S_3.$$

Hence if $\text{Res}_{\widetilde{\text{Spin}}(4,4)}^{\widetilde{F}_{4,4}} \sigma(\mathbb{C}[k])$ contains $\sigma(\text{Spin}(4, 4), S(a, b, c)[k])$, then it will also contain

$${}^s \sigma = \sigma \left(\text{Spin}(4, 4), S_{s(A)}^a \otimes S_{s(B)}^b \otimes S_{s(C)}^c [k] \right),$$

where $s \in S_3$.

6.4. By (31) $\text{Spin}(4, 4) \rtimes S_3 \subset F_{4,4}$ and $M''(\mathbb{C}) = SL_2^3(\mathbb{C}) \rtimes S_3$. Set $\mathcal{K} = \mathbb{C}$ and we recall the definition of \mathcal{J}_i and W_i ($i = 1, 2, 3$) in (26). The W_i 's give the standard representations of each of the 3 factor groups of $SU_2^3 \subset M''$. $V_0 = W_1 \oplus W_2 \oplus W_3$. The outer automorphism group S_3 acts on V_0 by permuting the W_i 's. Then $\mathbb{P}V_0$ has a dense $M''(\mathbb{C})$ -orbit. There are two more orbits and their closures are

$$\begin{aligned} X_2 &:= \mathbb{P}(W_1 \oplus W_2) \cup \mathbb{P}(W_2 \oplus W_3) \cup \mathbb{P}(W_3 \oplus W_1) \\ X_1 &:= \mathbb{P}W_1 \cup \mathbb{P}W_2 \cup \mathbb{P}W_3 \subset X_2. \end{aligned}$$

The homogeneous ideal of X_2 in $\mathbb{P}V_0$ is generated by

$$(32) \quad W_0 = W_1 \otimes W_2 \otimes W_3 \subset S^3(V_0)$$

and it has coordinate ring

$$A^n(X_2) = \sum_{a,b,c} S^a(W_1) \otimes S^b(W_2) \otimes S^c(W_3)$$

where the sum is taken over all nonnegative integers a, b, c such that $a + b + c = n$ and $abc = 0$.

Lemma 6.4.1.

- (a) $\mathbb{P}V_0 \cap X = \mathbb{P}V_0$.
- (b) $\mathbb{P}V_0 \cap Y = X_2$.
- (c) $\mathbb{P}V_0 \cap Z$ is the empty set.

Proof. Let $J_1 = (0, 0, 0; 0, 0, 1)$, $J_2 = (0, 0, 0; 0, 1, 1)$, $J_3 = (0, 0, 0; 1, 1, 1)$ and $v_i = (0, J_i, 0, 0) \in X$ for $i = 1, 2, 3$. $p(v_3) \in (X \cap \mathbb{P}V_0) \setminus X_2$ so $\mathbb{P}V_0 \cap X$ strictly contains X_2 and thus equals $\mathbb{P}V_0$. This proves (a). Note that $p(v_3) \notin Y$ so $\mathbb{P}V_0 \cap Y \subset X_2$. On the other hand $p(v_2) \in Y \cap X_2$. This proves (b). Finally $p(v_3) \in X_1$ but it does not lie in Z . This proves (c). □

6.5. Consider $\text{Spin}(5, 4) \subset F_{4,4}$ and $M' = SU_2 \times \text{Spin}(5) = SU_2 \times Sp_4$. Then $V'_0 = W_2 \oplus W_3$ gives the standard representation of Sp_4 . Note that $\mathbb{P}V'_0$ is a single Sp_4 orbit.

Lemma 6.5.1.

- (a) $\mathbb{P}V'_0 \cap X = \mathbb{P}V'_0 \cap Y = \mathbb{P}V'_0$.
- (b) $\mathbb{P}V'_0 \cap Z$ is the empty set.

Proof. Since $V'_0 \subset V_0$, the lemma follows from Lemma 6.4.1. □

6.6. *Proof of Theorems 4.2.1(c)(d) and 4.3.1(c)(d).* Part (d) follows easily from Proposition 3.3.1 since $S^n V'_0 = V_{0,n}$ and $S^n V_0 = \sum_{a+b+c=n} S(a, b, c)$ respectively.

(c) By (20) we have

$$0 \rightarrow \text{Res } \sigma_X \rightarrow \sum_{n=0}^{\infty} \mathbf{H}(\text{Spin}(5, 4), V_{0,n}[d - 1 + n])$$

$$\rightarrow \sum_{n=0}^{\infty} \mathbf{H}(\text{Spin}(5, 4), V_{0,n}[d + 3 + n]) \rightarrow 0.$$

Considering the infinitesimal characters of the summands gives (13).

The restriction of σ_X to $\text{Spin}(4, 4)$ and (14) have been proven in Prop. 4.6.1 of [L2]. This proves (c). □

Proof of Theorem 4.2.1(b). $r_3 : V_M \rightarrow S^3(V'_0)$ and since the codomain is irreducible, r_3 is either the zero map or a surjection. However the image of r_3 has to cut out the empty set so $r_3 = 0$. This implies $r_n = 0$ and $R_n = S^n(V_0)$ for all n . □

6.7. *Proof of Theorem 4.3.1(b).* $r_3 : V_M \rightarrow S^3(V_0)$ and

$$\begin{aligned} V_M &= S_A^1 + S_B^1 + S_C^1 + (S_A^1 \otimes S_B^1 \otimes S_C^1) \\ S^3(V_0) &= \sum_{a+b+c=3} S(a, b, c) \end{aligned}$$

as representations of M'' . Thus image of r_3 is either 0 or W_0 (cf. (32)). By Lemma 6.4.1 the image has to cut out X_2 so it is W_0 . Moreover W_0 generates the ideal of X_2 and thus $R_n = A^n(X_2)$. □

6.8. Recall that the minimal representation σ_Z of $\widetilde{F}_{4,4}$ has K -types

$$(33) \quad \sum_{n=0}^{\infty} S_{\alpha}^{n+1}(\mathbb{C}^2) \otimes \pi_{Sp_6}(n\varpi_3).$$

Suppose

$$\text{Res}_{\widetilde{Spin}(5,4) \times C_1}^{\widetilde{F}_{4,4}} \sigma_Z = \sum_{\chi \in C_1^{\wedge}} \Theta'(\chi) \otimes \chi.$$

The center of $\widetilde{F}_{4,4}$ acts nontrivially on σ_Z and hence C_1 only acts by the characters $\chi(1, 0)$ and $\chi(1, 1)$. Theorem 4.2.1(a) is a consequence of the following theorem:

Theorem 6.8.1. *Let $\epsilon = 0, 1$. Then $\Theta'(\chi(1, \epsilon)) = \sigma(\widetilde{Spin}(5, 4), V_{0,\epsilon}[4])$ and it has K -types $(K = SU_2(\tilde{\alpha}) \times_{\mu_2} (SU_2 \times Spin(5)))$*

$$\sum_{a,b \geq 0} S_{\alpha}^{a+2b+1+\epsilon}(\mathbb{C}^2) \otimes S^a(\mathbb{C}^2) \otimes V_{a,2b+\epsilon}.$$

Proof. By Lemma 6.5.1, \mathcal{O}' is empty so $r'_2 : S^2V'_0 \rightarrow S^2V'_0$ is not the zero map. Hence r'_2 is a surjection and Theorem 3.5.1(a) applies. The following lemma proves the claim about the K -types and completes the proof of the theorem:

Lemma 6.8.2.

$$\text{Res}_{SU_2 \times Spin(5) \times C'}^{Sp_6} \pi(n\varpi_3) = \sum_{a+b=n} S^a(\mathbb{C}) \otimes \pi_{Spin(5)}(a\varpi_1 + b\varpi_2) \otimes \chi(a, b).$$

We omit the proof of the lemma since branching law of Sp_6 is known (see Equation (25.27) [FH]). □

6.9. Suppose

$$\text{Res}_{\widetilde{Spin}(4,4) \times C}^{\widetilde{F}_{4,4}} \sigma_Z = \sum_{\chi \in C^{\wedge}} \Theta(\chi) \otimes \chi.$$

Since the center of $\widetilde{F}_{4,4}$ acts nontrivially on σ_Z , C can only act by the characters $\chi(s_1, s_2, s_3)$ where $s_1 + s_2 + s_3$ is an odd integer.

Theorem 4.5.1(a) is a consequence of the following theorem:

Theorem 6.9.1.

- (a) $\Theta(\chi(1, 1, 1)) = \sigma(\widetilde{\text{Spin}}(4, 4), \mathbb{C}[3])$.
 - (b) $\Theta(\chi(1, 0, 0)) = \sigma(\widetilde{\text{Spin}}(4, 4), S(1, 0, 0)[4])$.
 - (c) $\Theta(\chi(0, 1, 0)) = \sigma(\widetilde{\text{Spin}}(4, 4), S(0, 1, 0)[4])$.
 - (d) $\Theta(\chi(0, 0, 1)) = \sigma(\widetilde{\text{Spin}}(4, 4), S(0, 0, 1)[4])$.
- $\Theta(\chi(1, 1, 1))$ (resp. $\Theta(\chi(1, 0, 0))$) has K -types ($K = SU_2 \times SU_2^3$)

$$(34) \quad \sum_{n,a,b,c \geq 0} S_{\alpha}^{1+n}(\mathbb{C}) \otimes S(a, b, c)$$

where the sum is taken over all nonnegative integers n, a, b, c such that $a + b + c \geq n$, $a + b - c \leq n$, $a - b + c \leq n$, $b + c - a \leq n$ and $n \equiv a \equiv b \equiv c \pmod{2}$ (resp. $n \equiv a \not\equiv b \equiv c \pmod{2}$).

The K -types of $\Theta(\chi(0, 1, 0))$ and $\Theta(\chi(0, 0, 1))$ differ from that of $\Theta(\chi(1, 0, 0))$ by permuting a, b, c in (34) accordingly under the action of S_3 (cf. §6.3).

Proof. By the action of S_3 , it suffices to prove (a) and (b). By the see-saw pair (30), we note that

$$(35) \quad \Theta'(\chi(1, 0)) = \Theta(\chi(1, 1, 1)) + \Theta(\chi(1, 0, 0)).$$

The branching rule from $\text{Spin}(5)$ to $\text{Spin}(4) = SU_2^2$ is well-known (see eqn (25.34) [FH]). Applying this to Theorem 6.8.1 shows that the sum of the K -types of (a) and (b) agrees with the K -types in (35). A K -type

$$S^n(\mathbb{C}^2) \otimes S(a, b, c)$$

in (35) will belong to $\Theta(\chi(1, 1, 1))$ (resp. $\Theta(\chi(1, 0, 0))$) if and only if $a - b$ is an even (resp. odd) integer. This proves (34).

By naturality, the composition of the map

$$I^2(Z) \xrightarrow{r_2} S^2V_0 \rightarrow S^2V'_0 = S^2_B + S^2_C + \mathbb{C}_B \otimes \mathbb{C}_C$$

is the map r'_2 in the proof of Theorem 6.8.1 and it is surjective. This implies that the image contains $S^2_B + S^2_C + \mathbb{C}_B \otimes \mathbb{C}_C$. Since r_2 commutes with the action of S_3 (cf. §6.3), r_2 is surjective and the theorem follows from Theorem 3.5.1(c). □

7. Dual pairs in $E_{6,4}$.

7.1. Let $G' \subset G$ be one of the dual pairs (5) to (7). We set $s = 2, 4, 8$ and we define

$$\mathcal{J}_s^p = \{(0, 0, 0; x_1, x_2, x_3) \in \mathcal{J}(\mathcal{K}_s) : x_i \in \mathcal{K}_s, \langle x_i, z \rangle = 0 \text{ for all } z \in \mathcal{K}_{s/2}\}$$

then

$$(36) \quad V_0 = \{(0, J, J', 0) \in V_M : J, J' \in \mathcal{J}_s^p\}.$$

It is known that $M'(\mathbb{C})$ has a dense orbit in $\mathbb{P}V_0$ [SK]. The orbits and their coordinate rings have been extensively studied and they are documented in §7 [GW2]. We will make use of these results to determine R_\bullet .

7.2. Consider $G = \tilde{E}_{6,4} \rtimes \mathbb{Z}/2\mathbb{Z} \supset G' = \tilde{F}_{4,4} \times \mathbb{Z}/2\mathbb{Z}$. Set $s = 2$. Then $V_0 = \mathbb{C}^6$ in (36) is the standard representation of $M'(\mathbb{C}) = Sp_6(\mathbb{C}) \times \mathbb{Z}/2\mathbb{Z}$. It is well-known that $\mathbb{P}V_0$ is a single orbit of $M'(\mathbb{C})$.

Lemma 7.2.1.

- (a) $\mathbb{P}V_0 \cap X = \mathbb{P}V_0 \cap Y = \mathbb{P}V_0$.
- (b) $\mathbb{P}V_0 \cap Z$ is an empty set.

Proof. (a) It suffices to show that $\mathbb{P}V_0 \cap Y$ is nonempty since $X \supset Y$ and $\mathbb{P}V_0$ is a single orbit of $M'(\mathbb{C})$. Let $J = (0, 0, 0; s, 0, 0) \in \mathcal{J}(\mathcal{K}_2)$ where $s = \text{diag}(1, -1) \in \mathcal{K}_2$. By (36) $(0, J, 0, 0) \in V_0$. Since $\det(J) = 0$, $p(J) \in Y$ by §5.7.

(b) We will prove this by contradiction. Suppose on the contrary $\mathbb{P}V_0 \cap Z$ is nonempty and it equals $\mathbb{P}V_0$. Let $v = (0, J, J', 0) \in V_0$ as given in (36) and assume that $J \neq 0$. Since $p(v) \in Z$, by Lemma 5.5.1 and (27) J has rank at most 1. By (24), $J = 0$. This yields the contradiction. \square

Proof of Theorem 4.5.1. (a) By Lemma 7.2.1(b), $r_2 : \mathfrak{sl}_6 \rightarrow S^2(\mathbb{C}^6)$ is non-zero. Since the image is irreducible r_2 is a surjection. (a) follows from Theorem 3.5.1(c).

(b) and (c) follow from Lemma 7.2.1(a) and Theorem 3.5.1(a) since $R_n = S^n(\mathbb{C}^6)$.

By (20) we have

$$\begin{aligned}
 0 \rightarrow \text{Res } \sigma_X &\rightarrow \sum_{n=0}^{\infty} \mathbf{H}(\tilde{F}_{4,4}, S^n(\mathbb{C}^6)[9+n]) \otimes \chi^n \\
 &\rightarrow \sum_{n=0}^{\infty} \mathbf{H}(\tilde{F}_{4,4}, S^n(\mathbb{C}^6)[13+n]) \otimes \chi^n \rightarrow 0.
 \end{aligned}$$

Finally considering the infinitesimal characters of the above summands gives (15). \square

7.3. Consider $G_0 = E_{6,4} \rtimes \mathbb{Z}/2\mathbb{Z} \supset G'' = \text{Spin}(4, 4) \times_{\mu_2^2} (U_1^2 \times \mathbb{Z}/2\mathbb{Z})$ and $M'' = SU_2^3 \times_{\mu_2^2} (U_1^2 \times \mathbb{Z}/2\mathbb{Z})$. Here we identify $U_1^2 = \{(t_1, t_2, t_3) : t_i \in U_1, t_1 t_2 t_3 = 1\}$.

Let $\chi_0(a_1, a_2, a_3) : (t_1, t_2, t_3) \mapsto t_1^{a_1} t_2^{a_2} t_3^{a_3}$ be a character of U_1^2 where $a_i \in \mathbb{Z}$. Let $\chi(a_1, a_2, a_3)$ be the unique irreducible representation of $U_1^2 \times \mathbb{Z}/2\mathbb{Z}$ containing $\chi_0(a_1, a_2, a_3)$. Note that $\chi(a, a, a) = \mathbb{C}$ and if not all the a_i 's are the same, then $\chi(a_1, a_2, a_3) = \chi_0(a_1, a_2, a_3) + \chi_0(-a_1, -a_2, -a_3)$. Clearly

$\chi(a_1, a_2, a_3) = \chi(a_1 - a, a_2 - a, a_3 - a) = \chi(-a_1, -a_2, -a_3)$. Therefore we may assume that (a_1, a_2, a_3) takes values from the set

$$T := \{(a_1, a_2, a_3) \in \mathbb{Z}^3 : a_i > a_{i+1} = 0 \geq a_{i+2} \text{ for some } i = 1, 2, 3\} \cup \{(0, 0, 0)\}.$$

Set $\mathcal{K} = \mathcal{K}_2$ and we define \mathcal{J}_i and W_i ($i = 1, 2, 3$) using (26). Then $V_0 = W_1 \oplus W_2 \oplus W_3$. $W_i = W_{i1} \oplus W_{i2}$ where W_{i1} and W_{i2} are the standard representations of SU_2 . $(t_1, t_2, t_3) \in U_1^2$ acts on W_{i1} (resp. W_{i2}) by multiplication by t_i (resp. t_i^{-1}).

Write $V_0 = \sum_{i,j} W_{ij}$ and we denote an element of V_0 by (w_{ij}) where $w_{ij} \in W_{ij}$. Let \mathcal{V} denote the subset of V_0 consisting of (w_{ij}) satisfying the following:

1. For each $j = 1, 2$, at least 2 of w_{1j}, w_{2j}, w_{3j} is zero.
2. For each $i = 1, 2, 3$, either $w_{i1} = 0$ or $w_{i2} = 0$.

Then one can show that $Z \cap \mathbb{P}V_0 = \mathbb{P}\mathcal{V}$ and the ideal of $\mathbb{P}\mathcal{V}$ is generated by degree 2 polynomials. Its coordinate ring is

$$A^n(\mathbb{P}\mathcal{V}) = \sum S(|a_1|, |a_2|, |a_3|) \otimes \chi(a_1, a_2, a_3)$$

where the sum is taken over all $(a_1, a_2, a_3) \in T$ satisfying $|a_1| + |a_2| + |a_3| = n$. By Theorem 3.5.1(c) we get the dual pair correspondence of G'' in G .

Theorem 7.3.1.

$$\begin{aligned} & \text{Res}_{\text{Spin}(4,4) \times (U_1^2 \times \mathbb{Z}/2\mathbb{Z})}^{E_{6,4} \times \mathbb{Z}/2\mathbb{Z}} \sigma_Z \\ &= \sum_{(a_1, a_2, a_3) \in T} \sigma(\text{Spin}(4,4), S(|a_1|, |a_2|, |a_3|)[|a_1| + |a_2| + |a_3| + 4]) \otimes \chi(a_1, a_2, a_3). \end{aligned}$$

The K -types of the summands could be calculated using (12) by applying the branching rule from $M = SU_6$ to $SU_2^3 \times U_1^2 \subset M''$. This in turn could be computed using the Littlewood-Richardson rule (see [FH, p. 456]).

Note that the summands of the above correspondence also appear in Theorem 4.3.1(b). This follows from the fact that the dual pairs G''_0 and $F_{4,4} \times \mathbb{Z}/2\mathbb{Z}$ form a see-saw pair in G_0 . Similarly restrictions of σ_Y and σ_X to G''_0 will yield representations of $\widetilde{\text{Spin}}(4,4)$ appearing in Theorem 4.3.1(c) and (d).

8. Dual pair in $E_{7,4}$.

8.1. Consider $G_0 = E_{7,4} \supset G'_0 = (E_{6,4} \times_{\mu_3} U_1) \times \mathbb{Z}/2\mathbb{Z}$ and $M'(\mathbb{C}) = GL_6 \times \mathbb{Z}/2\mathbb{Z}$. The nontrivial element in $\mathbb{Z}/2\mathbb{Z}$ acts on $E_{6,4}$ as the outer automorphism. It also acts on U_1 by sending $z \mapsto z^{-1}$.

Set $s = 4$ and define V_0 as in (36). $z \in U_1(\mathbb{C}) = \mathbb{C}^*$ will act on V_0 by

$$(0, J, J', 0) \rightarrow (0, zJ, z^{-1}J', 0).$$

Recall $\mathcal{K}_4 = M_2(\mathbb{C})$ and let \mathcal{K}_u (resp. \mathcal{K}_l) denote the subspace of strictly upper (resp. lower) triangular matrices. Define

$$\begin{aligned} \mathcal{J}_u &= \{(0, 0, 0; c_1, c_2, c_3) \in \mathcal{J}(\mathcal{K}_4) : c_i \in \mathcal{K}_u\} \\ V_u &= \{(0, J, J', 0) \in V_0 : J, J' \in \mathcal{J}_u\}. \end{aligned}$$

Similarly we define \mathcal{J}_l and V_l by replacing \mathcal{K}_u with \mathcal{K}_l . V_u (resp. V_l) gives the standard (resp. dual) representation of GL_6 . Thus $V_0 = \mathbb{C}^6 \oplus (\mathbb{C}^6)^*$ as a representation of $M'(\mathbb{C}) = GL_6 \rtimes \mathbb{Z}/2\mathbb{Z}$.

There are only two nontrivial proper orbits of M' in $\mathbb{P}V_0$ (cf. §6 [GW2]). Their closures are

$$\begin{aligned} X_1 &= \mathbb{P}\mathbb{C}^6 \cup \mathbb{P}(\mathbb{C}^6)^* \\ \mathcal{F} &= \{(v, v^*) \in \mathbb{P}V_0 : f_0 := \langle v, v^* \rangle = 0\}. \end{aligned}$$

Note that $X_1 \subset \mathcal{F}$. The inner product f_0 defining \mathcal{F} is an M' -invariant quadratic form in S^2V_0 and this in turn induces an inclusion

$$S^n V_0 \cdot f_0 \hookrightarrow S^{n+2} V_0.$$

The coordinate ring $A^n(\mathcal{F})$ is the quotient of the above inclusion. Recall §4.7 where we define $V_{a,b} = \pi_{SL_6}(a\varpi_1 + b\varpi_6)$. Then

$$\begin{aligned} (37) \quad A^n(\mathcal{F}) &= \sum_{a+b=n} V_{a,b} \otimes \chi_1^{a-b} \\ A^n(\mathbb{P}V_0) &= A^n(\mathcal{F})[f_0] = \sum_{a+b+2m=n} V_{a,b} \otimes \chi_1^{a-b} \end{aligned}$$

as a representation of $GL_6 = SL_6 \times_{\mu_6} U_1$.

Lemma 8.1.1.

- (a) $\mathbb{P}V_0 \cap Z = \mathbb{P}\mathbb{C}^6 \cup \mathbb{P}(\mathbb{C}^6)^*$.
- (b) $\mathcal{F} = \mathbb{P}V_0 \cap X = \mathbb{P}V_0 \cap Y$.

Proof. Define

$$\begin{aligned} s &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathcal{K}_4 = M_2(\mathbb{C}), \\ x_1 &= (0, 0, 0; s, 0, 0), \quad x_2 = (0, 0, 0; t, 0, 0) \in \mathcal{J}(\mathcal{K}_4), \\ T_1 &= (0, x_1, 0, 0), \quad T_2 = (0, x_1 + x_2, 0, 0), \quad T_3 = (0, x_1, x_2, 0) \in V_0. \end{aligned}$$

By Lemma 5.5.1 $p(T_1) \in Z \cap \mathbb{P}V_0$. Hence $Z \cap \mathbb{P}V_0$ is nontrivial and it must contain X_1 . $x_1 + x_2$ has rank 2 so $p(T_2) \in Y \setminus Z$ (cf. §5.7) and $p(T_2) \notin X_1$. This proves (a) and implies that $\mathbb{P}V_0 \cap Y \supseteq \mathcal{F}$.

Next $p(T_3) \in \mathbb{P}V_0 \setminus \mathcal{F}$. $f_4(T_3) \neq 0$ so $T_3 \notin X$. Since \mathcal{F} is maximal proper, $\mathcal{F} \supseteq X \cap \mathbb{P}V_0 \supseteq Y \cap \mathbb{P}V_0$. This proves (b). □

Proof of Theorem 4.7.1. (a) The image of r_2 must be cut out X_1 so it is either $V_{1,1}$ or $\mathbb{C}f_0 + V_{1,1}$ in S^2V_0 .

We claim that the image cannot be $V_{1,1}$. Indeed if otherwise by Theorem 3.5.1(b) the restriction of σ_Z will contain $\sigma' = \sigma(E_{6,4}, \mathbb{C}[6])$. σ' contains the lowest K' -type $\tau = S_{\tilde{\alpha}}^4(\mathbb{C}^2) \otimes \mathbb{C}$ ($K' = SU_2(\tilde{\alpha}) \times SU_6$). Therefore τ is a subrepresentation of the K -type

$$S_{\tilde{\alpha}}^4(\mathbb{C}^2) \otimes \pi_{\text{Spin}(12)}(2\varpi_6)$$

in σ_Z . However the tables of [KP] show that the above does not contain τ . This proves the claim.

Finally since $V_{1,1} + \mathbb{C}f_0$ generates the homogeneous ideal of X_1 so $R_{\bullet} = A^{\bullet}(\mathcal{F})$ (cf. (37)). This proves (a).

(b) Similar to (a) the image of r_3 cuts out \mathcal{F} and it has to be $V_0 \cdot f_0 \subset S^3V_0$. The ideal generated by V_0f_0 contains all homogeneous polynomials vanishing on \mathcal{F} except $\mathbb{C}f_0$. Thus $R_{\bullet} = A^{\bullet}(\mathcal{F}) + \mathbb{C}f_0$. Finally we note that $\mathbf{H}(E_{6,4}, \mathbb{C}[12])$ is irreducible (cf. §3.3).

(c) By (20) we get

$$\begin{aligned} (38) \quad 0 \rightarrow \text{Res } \sigma_X &\rightarrow \sum_{a,b,n} \mathbf{H}(\tilde{E}_{6,4}, V_{a,b}[15 + 2n + a + b]) \otimes \chi_1^{a-b} \\ &\rightarrow \sum_{a,b,n} \mathbf{H}(\tilde{E}_{6,4}, V_{a,b}[19 + 2n + a + b]) \otimes \chi_1^{a-b} \rightarrow 0. \end{aligned}$$

The summands on the right are all irreducible and unitary, and they also appear as summands in the middle term. Therefore removing these representations from (38) gives

$$0 \rightarrow \text{Res } \sigma_X \rightarrow \sum_{n=0,1} \sum_{a,b} \mathbf{H}(\tilde{E}_{6,4}, V_{a,b}[15 + 2n + a + b]) \otimes \chi_1^{a-b} \rightarrow 0.$$

This completes the proof of (c). □

Note that in (c) the image of r_4 is one dimensional and it cuts out \mathcal{F} and it has to be $(f_0)^2 \subset S^4V_0$. Therefore R_{\bullet} is not reduced. This example shows that the containment in Theorem 3.5.1(b) may be proper.

9. Dual pairs in $E_{8,4}$.

9.1. In this section we consider $\widetilde{E}_{8,4} \supset \widetilde{E}_{7,4} \times_{\mu_3} SU_2$ and $M' = \text{Spin}(12) \times SU_2$. Set $s = 8$ and define V_0 by (36).

9.2. Recall U_0 in §5.3. Define $U_e := \mathbb{C}e_2 \oplus \mathbb{C}e_3 \subset U_0$. By (22) $SL(U_e)$ acts on \mathcal{K}_8 . This induces an action of $SL(U_e)$ on $(0, J, J', 0) \in V_0$ where $SL(U_e)$ acts uniformly on each of the nonzero entries of J and J' .

For $i = 1, 2$, define $J_i = (0, 0, 0; x_{i1}, x_{i2}, x_{i3}) \in \mathcal{J}$ where $x_{ij} = (0, 0; w_{ij1}e_2, -w_{ij2}e_3)$ in \mathcal{K}_8 (cf. (21)) and $w_{ijk} \in \mathbb{C}$. Let $v = (0, J_1, J_2, 0) \in V_0$. Denote the set of such vectors v in V_0 by V_1 .

Similarly for $i = 1, 2$, define $J'_i = (0, 0, 0; x'_{i1}, x'_{i2}, x'_{i3}) \in \mathcal{J}$ where $x'_{ij} = (0, 0; w'_{ij1}e_3, w'_{ij2}e_2) \in \mathcal{K}_8$ and $w'_{ijk} \in \mathbb{C}$. Let $v' = (0, J'_1, J'_2, 0) \in V_0$. Denote the set of such vectors v' in V_0 by V_2 .

Note that $V_0 = V_1 \oplus V_2$. Both V_1 and V_2 give the standard representations of $\text{Spin}(12)$ in $M(\mathbb{C})$. The invariant quadratic forms on V_1 and V_2 are given respectively by

$$q_1(w_{ijk}) = \sum_{j=1}^3 (w_{1j1}w_{2j2} - w_{1j2}w_{2j1})$$

$$q_2(w'_{ijk}) = \sum_{j=1}^3 (w'_{1j1}w'_{2j2} - w'_{1j2}w'_{2j1}).$$

Let $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(U_e)$, $(w_{ijk}) \in V_1$ and $(w'_{ijk}) \in V_2$. The action of ω on V_0 commutes with that of $\text{Spin}(12)$ and $\omega V_2 = V_1$ by sending $(w'_{ijk}) \mapsto (w_{ijk})$ where $w_{ijk} = w'_{ijk}$. Let $w_1, w_2 \in V_1$, define $\phi : V_1 \otimes U_e \rightarrow V_0$ by

$$\phi : w_1 \otimes e_2 + w_2 \otimes e_3 \rightarrow w_1 + \omega^{-1}w_2.$$

Then ϕ is an isomorphism of representations of $M'(\mathbb{C}) = \text{Spin}(12) \times SL(U_e)$.

9.3. We will describe the orbits of $M'(\mathbb{C})$ on $\mathbb{P}V_0$. $\mathbb{P}V_0$ has a dense $M'(\mathbb{C})$ orbit. It contains four additional orbits and we denote their closures by X_1, Y_1, Y_2, Z_1 . Let $\langle \cdot, \cdot \rangle$ denote the inner product induced by quadratic form q_1 on V_1 and let $v = w_1 \otimes e_2 + w_2 \otimes e_2$. Then X_1 is a hypersurface whose ideal is generated by a degree 4 polynomial f'_4

$$f'_4(v) = \det \begin{pmatrix} \langle w_1, w_1 \rangle & \langle w_1, w_2 \rangle \\ \langle w_2, w_1 \rangle & \langle w_2, w_2 \rangle \end{pmatrix}.$$

Y_1 is the complete intersection of the 3 quadrics

$$(39) \quad \langle w_1, w_1 \rangle = \langle w_1, w_2 \rangle = \langle w_2, w_2 \rangle = 0.$$

$Y_2 \subset X_1$ is the subvariety $\mathbb{P}W \times \mathbb{P}U$. Let $Q \subset \mathbb{P}W$ be defined by $\langle w_1, w_1 \rangle = 0$. Then $Z_1 = Q \times \mathbb{P}U = Y_1 \cap Y_2$ is the unique minimal closed orbit in $\mathbb{P}V_0$. $Y_1 \cup Y_2$ is cut out by cubics

$$(40) \quad \begin{aligned} f_a(v) &:= \det \begin{pmatrix} \langle w_1, w_1 \rangle & \langle w_1, a \rangle \\ \langle w_2, w_1 \rangle & \langle w_2, a \rangle \end{pmatrix} = 0, \\ f'_b(v) &:= \det \begin{pmatrix} \langle b, w_1 \rangle & \langle b, w_2 \rangle \\ \langle w_2, w_1 \rangle & \langle w_2, w_2 \rangle \end{pmatrix} = 0 \end{aligned}$$

for all $a, b \in W$. Let I_0 be the homogeneous ideal generated by $\{f_a, f'_b : a, b \in W\}$. We claim that I_0 is the homogeneous ideal of $Y_1 \cup Y_2$. Indeed suppose f vanishes on $Y_1 \cup Y_2$, by (39) we may assume that modulo I_0

$$f(w_1, w_2) = h_1(w_1)\langle w_1, w_1 \rangle + h_2(w_1, w_2)\langle w_1, w_2 \rangle + h_3(w_2)\langle w_2, w_2 \rangle$$

where h_1, h_2, h_3 are polynomials on V_1, W and V_2 respectively. Since $f(w_1, w_2) = 0$ whenever w_1 is parallel to w_2 , we get $h_1 = h_3 = 0$ and h_2 vanishes on Y_2 . Thus $f \in I_0$ and this proves the claim.

Recall §4.8 where $V_{a,b} = \pi_{\text{Spin}(12)}(a\varpi_1 + b\varpi_2)$. Then the coordinate rings are:

$$(41) \quad \begin{aligned} A^\bullet(\mathbb{P}V_0) &= A^\bullet(X_1)[f'_4] \\ A^n(X_1) &= \sum^* V_{a+2d,c} \otimes S^{a+2b}(U) \end{aligned}$$

where the sum \sum^* is taken over a, b, c, d satisfying (16).

$$(42) \quad A^n(Y_1) = \sum_{a+2c=n} V_{a,c} \otimes S^a(U).$$

$$(43) \quad A^n(Y_2) = S^n(W) \otimes S^n(U) = \sum_{a+2b=n} V_{a,0} \otimes S^n(U).$$

$$(44) \quad A^n(Y_1 \cup Y_2) = \sum_{a+2b+2c=n, bc=0} V_{a,c} \otimes S^{a+2b}(U).$$

$$(45) \quad A^n(Z_1) = V_{n,0} \otimes S^n(U).$$

The coordinate rings except (44) are given in §5, §6 [GW2]. Since $A^n(Y_1 \cup Y_2)$ is a quotient of $A^n(X_1)$ which is multiplicity free, (44) follows from (42) and (43).

Lemma 9.3.1.

- (a) $X \cap \mathbb{P}V_0 = X_1$.
- (b) $Y \cap \mathbb{P}V_0 = Y_1 \cup Y_2$.
- (c) $Z \cap \mathbb{P}V_0 = Z_1$.

Proof. For the ease of notations, suppose $J_i = (0, 0, 0; c_{i1}, c_{i2}, c_{i3}) \in \mathcal{J}$ ($i = 1, 2$), then we denote $(0, J_1, J_2, 0) \in V_0$ by $(c_{11}, c_{12}, c_{13} | c_{11}, c_{12}, c_{13})$. For $i = 2, 3$, let $x_i = (0, 0; e_i, e_i), y_i = (0, 0; e_i, 0) \in \mathcal{K}_8$ and (cf. (21)).

(a) Set $v_1 := (x_2, x_2, x_2 | 0, 0, 0) \in V_0$. Then $f_4(v_1) \neq 0$. Hence $X \cap \mathbb{P}V_0$ is a hypersurface in $\mathbb{P}V_0$ of degree 4 and it has to be X_1 .

(b) $J_4 = (0, 0, 0; x_2, x_2, x_2)$ has rank 3 so $p(0, J_4, 0, 0) \in (X \setminus Y) \cap \mathbb{P}V_0$. Hence $Y \cap \mathbb{P}V_0 \subseteq Y_1 \cup Y_2$. $J_5 = (0, 0, 0; x_2, 0, 0)$ has rank 2 so $p(0, J_5, 0, 0) \in (Y \setminus Z) \cap (Y_1 \setminus Z_1)$. Hence $Z \cap \mathbb{P}V_0 \subseteq Y_2$ and $Y_1 \subseteq Y \cap \mathbb{P}V_0$.

Let $v_3 = (y_2, 0, 0 | y_3, 0, 0)$ so that $pv_3 \in Y_2 \setminus Z_1$. It is easy to check that v_3 satisfies $\partial f_4 / \partial v = 0$ in §5.7 so $pv_2 \in Y$. This implies that $Y \supseteq Y_2$.

(c) We have seen in (b) that $Z \cap \mathbb{P}V_0 \subseteq Y_2$. $J_6 = (0, 0, 0; y_2, 0, 0)$ has rank 1 so $p(0, J_6, 0, 0) \in Z \cap \mathbb{P}V_0$. Hence $Z \cap \mathbb{P}V_0 \supseteq Z_1$. To complete the proof it suffices to show that $pv_3 \notin Z$. Indeed otherwise, by Lemma 5.5.1, $v_3 = p_B(0, J_6, 0, 0)$ for some $B = (\beta_i; d_i) \in \mathcal{J}$ (cf. (27)). Computations show that $p_B(J_6) = (0, J_6, 2B \times J_6, 0)$. However $2B \times J_6 = (*, *, *, -\beta_1 y_2, *, *) \neq (0, 0, 0; y_3, 0, 0)$. Hence $pv_3 \notin Z$. \square

Proof of Theorem 4.8.1. (a) This is determined by the map

$$\begin{aligned} r_2 \quad : \quad \mathfrak{e}_7 = \mathbb{C} \otimes S^2U + V_{0,1} + U \otimes \pi_{\text{Spin}_{12}}(\varpi_6) &\rightarrow \\ &\rightarrow S^2(W \otimes U) = V_{2,0} \otimes S^2U + S^2(U) + V_{0,1}. \end{aligned}$$

The image of r_2 is nonzero so it is either S^2U , $V_{0,1}$ or the sum. By (42) (resp. (43)) S^2U (resp. $V_{0,1}$) vanishes on Y_1 (resp. Y_2). By Lemma 9.3.1(c) the image cuts out Z_1 and hence it is must be the sum. Since $I^\bullet(Z_1)$ is generated by degree 2 polynomials, $R_n = A^n(Z_1)$.

(b)

$$\begin{aligned} r_3 \quad : \quad V_M = W \otimes U + \pi_{\text{Spin}_{12}}(\varpi_5) &\rightarrow \\ &\rightarrow S^3(V_0) = V_{3,0} \otimes S^3(U) + W \otimes U + W \otimes S^3(U) + V_{1,2} \otimes U. \end{aligned}$$

Since r_3 is nontrivial, the image has to be $W \otimes U$ and they are the set of cubics in (40). We have shown that the set of cubics generates the ideal of $Y_1 \cup Y_2$ and (b) follows from Theorem 3.5.1(b) and (44).

(c) By (20) and (41) we get

$$\begin{aligned} (46) \quad 0 \rightarrow \text{Res } \sigma_X &\rightarrow \sum_{n,m} \mathbf{H}(G', A^m(X_1)[27 + m + 4n]) \\ &\rightarrow \sum_{n,m} \mathbf{H}(G', A^m(X_1)[31 + m + 4n]) \rightarrow 0. \end{aligned}$$

The summands on the right also appear in the middle term. Therefore by removing these representations from (46) we get

$$0 \rightarrow \text{Res } \sigma_X \rightarrow \sum_{m=0}^{\infty} \mathbf{H}(G', A^m(X_1)[27 + m + n]) \rightarrow 0.$$

This completes the Proof of Theorem 4.8.1(c).

(d) This follows from Proposition 3.3.1 and (41). □

9.4. *Proof of Corollary 4.10.1.* First we recall a well-known fact [FH].

Lemma 9.4.1. *The 1 dimensional character \det^{a-b} of U_2 is the only SU_2 fixed vector in $\pi_{SU_3}(a\varpi_1 + b\varpi_2)$.*

Consider the see-saw pair

$$(47) \quad \begin{array}{ccc} E_{7,4} & & SU_3 \\ & \bowtie & \\ E_{6,4} & & SU_2 \end{array}$$

By Theorem 4.8.1(a), the trivial representation of SU_2 corresponds to the representation σ_Y of $E_{7,4}$. Applying Lemma 9.4.1 to the see-saw pair (47) gives

$$(48) \quad \begin{aligned} & \sum_{a,b \geq 0} \Theta(a,b) \otimes \chi_1^{a-b} \\ &= \text{Res}_{E_{6,4} \times U_1}^{E_{7,4}} \sigma_Y \\ &= \mathbf{H}(E_{6,4}, \mathbb{C}[12]) \otimes \chi_1^0 + \sum_{a,b \geq 0} \sigma(E_{6,4}, \pi_{SU_6}(a\varpi_1 + b\varpi_5)[a + b + 10]) \otimes \chi_1^{a-b}. \end{aligned}$$

The second equality is Theorem 4.7.1(b). By Table 1B of the Appendix to [L3], $\Theta(a,b)$ contains the right-hand side of (17). Alternatively, one can deduce this by considering the correspondence of the infinitesimal characters [Li3]. By (48) the containment is an equality. This proves Corollary 4.10.1. □

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