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In this paper we give some characterizations of M. Hamana's injective envelope $I(A)$ of a C^*-algebra A in the setting of operator spaces and completely bounded maps. These characterizations lead to simplifications and generalizations of some known results concerning completely bounded projections onto C^*-algebras. We prove that $I(A)$ is rigid for completely bounded A-module maps. This rigidity yields a natural representation of many kinds of multipliers as multiplications by elements of $I(A)$. In particular, we prove that the(n times iterated) local multiplier algebra of A embeds into $I(A)$.

1. Introduction.

Let A denote a unital C^*-algebra. M. Hamana [**13**, **14**, **16**] introduced the injective envelope of A, $I(A)$, as a "minimal" injective operator system containing A and established various characterizations and properties of $I(A)$ in the setting of completely positive mappings and operator systems.

In recent years attention has shifted from completely positive maps and operator systems to completely bounded maps, operator algebras and operator spaces. In particular a theory has evolved of operator spaces that are completely contractive as modules over operator algebras. See for example, [**3**, **20**].

This theory gives a new categorical framework where one can examine injective envelopes. While other author's have pursued this viewpoint they have generally defined injectivity and rigidity in terms of completely contractive maps. For example, defining injectivity by requiring completely contractive maps to have completely contractive extensions. This is equivalent to requiring that completely bounded maps have completely bounded extensions of the same completely bounded norm. Since unital completely contractive maps on C^*-algebras are completely positive this approach generally reduces to M. Hamana's results in the C^*-algebra setting.

Our approach is different in that we are interested in a setting where our objects are A-modules and injectivity is defined by requiring that completely bounded A-module maps have completely bounded A-module extensions, but not necessarily of the same norm. We show for example that, as well as being a minimal injective operator system containing A, that $I(A)$ is in a certain sense the "minimal" injective left operator A-module containing A. This immutability of the "injective hull" of C^*-algebras under change of category has some immediate applications to completely bounded projections and multiplier algebras.

In this paper our primary focus is on the new properties of $I(A)$ and their applications. For this reason we take the quickest approach, which is to restrict our attention to the unital case and use M. Hamana's results to deduce these new properties by our off-diagonalization trick.

Many of our results do carry over to the case of a non-unital C^*-algebra B by the simple device of adjoining a unity to B, letting A denote this unital C^*-algebra and observing that any B-modules are automatically A-modules. For a greater development of the non-unital case we refer the reader to the subsequent paper [**4**].

2. Mapping properties of $I(A)$.

Throughout this section A will denote a unital C^*-algebra and $I(A)$ will denote its injective envelope as defined in [**13**, Def. 2.1, Th. 4.1]. We assume that the reader is familiar with the definitions and elementary properties of completely bounded and completely positive maps as presented in [**20**] or [**23**].

One of M. Hamana's fundamental results about $I(A)$ was his rigidity theorem. This theorem says that if $\varphi : I(A) \to I(A)$ is completely positive with $\varphi(a) = a$ for all a in A then $\varphi(x) = x$ for all x in $I(A)$.

A direct analog of this result is false for general completely bounded maps. If $A \neq I(A)$, then there exists a nonzero bounded linear functional $f : I(A) \to \mathbb{C}$ with $f(A) = \{0\}$. Defining the map $\varphi : I(A) \to I(A)$ via $\varphi(x) = x + f(x)1$ yields a completely bounded map with $\varphi(a) = a$ for all a in A, but $\varphi(x) \neq x$ for all x in $I(A)$. However, if one recalls that completely positive maps that fix A are automatically A-bimodule maps [**6**], [**20**, Exercise 4.3] then one is led to the appropriate generalization of M. Hamana's rigidity. Surprisingly one does not need bimodules, only left or right A-modules as the following results show.

Theorem 2.1. *Let $E \subseteq I(A)$ be a subspace such that $AE \subseteq E$ (respectively, $EA \subseteq E$) and let $\varphi : E \to I(A)$ be a completely bounded left (resp., right) A-module map. Then there exists an element y in $I(A)$ such that φ is right (resp., left) multiplication by y, i.e., $\varphi(e) = ey$ ($\varphi(e) = ye$) for all e in E and $\|y\| = \|\varphi\|_{cb}$. When $AEA \subseteq E$ and φ is a bimodule map, then y may be taken in the center of $I(A)$.*

In particular, φ extends to a completely bounded, left (resp., right) A-module map $\psi : I(A) \to I(A)$ such that $\psi|_E = \varphi$ and $\|\varphi\|_{cb} = \|\psi\|_{cb}$. If

$E \subseteq I(A)$ contains an invertible element of $I(A)$, then the element $y \in I(A)$ and consequently the extension ψ are unique.

Proof. It will suffice to assume that $\|\varphi\|_{cb} \leq 1$. Let $\mathcal{S} \subseteq M_2(I(A))$ be defined as

$$\mathcal{S} = \left\{ \begin{pmatrix} a & e \\ f^* & \lambda \end{pmatrix} : a \in A, e, f, \in E, \lambda \in \mathbb{C} \right\}$$

and $\Phi : \mathcal{S} \to M_2(I(A))$ by

$$\Phi\left(\begin{pmatrix} a & e \\ f^* & \lambda \end{pmatrix} \right) = \begin{pmatrix} a & \varphi(e) \\ \varphi(f)^* & \lambda \end{pmatrix}.$$

Arguing as in [**20**] or [**28**] one sees that Φ is completely positive and hence can be extended to a completely positive map on all of $M_2(I(A))$ which we still denote by Φ. Using the fact that Φ fixes $A \oplus \mathbb{C} = \left\{ \begin{pmatrix} a & 0 \\ 0 & \lambda \end{pmatrix} : a \in A, \lambda \in \mathbb{C} \right\}$ and again arguing as in [**28**], we see that there exists $\varphi_i : I(A) \to I(A), i = 1, 2, 3, 4$ such that

$$\Phi\left(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right) = \begin{pmatrix} \varphi_1(x_1) & \varphi_2(x_2) \\ \varphi_3(x_3) & \varphi_4(x_4) \end{pmatrix}.$$

Clearly, φ_1 and φ_4 must be completely positive and φ_2 extends φ.

Since $\varphi_1(a) = a$ for all a in A, by M. Hamana's rigidity result $\varphi_1(x) = x$ for all x in $I(A)$. Thus, Φ fixes the C^*-subalgebra, $I(A) \oplus \mathbb{C}$ and so by [**6**] (see also [**20**]) Φ must be a bimodule map over this algebra. Thus, $\begin{pmatrix} 0 & \varphi_2(x) \\ 0 & 0 \end{pmatrix} = \Phi\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) = \Phi\left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \varphi_2(1) \\ 0 & 0 \end{pmatrix}$ and we have that $\varphi_2(x) = x \cdot \varphi_2(1)$. Finally, $\|\varphi\|_{cb} = \|\varphi_2\|_{cb} = \|\varphi_2(1)\|$.

The proof for right A-module maps is similar. For the bimodule case let $\mathcal{S} = \left\{ \begin{pmatrix} a & e \\ f^* & b \end{pmatrix} : a, b \in A, e, f \in E \right\}$ and deduce that φ_2 is an $I(A)$-bimodule map. If E contains an invertible element e, then $y = e^{-1}\varphi(e)$ and so y is unique. ☐

In particular, the above results show that every completely bounded left (resp., right or bi-) A-module map of A into $I(A)$ admits a unique extension to a completely bounded left (resp., right or bi-) A-module map of $I(A)$ into itself and this extension has the same completely bounded norm.

Corollary 2.2 (Rigidity)**.** *Let A be a unital C^*-algebra and $I(A)$ be its injective envelope C^*-algebra. Let E be a subspace of $I(A)$ with $A \subseteq E$ and $AE \subseteq E$ (respectively, $EA \subseteq E$) and let $\varphi : E \to I(A)$ be a completely bounded left (resp., right) A-module map. If $\varphi(a) = a$ for all a in A, then $\varphi(e) = e$ for all e in E.*

Proof. There exists y in $I(A)$ with $\varphi(e) = e \cdot y$ for all e in E. Since $\varphi(1) = 1$, $y = 1$ and hence, $\varphi(e) = e$ for all e in E. ☐

3. Injective bimodule extensions of C*-algebras A and the injective envelope $I(A)$.

Let A and B be unital C^*-algebras. Recall the definition of *operator A-B-bimodules.* These are operator spaces E which are A-B-bimodules and such that the trilinear module pairing $A \times E \times B \to E,\ (a, e, b) \to aeb$ is completely contractive in the sense of E. Christensen and A. Sinclair [**7**]. This is equivalent to requiring that for matrices $(a_{i,j}), (e_{i,j})$ and $(b_{i,j})$ of the appropriate sizes, the induced matricial module product is contractive, i.e.,

$$\left\| \left(\sum_{k,m} a_{i,k} e_{k,m} b_{m,j} \right) \right\| \leq \|(a_{i,j})\| \|(e_{i,j})\| \|(b_{i,j})\| .$$

These are the objects of the category ${}_A\mathcal{O}_B$, [**3**], [**21**] and the morphisms between two operator A-B-bimodules in this category are the completely bounded A-B-bimodules maps. When we want to restrict the morphisms to be completely contractive A-B-bimodule maps we will denote the category by ${}_A\mathcal{O}^1_B$.

We assume that all module actions are unital, i.e., $1 \cdot e \cdot 1 = e$. We set ${}_A\mathcal{O} = {}_A\mathcal{O}_{\mathbb{C}}$ and call these *left operator A-modules*, and $\mathcal{O}_A = {}_{\mathbb{C}}\mathcal{O}_A$ and call these *right operator A-modules.*

Definition 1. An operator A-B-bimodule I is *A-B-injective*, if whenever $E \subseteq F$ are operator A-B-bimodules, then every completely bounded A-B-bimodule map from E into I has a completely bounded A-B-bimodule extension to F. Note that we do *not* require that the *cb*-norm of the extension is the same as the *cb*-norm of the original map. When this is the case we will call I a *tight A-B-injective A-B-bimodule.*

Some comments on terminology are helpful. Our definition of A-B-injective is the usual definition of injectivity in the category ${}_A\mathcal{O}_B$, while what we are calling tight A-B-injective is the corresponding definition of injectivity in the category ${}_A\mathcal{O}^1_B$.

If A and B are both C^*-subalgebras of $B(H)$, then by the bimodule version of G. Wittstock's extension theorem [**29**, Thm. 4.1] (see also [**28**]) $B(H)$ is a tight A-B-injective. Thus, if $M \subseteq B(H)$ is the range of a completely bounded projection $\varphi : B(H) \to M$, which is also an A-B-bimodule map, then M is A-B-injective, but it is not evidently tight A-B-injective. A C^*-subalgebra $I \subseteq B(H)$ is generally called "injective" if it is the range of a completely positive projection. This term is so widespread we continue to use it here. Note that such a map is also automatically an I-bimodule map. Thus, such an I is a tight A-B-injective for any C^*-subalgebras A and B of I.

In particular, M. Hamana's injective envelope $I(A)$ is a tight A-A-injective A-A-bimodule, a tight A-$\mathbb{C}$-injective left A-module and a tight $\mathbb{C}$-A-injective right A-module.

On the other hand there are many $\mathbb{C}$-$\mathbb{C}$-injectives which are not tight, i.e., not injective in the usual sense. For example, for any subspace E of $B(H)$ of finite codimension, it is easy to show that there exists a completely bounded projection from $B(H)$ onto E and hence E is $\mathbb{C}$-$\mathbb{C}$-injective.

T. Huruya [**17**] has given an example of a unital C^*-subalgebra of an injective C^*-algebra of finite codimension that is not injective. By the above argument, this algebra is the range of a completely bounded projection and hence is $\mathbb{C}$-$\mathbb{C}$-injective. Thus there exist C^*-algebras that are $\mathbb{C}$-$\mathbb{C}$-injective, but are not injective in the usual sense.

In our terminology, M. Hamana's rigidity result implies that if $A \subseteq E \subseteq I(A)$ and E is a tight $\mathbb{C}$-$\mathbb{C}$-injective, then $E = I(A)$. We prove this fact in the remarks following the theorem.

Theorem 3.1. *Let A be a unital C^*-algebra and let $A \subseteq E \subseteq I(A)$. Then the following are equivalent:*

a) *E is A-$\mathbb{C}$-injective,*
b) *E is $\mathbb{C}$-A-injective,*
c) *E is A-A-injective,*
d) *$E = I(A)$.*

Proof. By the Hahn-Banach extension theorem for completely bounded A-B-bimodule maps [**29**, Thm. 4.1] (see also [**28**], [**20**]), it follows that $I(A)$ is A-$\mathbb{C}$-injective, $\mathbb{C}$-A-injective and A-A-injective. Thus, d) implies a), b) and c). It will suffice to prove that a) implies d), the other implications are similar.

If E is A-$\mathbb{C}$-injective then the identity map from E to E extends to a completely bounded left A-module projection from $I(A)$ to E. Letting $I(A)$ play the role of E in the rigidity theorem yields the result. $\square$

The module actions are necessary in the above theorem. Since there always exists a completely bounded projection from any operator space onto a subspace of finite codimension, if $A \subseteq E \subseteq I(A)$ with E a subspace of finite codimension then E is $\mathbb{C}$-$\mathbb{C}$-injective but $E \neq I(A)$.

On the other hand, if we required E to be tight, then there would exist a completely contractive projection φ onto E. Since 1 belongs to E we would have $\varphi(1) = 1$ and, consequently, this projection would be completely positive. Hence E would be an operator system. Thus, $E = I(A)$ by M. Hamana's rigidity theorem and we would be adding nothing new.

We now are in a position to clarify the relationship between these new notions of injectivity and injectivity in the usual sense for C*-algebras.

Theorem 3.2. *Let A be a unital C^*-algebra. Then the following are equivalent:*

a) *A is an injective C^*-algebra (in the usual sense),*
b) *A is a tight A-$\mathbb{C}$-injective module,*
c) *A is a A-$\mathbb{C}$-injective module,*
d) *A is a tight $\mathbb{C}$-A-injective module,*
e) *A is a $\mathbb{C}$-A-injective module,*
f) *A is a tight A-A-injective module,*
g) *A is a A-A-injective module.*

Proof. We prove the equivalence of a), b) and c), the remaining arguments are similar. We have that a) implies b) by Wittstock's Hahn-Banach extension theorem for module maps. Clearly, b) implies c). We now prove that c) implies a). Since A is a A-$\mathbb{C}$-injective module, the identity map on A extends to a completely bounded left A-module map from $I(A)$ into A. But by the Rigidity Theorem, this extended map must be the identity map on $I(A)$ and hence $I(A) = A$. Thus, A is injective. $\square$

Definition 2. Let M be an operator A-B-bimodule. Call I a *minimal A-B-injective extension* of M, if $M \subseteq I$ and $M \subseteq E \subseteq I$ with E $A-B$-injective implies $E = I$.

By Theorem 3.1, $I(A)$ is a minimal A-$\mathbb{C}$-injective extension of A and also a minimal $\mathbb{C}$-A and A-A injective extension of A.

We call a map φ a *completely bounded isomorphism* if both φ and φ^{-1} are completely bounded.

Theorem 3.3. *Let A be a unital C^*-algebra and let I be a minimal A-$\mathbb{C}$-injective extension of A, then there exists a completely bounded left A-module isomorphism $\varphi : I(A) \to I$ with $\varphi(a) = a$ for all a in A. If we require that I is also a tight A-$\mathbb{C}$-injective then φ may be taken to be a complete isometry. Analogous statements hold for right modules and bimodules.*

Proof. Since I and $I(A)$ are A-$\mathbb{C}$-injective, there exist completely bounded left A-module maps $\varphi : I(A) \to I$ and $\psi : I \to I(A)$ which fix A. By the rigidity of $I(A)$, $\psi \circ \varphi$ is the identity on $I(A)$ and hence $\varphi \circ \psi$ is the identity restricted to $E =$ range (φ). This makes E an A-$\mathbb{C}$-injective module and hence $E = I$ and $\psi = \varphi^{-1}$. $\square$

If we required I to be tight and only minimal among all tight injectives, then as in the remark following Theorem 3.1, our result would reduce to M. Hamana's theory.

We now turn to some applications to projections. In [**5**] it was shown that if $M \subseteq B(H)$ is a von Neumann algebra and there exists a bounded M-bimodule projection, $\varphi : B(H) \to M$, then M is injective. Such a map φ is easily shown to be automatically completely bounded.

In [**21**] the same result was shown to hold for C^*-algebras. The above results on injective envelopes allow us to extend these results a bit. Perhaps, more importantly, the new proof is much simpler than the proof in [**21**].

Theorem 3.4. *Let $A \subseteq B(H)$ be a unital C^*-algebra. If there exists a completely bounded left (or right) A-module projection of $B(H)$ onto A, then A is injective.*

Proof. Since $B(H)$ is A-A-injective the identity map on A extends to a completely bounded A-bimodule map from $I(A)$ to $B(H)$. Composing with the projection onto A gives a completely bounded left A-module map from $I(A)$ to A which is the identity on A. By rigidity (Corollary 2.2) $I(A) = A$ and hence A is injective. □

The following example gives an indication of the obstacles that arise in attempting to generalize Theorem 2.1, Corollary 2.2 and Theorem 3.4 to the case of non-involutive operator algebras in $B(H)$. Consider the operator algebra $A \subset M_2(\mathbb{C})$ defined by

$$A = \left\{ X \in M_2(\mathbb{C}) \,:\, X = S^{-1} \cdot \mathrm{diag}(a, b) \cdot S \,,\, a, b \in \mathbb{C} \right\}, \quad S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

If $\omega : M_2(\mathbb{C}) \to \mathbb{C} \oplus \mathbb{C} \subset M_2(\mathbb{C})$ is the canonical conditional expectation on $M_2(\mathbb{C})$ preserving the diagonal and mapping off-diagonal elements to zero, then the map $\phi : M_2(\mathbb{C}) \to A$ defined by the rule $\phi(X) = S^{-1} \cdot \omega(SXS^{-1}) \cdot S$ is a completely bounded A-bimodule projection. However, the smallest C^*-subalgebra of $M_2(\mathbb{C})$ generated by A is $M_2(\mathbb{C})$ itself, and since $M_2(\mathbb{C})$ is injective we obtain $I(A + A^*) = M_2(\mathbb{C})$. Consequently, Theorem 2.1 and Corollary 2.2 cannot be extended to this situation, and Theorem 3.4 is not true for the described operator algebra A.

In [**8**] and [**24**] it was proven that if $M \subseteq B(H)$ is a von Neumann algebra and if there exists a completely bounded projection of $B(H)$ onto M then M is injective, cf. [**25**]. The direct analogue of this result is false for C^*-algebras. T. Huruya [**17**] gave an example of a non-injective C^*-subalgebra of codimension 1 of an injective C^*-algebra. It is easily shown that any time Huruya's algebra is represented as a C^*-subalgebra of $B(H)$, then there will exist a completely bounded projection of $B(H)$ onto this non-injective algebra. Thus, to generalize the results of [**8**] and [**24**], we will need an additional condition.

Definition 3. An operator A-B-bimodule R is *relatively A-B-injective* if whenever $E \subseteq F$ are operator A-B-bimodules such that there exists a completely bounded projection of F onto E, then every completely bounded A-B-bimodule map of E into R has a completely bounded A-B-bimodule extension to F. It is important to note that we do not require that the projection of F onto E is an A-B-bimodule map.

The concept of relative injectivity was introduced in [**21**] with slightly different notation, relative A-B-injective was denoted $(A\text{-}B, \mathbb{C}\text{-}\mathbb{C})$-injective. A C^*-algebra A is A-A-injective if and only if A is injective in the usual sense. In contrast, [**21**] showed that every von Neumann algebra M is relatively M-M-injective, M-$\mathbb{C}$-injective and $\mathbb{C}$-M-injective.

Theorem 3.5. *Let $A \subseteq B(H)$ be a unital C^*-subalgebra. If there exists a completely bounded projection of $B(H)$ onto A and A is relatively A-$\mathbb{C}$-injective* (*or $\mathbb{C}$-A-injective*), *then A is injective.*

Proof. Since A is A-$\mathbb{C}$-injective, the identity map from A to A has a completely bounded left A-module extension to $B(H)$. This map is clearly a projection. Hence A is injective by Theorem 3.4. □

Because von Neumann algebras are relatively injective [**21**], Theorem 3.5 implies the result of [**8**] and [**24**].

Corollary 3.6. *Let $M \subseteq B(H)$ be a von Neumann algebra. If there exists a completely bounded projection of $B(H)$ onto M, then M is injective.*

Relative injectivity was shown in [**21**] to be equivalent to the vanishing of certain completely bounded "Ext" groups, which in turn implied the vanishing of completely bounded Hochschild cohomology. Thus, relative injectivity captures both the vanishing of cohomology and these projection results. It is still unknown which C^*-algebras A are relatively A-$\mathbb{C}$-injective. By Theorem 3.5, T. Huruya's C^*-algebra A, cannot be relatively A-$\mathbb{C}$-injective.

4. Local multiplier algebras, injective envelopes and regular completions.

We close this paper with some applications to multiplier algebras. Our main point is that by invoking Theorem 2.1, we will see immediately that multipliers are "naturally" represented as multiplication by elements in $I(A)$. This concrete representation of multipliers can be used to simplify some arguments. Thus, Theorem 2.1 provides an alternative starting point for developing the theory of multipliers.

In particular, we show that $I(A)$ contains the local multiplier algebra of A, $M_{\mathrm{loc}}(A)$, intrinsically as a C^*-subalgebra. Recall that a closed 2-sided ideal J of A is called *essential* if $J \cap K \neq \{0\}$ for every nontrivial 2-sided ideal K. All ideals in this section are norm-closed.

The *left multiplier algebra $LM(J)$ of J* is just the set of right A-module maps $\psi : J \to J$. Such a map is automatically (completely) bounded and $\|\psi\| = \|\psi\|_{cb}$. The *right multiplier algebra $RM(J)$* is defined similarly. The *multiplier algebra $M(J)$* consists of pairs of linear maps $\varphi, \psi : J \to J$ satisfying $\varphi(j_1)j_2 = j_1\psi(j_2)$. This identity implies that $\psi \in LM(J)$, $\varphi \in RM(J)$ and $\|\varphi\| = \|\psi\| = \|\varphi\|_{cb} = \|\psi\|_{cb}$.

The *local multiplier algebra* $M_{\text{loc}}(A)$ is defined by taking a direct limit of $M(J)$ over all essential ideals J of A ordered by reverse inclusion. See [**1**] for details.

Lemma 4.1. *Let A be a unital C^*-algebra and let J be a 2-sided essential ideal of A and let $\varphi : J \to I(A)$ be a completely bounded left (resp., right) A-module map. Then there exists a unique element x in $I(A)$ such that φ is right (resp., left) multiplication by x. Moreover, $\|x\| = \|\varphi\| = \|\varphi\|_{cb}$.*

Proof. By Theorem 2.1 such an x exists, it remains to show that x is unique. To this end consider, $F = \{y \in I(A) : Jy = 0\}$ which is clearly a right A-submodule of $I(A)$. It will suffice to show that $F = \{0\}$. Let $\{e_\alpha\}$ be a contractive approximate identity for J. For $a \in A,\ y \in F$, we have,

$$\|a - y\| \geq \sup_\alpha \|e_\alpha(a - y)\| = \sup_\alpha \|e_\alpha a\| = \|a\|$$

with the last equality using the fact that J is essential. The same calculation for matrices shows that the quotient map $q : I(A) \to I(A)/F$ is a complete isometry on A and a right A-module map. Now since $I(A)$ is injective, there exists a completely contractive right A-module map $\varphi : I(A)/F \to I(A)$. Hence by rigidity $\varphi \circ q(b) = b$ for all b in $I(A)$, and it follows that $F = \{0\}$. $\square$

The fact that F must be $\{0\}$ is related to the fact that $I(A)$ is in a certain sense an "essential extension" of A.

Theorem 4.2. *Let A be a unital C^*-algebra, let J be a 2-sided essential ideal in A and let (φ, ψ) be in $M(J)$. Then there exists a unique element x in $I(A)$ such that $\varphi(j) = jx, \psi(j) = xj$ for all j in J.*

Proof. By Lemma 4.1, there exist unique elements x_1, x_2 in J such that $\varphi(j_1) = j_1 x_1,\ \psi(j_2) = x_2 j_2$ for all j_1, j_2 in J. But $j_1 x_1 j_2 = \varphi(j_1) j_2 = j_1 \psi(j_2) = j_1 x_2 j_2$ and so $j_1(x_1 - x_2) j_2 = 0$ for all j_1. Applying Lemma 4.1 we conclude that $(x_1 - x_2) j_2 = 0$ for all j_2 and so $x_1 = x_2$. $\square$

Corollary 4.3. *The inclusion of A into $I(A)$ extends in a unique way to a $*$-monomorphism of $M_{\text{loc}}(A)$ into $I(A)$. The image of $M_{\text{loc}}(A)$ under this map is the norm closure of the set*

$$\{x \in I(A) : xJ \subseteq J \text{ and } Jx \subseteq J \text{ for some essential ideal } J\}.$$

Proof. For each (φ, ψ) in $M(J)$ there exists a unique x in $I(A)$ implementing (φ, ψ). By this uniqueness the map $(\varphi, \psi) \to x$ must be a $*$-monomorphism on $M(J)$. Furthermore, let J_i be essential ideals, and let (φ_i, ψ_i) in $M(J_i)$ be implemented by x_i. If $\varphi_1 = \varphi_2$ and $\psi_1 = \psi_2$ on $J_1 \cap J_2$ then, since $J_1 \cap J_2$ is essential, we must have $x_1 = x_2$.

This shows that the inclusions of $M(J_i)$ into $I(A)$ are coherent and allows us to extend these $*$-monomorphism to the direct limit, $M_{\text{loc}}(A)$.

Now assume that $\pi : M_{\text{loc}}(A) \to I(A)$ is any $*$-monomorphism with $\pi(a) = a$ for all a in A. Then for (φ, ψ) in $M(J)$

$$\pi((\varphi, \psi))j = \pi((\varphi, \psi)j) = \pi(\psi(j)) = \psi(j),$$

and $j\pi((\varphi, \psi)) = \varphi(j)$ from which it follows that $\pi((\varphi, \psi))$ is the unique element implementing (φ, ψ).

Finally, since $\{x \in I(A) : xJ \subseteq J \text{ and } Jx \subseteq J\}$ is exactly the image of $M(J)$ we have the last claim. □

Remark. The above results allow one to define a *local left* (resp., *right*) *multiplier algebra of* A easily, which we have not seen in the literature. Indeed, if $LM_{\text{loc}}(A) = \{x \in I(A) : xJ \subseteq J \text{ for some essential ideal } J\}^{--}$, then this set is easily seen to be completely isometrically isomorphic to the direct limit of $LM(J)$. It is interesting to note that if J_1 and J_2 are essential ideals and $\varphi \in LM(J_1)$ then $\varphi(J_1 \cap J_2) \subseteq J_1 \cap J_2$ and by Lemma 4.1, $\|\varphi\| = \|\varphi\,|_{J_1 \cap J_2}\|$. We define the local right multiplier algebra $RM_{\text{loc}}(A)$ of A, analogously.

To define the *local quasi-multiplier space* $QM_{\text{loc}}(A)$ *of* A we have to recall that the injective envelope $I(A)$ of A is a monotone complete C^*-algebra and, hence, an AW^*-algebra. On the other hand norm-closed two-sided ideals J of C^*-algebras are automatically hereditary, and so we can apply [**10**, Cor. 1.4]: For every norm-closed two-sided ideal $J \subseteq A$ and every quasi-multiplier $x \in QM(J)$ there exists an element $\overline{x} \in I(A)$ such that $j_1 x j_2 = j_1 \overline{x} j_2$ for any $j_1, j_2 \in J$ and $\|\overline{x}\|$ equals the norm of x estimated in the bidual von Neumann algebra A^{**}. For essential ideals J the element $\overline{x}$ has to be unique, in fact it can be found as a quasi-strict limit of nets of J. Since inclusion relations of essential ideals and their corresponding quasi-multiplier spaces are respected inside $I(A)$ we can define $QM_{\text{loc}}(A) = \{x \in I(A) : JxJ \subseteq J \text{ for some essential ideal } J\}^{--}$, to be the local quasi-multiplier space of A.

Note that in any situation where $M_{\text{loc}}(A) \not\equiv LM_{\text{loc}}(A)$ then necessarily $M_{\text{loc}}(A) \not\equiv I(A)$. (In general, the conditions $M_{\text{loc}}(A) \not\equiv LM_{\text{loc}}(A)$, $M_{\text{loc}}(A) \not\equiv QM_{\text{loc}}(A)$ and $LM_{\text{loc}}(A) \not\equiv QM_{\text{loc}}(A)$ are equivalent by general multiplier theory.) If A is any simple, unital, non-injective C^*-algebra like a non-injective Type II_1 or Type III von Neumann factor then $A = M_{\text{loc}}(A) = LM_{\text{loc}}(A) \neq I(A)$. However, in some cases we obtain the coincidence of the C^*-algebras $M_{\text{loc}}(A) = I(A)$.

Proposition 4.4. *Let A be a unital C^*-algebra, $K \subseteq A \subseteq B(H)$ where K denotes the ideal of compact operators, then $M_{\text{loc}}(A) = I(A) = B(H)$, $*$-isomorphically.*

Proof. Since K is necessarily an essential ideal of A and $M(K) = B(H)$ we have $B(H) \subseteq M_{\text{loc}}(A)$. By Corollary 4.3 we have a $*$-monomorphism π of

$M_{\text{loc}}(A)$ into $I(A)$. Hence, $A \subseteq B(H) \subseteq M_{\text{loc}}(A) \subseteq I(A)$, as C^*-algebras. Since $B(H)$ is A-A-injective, by Theorem 3.1, we have $B(H) = I(A)$ and the result follows. □

The fact that $I(A) = B(H)$ is due to M. Hamana [**13**] with a different proof.

Theorem 4.5. *Let A be a commutative unital C^*-algebra, then $M_{\text{loc}}(A) = I(A)$, $*$-isomorphically.*

Proof. By [**1**, Thm. 1] $M_{\text{loc}}(A)$ is a commutative AW^*-algebra. However, commutative AW^*-algebras are injective by [**26**, Th. 25.5.1] since bounded linear maps between C*-algebras are positive whenever their norm equals their evaluation at the identity of the C*-algebra. Consequently, the $*$-monomorphism of $M_{\text{loc}}(A)$ into $I(A)$ must be onto. □

In the theory of local multiplier C^*-algebras the problem of whether $M_{\text{loc}}(A)$ coincides with $M_{\text{loc}}(M_{\text{loc}}(A))$ for any C^*-algebra A is one of the main open questions, cf. [**1**, **27**]. Set $M_{\text{loc}}^{k+1}(A) = M_{\text{loc}}(M_{\text{loc}}^k(A))$, which is called the $(k+1)$-order local multiplier algebra of A. We show that any higher order local multiplier C^*-algebra of a given C^*-algebra A is contained in its injective envelope $I(A)$ and, what is more, that the injective envelopes $I(A)$ and $I(M_{\text{loc}}^k(A))$ coincide for any C^*-algebra A. The latter is of special interest since general C^*-subalgebras A of injective C^*-algebras B might not admit an embedding of their injective envelopes $I(A)$ as a C^*-subalgebra of B that extends the given embedding of A into B, see [**14**, Rem. 3.9] for an example.

Theorem 4.6. *Let A be a unital C^*-algebra and $M_{\text{loc}}(A)$ be its local multiplier C^*-algebra. Then the injective envelope $I(A)$ of A is the injective envelope $I(M_{\text{loc}}(A))$ of $M_{\text{loc}}(A)$ and consequently, $M_{\text{loc}}^k(A)$ is contained in $I(A)$ for all k.*

Proof. Since $M_{\text{loc}}(A)$ is $*$-isomorphically embedded into $I(A)$ extending the canonical $*$-monomorphism of A into $I(A)$ by Theorem 2.1, the C^*-algebra $I(A)$ serves as an injective extension of the C^*-algebra $M_{\text{loc}}(A)$, cf. [**13**]. However, the identity map on $M_{\text{loc}}(A)$ admits a unique extension to a completely positive map of $I(A)$ into itself with the same completely bounded norm one since $A \subseteq M_{\text{loc}}(A) \subseteq I(A)$ by construction and $I(A)$ is the injective envelope of A. So $I(A)$ has to be the injective envelope of $M_{\text{loc}}(A)$, too. □

Problem. Characterize the C^*-algebras A for which the local multiplier C^*-algebra $M_{\text{loc}}(A)$ of A coincides with the injective envelope $I(A)$ of A or at least with the regular monotone completion $\overline{A}$ of A in $I(A)$.

This question is surely difficult to answer: If A is an AW^*-algebra then the local multiplier algebra $M_{\text{loc}}(A)$ of A coincides with A by [**22**]. However,

A coincides with its regular monotone completion $\overline{A}$ if and only if A is monotone complete. So we arrive at a long standing open problem of C^*-theory dating back to the work of I. Kaplansky in 1951 ([**18**]): Are all AW^*-algebras monotone complete, or do there exist counterexamples?

Remark. If A is a non-unital C^*-algebra and B denotes its unitization, then A is a 2-sided essential ideal in B. Hence, by Theorem 4.2, $M(A) \subseteq I(B)$. However, in [**4**], it is observed that $I(A) = I(B)$, and so the hypothesis that A is unital can be removed from Theorem 4.2. Similarly, every 2-sided essential ideal in A is an essential ideal in B, so that Corollary 4.3 applies for non-unital A as well. Similar arguments show that the unital hypothesis can be dropped in Proposition 4.4, Theorem 4.5 and Theorem 4.6.

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