Pacific Journal of Mathematics

ENDOCOHERENT MODULES

Lidia Angeleri-Hügel

Volume 212 No. 1 November 2003

PACIFIC JOURNAL OF MATHEMATICS
Vol. 212, No. 1, 2003

ENDOCOHERENT MODULES

Lidia Angeleri-Hügel

We study coherence properties of a module M over its endomorphism ring. Hereby we extend to modules known characterizations of coherent and π-coherent rings. Moreover, we discuss the case that the category add$M$ is covariantly, respectively contravariantly, finite in mod$R$. Finally, we give a new characterization of endofinite modules.

A left module ${}_SM$ over a ring S is **coherent** if it is finitely presented and every finitely generated submodule of ${}_SM$ is finitely presented. Inspired by Lenzing's and Camillo's work on a special class of coherent rings [15] and [7], we will further say that ${}_SM$ is **π-coherent** if it is finitely presented and every finitely generated left S-module which is cogenerated by ${}_SM$ is finitely presented. Then the ring S is left π-coherent in the sense of [7] if and only if the regular left module ${}_SS$ is π-coherent.

In this note, we consider the case that M *is a right module over a ring* R *with endomorphism ring* S and study coherence as well as π-coherence of ${}_SM$. We prove the following results which extend to modules known characterizations of coherent and π-coherent rings [15], [7], [18, 5.3] and [9, 5.1].

Theorem 1. *The following statements are equivalent:*

(1) *Every finitely generated left S-module which is cogenerated by ${}_SM$ is finitely presented.*
(2) *Every finitely M-generated right R-module has an* add M*-preenvelope.*
(3) *For every $n \in \mathbb{N}$ and every subset $X \subset M^n$ the annihilator $ann_{S^{n\times n}}(X)$ of X in the matrix ring $S^{n\times n}$ is a finitely generated left ideal.*

Theorem 2.

(1) *If ${}_SM$ is π-coherent, then every finitely generated module has an* add M*-preenvelope. The converse holds if M_R is finitely generated.*
(2) *If ${}_SM$ is coherent, then every finitely presented module has an* add M*-preenvelope. The converse holds if M_R is finitely presented.*

In particular, we see that a finitely presented module M_R is coherent over its endomorphism ring if and only if the category add M is covariantly finite in mod R. We also prove a dual result characterizing the case that add M

is contravariantly finite in $\operatorname{mod} R$ (Corollary 11). Finally, we employ our results to give a new characterization of endofinite modules (Corollary 9).

Let us start with some notation. For an arbitrary ring R, we write **Mod *R*** and **mod *R*** for the categories of all, respectively of the finitely presented, right R-modules. By a subcategory we always mean a full subcategory.

Let $\mathcal{X} \subset \operatorname{Mod} R$ and A be a right R-module. Following [**9**], we say that a homomorphism $a : A \to X$ is an **$\mathcal{X}$-preenvelope** if $X \in \mathcal{X}$ and the abelian group homomorphism $\operatorname{Hom}_R(a, X') : \operatorname{Hom}_R(X, X') \to \operatorname{Hom}_R(A, X')$ is surjective for each $X' \in \mathcal{X}$. A homomorphism $a : A \to X$ is said to be **left minimal** if every endomorphism $h : X \to X$ such that $h\,a = a$ is an isomorphism. Left minimal preenvelopes are called **envelopes** and are uniquely determined up to isomorphism. **(Pre)covers** are defined dually. In the representation theory of artin algebras, the usual terminology is (minimal) left or right $\mathcal{X}$-approximation.

Given a module M_R, we denote by **Add *M*** (respectively, **add *M***) the category consisting of all modules isomorphic to direct summands of (finite) direct sums of copies of M. Throughout the paper, we will freely use the fact that for a finitely generated module the existence of an $\operatorname{add} M$-preenvelope is equivalent to the existence of an $\operatorname{Add} M$-preenvelope.

If M_R is finitely presented, then $\operatorname{add} M$ is a subcategory of $\operatorname{mod} R$, and it is said to be **covariantly finite** in $\operatorname{mod} R$ if every finitely presented module has an $\operatorname{add} M$-preenvelope. Dually, one says that $\operatorname{add} M$ is **contravariantly finite** in $\operatorname{mod} R$ if every finitely presented module has an $\operatorname{add} M$-precover [**5**].

The following easy observation will be very useful:

Lemma 3. *Let R be a ring and M_R a module with endomorphism ring S.*

(1) *A_R has an $\operatorname{add} M$-preenvelope if and only if the left S-module ${}_S\operatorname{Hom}_R(A, M)$ is finitely generated.*

(2) *C_R has an $\operatorname{add} M$-precover if and only if the right S-module $\operatorname{Hom}_R(M, C)_S$ is finitely generated.*

Proof. (1) If ${}_S\operatorname{Hom}_R(A, M)$ is finitely generated, one can easily check that the map $c : A \to M^n$ induced by an S-generating set $c_k : A \to M$, $1 \leq k \leq n$, of $\operatorname{Hom}_R(A, M)$ is an $\operatorname{add} M$-preenvelope of A. Conversely, if $a : A \to X$ is an $\operatorname{add} M$-preenvelope, then we can assume w.l.o.g. that $X = M^n$ for some n, and applying the functor $\operatorname{Hom}_R(\ , M) : \operatorname{Mod} R \longrightarrow S\operatorname{Mod}$ on a, we immediately obtain the claim.

(2) is proven dually. □

The above lemma suggests that the existence of $\operatorname{add} M$-preenvelopes is related to the behaviour of the contravariant functor $\operatorname{Hom}_R(\ , M) : \operatorname{Mod} R \longrightarrow S\operatorname{Mod}$. We now investigate this connection more closely.

Let ${}_BQ_A$ be a bimodule. Recall that a module X_A is said to be Q_A-**reflexive** if the evaluation morphism $\boldsymbol{\delta}_X : X \to \operatorname{Hom}_B(\operatorname{Hom}_A(X, Q_A), {}_BQ)$ given by $\delta_X(x) : \alpha \mapsto \alpha(x)$ is an isomorphism. Of course, since $\operatorname{Ker}\delta_X$ coincides with the **reject** $\operatorname{Rej}_Q(X)$ of Q in X, all reflexive modules are in the category $\mathbf{Cogen}\,\boldsymbol{Q}$ of Q-cogenerated modules. We denote further by $\mathbf{cogen}\,\boldsymbol{Q}$ the category of all finitely Q-cogenerated modules, by $\mathbf{copres}\,\boldsymbol{Q}$ (respectively, by $\mathbf{sfcopres}\,\boldsymbol{Q}$) the category of all **finitely** (respectively, **semi-finitely**) $\boldsymbol{Q}$**-copresented modules**, that is, of all modules X admitting an exact sequence $0 \longrightarrow X \longrightarrow Q^n \longrightarrow L \longrightarrow 0$ where $n \in \mathbb{N}$ and L is finitely Q-cogenerated (respectively, Q-cogenerated). Dually, we write $\mathbf{gen}\,\boldsymbol{Q}$ for the category of all finitely Q-generated modules, and $\mathbf{pres}\,\boldsymbol{Q}$ for the category of all **finitely** $\boldsymbol{Q}$**-presented modules**, that is, of all modules X admitting an exact sequence $0 \longrightarrow K \longrightarrow Q^n \longrightarrow X \longrightarrow 0$ where $n \in \mathbb{N}$ and K is finitely Q-generated. Finally, we denote by $\boldsymbol{\mathcal{K}}(\boldsymbol{Q}_A)$ the subcategory of $\operatorname{Mod} A$ consisting of all modules K_A which admit an exact sequence $0 \longrightarrow K \longrightarrow A^n \longrightarrow Y_A \longrightarrow 0$ where $n \in \mathbb{N}$ and Y_A is Q_A-cogenerated, and by $\boldsymbol{\mathcal{K}}({}_B\boldsymbol{Q})$ the corresponding subcategory of $B\operatorname{Mod}$.

We are interested in the special case where Q is our bimodule ${}_SM_R$ with $S = \operatorname{End}_R M$. Then S is obviously ${}_SM$-reflexive, and we have the following result:

Lemma 4.

1. ${}_S\operatorname{Hom}_R(A, M) \in \operatorname{sfcopres}{}_SM$ *for all finitely generated modules* A_R, *and* ${}_S\operatorname{Hom}_R(A, M) \in \operatorname{copres}{}_SM$ *for all finitely presented modules* A_R.
2. *The functor* $\operatorname{Hom}_R(\ , M) : \operatorname{Mod} R \longrightarrow S\operatorname{Mod}$ *induces dense functors* $\operatorname{gen} M_R \longrightarrow \mathcal{K}({}_SM)$ *and* $\operatorname{pres} M_R \longrightarrow \operatorname{copres}{}_SS$.

Proof. (1) Let A_R be finitely generated with an exact sequence $0 \longrightarrow K \xrightarrow{f} R^n \longrightarrow A \longrightarrow 0$. We then have an exact sequence $0 \to {}_S\operatorname{Hom}_R(A, M) \to {}_S\operatorname{Hom}_R(R^n, M) \xrightarrow{\operatorname{Hom}_R(f,M)} {}_S\operatorname{Hom}_R(K, M)$ where ${}_S\operatorname{Hom}_R(K, M)$ is a submodule of ${}_S\operatorname{Hom}_R(R^{(J)}, M) \simeq {}_SM^J$ for some set J. Further, if A_R is finitely presented, then K is finitely generated, and ${}_S\operatorname{Hom}_R(K, M)$ is a submodule of ${}_S\operatorname{Hom}_R(R^m, M) \simeq {}_SM^m$ for some $m \in \mathbb{N}$.

(2) As in (1), we show that $A \in \operatorname{gen} M_R$ gives rise to an exact sequence $0 \to {}_S\operatorname{Hom}_R(A, M) \to {}_S\operatorname{Hom}_R(M^n, M) \to {}_S\operatorname{Hom}_R(K, M)$ where ${}_S\operatorname{Hom}_R(K, M)$ is ${}_SM$-cogenerated, and moreover, that we can assume ${}_S\operatorname{Hom}_R(K, M)$ finitely cogenerated by S provided that $A \in \operatorname{pres} M_R$. So, it remains to prove that the functors are dense. Any exact sequence $0 \longrightarrow K \longrightarrow S^n \longrightarrow {}_SY \longrightarrow 0$ with $Y \in \operatorname{Cogen}{}_SM$ yields an exact sequence $0 \to \operatorname{Hom}_S(Y, M) \to \operatorname{Hom}_S(S^n, M) \xrightarrow{g} \operatorname{Hom}_S(K, M)$ where $L_R = \operatorname{Im} g$ is an epimorphic image of M^n. We obtain the commutative diagram

$$\begin{array}{ccccccc}
0\to & K & \longrightarrow & S^n & \longrightarrow & Y & \to 0\\
& \downarrow\alpha & & \delta_{S^n}\downarrow & & \downarrow\delta_Y & \\
0\to & \mathrm{Hom}_R\,(L,M) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(S^n,M),M) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(Y,M),M) &
\end{array}$$

where α and δ_Y are monomorphisms and δ_{S^n} is an isomorphism. Then by the snake lemma α is an isomorphism, hence ${}_SK\cong \mathrm{Hom}_R\,(L,M)$ with $L\in \mathrm{gen}\,M_R$.

Assume further that there is a monomorphism $i:Y\to S^m$ for some $m\in\mathbb{N}$. Then we also have a map $f=\mathrm{Hom}_S\,(i,M):\mathrm{Hom}_S\,(S^m,M)\to \mathrm{Hom}_S\,(Y,M)$ with $A_R=\mathrm{Im}\,f\in \mathrm{gen}\,M_R$ and a commutative diagram

$$\begin{array}{ccccccc}
0\longrightarrow & A & \longrightarrow & \mathrm{Hom}_S\,(S^n,M) & \longrightarrow & L' & \longrightarrow 0\\
& \bigcap e & & \| & & \downarrow & \\
0\longrightarrow & \mathrm{Hom}_R\,(Y,M) & \longrightarrow & \mathrm{Hom}_S(S^n,M) & \longrightarrow & L & \longrightarrow 0
\end{array}$$

where $L'\in \mathrm{pres}\,M_R$. Since δ is a natural transformation, $\mathrm{Hom}_R\,(f,M)\,\delta_Y=\delta_{S^m}\,i$ is a monomorphism, and therefore $\mathrm{Hom}_R\,(e,M)\,\delta_Y$ is a monomorphism as well. So, we conclude as above from the commutative diagram

$$\begin{array}{ccccccc}
0\to & K & \longrightarrow & S^n & \longrightarrow & Y & \to 0\\
& \downarrow\alpha & & \delta_{S^n}\downarrow & & \downarrow\delta_Y & \\
0\to & \mathrm{Hom}_R\,(L,M) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(S^n,M),M) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(Y,M),M) & \\
& \downarrow\beta & & \| & & \downarrow \mathrm{Hom}_R\,(e,M) & \\
0\to & \mathrm{Hom}_R\,(L',M) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(S^n,M),M) & \longrightarrow & \mathrm{Hom}_R(A,M) &
\end{array}$$

that $\beta\alpha$ is an isomorphism, hence ${}_SK\cong \mathrm{Hom}_R\,(L',M)$ with $L'\in \mathrm{pres}\,M_R$. □

Let us remark that if ${}_SM_R$ is faithfully balanced, then by similar arguments, the functor $\mathrm{Hom}_R\,(\;,M):\mathrm{Mod}\,R\longrightarrow S\,\mathrm{Mod}$ induces dense functors $\mathrm{gen}\,R\longrightarrow \mathrm{sfcopres}\,{}_SM$ and $\mathrm{mod}\,R\longrightarrow \mathrm{copres}\,{}_SM$.

We now obtain a characterization of left coherent endomorphism rings, see also [**10**]. Moreover, we prove the equivalence of the first two conditions in Theorem 1.

Proposition 5.

(1) *S is left coherent if and only if every $A\in \mathrm{pres}\,M_R$ has an $\mathrm{add}\,M$-preenvelope.*

(2) *Every finitely generated left S-module which is cogenerated by ${}_SM$ is finitely presented if and only if every $A \in \operatorname{gen} M_R$ has an $\operatorname{add} M$-preenvelope.*

Proof. (1) Of course, S is left coherent if and only if every module in $\operatorname{copres}{}_SS$ is finitely generated over S. By Lemma 4 the latter means that ${}_S\mathrm{Hom}_R(A, M)$ is finitely generated for all modules $A_R \in \operatorname{pres} M$. Combining this with Lemma 3, we obtain the claim.

(2) is proven similarly. □

Note that Lenzing has described left coherence in terms of annihilators of matrix rings [**15**, §4, Korollar 1]. More precisely, denoting by $R^{n\times n}$ the $n \times n$ matrix ring over R, he has proven that R is left coherent if and only if for every $n \in \mathbb{N}$ and every $A \in R^{n\times n}$ the left annihilator of A in $R^{n\times n}$ is a finitely generated left ideal. Moreover, he has shown in [**15**, Satz 4] that R is left π-coherent if and only if for every $n \in \mathbb{N}$ all left annihilators in $R^{n\times n}$ are finitely generated left ideals, see also [**7**]. We now establish a corresponding result for modules and complete the proof of Theorems 1 and 2.

Proposition 6. *The following statements are equivalent:*

(1) *Every finitely generated left S-module which is cogenerated by ${}_SM$ is finitely presented.*

(2) *For every $n \in \mathbb{N}$ and every subset $X \subset M^n$ the annihilator $\operatorname{ann}_{S^{n\times n}}(X)$ of X in $S^{n\times n}$ is a finitely generated left ideal.*

Proof. (1)⇒(2): Let $X \subset M^n$, put $K = X \cdot R$ and $A_R = M^n/K$, and denote by $\nu : M^n \to A$ the canonical surjection. By assumption and the above proposition, $A_R \in \operatorname{gen} M$ has an $\operatorname{add} M$-preenvelope $a : A \to M^m$, and we can consider the maps $f_i : M^n \xrightarrow{\nu} A \xrightarrow{a} M^m \xrightarrow{\mathrm{pr}_i} M \xrightarrow{\iota} M^n$, $1 \le i \le m$, where pr_i and ι denote the canonical projections and a canonical injection, respectively. Obviously, $f_1, \ldots, f_m$ are contained in $\operatorname{ann}_{S^{n\times n}}(X)$, and since every other map $h \in \operatorname{ann}_{S^{n\times n}}(X)$ factors through ν and hence through $a\nu$, they are generators of $\operatorname{ann}_{S^{n\times n}}(X)$ over $S^{n\times n}$.

(2)⇒(1): We again apply Proposition 5 and show that every $A \in \operatorname{gen} M$ has an $\operatorname{add} M$-preenvelope. Consider an exact sequence $0 \longrightarrow K \longrightarrow M^n \xrightarrow{g} A \longrightarrow 0$ and a generating set $f_1, \ldots, f_m$ of $\operatorname{ann}_{S^{n\times n}}(K)$ over $S^{n\times n}$. Then K is contained in the kernel of the product map $f : M^n \to M^{nm}$ induced by the f_i, and so there is a map $a : A \to M^{nm}$ such that $f = a\,g$. Let us verify that a is an $\operatorname{add} M$-preenvelope. In fact, if we denote again by $M \xrightarrow{\iota} M^n$ a canonical injection, then for every homomorphism $h : A \to M$ the composition $\iota\,h\,g$ lies in $\operatorname{ann}_{S^{n\times n}}(K)$ and therefore has the form $\sum_{i=1}^m t_i\,f_i$ for some $t_1, \ldots, t_m \in S^{n\times n}$. This shows that $h\,g$ factors through $a\,g$, and hence h factors through a. □

Proof of Theorem 2. (1) If A_R is finitely generated, then by Lemma 4 there is an exact sequence $0 \longrightarrow {}_S\mathrm{Hom}_R\,(A, M) \longrightarrow {}_SM^n \longrightarrow L \longrightarrow 0$ where $n \in \mathbb{N}$ and $L \in \mathrm{Cogen}\,{}_SM$. By assumption L is then finitely generated and even finitely presented, so ${}_S\mathrm{Hom}_R\,(A, M)$ is finitely generated, and A has an $\mathrm{add}\,M$-preenvelope by Lemma 3. Conversely, if M_R is finitely generated and every finitely generated module has an $\mathrm{add}\,M$-preenvelope, then we deduce that R and every $A \in \mathrm{gen}\,M$ have an $\mathrm{add}\,M$-preenvelope. But this implies by Lemma 3 and Proposition 5(2) that ${}_SM$ is π-coherent.

(2) We show as in (1) that Lemma 4 and Lemma 3 yield the existence of an $\mathrm{add}\,M$-preenvelope for every finitely presented module A_R. Conversely, if M_R is finitely presented and every finitely presented module has an $\mathrm{add}\,M$-preenvelope, then we deduce that R and every $A \in \mathrm{pres}\,M$ have an $\mathrm{add}\,M$-preenvelope. In particular, S is then left coherent by Proposition 5(1). Moreover, if $a : R \to M^n$ is an $\mathrm{add}\,M$-preenvelope with cokernel L, then also L_R is finitely presented, and therefore ${}_S\mathrm{Hom}_R\,(L, M)$ is finitely generated by Lemma 3. So, we infer from the exact sequence $0 \longrightarrow {}_S\mathrm{Hom}_R\,(L, M) \longrightarrow {}_S\mathrm{Hom}_R\,(M^n, M) \longrightarrow {}_S\mathrm{Hom}_R\,(R, M) \longrightarrow 0$ that ${}_SM$ is finitely presented and hence coherent. □

Assume that R is **semiregular**, that is, idempotents lift modulo the Jacobson radical $J(R)$ and $R/J(R)$ is von Neumann regular. Then we know from [**3**, Corollary 3] and [**18**, Corollary 5.4] that R being left (π-)coherent even implies the existence of projective envelopes for the finitely presented (respectively, finitely generated) modules. Also these results can be extended to modules.

Corollary 7. *Let S be semiregular.*

1. *If ${}_SM$ is π-coherent, then every finitely generated module has an* $\mathrm{add}\,M$*-envelope.*
2. *If ${}_SM$ is coherent and M_R is finitely presented, then every finitely presented module has an* $\mathrm{add}\,M$*-envelope.*

Proof. From Theorem 2 we obtain the existence of an $\mathrm{add}\,M$-preenvelope $f : A \to M^n$ with A finitely generated or finitely presented, respectively. Note that in both cases the cokernel $L = \mathrm{Coker}\,f$ has an $\mathrm{add}\,M$-preenvelope $g : L \to M^m$, too. Indeed, in Case (1) this follows from Proposition 5(2) and the fact that $L \in \mathrm{gen}\,M$, and in Case (2) we have only to remind that M_R, and therefore also L_R, are finitely presented. Set $E = \mathrm{End}_R\,M^n$. From the exact sequence ${}_E\mathrm{Hom}_R\,(M^m, M^n) \longrightarrow {}_EE \xrightarrow{\mathrm{Hom}_R\,(f, M^n)} {}_E\mathrm{Hom}_R\,(A, M^n) \longrightarrow 0$ we deduce that the annihilator $\mathrm{ann}_E(f)$ is a finitely generated left ideal of E. Since E is semiregular by [**16**, 2.7], we know from [**17**, Satz 1.2] that there is a left ideal $\mathcal{I}$ which satisfies $\mathrm{ann}_E(f) + \mathcal{I} = E$ and is minimal with respect to this property. Then $\mathrm{ann}_E(f) \cap \mathcal{I}$ is superfluous in $\mathcal{I}$ and therefore also in E. So, we have verified that:

(i) There is a left ideal $\mathcal{I}$ in E such that $\mathrm{ann}_E(f)+\mathcal{I}=E$ and $\mathrm{ann}_E(f)\cap\mathcal{I}\subset J(E)$; and
(ii) idempotents lift modulo $J(E)$.

Thus we can apply a result of Zimmermann [21] asserting that under these conditions f has a left minimal version, that is, there is a decomposition $M^n = X\oplus K$ such that the composition of f with the canonical projection $p: M^n \to X$ gives rise to an add M-envelope. □

Let us now compare different notions of coherence. Recall that a ring R is said to be **left strongly coherent** if products of projective right R-modules are locally projective [19] and [11]. Such rings are characterized by the property that every matrix subgroup of the right module R_R is a finitely generated left ideal. Moreover, as observed in [20], they are always left π-coherent.

More generally, if M_R is a finitely generated module with all matrix subgroups being finitely generated over the endomorphism ring S, then we can prove as in [2, 3.1] that every finitely generated module has an add M-preenvelope, and so it follows immediately from Theorem 2 that ${}_SM$ is π-coherent and in particular coherent.

Examples for the failure of the converse implications even in the case $M = R$ are given in [20, Example 29], [11, Example 5.2] and [7]. In particular, every commutative von Neumann regular ring which is not self-injective is coherent but not π-coherent, and the ring $R = K[X_1, X_2, \dots]$ over a field K is π-coherent but not strongly coherent.

Next, we investigate the gap between π-coherence and coherence. To this end, we recall the notion of an R-Mittag-Leffler (or finitely pure-projective) module studied in [12], [8], [13] and [6]. A module X_R is said to be an ***R*-Mittag-Leffler module** if the canonical map $X\otimes_R R^J \to X^J$ is a monomorphism for every set J, or equivalently, if for every finitely generated submodule A_R the embedding $A\subset X$ factors through a finitely presented module. Jones showed in [13, p. 104] that a ring is left π-coherent if and only if it is left coherent and all products of copies of R (on either side) are R-Mittag-Leffler modules. Note that since the class of R-Mittag-Leffler modules is closed under pure submodules [6, Proposition 9], the latter property amounts to saying that all products of projective modules are R-Mittag-Leffler modules. We now prove the general statement for modules.

Corollary 8. *The following statements are equivalent:*

(1) ${}_SM$ *is π-coherent.*
(2) S *is left (π-)coherent,* ${}_SM$ *is finitely presented, and all products of copies of* ${}_SM$ *are S-Mittag-Leffler modules.*

If M_R is finitely presented, the following statement is further equivalent:

(3) *S is left (π-)coherent, ${}_SM$ is finitely presented, and all products of copies of M_R are R-Mittag-Leffler modules.*

Proof. (1)⇒(2): Any epimorphism $R^{(K)} \to M$ gives rise to a monomorphism ${}_SS \simeq \mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(R^{(K)}, M) \simeq {}_SM^K$, showing that S is left (π-)coherent. Moreover, all finitely generated submodules of products of copies of ${}_SM$ are finitely presented by definition, and so the claim is proven.

(2)⇒(1): Let ${}_SA$ be a finitely generated submodule of a product of copies of ${}_SM$. By assumption, ${}_SA$ is contained in a finitely presented module ${}_SY$, which is coherent since so is the ring S. Hence ${}_SA$ is finitely presented, and we have verified that ${}_SM$ is π-coherent.

(1)⇒(3): Let A_R be a finitely generated submodule of M^J for some set J. By Theorem 2, the embedding $A \subset M^J$ factors through an $\mathrm{add}\, M$-preenvelope $A \to M^n$, and M^n is finitely presented if so is M_R.

(3)⇒(1): We claim that every finitely generated module has an $\mathrm{add}\, M$-preenvelope. The claim then follows from Theorem 2 whenever M_R is finitely generated. So, let A_R be finitely generated. By possibly considering $A/\mathrm{Rej}_M(A)$, we can assume without loss of generality that A is M-cogenerated. Then the product map $f : A \to M^J$ induced by all maps in $J = \mathrm{Hom}_R(A, M)$ is a monomorphism and therefore factors through a homomorphism $f' : A \to F$ where F is finitely presented. But since ${}_SM$ is coherent by assumption, we obtain from Theorem 2 the existence of an $\mathrm{add}\, M$-preenvelope $a : F \to M^n$. Now it is easy to check that the composition $a\, f' : A \to M^n$ is an $\mathrm{add}\, M$-preenvelope as well. □

Here is a further application of Theorem 2. Recall that M is said to be **endonoetherian**, respectively **endofinite**, if ${}_SM$ is noetherian, respectively a module of finite length. We will moreover call M **endocoherent** if ${}_SM$ is coherent, and **endocoperfect** if it satisfies the descending chain condition for cyclic S-submodules. We explore the relationship between these finiteness conditions over the endomorphism ring.

Corollary 9. *The following statements are equivalent:*

(1) *M is endofinite.*

(2) *M is endocoperfect, and for all direct summands M' of M and all finitely presented modules A_R, there exists an* $\mathrm{add}\, M'$*-preenvelope.*

If M_R is finitely generated, then (1) is further equivalent to:

(3) *M is endocoperfect and all its direct summands are endocoherent.*

Proof. (1)⇔(2): Assume that M is endofinite. Then M is Σ-pure-injective and therefore satisfies the descending chain condition for cyclic S-submodules. Moreover, M is endonoetherian, and it is well-known that its direct summands are then endonoetherian as well. Now, we have shown in [**2**, 3.1] that all finitely presented modules A_R have an $\mathrm{add}\, M'$-preenvelope if and only if certain endo-submodules of M', namely the finite matrix subgroups, are finitely generated over $\mathrm{End}_R M'$. Thus (1) implies (2). For the converse implication, we use that M is endofinite if and only if every direct summand of M is product-complete [**14**]. Observe that by [**2**, 5.1] a module M' is product-complete if and only if it is endocoperfect and all finite matrix subgroups of M' are finitely generated over $\mathrm{End}_R M'$. Since endocoperfectness is inherited to direct summands, we have verified (2)⇒(1).

(3)⇒(2) follows immediately from Theorem 2.

(1)⇒(3): The direct summands of M are finitely generated and endonoetherian, so their endomorphism rings are left noetherian. Thus they are also endocoherent. □

We close the paper with some dual considerations. We have seen above that the existence of $\mathrm{add}\, M$-preenvelopes is related to coherence properties of ${}_SM$. Dually, we can describe the existence of $\mathrm{add}\, M$-precovers in terms of coherence properties of the dual module $\boldsymbol{M^*}_S = \mathrm{Hom}_R(M, W)_S$, where $\boldsymbol{W_R}$ denotes a minimal injective cogenerator of $\mathrm{Mod}\, R$. We refer to [**1**] for details and only mention the main results.

Theorem 10.

(1) *If M^*_S is π-coherent, then every finitely W-cogenerated module has an* $\mathrm{add}\, M$*-precover. The converse holds if M_R is finitely W-cogenerated.*

(2) *If M^*_S is coherent, then every finitely W-copresented module has an* $\mathrm{add}\, M$*-precover. The converse holds if M_R is finitely W-copresented.*

If R is **a right Morita ring**, that is, if R is a right artinian ring and W_R is finitely generated, then we obtain a characterization of contravariantly finiteness. This and other consequences are collected in the following corollary. Observe that the last statement generalizes a result proven by Auslander for finitely generated projective modules [**4**, 6.6].

Corollary 11.

(1) *Assume that M is a finitely generated module over a right Morita ring R. Then M^*_S is (π-)coherent if and only if* $\mathrm{add}\, M$ *is contravariantly finite in* $\mathrm{mod}\, R$.

(2) *Assume that M_R is a finitely generated module over a right noetherian ring R. If* $\mathrm{add}\, M$ *is contravariantly finite in* $\mathrm{mod}\, R$*, then every finitely generated right S-module which is cogenerated by M^*_S is finitely presented. In particular, S is then a right π-coherent ring.*

(3) *Assume that M_R is a coherent module. If all finitely generated modules have an* $\mathrm{add}\, M$*-precover, then S is a right coherent ring.*

Proof. (1) By assumption every finitely generated module is finitely W-copresented and therefore has an $\mathrm{add}\, M$-precover provided that M^*_S is coherent. Conversely, assume that $\mathrm{add}\, M$ is contravariantly finite in $\mathrm{mod}\, R$. Then every finitely W-cogenerated module, being finitely presented by assumption, has an add M-precover. Moreover, the finitely generated module M_R is finitely W-cogenerated, and we conclude from Theorem 10 that M^*_S is π-coherent.

(2) Under the given assumptions, all modules in cogen M are finitely presented and therefore have an add M-precover whenever add M is contravariantly finite in mod R. The claim then follows from the dual version of Proposition 5(2). That S is right π-coherent follows from the fact that S_S is M^*_S-cogenerated.

(3) Under the given assumption, all modules in copres M are finitely generated and therefore have an add M-precover. The claim then follows from the dual version of Proposition 5(1), see also [**10**]. □

References

[1] L. Angeleri-Hügel, *On Some Precovers and Preenvelopes*, Habilitationsschrift, München, 2000.

[2] ———, *Covers and envelopes via endoproperties of modules*, Proc. London Math. Soc., **86** (2003), 649-665, CMP 1 974 393.

[3] J. Asensio Mayor and J.M. Hernandez, *Flat envelopes in commutative rings*, Israel J. Math., **62** (1988), 123-128, MR 89i:13016, Zbl 0667.13005.

[4] M. Auslander, *Representation theory of Artin algebras* I, Comm. Algebra, **1** (1974), 177-268, MR 50 #2240, Zbl 0285.16028.

[5] M. Auslander and S.O. Smalø, *Preprojective modules over Artin algebras*, J. Algebra, **66** (1980), 61-122, MR 83a:16039, Zbl 0477.16013.

[6] G. Azumaya, *Finite splitness and finite projectivity*, J. Algebra, **106** (1987), 114-134, MR 89a:16034, Zbl 0607.16017.

[7] V. Camillo, *Coherence for polynomial rings*, J. Algebra, **132** (1990), 72-76, MR 91c:16018, Zbl 0701.16023.

[8] T.G. Clarke, *On $\aleph_1$-Projective Modules*, Ph.D. Thesis, Kent State University, 1976.

[9] E.E. Enochs, *Injective and flat covers, envelopes and resolvents*, Israel J. Math., **39** (1981), 189-209, MR 83a:16031, Zbl 0464.16019.

[10] J.L. García, J.M. Hernández and P.L.G. Sánchez, *Endomorphism rings and the category of finitely generated projective modules*, Comm. Algebra, **28** (2000), 3837-3852, MR 2001e:16049, Zbl 0958.16007.

[11] G.S. Garfinkel, *Universally torsionless and trace modules*, Trans. Amer. Math. Soc., **215** (1976), 119-144, MR 53 #8136, Zbl 0319.16020.

[12] K.R. Goodearl, *Distributing tensor product over direct product*, Pacific J. Math., **43** (1972), 107-110, MR 47 #276, Zbl 0237.13007.

[13] M.F. Jones, *Flatness and f-projectivity of torsion-free modules and injective modules*, in 'Advances of Non-commutative Ring Theory', Lecture Notes in Math., **951** (1982), 94-116, Zbl 0498.16020.

[14] H. Krause and M. Saorín, *On minimal approximations of modules*, in 'Trends in the representation theory of finite dimensional algebras' (ed. by E.L. Green and

B. Huisgen-Zimmermann), Contemp. Math., **229** (1998), 227-236, MR 99m:16002, Zbl 0959.16003.

[15] H. Lenzing, *Kohärente Endomorphismenringe und Produkte projektiver Moduln*, unpublished manuscript, 1970.

[16] W.K. Nicholson, *Semiregular modules and rings*, Can. J. Math., **XXVIII** (1976), 1105-1120, MR 54 #10333, Zbl 0317.16005.

[17] U. Oberst and H.J. Schneider, *Die Struktur von projektiven Moduln*, Invent. Math., **13** (1971), 295-304, MR 45 #319, Zbl 0232.16020.

[18] J. Rada and M. Saorín, *Rings characterized by (pre)envelopes and (pre)covers of their modules*, Comm. Algebra, **26** (1998), 899-912, MR 99f:16002, Zbl 0908.16003.

[19] B. Zimmermann-Huisgen, *Pure submodules of direct products of free modules*, Math. Ann., **224** (1976), 233-245, MR 54 #12835, Zbl 0321.16015.

[20] ———, *Direct Products of Modules and Algebraic Compactness*, Habilitationsschrift, Technische Universität München, 1980.

[21] W. Zimmermann, Lectures, University Munich, 1998.

Received April 22, 2002. This research supported by a HWP-grant of the University of Munich.

Mathematisches Institut der Universität
Theresienstrasse 39
D-80333 München
Germany
E-mail address: angeleri@rz.mathematik.uni-muenchen.de

Dipartimento di Informatica e Comunicazione
Universita' dell'Insubria
via Mazzini 5
I-21100 Varese
Italy