NONWANDERING, NONRECURRENT FATOU COMPONENTS IN $P^2$

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Let $\Omega$ be a nonwandering, nonrecurrent Fatou component for a holomorphic self-map $f$ of $\mathbb{P}^2$ of degree $d \geq 2$, and let $h$ be a normal limit of the family of iterates of $f$. We prove that $\Sigma := h(\Omega)$ is either a fixed point of $f$ or its normalization is a hyperbolic Riemann surface, so that the dynamics of $f|_{\Sigma}$ may be lifted to the unit disk. We also show that basins of attraction for holomorphic self-maps of $\mathbb{P}^k$ of degree $d \geq 2$ are taut.

1. Introduction.

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be holomorphic. By definition, therefore, there exists a homogeneous polynomial mapping $\tilde{f} : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{C}^{k+1} \setminus \{0\}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{C}^{k+1} \setminus \{0\} & \xrightarrow{\tilde{f}} & \mathbb{C}^{k+1} \setminus \{0\} \\
p & \downarrow & \downarrow p \\
\mathbb{P}^k & \xrightarrow{f} & \mathbb{P}^k.
\end{array}
$$

Here $p$ denotes the standard projection from $\mathbb{C}^{k+1} \setminus \{0\}$ onto $\mathbb{P}^k$. The degree $d$ of $f$ is by definition the degree of $\tilde{f}$. Throughout this paper we assume that $d > 1$.

The Fatou set $\mathcal{F}(f)$ is the largest open subset of $\mathbb{P}^k$ on which the family $\{f^n\}_{n \in \mathbb{N}}$ is normal. In [7], Ueda shows that $\tilde{f}$ has a bounded basin of attraction $A$ to the origin. Let $\Omega$ be any connected component of $\mathcal{F}(f)$. Ueda shows that there exists a set $\tilde{\Omega} \subset \partial A$ such that the restriction of $p$ to $\tilde{\Omega}$ is a holomorphic covering map onto $\Omega$. A corollary of this construction is the Kobayashi hyperbolicity of $\Omega$. Fornaess and Sibony have exploited this fact in their classification of recurrent Fatou components for holomorphic maps on $\mathbb{P}^2$ ([4]).

Suppose now that $\Omega$ is a fixed, nonrecurrent Fatou component; that is, $\Omega$ satisfies $f(\Omega) = \Omega$ and $f^n(z) \to \partial \Omega$ for all $z \in \Omega$. Let $h$ be a normal limit of some subsequence of $\{f^n\}$, so that $f^{n_i} \to h$ locally uniformly on $\Omega$ as
Then \( \Sigma := h(\Omega) \subset \partial \Omega \). The principal aim of this paper is to prove the following result:

**Theorem 1.** Suppose that \( f : \mathbb{P}^2 \to \mathbb{P}^2 \) is holomorphic, and \( \Omega \) a fixed, nonrecurrent Fatou component for \( f \). Let \( \Sigma \) be as described above. Then \( \Sigma \) is either a fixed point of \( f \), or there exists a locally injective holomorphic mapping \( \sigma : \Delta \to \Sigma \), where \( \Delta \subset \mathbb{C} \) is the unit disk, and a holomorphic function \( F : \Delta \to \Delta \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\Delta & \xrightarrow{F} & \Delta \\
\sigma \downarrow & & \downarrow \sigma \\
\Sigma & \xrightarrow{f} & \Sigma.
\end{array}
\]

In the latter case, \( F \) must either be conjugate to an irrational rotation, or \( F^n(z) \to \partial \Delta \) for all \( z \in \Delta \).

The proof is given in Section 2.

**Remark 1.** A more general theorem was stated by Fornaess and Sibony in [3], but the proof seems incomplete.

A complex manifold \( M \) is called **taut** if the family of maps from the unit disk \( \Delta \) to \( M \) is normal. Abate has asked ([1]) whether Fatou components for holomorphic self-maps of \( \mathbb{P}^k \) are taut. In Section 3, we prove the following:

**Theorem 2.** Let \( \Omega \) be a Fatou component for \( f : \mathbb{P}^k \to \mathbb{P}^k \) which is preperiodic to a basin of attraction. Then \( \Omega \) is taut.

## 2. Proof of Theorem 1.

Let \( f \) be a holomorphic self-map of \( \mathbb{P}^k \), and \( \Omega \) a fixed, nonrecurrent Fatou component. Choose and fix some subsequence \( f^{n_i} \) which converges locally uniformly on \( \Omega \). Let \( h = \lim_{i \to \infty} f^{n_i} \), and let \( \Sigma = h(\Omega) \). Then \( \Sigma \subset \partial \Omega \).

**Lemma 1.** Let \( \Sigma \) be as above. Then \( f(\Sigma) = \Sigma \).

*Proof.* Since \( h = \lim_{i \to \infty} f^{n_i} \), \( h \) commutes with \( f \) on \( \Omega \). Let \( z \in \Sigma, x \in h^{-1}(z) \). Let \( y \in f^{-1}(x) \cap \Omega \). Then \( f(z) = f(h(x)) = h(f(x)) \in \Sigma \), so \( f(\Sigma) \subset \Sigma \). And \( h(y) \in \Sigma \) with \( f(h(y)) = h(f(y)) = h(x) = z \). Thus \( f(\Sigma) = \Sigma \). \( \square \)

Let \( p \) be the natural projection from \( \mathbb{C}^{k+1} \setminus \{0\} \) to \( \mathbb{P}^k \), and \( \tilde{f} : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{C}^{k+1} \setminus \{0\} \) the homogeneous polynomial lift of \( f \) by \( p \). It was shown by Ueda ([7]) that any homogeneous polynomial self-map of \( \mathbb{C}^k \setminus \{0\} \) has a bounded basin of attraction to the origin. Let \( A \) be the bounded basin of attraction to the origin for \( \tilde{f} \). Ueda showed further the existence of a set
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Let $\Omega$ be such that the restriction of $p$ to $\tilde{\Omega}$ is a holomorphic covering map onto $\Omega$.

**Lemma 2.** Let $U$ be an open subset of $\Omega$ sufficiently small that a local inverse $q : U \to \tilde{\Omega}$ of $p|_{\tilde{\Omega}}$ may be defined. Then there exists $\hat{h} : U \to \partial A$ holomorphic (as a mapping into $\mathbb{C}^{k+1}$) such that $p \circ \hat{h} = h$. Furthermore, if $\hat{h}_1$ is one such lift, then $\hat{h}_2$ is another if and only if $\hat{h}_2 = e^{i\theta} \hat{h}_1$ for some real $\theta$.

**Proof.** Write $h = \lim f^{n_i}$. On $U$, we have $p \circ (\tilde{f}^{n_i} \circ q) = f^{n_i}$. Since $\{\tilde{f}^{n_i} \circ q\}$ is uniformly bounded as a family of mappings into $\mathbb{C}^{k+1}$, by passing to a subsequence, if necessary, we may assume that it has a holomorphic limit $\hat{h}$ on $U$. Taking limits of both sides of $p \circ (\tilde{f}^{n_i} \circ q) = f^{n_i}$ gives

$$p \circ \hat{h} = h.$$  

To prove the second statement, note that $\hat{h}_1(z)$ and $\hat{h}_2(z)$ are in the same fiber of $p$ for all $z \in U$; i.e., in the same complex line in $\mathbb{C}^{k+1}$. Thus

$$h_1(z) = \lambda(z) \hat{h}_2(z)$$

for $z \in U$, $\lambda : U \to \mathbb{C}$ holomorphic. Recall also that $h_1(z)$, $h_2(z)$ are contained in $\partial A$. If $G$ is the Green’s function for $A$, we have $\partial A = \{G = 0\}$. It is shown in [7] that for $\lambda \in \mathbb{C}$, $G$ satisfies

$$G(\lambda z) = G(z) + \log |\lambda|.$$  

Thus

$$0 = G(\hat{h}_1(z)) = G(\hat{h}_2(z)) = G(\lambda(z) \hat{h}_1(z)) = G(\hat{h}_1(z)) + \log |\lambda(z)|.$$  

Thus $|\lambda(z)| = 1$ for all $z \in U$. Since $\lambda$ is holomorphic, this gives $\lambda = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

This shows that any two lifts $\hat{h}$ of $h$ differ by a multiplicative constant of absolute value one. Conversely, it is easy to check that if $\hat{h} : U \to \partial A$ is a lift of $h$, then so is $e^{i\theta} \hat{h} : U \to \partial A$.

The next lemma is part of the classical construction of the desingularization of a Riemann surface; see [5]. We omit the proof.

**Lemma 3.** Let $f$ be a germ at 0 of a nonconstant holomorphic mapping from $\mathbb{C}$ to $\mathbb{C}^n$. Then there exists another germ $g$ at 0 of a holomorphic mapping from $\mathbb{C}$ to $\mathbb{C}^n$ such that $g$ is injective in a neighborhood of 0, and such that the images of $f$ and $g$ agree as germs.
Lemma 4. Given $z \in \Sigma$, let $x \in h^{-1}(z)$, and let $L$ be a complex line through $x$ such that $h|_L$ is not constant. Then there exists a ball $U$ centered at $x$ such that the restriction of $p$ to $\hat{h}(L \cap U)$ is injective.

Proof. Let $U$ be sufficiently small that we may define $\hat{h}$ on $U$, as in Lemma 2. By shrinking $U$, if necessary, we may assume that $x$ is the only critical point of both $\hat{h}$ and of $p \circ \hat{h}$ in $L \cap U$. Let $D = L \cap U$, and $D^* = D \setminus \{x\}$. By Lemma 3, shrinking $U$ further, we may assume that both $\hat{h}(D^*)$ and $p \circ \hat{h}(D^*)$ are biholomorphic to punctured disks. Thus if $p|_{\hat{h}(D)}$ is not injective, we may assume, making the Böttcher coordinate change, that it is of the form $w \mapsto w^s$ for some $s \geq 2$.

But then we can replace $\hat{h}$ by another lift $g \circ \hat{h}$, where $g$, in the appropriate coordinates, is a nontrivial rotation of $\hat{h}(D)$ about $\hat{h}(x)$. In particular, $g \circ \hat{h}(x) = \hat{h}(x)$. But by Lemma 2, $g \circ \hat{h}$ must be of the form $e^{i\theta} \hat{h}$. Furthermore, $\hat{h}(x) \neq 0$, since it is in $\partial A$. Thus $e^{i\theta} = 1$, and $g$ is the trivial rotation. This contradiction establishes the lemma. \qed

For the remainder of this section, we will assume that $k = 2$, and that $h : \Omega \to \partial S$ is nonconstant. In this case, for $x \in \Omega$, there is an irreducible piece $\Sigma_x$ of a Riemann surface, possibly with singularities, and a neighborhood $U(x)$ such that $h(U(x)) = \Sigma_x$. We define $R$ to be the abstract union $\bigcup \Sigma_{x_i}$ for a covering $\{U(x_i)\}$ of $\Omega$, with identifications of $z_i \in \Sigma_{x_i}$ to $z_j \in \Sigma_{x_j}$ if the images under $h$ agree there as germs. $R$ is Hausdorff, by the identity theorem. It is a one-dimensional Riemann surface, possibly with singularities. Let $S$ be its smooth normalization. The map $h$ factors naturally as $\pi_1 \circ h_1$, where $h_1 : \Omega \to S$ and $\pi_1 : S \to \Sigma$.

Near a regular value of $h_1$, $h_1$ has an inverse $q$ onto some linear disk in $\Omega$. Define $f_1$ locally by $f_1 = h_1 \circ f \circ q$. It is straightforward to check that $f_1$ is thereby well-defined and holomorphic away from critical values of $h_1$, and may be extended continuously to $\Sigma$. Thus $f : \Sigma \to \Sigma$ lifts naturally by $\pi_1$ to $f_1 : S \to S$.

Lemma 5. The Riemann surface $S$ described above is hyperbolic.

Proof. Given $z_0 \in S$, let $U$ be a neighborhood of $z_0$ sufficiently small that $\pi_2(U) \subset R$ contains at most one singular point, $w_0 := \pi_2(z_0)$. Assume also that $U$ is small enough that there exists a linear disk $L \subset \Omega$ such that $p$ maps $\hat{h}(L)$ injectively onto some set containing $\pi_1 \circ \pi_2(U)$, as in Lemma 4.

Let $z_1 \in U \setminus \{z_0\}$. Then there exists a neighborhood $V$ of $z_1$ and an open subset $W \subset L$ such that $g := h|_W$ is a biholomorphism onto $\pi_1 \circ \pi_2(V)$. Consider

$$\phi : V \to \partial A$$

$$z \mapsto \hat{h} \circ g^{-1} \circ \pi_1 \circ \pi_2(z).$$
Then $\phi$ is holomorphic, and $p \circ \phi = \pi_1 \circ \pi_2$. Any other choice of $\phi$ (obtained by choosing a different subset $W \subset L$) must therefore differ from the first by a multiplicative constant of absolute value one. Since $z_1$ was arbitrary, $\phi$ may therefore be extended along any path in $U \setminus \{z_0\}$. Since $p|_{\hat{h}(L)}$ is injective, this extension gives rise to a single-valued holomorphic mapping, of which $z_0$ is a removable singularity. Thus $\phi$ is holomorphic on $U$, with $p \circ \phi = \pi_1 \circ \pi_2$. Again, any other choice of $\phi$ must differ from this one by a multiplicative constant of absolute value one; and since $z_0$ was arbitrary, $\phi$ may therefore be extended along any path in $S$. But this defines a covering surface $\tilde{S} \subset \partial A$ of $S$. Since $S$ is covered by a bounded subset of $\mathbb{C}^3$, it is hyperbolic.

□

There are four a priori possibilities for $f_2 : S \to S$ (see [6]):

1. Some iterate of $f_2$ is the identity.
2. There exists $a \in \mathbb{R}$ such that $f_2^n(z) \to a$ for all $z \in S$.
3. $f_2^n(z)$ diverges to infinity with respect to the Poincaré metric on $S$ for all $z \in S$.
4. $S$ is conformally a disk, punctured disk, or annulus, and the action of $f_2$ on $S$ is conjugate to irrational rotation.

In our case, (1) is impossible, since then some iterate of $f$ would fix $\Sigma$. But by Bezout’s theorem the number of fixed points of a holomorphic self-map of complex projective space is finite. In Case (2), the point $a$ would be an attractive or semi-attractive fixed point of $f$. But then the topological dynamics in a neighborhood $U$ of $a$ are well understood. In both cases, if $U$ is sufficiently small, points in $F(f) \cap U$ cannot converge to $\Sigma \setminus \{a\}$. But this contradicts our assumption that $h$ is nonconstant. Thus (2) is also impossible.

Now, we note that $f_2$ can in turn be lifted to a holomorphic self-map $F$ of the unit disk, $\Delta$. Cases (3) and (4) above give the following possibilities for $F$:

1. $F^n(z) \to \partial \Delta$ locally uniformly on $\Delta$.
2. $F$ is an irrational rotation of $\Delta$.

Collecting the preceding lemmas gives us the following theorem:

**Theorem 1.** If $h$ is a limit of some subsequence $f^{n_i}$ on $\Omega$ and $\Sigma := h(\Omega)$, then either $\Sigma$ is a fixed point of $f$ or there exists a surjective, locally injective holomorphic mapping $\sigma : \Delta \to \Sigma := h(\Omega)$, and a holomorphic self-map $F$ of $\Delta$ satisfying (1) or (2) above, such that the following diagram commutes:

$$
\begin{array}{ccc}
\Delta & \xrightarrow{F} & \Delta \\
\downarrow{\sigma} & & \downarrow{\sigma} \\
\Sigma & \xrightarrow{f} & \Sigma
\end{array}
$$
Since $\Sigma$ does not contain an entire curve of singularities, Case (2) gives that $\Sigma$ is a disk, punctured disk, or annulus, with at most one singularity, at the fixed point. An example of this type of behavior is the following: Take $f : \mathbb{P}^2 \to \mathbb{P}^2$

$$[z : w : t] \mapsto [zt + z^2 : \lambda wt + w^2 : t^2],$$

where $\lambda = e^{2\pi i \theta}$ and $\theta$ satisfies a Diophantine condition. Let $S$ be the Siegel disk centered at 0 for the mapping $w \mapsto \lambda w + w^2$. Then $\{f^n\}$ is compactly divergent on the Fatou component containing the point $[-1 : 0 : 1]$, any uniform limit $h$ satisfies

$$\Sigma = h(\Omega) = \{[0, w, 1] : w \in S\}$$

(note that $\Sigma$ is conformally a disk), and $f|_{\Sigma}$ is conjugate to multiplication by $\lambda$.

In Case (1) above, the mapping $\sigma$ may be very complicated. I have no example of this type of behavior, nor a proof that it cannot occur.

3. Proof of Theorem 2.

**Theorem 2.** Let $\Omega$ be a Fatou component for $f : \mathbb{P}^k \to \mathbb{P}^k$ which is preperiodic to a basin of attraction. Then $\Omega$ is taut.

**Proof.** Replacing $f$ by an iterate, we may assume that $\Omega$ is an invariant basin of attraction to $q \in \Omega$. Assume, to get a contradiction, that there exists a sequence of holomorphic mappings $\{g_i : \Delta \to \Omega\}$ with no convergent subsequence. Since $\Omega$ is covered by a bounded set in $\mathbb{C}^{k+1}$, the family $\{g_i\}$ is normal as a family of maps from $\Delta$ into $\mathbb{P}^k$. Thus, passing to a subsequence if necessary, we may assume that

$$g_i \to g : \Delta \to \overline{\Omega}.$$  

But, by assumption, $g(\Delta) \not\subset \Omega$ and $g(\Delta) \not\subset \partial \Omega$.

For each $i$, let $\tilde{g}_i : \Delta \to \partial A$ be a lift of $g_i$. Then $\{\tilde{g}_i\}$ is uniformly bounded as a family of maps into $\mathbb{C}^{k+1}$, so it is normal. By passing to a subsequence if necessary, we may assume that

$$\tilde{g}_i \to \tilde{g} : \Delta \to \partial A.$$  

Taking limits of both sides of

$$p \circ \tilde{g}_i = g_i$$

gives

$$p \circ \tilde{g} = g.$$  

Now,

$$p \circ f^n \circ \tilde{g}_i = f^n \circ p \circ \tilde{g}_i = f^n \circ g_i.$$
Thus, for each \( n \) and each \( i \), \( \tilde{f}^n \circ \tilde{g}_i \) is a lift of \( f^n \circ g_i \). Taking limits with respect to \( i \) gives

\[
p \circ \tilde{f}^n \circ \tilde{g} = f^n \circ g.
\]

But \( \{ \tilde{f}^n \circ \tilde{g} \} \) is uniformly bounded as a family of mappings into \( C^{k+1} \). Thus it is normal, and so therefore is \( \{ f^n \circ g \} \). Let \( h \) be a normal limit of \( \{ f^n \circ g \} \). Then \( h \equiv q \) on \( g^{-1}(g(\Delta) \cap \Omega) \), so \( h \equiv q \) on \( \Delta \). But this is impossible, since

\[ f^n \circ g(\tilde{z}) \in \partial \Omega \text{ for all } \tilde{z} \in g^{-1}(g(\Delta) \cap \partial \Omega). \]

\( \square \)

References


Received September 17, 2002.

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