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ENDOCOHERENT MODULES

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We study coherence properties of a module M over its endomorphism ring. Hereby we extend to modules known characterizations of coherent and π-coherent rings. Moreover, we discuss the case that the category add$M$ is covariantly, respectively contravariantly, finite in mod$R$. Finally, we give a new characterization of endofinite modules.

A left module ${}_SM$ over a ring S is **coherent** if it is finitely presented and every finitely generated submodule of ${}_SM$ is finitely presented. Inspired by Lenzing's and Camillo's work on a special class of coherent rings [**15**] and [**7**], we will further say that ${}_SM$ is **π-coherent** if it is finitely presented and every finitely generated left S-module which is cogenerated by ${}_SM$ is finitely presented. Then the ring S is left π-coherent in the sense of [**7**] if and only if the regular left module ${}_SS$ is π-coherent.

In this note, we consider the case that *M is a right module over a ring R with endomorphism ring S* and study coherence as well as π-coherence of ${}_SM$. We prove the following results which extend to modules known characterizations of coherent and π-coherent rings [**15**], [**7**], [**18**, 5.3] and [**9**, 5.1].

Theorem 1. *The following statements are equivalent:*

(1) *Every finitely generated left S-module which is cogenerated by ${}_SM$ is finitely presented.*
(2) *Every finitely M-generated right R-module has an* add M*-preenvelope.*
(3) *For every $n \in \mathbb{N}$ and every subset $X \subset M^n$ the annihilator $ann_{S^{n\times n}}(X)$ of X in the matrix ring $S^{n\times n}$ is a finitely generated left ideal.*

Theorem 2.

(1) *If ${}_SM$ is π-coherent, then every finitely generated module has an* add M*-preenvelope. The converse holds if M_R is finitely generated.*
(2) *If ${}_SM$ is coherent, then every finitely presented module has an* add M*-preenvelope. The converse holds if M_R is finitely presented.*

In particular, we see that a finitely presented module M_R is coherent over its endomorphism ring if and only if the category add M is covariantly finite in mod R. We also prove a dual result characterizing the case that add M

is contravariantly finite in $\operatorname{mod} R$ (Corollary 11). Finally, we employ our results to give a new characterization of endofinite modules (Corollary 9).

Let us start with some notation. For an arbitrary ring R, we write **Mod** $\boldsymbol{R}$ and **mod** $\boldsymbol{R}$ for the categories of all, respectively of the finitely presented, right R-modules. By a subcategory we always mean a full subcategory.

Let $\mathcal{X} \subset \operatorname{Mod} R$ and A be a right R-module. Following [9], we say that a homomorphism $a : A \to X$ is an $\boldsymbol{\mathcal{X}}$**-preenvelope** if $X \in \mathcal{X}$ and the abelian group homomorphism $\operatorname{Hom}_R(a, X') : \operatorname{Hom}_R(X, X') \to \operatorname{Hom}_R(A, X')$ is surjective for each $X' \in \mathcal{X}$. A homomorphism $a : A \to X$ is said to be **left minimal** if every endomorphism $h : X \to X$ such that $h\,a = a$ is an isomorphism. Left minimal preenvelopes are called **envelopes** and are uniquely determined up to isomorphism. **(Pre)covers** are defined dually. In the representation theory of artin algebras, the usual terminology is (minimal) left or right $\mathcal{X}$-approximation.

Given a module M_R, we denote by **Add** $\boldsymbol{M}$ (respectively, **add** $\boldsymbol{M}$) the category consisting of all modules isomorphic to direct summands of (finite) direct sums of copies of M. Throughout the paper, we will freely use the fact that for a finitely generated module the existence of an $\operatorname{add} M$-preenvelope is equivalent to the existence of an $\operatorname{Add} M$-preenvelope.

If M_R is finitely presented, then $\operatorname{add} M$ is a subcategory of $\operatorname{mod} R$, and it is said to be **covariantly finite** in $\operatorname{mod} R$ if every finitely presented module has an $\operatorname{add} M$-preenvelope. Dually, one says that $\operatorname{add} M$ is **contravariantly finite** in $\operatorname{mod} R$ if every finitely presented module has an $\operatorname{add} M$-precover [5].

The following easy observation will be very useful:

Lemma 3. *Let R be a ring and M_R a module with endomorphism ring S.*

(1) *A_R has an $\operatorname{add} M$-preenvelope if and only if the left S-module ${}_S\operatorname{Hom}_R(A, M)$ is finitely generated.*

(2) *C_R has an $\operatorname{add} M$-precover if and only if the right S-module $\operatorname{Hom}_R(M, C)_S$ is finitely generated.*

Proof. (1) If ${}_S\operatorname{Hom}_R(A, M)$ is finitely generated, one can easily check that the map $c : A \to M^n$ induced by an S-generating set $c_k : A \to M$, $1 \leq k \leq n$, of $\operatorname{Hom}_R(A, M)$ is an $\operatorname{add} M$-preenvelope of A. Conversely, if $a : A \to X$ is an $\operatorname{add} M$-preenvelope, then we can assume w.l.o.g. that $X = M^n$ for some n, and applying the functor $\operatorname{Hom}_R(\ , M) : \operatorname{Mod} R \longrightarrow S\operatorname{Mod}$ on a, we immediately obtain the claim.

(2) is proven dually. □

The above lemma suggests that the existence of $\operatorname{add} M$-preenvelopes is related to the behaviour of the contravariant functor $\operatorname{Hom}_R(\ , M) : \operatorname{Mod} R \longrightarrow S\operatorname{Mod}$. We now investigate this connection more closely.

Let ${}_BQ_A$ be a bimodule. Recall that a module X_A is said to be Q_A-**reflexive** if the evaluation morphism $\boldsymbol{\delta}_X : X \to \operatorname{Hom}_B(\operatorname{Hom}_A(X, Q_A), {}_BQ)$ given by $\delta_X(x) : \alpha \mapsto \alpha(x)$ is an isomorphism. Of course, since $\operatorname{Ker}\delta_X$ coincides with the **reject** $\operatorname{Rej}_Q(X)$ of Q in X, all reflexive modules are in the category $\mathbf{Cogen}\,\boldsymbol{Q}$ of Q-cogenerated modules. We denote further by $\mathbf{cogen}\,\boldsymbol{Q}$ the category of all finitely Q-cogenerated modules, by $\mathbf{copres}\,\boldsymbol{Q}$ (respectively, by $\mathbf{sfcopres}\,\boldsymbol{Q}$) the category of all **finitely** (respectively, **semi-finitely**) $\boldsymbol{Q}$**-copresented modules**, that is, of all modules X admitting an exact sequence $0 \longrightarrow X \longrightarrow Q^n \longrightarrow L \longrightarrow 0$ where $n \in \mathbb{N}$ and L is finitely Q-cogenerated (respectively, Q-cogenerated). Dually, we write $\mathbf{gen}\,\boldsymbol{Q}$ for the category of all finitely Q-generated modules, and $\mathbf{pres}\,\boldsymbol{Q}$ for the category of all **finitely** $\boldsymbol{Q}$**-presented modules**, that is, of all modules X admitting an exact sequence $0 \longrightarrow K \longrightarrow Q^n \longrightarrow X \longrightarrow 0$ where $n \in \mathbb{N}$ and K is finitely Q-generated. Finally, we denote by $\boldsymbol{\mathcal{K}}(\boldsymbol{Q}_A)$ the subcategory of $\operatorname{Mod} A$ consisting of all modules K_A which admit an exact sequence $0 \longrightarrow K \longrightarrow A^n \longrightarrow Y_A \longrightarrow 0$ where $n \in \mathbb{N}$ and Y_A is Q_A-cogenerated, and by $\boldsymbol{\mathcal{K}}({}_B\boldsymbol{Q})$ the corresponding subcategory of $B\operatorname{Mod}$.

We are interested in the special case where Q is our bimodule ${}_SM_R$ with $S = \operatorname{End}_R M$. Then S is obviously ${}_SM$-reflexive, and we have the following result:

Lemma 4.

(1) ${}_S\operatorname{Hom}_R(A, M) \in \operatorname{sfcopres}{}_SM$ *for all finitely generated modules* A_R*, and* ${}_S\operatorname{Hom}_R(A, M) \in \operatorname{copres}{}_SM$ *for all finitely presented modules* A_R*.*

(2) *The functor* $\operatorname{Hom}_R(\ , M) : \operatorname{Mod} R \longrightarrow S\operatorname{Mod}$ *induces dense functors* $\operatorname{gen} M_R \longrightarrow \mathcal{K}({}_SM)$ *and* $\operatorname{pres} M_R \longrightarrow \operatorname{copres}{}_SS$*.*

Proof. (1) Let A_R be finitely generated with an exact sequence $0 \longrightarrow K \xrightarrow{f} R^n \longrightarrow A \longrightarrow 0$. We then have an exact sequence $0 \to {}_S\operatorname{Hom}_R(A, M) \to {}_S\operatorname{Hom}_R(R^n, M) \xrightarrow{\operatorname{Hom}_R(f,M)} {}_S\operatorname{Hom}_R(K, M)$ where ${}_S\operatorname{Hom}_R(K, M)$ is a submodule of ${}_S\operatorname{Hom}_R(R^{(J)}, M) \simeq {}_SM^J$ for some set J. Further, if A_R is finitely presented, then K is finitely generated, and ${}_S\operatorname{Hom}_R(K, M)$ is a submodule of ${}_S\operatorname{Hom}_R(R^m, M) \simeq {}_SM^m$ for some $m \in \mathbb{N}$.

(2) As in (1), we show that $A \in \operatorname{gen} M_R$ gives rise to an exact sequence $0 \to {}_S\operatorname{Hom}_R(A, M) \to {}_S\operatorname{Hom}_R(M^n, M) \to {}_S\operatorname{Hom}_R(K, M)$ where ${}_S\operatorname{Hom}_R(K, M)$ is ${}_SM$-cogenerated, and moreover, that we can assume ${}_S\operatorname{Hom}_R(K, M)$ finitely cogenerated by S provided that $A \in \operatorname{pres} M_R$. So, it remains to prove that the functors are dense. Any exact sequence $0 \longrightarrow K \longrightarrow S^n \longrightarrow {}_SY \longrightarrow 0$ with $Y \in \operatorname{Cogen}{}_SM$ yields an exact sequence $0 \to \operatorname{Hom}_S(Y, M) \to \operatorname{Hom}_S(S^n, M) \xrightarrow{g} \operatorname{Hom}_S(K, M)$ where $L_R = \operatorname{Im} g$ is an epimorphic image of M^n. We obtain the commutative diagram

$$\begin{array}{ccccccc}
0 \to & K & \longrightarrow & S^n & \longrightarrow & Y & \to 0 \\
 & \downarrow \alpha & & \delta_{S^n} \downarrow & & \downarrow \delta_Y & \\
0 \to & \mathrm{Hom}_R\,(L, M) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(S^n, M), M) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(Y, M), M) &
\end{array}$$

where α and δ_Y are monomorphisms and δ_{S^n} is an isomorphism. Then by the snake lemma α is an isomorphism, hence ${}_SK \cong \mathrm{Hom}_R\,(L, M)$ with $L \in \mathrm{gen}\, M_R$.

Assume further that there is a monomorphism $i : Y \to S^m$ for some $m \in \mathbb{N}$. Then we also have a map $f = \mathrm{Hom}_S\,(i, M) : \mathrm{Hom}_S\,(S^m, M) \to \mathrm{Hom}_S\,(Y, M)$ with $A_R = \mathrm{Im}\, f \in \mathrm{gen}\, M_R$ and a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & \mathrm{Hom}_S\,(S^n, M) & \longrightarrow & L' & \longrightarrow & 0 \\
 & & \bigcap e & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R\,(Y, M) & \longrightarrow & \mathrm{Hom}_S(S^n, M) & \longrightarrow & L & \longrightarrow & 0
\end{array}$$

where $L' \in \mathrm{pres}\, M_R$. Since δ is a natural transformation, $\mathrm{Hom}_R\,(f, M)\, \delta_Y = \delta_{S^m}\, i$ is a monomorphism, and therefore $\mathrm{Hom}_R\,(e, M)\, \delta_Y$ is a monomorphism as well. So, we conclude as above from the commutative diagram

$$\begin{array}{ccccccc}
0 \to & K & \longrightarrow & S^n & \longrightarrow & Y & \to 0 \\
 & \downarrow \alpha & & \delta_{S^n} \downarrow & & \downarrow \delta_Y & \\
0 \to & \mathrm{Hom}_R\,(L, M) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(S^n, M), M) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(Y, M), M) & \\
 & \downarrow \beta & & \| & & \downarrow \mathrm{Hom}_R\,(e, M) & \\
0 \to & \mathrm{Hom}_R\,(L', M) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(S^n, M), M) & \longrightarrow & \mathrm{Hom}_R(A, M) &
\end{array}$$

that $\beta\alpha$ is an isomorphism, hence ${}_SK \cong \mathrm{Hom}_R\,(L', M)$ with $L' \in \mathrm{pres}\, M_R$. □

Let us remark that if ${}_SM_R$ is faithfully balanced, then by similar arguments, the functor $\mathrm{Hom}_R\,(\ , M) : \mathrm{Mod}\, R \longrightarrow S\,\mathrm{Mod}$ induces dense functors $\mathrm{gen}\, R \longrightarrow \mathrm{sfcopres}\, {}_SM$ and $\mathrm{mod}\, R \longrightarrow \mathrm{copres}\, {}_SM$.

We now obtain a characterization of left coherent endomorphism rings, see also [**10**]. Moreover, we prove the equivalence of the first two conditions in Theorem 1.

Proposition 5.

(1) *S is left coherent if and only if every $A \in \mathrm{pres}\, M_R$ has an $\mathrm{add}\, M$-preenvelope.*

(2) *Every finitely generated left S-module which is cogenerated by ${}_SM$ is finitely presented if and only if every $A \in \operatorname{gen} M_R$ has an $\operatorname{add} M$-preenvelope.*

Proof. (1) Of course, S is left coherent if and only if every module in $\operatorname{copres}{}_SS$ is finitely generated over S. By Lemma 4 the latter means that ${}_S\mathrm{Hom}_R(A, M)$ is finitely generated for all modules $A_R \in \operatorname{pres} M$. Combining this with Lemma 3, we obtain the claim.

(2) is proven similarly. □

Note that Lenzing has described left coherence in terms of annihilators of matrix rings [**15**, §4, Korollar 1]. More precisely, denoting by $R^{n\times n}$ the $n \times n$ matrix ring over R, he has proven that R is left coherent if and only if for every $n \in \mathbb{N}$ and every $A \in R^{n\times n}$ the left annihilator of A in $R^{n\times n}$ is a finitely generated left ideal. Moreover, he has shown in [**15**, Satz 4] that R is left π-coherent if and only if for every $n \in \mathbb{N}$ all left annihilators in $R^{n\times n}$ are finitely generated left ideals, see also [**7**]. We now establish a corresponding result for modules and complete the proof of Theorems 1 and 2.

Proposition 6. *The following statements are equivalent:*

(1) *Every finitely generated left S-module which is cogenerated by ${}_SM$ is finitely presented.*
(2) *For every $n \in \mathbb{N}$ and every subset $X \subset M^n$ the annihilator $\operatorname{ann}_{S^{n\times n}}(X)$ of X in $S^{n\times n}$ is a finitely generated left ideal.*

Proof. (1)⇒(2): Let $X \subset M^n$, put $K = X \cdot R$ and $A_R = M^n/K$, and denote by $\nu : M^n \to A$ the canonical surjection. By assumption and the above proposition, $A_R \in \operatorname{gen} M$ has an $\operatorname{add} M$-preenvelope $a : A \to M^m$, and we can consider the maps $f_i : M^n \xrightarrow{\nu} A \xrightarrow{a} M^m \xrightarrow{\mathrm{pr}_i} M \xrightarrow{\iota} M^n$, $1 \le i \le m$, where pr_i and ι denote the canonical projections and a canonical injection, respectively. Obviously, $f_1, \dots, f_m$ are contained in $\operatorname{ann}_{S^{n\times n}}(X)$, and since every other map $h \in \operatorname{ann}_{S^{n\times n}}(X)$ factors through ν and hence through $a\nu$, they are generators of $\operatorname{ann}_{S^{n\times n}}(X)$ over $S^{n\times n}$.

(2)⇒(1): We again apply Proposition 5 and show that every $A \in \operatorname{gen} M$ has an $\operatorname{add} M$-preenvelope. Consider an exact sequence $0 \longrightarrow K \longrightarrow M^n \xrightarrow{g} A \longrightarrow 0$ and a generating set $f_1, \dots, f_m$ of $\operatorname{ann}_{S^{n\times n}}(K)$ over $S^{n\times n}$. Then K is contained in the kernel of the product map $f : M^n \to M^{nm}$ induced by the f_i, and so there is a map $a : A \to M^{nm}$ such that $f = a\,g$. Let us verify that a is an $\operatorname{add} M$-preenvelope. In fact, if we denote again by $M \xrightarrow{\iota} M^n$ a canonical injection, then for every homomorphism $h : A \to M$ the composition $\iota\, h\, g$ lies in $\operatorname{ann}_{S^{n\times n}}(K)$ and therefore has the form $\sum_{i=1}^m t_i\, f_i$ for some $t_1, \dots, t_m \in S^{n\times n}$. This shows that $h\,g$ factors through $a\,g$, and hence h factors through a. □

Proof of Theorem 2. (1) If A_R is finitely generated, then by Lemma 4 there is an exact sequence $0 \longrightarrow {}_S\mathrm{Hom}_R(A, M) \longrightarrow {}_SM^n \longrightarrow L \longrightarrow 0$ where $n \in \mathbb{N}$ and $L \in \mathrm{Cogen}\, {}_SM$. By assumption L is then finitely generated and even finitely presented, so ${}_S\mathrm{Hom}_R(A, M)$ is finitely generated, and A has an $\mathrm{add}\, M$-preenvelope by Lemma 3. Conversely, if M_R is finitely generated and every finitely generated module has an $\mathrm{add}\, M$-preenvelope, then we deduce that R and every $A \in \mathrm{gen}\, M$ have an $\mathrm{add}\, M$-preenvelope. But this implies by Lemma 3 and Proposition 5(2) that ${}_SM$ is π-coherent.

(2) We show as in (1) that Lemma 4 and Lemma 3 yield the existence of an $\mathrm{add}\, M$-preenvelope for every finitely presented module A_R. Conversely, if M_R is finitely presented and every finitely presented module has an $\mathrm{add}\, M$-preenvelope, then we deduce that R and every $A \in \mathrm{pres}\, M$ have an $\mathrm{add}\, M$-preenvelope. In particular, S is then left coherent by Proposition 5(1). Moreover, if $a : R \to M^n$ is an $\mathrm{add}\, M$-preenvelope with cokernel L, then also L_R is finitely presented, and therefore ${}_S\mathrm{Hom}_R(L, M)$ is finitely generated by Lemma 3. So, we infer from the exact sequence $0 \longrightarrow {}_S\mathrm{Hom}_R(L, M) \longrightarrow {}_S\mathrm{Hom}_R(M^n, M) \longrightarrow {}_S\mathrm{Hom}_R(R, M) \longrightarrow 0$ that ${}_SM$ is finitely presented and hence coherent. □

Assume that R is **semiregular**, that is, idempotents lift modulo the Jacobson radical $J(R)$ and $R/J(R)$ is von Neumann regular. Then we know from [**3**, Corollary 3] and [**18**, Corollary 5.4] that R being left (π-)coherent even implies the existence of projective envelopes for the finitely presented (respectively, finitely generated) modules. Also these results can be extended to modules.

Corollary 7. *Let S be semiregular.*

(1) *If ${}_SM$ is π-coherent, then every finitely generated module has an* $\mathrm{add}\, M$*-envelope.*
(2) *If ${}_SM$ is coherent and M_R is finitely presented, then every finitely presented module has an* $\mathrm{add}\, M$*-envelope.*

Proof. From Theorem 2 we obtain the existence of an $\mathrm{add}\, M$-preenvelope $f : A \to M^n$ with A finitely generated or finitely presented, respectively. Note that in both cases the cokernel $L = \mathrm{Coker}\, f$ has an $\mathrm{add}\, M$-preenvelope $g : L \to M^m$, too. Indeed, in Case (1) this follows from Proposition 5(2) and the fact that $L \in \mathrm{gen}\, M$, and in Case (2) we have only to remind that M_R, and therefore also L_R, are finitely presented. Set $E = \mathrm{End}_R\, M^n$. From the exact sequence ${}_E\mathrm{Hom}_R(M^m, M^n) \longrightarrow {}_EE \xrightarrow{\mathrm{Hom}_R(f, M^n)} {}_E\mathrm{Hom}_R(A, M^n) \longrightarrow 0$ we deduce that the annihilator $\mathrm{ann}_E(f)$ is a finitely generated left ideal of E. Since E is semiregular by [**16**, 2.7], we know from [**17**, Satz 1.2] that there is a left ideal $\mathcal{I}$ which satisfies $\mathrm{ann}_E(f) + \mathcal{I} = E$ and is minimal with respect to this property. Then $\mathrm{ann}_E(f) \cap \mathcal{I}$ is superfluous in $\mathcal{I}$ and therefore also in E. So, we have verified that:

(i) There is a left ideal $\mathcal{I}$ in E such that $\mathrm{ann}_E(f)+\mathcal{I}=E$ and $\mathrm{ann}_E(f)\cap\mathcal{I}\subset J(E)$; and
(ii) idempotents lift modulo $J(E)$.

Thus we can apply a result of Zimmermann [21] asserting that under these conditions f has a left minimal version, that is, there is a decomposition $M^n = X\oplus K$ such that the composition of f with the canonical projection $p: M^n \to X$ gives rise to an $\mathrm{add}\, M$-envelope. □

Let us now compare different notions of coherence. Recall that a ring R is said to be **left strongly coherent** if products of projective right R-modules are locally projective [19] and [11]. Such rings are characterized by the property that every matrix subgroup of the right module R_R is a finitely generated left ideal. Moreover, as observed in [20], they are always left π-coherent.

More generally, if M_R is a finitely generated module with all matrix subgroups being finitely generated over the endomorphism ring S, then we can prove as in [2, 3.1] that every finitely generated module has an $\mathrm{add}\, M$-preenvelope, and so it follows immediately from Theorem 2 that ${}_SM$ is π-coherent and in particular coherent.

Examples for the failure of the converse implications even in the case $M=R$ are given in [20, Example 29], [11, Example 5.2] and [7]. In particular, every commutative von Neumann regular ring which is not self-injective is coherent but not π-coherent, and the ring $R=K[X_1,X_2,\dots]$ over a field K is π-coherent but not strongly coherent.

Next, we investigate the gap between π-coherence and coherence. To this end, we recall the notion of an R-Mittag-Leffler (or finitely pure-projective) module studied in [12], [8], [13] and [6]. A module X_R is said to be an ***R*-Mittag-Leffler module** if the canonical map $X\otimes_R R^J \to X^J$ is a monomorphism for every set J, or equivalently, if for every finitely generated submodule A_R the embedding $A\subset X$ factors through a finitely presented module. Jones showed in [13, p. 104] that a ring is left π-coherent if and only if it is left coherent and all products of copies of R (on either side) are R-Mittag-Leffler modules. Note that since the class of R-Mittag-Leffler modules is closed under pure submodules [6, Proposition 9], the latter property amounts to saying that all products of projective modules are R-Mittag-Leffler modules. We now prove the general statement for modules.

Corollary 8. *The following statements are equivalent:*

(1) ${}_SM$ *is π-coherent.*
(2) S *is left (π-)coherent,* ${}_SM$ *is finitely presented, and all products of copies of* ${}_SM$ *are S-Mittag-Leffler modules.*

If M_R is finitely presented, the following statement is further equivalent:

(3) *S is left (π-)coherent, ${}_SM$ is finitely presented, and all products of copies of M_R are R-Mittag-Leffler modules.*

Proof. (1)⇒(2): Any epimorphism $R^{(K)} \to M$ gives rise to a monomorphism ${}_SS \simeq \mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(R^{(K)}, M) \simeq {}_SM^K$, showing that S is left (π-)coherent. Moreover, all finitely generated submodules of products of copies of ${}_SM$ are finitely presented by definition, and so the claim is proven.

(2)⇒(1): Let ${}_SA$ be a finitely generated submodule of a product of copies of ${}_SM$. By assumption, ${}_SA$ is contained in a finitely presented module ${}_SY$, which is coherent since so is the ring S. Hence ${}_SA$ is finitely presented, and we have verified that ${}_SM$ is π-coherent.

(1)⇒(3): Let A_R be a finitely generated submodule of M^J for some set J. By Theorem 2, the embedding $A \subset M^J$ factors through an $\mathrm{add}\,M$-preenvelope $A \to M^n$, and M^n is finitely presented if so is M_R.

(3)⇒(1): We claim that every finitely generated module has an $\mathrm{add}\,M$-preenvelope. The claim then follows from Theorem 2 whenever M_R is finitely generated. So, let A_R be finitely generated. By possibly considering $A/\mathrm{Rej}_M(A)$, we can assume without loss of generality that A is M-cogenerated. Then the product map $f : A \to M^J$ induced by all maps in $J = \mathrm{Hom}_R(A, M)$ is a monomorphism and therefore factors through a homomorphism $f' : A \to F$ where F is finitely presented. But since ${}_SM$ is coherent by assumption, we obtain from Theorem 2 the existence of an $\mathrm{add}\,M$-preenvelope $a : F \to M^n$. Now it is easy to check that the composition $a\,f' : A \to M^n$ is an $\mathrm{add}\,M$-preenvelope as well. □

Here is a further application of Theorem 2. Recall that M is said to be **endonoetherian**, respectively **endofinite**, if ${}_SM$ is noetherian, respectively a module of finite length. We will moreover call M **endocoherent** if ${}_SM$ is coherent, and **endocoperfect** if it satisfies the descending chain condition for cyclic S-submodules. We explore the relationship between these finiteness conditions over the endomorphism ring.

Corollary 9. *The following statements are equivalent:*

(1) *M is endofinite.*

(2) *M is endocoperfect, and for all direct summands M' of M and all finitely presented modules A_R, there exists an* $\mathrm{add}\,M'$*-preenvelope.*

If M_R is finitely generated, then (1) is further equivalent to:

(3) *M is endocoperfect and all its direct summands are endocoherent.*

Proof. (1)⇔(2): Assume that M is endofinite. Then M is Σ-pure-injective and therefore satisfies the descending chain condition for cyclic S-submodules. Moreover, M is endonoetherian, and it is well-known that its direct summands are then endonoetherian as well. Now, we have shown in [**2**, 3.1] that all finitely presented modules A_R have an $\mathrm{add}\,M'$-preenvelope if and only if certain endo-submodules of M', namely the finite matrix subgroups, are finitely generated over $\operatorname{End}_R M'$. Thus (1) implies (2). For the converse implication, we use that M is endofinite if and only if every direct summand of M is product-complete [**14**]. Observe that by [**2**, 5.1] a module M' is product-complete if and only if it is endocoperfect and all finite matrix subgroups of M' are finitely generated over $\operatorname{End}_R M'$. Since endocoperfectness is inherited to direct summands, we have verified (2)⇒(1).

(3)⇒(2) follows immediately from Theorem 2.

(1)⇒(3): The direct summands of M are finitely generated and endonoetherian, so their endomorphism rings are left noetherian. Thus they are also endocoherent. □

We close the paper with some dual considerations. We have seen above that the existence of $\operatorname{add} M$-preenvelopes is related to coherence properties of ${}_S M$. Dually, we can describe the existence of $\operatorname{add} M$-precovers in terms of coherence properties of the dual module $\boldsymbol{M^*}_S = \operatorname{Hom}_R(M, W)_S$, where $\boldsymbol{W_R}$ denotes a minimal injective cogenerator of $\operatorname{Mod} R$. We refer to [**1**] for details and only mention the main results.

Theorem 10.

(1) *If M^*_S is π-coherent, then every finitely W-cogenerated module has an* $\operatorname{add} M$*-precover. The converse holds if M_R is finitely W-cogenerated.*

(2) *If M^*_S is coherent, then every finitely W-copresented module has an* $\operatorname{add} M$*-precover. The converse holds if M_R is finitely W-copresented.*

If R is **a right Morita ring**, that is, if R is a right artinian ring and W_R is finitely generated, then we obtain a characterization of contravariantly finiteness. This and other consequences are collected in the following corollary. Observe that the last statement generalizes a result proven by Auslander for finitely generated projective modules [**4**, 6.6].

Corollary 11.

(1) *Assume that M is a finitely generated module over a right Morita ring R. Then M^*_S is (π-)coherent if and only if* $\operatorname{add} M$ *is contravariantly finite in* $\operatorname{mod} R$.

(2) *Assume that M_R is a finitely generated module over a right noetherian ring R. If* $\operatorname{add} M$ *is contravariantly finite in* $\operatorname{mod} R$*, then every finitely generated right S-module which is cogenerated by M^*_S is finitely presented. In particular, S is then a right π-coherent ring.*

(3) *Assume that M_R is a coherent module. If all finitely generated modules have an* $\operatorname{add} M$*-precover, then S is a right coherent ring.*

Proof. (1) By assumption every finitely generated module is finitely W-copresented and therefore has an $\operatorname{add} M$-precover provided that M^*_S is coherent. Conversely, assume that $\operatorname{add} M$ is contravariantly finite in $\operatorname{mod} R$. Then every finitely W-cogenerated module, being finitely presented by assumption, has an $\operatorname{add} M$-precover. Moreover, the finitely generated module M_R is finitely W-cogenerated, and we conclude from Theorem 10 that M_S^* is π-coherent.

(2) Under the given assumptions, all modules in $\operatorname{cogen} M$ are finitely presented and therefore have an $\operatorname{add} M$-precover whenever $\operatorname{add} M$ is contravariantly finite in $\operatorname{mod} R$. The claim then follows from the dual version of Proposition 5(2). That S is right π-coherent follows from the fact that S_S is M_S^*-cogenerated.

(3) Under the given assumption, all modules in $\operatorname{copres} M$ are finitely generated and therefore have an $\operatorname{add} M$-precover. The claim then follows from the dual version of Proposition 5(1), see also [**10**]. □

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AREA, WIDTH, AND LOGARITHMIC CAPACITY OF CONVEX SETS

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For a planar convex compact set E, we describe the mutual range of its area, width, and logarithmic capacity. This result will follow from a more general theorem describing the mutual range of area, logarithmic capacity, and length of orthogonal projection onto a given axis of an arbitrary compact set, connected or not.

1. Introduction.

For a planar convex compact set E, let $A(E)$, $w(E)$, and $\operatorname{cap} E$ denote the area, width, and logarithmic capacity of E respectively. The width $w(E)$ is the minimal orthogonal projection of E, i.e.,

$$w(E) = \min_{0 \le \theta \le \pi} \operatorname{proj}_\theta E,$$

where $\operatorname{proj}_\theta E$ denotes the length of the orthogonal projection of E onto the line $l_\theta = \{z = te^{i\theta} : -\infty < t < \infty\}$. The logarithmic capacity $\operatorname{cap} E$ of a compact set E is defined by

$$-\log \operatorname{cap} E = \lim_{z \to \infty} (g(z) - \log |z|),$$

where $g(z)$ denotes Green's function of the unbounded component $\Omega(E)$ of $\overline{\mathbb{C}} \setminus E$ having singularity at $z = \infty$. This notion combines several characteristics of a compact set such as transfinite diameter, Chebyshev's constant, and outer radius, see [**3**, **4**, **7**, **8**, **10**] and [**12**].

How large can the area of E be if the width and logarithmic capacity of E are prescribed? — For convex sets, the answer to this question is given by:

Theorem 1.1. *For a planar convex compact set E, let $2h = w(E)/\operatorname{cap} E$. Then $0 \le h = h(E) \le 1$ and*

$$A(E) \le \operatorname{cap}^2 E \left(\pi\beta^2 + 4h\beta' \mathbf{E}(\beta', {\beta'}^{-1}) \right), \tag{1.1}$$

where $\mathbf{E}$ denotes the elliptic integral of the second kind, $\beta' = \sqrt{1-\beta^2}$, and $\beta = \beta(h)$ is a solution to the equation

$$h = \beta \mathbf{E}(\beta, \beta^{-1}) \tag{1.2}$$

unique in the interval $0 < \beta < 1$. *In addition, for a fixed* $\operatorname{cap} E = c$, *the right-hand side of* (1.1) *strictly increases from* 0 *to* πc^2 *as* h *runs from* 0 *to* 1.

Equality occurs in (1.1) *if and only if* E *coincides up to a linear transformation with the set* E^h, *symmetric w.r.t. the coordinate axes, complementary to the image* $f(\mathbb{U}^*)$ *of* $\mathbb{U}^* = \{z : |z| > 1\}$ *under a univalent conformal mapping* $w = f(z)$ *with* $f = g \circ \tau$, *where*

$$g(\tau) = h + \frac{1}{2}\int_2^\tau \frac{\tau + \sqrt{\tau^2 - 4\beta^2}}{\sqrt{\tau^2 - 4}}\, d\tau, \quad \tau = (1/2)(z + \sqrt{z^2 - 4}) \tag{1.3}$$

with the principal branches of the radicals.

Figure 1 displays extremal configurations for some typical values of h.

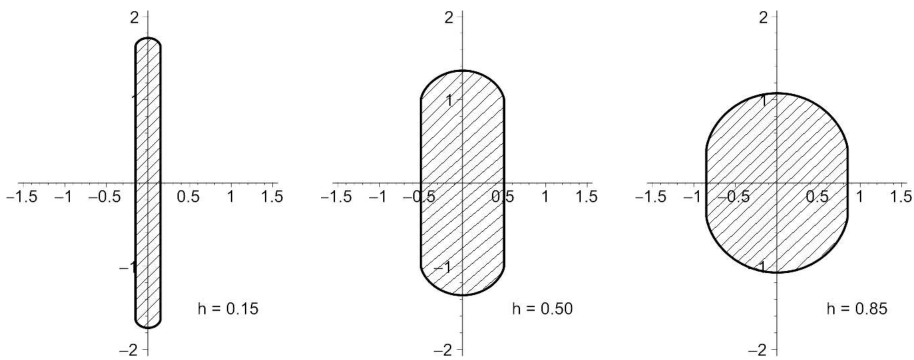

Figure 1. Typical extremal confugurations.

Let $A(h) = \max A(E)$, where the maximum is taken among all convex compact sets E such that $\operatorname{cap} E = 1$, $w(E) = 2h$. Then by Theorem 1.1, $A(h)$ equals the right-hand side of (1.1). The graph of $A(h)$ coincides with a part, for $0 \le h \le 1$, of the graph in Figure 2, which shows the maximal area among all compact sets with logarithmic capacity 1 and prescribed projection onto the real axis, as it is explained in Theorem 1.2.

The Proof of Theorem 1.1 given in Section 3 actually leads to a more general result: Inequality (1.1) holds true with the same uniqueness assertion for all compact sets E (connected or not) such that $0 \le h(E) \le 1$. However, we prefer to speak about convex sets since the inequalities $0 \le h(E) \le 1$ give the whole range of $h(E)$ over the family of all such sets with equalities $h(E) = 0$ and $h(E) = 1$ only for rectilinear segments and disks, respectively. This follows from the well-known isoperimetric inequalities:

$$w(E) \le \frac{1}{\pi}\int_0^\pi \operatorname{proj}_0 E\, d\theta = \frac{1}{\pi}\text{length}\,(\partial E),$$

$$\frac{1}{2\pi}\text{length}\,(\partial E) \le \left(\frac{\text{area}\, E}{\pi}\right)^{1/2} \le \operatorname{cap} E,$$

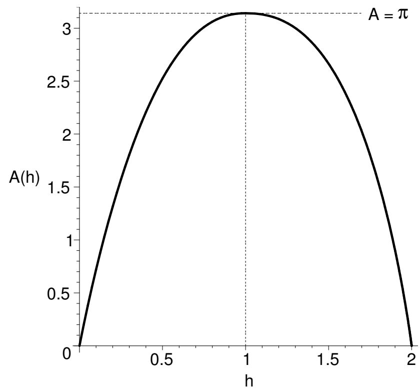

Figure 2. Graph of $A(h)$

cf. [**10**, pp. 8, 164].

In contrast, the range of $h(E)$ over the set of all continua (= connected compact sets) E is not known. There is an open question, first referenced by Erdös, Herzog and Piranian [**5**], and later commented on by Ch. Pommerenke [**11**] to find max $h(E)$. Erdös et al. conjectured that max $h(E)$ would be 1; however, Pommerenke gave an counterexample, E_6, the symmetric star with six rays, for which $h(E_6) > 1$. An easy computation shows that for E_3, the symmetric star with three rays, that $h(E_3) > h(E_6)$. However, counter to intuition, there are intermediate stars (between E_3 and E_6) which show that E_3 cannot be the extremal configuration for max $h(E)$. This remark points out that the problem on the maximal area of E among all continua E with prescribed $h(E) > 1$ is potentially quite difficult.

A characteristic of a compact set E, dual to the width, is the diameter of E which can be defined as

$$\operatorname{diam} E = \max_{0\le\theta\le\pi} \operatorname{proj}_\theta E.$$

In [**1**, Theorem 2], we found the maximal area $A(d) = \max A(E)$ among all continua E such that $\operatorname{cap} E = 1$, $\operatorname{diam} E = 2d$. The range of $d = d(E)$, if E is connected and $\operatorname{cap} E = 1$, is given by the classical inequalities $1 \le d \le 2$. The first of them is due to G. Pólya [**9**] and the second one is due to G. Faber [**6**]. The upper bound for d shows that the range of the length of projection of E onto a fixed axis, say on $\mathbb{R}$, is

$$0 \le \operatorname{proj}_0 E \le 4.$$

For a half of this range, when the projection is between 0 and 2, the arguments used to prove Theorem 1.1 show also that (1.1) holds true with the same uniqueness assertion for all compact sets E such that $\operatorname{cap} E = 1$ and $0 \le \operatorname{proj}_0 E \le 2$. This result combined with Theorem 2 in [**1**] gives:

Theorem 1.2. *Let E be a compact set in $\mathbb{C}$ such that* $\operatorname{cap} E = 1$ *and* $\operatorname{proj}_0 E = 2h$*, where* $0 \le h \le 2$*. Then*

$$A(E) \le \begin{cases} \pi\beta^2 + 4h\beta' \mathbf{E}(\beta', \beta'^{-1}), & \text{if} \quad 0 \le h \le 1, \\ \\ \pi\beta^2 - 2\pi h(\beta - 1), & \text{if} \quad 1 \le h \le 2, \end{cases} \tag{1.4}$$

where $\beta = \beta(h)$*,* $0 \le \beta \le 1$ *is defined by* (1.2) *in the first case and* $1 \le \beta \le 2$ *is the unique solution to the equation* $h = \beta - (\beta - 1)\log(\beta - 1)$ *in the second case. In addition, the right-hand side of* (1.4) *strictly increases from* 0 *to* π *as* h *runs from* 0 *to* 1 *and strictly decreases from* π *to* 0 *as* h *runs from* 1 *to* 2*.*

For $0 \le h \le 1$*, extremal configurations are described in Theorem* 1.1*. For* $1 \le h \le 2$*, equality occurs in* (1.4) *if and only if* E *coincides up to a linear transformation with the complement to the image* $f(\mathbb{U}^*)$ *of* $\mathbb{U}^*$ *under a conformal mapping* $f(z) = h + \int_1^z \varphi(z; h)\, dz$*, where* φ *maps* $\mathbb{U}^*$ *conformally onto the complement of the "double anchor"*

$$\begin{aligned} F(\beta, \psi) = [-i\beta, i\beta] &\cup \left\{ \beta e^{it} : \frac{\pi}{2} - \psi \le t \le \frac{\pi}{2} + \psi \right\} \\ &\cup \left\{ \beta e^{it} : \frac{3\pi}{2} - \psi \le t \le \frac{3\pi}{2} + \psi \right\} \end{aligned}$$

with $\psi = (1/2)\cos^{-1}(8\beta^{-1} - 8\beta^{-2} - 1)$*.*

For the right-hand side of (1.4) we will keep notation $A(h)$, where now $0 \le h \le 2$; in context of Theorem 1.1, $A(h)$ was defined only for $0 \le h \le 1$.

2. Geometry and closed form of the extremals.

In Lemma 2.1 we summarize well-known symmetrization results necessary for our main proofs, see [**3**, **7**, **2**] and [**1**].

Lemma 2.1. *For any compact set E, let E^{**} be the result of successive Steiner symmetrizations of E w.r.t. the real and imaginary axes, respectively. Then*

$$A(E^{**}) = A(E), \quad \operatorname{proj}_0 E^{**} = \operatorname{proj}_0 E, \quad \operatorname{cap} E^{**} \le \operatorname{cap} E \tag{2.1}$$

with the sign of equality in the third relation if and only if E^{**} coincides with E a.e. up to shifts in the directions of the coordinate axes.

It follows from (2.1) that in proving Theorem 1.2 we may restrict ourselves with continua possessing double Steiner symmetry w.r.t. the coordinate axes. Furthermore, since $\operatorname{cap} E$, $w(E)$, $\operatorname{proj}_\theta E$, and $(A(E))^{1/2}$ all change linearly w.r.t. scaling, we may assume in what follows that $\operatorname{cap} E = 1$. Then, $w(E)$ in Theorem 1.1 may vary in between 0 and 2, and $\operatorname{proj}_0 E$ in Theorem 1.2 varies in between 0 and 4.

If E is connected and Steiner symmetric, then $\Omega_E = \overline{\mathbb{C}}\setminus E$ is a simply connected domain containing the point $z = \infty$. Let f be a conformal mapping from $\mathbb{U}^*$ onto Ω_E. If $\operatorname{cap} E = 1$, we can normalize f such that

$$f(\zeta) = \zeta + a_0(f) + a_1(f)\zeta^{-1} + \cdots . \tag{2.2}$$

The set of all analytic functions univalent in $\mathbb{U}^*$ and normalized by (2.2) constitute the standard class Σ, see [**3**, **4**] and [**8**].

For $f \in \Sigma$, let $D_f = f(\mathbb{U}^*)$ and $E_f = \mathbb{C}\setminus D_f$. Our previous considerations show that the problem in Theorem 1.2 is equivalent to the problem on the maximal omitted area for the class Σ under the additional constraint

$$\operatorname{proj}_0 E_f = 2h,$$

$0 \le h \le 2$. The set of functions $f \in \Sigma$ such that $0 \in E_f$ and projection of E_f onto $\mathbb{R}$ coincides with the segment $[-h, h]$ will be denoted by Σ^h. The omitted area $A_f = A(E_f)$ can be computed as

$$A_f = \pi\left(1 - \sum_{n=1}^{\infty} n|a_n(f)|^2\right).$$

Let $A_\Sigma(h) = \sup_{f\in\Sigma^h} A_f$. Since the area functional A_f is lower semicontinuous, the existence of an extremal function, at least one for each h, easily follows from the compactness of the class Σ^h. Thus, the proof of Lemma 2.2 is standard (see [**1**] and [**2**]) and left to the reader.

Lemma 2.2. *For every $0 \le h \le 2$, there exists $f \in \Sigma^h$ such that $A_f = A_\Sigma(h)$. In addition, $A_\Sigma(h)$ is continuous in $0 \le h \le 2$.*

Let f be an extremal function in Σ^h, $0 < h < 2$. By Lemma 2.1, we may assume that E_f possesses Steiner symmetry w.r.t. the coordinate axes. This implies that the boundary $L_f = \partial E_f$ contains two “free” parts $L^+_{fr} = \{z \in \partial E_f : \Im z > 0, |\Re z| < h\}$ and $L^-_{fr} = \{z : \bar{z} \in L^+_{fr}\}$. The double symmetry of E_f and a standard subordination argument easily imply that L^+_{fr} is Jordan rectifiable, see similar considerations in [**1**].

For the “non-free” part of L_f there are two possibilities: Either it consists of two vertical segments (possibly degenerate) $I^\pm = \{w = \pm h + is : |s| \le s_f\}$, $0 \le s_f \le 2$, or it consists of two horizontal segments $I_\pm = \{w = \pm t : h_f \le t \le h\}$, $0 \le h_f \le h$.

Let $l^+_{fr} = \{e^{i\theta} : \theta_0 < \theta < \pi - \theta_0\}$ and $l^-_{fr} = \{e^{i\theta} : e^{-i\theta} \in l^+_{fr}\}$ be the “free arcs”, i.e., $l^\pm_{fr}$ are the preimages of $L^\pm_{fr}$ under the mapping f. Similarly, let $l^\pm_{nf} = f^{-1}(I^\pm)$ if the non-free boundary is vertical and $l^\pm_{nf} = f^{-1}(I_\pm)$ if it is horizontal.

Lemma 2.3. *For a fixed h, $0 \le h \le 2$, let $f \in \Sigma^h$ be extremal for $A_\Sigma(h)$ possessing Steiner symmetry w.r.t. the coordinate axes and having a vertical non-free boundary. Then:*

(i) $|f'(z)| = \beta$ *with some* $0 < \beta < 1$ *for all* $z \in l^{\pm}_{fr}$;
(ii) $|f'(e^{i\theta})|$ *strictly decreases from* $\rho = |f'(1)|$ *to* β *as* θ *runs from* 0 *to* θ_0.

Proof. First, we show that $|f'(z)|$ is constant a.e. on l^+_{fr}. Since L^+_{fr} is Jordan rectifiable it follows that the nonzero finite limit

$$f'(\zeta) = \lim_{z\to\zeta,\ z\in\overline{\mathbb{U}^*}} \frac{f(z)-f(\zeta)}{z-\zeta} \neq 0, \infty \tag{2.3}$$

exists a.e. on l_{fr}. This easily follows from [**12**, Theorem 6.8] applied to the univalent function $1/f(1/z)$. Assume that

$$0 < \beta_1 = |f'(e^{i\theta_1})| < |f'(e^{i\theta_2})| = \beta_2 < \infty \tag{2.4}$$

for $e^{i\theta_1}, e^{i\theta_2} \in l^+_{fr}$. Note that (2.3) and (2.4) allow us to apply the two point variational formulas of Lemma 5 in [**1**], see also [**2**, Lemma 10] for similar variational formulas for analytic functions univalent in the unit disk $\mathbb{U} = \{z : |z| < 1\}$. Namely, for fixed positive k_1, k_2 such that $0 < k_1 < 1 < k_2$ and $k_1\beta_1^{-1} > k_2\beta_2^{-1}$ and fixed $\varphi > 0$ small enough, we consider the two point variation $\widetilde{D}$ of D_f centered at $w_1 = f(e^{i\theta_1})$ and $w_2 = f(e^{i\theta_2})$ with inclinations φ and radii $\varepsilon_1 = k_1\varepsilon$, $\varepsilon_2 = k_2\varepsilon$ respectively, see Section 2 in [**1**]. Computing the change in the area by formula (2.11) [**1**], we find

$$\begin{aligned}\text{Area}\,(\mathbb{C}\setminus\widetilde{D}) - \text{Area}\,(\mathbb{C}\setminus D_f) &= \frac{2\pi\varphi - \sin 2\pi\varphi}{2\sin^2\pi\varphi}\varepsilon^2(k_2^2 - k_1^2) + o(\varepsilon^2)\\ &> 0\end{aligned} \tag{2.5}$$

for all $\varepsilon > 0$ small enough. Similarly, applying formula (2.10) [**1**], we get

$$\log\frac{R(\widetilde{D},\infty)}{R(D_f,\infty)} = \left[\frac{\varphi(2+\varphi)}{6(1+\varphi)^2}\frac{k_1^2}{\beta_1^2} - \frac{\varphi(2-\varphi)}{6(1-\varphi)^2}\frac{k_2^2}{\beta_2^2}\right]\varepsilon^2 + o(\varepsilon^2) > 0 \tag{2.6}$$

for all $\varepsilon > 0$ small enough and φ chosen such that the expression in the brackets is positive.

Inequalities (2.5) and (2.6), via a standard subordination argument, lead to a contradiction with the extremality of f for $A_\Sigma(h)$. Thus $|f'(e^{i\theta})| = \beta$ a.e. on $l^{\pm}_{fr}$ with some $\beta > 0$.

Since E_f is Steiner symmetric w.r.t. $\mathbb{R}$, the strict monotonicity of $|f'(e^{i\theta})|$ in $0 \le \theta < \theta_0$ follows from Lemma 3 [**1**]. To prove that $|f'(e^{i\theta})| > \beta$ for all $e^{i\theta} \in l^+_{nf}$, we assume that $\beta = |f'(e^{i\theta_1})| > |f'(e^{i\theta_2})| = \beta_2$ with $e^{i\theta_1} \in l^+_{fr}$ and some $e^{i\theta_2} \in l^+_{nf}$. Then applying the two point variation as above, we get inequalities (2.5) and (2.6), again, via a subordination argument, contradicting the extremality of f for $A_\Sigma(h)$. Hence, $|f'(e^{i\theta})| \ge \beta$ for all $e^{i\theta} \in l^{\pm}_{nf}$, which combined with the strict monotonicity property of $|f'|$ leads to the strict inequality $|f'(e^{i\theta})| > \beta$ for $e^{i\theta} \in l^{\pm}_{nf}$.

To prove that $|f'| = \beta$ everywhere on l^+_{fr}, we consider the function $g(z) = 1/f(1/z)$. The double symmetry property of Lemma 2.1 implies that $D_g = g(\mathbb{U})$ is Jordan rectifiable and starlike w.r.t. $w = 0$. Therefore, it is a Smirnov domain, see [**12**, p. 163]. This implies that $\log|g'(z)| = \log|f'(1/z)| - 2\log|zf(1/z)|$, and therefore $\log|f'(1/z)|$, can be represented by the Poisson integral

$$\log|f'(1/z)| = \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta - t)\log|f'(e^{-it})|\,dt \tag{2.7}$$

with boundary values defined a.e. on $\mathbb{T} = \{z : |z| = 1\}$, see [**12**, p. 155]. Equation (2.7) along with relations $|f'| = \beta$ a.e. on $l^{\pm}_{fr}$ and $|f'| > \beta$ everywhere on $l^{\pm}_{nf}$ shows that $1 = |f'(\infty)| \ge \beta$ with equality only for the function $f(z) \equiv z$.

If $l^+_n = \emptyset$ or consists of a single point, then the previous arguments show that $|f'| = \beta$ identically on $\mathbb{U}^*$. Therefore, $f(z) \equiv z$, which can happen only for $h = 1$. Hence, $l^+_n \ne \emptyset$ and therefore $0 < \theta_0 < \pi/2$ if $h \ne 1$. Let v be a bounded harmonic function on $\mathbb{U}$ with boundary values $\log(\beta)$ on $l^{\pm}_{fr}$ and $\log|f'(e^{-i\theta})|$ on $l^{\pm}_{nf}$. Then $v(z) - \log|f'(1/z)|$ has nontangential limit 0 a.e. on $\mathbb{T}$. Therefore, $v(z) - \log|f'(1/z)| \equiv 0$ on $\overline{\mathbb{U}}$. Hence, $|f'| = \beta$ everywhere on $l^{\pm}_{fr}$.

To show that f' is continuous at $\pm e^{\pm i\theta_0}$, we note that by the reflection principle, f can be continued analytically through l^+_{nf} and f' can be continued analytically through l^+_{fr}. This implies that f can be considered as a function analytic in a slit disk $\{z : |z - e^{i\theta_0}| < \varepsilon\} \setminus [(1-\varepsilon)e^{i\theta_0}, e^{\theta_0}]$ with $\varepsilon > 0$ small enough.

Using the Julia-Wolff lemma, see [**12**, Proposition 4.13], boundedness of $\log f'$, and well-known properties of the angular derivatives, see [**12**, Propositions 4.7, 4.9], one can prove that f' has a finite limit $f'(e^{i\theta_0})$, $|f'(e^{i\theta_0})| = \beta$, along any path in $\overline{\mathbb{U}^*}$ ending at $e^{i\theta_0}$ and by double symmetry at $-e^{\pm i\theta_0}$ and $e^{-i\theta}$. The details of this proof are similar to the arguments in Lemma 13 in [**2**].

Since $|f'|$ takes its minimal values on $\mathbb{T}$, it follows that $|f'(z)| > \beta$ for all $z \in \mathbb{U}^*$. In particular, $\beta < |f'(\infty)| = 1$. The proof is complete. $\square$

Summing up the results of this section we can prove the following lemma, which allows us to find a closed form for extremal functions.

Lemma 2.4. *Let $f \in \Sigma^h$, $0 \le h \le 2$, be extremal for $A_\Sigma(h)$ having the vertical non-free boundary. Then $\varphi(z) = zf'(z)$ maps $\mathbb{U}^*$ univalently onto a domain $\Omega(\beta,\rho) = \overline{\mathbb{C}} \setminus \{\overline{\mathbb{U}}_\beta \cup [-\rho,\rho]\}$ with $\rho = 1 + \sqrt{1-\beta^2}$ and some $\beta = \beta(h) \in (0,1)$.*

Proof. Considering boundary values of φ, we have $\arg\varphi(e^{i\theta}) = 0$ for $0 \le \theta \le \theta_0$ since $\Re f(e^{i\theta})$ is constant for such θ. Since $|\varphi(e^{i\theta})| = |f'(e^{i\theta})|$ strictly increases in $0 < \theta < \theta_0$, φ maps the arc $\{e^{i\theta} : 0 \le \theta \le \theta_0\}$ continuously and one-to-one onto the segment $\{w = t : \beta \le t \le \rho\}$ with $\rho = |f'(1)|$.

For $\theta_0 \le \theta \le \pi - \theta_0$, $|\varphi(e^{i\theta})| = \beta$. Since $|\varphi(z)| > \beta$ for all $z \in \mathbb{U}^*$ it follows that $\varphi'(e^{i\theta}) \ne 0$ for $\theta_0 < \theta < \pi - \theta_0$. Hence φ is locally univalent on l_{fr}^+ and therefore $\arg\varphi(e^{i\theta})$ strictly increases when θ runs from θ_0 to $\pi - \theta_0$.

Let $\vec{n}(\theta)$ be the inner unit normal to L_{fr}^+ at $f(e^{i\theta})$. Then $0 \le \arg\vec{n}(\theta) \le \pi$ for $\theta_0 \le \theta \le \pi - \theta_0$ since E_f is Steiner symmetric. Since $\arg\vec{n}(\theta) = \theta + \arg f'(r^{i\theta}) = \arg\varphi(e^{i\theta})$, the total variation of $\arg\varphi(e^{i\theta})$ on l_{fr}^+ is $< \pi$. This together with the equalities $\arg\varphi(e^{i\theta_0}) = 0$ and $\arg\varphi(-e^{-i\theta_0}) = \pi$ shows that φ maps l_{fr}^+ continuously and one-to-one onto the upper semicircle $\{\beta e^{i\psi} : 0 < \psi < \pi\}$.

Since E_f possesses double symmetry w.r.t. the coordinate axes it follows that φ maps $\mathbb{T}$ continuously and one-to-one in the sense of boundary correspondence onto the boundary of $\Omega(\beta,\rho)$. Now by the argument principle, φ maps $\mathbb{U}^*$ conformally and one-to-one onto $\Omega(\beta,\rho)$. Since $\varphi'(\infty) = f'(\infty) = 1$, an easy computation shows that $\rho = 1 + \sqrt{1-\beta^2}$. The lemma is proved. $\square$

3. Proof of the theorems.

Proof of Theorem 1.2. By Lemma 2.1, we may restrict ourselves to connected compact sets, which are Steiner symmetric w.r.t. the coordinate axes. Let E be such a continuum extremal for $A_\Sigma(h)$, $0 \le h \le 2$ and let $f \in \Sigma^h$ map $\mathbb{U}^*$ onto $\Omega(E)$.

First we consider the case when the non-free boundary is vertical. By Lemma 2.4, $\varphi = zf'$ maps $\mathbb{U}^*$ conformally onto $\Omega(\beta,\rho)$ with $\rho = 1 + \sqrt{1-\beta^2}$ and some $0 < \beta < 1$. The function φ can be represented as $\varphi = \beta(g^{-1}(\beta^{-1}g(z)))$, where $g(z) = z + 1/z$ is Joukowski's function. Therefore,

$$f(z) = h + \beta\int_1^z z^{-1}g^{-1}(\beta^{-1}g(z))\,dz.$$

Changing the variable of integration $\tau = g(z)$, we get

$$f(z) = h + \frac{1}{2}\int_2^\tau \frac{\tau + \sqrt{\tau^2 - 4\beta^2}}{\sqrt{\tau^2-4}}\,d\tau, \tag{3.1}$$

which gives (1.3). Since $\Re f(i) = 0$ and $\tau(i) = 0$, we find from (3.1),

$$h = \frac{1}{2}\Re\int_0^2 \frac{\tau + \sqrt{\tau^2 - 4\beta^2}}{\sqrt{\tau^2-4}}\,d\tau = \beta\int_0^\beta \sqrt{\frac{1-\beta^{-2}x^2}{1-x^2}}\,dx, \tag{3.2}$$

which is equivalent to (1.2). From (3.2) it is clear that $\beta\mathbf{E}(\beta,\beta^{-1})$ strictly increases in β. Since

$$\lim_{\beta\to 0^+}\beta\mathbf{E}(\beta,\beta^{-1})=0 \quad\text{and}\quad \beta\mathbf{E}(\beta,\beta^{-1})\big|_{\beta=1}=1,$$

it follows that for every fixed $0\le h\le 1$, (1.2) has exactly one solution in $0\le\beta\le 1$. Moreover, this shows that for $1<h\le 2$, (1.2) has no solutions and therefore extremal continua with the vertical non-free boundary can exist only for $0\le h\le 1$.

The case of extremal continua with horizontal non-free boundary was studied in [**1**, Theorem 2], which proves (1.4) for $1\le h\le 2$ and shows, in particular, that extremal continua with horizontal non-free boundary can exist only for $1\le h\le 2$. In addition, in case $h=1$ the unit disk $\overline{\mathbb{U}}$ is the unique extremal configuration of the problem under consideration.

In case $1\le h\le 2$, the maximal area $A(h)$ was found in [**1**, Theorem 2]. To compute $A(h)$ for $0\le h\le 1$, we consider the function $f\in\Sigma^h$, such that E_f is extremal for the problem under consideration and symmetric w.r.t. the coordinate axes. Applying the standard line integral formula to compute $A(h)=A(E_f)$, we get

$$\begin{aligned}A(E_f)&=\frac{1}{2}\Im\int_{\partial E_f}\overline{w}\,dw=\frac{1}{2}\Im\int_{L_{nf}}\overline{w}\,dw+\frac{1}{2}\Im\int_{L_{fr}}\overline{w}\,dw\\&=2hv_0+\frac{1}{2}\Im\int_{L_{fr}}\overline{w}\,dw,\end{aligned}$$

where

$$v_0=\Im f(e^{i\theta_0})=\frac{1}{2}\Im\int_2^{2\beta}\frac{\tau+\sqrt{\tau^2-4\beta^2}}{\sqrt{\tau^2-4}}\,d\tau=\int_\beta^1\frac{x+\sqrt{x^2-\beta^2}}{\sqrt{1-x^2}}\,dx.$$

Now, taking the condition $|f'(z)|=\beta$ for $z\in l_{fr}$ into account, we find the integral over the free boundary:

$$\begin{aligned}\frac{1}{2}\Im\int_{L_{fr}}\overline{w}\,dw&=\frac{1}{2}\Re\int_{l_{fr}}f(e^{i\theta})e^{-i\theta}\overline{f'(e^{i\theta})}\,d\theta\\&=\frac{\beta^2}{2}\Re\int_{-\pi}^{\pi}\frac{f(e^{i\theta})e^{i\theta}}{e^{2i\theta}f'(e^{i\theta})}\,d\theta-\frac{\beta^2}{2}\Re\int_{l_{nf}}\frac{f(e^{i\theta})}{e^{i\theta}f'(e^{i\theta})}\,d\theta\\&=\frac{\beta^2}{2}\Im\int_{\mathbb{T}}\frac{f(z)}{z^2f'(z)}\,dz-2\beta^2h\int_0^{\theta_0}\frac{d\theta}{|f'(e^{i\theta})|}\\&=\frac{\beta^2}{2}\Im\operatorname{Res}\left[\frac{f(z)}{z^2f'(z)},\infty\right]-2\beta^2h\int_0^{\theta_0}\frac{d\theta}{|f'(e^{i\theta})|}\\&=\pi\beta^2-2\beta^2h\int_0^{\theta_0}\frac{d\theta}{|f'(e^{i\theta})|}.\end{aligned}$$

To find $\int_0^{\theta_0} \frac{d\theta}{|f'(e^{i\theta})|}$, we change the variable of integration $z = (1/2)(\tau + \sqrt{\tau^2 - 4})$, then we get

$$\int_0^{\theta_0} \frac{d\theta}{|f'(e^{i\theta})|} = 2\int_{2\beta}^{2} \frac{d\tau}{\sqrt{4-\tau^2}(\tau + \sqrt{\tau^2 - 4\beta^2})}$$
$$= \beta^{-2}\int_{\beta}^{1} \frac{x - \sqrt{x^2 - \beta^2}}{\sqrt{1-x^2}}\, dx.$$

Combining all these computations, we obtain

$$A(h) = \pi\beta^2 + 4h\int_{\beta}^{1} \frac{\sqrt{x^2-\beta^2}}{\sqrt{1-x^2}}\, dx = \pi\beta^2 + 4h\beta'\mathbf{E}(\beta', \beta'^{-1}),$$

which proves (1.4) for $0 \le h \le 1$.

The monotonicity of $A(h)$ for $1 \le h \le 2$ was established in [**1**]. To prove that $A(h)$ is monotone in $0 \le h \le 1$, one can show by direct computation that $A'(h) > 0$ for $0 < h < 1$. Here we prefer another argument of a general nature. Since $\operatorname{cap} E = 1$, $\operatorname{diam} E \ge 2 > 2h$. Since ∂E^h is smooth, it follows that for every h', $h < h' \le 1$ there is $\theta' = \theta'(h')$, $0 < \theta' < \pi$, such that $\operatorname{proj}_{\theta'} E^h = 2h'$. This implies that the continuum $E^{h,\theta'} = \{z : e^{i\theta'}z \in E^h\}$ is admissible for the problem on $A_\Sigma(h')$ but not extremal since $E^{h,\theta'}$ clearly does not have Steiner symmetry w.r.t. $\mathbb{R}$. Therefore $A_\Sigma(h') > A(E^{h,\theta'}) = A_\Sigma(h)$. The Proof of Theorem 1.2 is complete. □

Proof of Theorem 1.1. Let E be a compact set such that $\operatorname{cap} E = 1$ and $w(E) = 2h$, $0 < h < 1$ and let E^h be the continuum from the Proof of Theorem 1.2 extremal for $A_\Sigma(h)$. It follows from Theorem 1.2 that $A(E) \le A(E^h)$ with the sign of equality only if E coincides a.e. with E^h up to a linear transformation. Note that $w(E^h) = 2h$. Indeed, if $w(E^h) = 2h' < 2h$, then $A(h) = A(E^h) \le A(h')$ contradicting the strict monotonicity property of $A(h)$. This shows that E^h has the maximal area among all compact sets, connected or not, with logarithmic capacity 1 and prescribed width $2h$.

To complete the Proof of Theorem 1.1, we consider the function $f \in \Sigma^h$, which maps U^* onto $\Omega(E^h)$. By Lemma 2.4, $\varphi = zf'$ maps $\mathbb{U}^*$ onto $\Omega(\beta, \rho)$ with certain $\rho \ge \beta \ge 0$. Since $\mathbb{C} \setminus \Omega(\beta, \rho)$ is starlike w.r.t. the origin, it follows from the classical Alexander's theorem, see [**4**, p. 43], that L_f is convex. Thus, E^h is a unique up to a linear transformation convex compact set, which maximizes the area among all such sets with $\operatorname{cap} E = 1$ and prescribed width $w(E) = 2h$. □

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PERTURBATION OF DIFFERENTIAL OPERATORS ADMITTING A CONTINUOUS LINEAR RIGHT INVERSE ON ULTRADISTRIBUTIONS

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Let P_m be a homogeneous polynomial of degree m in $n \geq 2$ variables for which the associated partial differential operator $P_m(D)$ admits a continuous linear right inverse on $C^\infty(\mathbb{R}^n)$. Examples suggest that then for each polynomial Q of degree less than m there exists a number $0 < \beta < 1$ such that the operator $(P_m+Q)(D)$ admits a continuous linear right inverse on the space of all ω_β-ultradifferentiable functions on $\mathbb{R}^n$, where $\omega_\beta(t) = (1+t)^\beta$. The main result of the present paper is to determine the optimal value of β for which the above holds for all perturbations Q of a given degree in the case $n = 3$. When $n > 3$ sufficient conditions as well as necessary conditions of this type are presented, but there is a gap between them. The results are illustrated by several examples.

1. Introduction.

The problem of determining when a given partial differential operator $P(D)$ with constant coefficients admits a continuous linear right inverse on the space $\mathcal{E}(G)$ (respectively $\mathcal{D}'(G)$) of all C^∞ functions (respectively distributions) on an open set G in $\mathbb{R}^n$ was solved in Meise, Taylor, and Vogt [**11**], where various equivalent characterizing conditions were given. In [**13**] these characterizations were extended to ω-ultradifferentiable functions $\mathcal{E}_{(\omega)}(G)$ and to ω-ultradistributions $\mathcal{D}'_{(\omega)}(G)$ of Beurling type. Since all of these equivalent characterizations are rather involved, several attempts were made to derive other characterizations in terms of the symbol P or its zero variety.

One way to attack this problem is based on a result from [**15**] which shows that if $P(D)$ admits a continuous linear right inverse on $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ then so does $P_m(D)$, where P_m is the principal part of P. Thus, one might treat P as a perturbation of its principal part P_m. In [**4**] this idea led to an explicit characterization of the homogeneous polynomials P_m of degree m in n variables for which $(P_m + Q)(D)$ admits a continuous linear right inverse on $\mathcal{E}(\mathbb{R}^n)$ (or on $\mathcal{D}'(\mathbb{R}^n)$) for each polynomial Q of degree at most $m-1$. The main part of this characterization is that — up to a complex multiple

— P_m is a real polynomial of principal type; i.e., P_m has real coefficients and $\operatorname{grad} P_m(x) \neq 0$ for $0 \neq x \in \mathbb{R}^n$.

Our aim in this paper is to refine the perturbation result just cited. The goal is to explain in terms of P_m and $l = \deg(Q) < m$ the optimal choice of $\beta = \beta(l, P_m)$ so that $(P_m + Q)(D)$ has a continuous linear right inverse on $\mathcal{E}_{(\omega_\beta)}(\mathbb{R}^n)$ for all polynomials Q of degree at most l. In dimension three, this is achieved in the following theorem:

Theorem 1.1. *Let $P_m \in \mathbb{C}[x, y, z]$ be homogeneous of degree $m \geq 2$ and let $\nu = \max\{\deg(P_m)_\theta : \theta \in V(P_m) \cap S^2\}$, where $(P_m)_\theta$ denotes the localization of P_m at θ (see Definition 3.4). For $0 \leq l < m$, let $\beta(l) := \max\{0, 1 - \frac{m-l}{\nu}\}$ and let $\mathcal{E}_{(\omega_0)}(\mathbb{R}^3) := \mathcal{E}(\mathbb{R}^3)$. Then $P_m(D)$ admits a continuous linear right inverse on $\mathcal{E}(\mathbb{R}^3)$ if and only if for each polynomial $Q \in \mathbb{C}[x, y, z]$, $\deg(Q) \leq l$, the operator $(P_m + Q)(D)$ admits a continuous linear right inverse on $\mathcal{E}_{(\omega_{\beta(l)})}(\mathbb{R}^3)$.*

Moreover, the number $\beta(l)$ is optimal in the following sense: If for some number $0 \leq \gamma < 1$ the operator $(P_m+Q)(D)$ admits a continuous linear right inverse on $\mathcal{E}_{(\omega_\gamma)}(\mathbb{R}^3)$ for each $Q \in \mathbb{C}[x, y, z]$ with $\deg Q \leq l$, then $\gamma \geq \beta(l)$.

The proof of the theorem is carried out by establishing three new results about the Phragmén-Lindelöf condition $\mathrm{PL}(\mathbb{R}^n, \omega)$ (see Definition 2.4) that was shown in [**13**] to characterize the existence of a continuous linear right inverse for $P(D)$. First, we show (Proposition 3.3) that the condition can be localized to cones about the real points in $V(P_m) \cap S^{n-1}$. Second, we use the lemma of Boutroux-Cartan and Rouche's theorem to derive a sufficient condition for such Phragmén-Lindelöf conditions to hold. From this, we then derive a sufficient condition which ensures that for a given homogeneous polynomial P_m in n variables and for all perturbations Q with $\deg(Q) \leq l < m$, the variety $V(P_m + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_\beta)$ where β is given by the formula in Theorem 1.1. Third, we use a result from [**4**] to show that $\gamma \geq \beta(l)$ when $(P_m + Q)(D)$ admits a continuous linear right inverse on $\mathcal{E}_{(\omega_\gamma)}(\mathbb{R}^n)$ for each polynomial Q of degree l, where $m - \nu \leq l < m$. The argument is based on the fact that the maximal degree of the localization of a homogeneous polynomial $P_m \in \mathbb{C}[z_1, \ldots, z_n]$ at the points in $V(P_m) \cap S^{n-1}$ greatly influences the existence of a continuous linear right inverse. The combination of these results then implies Theorem 1.1. We remark that the case $n = 2$ is much simpler and was already known.

Our results also imply that we can extend the perturbation theorem from [**4**] to ultradifferentiable functions. Further, we show that under additional hypotheses on the localization at a singular point $\xi \in V(P_m) \cap S^2$, a better result concerning Phragmén-Lindelöf conditions in cones can be obtained (Lemma 4.1). Finally, quite a number of examples are provided. They also show the known effect that for a fixed polynomial Q, the operator $(P_m + Q)(D)$ may do better than predicted by Theorem 1.1. Namely, it may have a continuous linear right inverse on $\mathcal{E}_{(\omega_\sigma)}(\mathbb{R}^n)$ for some $\sigma < \beta$. When $n = 3$, the optimal value for σ can be determined. However, this is a much more complicated procedure, based on hyperbolicity considerations, for which we refer to our forthcoming paper [6].

2. Preliminaries.

In this preliminary section we introduce the basic definitions, notation, and a few results which will be used subsequently.

Throughout this paper, $|.|$ denotes the Euclidean norm on $\mathbb{C}^n$ and $B^n(\xi, r)$ denotes the ball of center ξ and radius r in $\mathbb{C}^n$.

Definition 2.1. Let $\omega :]0,\infty[\to]0,\infty[$ be continuous and increasing and assume that it has the following properties:

(α) $\omega(2t) = O(\omega(t))$,
(β) $\int_1^\infty \frac{\omega(t)}{t^2} dt < \infty$,
(γ) $\log t = O(\omega(t))$ as t tends to infinity,
(δ) $x \mapsto \omega(e^x)$ is convex.

Then its radial extension to $\mathbb{C}^n$, defined by $\omega\colon z \mapsto \omega(|z|)$, $z \in \mathbb{C}^n$, will be called a *weight function.* Throughout this paper we assume that $\omega(0) \geq 1$. It is easy to check that this can be assumed without loss of generality.

Example 2.2. Examples of weight functions are

(a) $\omega_0(t) = \log(e + t)$,
(b) $\omega_\alpha(t) = (1 + t)^\alpha$ for $0 < \alpha < 1$.

Definition 2.3. Let V be an algebraic variety in $\mathbb{C}^n$ and Ω an open subset of V. A function $u\colon \Omega \to [-\infty,\infty[$ will be called *plurisubharmonic* if it is locally bounded above, plurisubharmonic in the usual sense on Ω_{reg}, the set of all regular points of V in Ω, and satisfies

$$u(z) = \limsup_{\xi \in \Omega_{\text{reg}}, \xi \to z} u(\xi)$$

at the singular points of V in Ω. By $\mathrm{PSH}(\Omega)$ we denote the set of all plurisubharmonic functions on Ω.

Definition 2.4. Let $V \subset \mathbb{C}^n$ be an algebraic variety and let ω be a weight function. Then V satisfies the condition $\mathrm{PL}(\mathbb{R}^n, \omega)$ if the following holds:

There exists $A \geq 1$ such that for each $\rho > 1$ there exists $B > 0$ such that each $u \in \mathrm{PSH}(V)$ satisfying (α) and (β) also satisfies (γ), where:

(α) $u(z) \leq |\mathrm{Im}\, z| + O(\omega(z))$, $z \in V$,
(β) $u(z) \leq \rho\, |\mathrm{Im}\, z|$, $z \in V$,
(γ) $u(z) \leq A\, |\mathrm{Im}\, z| + B\omega(z)$, $z \in V$.

Phragmén-Lindelöf conditions and continuous linear right inverses 2.5. To explain the significance of the condition $\mathrm{PL}(\mathbb{R}^n, \omega)$, let

$n \geq 2$, let $P(z) = \sum_{|\alpha| \leq m} a_\alpha z^\alpha$ be a complex polynomial of degree $m > 0$, and let

$$V(P) := \{z \in \mathbb{C}^n \colon P(-z) = 0\}$$

denote its zero variety. Then $V(P)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega)$ if and only if the linear partial differential operator

$$P(D) \colon \mathcal{E}_{(\omega)}(\mathbb{R}^n) \to \mathcal{E}_{(\omega)}(\mathbb{R}^n), \quad P(D)f := \sum_{|\alpha| \leq m} a_\alpha i^{-|\alpha|} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}$$

admits a continuous linear right inverse, where $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ is the Fréchet space of all ω-ultradifferentiable functions of Beurling type (see [**2**]). This follows from the general characterization in Meise, Taylor, and Vogt [**15**]. Recall that $\mathcal{E}_{(\omega_0)}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$ and that in this case the characterization of the existence of continuous linear right inverses was already obtained in Meise, Taylor, and Vogt [**12**]. Note also that Palamodov [**17**] proved that a differential complex of C^∞-functions over $\mathbb{R}^n$ splits if and only if the associated varieties satisfy $\mathrm{PL}(\mathbb{R}^n, \omega_0)$.

From [**12**], Lemma 2.9, we recall the following lemma:

Lemma 2.6. *For each $n \in \mathbb{N}$ the function $H : \mathbb{C}^n \to \mathbb{R}$, defined as $H(z) := \frac{1}{2}((\operatorname{Im} z)^2 - (\operatorname{Re} z)^2)$ is plurisubharmonic and has the following properties:*

(a) $H(z) \leq |\operatorname{Im} z|, \quad |z| \leq 1$,
(b) $H(z) \leq |\operatorname{Im} z| - \frac{1}{2}, \quad |z| = 1$,
(c) $H(x) \leq 0, \quad x \in \mathbb{R}^n$,
(d) $H(iy) \geq 0, \quad y \in \mathbb{R}^n$.

Definition 2.7. For $d = (d_1, \dots, d_n) \neq (0, \dots, 0)$ with $d_j \in \mathbb{N}_0$, $1 \leq j \leq n$, a nonzero polynomial $P \in \mathbb{C}[z_1, \dots, z_n]$ is said to be d-quasihomogeneous of d-degree $m \geq 0$ if

$$P(z) = \sum_{\langle d, \alpha \rangle = m} a_\alpha z^\alpha, \quad z \in \mathbb{C}^n,$$

where $\langle d, \alpha \rangle = \sum_{j=1}^n d_j \alpha_j$. The zero polynomial is considered to be d-quasihomogeneous of d-degree $-\infty$.

Combining Lemma 3.2 and Lemma 3.6 from [**4**] we get the following lemma:

Lemma 2.8. *For $n \geq 2$ let $P \in \mathbb{C}[z_1, \dots, z_n]$ be d-quasihomogeneous of d-degree m and let $Q \in \mathbb{C}[z_1, \dots, z_n]$ be the sum of d-quasihomogeneous polynomials of d-degrees less than m. Assume that for some k, $1 \leq k < n$, the following conditions are fulfilled*:

(1) *$d_1 = \dots = d_k < d_j$ for $j > k$,*
(2) *there exists $\zeta = (\zeta', \zeta'') \in \mathbb{C}^k \times \mathbb{R}^{n-k}$ satisfying $P(\zeta) = 0$ and $\zeta'' \neq 0$,*

(3) *if* $P(z', \zeta'') = 0$ *then* $\operatorname{Im} z' \neq 0$.

If $V(P+Q)$ *satisfies* $\mathrm{PL}(\mathbb{R}^n, \omega)$ *for some weight function* ω *and* $D := \max\{d_j : \zeta_j \neq 0\}$, *then* ω *satisfies* $t^{d_1/D} = O(\omega(t))$ *as* t *tends to infinity.*

3. Main results.

The aim of this section is to derive conditions which imply that for a homogeneous poynomial $P \in \mathbb{C}[z_1, \dots, z_n]$ for which $V(P)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_0)$ the variety $V(P+Q)$ satifies $\mathrm{PL}(\mathbb{R}^n, \omega_{\beta(l)})$ for all polynomials Q of degree $l < m$. The number $\beta(l)$ will be shown to be sharp.

Throughout this section we assume $n \geq 2$ unless other assumptions are made.

Instead of working with the property $\mathrm{PL}(\mathbb{R}^n, \omega)$ as it is given in Definition 2.4, it is often easier to consider the intersection of the variety V with cones. This will be made more precise in Proposition 3.3. To formulate this proposition, recall that for a point ξ in the unit sphere $S^{n-1} \subset \mathbb{R}^n$, a set M with $\overline{M} \subset B^n(0,1)$, and $r > 0$ the *cone* $\Gamma(\xi, M, r)$ around the ray generated by ξ with profile M, truncated at r, is defined as

$$\Gamma(\xi, M, r) := \bigcup_{t>r} t(\xi + M).$$

Definition 3.1. For $P \in \mathbb{C}[z_1, \dots, z_n] \setminus \mathbb{C}$ let $V := V(P)$, let ω be a weight function, and let $\Gamma := \Gamma(\xi, G, r)$ be a cone for which G is an open neighborhood of zero in $\mathbb{C}^n$. We say that V satisfies the condition $\mathrm{PL}(V(P), \Gamma, \omega)$ if there exist a compact set $K \subset G$ which is a neighborhood of zero and numbers $A_1 \geq 1$ and $r_1 \geq r$ such that for each $\rho > 0$ there exists B_ρ such that for each $u \in \mathrm{PSH}(V \cap \Gamma)$ the following two conditions:

(α) $u(z) \leq |z|$, $z \in V \cap \Gamma$,

(β) $u(z) \leq \rho\, |\operatorname{Im} z|$, $z \in V \cap \Gamma$,

imply

(γ) $u(z) \leq A_1\, |\operatorname{Im} z| + B_\rho \omega(z)$, $z \in V \cap \Gamma(\xi, K, r_1)$.

Lemma 3.2. *For a polynomial* $P \in \mathbb{C}[z_1, \dots, z_n] \setminus \mathbb{C}$ *denote by* P_m *its principal part and let* ω *be a weight function. If* $V(P)$ *satisfies the condition* $\mathrm{PL}(\mathbb{R}^n, \omega)$ *then for each* $\xi \in V(P_m) \cap S^{n-1}$, $r \geq 1$, *and each open zero neighborhood* G *with* $\overline{G} \subset B^n(0,1)$, *the variety* $V(P)$ *has the property* $\mathrm{PL}(V(P), \Gamma(\xi, G, r), \omega)$.

Proof. Fix $\xi \in V(P_m) \cap S^{n-1}$, $r \geq 1$, and G as in the statement of the lemma. Then fix a compact zero neighborhood $K \subset G$, choose $0 < \eta < 1$ so small that $K + B(0, 2\eta) \subset G$ and note that $\max\{|z| : z \in K\} \leq 1$. Next fix $u \in \mathrm{PSH}(V(P) \cap \Gamma(\xi, G, r))$ and assume that u satisfies Conditions 3.1 (α) and (β). Now fix

$$z_0 = t_0\xi + t_0 w_0 \in \Gamma(\xi, K, r)$$

and distinguish the following two cases:

Case 1: $|\mathrm{Im}\, z_0| \geq \eta t_0$.

Then $|z_0| \leq t_0\, |\xi + w_0| \leq 2t_0$, the present hypothesis, and Condition 3.1 (α) on u imply

$$u(z_0) \leq |z_0| \leq 2t_0 \leq \frac{2}{\eta}\, |\mathrm{Im}\, z_0|\, .$$

Case 2: $|\mathrm{Im}\, z_0| < \eta t_0$.

Then note that for each $z \in B(\mathrm{Re}\, z_0, \eta t_0)$ the present hypothesis and the choice of η imply

$$z - t_0\xi = z - \mathrm{Re}\, z_0 - i\,\mathrm{Im}\, z_0 + z_0 - t_0\xi \in t_0 B(0, 2\eta) + t_0 K \subset t_0 G.$$

Hence we can define $\varphi : V(P) \to [-\infty, \infty[$ by

$$\varphi(z) := \max\left\{\frac{\eta}{9}u(z) + \eta t_0 H\left(\frac{z - \mathrm{Re}\, z_0}{\eta t_0}\right),\ |\mathrm{Im}\, z|\right\},\ z \in V \cap B(\mathrm{Re}\, z_0, \eta t_0)$$

and by $\varphi(z) := |\mathrm{Im}\, z|$ elsewhere on V, where H denotes the function defined in Lemma 2.6. To see that φ is plurisubharmonic on V, note that for $z \in V \cap \partial B(\mathrm{Re}\, z_0, \eta t_0)$ the above estimate for z_0 implies

$$|z| = |z - \mathrm{Re}\, z_0 + \mathrm{Re}\, z_0| \leq \eta t_0 + |z_0| \leq 3t_0.$$

Hence Condition 3.1 (α) for u gives $u(z) \leq 3t_0$. By the properties of H, this implies

$$\frac{\eta}{9}u(z) + \eta t_0 H\left(\frac{z - \mathrm{Re}\, z_0}{\eta t_0}\right) \leq \frac{\eta}{3}t_0 + |\mathrm{Im}\, z| - \frac{\eta}{2}t_0 < |\mathrm{Im}\, z|$$

for each $z \in V(P) \cap \partial B(\mathrm{Re}\, z_0, \eta t_0)$. Thus, φ is plurisubharmonic on $V(P)$. From 3.1 (β) and the properties of H it follows that

$$\varphi(z) \leq \left(\frac{\eta}{9}\rho + 1\right) |\mathrm{Im}\, z|\, , \text{ and } \varphi(z) = |\mathrm{Im}\, z| + O(1), \quad z \in V.$$

Since $V(P)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega)$, we conclude from these estimates the existence of $A \geq 1$ depending only on $V(P)$, and of B, depending on ρ, such that

$$\varphi(z) \leq A\, |\mathrm{Im}\, z| + B\omega(z), \quad z \in V.$$

Evaluating this estimate at z_0 and using the properties of H together with the definition of φ, we get

$$A\, |\mathrm{Im}\, z_0| + B\omega(z_0) \geq \varphi(z_0) \geq \frac{\eta}{9}\, u(z_0) + \eta t_0 H\left(\frac{i\,\mathrm{Im}\, z_0}{\eta t_0}\right) \geq \frac{\eta}{9}\, u(z_0),$$

and hence

$$u(z_0) \le \frac{9A}{\eta} |\mathrm{Im}\, z_0| + \frac{9B}{\eta} \omega(z_0).$$

Both cases together show that u satisfies Condition 3.1 (γ) with $A_1 := \frac{9A}{\eta}$ and $B_\rho := \frac{9B}{\eta}$. ☐

Proposition 3.3. *Let $P \in \mathbb{C}[z_1, \dots, z_n]$ be a polynomial of degree $m \ge 1$ and denote by P_m its principal part. Then for a given weight function ω the following two conditions are equivalent*:

(a) *$V(P)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega)$.*
(b) *$V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_0)$ and for each $\xi \in V(P_m) \cap S^{n-1}$ there exist an open neighborhood G_ξ of zero and $r_\xi > 0$ such that $V(P)$ satisfies $\mathrm{PL}(V(P), \Gamma(\xi, G_\xi, r_\xi), \omega)$.*

Proof. (a) ⇒ (b): If $V(P)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega)$ then also $V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_0)$ by Meise, Taylor, and Vogt [**15**], Theorem 4.1. Hence the first condition of (b) is fulfilled. The second one holds by Lemma 3.2.

(b) ⇒ (a): Since $V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_0)$ by the present hypothesis, it follows from [**15**], Theorem 3.13, and Meise, Taylor, and Vogt [**12**], Theorem 5.1, that $V(P)$ satisfies Condition (RPL) defined in [**12**], 2.2. This means that there exists $A_0 \ge 1$ such that for each $\rho > 0$ there exists $B_\rho > 0$ such that each $u \in \mathrm{PSH}(V(P))$ satisfying

$$u(z) \le |z| + o(|z|) \text{ and } u(z) \le \rho\, |\mathrm{Im}\, z|\,, \ z \in V(P)$$

also satisfies

$$u(z) \le A_0\, |z| + B_\rho, \ z \in V(P). \tag{3.1}$$

From this we get in particular that each $u \in \mathrm{PSH}(V(P))$ which satisfies Conditions 2.4 (α) and (β) of $\mathrm{PL}(\mathbb{R}^n, \omega)$ already satisfies (3.1). Consequently, $v(z) := \frac{1}{A_0} u(z) - B_\rho$ satisfies Conditions 3.1 (α) and (β) with $\rho' := \frac{\rho}{A_0}$ in any cone $\Gamma(\xi, G_\xi, r_\xi)$, $\xi \in V(P_m) \cap S^{n-1}$. Therefore we can use the hypothesis and a compactness argument to conclude similarly as in the proof of Meise and Taylor [**10**], Proposition 4.5, that there exist $A_1 \ge 1$ and $C_\rho > 0$ such that u satisfies

$$u(z) \le A_1\, |\mathrm{Im}\, z| + C_\rho \omega(z), \quad z \in V(P).$$

Hence (a) holds. ☐

To apply Proposition 3.3 we will use the following lemma, which is the key step for our positive results. To formulate it we need the following definitions:

Definition 3.4. Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ and $\theta \in \mathbb{C}^n$ be given. Then the *localization* P_θ of P at θ is defined as the lowest order nonvanishing homogeneous polynomial in the Taylor series expansion of P at θ.

Definition 3.5. For $P \in \mathbb{C}[z_1, \ldots, z_n]$ the variety $V(P)$ is said to be *locally hyperbolic* at $\xi \in V(P) \cap \mathbb{R}^n$ if there exist a projection $\pi : \mathbb{C}^n \to \mathbb{C}^n$ and an open neighborhood U of ξ such that the following conditions are satisfied:

(a) $\ker \pi$ and $\operatorname{im} \pi$ are spanned by real vectors, $\dim \ker \pi = 1$, and $(\ker \pi) \cap V(P_\xi) = \{0\}$.
(b) Whenever $z \in V(P) \cap U$ and $\pi(z)$ is real then z is real.

Lemma 3.6. *Let $P_m \in \mathbb{C}[z_1, \ldots, z_n]$ be homogeneous of degree $m \geq 2$. Assume $P_m(\xi) = 0$, $\deg(P_m)_\xi = \mu$, and $(P_m)_\xi(0, \ldots, 0, 1) \neq 0$ for $\xi = (1, 0, \ldots, 0)$ and define $\beta(l) := \max(0, 1 - \frac{m-l}{\mu})$ for $0 \leq l < m$. Then for each $Q \in \mathbb{C}[z_1, \ldots, z_n]$, $\deg Q \leq l < m$, there exist $\eta, \sigma > 0$ and $R, C > 1$ such that the following holds*:

(a) *For each $(z_1, z') \in \mathbb{C}^{n-1}$ satisfying $\left|\frac{z_1}{|z_1|} - 1\right| < \eta$, $|z_1| > R$, and $|z'| < \eta\,|z_1|$, and for each $\zeta \in \mathbb{C}$ satisfying $|\zeta| < \sigma\,|z_1|$ and $(P_m + Q)(z_1, z', \zeta) = 0$, there exists $w \in \mathbb{C}$ satisfying $|w| < \sigma\,|z_1|$, $P_m(z_1, z', w) = 0$, and*

$$|\zeta - w| \leq C\,|z_1|^{\beta(l)}.$$

(b) *If $V(P_m)$ is locally hyperbolic at ξ with respect to the projection $\pi : (z'', z_n) \mapsto (z'', 0)$, then the parameters η, σ, R and C in* (a) *can be chosen in such a way that for each (z_1, z', ζ) satisfying $(P_m + Q)(z_1, z', \zeta) = 0$ and $\left|\frac{z_1}{|z_1|} - 1\right| < \eta$, $|z_1| > R$, $|z'| < \eta\,|z_1|$, $|\zeta| < \sigma\,|z_1|$, and (z_1, z') real, we have*

$$|\operatorname{Im} \zeta| \leq C\,|z_1|^{\beta(l)}.$$

Proof. The present hypotheses imply for the Taylor series expansion of P_m at ξ (see [3], Lemma 3.9)

$$P_m(z_1, z', z_n) = \sum_{j=\mu}^{m} z_1^{m-j} p_j(z', z_n),$$

where p_j is either homogeneous of degree j or identically zero, and where $(P_m)_\xi(z_1, z', z_n) = p_\mu(z', z_n)$. From this expansion and the hypotheses we get

$$P_m(1, 0', z_n) = \sum_{j=\mu}^{m} z_n^j p_j(0', 1) = z_n^\mu \left((P_m)_\xi(0, 0', 1) + \sum_{j=\mu+1}^{m} z_n^{j-\mu} p_j(0, 0', 1) \right).$$

Hence we can choose $\sigma > 0$ such that $z_n \mapsto P_m(1, 0', z_n)$ has exactly μ zeros in the disk $B^1(0, \sigma)$ and does not vanish on $\partial B^1(0, \sigma)$. Hence it follows from the Weierstraß preparation theorem that we can choose $\eta > 0$ such that for $(z_1, z', z_n) \in G := B^1(1,\eta) \times B^{n-2}(0,\eta) \times B^1(0,\sigma)$ we have

$$P_m(z_1, z', z_n) = U(z_1, z', z_n) \sum_{j=0}^{\mu} z_n^j c_j(z_1, z') = U(z_1, z', z_n) \prod_{j=1}^{\mu} (z_n - \beta_j(z_1, z')), \tag{3.2}$$

where U is a holomorphic function which does not vanish on G. In fact, shrinking η if necessary, we may assume that there exists $\alpha > 0$ such that $|U(z)| > \alpha$ for all $z \in G$. We also may assume $|\beta_j(z_1, z')| \le \sigma/2$ for $(z_1, z') \in B^1(1,\eta) \times B^{n-2}(0,\eta)$ and $1 \le j \le \mu$. Next note that by the homogeneity of P_m, U is also homogeneous and extends holomorphically to the cone

$$\Gamma := \left\{ (z_1, z', z_n) \in \mathbb{C}^n : \left| \frac{z_1}{|z_1|} - 1 \right| < \eta, \quad |z'| < \eta |z_1|, \quad |z_n| < \sigma |z_1|, \quad \text{and} \quad |z_1| > 1 \right\}.$$

For $z = (z_1, z', z_n) \in \Gamma$ we have

$$|U(z)| = |z_1|^{m-\mu} \left| U\left(\frac{z}{|z_1|} \right) \right| \ge \alpha |z_1|^{m-\mu}. \tag{3.3}$$

Also by homogeneity the functions β_j extend to the cone

$$\Gamma' := \left\{ (z_1, z') \in \mathbb{C}^{n-1} : \left| \frac{z_1}{|z_1|} - 1 \right| < \eta, \text{ and } |z'| < \eta |z_1| \right\}.$$

For $(z_1, z') \in \Gamma'$ define $F(z_1, z', z_n) := \prod_{j=1}^{\mu} (z_n - \beta_j(z_1, z'))$ and note that by the Lemma of Boutroux-Cartan (see Levin [9], Theorem I.10) the following holds: For each $(z_1, z') \in \Gamma'$ and each $\delta > 0$ there exist finitely many disks $D_l(z_1, z')$, $1 \le l \le d(z_1, z')$, for which the sum of the radii is at most 2δ, such that

$$|F(z_1, z', z_n)| \ge \left(\frac{\delta}{e} \right)^{\mu} \quad \text{whenever } z_n \in \mathbb{C} \setminus \bigcup_{l=1}^{d} D_l. \tag{3.4}$$

We may assume that the $D_l(z_1, z')$ are constructed as in the proof that is given in [9]; then each $D_l(z_1, z')$ contains at least one zero of $F(z_1, z', \cdot)$. Now fix $Q \in \mathbb{C}[z_1, \dots, z_n]$, $\deg Q \le l < m$. Then it is easy to check that there exists a constant $M > 1$ such that

$$|Q(z)| \le M |z_1|^l, \quad z \in \Gamma.$$

Choose $r > 1$ and $R > 1$ so large that $\alpha r^\mu > M$ and such that $4ert^{\beta(l)} < \frac{\sigma}{2}t$ for $t \geq R$. Next fix $(z_1, z') \in \Gamma'$ satisfying $|z_1| > R$ and let

$$\delta := er\,|z_1|^{\beta(l)}.$$

Then it follows from (3.2), (3.3), (3.4), and our choice of r that for each $z_n \in \mathbb{C} \setminus \bigcup_{l=1}^d D_l(z_1, z')$ satisfying $|z_n| < \sigma\,|z_1|$ we have

$$\begin{aligned} \left|P_m(z_1, z', z_n)\right| &\geq \alpha\,|z_1|^{m-\mu}\left(\frac{\delta}{e}\right)^\mu \\ &= \alpha r^\mu\,|z_1|^{m-\mu+\mu\beta(l)} \\ &> M\,|z_1|^l \geq \left|Q(z_1, z', z_n)\right|. \end{aligned} \tag{3.5}$$

Let now $\zeta \in B^1(0, \sigma\,|z_1|)$ with $(P_m + Q)(z_1, z', \zeta) = 0$ be given. Then

$$|P_m(z_1, z, \zeta)| = \left|Q(z_1, z', \zeta)\right| \leq M\,|z_1|^l,$$

and thus there is $l \leq d(z_1, z')$ with $\zeta \in D_l(z_1, z')$. Let $w \in D_l(z_1, z')$ be a zero of $F(z_1, z', \cdot)$. Then $|\zeta - w| < 4\delta$ since the sum of the radii of all disks $D_l(z_1, z')$ is at most 2δ. We have shown that the estimate in (a) holds. To see that $|w| < \sigma\,|z_1|$, note that $F(z_1, z', w) = 0$ implies the existence of j, $1 \leq j \leq \mu$, such that $w = \beta_j(z_1, z')$. Hence the choice of η implies

$$|w| \leq |z_1|\left|\beta_j(1, z'/\,|z_1|)\right| \leq \frac{\sigma\,|z_1|}{2}.$$

Thus the proof of Part (a) is complete.

To prove (b) we note first that under the present hypothesis we can choose η and σ so small that $B^1(1, \eta) \times B^{n-2}(0, \eta) \times B^1(0, \sigma)$ is contained in the set U which exists by local hyperbolicity. This implies that the zeros $\beta_j(z', z_n)$ are all real whenever $(z', z_n) \in \Gamma'$ is real. Hence the estimate in (a) implies the one in (b). □

Lemma 3.7. *Let V be an algebraic variety in $\mathbb{C}^n$ and ω a weight function. Assume that for $\xi = (1, 0, \ldots, 0)$ and $G = B^{n-1}(0, \delta) \times B^1(0, \sigma)$ $(0 < \delta, \sigma \leq 1)$ the map $\pi : V \cap \Gamma(\xi, G, r) \to \Gamma(\xi, G, r)$, $\pi(z', z_n) := (z', 0)$, is proper and satisfies the following condition:*

$$\begin{aligned} &\textit{There exists } C > 0 \textit{ such that } |\operatorname{Im} z_n| \leq C\omega(z) \\ &\textit{for each } z \in V \cap \Gamma(\xi, G, r) \textit{ with } \pi(z) \textit{ real.} \end{aligned} \tag{3.6}$$

Then V satisfies $\mathrm{PL}(V, \Gamma(\xi, G, r), \omega)$.

Proof. To show that there are a compact set $K \subset G$ and a constant $A_1 \geq 1$ such that V satisfies $\mathrm{PL}(V, \Gamma(\xi, G, r), \omega)$ let $K := \frac{1}{2}\overline{G}$, fix $u \in \mathrm{PSH}(V \cap \Gamma(\xi, G, R))$, and assume that u satisfies Conditions 3.1 (α) and (β). Then let $G' := B^{n-1}(0,\delta)$, fix $t > r$, and define

$$\varphi : G' \to [-\infty, \infty[,$$
$$z' \mapsto \max\{u(t\xi + tz) : t\xi + tz \in V,\ z \in G,\ \pi(z) = (z', 0)\}.$$

Then φ is plurisubharmonic outside the branch locus of π. Since π is proper by hypothesis, it follows from Hörmander [8], Lemma 4.4, that φ extends to a plurisubharmonic function on G'. Condition 3.1 (α) for u and $0 < \delta$, $\sigma \le 1$ imply

$$\varphi(z') \le \max\{|t(\xi + z)| : z \in G\} \le 2t,$$

while Condition 3.1 (β) together with (3.6) implies

$$\begin{aligned}\varphi(z') &\le \max\{\rho\, |\mathrm{Im}(t\xi + tz)| : t\xi + tz \in V,\ z \in G,\ \pi(z) = (z', 0)\}\\ &\le \max\{\rho C\omega(tz) : z \in G\} \le \rho C\omega(t).\end{aligned}$$

From these two estimates for φ and classical estimates of the harmonic measure of the half disk (see, e.g., Nevanlinna [16], Section 38) it now follows that there is a constant A_0, depending only on the dimension, so that

$$\varphi(z') \le \frac{A_0}{\delta} 2t \left|\mathrm{Im}\, z'\right| + \rho C\omega(t),\ z \in \overline{B^{n-1}(0, \delta/2)}.$$

To evaluate this further, note that for $k \in K$ we have

$$|t\xi + tk| \ge t(1 - |k|) \ge \frac{t}{2}.$$

Note also that our requirements on the weight functions imply the existence of a constant $L > 0$ such that $\omega(2s) \le L\omega(s)$ for $s \ge 0$. Therefore, the definition of φ and the previous estimates imply for $t\xi + tk \in V$

$$\begin{aligned}u(t\xi + tk) \le \varphi(k) &\le \frac{2A_0}{\delta} t\, |\mathrm{Im}\, k| + \rho C\omega(t)\\ &\le \frac{2A_0}{\delta} |\mathrm{Im}(t\xi + tk)| + \rho CL\omega(t\xi + tk).\end{aligned}$$

Since $k \in K$ and $t > r$ were chosen arbitrarily, this estimate shows that u satisfies 3.1 (γ) for $A_1 := \frac{2A_0}{\delta}$ and $B_\rho := \rho CL$. □

Theorem 3.8. *Let $P_m \in \mathbb{C}[z_1, \ldots, z_n]$ be homogeneous of degree $m \ge 2$ and assume that $V(P_m)$ satisfies* $\mathrm{PL}(\mathbb{R}^n, \omega_0)$. *Let*

$$\nu := \max\left\{\deg(P_m)_\theta : \theta \in V(P_m) \cap S^{n-1}\right\}$$

and define

$$\beta(l) := \max\left(0, 1 - \frac{m - l}{\nu}\right) \quad \text{for } 0 \le l < m. \tag{3.7}$$

If for each $\xi \in V(P_m)_{\mathrm{sing}} \cap S^{n-1}$ the variety $V(P_m)$ is locally hyperbolic at ξ, then for each $Q \in \mathbb{C}[z_1, \ldots, z_n]$ with $\deg Q \le l < m$ the variety $V(P_m + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_{\beta(l)})$, *where ω_β is defined in Example 2.2.*

Proof. Since $V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_0)$ by hypothesis, the theorem follows from Proposition 3.3 once we show that the second condition in 3.3 (b) is fulfilled. To show this, we first factorize $P_m = \prod_{j=1}^{s} q_j^{k_j}$, where the polynomials q_j are irreducible and where $\prod_{j=1}^{s} q_j$ is square-free. Since $V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_0)$, also $V(q_j)$ has this property for each j. By Meise, Taylor, and Vogt [**14**], Lemma 2, this implies that there exists $c_j \in \mathbb{C}$, $|c_j| = 1$, so that $c_j q_j$ has real coefficients. Hence it is no restriction to assume that each q_j has real coefficients.

Now fix a regular point $a \in V(P_m)$ of length 1. Then there exists an index i so that $q_i(a) = 0$. This implies $q_j(a) \neq 0$ for all $j \neq i$ by the following argument: If $q_j(a) = 0$ for some $j \neq i$ then $V(q_j)$ and $V(q_i)$ must coincide in a neighborhood of a since a is a regular point of $V(P_m)$. But then $V(q_j) = V(q_i)$ since both varieties are irreducible. Hence q_j and q_i are proportional, in contradiction to $\prod_{l=1}^{s} q_l$ being square-free.

Next note that

$$(P_m)_a = ((q_i)_a)^{k_i} \prod_{j \neq i} (q_j(a))^{k_j}.$$

Since a is a regular point of $V(P_m)$ and hence of $V(q_i)$, the localization satisfies $(q_i)_a(z) = \sum_{j=1}^{n} \frac{\partial q_i}{\partial z_j}(a) z_j$, which implies $\deg(P_m)_a = k_i \leq \nu$. After a real linear change of variables we may assume $a = (1, 0, \ldots, 0)$ and $\frac{\partial q_i}{\partial z_n}(a) \neq 0$. Then the real and the complex implicit function theorem imply the existence of a neighborhood U of $(1, 0, \ldots, 0) \in \mathbb{C}^{n-1}$, of $\delta > 0$, and of a holomorphic function $\beta : U \to B^1(0, \delta)$ which is real over real points so that

$$V(P_m) \cap (U \times W) = \{(z', \beta(z')) : z' \in U\}.$$

Hence $V(P_m)$ is locally hyperbolic at a with respect to $\pi(z', z_n) := (z', 0)$ in these coordinates.

If $a \in V(P_m) \cap S^{n-1}$ is a singular point of $V(P_m)$, then $V(P_m)$ is locally hyperbolic at a by hypothesis. Then we can perform a real linear change of coordinates so that in the new coordinates $a = (1, 0, \ldots, 0)$ and $\pi : (z', z_n) \mapsto (z', 0)$ is the projection which exists by local hyperbolicity. If we let $\mu = \deg(P_m)_a$, then in both cases the hypotheses of Lemma 3.6 (b) are fulfilled. Now Lemma 3.6 implies that the hypotheses of Lemma 3.7 are fulfilled in a suitable cone $\Gamma(a, G_a, r_a)$ for $\omega = \omega_{\beta(l,a)}$, where $\beta(l, a) = \max(0, 1 - \frac{m-l}{\mu})$. By the definition of μ we have $\mu \leq \nu$ and hence $\beta(l, a) \leq \beta(l)$. Thus, the second condition of Proposition 3.3 (b) holds with $\omega = \omega_{\beta(l)}$, which completes the proof of the theorem. $\square$

Remark. Theorem 3.8 also holds if we replace the hypothesis "$V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_0)$" by the following one: "Each irreducible factor of P_m has real coefficients up to a complex factor and is not elliptic". Under this hypothesis, the present proof shows that $V(P_m)$ is locally hyperbolic at each real regular point of $V(P_m) \cap S^{n-1}$. Hence the hypotheses imply that this property holds at each point of $V(P_m) \cap S^{n-1}$. By Hörmander [**8**], Theorem 6.5, this implies that $V(P_m)$ satisfies Condition (HPL) and therefore it follows from Meise, Taylor, and Vogt [**15**], Corollary 3.14 that $V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_0)$, which is needed for the application of Proposition 3.3. Otherwise the proof remains unchanged.

Remark. Note that for $n \geq 4$ there are homogeneous polynomials $P_m \in \mathbb{C}[z_1, \ldots, z_n]$ for which $V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_0)$ but which are not locally hyperbolic at some singular points of $V(P_m) \cap S^{n-1}$. When $n = 3$, this cannot happen, as a result of Hörmander [**8**] shows. This fact will be used in Corollary 3.12 below.

As a corollary of Theorem 3.8 we get:

Corollary 3.9. *Let $k_j \in \mathbb{N}$ and $P_j \in \mathbb{R}[z_1, \ldots, z_n]$, $1 \leq j \leq s$, be given. Assume that each P_j is irreducible, homogeneous of degree q_j, and not elliptic and that $\prod_{j=1}^s P_j$ is square-free. Set $m := \sum_{j=1}^s q_j k_j$, $P := \prod_{j=1}^s P_j^{k_j}$, $k := \max_{1 \leq j \leq s} k_j$, assume $m \geq 2$, and let $\beta(l)$ be as in (3.7) with $\nu = k$. If all points in $V(P) \cap S^{n-1}$ are regular points of $V(P)$, then for each $Q \in \mathbb{C}[z_1, \ldots, z_n]$ with $\deg Q \leq l$ the variety $V(P + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_{\beta(l)})$.*

Proof. Since the localization of a product equals the product of the localizations of its factors, the present hypotheses imply

$$\nu = \max\{\deg(P)_\theta : \theta \in V(P_m) \cap S^{n-1}\} = \max_{j=1,\ldots,s} k_j = k.$$

Since all points of $V(P) \cap S^{n-1}$ are regular points of $V(P)$, the corollary follows from Theorem 3.8. $\square$

As an obvious consequence of Corollary 3.9 we get the following result which is a reformulation of [**4**], Corollary 4.7:

Corollary 3.10. *Let $P \in \mathbb{R}[z_1, \ldots, z_n]$ be homogeneous of degree μ and assume that $\operatorname{grad} P(x) \neq 0$ for all $x \in V(P) \cap S^{n-1}$. Let $k \in \mathbb{N}$ be given so that $k\mu \geq 2$. Then for each $Q \in \mathbb{C}[z_1, \ldots, z_n]$ with $\deg Q =: l < k\mu$ the variety $V(P^k + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_{\beta(l)})$ for $\beta(l)$ as in (3.7).*

It will be shown in Theorem 3.14 that in Corollary 3.10 the condition $\operatorname{grad} P(x) \neq 0$ for all $x \in V(P) \cap S^{n-1}$ is in fact necessary. To prove this result, we use the following lemma:

Lemma 3.11. *Let $P \in \mathbb{R}[z_1, \ldots, z_n]$ be homogeneous of degree $m \geq 2$, let*

$$\nu := \max\{\deg P_\theta : \theta \in V(P) \cap S^{n-1}\},$$

and fix $p \in \mathbb{N}$ with $1 \leq p < \nu$ and a weight function ω. If for each $Q \in \mathbb{C}[z_1, \ldots, z_n]$ with $\deg Q \leq m - \nu + p$ the variety $V(P+Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega)$, then $t^{p/\nu} = O(\omega(t))$ as t tends to infinity.

Proof. Fix $\xi \in S^{n-1}$ with $\deg P_\xi = \nu$. After a real linear change of variables we may assume $\xi = (0, \ldots, 0, 1)$. Hence [3], Lemma 3.9, implies

$$P(z', z_n) = \sum_{j=\nu}^{m} z_n^{m-j} Q_j(z'),$$

where the polynomials $Q_j \in \mathbb{C}[z_1, \ldots, z_{n-1}]$ are either zero or homogeneous of degree j and where $Q_\nu(z') = P_\xi(z', z_n)$. Now let $S := P + iz_n^{m-\nu+p}$ and note that $V(S)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega)$ by the present hypothesis. To apply Lemma 2.8, let $d := (p, \ldots, p, \nu)$. Then

$$q(z', z_n) := z_n^{m-\nu} Q_\nu(z') + iz_n^{m-\nu+p}$$

has d-degree $(m - \nu + p)\nu$. For $\nu + 1 \leq j \leq m$ the term $z_n^{m-j} Q_j(z')$ has d-degree $(m - j)\nu + jp$. Since

$$(m - \nu + p)\nu - ((m - j)\nu + jp) = (j - \nu)(\nu - p) > 0$$

the polynomial q is the term in S with the highest d-degree. Next choose $a \in S^{n-1}$ such that $Q_\nu(a) \neq 0$ and consider the polynomial

$$\lambda \mapsto q(\lambda a, 1) = Q_\nu(\lambda a) + i = \lambda^\nu Q_\nu(a) + i.$$

Since Q_ν has real coefficients by hypothesis, we can choose $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ such that $\zeta' := \lambda_0 a \in \mathbb{C}^{n-1} \setminus \mathbb{R}^{n-1}$ satisfies $q(\zeta', 1) = 0$. Finally, note that the equation

$$0 = q(z', 1) = Q_\nu(z') + i$$

has no real solutions since Q_ν has real coefficients. Thus we have shown that all conditions of Lemma 2.8 are fulfilled. Therefore, the present lemma follows from Lemma 2.8. □

Corollary 3.12. *Let $P_m \in \mathbb{C}[x, y, z]$ be homogeneous of degree $m \geq 2$, let*

$$\nu := \max\{\deg(P_m)_\theta : \theta \in V(P_m) \cap S^2\}$$

and define $\beta(l)$ as in (3.7). Then the following assertions are equivalent:

(a) *$V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^3, \omega_0)$.*

(b) *For each $0 \leq l < m$ and for each $Q \in \mathbb{C}[x, y, z]$, $\deg Q \leq l$, the variety $V(P_m + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^3, \omega_{\beta(l)})$.*

(c) *There exist $Q \in \mathbb{C}[x, y, z]$, $\deg Q < m$, and a weight function ω such that the variety $V(P_m + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^3, \omega)$.*

Moreover, the numbers $\beta(l)$ are optimal in the following sense: If l satisfies $m - \nu \leq l < m$ and if for some weight function ω and each polynomial Q of degree at most l the variety $V(P + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^3, \omega)$, then $t^{\beta(l)} = O(\omega(t))$ as t tends to infinity.

Proof. (b) ⇒ (c): This holds obviously.

(c) ⇒ (a): This holds by Meise, Taylor, and Vogt [15], Theorem 4.1.

(a) $\Rightarrow$ (b): Since $V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^3, \omega_0)$ it also satisfies the Phragmén-Lindelöf condition $\mathrm{HPL}(\mathbb{R}^3)$, considered in Hörmander [**8**] (by Meise, Taylor, and Vogt [**15**], Proposition 3.9). By Hörmander [**8**], Theorem 6.5, this implies that $V(P_m)$ is locally hyperbolic at each $\xi \in V(P_m) \cap S^2$. Hence (b) follows from Theorem 3.8.

The additional assertion obviously follows from Lemma 3.11. $\square$

Remark. Note that Corrollary 3.12 implies Theorem 1.1 by the results of Meise, Taylor, and Vogt [**13**], mentioned in 2.5.

From Hörmander [**7**], 10.4.11, we recall the following definition:

Definition 3.13. A polynomial $P \in \mathbb{C}[z_1, \dots, z_n]$ is said to be of *principal type* if its principal part P_m satisfies

$$\sum_{j=1}^{n} \left| \frac{\partial P_m}{\partial x_j}(x) \right| \neq 0 \quad \text{for each } x \in \mathbb{R}^n \setminus \{0\}.$$

Note that by Euler's rule $\langle x, \operatorname{grad} P_m(x) \rangle = m P_m(x)$, so P is of principal type if and only if $\operatorname{grad} P_m(x) \neq 0$ for each $x \in V(P_m) \cap \mathbb{R}^n \setminus \{0\}$.

Theorem 3.14. *For a homogeneous polynomial $P \in \mathbb{C}[z_1, \dots, z_n]$ of degree $\mu \geq 1$ and $k \in \mathbb{N}$ satisfying $\mu k \geq 2$ the following assertions are equivalent (for the definition of the weights ω_α see Example 2.2):*

(a) *$V(P^k + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_0)$ for each $Q \in \mathbb{C}[z_1, \dots, z_n]$ with $\deg Q \leq (\mu - 1)k$.*

(b) *For each $p \in \mathbb{N}_0$, $0 \leq p < k$, and each $Q \in \mathbb{C}[z_1, \dots, z_n]$ with $\deg Q \leq (\mu - 1)k + p$ the variety $V(P^k + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_{p/k})$.*

(c) *There exists $p \in \mathbb{N}_0$, $0 \leq p < k$, such that the assertion in (b) holds.*

(d) *P is of principal type and real up to a complex factor, and each irreducible factor of P admits a real zero $\xi \neq 0$.*

Proof. From (d) we get (a) and (b) by Corollary 3.10. Obviously, (a) implies (c) and also (b) implies (c). Hence it suffices to prove that (c) implies (d). To do so, note first that $V(P^k)$ and hence $V(P)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_{p/k})$. Since P is homogeneous, it follows from Meise, Taylor, and Vogt [**15**], Theorem 3.3, that $V(P)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_0)$. From this and [**15**], Theorem 3.13, we get that for each irreducible factor q of P we have $\dim_{\mathbb{R}} V(q) \cap \mathbb{R}^n = n - 1$. Thus the last condition in (d) is fulfilled. Since $V(P)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_0)$, Lemma 2 in Meise, Taylor, and Vogt [**14**] implies the existence of $\lambda \in \mathbb{C} \setminus \{0\}$ such that λP has real coefficients. Hence the second condition in (d) holds, and we may assume that P has real coefficients. To show that P is of principal type we argue by contradiction and assume that for some $a \in V(P) \cap S^{n-1}$ we have $\operatorname{grad} P(a) = 0$. This implies $\deg P_a \geq 2$. Since the localization of a product is the product of the localizations of its factors it follows that

$$\nu := \max\{\deg(P^k)_\theta : \theta \in V(P^k) \cap S^{n-1}\} \geq 2k.$$

Now let $s := \nu - k + p$, where $0 \le p < k$ is chosen according to (c). Then

$$\frac{s}{\nu} = 1 - \nu^{-1}(k-p) > 1 - k^{-1}(k-p) = \frac{p}{k} \text{ and } \mu k - \nu + s = (\mu - 1)k + p.$$

Hence (c) implies that for each $Q \in \mathbb{C}[z_1, \ldots, z_n]$ with $\deg Q \le \mu k - \nu + s$ the variety $V(P^k + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega_{p/k})$. By Lemma 3.11 this implies

$$t^{s/\nu} = O(\omega_{p/k}(t)) = O(t^{p/k}).$$

Since $\frac{s}{\nu} > \frac{p}{k}$, this contradiction completes the proof. $\square$

Remark. Theorem 3.14 extends [4], Theorem 4.3 and Corollary 4.7.

4. Further results and examples.

In this section we first indicate that there are further variants of Lemma 3.6 which may be helpful in considering examples. Then we provide several examples to illustrate the results of the previous section and to explain the difficulties that one encounters in proving perturbation results.

Lemma 4.1. *Let $P_m \in \mathbb{C}[x, y, z]$ be homogeneous of degree $m \ge 2$, let $\xi \in V(P_m) \cap S^2$ satisfy $\deg(P_m)_\xi =: \mu \ge 2$, and assume that $V(P_m)$ is locally hyperbolic at ξ and that $(P_m)_\xi$ is square-free. Let $Q \in \mathbb{C}[x, y, z]$ with $\deg Q < m$ be given. Decompose Q as $Q = \sum_{j=\mu}^{m-1} q_j$, where q_j is either homogeneous of degree j or zero. If $\deg(q_j)_\xi \ge \mu$ for each j, then $V(P_m + Q)$ satisfies $\mathrm{PL}(V(P_m + Q), \Gamma(\xi, G_\xi, r_\xi), \omega_0)$ in a suitable cone $\Gamma(\xi, G_\xi, r_\xi)$.*

Proof. After a real linear change of variables we may assume $\xi = (1, 0, 0)$. By [3], Lemma 3.9, we then have in these coordinates

$$P_m(x, y, z) = \sum_{j=\mu}^{m} x^{m-j} p_j(y, z),$$

where p_j is either homogeneous of degree j or identically zero and where $p_\mu(y, z) = (P_m)_\xi(x, y, z)$. We may also assume that the coordinates have been chosen so that $\pi(x, y, z) := (x, y, 0)$ is a projection for which the local hyperbolicity condition holds. Then we get

$$p_\mu(y, z) = c \prod_{j=1}^{\mu} (z - a_j y)$$

for suitable numbers $c, a_1, \ldots, a_\mu \in \mathbb{C}$. Since p_μ is square-free by hypothesis, we have $a_i \ne a_j$ for $i \ne j$ and hence

$$\delta := \min\{|a_i - a_j| : 1 \le i, j \le \mu,\ i \ne j\} > 0.$$

By Braun [1], Corollary 12, the local hyperbolicity of $V(P_m)$ at ξ implies the existence of $\sigma > 0$ and $0 < \eta < \frac{1}{2}$ and of holomorphic functions $\beta_j : B(1, \eta) \times B(0, \eta) \to B(0, \sigma)$, $1 \le j \le \mu$ so that

$$V(P_m) \cap (B(1,\eta) \times B(0,\eta) \times B(0,\sigma)) = \bigcup_{j=1}^{\mu} \{(x, y, \beta_j(x,y)) : (x,y) \in B(1,\eta) \times B(0,\eta)\}$$

and so that $\beta_j(x,y)$ is real for real (x,y). By the homogeneity of P_m we get

$$\beta_j(x,y) = x\,\beta_j\Big(1, \frac{y}{x}\Big) = x \sum_{k=1}^{\infty} b_{j,k} \left(\frac{y}{x}\right)^k, \quad 1 \le j \le \mu,$$

where $\{b_{j,1} : 1 \le j \le \mu\} = \{a_j : 1 \le j \le \mu\}$. Hence we may assume $a_j = b_{j,1}$ for $1 \le j \le \mu$. It is no restriction to assume η to be so small that

$$\left| \sum_{k=2}^{\infty} b_{j,k} \left(\frac{y}{x}\right)^k \right| \le \delta \quad \text{if } (x,y) \in B(1,\eta) \times B(0,\eta).$$

Arguing as in the proof of Lemma 3.6 we get for $\eta > 0$ small enough with

$$\Gamma := \left\{ (x,y,z) \in \mathbb{C}^3 : \left| \frac{x}{|x|} - 1 \right|,\ |x| > R,\ \left|\frac{y}{x}\right| < \eta,\ \left|\frac{z}{x}\right| < \sigma \right\}$$

and $\Gamma' := \{(x,y) \in \mathbb{C}^3 : (x,y,0) \in \Gamma\}$ that

$$P_m(x,y,z) = U(x,y,z)F(x,y,z), \quad (x,y,z) \in \Gamma,$$

where $F(x,y,z) = \prod_{j=1}^{\mu}(z - \beta_j(x,y))$. We also get the existence of $\alpha > 0$ such that

$$|U(x,y,z)| \ge \alpha\,|x|^{m-\mu},\ (x,y,z) \in \Gamma.$$

Choosing η small enough, we get

$$\left| \beta_j\left(1, \frac{y}{x}\right) - a_j \frac{y}{x} \right| = \left| \sum_{k=2}^{\infty} b_{j,k} \left(\frac{y}{x}\right)^k \right| \le 2\delta \left|\frac{y}{x}\right|,\ 1 \le j \le \mu.$$

Next fix $1 \le i \le \mu$ and define for $(x,y) \in \Gamma'$

$$z(x,y,\lambda) := \beta_i(x,y) + \lambda$$

where for some $\rho > 1$ (to be determined later), $\lambda \in \mathbb{C}$ satisfies

$$|\lambda| = \min(\rho, \delta\,|y|).$$

For $j \ne i$ the previous choices imply

$$\begin{aligned} |z(x,y,\lambda) - \beta_j(x,y)| &= \left| (a_i - a_j)y + y \sum_{k=2}^{\infty} (b_{i,k} - b_{j,k}) \left(\frac{y}{x}\right)^k + \lambda \right| \\ &\ge 4\delta\,|y| - 2\delta\,|y| - |y| = \delta\,|y|\,. \end{aligned}$$

Moreover, we get

$$
\begin{aligned}
\left|\frac{z(x,y,\lambda)}{x}\right| &= \left|a_i\frac{y}{x}+\sum_{k=2}^{\infty} b_{i,k}\left(\frac{y}{x}\right)^k+\frac{\lambda}{x}\right| \\
&\le (|a_i|+2\delta+\delta)\left|\frac{y}{x}\right| \\
&\le \max_{1\le j\le\mu}(|a_j|+3\delta)\eta<\sigma,
\end{aligned}
\tag{4.1}
$$

provided that η is small enough. All together we get for $(x,y)\in\Gamma'$

$$
|P_m(x,y,z(x,y,\lambda))| = |(UF)(x,y,z(x,y,\lambda))| \ge \alpha\,|x|^{m-\mu}\,|\lambda|\,(\delta\,|y|)^{\mu-1}.
$$

Now fix $Q=\sum_{j=\mu}^{m-1} q_j$ as in the hypothesis. For $\kappa_j := \deg(q_j)_\xi$ there is M_j with

$$
|q_j(1,\eta,\zeta)| \le M_j\,|(\eta,\zeta)|^{\kappa_j} \quad \text{if } |(\eta,\zeta)|\le 1.
$$

Hence (4.1) implies

$$
\begin{aligned}
|q_j(x,y,z(x,y,\lambda))| &= \left|q_j\left(1,\frac{y}{x},\frac{z(x,y,\lambda)}{x}\right)\right|\,|x|^j \\
&\le M_j(1+|a_j|+3\delta)^{\kappa_j}\left|\frac{y}{x}\right|^{\kappa_j}|x|^j.
\end{aligned}
$$

Since $\kappa_j\ge\mu$ by hypothesis, the last estimate implies the existence of M such that

$$
|Q(x,y,z(x,y,\lambda))| \le M\,|x|^{m-1}\left|\frac{x}{y}\right|^{\mu}.
$$

Now we claim that we can choose $R>1$ and $\rho>1$ so that

$$
M<\alpha\,|\lambda|\left|\frac{x}{y}\right|\delta^{\mu-1} \quad \text{for } |x|>R,\ (x,y)\in\Gamma'. \tag{4.2}
$$

To see this, assume first $|\lambda|=\delta\,|y|$. Then

$$
\alpha\,|\lambda|\left|\frac{x}{y}\right|\delta^{\mu-1} = \alpha\delta^{\mu}\,|y|\left|\frac{x}{y}\right| = \alpha\delta^\mu\,|x|>M \quad \text{if } |x|>R=\max\Bigl(1,\ \frac{M}{\delta^\mu\alpha}\Bigr).
$$

If $|\lambda|=\rho$ then

$$
\alpha\,|\lambda|\left|\frac{x}{y}\right|\delta^{\mu-1} = \gamma\rho\left|\frac{x}{y}\right|\delta^{\mu-1} > \alpha\rho\frac{1}{\eta}\delta^{\mu-1}>M
$$

if we choose $\rho>\frac{\eta M}{\alpha\delta^{\mu-1}}$. Hence (4.2) holds. From it and $\kappa_j\le\mu$ we now get

$$\begin{aligned} (4.3) \qquad |Q(x,y,z(x,y,\lambda))| &\le M\,|x|^{m-1}\left|\frac{y}{x}\right|^{\mu} \\ &< \alpha\,|\lambda|\left|\frac{x}{y}\right|\delta^{\mu-1}\,|x|^{m-1}\left|\frac{y}{x}\right|^{\mu} \\ &= \alpha\,|\lambda|\,|x|^{m-1}\,\delta^{\mu-1}\left|\frac{y}{x}\right|^{\mu-1} \\ &\le \alpha\,|\lambda|\,|x|^{m-1}\,\delta^{\mu-1}\left|\frac{y}{x}\right|^{\mu-1} \\ &\le |P_m(x,y,z(x,y,\lambda))|, \end{aligned}$$

since $\left|\frac{y}{x}\right| < 1$. From this estimate and the theorem of Rouché it follows for $(x,y) \in \Gamma'$, $y \neq 0$, $1 \le i \le \mu$, that the functions $\zeta \mapsto P_m(x,y,\zeta)$ and $\zeta \mapsto (P_m+Q)(x,y,\zeta)$ have the same number of zeros in the disk $|\zeta - \beta_i(x,y)| < r$, where $r = \min(\rho, \delta\,|y|)$. Since P_m has exactly one zero in this disk, we get that $(P_m+Q)(x,y,\cdot)$ also has exactly one zero in that disk.

A similar application of the theorem of Rouché shows that for each $(x,y) \in \Gamma'$ the functions $\zeta \mapsto P_m(x,y,\zeta)$ and $(P_m+Q)(x,y,\zeta)$ have the same number of zeros in the disk of radius $\sigma\,|x|$. All together we have shown that under the present hypotheses the conclusion of Lemma 3.6 holds with $\beta(l) = 0$. Since $\beta_j(x,y)$ is real for each $(x,y) \in \Gamma'$ and $1 \le j \le \mu$, this implies the existence of a constant $C > 0$ such that for each $(x,y,\zeta) \in \Gamma$ satisfying $(P_m+Q)(x,y,\zeta) = 0$ we have the estimate

$$|\operatorname{Im}\zeta| \le C.$$

Therefore, the assertion of the lemma follows from Lemma 3.6. □

Lemma 4.1 does not hold without the hypothesis "$(P_m)_\xi$ is square-free". To provide an example for this fact, we will use the following lemma. Since its proof uses only basic calculus, we omit it.

Lemma 4.2. *For $t > 0$ and $a \in \mathbb{R}$ consider the polynomial*

$$p(z;t,a) := (z^2 - t^2)(z - 2t) + a.$$

Then for each $t > 0$ and each a satisfying $|a| \le \frac{1}{2}t^3$, all zeros of $p(\cdot;t,a)$ are real.

Example 4.3. Define P_6, $Q \in \mathbb{R}[x,y,z]$ by

$$P_6(x,y,z) := (xz - y^2)(xz + y^2)(xz - 2y^2), \quad Q(x,y,z) := x^2y^3.$$

Then the following assertions hold:

(a) For $\xi = (\pm 1, 0, 0)$, the variety $V(P_6+Q)$ satisfies $\mathrm{PL}(V(P_6+Q),$ $\Gamma(\xi, G_\xi, r_\xi), \omega_{1/3})$.
(b) For each $\xi \in V(P_6) \cap S^2$, the variety $\xi \neq (\pm 1, 0, 0)$, $V(P_6+Q)$ satisfies $\mathrm{PL}(V(P_6+Q),\ \Gamma(\xi, G_\xi, r_\xi),\ \omega_0)$.
(c) $V(P_6+Q)$ satisfies $\mathrm{PL}(\mathbb{R}^3, \omega_{1/3})$.

(d) If $V(P_6+Q)$ satisfies $\mathrm{PL}(\mathbb{R}^3,\omega)$ for some weight function ω then $t^{1/3} = O(\omega(t))$ as t tends to infinity.

Proof. (a) Let $\xi := (1,0,0)$, define

$$\Gamma := \left\{(x,y,z) \in \mathbb{C}^3 : \left|\frac{x}{|x|} - 1\right| < \frac{1}{4},\ |y| < \frac{1}{4}\,|x|\,,\ |z| < \frac{1}{4}\,|x|\,,\ |x| > 1\right\},$$

and let $\Gamma' = \{(x,y) \in \mathbb{C}^2 : (x,y,0) \in \Gamma\}$. Fix $(x,y) \in \Gamma' \cap \mathbb{R}^2$ and assume first that $|y| > (2x)^{2/3}$. Then we have

$$\left|\frac{y^3}{x}\right| < \frac{1}{2}\left|\frac{y^2}{x}\right|^3.$$

Hence Lemma 4.2 implies that all zeros of the equation

$$(P_6 + Q)(x,y,z) = x^3\left(\left(z^2 - \left(\frac{y^2}{x}\right)^2\right)\left(z - 2\frac{y^2}{x}\right) + \frac{y^3}{x}\right)$$

are real.

Assume next that $(x,y) \in \Gamma' \cap \mathbb{R}^2$ satisfies $|y| < 2\,|x|^{2/3}$. If $y = 0$ then obviously, $(P_6 + Q)(x,0,z)$ has a zero of order 3 at the origin. Hence we may assume $|y| > 0$. Then let $\delta := 2^{1/3}e\,|y|\,|x|^{-1/3}$ and apply the Lemma of Boutroux-Cartan to get

$$\left|\left(z^2 - \left(\frac{y^2}{x}\right)^2\right)\left(z - 2\frac{y^2}{x}\right)\right| \geq \left(\frac{\delta}{e}\right)^3 = 2\left|\frac{y^3}{x}\right| > \left|\frac{y^3}{x}\right|$$

outside a finite number of disks for which the sum of their radii is at most 2δ. Applying the theorem of Rouché at the boundary of these disks we get that for each zero ζ of the equation $(P_6 + Q)(x,y,\zeta) = 0$ we have

$$|\mathrm{Im}\,\zeta| \leq 4e2^{1/3}\left|\frac{y^3}{x}\right|^{1/3} \leq 4e2^{1/3}\left(\frac{2^3x^2}{x}\right)^{1/3} \leq e2^4\,|x|^{1/3}.$$

Combining this estimate with Lemma 3.7, we get (a) for ξ. Since it is easy to check that the same arguments apply also for $-\xi$, the proof of (a) is complete.

(b) Whenever $\xi \in V(P_6) \cap S^2$ and $\xi \neq (\pm 1,0,0)$ and $\xi \neq (0,0,\pm 1)$ then $\operatorname{grad} P_6(\xi) \neq 0$. Hence a real linear change of coordinates shows that we can apply Lemma 3.6 with $\mu = 1$. Hence in this case the assertion follows from Lemma 3.6 and 3.7. If $\xi = (0,0,1)$ then let

$$\Gamma' := \left\{(y,z) \in \mathbb{C}^2 : \left|\frac{z}{|z|} - 1\right| < \frac{1}{4}, \quad |z| > 2,\ |y| < \frac{1}{4}\,|z|\right\}$$

and note that

$$(P_6 + Q)(x,0,z) = x^3z^3$$

has a zero of order 3 at the origin for each $|z| > 0$. For $(y,z) \in \Gamma' \cap \mathbb{R}^2$ with $y \neq 0$ we have

$$(P_6 + Q)(0, y, z) = 2y^6 > 0,$$

$$(P_6 + Q)\left(\frac{3}{2}\,\frac{y^2}{z}, y, z\right) = -\frac{5}{8}y^6 + \frac{9}{4}\,\frac{y^7}{z^2} = -y^6\left(\frac{5}{8} - \frac{9}{4}\,\frac{y}{z^2}\right) < 0,$$

$$(P_6 + Q)\left(-2\frac{y^2}{z}, y, z\right) = -12y^6 + 4\frac{y^7}{z^2} = -y^6\left(12 + \frac{4y}{z^2}\right) < 0.$$

Since $(P_6 + Q)(\cdot, y, z)$ has degree three and real coefficients, this implies that for each $(y,z) \in \Gamma' \cap \mathbb{R}^2$, all zeros of $x \mapsto (P_6 + Q)(x, y, z)$ are real. Hence Lemma 3.7 implies the assertion of (b) also in this case. The same arguments apply for $\xi = (0, 0, -1)$.

(c) This assertion follows from Proposition 3.3 since the second condition in 3.3 (b) holds by the present parts (a) and (b) and since $V(P_6)$ satisfies $\mathrm{PL}(\mathbb{R}^3, \omega_0)$. The latter assertion follows from the fact that P_6 is the product of three polynomials, each of which defines a wave operator.

(d) It is easy to check that $P_6 + Q$ is $(3,2,1)$-quasihomogeneous and that the polynomial $z^3 - 2z^2 - z + 3$ has a zero τ which is not real. Then $\zeta := (1, 1, \tau)$ satisfies

$$(P_6 + Q)(\zeta) = (\tau - 1)(\tau + 1)(\tau - 2) + 1 = \tau^3 - 2\tau^2 - \tau + 3 = 0.$$

Hence $P := P_6 + Q$, ζ and $d = (3,2,1)$ satisfy the hypotheses of [**4**], Lemma 3.2. Hence (d) follows from this lemma. $\square$

The following example shows how Lemma 4.1 can be applied.

Example 4.4. Define the polynomial P_5 by

$$P_5(x,y,z) := z^2 y(x^2 - y^2) + x^5 + (x - y)^2 x^3 + y^5.$$

Let q_3, q_4, and p_3 be polynomials in $\mathbb{C}[x,y]$, each of which is homogeneous of degree 3, 4, and 3 respectively or identically zero, and define

$$Q(x,y,z) := q_3(x,y) + q_4(x,y) + z p_3(x,y).$$

Then $V(P_5 + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^3, \omega_0)$.

An interesting example of a perturbation Q satisfying the above conditions is given by $Q(x,y,z) := x^4 - xy^2$.

Proof. To derive the assertion from Proposition 3.3, let $P := P_5 + Q$, so that P_5 is the principal part of P. Some computation shows that $\operatorname{grad} P_5$ vanishes only on $V(P_5) \cap \{(0,0,t) : t \in \mathbb{C}\}$ so that

$$V(P_5)_{\mathrm{sing}} \cap S^2 = \{(0,0,1), (0,0,-1)\}.$$

Since P_5 is irreducible, it follows from Meise, Taylor, and Vogt [**15**], Corollary 3.14, that $V(P_5)$ satisfies $\mathrm{PL}(\mathbb{R}^3, \omega_0)$ if and only if $V(P_5)$ satisfies

Hörmander's Phragmén-Lindelöf condition, which by Hörmander [8], Theorem 6.5, is equivalent to $V(P_5)$ being locally hyperbolic at each $\xi \in V(P_5) \cap S^2$. Since P_5 has real coefficients, this condition obviously holds at each regular point $\xi \in V(P_5) \cap S^2$. At the singular points $\xi_\pm := (0, 0, \pm 1)$, it holds by the following observation: The reduction of P_5 at $\xi_\pm$, defined by

$$q_\pm(x, y) := P_5(x, y, \pm 1) = y(x^2 - y^2) + x^5 + (x^2 - y^2)x^3 + y^5,$$

has a zero variety which is locally hyperbolic at the origin. Therefore it follows from [5], Lemma 6.1, that $V(P_5)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(\xi)$ and hence $V(P_5)$ is locally hyperbolic at $\xi_\pm$, by Braun [1], Corollary 12. Thus we have shown that $V(P_5)$ satisfies $\mathrm{PL}(\omega_0)$, i.e., the first condition of Proposition 3.3 (b) holds. To show that also the second one is fulfilled, note that $V(P)$ satisfies $\mathrm{PL}(V(P), \Gamma(\xi, G_\xi, r_\xi), \omega_0)$ for each $\xi \in V(P_5) \cap S^2 \setminus \{\pm\xi\}$ and a suitable cone $\Gamma(\xi, G_\xi, r_\xi)$ because of Lemma 3.6 and Lemma 3.7, since $\operatorname{grad} P_5(\xi)$ is not zero. To show that the same condition also holds at $\xi = \xi_\pm$, note that $\deg(P_5)_{\xi_\pm} = 3$ and that $(P_5)_{\xi_\pm}(x, y, z) = y(x^2 - y^2)$ is square-free. The decomposition of Q into homogeneous components is $Q = Q_3 + Q_4$ with $Q_3(x, y, z) = q_3(x, y)$ and $Q_4(x, y, z) = q_4(x, y) + zp_3(x, y)$. Hence $\deg(Q_j)_{\xi_\pm} \geq 3 = \deg(P_5)_{\xi_\pm}$ for $j = 3, 4$, and it follows from Lemma 4.1 that $V(P)$ satisfies $\mathrm{PL}(V(P), \Gamma(\xi_\pm, G_{\xi_\pm}, r_{\xi_\pm}), \omega_0)$ for suitable cones $\Gamma(\xi_\pm, G_{\xi_\pm}, r_{\xi_\pm})$. This shows that also the second condition of 3.3 (b) is fulfilled. Therefore, the assertion follows from Proposition 3.3. $\square$

Example 4.5. Let $P \in \mathbb{R}[x, y, z]$ be defined as

$$P(x, y, z) := z^2(x^2 + y^2 - z^2)^3.$$

Then for each $Q \in \mathbb{C}[x, y, z]$ with $\deg Q \leq 5$ the operator $(P+Q)(D)$ admits a continuous linear right inverse on $\mathcal{D}'(\mathbb{R}^3)$. If $\deg Q = 6$ or $\deg Q = 7$, then $(P + Q)(D)$ admits a continuous linear right inverse on $\mathcal{D}'_{(\omega_{1/3})}(\mathbb{R}^3)$ or $\mathcal{D}'_{(\omega_{2/3})}(\mathbb{R}^3)$, respectively. This follows from 2.5 and Proposition 3.9.

Remark. In Lemma 3.11 and Theorem 3.14 the statements are optimal if perturbations by arbitrary polynomials of a given degree are considered. For an individual polynomial it may happen that $(P + Q)(D)$ admits a continuous linear right inverse on $\mathcal{D}'_{(\sigma)}(\mathbb{R}^n)$ for a weight function σ which grows more slowly than indicated by 3.9 or 3.14. Such examples can be constructed easily from our results, as we show next.

Example 4.6. Let $P(x, y, z) := (x^2 + y^2 - z^2)^2$. Then 2.5 and Theorem 3.14 imply that $(P + Q)(D)$ admits a continuous linear right inverse on $\mathcal{D}'_{(\omega_{1/2})}(\mathbb{R}^3)$ whenever $\deg Q \leq 3$ and that for each ω satisfying $\omega(t) = o(t^{1/2})$ there is a polynomial Q of degree 3 such that $(P + Q)(D)$ does not admit a continuous linear right inverse on $\mathcal{D}'_{(\omega)}(\mathbb{R}^3)$.

Nevertheless, in special cases like, e.g., $Q_0(x, y, z) := (x^2+y^2-z^2)y$, more can be said. For this example

$$P + Q_0 = ST \quad \text{for } S := x^2 + y^2 - z^2 \text{ and } T := x^2 + y^2 - z^2 + y.$$

Since $S(D)$ is the wave operator, $S(D)$ admits a continuous linear right inverse on $\mathcal{D}'(\mathbb{R}^3)$. By Proposition 3.9, the same holds for $T(D)$, and thus also for the product $(P + Q_0)(D)$.

Remark 4.7. It has been known for some time that the conclusion of Proposition 3.9 does not hold in general if there are singular points of $V(P_m)$ in S^{n-1}. The simplest example is provided by the polynomial $P_2 \in \mathbb{R}[x, y, z]$,

$$P_2(x, y, z) = xy.$$

For $Q(x, y, z) := iz$, the operator $(P_2 + Q)(D)$ does not admit a continuous linear right inverse in $\mathcal{D}'_{(\omega)}(\mathbb{R}^3)$ if the weight function ω satisfies $\omega(t) = o(t^{1/2})$. This was shown in Meise, Taylor, and Vogt [**15**], Example 4.9, but can also be derived from Lemma 3.11.

In [**6**] we derive new necessary conditions for a given polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ to satisfy $\mathrm{PL}(\mathbb{R}^n, \omega)$ for a given weight function and we show that these conditions are characterizing when $n = 3$. To achieve this, a more refined analysis of the behavior of $V(P)$ in conoids is necessary and ω-hyperbolicity conditions in these conoids play a crucial role.

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COMPACT HYPERSURFACES IN A UNIT SPHERE WITH INFINITE FUNDAMENTAL GROUP

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It is our purpose to study curvature structures of compact hypersurfaces in the unit sphere $S^{n+1}(1)$. We proved that the Riemannian product $S^1(\sqrt{1-c^2})\ \times S^{n-1}(c)$ is the only compact hypersurfaces in $S^{n+1}(1)$ with infinite fundamental group, which satisfy $r \geq \frac{n-2}{n-1}$ and $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, where $n(n-1)r$ is the scalar curvature of hypersurfaces and $c^2 = \frac{n-2}{nr}$. In particular, we obtained that the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ is the only compact hypersurfaces with infinite fundamental group in $S^{n+1}(1)$ if the sectional curvatures are nonnegative.

1. Introduction.

Let M be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n+1$. It is well-known that the investigation on curvature structures of compact hypersurfaces is important and interesting. In 1977, S.Y. Cheng and Yau [4] studied compact hypersurfaces with constant scalar curvature in the unit sphere $S^{n+1}(1)$. They proved that let M be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, if $r \geq 1$ and the sectional curvatures of M are nonnegative, then M is isometric to the totally umbilical hypersurface $S^n(c)$ or the Riemannian product $S^k(c_1) \times S^{n-k}(c_2)$ $1 \leq k \leq n-1$, where $S^k(c)$ denote the sphere of radius c. In order to prove this theorem, they introduced a differential operator $\square$ defined by

$$\square f = \sum_{i,j=1}^{n} (nH\delta_{ij} - h_{ij})\nabla_i \nabla_j f,$$

for any C^2-function f on M. Where h_{ij} and H are components of the second fundamental form and the mean curvature of M, respectively. We should notice the following:

(1) The differential operator $\square$ is self-adjoint.
(2) The differential operator $\square$ is degenerate elliptic if $r \geq 1$.

Therefore, in order to prove their theorem, they must make use of the properties that the differential operator $\square$ is self-adjoint and degenerate elliptic.

And in order to obtain the estimate $\sum_{i,j,k=1}^{n} h_{ijk}^2 \geq n^2|\mathrm{grad}H|^2$, which is very important in the proof of their theorem, the condition of $r \geq 1$ and the assumption of constant scalar curvature is essential. Where h_{ijk}'s are components of the covariant differentiation of the second fundamental form. Hence, the condition $r \geq 1$ and the assumption of constant scalar curvature play an essential role in the proof of their theorem. Further, by making use of the similar method which was used by Nakagawa and the author in [3] and the differential operator introduced by S.Y. Cheng and Yau, Li [5] proved that let M be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, if $r \geq 1$ and $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then M is isometric to either the totally umbilical hypersurface or the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ with $c^2 = \frac{n-2}{nr} \leq \frac{n-2}{n}$, where S is the squared norm of the second fundamental form of M. These properties that the differential operator $\Box$ is self-adjoint and degenerate elliptic are indispensable in the proof of his theorem again. And the estimate $\sum_{i,j,k=1}^{n} h_{ijk}^2 \geq n^2|\mathrm{grad}H|^2$ is also essential in the proof of his theorem.

On the other hand, for any $0 < c < 1$, by considering the standard immersions $S^{n-1}(c) \subset \mathbf{R}^n$, $S^1(\sqrt{1-c^2}) \subset \mathbf{R}^2$ and taking the Riemannian product immersion $S^1(\sqrt{1-c^2}) \times S^{n-1}(c) \hookrightarrow \mathbf{R}^2 \times \mathbf{R}^n$, we obtain a compact hypersurface $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$, where $r = \frac{n-2}{nc^2} > 1 - \frac{2}{n}$. Hence, some of Riemannian products $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ do not appear in these results of S.Y. Cheng and Yau [4] and Li [5]. Moreover, Cheng [2] proved:

Theorem C (Cheng [2]). *Let M be an n-dimensional complete hypersurface with constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$. If M has only two distinct principal curvatures one of which is simple, then, $r > 1 - \frac{2}{n}$ holds and M is isometric to $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ if $r \neq \frac{n-2}{n-1}$ and $S \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, where $c^2 = \frac{n-2}{nr}$.*

From the assertion above, it is natural and interesting to generalize these results due to S.Y. Cheng and Yau [4] and Li [5] to the case $r > 1 - \frac{2}{n}$. That is, it is interesting to prove the following:

Problem 1 (cf. Cheng [2]). Let M be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$. If $r > 1 - \frac{2}{n}$ and $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then M is isometric to the totally umbilical hypersurface or the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$.

It is our purpose to try to solve this problem above. Since the problem seems to be a very hard problem, we shall try to solve it under a topological condition. It is known that $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ has infinite fundamental group. Hence, we shall consider compact hypersurfaces with infinite fundamental group in the unit sphere $S^{n+1}(1)$. The following theorems will be proved.

Theorem 1. *Let M be an n-dimensional compact hypersurface with infinite fundamental group in $S^{n+1}(1)$. If $r \geq \frac{n-2}{n-1}$ and $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$, where $n(n-1)r$ is the scalar curvature of M and $c^2 = \frac{n-2}{nr}$.*

Theorem 2. *Let M be an n-dimensional compact hypersurface with infinite fundamental group in $S^{n+1}(1)$. If the sectional curvatures are nonnegative, then M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$.*

Remark. In our Theorems 1 and 2, we do not assume that the scalar curvature is constant. And in our Theorem 2, we do not assume any condition on scalar curvature.

2. Proofs of theorems.

Let M be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$. We take a local orthonormal frame field $\{e_1, \ldots, e_{n+1}\}$ in $S^{n+1}(1)$, restricted to M, so that $e_1, \ldots, e_n$ are tangent to M. Let $\omega_1, \cdots, \omega_{n+1}$ denote the dual coframe fields in $S^{n+1}(1)$. We shall make use of the following convention on the ranges of indices:

$$1 \leq A, B, C, \cdots, \leq n+1; \quad 1 \leq i, j, k, \cdots, \leq n.$$

Then the structure equations of $S^{n+1}(1)$ are given by

$$d\omega_A = \sum_{B=1}^{n+1} \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = \sum_{C=1}^{n+1} \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}, \quad \Omega_{AB} = -\frac{1}{2} \sum_{C,D=1}^{n+1} R_{ABCD}\omega_C \wedge \omega_D,$$

$$R_{ABCD} = (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}),$$

where Ω_{AB} (resp. R_{ABCD}) denotes the curvature form (resp. the components of the curvature tensor) of $S^{n+1}(1)$. Then, in M,

$$\omega_{n+1} = 0.$$

It follows from Cartan's Lemma that

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.1}$$

The second fundamental form α and the mean curvature of M are defined by

$$\alpha = \sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1} \quad \text{and} \quad nH = \sum_i h_{ii},$$

respectively. M is said to be totally umbilical if the h_{ij} can be expressed as $h_{ij} = H\delta_{ij}$. The structure equations of M are given by

$$d\omega_i = \sum_{k=1}^{n} \omega_{ik} \wedge \omega_k, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.2}$$

$$d\omega_{ij} = \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \quad \Omega_{ij} = -\frac{1}{2}\sum_{k,l=1}^{n} R_{ijkl}\omega_k \wedge \omega_l,$$

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}),$$

where Ω_{ij} (resp. R_{ijkl}) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor) of M. From the above equation, we have

$$R_{ij} = (n-1)\delta_{ij} + nHh_{ij} - \sum_{k=1}^{n} h_{ik}h_{kj}, \tag{2.3}$$

$$n(n-1)r = n(n-1) + n^2H^2 - S, \tag{2.4}$$

where R_{ij} and $n(n-1)r$ are components of the Ricci curvature tensor and the scalar curvature of M, respectively, and $S = \sum_{ij=1}^{n} h_{ij}^2$ is the squared norm of the second fundamental form of M.

Proof of Theorem 1. Since $r \geq \frac{n-2}{n-1}$ and $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, we infer

$$n + 2(n-1)(r-1) - \frac{n-2}{n}S \geq 0. \tag{2.5}$$

In fact,

$$\begin{aligned}
&n + 2(n-1)(r-1) - \frac{n-2}{n}S \\
&= n + 2(n-1)(r-1) - \frac{n-2}{n}\left\{(n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}\right\} \\
&\quad + \frac{n-2}{n}\left\{(n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2} - S\right\} \\
&\geq n + 2(n-1)(r-1) - \frac{n-2}{n}\left\{(n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}\right\} \\
&= \frac{(n-2)^2}{n} + \frac{n-1}{n}\{n(r-1)+2\} - \frac{(n-2)^2}{n\{n(r-1)+2\}}.
\end{aligned}$$

We consider function $f(t) = \frac{(n-2)^2}{n} + \frac{n-1}{n}t - \frac{(n-2)^2}{nt}$. By a direct computation, we have $f(t) \geq 0$ if $t \geq \frac{n-2}{n-1}$. Since $r \geq \frac{n-2}{n-1}$, we have $n(r-1)+2 \geq \frac{n-2}{n-1}$. Thus, we infer

$$n + 2(n-1)(r-1) - \frac{n-2}{n}S \geq 0.$$

Therefore, when $r \geq \frac{n-2}{n-1}$, we know that

$$S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2} \tag{2.6}$$

is equivalent to

$$\begin{aligned} n + 2(n-1)(r-1) - \frac{n-2}{n}S \\ \geq \frac{n-2}{n}\sqrt{\{n(n-1)(r-1)+S\}\{S-n(r-1)\}}. \end{aligned} \tag{2.7}$$

Indeed, we can prove that

$$S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$$

holds if and only if

$$\begin{aligned} &\left\{n + 2(n-1)(r-1) - \frac{n-2}{n}S\right\}^2 \\ &\geq \frac{(n-2)^2}{n^2}\{n(n-1)(r-1)+S\}\{S-n(r-1)\}. \end{aligned}$$

Since (2.5) holds, we obtain that (2.7) is true.

From Gauss equation $n(n-1)r = n(n-1) + n^2H^2 - S$, we conclude

$$\{S - n(r-1)\} = \frac{n}{n-1}(S - nH^2). \tag{2.8}$$

Hence, from (2.7) and (2.8), we obtain

$$n + 2nH^2 - S \geq \frac{n-2}{\sqrt{n(n-1)}}\sqrt{n^2H^2(S-nH^2)}. \tag{2.9}$$

For any point p and any unit vector $\vec{u} \in T_pM$, we choose a local orthonormal frame field $\{e_1, \cdots, e_n\}$ such that $e_n = \vec{u}$, we have, from Gauss equation (2.3),

$$\mathrm{Ric}(\vec{u}) = (n-1) + nHh_{nn} - \sum_{i=1}^{n} h_{in}^2. \tag{2.10}$$

Since

$$(nH - h_{nn})^2 = \left(\sum_{i=1}^{n-1} h_{ii}\right)^2 \leq (n-1)\sum_{i=1}^{n-1} h_{ii}^2,$$

we have

$$n^2H^2 - (n-1)\sum_{i=1}^n h_{ii}^2 + nh_{nn}^2 - 2nHh_{nn} \leq 0.$$

From $\sum_i (h_{ii} - H) = 0$ and $\sum_{i=1}^n (h_{ii} - H)^2 = \sum_{i=1}^n h_{ii}^2 - nH^2$, we have, for any i,

$$(h_{ii} - H)^2 \leq \frac{n-1}{n}\left(\sum_{i=1}^n h_{ii}^2 - nH^2\right).$$

$$\begin{aligned} 0 &\geq n(h_{nn}^2 - nHh_{nn}) + (n-2)nH(h_{nn} - H) \\ &\quad + 2(n-1)nH^2 - (n-1)\sum_{i=1}^n h_{ii}^2 \\ &\geq n(h_{nn}^2 - nHh_{nn}) - (n-2)n|H|\sqrt{\frac{n-1}{n}\left(\sum_{i=1}^n h_{ii}^2 - nH^2\right)} \\ &\quad + 2(n-1)nH^2 - (n-1)\sum_{i=1}^n h_{ii}^2, \end{aligned}$$

namely,

(2.11)

$$\begin{aligned} &(h_{nn}^2 - nHh_{nn}) \\ &\leq (n-2)|H|\sqrt{\frac{n-1}{n}\left(\sum_{i=1}^n h_{ii}^2 - nH^2\right)} - 2(n-1)H^2 + \frac{n-1}{n}\sum_{i=1}^n h_{ii}^2. \end{aligned}$$

From (2.10) and (2.11), we have

$$\begin{aligned} \mathrm{Ric}(\vec{u}) \geq (n-1) - (n-2)|H|&\sqrt{\frac{n-1}{n}\left(\sum_{i=1}^n h_{ii}^2 - nH^2\right)} \\ &+ 2(n-1)H^2 - \frac{n-1}{n}\sum_{i,j=1}^n h_{ij}^2 \end{aligned}$$

because of $\frac{n-1}{n} > \frac{1}{2}$. Thus, we obtain, from the above inequality and $S = \sum_{i,j=1}^n h_{ij}^2$,

$$\text{(2.12)} \quad \mathrm{Ric}(\vec{u}) \geq \frac{n-1}{n}\left\{n + 2nH^2 - S - \frac{n-2}{\sqrt{n(n-1)}}\sqrt{n^2H^2(S - nH^2)}\right\}.$$

From (2.9), we have $\mathrm{Ric}(\vec{u}) \geq 0$. In particular, from the assertions above, we know that if $S < (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$ holds, then $\mathrm{Ric}(\vec{u}) > 0$. Thus, if there exists point p in M such that $S < (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then, at the point p, the Ricci curvature is positive. From the following Lemma due to Aubin [**1**], we know that there exists a metric on M such that the Ricci curvature is positive on M. According to Myers theorem, we know that the fundamental group is finite. This is impossible because M has infinite fundamental group.

Lemma (cf. Aubin [**1**, p. 344])**.** *If the Ricci curvature of a compact Riemannian manifold is nonnegative and positive at somewhere, then the manifold carries a metric with positive Ricci curvature.*

Thus, we must have $S = (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$. And at each point, there exists a unit vector $\vec{u}$ such that $\mathrm{Ric}(\vec{u}) = 0$. From the assertions above, we infer that these inequalities above are equalities. That is, we must have $h_{ij} = 0$ if $i \neq j$ and $h_{11} = \cdots = h_{n-1n-1}$,

$$(h_{nn} - H)^2 = \frac{n-1}{n}\left(\sum_{i=1}^{n} h_{ii}^2 - nH^2\right) = \frac{n-1}{n}(S - nH^2)$$

and

$$(h_{11} - H)^2 = \cdots = (h_{n-1n-1} - H)^2 = \frac{1}{n(n-1)}(S - nH^2).$$

Hence, we conclude that M has only two distinct principal curvatures one of which is simple. Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame field such that $h_{ij} = \lambda_i\delta_{ij}$, where λ_i's are principal curvatures on M. Without loss of generality, we can assume $\mu = \lambda_n$, $\lambda = \lambda_1 = \cdots = \lambda_{n-1}$. From Gauss Equation (2.2) and the definition of the Ricci curvature, we have $1 + \mu\lambda = 0$ because of $1 + \lambda_i\lambda_j = 1 + \lambda^2 > 0$, for any $i, j = 1, \cdots, n-1$. From (2.4), we have

$$\mu = \frac{n(r-1)}{2\lambda} - \frac{n-2}{2}\lambda.$$

Hence $\lambda^2 = \frac{n(r-1)+2}{n-2}$ and $\mu^2 = \frac{n-2}{n(r-1)+2}$.

We consider the integral submanifold of the corresponding distribution of the space of principal vectors corresponding to the principal curvature λ. Since the multiplicity of the principal curvature λ is greater than 1, we know that the principal curvature λ is constant on this integral submanifold (cf. Otsuki [**6**]). From $\lambda^2 = \frac{n(r-1)+2}{n-2}$ and $\mu^2 = \frac{n-2}{n(r-1)+2}$, we know that the scalar curvature $n(n-1)r$ and the principal curvature μ are constant. Thus, we obtain that M is isoparametric. Therefore, M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ because $S = (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$ holds. This completes the Proof of Theorem 1.

Proof of Theorem 2. Since the sectional curvatures are nonnegative, we have that the Ricci curvature is nonnegative. From the arguments in the

Proof of Theorem 1, we infer that at each point, there exists a unit vector $\vec{u}$ such that $\mathrm{Ric}(\vec{u}) = 0$.

Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame field such that $h_{ij} = \lambda_i \delta_{ij}$, where λ_i's are principal curvatures on M. Then, from Gauss Equation (2.2), we have $1 + \lambda_i \lambda_j \geq 0$ for $i \neq j$. Further, there exists an i such that $\sum_{j \neq i}(1 + \lambda_i \lambda_j) = 0$ from the definition of Ricci curvature. Hence, we must have $1 + \lambda_i \lambda_j = 0$ for $j \neq i$. Therefore, M has only two distinct principal curvatures one of which is simple. Let $\mu = \lambda_i$ and $\lambda = \lambda_j$ for $j \neq i$. From (2.4), we have

$$\mu = \frac{n(r-1)}{2\lambda} - \frac{n-2}{2}\lambda. \tag{2.13}$$

Since $1 + \mu\lambda = 0$ and (2.13) hold, we have $\lambda^2 = \frac{n(r-1)+2}{n-2}$ and $\mu^2 = \frac{n-2}{n(r-1)+2}$. Hence, we have

$$S = (n-1)\lambda^2 + \mu^2 = (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}.$$

By making use of the same assertion as in the Proof of Theorem 1, we infer that M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$. This completes the Proof of Theorem 2.

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INJECTIVE ENVELOPES OF C^*-ALGEBRAS AS OPERATOR MODULES

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In this paper we give some characterizations of M. Hamana's injective envelope $I(A)$ of a C^*-algebra A in the setting of operator spaces and completely bounded maps. These characterizations lead to simplifications and generalizations of some known results concerning completely bounded projections onto C^*-algebras. We prove that $I(A)$ is rigid for completely bounded A-module maps. This rigidity yields a natural representation of many kinds of multipliers as multiplications by elements of $I(A)$. In particular, we prove that the(n times iterated) local multiplier algebra of A embeds into $I(A)$.

1. Introduction.

Let A denote a unital C^*-algebra. M. Hamana [13, 14, 16] introduced the injective envelope of A, $I(A)$, as a "minimal" injective operator system containing A and established various characterizations and properties of $I(A)$ in the setting of completely positive mappings and operator systems.

In recent years attention has shifted from completely positive maps and operator systems to completely bounded maps, operator algebras and operator spaces. In particular a theory has evolved of operator spaces that are completely contractive as modules over operator algebras. See for example, [3, 20].

This theory gives a new categorical framework where one can examine injective envelopes. While other author's have pursued this viewpoint they have generally defined injectivity and rigidity in terms of completely contractive maps. For example, defining injectivity by requiring completely contractive maps to have completely contractive extensions. This is equivalent to requiring that completely bounded maps have completely bounded extensions of the same completely bounded norm. Since unital completely contractive maps on C^*-algebras are completely positive this approach generally reduces to M. Hamana's results in the C^*-algebra setting.

Our approach is different in that we are interested in a setting where our objects are A-modules and injectivity is defined by requiring that completely bounded A-module maps have completely bounded A-module extensions, but not necessarily of the same norm. We show for example that, as well as being a minimal injective operator system containing A, that $I(A)$ is in a certain sense the "minimal" injective left operator A-module containing A. This immutability of the "injective hull" of C^*-algebras under change of category has some immediate applications to completely bounded projections and multiplier algebras.

In this paper our primary focus is on the new properties of $I(A)$ and their applications. For this reason we take the quickest approach, which is to restrict our attention to the unital case and use M. Hamana's results to deduce these new properties by our off-diagonalization trick.

Many of our results do carry over to the case of a non-unital C^*-algebra B by the simple device of adjoining a unity to B, letting A denote this unital C^*-algebra and observing that any B-modules are automatically A-modules. For a greater development of the non-unital case we refer the reader to the subsequent paper [**4**].

2. Mapping properties of $I(A)$.

Throughout this section A will denote a unital C^*-algebra and $I(A)$ will denote its injective envelope as defined in [**13**, Def. 2.1, Th. 4.1]. We assume that the reader is familiar with the definitions and elementary properties of completely bounded and completely positive maps as presented in [**20**] or [**23**].

One of M. Hamana's fundamental results about $I(A)$ was his rigidity theorem. This theorem says that if $\varphi : I(A) \to I(A)$ is completely positive with $\varphi(a) = a$ for all a in A then $\varphi(x) = x$ for all x in $I(A)$.

A direct analog of this result is false for general completely bounded maps. If $A \neq I(A)$, then there exists a nonzero bounded linear functional $f : I(A) \to \mathbb{C}$ with $f(A) = \{0\}$. Defining the map $\varphi : I(A) \to I(A)$ via $\varphi(x) = x + f(x)1$ yields a completely bounded map with $\varphi(a) = a$ for all a in A, but $\varphi(x) \neq x$ for all x in $I(A)$. However, if one recalls that completely positive maps that fix A are automatically A-bimodule maps [**6**], [**20**, Exercise 4.3] then one is led to the appropriate generalization of M. Hamana's rigidity. Surprisingly one does not need bimodules, only left or right A-modules as the following results show.

Theorem 2.1. *Let $E \subseteq I(A)$ be a subspace such that $AE \subseteq E$ (respectively, $EA \subseteq E$) and let $\varphi : E \to I(A)$ be a completely bounded left (resp., right) A-module map. Then there exists an element y in $I(A)$ such that φ is right (resp., left) multiplication by y, i.e., $\varphi(e) = ey$ ($\varphi(e) = ye$) for all e in E and $\|y\| = \|\varphi\|_{cb}$. When $AEA \subseteq E$ and φ is a bimodule map, then y may be taken in the center of $I(A)$.*

In particular, φ extends to a completely bounded, left (resp., right) A-module map $\psi : I(A) \to I(A)$ such that $\psi|_E = \varphi$ and $\|\varphi\|_{cb} = \|\psi\|_{cb}$. If

$E \subseteq I(A)$ contains an invertible element of $I(A)$, then the element $y \in I(A)$ and consequently the extension ψ are unique.

Proof. It will suffice to assume that $\|\varphi\|_{cb} \le 1$. Let $\mathcal{S} \subseteq M_2(I(A))$ be defined as

$$\mathcal{S} = \left\{ \begin{pmatrix} a & e \\ f^* & \lambda \end{pmatrix} : a \in A, e, f, \in E, \lambda \in \mathbb{C} \right\}$$

and $\Phi : \mathcal{S} \to M_2(I(A))$ by

$$\Phi\left(\begin{pmatrix} a & e \\ f^* & \lambda \end{pmatrix} \right) = \begin{pmatrix} a & \varphi(e) \\ \varphi(f)^* & \lambda \end{pmatrix}.$$

Arguing as in [**20**] or [**28**] one sees that Φ is completely positive and hence can be extended to a completely positive map on all of $M_2(I(A))$ which we still denote by Φ. Using the fact that Φ fixes $A \oplus \mathbb{C} = \left\{ \begin{pmatrix} a & 0 \\ 0 & \lambda \end{pmatrix} : a \in A, \lambda \in \mathbb{C} \right\}$ and again arguing as in [**28**], we see that there exists $\varphi_i : I(A) \to I(A), i = 1, 2, 3, 4$ such that

$$\Phi\left(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right) = \begin{pmatrix} \varphi_1(x_1) & \varphi_2(x_2) \\ \varphi_3(x_3) & \varphi_4(x_4) \end{pmatrix}.$$

Clearly, φ_1 and φ_4 must be completely positive and φ_2 extends φ.

Since $\varphi_1(a) = a$ for all a in A, by M. Hamana's rigidity result $\varphi_1(x) = x$ for all x in $I(A)$. Thus, Φ fixes the C^*-subalgebra, $I(A) \oplus \mathbb{C}$ and so by [**6**] (see also [**20**]) Φ must be a bimodule map over this algebra. Thus, $\begin{pmatrix} 0 & \varphi_2(x) \\ 0 & 0 \end{pmatrix} = \Phi\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right) = \Phi\left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \varphi_2(1) \\ 0 & 0 \end{pmatrix}$ and we have that $\varphi_2(x) = x \cdot \varphi_2(1)$. Finally, $\|\varphi\|_{cb} = \|\varphi_2\|_{cb} = \|\varphi_2(1)\|$.

The proof for right A-module maps is similar. For the bimodule case let $\mathcal{S} = \left\{ \begin{pmatrix} a & e \\ f^* & b \end{pmatrix} : a, b \in A, e, f \in E \right\}$ and deduce that φ_2 is an $I(A)$-bimodule map. If E contains an invertible element e, then $y = e^{-1}\varphi(e)$ and so y is unique. ☐

In particular, the above results show that every completely bounded left (resp., right or bi-) A-module map of A into $I(A)$ admits a unique extension to a completely bounded left (resp., right or bi-) A-module map of $I(A)$ into itself and this extension has the same completely bounded norm.

Corollary 2.2 (Rigidity)**.** *Let A be a unital C^*-algebra and $I(A)$ be its injective envelope C^*-algebra. Let E be a subspace of $I(A)$ with $A \subseteq E$ and $AE \subseteq E$ (respectively, $EA \subseteq E$) and let $\varphi : E \to I(A)$ be a completely bounded left (resp., right) A-module map. If $\varphi(a) = a$ for all a in A, then $\varphi(e) = e$ for all e in E.*

Proof. There exists y in $I(A)$ with $\varphi(e) = e \cdot y$ for all e in E. Since $\varphi(1) = 1$, $y = 1$ and hence, $\varphi(e) = e$ for all e in E. ☐

3. Injective bimodule extensions of C*-algebras A and the injective envelope $I(A)$.

Let A and B be unital C^*-algebras. Recall the definition of *operator A-B-bimodules*. These are operator spaces E which are A-B-bimodules and such that the trilinear module pairing $A \times E \times B \to E$, $(a, e, b) \to aeb$ is completely contractive in the sense of E. Christensen and A. Sinclair [7]. This is equivalent to requiring that for matrices $(a_{i,j}), (e_{i,j})$ and $(b_{i,j})$ of the appropriate sizes, the induced matricial module product is contractive, i.e.,

$$\left\| \left(\sum_{k,m} a_{i,k} e_{k,m} b_{m,j} \right) \right\| \leq \|(a_{i,j})\| \|(e_{i,j})\| \|(b_{i,j})\| .$$

These are the objects of the category ${}_A\mathcal{O}_B$, [3], [21] and the morphisms between two operator A-B-bimodules in this category are the completely bounded A-B-bimodules maps. When we want to restrict the morphisms to be completely contractive A-B-bimodule maps we will denote the category by ${}_A\mathcal{O}^1_B$.

We assume that all module actions are unital, i.e., $1 \cdot e \cdot 1 = e$. We set ${}_A\mathcal{O} = {}_A\mathcal{O}_{\mathbb{C}}$ and call these *left operator A-modules*, and $\mathcal{O}_A = {}_{\mathbb{C}}\mathcal{O}_A$ and call these *right operator A-modules*.

Definition 1. An operator A-B-bimodule I is *A-B-injective*, if whenever $E \subseteq F$ are operator A-B-bimodules, then every completely bounded A-B-bimodule map from E into I has a completely bounded A-B-bimodule extension to F. Note that we do *not* require that the *cb*-norm of the extension is the same as the *cb*-norm of the original map. When this is the case we will call I a *tight A-B-injective A-B-bimodule.*

Some comments on terminology are helpful. Our definition of A-B-injective is the usual definition of injectivity in the category ${}_A\mathcal{O}_B$, while what we are calling tight A-B-injective is the corresponding definition of injectivity in the category ${}_A\mathcal{O}^1_B$.

If A and B are both C^*-subalgebras of $B(H)$, then by the bimodule version of G. Wittstock's extension theorem [29, Thm. 4.1] (see also [28]) $B(H)$ is a tight A-B-injective. Thus, if $M \subseteq B(H)$ is the range of a completely bounded projection $\varphi : B(H) \to M$, which is also an A-B-bimodule map, then M is A-B-injective, but it is not evidently tight A-B-injective. A C^*-subalgebra $I \subseteq B(H)$ is generally called "injective" if it is the range of a completely positive projection. This term is so widespread we continue to use it here. Note that such a map is also automatically an I-bimodule map. Thus, such an I is a tight A-B-injective for any C^*-subalgebras A and B of I.

In particular, M. Hamana's injective envelope $I(A)$ is a tight A-A-injective A-A-bimodule, a tight A-$\mathbb{C}$-injective left A-module and a tight $\mathbb{C}$-A-injective right A-module.

On the other hand there are many $\mathbb{C}$-$\mathbb{C}$-injectives which are not tight, i.e., not injective in the usual sense. For example, for any subspace E of $B(H)$ of finite codimension, it is easy to show that there exists a completely bounded projection from $B(H)$ onto E and hence E is $\mathbb{C}$-$\mathbb{C}$-injective.

T. Huruya [**17**] has given an example of a unital C^*-subalgebra of an injective C^*-algebra of finite codimension that is not injective. By the above argument, this algebra is the range of a completely bounded projection and hence is $\mathbb{C}$-$\mathbb{C}$-injective. Thus there exist C^*-algebras that are $\mathbb{C}$-$\mathbb{C}$-injective, but are not injective in the usual sense.

In our terminology, M. Hamana's rigidity result implies that if $A \subseteq E \subseteq I(A)$ and E is a tight $\mathbb{C}$-$\mathbb{C}$-injective, then $E = I(A)$. We prove this fact in the remarks following the theorem.

Theorem 3.1. *Let A be a unital C^*-algebra and let $A \subseteq E \subseteq I(A)$. Then the following are equivalent:*

a) *E is A-$\mathbb{C}$-injective,*
b) *E is $\mathbb{C}$-A-injective,*
c) *E is A-A-injective,*
d) $E = I(A)$.

Proof. By the Hahn-Banach extension theorem for completely bounded A-B-bimodule maps [**29**, Thm. 4.1] (see also [**28**], [**20**]), it follows that $I(A)$ is A-$\mathbb{C}$-injective, $\mathbb{C}$-A-injective and A-A-injective. Thus, d) implies a), b) and c). It will suffice to prove that a) implies d), the other implications are similar.

If E is A-$\mathbb{C}$-injective then the identity map from E to E extends to a completely bounded left A-module projection from $I(A)$ to E. Letting $I(A)$ play the role of E in the rigidity theorem yields the result. $\square$

The module actions are necessary in the above theorem. Since there always exists a completely bounded projection from any operator space onto a subspace of finite codimension, if $A \subseteq E \subseteq I(A)$ with E a subspace of finite codimension then E is $\mathbb{C}$-$\mathbb{C}$-injective but $E \neq I(A)$.

On the other hand, if we required E to be tight, then there would exist a completely contractive projection φ onto E. Since 1 belongs to E we would have $\varphi(1) = 1$ and, consequently, this projection would be completely positive. Hence E would be an operator system. Thus, $E = I(A)$ by M. Hamana's rigidity theorem and we would be adding nothing new.

We now are in a position to clarify the relationship between these new notions of injectivity and injectivity in the usual sense for C*-algebras.

Theorem 3.2. *Let A be a unital C^*-algebra. Then the following are equivalent:*

a) *A is an injective C^*-algebra (in the usual sense),*
b) *A is a tight A-$\mathbb{C}$-injective module,*
c) *A is a A-$\mathbb{C}$-injective module,*
d) *A is a tight $\mathbb{C}$-A-injective module,*
e) *A is a $\mathbb{C}$-A-injective module,*
f) *A is a tight A-A-injective module,*
g) *A is a A-A-injective module.*

Proof. We prove the equivalence of a), b) and c), the remaining arguments are similar. We have that a) implies b) by Wittstock's Hahn-Banach extension theorem for module maps. Clearly, b) implies c). We now prove that c) implies a). Since A is a A-$\mathbb{C}$-injective module, the identity map on A extends to a completely bounded left A-module map from $I(A)$ into A. But by the Rigidity Theorem, this extended map must be the identity map on $I(A)$ and hence $I(A) = A$. Thus, A is injective. □

Definition 2. Let M be an operator A-B-bimodule. Call I a *minimal A-B-injective extension* of M, if $M \subseteq I$ and $M \subseteq E \subseteq I$ with E $A-B$-injective implies $E = I$.

By Theorem 3.1, $I(A)$ is a minimal A-$\mathbb{C}$-injective extension of A and also a minimal $\mathbb{C}$-A and A-A injective extension of A.

We call a map φ a *completely bounded isomorphism* if both φ and φ^{-1} are completely bounded.

Theorem 3.3. *Let A be a unital C^*-algebra and let I be a minimal A-$\mathbb{C}$-injective extension of A, then there exists a completely bounded left A-module isomorphism $\varphi : I(A) \to I$ with $\varphi(a) = a$ for all a in A. If we require that I is also a tight A-$\mathbb{C}$-injective then φ may be taken to be a complete isometry. Analogous statements hold for right modules and bimodules.*

Proof. Since I and $I(A)$ are A-$\mathbb{C}$-injective, there exist completely bounded left A-module maps $\varphi : I(A) \to I$ and $\psi : I \to I(A)$ which fix A. By the rigidity of $I(A)$, $\psi \circ \varphi$ is the identity on $I(A)$ and hence $\varphi \circ \psi$ is the identity restricted to $E =$ range (φ). This makes E an A-$\mathbb{C}$-injective module and hence $E = I$ and $\psi = \varphi^{-1}$. □

If we required I to be tight and only minimal among all tight injectives, then as in the remark following Theorem 3.1, our result would reduce to M. Hamana's theory.

We now turn to some applications to projections. In [**5**] it was shown that if $M \subseteq B(H)$ is a von Neumann algebra and there exists a bounded M-bimodule projection, $\varphi : B(H) \to M$, then M is injective. Such a map φ is easily shown to be automatically completely bounded.

In [**21**] the same result was shown to hold for C^*-algebras. The above results on injective envelopes allow us to extend these results a bit. Perhaps, more importantly, the new proof is much simpler than the proof in [**21**].

Theorem 3.4. *Let $A \subseteq B(H)$ be a unital C^*-algebra. If there exists a completely bounded left (or right) A-module projection of $B(H)$ onto A, then A is injective.*

Proof. Since $B(H)$ is A-A-injective the identity map on A extends to a completely bounded A-bimodule map from $I(A)$ to $B(H)$. Composing with the projection onto A gives a completely bounded left A-module map from $I(A)$ to A which is the identity on A. By rigidity (Corollary 2.2) $I(A) = A$ and hence A is injective. □

The following example gives an indication of the obstacles that arise in attempting to generalize Theorem 2.1, Corollary 2.2 and Theorem 3.4 to the case of non-involutive operator algebras in $B(H)$. Consider the operator algebra $A \subset M_2(\mathbb{C})$ defined by

$$A = \left\{ X \in M_2(\mathbb{C}) \,:\, X = S^{-1} \cdot \mathrm{diag}(a,b) \cdot S\,,\ a,b \in \mathbb{C} \right\},\ \ S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

If $\omega : M_2(\mathbb{C}) \to \mathbb{C} \oplus \mathbb{C} \subset M_2(\mathbb{C})$ is the canonical conditional expectation on $M_2(\mathbb{C})$ preserving the diagonal and mapping off-diagonal elements to zero, then the map $\phi : M_2(\mathbb{C}) \to A$ defined by the rule $\phi(X) = S^{-1} \cdot \omega(SXS^{-1}) \cdot S$ is a completely bounded A-bimodule projection. However, the smallest C^*-subalgebra of $M_2(\mathbb{C})$ generated by A is $M_2(\mathbb{C})$ itself, and since $M_2(\mathbb{C})$ is injective we obtain $I(A + A^*) = M_2(\mathbb{C})$. Consequently, Theorem 2.1 and Corollary 2.2 cannot be extended to this situation, and Theorem 3.4 is not true for the described operator algebra A.

In [**8**] and [**24**] it was proven that if $M \subseteq B(H)$ is a von Neumann algebra and if there exists a completely bounded projection of $B(H)$ onto M then M is injective, cf. [**25**]. The direct analogue of this result is false for C^*-algebras. T. Huruya [**17**] gave an example of a non-injective C^*-subalgebra of codimension 1 of an injective C^*-algebra. It is easily shown that any time Huruya's algebra is represented as a C^*-subalgebra of $B(H)$, then there will exist a completely bounded projection of $B(H)$ onto this non-injective algebra. Thus, to generalize the results of [**8**] and [**24**], we will need an additional condition.

Definition 3. An operator A-B-bimodule R is *relatively A-B-injective* if whenever $E \subseteq F$ are operator A-B-bimodules such that there exists a completely bounded projection of F onto E, then every completely bounded A-B-bimodule map of E into R has a completely bounded A-B-bimodule extension to F. It is important to note that we do not require that the projection of F onto E is an A-B-bimodule map.

The concept of relative injectivity was introduced in [**21**] with slightly different notation, relative A-B-injective was denoted $(A\text{-}B, \mathbb{C}\text{-}\mathbb{C})$-injective. A C^*-algebra A is A-A-injective if and only if A is injective in the usual sense. In contrast, [**21**] showed that every von Neumann algebra M is relatively M-M-injective, M-$\mathbb{C}$-injective and $\mathbb{C}$-M-injective.

Theorem 3.5. *Let $A \subseteq B(H)$ be a unital C^*-subalgebra. If there exists a completely bounded projection of $B(H)$ onto A and A is relatively A-$\mathbb{C}$-injective (or $\mathbb{C}$-A-injective), then A is injective.*

Proof. Since A is A-$\mathbb{C}$-injective, the identity map from A to A has a completely bounded left A-module extension to $B(H)$. This map is clearly a projection. Hence A is injective by Theorem 3.4. □

Because von Neumann algebras are relatively injective [**21**], Theorem 3.5 implies the result of [**8**] and [**24**].

Corollary 3.6. *Let $M \subseteq B(H)$ be a von Neumann algebra. If there exists a completely bounded projection of $B(H)$ onto M, then M is injective.*

Relative injectivity was shown in [**21**] to be equivalent to the vanishing of certain completely bounded "Ext" groups, which in turn implied the vanishing of completely bounded Hochschild cohomology. Thus, relative injectivity captures both the vanishing of cohomology and these projection results. It is still unknown which C^*-algebras A are relatively A-$\mathbb{C}$-injective. By Theorem 3.5, T. Huruya's C^*-algebra A, cannot be relatively A-$\mathbb{C}$-injective.

4. Local multiplier algebras, injective envelopes and regular completions.

We close this paper with some applications to multiplier algebras. Our main point is that by invoking Theorem 2.1, we will see immediately that multipliers are "naturally" represented as multiplication by elements in $I(A)$. This concrete representation of multipliers can be used to simplify some arguments. Thus, Theorem 2.1 provides an alternative starting point for developing the theory of multipliers.

In particular, we show that $I(A)$ contains the local multiplier algebra of A, $M_{\mathrm{loc}}(A)$, intrinsically as a C^*-subalgebra. Recall that a closed 2-sided ideal J of A is called *essential* if $J \cap K \neq \{0\}$ for every nontrivial 2-sided ideal K. All ideals in this section are norm-closed.

The *left multiplier algebra $LM(J)$ of J* is just the set of right A-module maps $\psi : J \to J$. Such a map is automatically (completely) bounded and $\|\psi\| = \|\psi\|_{cb}$. The *right multiplier algebra $RM(J)$* is defined similarly. The *multiplier algebra $M(J)$* consists of pairs of linear maps $\varphi, \psi : J \to J$ satisfying $\varphi(j_1)j_2 = j_1\psi(j_2)$. This identity implies that $\psi \in LM(J)$, $\varphi \in RM(J)$ and $\|\varphi\| = \|\psi\| = \|\varphi\|_{cb} = \|\psi\|_{cb}$.

The *local multiplier algebra* $M_{\mathrm{loc}}(A)$ is defined by taking a direct limit of $M(J)$ over all essential ideals J of A ordered by reverse inclusion. See [**1**] for details.

Lemma 4.1. *Let A be a unital C^*-algebra and let J be a 2-sided essential ideal of A and let $\varphi : J \to I(A)$ be a completely bounded left (resp., right) A-module map. Then there exists a unique element x in $I(A)$ such that φ is right (resp., left) multiplication by x. Moreover, $\|x\| = \|\varphi\| = \|\varphi\|_{cb}$.*

Proof. By Theorem 2.1 such an x exists, it remains to show that x is unique. To this end consider, $F = \{y \in I(A) : Jy = 0\}$ which is clearly a right A-submodule of $I(A)$. It will suffice to show that $F = \{0\}$. Let $\{e_\alpha\}$ be a contractive approximate identity for J. For $a \in A,\ y \in F$, we have,

$$\|a - y\| \geq \sup_\alpha \|e_\alpha(a - y)\| = \sup_\alpha \|e_\alpha a\| = \|a\|$$

with the last equality using the fact that J is essential. The same calculation for matrices shows that the quotient map $q : I(A) \to I(A)/F$ is a complete isometry on A and a right A-module map. Now since $I(A)$ is injective, there exists a completely contractive right A-module map $\varphi : I(A)/F \to I(A)$. Hence by rigidity $\varphi \circ q(b) = b$ for all b in $I(A)$, and it follows that $F = \{0\}$. □

The fact that F must be $\{0\}$ is related to the fact that $I(A)$ is in a certain sense an "essential extension" of A.

Theorem 4.2. *Let A be a unital C^*-algebra, let J be a 2-sided essential ideal in A and let (φ, ψ) be in $M(J)$. Then there exists a unique element x in $I(A)$ such that $\varphi(j) = jx, \psi(j) = xj$ for all j in J.*

Proof. By Lemma 4.1, there exist unique elements x_1, x_2 in J such that $\varphi(j_1) = j_1x_1,\ \psi(j_2) = x_2j_2$ for all j_1, j_2 in J. But $j_1x_1j_2 = \varphi(j_1)j_2 = j_1\psi(j_2) = j_1x_2j_2$ and so $j_1(x_1 - x_2)j_2 = 0$ for all j_1. Applying Lemma 4.1 we conclude that $(x_1 - x_2)j_2 = 0$ for all j_2 and so $x_1 = x_2$. □

Corollary 4.3. *The inclusion of A into $I(A)$ extends in a unique way to a $*$-monomorphism of $M_{\mathrm{loc}}(A)$ into $I(A)$. The image of $M_{\mathrm{loc}}(A)$ under this map is the norm closure of the set*

$$\{x \in I(A) : xJ \subseteq J \text{ and } Jx \subseteq J \text{ for some essential ideal } J\}.$$

Proof. For each (φ, ψ) in $M(J)$ there exists a unique x in $I(A)$ implementing (φ, ψ). By this uniqueness the map $(\varphi, \psi) \to x$ must be a $*$-monomorphism on $M(J)$. Furthermore, let J_i be essential ideals, and let (φ_i, ψ_i) in $M(J_i)$ be implemented by x_i. If $\varphi_1 = \varphi_2$ and $\psi_1 = \psi_2$ on $J_1 \cap J_2$ then, since $J_1 \cap J_2$ is essential, we must have $x_1 = x_2$.

This shows that the inclusions of $M(J_i)$ into $I(A)$ are coherent and allows us to extend these $*$-monomorphism to the direct limit, $M_{\mathrm{loc}}(A)$.

Now assume that $\pi : M_{\rm loc}(A) \to I(A)$ is any $*$-monomorphism with $\pi(a) = a$ for all a in A. Then for (φ, ψ) in $M(J)$

$$\pi((\varphi,\psi))j = \pi((\varphi,\psi)j) = \pi(\psi(j)) = \psi(j),$$

and $j\pi((\varphi,\psi)) = \varphi(j)$ from which it follows that $\pi((\varphi,\psi))$ is the unique element implementing (φ,ψ).

Finally, since $\{x \in I(A) : xJ \subseteq J \text{ and } Jx \subseteq J\}$ is exactly the image of $M(J)$ we have the last claim. □

Remark. The above results allow one to define a *local left* (resp., *right*) *multiplier algebra of* A easily, which we have not seen in the literature. Indeed, if $LM_{\rm loc}(A) = \{x \in I(A) : xJ \subseteq J \text{ for some essential ideal } J\}^{--}$, then this set is easily seen to be completely isometrically isomorphic to the direct limit of $LM(J)$. It is interesting to note that if J_1 and J_2 are essential ideals and $\varphi \in LM(J_1)$ then $\varphi(J_1 \cap J_2) \subseteq J_1 \cap J_2$ and by Lemma 4.1, $\|\varphi\| = \|\varphi\,|_{J_1\cap J_2}\|$. We define the local right multiplier algebra $RM_{\rm loc}(A)$ of A, analogously.

To define the *local quasi-multiplier space* $QM_{\rm loc}(A)$ *of* A we have to recall that the injective envelope $I(A)$ of A is a monotone complete C^*-algebra and, hence, an AW^*-algebra. On the other hand norm-closed two-sided ideals J of C^*-algebras are automatically hereditary, and so we can apply [**10**, Cor. 1.4]: For every norm-closed two-sided ideal $J \subseteq A$ and every quasi-multiplier $x \in QM(J)$ there exists an element $\overline{x} \in I(A)$ such that $j_1xj_2 = j_1\overline{x}j_2$ for any $j_1, j_2 \in J$ and $\|\overline{x}\|$ equals the norm of x estimated in the bidual von Neumann algebra A^{**}. For essential ideals J the element $\overline{x}$ has to be unique, in fact it can be found as a quasi-strict limit of nets of J. Since inclusion relations of essential ideals and their corresponding quasi-multiplier spaces are respected inside $I(A)$ we can define $QM_{\rm loc}(A) = \{x \in I(A) : JxJ \subseteq J \text{ for some essential ideal } J\}^{--}$, to be the local quasi-multiplier space of A.

Note that in any situation where $M_{\rm loc}(A) \not\equiv LM_{\rm loc}(A)$ then necessarily $M_{\rm loc}(A) \not\equiv I(A)$. (In general, the conditions $M_{\rm loc}(A) \not\equiv LM_{\rm loc}(A)$, $M_{\rm loc}(A) \not\equiv QM_{\rm loc}(A)$ and $LM_{\rm loc}(A) \not\equiv QM_{\rm loc}(A)$ are equivalent by general multiplier theory.) If A is any simple, unital, non-injective C^*-algebra like a non-injective Type II_1 or Type III von Neumann factor then $A = M_{\rm loc}(A) = LM_{\rm loc}(A) \neq I(A)$. However, in some cases we obtain the coincidence of the C^*-algebras $M_{\rm loc}(A) = I(A)$.

Proposition 4.4. *Let A be a unital C^*-algebra, $K \subseteq A \subseteq B(H)$ where K denotes the ideal of compact operators, then $M_{\rm loc}(A) = I(A) = B(H)$, $*$-isomorphically.*

Proof. Since K is necessarily an essential ideal of A and $M(K) = B(H)$ we have $B(H) \subseteq M_{\rm loc}(A)$. By Corollary 4.3 we have a $*$-monomorphism π of

$M_{\mathrm{loc}}(A)$ into $I(A)$. Hence, $A \subseteq B(H) \subseteq M_{\mathrm{loc}}(A) \subseteq I(A)$, as C^*-algebras. Since $B(H)$ is A-A-injective, by Theorem 3.1, we have $B(H) = I(A)$ and the result follows. □

The fact that $I(A) = B(H)$ is due to M. Hamana [**13**] with a different proof.

Theorem 4.5. *Let A be a commutative unital C^*-algebra, then $M_{\mathrm{loc}}(A) = I(A)$, $*$-isomorphically.*

Proof. By [**1**, Thm. 1] $M_{\mathrm{loc}}(A)$ is a commutative AW^*-algebra. However, commutative AW^*-algebras are injective by [**26**, Th. 25.5.1] since bounded linear maps between C*-algebras are positive whenever their norm equals their evaluation at the identity of the C*-algebra. Consequently, the $*$-monomorphism of $M_{\mathrm{loc}}(A)$ into $I(A)$ must be onto. □

In the theory of local multiplier C^*-algebras the problem of whether $M_{\mathrm{loc}}(A)$ coincides with $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$ for any C^*-algebra A is one of the main open questions, cf. [**1**, **27**]. Set $M^{k+1}_{\mathrm{loc}}(A) = M_{\mathrm{loc}}(M^k_{\mathrm{loc}}(A))$, which is called the $(k+1)$-order local multiplier algebra of A. We show that any higher order local multiplier C^*-algebra of a given C^*-algebra A is contained in its injective envelope $I(A)$ and, what is more, that the injective envelopes $I(A)$ and $I(M^k_{\mathrm{loc}}(A))$ coincide for any C^*-algebra A. The latter is of special interest since general C^*-subalgebras A of injective C^*-algebras B might not admit an embedding of their injective envelopes $I(A)$ as a C^*-subalgebra of B that extends the given embedding of A into B, see [**14**, Rem. 3.9] for an example.

Theorem 4.6. *Let A be a unital C^*-algebra and $M_{\mathrm{loc}}(A)$ be its local multiplier C^*-algebra. Then the injective envelope $I(A)$ of A is the injective envelope $I(M_{\mathrm{loc}}(A))$ of $M_{\mathrm{loc}}(A)$ and consequently, $M^k_{\mathrm{loc}}(A)$ is contained in $I(A)$ for all k.*

Proof. Since $M_{\mathrm{loc}}(A)$ is $*$-isomorphically embedded into $I(A)$ extending the canonical $*$-monomorphism of A into $I(A)$ by Theorem 2.1, the C^*-algebra $I(A)$ serves as an injective extension of the C^*-algebra $M_{\mathrm{loc}}(A)$, cf. [**13**]. However, the identity map on $M_{\mathrm{loc}}(A)$ admits a unique extension to a completely positive map of $I(A)$ into itself with the same completely bounded norm one since $A \subseteq M_{\mathrm{loc}}(A) \subseteq I(A)$ by construction and $I(A)$ is the injective envelope of A. So $I(A)$ has to be the injective envelope of $M_{\mathrm{loc}}(A)$, too. □

Problem. Characterize the C^*-algebras A for which the local multiplier C^*-algebra $M_{\mathrm{loc}}(A)$ of A coincides with the injective envelope $I(A)$ of A or at least with the regular monotone completion $\overline{A}$ of A in $I(A)$.

This question is surely difficult to answer: If A is an AW^*-algebra then the local multiplier algebra $M_{\mathrm{loc}}(A)$ of A coincides with A by [**22**]. However,

A coincides with its regular monotone completion $\overline{A}$ if and only if A is monotone complete. So we arrive at a long standing open problem of C^*-theory dating back to the work of I. Kaplansky in 1951 ([**18**]): Are all AW^*-algebras monotone complete, or do there exist counterexamples?

Remark. If A is a non-unital C^*-algebra and B denotes its unitization, then A is a 2-sided essential ideal in B. Hence, by Theorem 4.2, $M(A) \subseteq I(B)$. However, in [**4**], it is observed that $I(A) = I(B)$, and so the hypothesis that A is unital can be removed from Theorem 4.2. Similarly, every 2-sided essential ideal in A is an essential ideal in B, so that Corollary 4.3 applies for non-unital A as well. Similar arguments show that the unital hypothesis can be dropped in Proposition 4.4, Theorem 4.5 and Theorem 4.6.

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A FAMILY OF ARITHMETIC SURFACES OF GENUS 3

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The aim of this paper is the study of the genus 3 curves

$$C_n : Y^4 = X^4 - (4n-2)X^2 + 1,$$

from the Arakelov viewpoint. The Jacobian of the curves C_n splits as a product of elliptic curves, and this fact gives enough arithmetical datum to determine the stable model and the canonical sheaf of the curves. We use this information to look for explicit expressions of the modular height and the self-intersection of the dualizing sheaf of the curves C_n.

1. Introduction.

The study of a curve from the arithmetical or the Arakelov viewpoints is a hard task, since it involves a very good knowledge of its geometry (differential forms, periods), its arithmetic (locus of bad reduction, stable models) and its analysis (Green function). Classically, two families of curves have been extensively studied: Fermat curves and modular curves. The study of these curves is feasible because they have a large automorphism group. On the other hand, the curves in these families have variable genus. If one wants to study the behaviour of some arithmetical or Arakelov invariants on the moduli space of curves of a given genus, these families are not useful.

The Arakelov invariants of elliptic curves are completely determined ([**Fa84**]). Some concrete examples of Arakelov invariants for curves of genus 2 were provided by Bost, Mestre and Moret-Bailly in [**Bo-M-M90**]. We present here the study of a family of curves of genus 3.

Let $n \in \mathbb{N}, n \geq 2$ be a natural number such that $n \equiv 2 \pmod 3$ and $n \not\equiv 0, 1 \pmod{2^5}$, and consider the projective curve C_n given by the equation

$$Y^4 = X^4 - (4n-2)X^2Z^2 + Z^4.$$

We have studied the geometry of the curves C_n in [**Gu01**]. They are nonsingular curves of genus 3. They have a large group of automorphisms, which gives the chance of performing a great deal of calculations on them. In this article we study the curves C_n from the arithmetical and Arakelov viewpoints. We find the stable models of the arithmetic surfaces given by them. Combining both the geometric and the arithmetic information compiled about the curves C_n, we initiate the study of their Arakelov invariants: Their modular height and the self-intersection of their canonical sheaf.

The conditions $n \not\equiv 0, 1 \pmod{2^5}$, $n \equiv 2 \pmod 3$ are assumed only for technical reasons, to simplify the exposition. They are necessary essentially for the results about the reduction of the curves C_n at primes over 2, and to avoid some particular cases of supersingularity.

2. Automorphisms. Splitting of the Jacobian.

The following results are proved in [**Gu01**]:

Proposition 2.1. *The automorphisms of* C_n $(n > 2)$ *are the restrictions of the following projectivities of* $\mathbb{P}^2(\mathbb{C})$:

$$\varphi_{0k} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i^k & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varphi_{1k} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i^k & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\varphi_{2k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & i^k & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \varphi_{3k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & i^k & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

for $k = 0, 1, 2, 3$. *The automorphisms* $\alpha = \varphi_{01}, \beta = \varphi_{12}, \gamma = \varphi_{20}$ *form a system of generators for* $\mathrm{Aut}(C_n)$, *with relations* $\alpha^4 = \beta^2 = \gamma^2 = \mathrm{Id}$, $\alpha\beta = \beta\alpha$, $\alpha\gamma = \gamma\alpha$, $\beta\gamma = \gamma\beta\alpha^2$. *The group* $\mathrm{Aut}(C_n)$ *is isomorphic to a semidirect product* $\mathbb{Z}/4\mathbb{Z} \ltimes V_4$, *where* V_4 *denotes the Klein group.*

Remark. The curve C_2 has a larger group of automorphisms, because it is isomorphic to the Fermat curve of fourth degree. For our purposes, we only need to know that the above matrices also give automorphisms of C_2.

The automorphisms $\beta\alpha^2, \beta$ and α^2 provide three elliptic quotients of C_n. We denote the subgroups that they generate by $G_1 = \langle \beta\alpha^2 \rangle$, $G_2 = \langle \beta \rangle$, $G_3 = \langle \alpha^2 \rangle$.

Notation 2.2. We will use the following notation for the rest of the paper:

$$a = \sqrt{n} + \sqrt{n-1}, \qquad \mu = i\sqrt[4]{4n(n-1)},$$

$$m = 2n - 1, \qquad \zeta = \frac{1+i}{\sqrt{2}},$$

$$K_0 = \mathbb{Q}(a, \zeta\mu),$$

$$E_1 = E_2 = E : Y^2Z = X^3 - XZ^2,$$

$$E_3 = E_{(n)} : Y^2Z = X(X-Z)(X-nZ).$$

Proposition 2.3. *The degree* 2 *maps*

$$\begin{array}{llll} \psi_1: & C_n & \longrightarrow & E \\ & (x,y,z) & \longrightarrow & (i(y^2-\zeta^2\mu^2x^2),(i-1)(z^2-mx^2),(y-\zeta\mu x)^2), \end{array}$$

$$\begin{array}{llll} \psi_2: & C_n & \longrightarrow & E \\ & (x,y,z) & \longrightarrow & (i(y^2-\zeta^2\mu^2z^2),(i-1)(x^2-mz^2),(y-\zeta\mu z)^2), \end{array}$$

$$\begin{array}{llll} \psi_n: & C_n & \longrightarrow & E_{(n)} \\ & (x,y,z) & \longrightarrow & (2(a^2+1)(z-ax)(az-x),(a^4-1)y^2,4a(z-ax)^2) \end{array}$$

are the quotient maps $C_n \longrightarrow C_n/G_1$, $C_n \longrightarrow C_n/G_2$, $C_n \longrightarrow C_n/G_3$, *respectively. They induce an isogeny of degree* 8

$$\Psi : J(C_n) \longrightarrow E\times E\times E_{(n)}, \tag{1}$$

defined over the field K_0.

This isogeny will be the key of our calculations, because it relates the type of reduction of the curve C_n with that of the elliptic curves $E, E_{(n)}$, which is easy to determine.

An automorphism φ of C_n which commutes with G_i induces an automorphism φ^{E_i} of the elliptic curve E_i. Let us compute these induced automorphisms.

Proposition 2.4. *The automorphisms of* C_n *induce the following automorphisms of* E *through the quotient map* ψ_2*:*

$$\begin{aligned} \varphi_{00}^E &= \varphi_{12}^E = \mathrm{Id}_E, \\ \varphi_{01}^E &= \varphi_{13}^E = [-i]_E + (1,0,1), \\ \varphi_{02}^E &= \varphi_{10}^E = [-1]_E + (0,0,1), \\ \varphi_{03}^E &= \varphi_{11}^E = [i]_E + (-1,0,1). \end{aligned}$$

Proof. It is enough to consider the affine part $Z = 1$. Put $x = X/Z$, $y = Y/Z$, so that

$$(u,v) := \psi_2(x,y) = \left(i\frac{y+\zeta\mu}{y-\zeta\mu},(i-1)\frac{x^2-m}{(y-\zeta\mu)^2}\right).$$

After some algebraic manipulation we obtain that

$$\varphi_{01}^E(u,v) = \left(\frac{u-1}{u+1}, -\frac{2iv}{(u+1)^2}\right).$$

We will now identify this map. Let us consider the associated map $f(Q) = \varphi_{01}^E(Q) + P_1$, where $P_1 = \varphi_{01}^E(O_E) = (1,0)$ (O_E denotes the point at infinity of E). We may use the addition formulas for E to calculate the equations which define f. We find that:

$$f(u,v) = (-u, iv).$$

Thus, the map f is multiplication by $-i$ on E, and hence it follows that $\varphi_{01}^{E} = [-i]_E + P_1$. The rest of the cases are solved in a similar way. Note that they can be grouped in pairs of the form $\varphi, \varphi \circ \beta$. $\square$

The same kind of calculations yields:

Proposition 2.5. *The automorphisms of C_n induce the following automorphisms of $E_{(n)}$ through the quotient map ψ_3:*

$$\begin{aligned}
\varphi_{00}^{E_{(n)}} &= \varphi_{02}^{E_{(n)}} = \mathrm{Id}_{E_{(n)}}, & \varphi_{01}^{E_{(n)}} &= \varphi_{03}^{E_{(n)}} = [-1]_{E_{(n)}},\\
\varphi_{11}^{E_{(n)}} &= \varphi_{13}^{E_{(n)}} = \mathrm{Id}_{E_{(n)}} + P_n, & \varphi_{10}^{E_{(n)}} &= \varphi_{12}^{E_{(n)}} = [-1]_{E_{(n)}} + P_n,\\
\varphi_{21}^{E_{(n)}} &= \varphi_{23}^{E_{(n)}} = \mathrm{Id}_{E_{(n)}} + P_0, & \varphi_{20}^{E_{(n)}} &= \varphi_{22}^{E_{(n)}} = [-1]_{E_{(n)}} + P_0,\\
\varphi_{31}^{E_{(n)}} &= \varphi_{33}^{E_{(n)}} = \mathrm{Id}_{E_{(n)}} + P_1, & \varphi_{30}^{E_{(n)}} &= \varphi_{32}^{E_{(n)}} = [-1]_{E_{(n)}} + P_1,
\end{aligned}$$

where $P_t = (t, 0, 1)$.

3. Reduction of the curves C_n.

We now begin the arithmetical study of the curves C_n. For the definitions and basic results concerning reduction of curves, we refer to [**Ds81**].

For $n \not\equiv 0, 1 \pmod{2^5}$, let us denote by $\mathcal{X}_n$ the arithmetic surface

$$\mathrm{Proj}\,(\mathbb{Z}[X, Y, Z]/(Y^4 - X^4 + (4n-2)X^2Z^2 - Z^4)).$$

This surface is a model over $\mathbb{Z}$ of the curve C_n. Hence, the generic fibre of $\mathcal{X}_n$ is smooth, but $\mathcal{X}_n$ has singularities in some special fibres.

Proposition 3.1. *The arithmetic surface $\mathcal{X}_n$ has good reduction outside the primes dividing $n(n-1)$.*

Proof. The special fibre of $\mathcal{X}_n$ in a prime p is given by the reduction mod p of the equation $Y^4 = X^4 - (4n-2)X^2Z^2 + Z^4$. This reduction is singular only at the double roots of the polynomial $X^4 - (4n-2)X^2Z^2 + Z^4$. The discriminant of this polynomial is $2^{12}n^2(n-1)^2$. $\square$

We must centre our attention on the primes dividing $n(n-1)$. It is well-known that there exists a stable model of C_n over the ring of integers of a certain number field K_n. We will determine this field and the stable model of C_n, taking into account the isogeny Ψ. We introduce some more notations:

Notation 3.2. Let $\alpha = \sqrt[3]{18 - 6\sqrt{3}}$, and let α_n be a root of the equation

$$t^3(t^3 - 24)^3 - 2^8\frac{(n^2 - n + 1)^3}{n^2(n-1)^2}(t^3 - 27) = 0.$$

We put $K_n = K_0(\alpha, \alpha_n)$, and we denote by $\mathcal{O}_n$ the ring of integers of K_n, and write $S_n = \mathrm{Spec}(\mathcal{O}_n)$.

We enumerate the reduction properties of the elliptic curves $E, E_{(n)}$ in the following two propositions. Note that the algebraic numbers α, α_n allow to define the Deuring normal form of the curves (cf. [**Si85**]). The technical condition $n \not\equiv 0, 1 (\text{mod } 2^5)$ is necessary to assure the potentially good reduction at $p = 2$ of the curve $E_{(n)}$.

Proposition 3.3.

a) *The elliptic curve $E : Y^2 = X^3 - X$ has good reduction outside $p = 2$, where it has potentially good reduction.*
b) *The curve E has good reduction everywhere over the field $\mathbb{Q}(\alpha)$. Its Deuring normal form $Y^2 + \alpha XY + Y = X^3$ has good reduction at the primes dividing 2.*
c) *The curve E is supersingular for a prime p if and only if $p = 2$ or $p \equiv 3 \pmod 4$.*

Proposition 3.4.

a) *The j-invariant of the curve $E_{(n)}$ is*
$$j_n = 2^8 \frac{(n^2 - n + 1)^3}{n^2(n-1)^2},$$
and hence the curve has good reduction outside the primes dividing $n(n-1)$.
b) *If $p | n(n-1)$ is an odd prime, $E_{(n)}$ has multiplicative reduction at p.*
c) *The curve $E_{(n)}$ has potentially good reduction at $p = 2$. Over the field $\mathbb{Q}(\alpha_n)$, the curve $E_{(n)}$ has good reduction at the primes dividing 2. Its Deuring normal form $Y^2 + \alpha_n XY + Y = X^3$ has good reduction at these primes.*
d) *The curve $E_{(n)}$ is supersingular at $p = 2$. It is supersingular at $p = 3$ if and only if $n \equiv 2 \pmod 3$. If $E_{(n)}$ is supersingular at a prime $p > 3$ then $p \equiv 3 \pmod 4$.*

The assertions concerning the supersingularity of E and E_n follow from the characterization of supersingular elliptic curves, and from the fact that $E_n(\mathbb{F}_p)$ has an evident subgroup of order 4 whenever $p > 2$.

Theorem 3.5. *Let $J_n = \mathrm{Pic}^0_{\mathcal{X}_n/K_n}$ be the Jacobian of $\mathcal{X}_{n/K_n} = C_n$, and let $\mathcal{N}$ be its Néron model over $\mathcal{O}_n$.*

a) *The curve C_n has a stable model $\mathcal{X}_n^{\mathrm{st}}$ and a semistable minimal regular model $\mathcal{X}_n^{\mathrm{reg}}$ over $\mathcal{O}_n$.*
b) *At the primes $\mathfrak{p}$ in $\mathcal{O}_n$ which divide 2, $\mathcal{N}$ has abelian reduction.*
c) *At the odd primes $\mathfrak{p}$ in $\mathcal{O}_n$ which divide $n(n-1)$, $\mathcal{N}$ has semi-abelian reduction, and its toric part has dimension 1.*
d) *At any prime $\mathfrak{p}$ in $\mathcal{O}_n$ we have canonical isomorphisms over the residual field $k_{\mathfrak{p}}$:*
$$\mathrm{Pic}^0_{\mathcal{X}_n^{\mathrm{st}}/k_{\mathfrak{p}}} \simeq \mathrm{Pic}^0_{\mathcal{X}_n^{\mathrm{reg}}/k_{\mathfrak{p}}} \simeq \mathcal{N}^0_{\mathfrak{p}}.$$

Proof. The elliptic curves $E, E_{(n)}$ have semi-stable reduction over K_n, so that the isogeny Ψ guarantees that J_n also has semi-stable reduction over K_n, by the Néron-Ogg-Safarevic criterion ([**Se-Ta68**]). But this is equivalent to the existence of a stable model $\mathcal{X}_n^{\mathrm{st}}$ for C_n over K_n (cf. [**De-Mu69**]). Blowing up the singular points of $\mathcal{X}_n^{\mathrm{st}}$ we obtain a semistable minimal regular model.

We can extend the isogeny Ψ to the Néron models of $J_n, E, E_{(n)}$, and again the criterion of Néron-Ogg-Safarevic translates Propositions 3.3 and 3.4 into Parts b) and c).

Finally, the second isomorphism in Part d) is well-known (cf. [**BLR90**]), while the first isomorphism is given by (the reduction of) the map from $\mathcal{X}_n^{\mathrm{reg}}$ onto $\mathcal{X}_n^{\mathrm{st}}$ which blows down the rational components with self-intersection -2 in the special fibre. □

4. Height of the curves C_n.

We are already in position to calculate the modular height of the curves C_n. For the definition of the modular height and the remaining basic concepts of Arakelov geometry, we refer to [**MB85**] or [**La88**].

The modular height of abelian varieties has a good behaviour with respect to isogenies: Raynaud proved the following result:

Proposition 4.1 ([**Ra85**], Cor. 2.1.4)**.** *Let A_K, B_K be two abelian varieties over a number field K, and let $\mathcal{A}, \mathcal{B}$ be their Néron models over $\mathcal{O}_K$. If there exists an isogeny $\phi : A_K \longrightarrow B_K$ of degree p^e, with p a prime, then:*

$$h(\mathcal{A}) = h(\mathcal{B}) + k \log p,$$

for some $k \in \mathbb{Z}, |k| \leq e/2$.

Being clear that the height of a product of abelian schemes is the sum of the heights of the factors, we will have a good approximation to the height of the curves C_n once we have the height of the elliptic curves $E, E_{(n)}$. The computation of the height of an elliptic curve amounts to the computation of its period lattice and of its reduction, following Tate's algorithm. We obtain:

Proposition 4.2. *The height of the elliptic curve $E_{(n)}$ is given by*

$$h(E_{(n)}) = \frac{1}{6} \log n(n-1) - \frac{1}{6} v_2(n(n-1)) \log 2 - \frac{1}{12} \log(|\Delta(\tau_n)|(\operatorname{Im} \tau_n)^6),$$

where

$$\tau_n = iK(\sqrt{1 - 1/n})/K(1/\sqrt{n})$$

is the fundamental period of $E_{(n)}$, $K()$ denotes the complete elliptic integral of the first kind, v_2 is the 2-adic valuation and Δ is the discriminant modular form.

Remark. For $n = 2$ this formula gives $h(E) = \log \dfrac{\Gamma(3/4)\sqrt{2}}{\Gamma(1/4)\sqrt{\pi}}$, as in [**De85**].

The modular height of a curve is that of the Néron model of its Jacobian. Combining Propositions 4.1 and 4.2 we arrive at:

Theorem 4.3. *The height of the curve C_n is given by*

$$h(C_n) = 2\log \frac{\Gamma(3/4)\sqrt{2}}{\Gamma(1/4)\sqrt{\pi}} + \frac{1}{6}\log \frac{n(n-1)}{2^{v_2(n(n-1))}} - \frac{1}{12}\log(|\Delta(\tau_n)|(\operatorname{Im}\tau_n)^6) + k\log 2,$$

for a certain $k \in \{0, \pm 1\}$ (which depends on n).

We can draw a graphic of the height $h(C_n)$ as a function of n, where its logarithmic behaviour when n approaches the singular curves of the family $(n = 1, n \to \infty)$ will become evident:

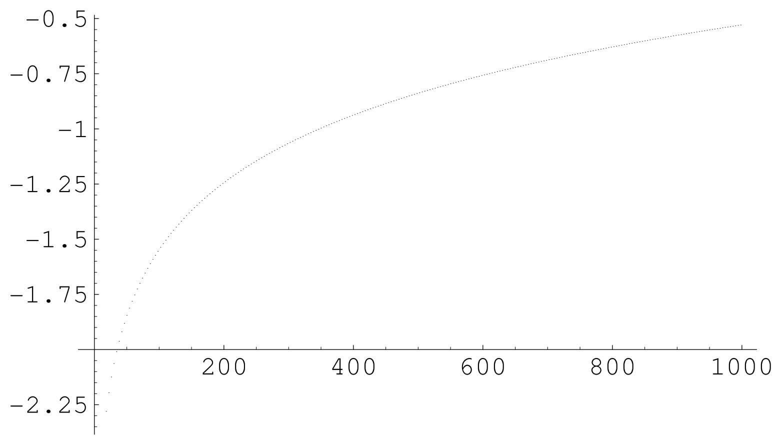

Figure 1.

5. The stable model of the curves C_n over odd primes.

Let $p|n(n-1)$ be an odd prime, and let $\mathfrak{p}$ be a prime divisor of p in $\mathcal{O}_n$. We will now determine the fibre over $\mathfrak{p}$ of $\mathcal{X}_n^{\mathrm{st}}$.

Lemma 5.1. *The irreducible components of $\mathcal{X}^{st}_{n/k_{\mathfrak{p}}}$ are smooth.*

Proof. If we reduce the equation of C_n mod p

$$Y^4 = X^4 - (4n-2)X^2Z^2 + Z^4 = (X^2+Z^2)^2 - 4nX^2Z^2 = (X^2-Z^2)^2 - 4(n-1)X^2Z^2,$$

we have that the fibre at $\mathfrak{p}$ has equation $(Y^2-X^2-Z^2)(Y^2+X^2+Z^2)=0$ if p divides n, and $(Y^2-X^2+Z^2)(Y^2+X^2-Z^2)=0$ if p divides $n-1$. In both cases, the fibre is reduced and its components are smooth and rational. Thus, we can obtain a semi-stable minimal regular model for C_n blowing-up, normalizing and blowing down exceptional components. None of these operations introduces singular components. The stable model is obtained from the minimal regular model contracting the rational components with self-intersection -2, but again this does not introduce singularities ([**Li69**]). □

Lemma 5.2. $\mathcal{X}^{st}_{n/k_{\mathfrak{p}}}$ *is not irreducible.*

Proof. The irreducibility of $\mathcal{X}^{st}_{n/k_{\mathfrak{p}}}$ would imply that it is a nonsingular curve of genus 3, and then $\mathrm{Pic}^0_{\mathcal{X}^{st}_n/k_{\mathfrak{p}}}$ would be an abelian variety of dimension 3, contradicting Theorem 3.5. □

These results drastically reduce the number of possible configurations for $\mathcal{X}^{st}_{n/k_{\mathfrak{p}}}$. The dimension of the toric part of $\mathrm{Pic}^0_{\mathcal{X}^{\mathrm{st}}_n/k_{\mathfrak{p}}}$ being 1 (Theorem 3.5), the geometric configuration of the fibre must form exactly one cycle (cf. [**BLR90**], pp. 245-249).

Suppose that $\mathcal{X}^{st}_{n/k_{\mathfrak{p}}}$ has a component of genus 2. Then, the other components should be rational, and they should form a cycle. But this is impossible, since the rational components must intersect the other components at least at three points. Hence, the components of $\mathcal{X}^{st}_{n/k_{\mathfrak{p}}}$ must be rational or elliptic. It remains only one possibility: $\mathcal{X}^{st}_{n/k_{\mathfrak{p}}}$ must have two elliptic components, intersecting at two points:

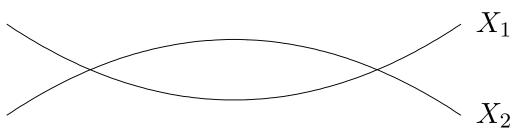

Figure 2.

Theorem 5.3. *The special fibre of* $\mathcal{X}^{\mathrm{st}}_n$ *at an odd prime* $\mathfrak{p}$ *in* $\mathcal{O}_n$ *dividing* $n(n-1)$ *has two elliptic components* $X_{1,\mathfrak{p}}, X_{2,\mathfrak{p}}$, *intersecting at two different points.*

6. Automorphisms of $\mathcal{X}^{st}_n$.

The automorphisms of C_n can be extended to automorphisms of the stable model $\mathcal{X}^{\mathrm{st}}_n$ (cf. [**Ra90**]), which we will denote by the same letters. The quotients $\mathcal{E}_i^0 := \mathcal{X}_n^{st}/G_i$ exist and are semi-stable curves (cf. [**Ra90**]). The generic fibre of $\mathcal{E}_i^0$ is the elliptic curve $E_i = E$ for $i = 1, 2$, and the elliptic curve $E_3 = E_{(n)}$ for $i = 3$.

An automorphism φ of $\mathcal{X}_n^{\mathrm{st}}$ which commutes with G_i induces an automorphism $\varphi^{\mathcal{E}_i^0}$ of $\mathcal{E}_i^0$, leaving the smooth part invariant, and hence determines an automorphism $\varphi^{\mathcal{E}_i}$ of the Néron model $\mathcal{E}_i$ of the elliptic curve E_i. We will denote by $\varphi^{E_{i/k_\mathfrak{p}}}$ the reduction of this automorphism mod a prime $\mathfrak{p}$, i.e., the automorphism induced by φ in the special fibre of $\mathcal{E}_i$. We remark that at good reduction primes the special fibres of $\mathcal{E}_i$ and $\mathcal{E}_i^0$ are the same, and we have a canonical isomorphism between them.

When E_i has good reduction at $\mathfrak{p}$, we can read the maps $\varphi^{E_{i/k_\mathfrak{p}}}$ from the maps φ^{E_i} calculated in Propositions 2.4 and 2.5, thanks to the universal property of the Néron model. We also have to control the reduction of the points appearing on those propositions, i.e., we have to control the supersingular character of E_i at $\mathfrak{p}$, which we know from Propositions 3.3 and 3.4. We obtain:

Theorem 6.1. *Let $\mathfrak{p}$ be a prime in $\mathcal{O}_n$ dividing 2.*

a) *The automorphisms of C_n induce the following automorphisms in the special fibre $E_{/k_\mathfrak{p}}$ of $\mathcal{E}$ at $\mathfrak{p}$:*

$$\varphi_{00}^{E_{/k_\mathfrak{p}}} = \varphi_{10}^{E_{/k_\mathfrak{p}}} = \mathrm{Id}_{E_{/k_\mathfrak{p}}}, \qquad \varphi_{02}^{E_{/k_\mathfrak{p}}} = \varphi_{12}^{E_{/k_\mathfrak{p}}} = [-1]_{E_{/k_\mathfrak{p}}},$$

$$\varphi_{01}^{E_{/k_\mathfrak{p}}} = \varphi_{11}^{E_{/k_\mathfrak{p}}} = [-i]_{E_{/k_\mathfrak{p}}}, \qquad \varphi_{03}^{E_{/k_\mathfrak{p}}} = \varphi_{13}^{E_{/k_\mathfrak{p}}} = [i]_{E_{/k_\mathfrak{p}}}.$$

b) *The automorphisms of C_n induce the following automorphisms in the special fibre $E_{(n)/k_\mathfrak{p}}$ of $\mathcal{E}_n$ at $\mathfrak{p}$:*

$$\varphi_{00}^{E_{(n)/k_\mathfrak{p}}} = \varphi_{02}^{E_{(n)/k_\mathfrak{p}}} = \varphi_{11}^{E_{(n)/k_\mathfrak{p}}} = \varphi_{13}^{E_{(n)/k_\mathfrak{p}}} =$$

$$\varphi_{21}^{E_{(n)/k_\mathfrak{p}}} = \varphi_{23}^{E_{(n)/k_\mathfrak{p}}} = \varphi_{31}^{E_{(n)/k_\mathfrak{p}}} = \varphi_{33}^{E_{(n)/k_\mathfrak{p}}} = \mathrm{Id}_{E_{(n)/k_\mathfrak{p}}},$$

$$\varphi_{01}^{E_{(n)/k_\mathfrak{p}}} = \varphi_{03}^{E_{(n)/k_\mathfrak{p}}} = \varphi_{10}^{E_{(n)/k_\mathfrak{p}}} = \varphi_{12}^{E_{(n)/k_\mathfrak{p}}} =$$

$$\varphi_{20}^{E_{(n)/k_\mathfrak{p}}} = \varphi_{22}^{E_{(n)/k_\mathfrak{p}}} = \varphi_{30}^{E_{(n)/k_\mathfrak{p}}} = \varphi_{32}^{E_{(n)/k_\mathfrak{p}}} = [-1]_{E_{(n)/k_\mathfrak{p}}}.$$

Note that this implies that the points in $E_{/k_\mathfrak{p}}$ fixed by the induced automorphisms must be 2-torsion points. The same kind of reasoning proves that, at the odd primes, the points in the special fibre of $E_{/k_\mathfrak{p}}$ fixed by the induced automorphisms must be 4 torsion points, a fact that will be used in Proposition 9.2.

7. The stable model of the curves C_n over $p = 2$.

We now determine the fibre of $\mathcal{X}_n^{\mathrm{st}}$ over a prime $\mathfrak{p}$ of $\mathcal{O}_n$ dividing 2. In analogy with Lemmas 5.1 and 5.2, we have:

Lemma 7.1.

a) $\mathrm{Pic}^0_{\mathcal{X}_n^{\mathrm{st}}/k_{\mathfrak{p}}}$ *is an abelian variety of dimension* 3.

b) *The geometric configuration of* $\mathcal{X}^{st}_{n/k_{\mathfrak{p}}}$ *contains no cycle.*

c) *The irreducible components of* $\mathcal{X}^{st}_{n/k_{\mathfrak{p}}}$ *are smooth.*

Proof. The first assertion follows from the isogeny Ψ. Parts b) and c) are consequence of a) by [**BLR90**], Cor. 9.12. □

The study of the special fibre of $\mathcal{X}^{st}_{n/k_{\mathfrak{p}}}$ is not so easy in this case, since the dimension of the Jacobian does not provide further information. We are now obliged to use the results obtained in previous section.

Proposition 7.2. $\mathcal{X}^{st}_{n/k_{\mathfrak{p}}}$ *is not irreducible.*

Proof. The reduction of the quotient maps $\psi_i : \mathcal{X}^{st}_{n/k_{\mathfrak{p}}} \longrightarrow \mathcal{E}_i^0$ yields an isogeny of degree 8

$$\Psi_{\mathfrak{p}} : \mathrm{Pic}^0_{\mathcal{X}^{st}_{n/k_{\mathfrak{p}}}} \longrightarrow E_{/k_{\mathfrak{p}}} \times E_{/k_{\mathfrak{p}}} \times E_{(n)/k_{\mathfrak{p}}}.$$

As $\mathrm{Pic}^0_{\mathcal{X}^{st}_{n/k_{\mathfrak{p}}}}$ is an abelian variety, we can consider the dual isogeny Φ_p of $\Psi_{\mathfrak{p}}$. The Hasse-Witt invariant of the elliptic curves $E, E_{(n)}$ is 0, since its j-invariant is 0. Hence, their product admits no separable isogenies of degree a power of 2. Thus, the composition $\Psi_{\mathfrak{p}} \circ \Phi_{\mathfrak{p}} = [2]$ must be purely inseparable, which implies that also $\Psi_{\mathfrak{p}}$ is purely inseparable.

Suppose that $\mathcal{X}^{st}_{n/k_{\mathfrak{p}}}$ is irreducible. It should be smooth, since its Jacobian has dimension 3. As the maps ψ have degree 2, they must be separable or purely inseparable. In the last case, $\mathcal{X}^{st}_{n/k_{\mathfrak{p}}}$ would be isomorphic to the elliptic curve $E_{/k_{\mathfrak{p}}}$, which is not possible. Hence, the three maps ψ_1, ψ_2, ψ_3 should be separable. But the isogeny $\Psi_{\mathfrak{p}}$ that they induce would be also separable (this can be seen in terms of differential forms), and we have seen that this is not the case. □

The reduction of the maps ψ_i being exhaustive, we must have non-rational irreducible components X_1, X_2, $X_{(n)}$ such that $\psi_i|_{X_i} : X_i \longrightarrow E_{i/k_{\mathfrak{p}}}$ is a degree 2 map. A priori, we do not know whether these components are really different or they coincide. In any case, we know that $\mathcal{X}^{st}_{n/k_{\mathfrak{p}}}$ cannot have more than three non-rational components, since it has genus 3.

Let X_i be one of the non-rational components of $\mathcal{X}^{st}_{n/k_{\mathfrak{p}}}$. If $\psi_i(X_i)$ is not a point, then $\psi_i|_{X_i}$ must be purely inseparable (otherwise the isogeny $\Psi_{\mathfrak{p}}$

would not be purely inseparable), i.e., it must be a degree 2 Frobenius map. In particular, X_i must have genus 1.

Let us consider the reduction of the automorphism φ_{01} over $\mathcal{X}^{st}_{n/k_\mathfrak{p}}$, and its action on $E_{1/k_\mathfrak{p}}$ and $E_{(n)/k_\mathfrak{p}}$:

$$\varphi_{01}^{E_{1/k_\mathfrak{p}}} = [-i]_{E_{1/k_\mathfrak{p}}}, \qquad \varphi_{01}^{E_{(n)/k_\mathfrak{p}}} = [-1]_{E_{(n)/k_\mathfrak{p}}}.$$

As $\psi_1|X_1$ and $\psi_3|X_n$ are bijective maps, we see that $\varphi_{01}(X_1)=X_1$, $\varphi_{01}(X_{(n)}) = X_{(n)}$. Moreover,

$$\psi_{01}|X_1 = \psi_{1/k_\mathfrak{p}}^{-1} \circ \varphi_{01}^{E_{1/k_\mathfrak{p}}} \psi_{1/k_\mathfrak{p}} = \psi_{1/k_\mathfrak{p}}^{-1} \circ [-i]_{E_{1/k_\mathfrak{p}}} \psi_{1/k_\mathfrak{p}},$$

and hence $\psi_{01}|X_1 = [-1]_{X_1}$. We can see in the same way that $\psi_{01}|X_{(n)} = \mathrm{Id}_{X_{(n)}}$. This fact ensures that $X_1 \neq X_{(n)}$. We can argue similarly to prove that $X_2 \neq X_{(n)}$ and $X_1 \neq X_2$. At this moment we have seen that the special fibre $\mathcal{X}^{st}_{n/k_\mathfrak{p}}$ has exactly three elliptic components, and the rest of the components must be rational. We know that $\psi_2 = \psi_1 \circ \gamma$, so that $X_2 = \gamma(X_1)$, and this forces $\gamma(X_{(n)}) = X_{(n)}$. Moreover, $X_1 \cap X_{(n)} \neq \emptyset$ if and only if $X_2 \cap X_{(n)} \neq \emptyset$. Gathering all these restrictions on the special fibre $\mathcal{X}^{st}_{n/k_\mathfrak{p}}$, only two possibilities remain:

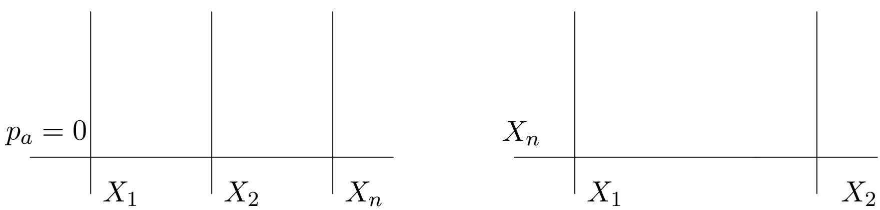

Figure 3.

Let us denote by O_i the neutral point of the group law on the elliptic component X_i.

Lemma 7.3. *The components $X_1, X_{(n)}$ can intersect only at their neutral point. The same is true for the components $X_2, X_{(n)}$.*

Proof. Let $P \in X_1 \cap X_{(n)}$. We know that $\varphi_{12}|X_{(n)} = \mathrm{Id}_{X_{(n)}}$, so that $\varphi_{12}(P) = P$. On the other hand, $\varphi_{12}|X_1 = [-1]_{X_1}$ implies $\varphi_{12}(P) = -P$, where we denote by $-P$ the opposite of P with respect to the group law on X_1. The component X_1 is isomorphic to the elliptic curve $E_{/k_\mathfrak{p}}$, which is supersingular, so that there are no 2-torsion points on X_1, that is, we must have $P = O_1$. Using the automorphism φ_{01} we see that we must also have $P = O_{(n)}$. The second assertion is proved using the automorphism γ. $\square$

The lemma excludes the second option in Figure 3. We have finally reached:

Theorem 7.4. *The special fibre of the stable model of C_n at a prime $\mathfrak{p}$ in $\mathcal{O}_n$ dividing 2 consists of three elliptic components $X_{1,\mathfrak{p}}, X_{2,\mathfrak{p}}, X_{3,\mathfrak{p}}$, which intersect a rational component $X_{0,\mathfrak{p}}$ at three different points. The map $\psi_i : \mathcal{X}^{st}_{n/k_\mathfrak{p}} \longrightarrow E_{/k_\mathfrak{p}}$ restricted to the component $X_{i,\mathfrak{p}}$ is the Frobenius map, and contracts the other components.*

8. A lower bound for Arakelov self-intersection.

We now begin the study of the self-intersection of the Arakelov dualizing sheaf of the arithmetic surfaces given by the curves C_n. For the definitions, we refer again to [**MB85**] or [**La88**].

In this section we use a result of Moriwaki to precise a lower bound for the self-intersection of the canonical sheaf of the curves C_n.

Proposition 8.1 ([**Mo96**]). *Let $\mathcal{X} \longrightarrow S = \operatorname{Spec}(\mathcal{O}_K)$ a stable arithmetic surface of genus $g \geq 2$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the primes in $\mathcal{O}_K$ where the fibres of $\mathcal{X}$ are reducible. We have that*

$$(\overline{\omega}_{\mathcal{X}/S}, \overline{\omega}_{\mathcal{X}/S})_{\mathrm{Ar}} \geq \frac{1}{6(g-1)} \sum_{i=1}^{t} \log N_{K/\mathbb{Q}}(\mathfrak{p}_i).$$

We apply this result to the arithmetic surface given by the curve C_n to obtain:

Proposition 8.2. *For the stable model $\mathcal{X}_n^{\mathrm{st}} \longrightarrow S_n$ of the curve C_n over the ring of integers of the field K_n, we have:*

$$\left(\overline{\omega}_{\mathcal{X}_n^{\mathrm{st}}/S_n}, \overline{\omega}_{\mathcal{X}_n^{\mathrm{st}}/S_n}\right)_{\mathrm{Ar}} \geq \frac{1}{12} \log \prod_{\mathfrak{p} \in S_n, \mathfrak{p} | n(n-1)} N_{K_n/\mathbb{Q}}(\mathfrak{p}).$$

The normalized Arakelov self-intersection of a curve C_K defined over a number field K is

$$e(C_K) := \frac{1}{[L:K]} (\omega_{\mathcal{C}^{st}}, \omega_{\mathcal{C}^{st}})_{\mathrm{Ar}},$$

where L/K is an extension over which C_K has a stable model $\mathcal{C}^{st}$. The curves C_n provide examples of curves with normalized Arakelov self-intersection as large as desired:

Theorem 8.3. *For any $H > 0$, there exist infinitely many curves $C_\mathbb{Q}$ defined over $\mathbb{Q}$ such that $e(C_\mathbb{Q}) > H$.*

Proof. We know from Sections 5 and 7 that the primes where the stable model of the curve C_n has reducible fibres are those dividing $n(n-1)$.

Hence, the curves C_n satisfy

$$e(C_n) \geq \frac{1}{12[K_n : \mathbb{Q}]} \log \prod_{p \mid n(n-1)} p.$$

The degree of the extension $K_n/\mathbb{Q}$ is always less than or equal to 1152, while we can take infinitely many values of n which make the product $\prod_{p|n(n-1)} p$ as large as desired. □

9. Canonical divisors.

We will ultimately give an explicit formula for the self-intersection of the Arakelov dualizing sheaf of the arithmetic surfaces given by the curves C_n. The purpose of this section is the determination of two canonical divisors on the minimal model of these curves. We shall need to make a base extension to determine such divisors.

Let us consider the following points on the (affine part of the) curve $C_n(\overline{\mathbb{Q}})$:

$$P_1 = \left(\sqrt{2n - 2u\sqrt{n(n-1)} - 1}, \tfrac{\sqrt[4]{12} - \sqrt{3} - 1}{2} \zeta\mu\right),$$

$$P_2 = \left(-\sqrt{2n - 2u\sqrt{n(n-1)} - 1}, \tfrac{\sqrt[4]{12} - \sqrt{3} - 1}{2} \zeta\mu\right),$$

$$P_3 = \left(\sqrt{2n + 2u\sqrt{n(n-1)} - 1}, \tfrac{\sqrt[4]{12} - \sqrt{3} - 1}{2} \zeta\mu\right),$$

$$P_4 = \left(-\sqrt{2n + 2u\sqrt{n(n-1)} - 1}, \tfrac{\sqrt[4]{12} - \sqrt{3} - 1}{2} \zeta\mu\right),$$

where $u = \sqrt{-6 + 4\sqrt[4]{12} - 4\sqrt{3} + 2\sqrt[4]{108}}$. The images of these points through the map ψ_2 are two different nontrivial 3-torsion points $R_1 = \psi_2(P_1) = \psi_2(P_2)$, $R_2 = -R_1 = \psi_2(P_3) = \psi_2(P_4)$ of the elliptic curve E.

In the proof of Lemma 10.2, we shall need the following result:

Lemma 9.1. *The pair of points P_1, P_2 is conjugate over $\mathbb{Q}$ to the pair of points P_3, P_4.*

Proof. It is enough to see that the pairs are conjugate over the larger field $L = \mathbb{Q}(b\zeta\mu)$, where $b = \frac{\sqrt[4]{12} - \sqrt{3} - 1}{2}$. This, in turn, reduces to proving that $u^2 n(n-1)$ is not a square in L. Since $\mathbb{Q}(b) = \mathbb{Q}(b^4) = \mathbb{Q}(\sqrt[4]{12}) \subset L = \mathbb{Q}(\sqrt[4]{12}, \sqrt[4]{-4n(n-1)})$, we have $\zeta\mu \in L$, and $(\zeta\mu)^4 = -4n(n-1)$ is a square in L. Hence, we must check that $-u^2 = b^4 - 1 \notin L^{*2}$. Let $F := \mathbb{Q}(\sqrt{b^4 - 1})$; we want to see that $[FL : L] = 2$, or equivalently, that $[FL : F] = 4$, so that it suffices to prove that the polynomial $X^4 - (\zeta\mu)^4 = X^4 + 4n(n-1)$ is irreducible over F. This can only happen if $-4n(n-1)$ is a square or -4 times a fourth power in F. The first possibility is excluded, since $F \subset \mathbb{R}$. So, it remains to prove that $H := \mathbb{Q}(\sqrt[4]{n(n-1)})$ may not be embedded in F. Using PARI ([**BBBCO99**]), we see that the only quartic subfield of F is $M = \mathbb{Q}(\sqrt[4]{12})$. If $H = M$ the discriminants of the polynomials $X^4 - 12$ and $X^4 - n(n-1)$ should agree up to a square, i.e., $3n(n-1)$ should be a rational square, which is not possible since we are assuming that $n \equiv 2 \pmod 3$. □

Let L_n be a finite extension of the field $K_n(P_1, P_3)$ satisfying the following condition: The prime ideals in $K_n(P_1, P_3)$ which divide $n(n-1)$ become principal in L_n (the convenience of this condition will be clear later, just before Theorem 9.4). We will denote by T_n the spectrum of the ring of integers $\mathcal{O}'_n$ of L_n, and by $\ell_{\mathfrak{p}}$ the residual field of L_n at a prime $\mathfrak{p}$ in $\mathcal{O}'_n$. Since the stable model is stable under base extensions, the stable model $\mathcal{V}_n$ of C_n over T_n is the base change of $\mathcal{X}_n^{\mathrm{st}}$ to $\mathcal{O}'_n$. In particular, the configuration of the special fibres of $\mathcal{V}_n$ is exactly the same of the special fibres of $\mathcal{X}_n^{\mathrm{st}}$ (so that we are going to use the same notation for them).

Proposition 9.2. *Let U be the smooth part of $\mathcal{V}_n$, and let U_1 be the complement of the bad fibres of $\mathcal{V}_n$.*

a) *The points P_1, P_2, P_3, P_4 extend to sections M_1, M_2, M_3, M_4 on $\mathcal{V}_n$ contained in U.*
b) *Let $D_1^0 = M_1 + M_2 + M_3 + M_4$. We have a canonical isomorphism:*

$$\omega_{\mathcal{V}_n/T_n}|U_1 \simeq \mathcal{O}_{\mathcal{V}_n/T_n}(D_1^0)|U_1.$$

Proof. a) The non-smooth points in $\mathcal{V}_n$ are the singular points of the fibres at primes of bad reduction. At an odd prime of bad reduction, any automorphism φ^2 leaves the singular points of the fibre invariant, so that the corresponding induced automorphism on E leaves invariants the images of these singular points through ψ_2. Hence, these points must be 4-torsion points (cf. remark at the end of Section 6). In the fibres over even primes, a similar argument shows that the singular points must be the neutral elements of the elliptic components. The reductions of the sections M_i are 3-torsion points on the special fibre, which can only be trivial at places of characteristic 3, (where E is supersingular), but these are primes of good reduction, since $n \equiv 2 \pmod 3$. Thus, the sections M_i do not pass through the non-smooth points of $\mathcal{V}_n$.

b) On U_1 we have an isomorphism $\omega_{\mathcal{V}_n/T_n}|U_1 \simeq \Omega_{\mathcal{V}_n/T_n}|U_1$, where $\Omega_{\mathcal{V}_n/T_n}$ denotes the sheaf of relative differential forms. On the generic fibre C_n we have

$$\Omega_{C_n} \simeq \mathcal{O}_{C_n}(P_1 + P_2 + P_3 + P_4),$$

since the genus of C_n is 3, every plane embedding of C_n is canonical, and the points P_i are collinear. This isomorphism extends to an isomorphism

$\Omega_{\mathcal{V}_n/T_n}|U_1 \simeq \mathcal{O}_{\mathcal{V}_n/T_n}(D_1^0)|U_1$, since the surface $\mathcal{V}_n|U_1$ is described by a global nonsingular plane quartic, and the line through the points P_i is globally defined over $\mathbb{Z}$, since it is of the form $y = \lambda$, with λ and algebraic integer. □

In order to extend the above isomorphism to the whole surface $\mathcal{V}_n$, we must add some vertical components of the bad fibres to the divisor D_1^0. To control these extra components, we have to determine the intersection of D_1^0 with the bad fibres.

Lemma 9.3.

a) *Let $\mathfrak{p}$ an odd prime in $\mathcal{O}'_n$ dividing $n(n-1)$. The divisor D_1^0 intersects each of the two components of $\mathcal{V}_{n,\mathfrak{p}}$ at two points (which may coincide).*

b) *Let $\mathfrak{p}$ a prime in $\mathcal{O}'_n$ dividing 2. The divisor D_1^0 intersects $X_{2,\mathfrak{p}}$ at two points with multiplicity 2, and does not cut any other component of the fibre.*

Proof. a) From $\varphi_{12}(P_1) = P_2$, we deduce that $\varphi_{12}(M_1) = M_2$. Moreover, φ_{12} permutes $X_{1,\mathfrak{p}}$ and $X_{2,\mathfrak{p}}$, because $\psi_1 = \psi_2 \circ \varphi_{12}$. Thus, M_1, M_2 must intersect $\mathcal{V}_{n,\mathfrak{p}}$ at different components. The same is true for M_3, M_4.

b) The reduced map $\psi_{2,\mathfrak{p}}|X_{2,\mathfrak{p}}$ is a Frobenius map of degree 2, so that is totally ramified. The images of the reduction of the sections M_i are two different 3-torsion points in the special fibre of E, and thus the M_i must pass through two points on $X_{2,\mathfrak{p}}$. □

From now on, we will work on the minimal regular model $\mathcal{V}'_n$ of the curves C_n. It may be obtained blowing up the singularities of the stable model. Let us introduce some notation to describe the bad fibres of $\mathcal{V}'_n$.

At a prime $\mathfrak{p}$ in $\mathcal{O}'_n$ dividing $n(n-1)$, we write $\mathcal{V}_{n,\mathfrak{p}} = X_{1,\mathfrak{p}} + X_{2,\mathfrak{p}}$, and denote by $W_{1,\mathfrak{p}}, W_{2,\mathfrak{p}}$ the intersection points of these two components; these points may be singular points on the surface $\mathcal{V}_n$; we call $s_{1,\mathfrak{p}}, s_{2,\mathfrak{p}}$ respectively the number of blow-ups necessary to desingularize these points. The fibre of $\mathcal{V}'_n$ consists of the two elliptic components $X_{1,\mathfrak{p}}, X_{2,\mathfrak{p}}$, linked by rational components $A_{1,\mathfrak{p}}, \dots, A_{s_{1,\mathfrak{p}},\mathfrak{p}}$ (lying above $W_{1,\mathfrak{p}}$), $B_{1,\mathfrak{p}}, \dots, B_{s_{2,\mathfrak{p}},\mathfrak{p}}$ (lying above $W_{2,\mathfrak{p}}$).

At a prime $\mathfrak{p}$ in $\mathcal{O}'_n$ dividing 2, we write $\mathcal{V}_{n,\mathfrak{p}} = X_{0,\mathfrak{p}} + X_{1,\mathfrak{p}} + X_{2,\mathfrak{p}} + X_{3,\mathfrak{p}}$, where $X_{0,\mathfrak{p}}$ is a rational component, and $X_{2,\mathfrak{p}}$ is the elliptic component that dominates the special fibre of E through the map ψ_2. Again, the intersection points $Y_{1,\mathfrak{p}}, Y_{2,\mathfrak{p}}, Y_{3,\mathfrak{p}}$ of the three elliptic components with the rational component may be singular points on $\mathcal{V}_n$; we will denote by $r_\mathfrak{p}, s_\mathfrak{p}, t_\mathfrak{p}$ the number of blow-ups needed to desingularize these points. Since the automorphism γ permutes $X_{1,\mathfrak{p}}$ and $X_{2,\mathfrak{p}}$, we know that $r_\mathfrak{p} = s_\mathfrak{p}$. The fibre of $\mathcal{V}'_n$ consists of the three elliptic components, linked to $X_{0,\mathfrak{p}}$ by rational components $A_{1,\mathfrak{p}}, \dots, A_{r_\mathfrak{p},\mathfrak{p}}$ (lying above $Y_{1,\mathfrak{p}}$), $B_{1,\mathfrak{p}}, \dots, B_{r_\mathfrak{p},\mathfrak{p}}$ (lying above $Y_{2,\mathfrak{p}}$), and $C_{1,\mathfrak{p}}, \dots, C_{s_\mathfrak{p},\mathfrak{p}}$ (lying above $Y_{3,\mathfrak{p}}$). All these rational components are numbered starting from the corresponding elliptic component and ending in $X_{2,\mathfrak{p}}$.

We shall determine a canonical divisor on $\mathcal{V}_n'$, that is, a divisor $D = D_1^0 + \sum_{\mathfrak{p}} D_{\mathfrak{p}}$ such that $\omega_{\mathcal{V}_n'/T_n} \simeq \mathcal{O}_{\mathcal{V}_n'}(D)$. We write $D_{\mathfrak{p}} = \sum_i a_{i,\mathfrak{p}} X_{i,\mathfrak{p}} + E_{\mathfrak{p}}$, with the $a_{i,\mathfrak{p}}$ integer coefficients, and $E_{\mathfrak{p}}$ a sum of rational components coming from blow-ups. The advantage of working on a regular model is that we may use the adjunction formula for every component X of a fibre $\mathcal{V}_{n,\mathfrak{p}}'$:

$$2g_X - 2 = \omega_{\mathcal{V}_n'/T_n}|X + X^2 = (D_1^0, X) + (D_{\mathfrak{p}}, X) + X^2,$$

to obtain a linear system of equations on the coefficients of $D_{\mathfrak{p}}$. By Proposition 9.2, we only have to deal with the primes of bad reduction.

Let $\mathfrak{p}$ be an odd prime in $\mathcal{O}_n'$ of bad reduction. We write $D_{\mathfrak{p}} = a_{1,\mathfrak{p}} X_{1,\mathfrak{p}} + a_{2,\mathfrak{p}} X_{2,\mathfrak{p}} + \sum_i \alpha_i A_{i,\mathfrak{p}} + \sum_j \beta_j B_{j,\mathfrak{p}}$. The adjunction formula applied to every component of $\mathcal{V}_{n,\mathfrak{p}}'$ yields a (degenerate) linear system of equations in the coefficients of D; solving this system we see that $D_{\mathfrak{p}}$ must be an integral multiple of the whole fibre $\mathcal{V}_{n,\mathfrak{p}}'$.

In a prime $\mathfrak{p}$ dividing 2, we write $D_{\mathfrak{p}} = \sum_{k=0}^{3} a_{k,\mathfrak{p}} X_{k,\mathfrak{p}} + \sum_i \alpha_i A_{i,\mathfrak{p}} + \sum_j \beta_j B_{j,\mathfrak{p}} + \sum_k \gamma_k C_{k,\mathfrak{p}}$. The solutions of the linear system provided by the adjunction formula are

$$\begin{array}{lll} \alpha_i = a_{1,\mathfrak{p}} + i, & \beta_j = a_{2,\mathfrak{p}} - 3j, & \gamma_k = a_{3,\mathfrak{p}} + j, \\ a_{1,\mathfrak{p}} = a_0 - r_{\mathfrak{p}} - 1, & a_{2,\mathfrak{p}} = a_0 + 3(r_{\mathfrak{p}} + 1), & a_{3,\mathfrak{p}} = a_0 - s_{\mathfrak{p}} - 1. \end{array}$$

We may take $a_{0,\mathfrak{p}} = r_{\mathfrak{p}} + s_{\mathfrak{p}} + 1$, and hence, up to integral multiples of the fibre:

$$D_{\mathfrak{p}} = (r_{\mathfrak{p}} + s_{\mathfrak{p}} + 1) X_{0,\mathfrak{p}} + s_{\mathfrak{p}} X_1 + (s_{\mathfrak{p}} + 4(r_{\mathfrak{p}} + 1)) X_{2,\mathfrak{p}} + r_{\mathfrak{p}} X_3 + E_{\mathfrak{p}},$$

where $E_{\mathfrak{p}} = \sum_{i=1}^{r_{\mathfrak{p}}} (s_{\mathfrak{p}} + i) A_{i,\mathfrak{p}} + \sum_{i=1}^{s_{\mathfrak{p}}} (r_{\mathfrak{p}} + i) C_{i,\mathfrak{p}} + \sum_{j=1}^{r_{\mathfrak{p}}} (s_{\mathfrak{p}} + 4(r_{\mathfrak{p}} + 1) - 3j) B_{j,\mathfrak{p}}$. From now on, we shall denote by F_2 the sum of all the $D_{\mathfrak{p}}$ for $\mathfrak{p}$ dividing 2:

$$F_2 := \sum_{\mathfrak{p}|2} \Big((r_{\mathfrak{p}} + s_{\mathfrak{p}} + 1) X_{0,\mathfrak{p}} + s_{\mathfrak{p}} X_1 + (s_{\mathfrak{p}} + 4(r_{\mathfrak{p}} + 1)) X_{2,\mathfrak{p}} + r_{\mathfrak{p}} X_3 + E_{\mathfrak{p}} \Big).$$

We have thus seen that, for certain divisor V on $\mathcal{V}_n$ coming from an ideal I in $\mathcal{O}_n'$ dividing $n(n-1)$,

$$\omega_{\mathcal{V}_n/T_n} \simeq \mathcal{O}_{\mathcal{V}_n}(M_1 + M_2 + M_3 + M_4 + F_2 + V).$$

In order to avoid the determination of the divisor V coming from the base, we have imposed the condition defining the field L_n at the beginning of the section. Then, the divisor V becomes a principal ideal, and we can ignore it [1] .

[1] Anyway, the computations in Section 10 could be carried on without additional effort if we define $L_n = K(P_1, P_3)$ and take into account the (indeterminate) divisor V; we have chosen our definition of L_n to simplify the expressions to be obtained later on.

Theorem 9.4. *The divisor $D_1 = M_1 + M_2 + M_3 + M_4 + F_2$ is a canonical divisor on $\mathcal{V}'_n$:*

$$\omega_{\mathcal{V}'_n/T_n} \simeq \mathcal{O}_{\mathcal{V}'_n}(D_1). \tag{2}$$

We note that for any automorphism σ of C_n, $\sigma(D_1)$ is also a canonical divisor on $\mathcal{V}'_n$. Let us take $D_2 = \varphi_{01}(D_1)$. Putting $Q_i = \varphi_{01}(P_i)$, $N_i = \varphi_{01}(M_i)$, we have that $D_2 = N_1 + N_2 + N_3 + N_4 + F_2$, since φ_{01} leaves the components of the fibres at even primes invariant, as we have seen in Section 7. Since $Q_i \neq P_j$, the divisors D_1 and D_2 are disjoint on the generic fibre, so that we can use them to compute the self-intersection of the canonical sheaf.

Let us write $R_1 = \psi_2(P_1) = \psi_2(P_2), R_2 = \psi_2(P_3) = \psi_2(P_4)$, $T_1 = \psi_2(Q_1) = \psi_2(Q_2)$, $T_2 = \psi_2(Q_3) = \psi_2(Q_4)$. We know that φ_{01} induces the automorphism $\varphi_{01}^E = [-i]_E + (1,0)$ on E through ψ_2. Thus $T_k = [-i]_E R_k + (1,0)$, and $3(T_j - R_k) = (1,0)$ has exact order 2; in particular $T_j \neq R_k$. Let us denote by $R_{j,\mathfrak{p}}$, $T_{k,\mathfrak{p}}$ the reductions of the points R_j, T_k modulo a prime $\mathfrak{p}$. It is clear that if these reductions do not coincide, the corresponding sections M_j, N_k do not intersect on the fibre at $\mathfrak{p}$. If $\mathfrak{p}$ is an odd prime, the elliptic curve E has good reduction at $\mathfrak{p}$ and everything reduces properly, so that $R_{j,\mathfrak{p}} \neq T_{k,\mathfrak{p}}$. If $\mathfrak{p}|2$, the reduction of φ_{01}^E is $[-i]_{E/\ell_\mathfrak{p}}$, and we have $T_{1,\mathfrak{p}} = -T_{2,\mathfrak{p}} = [-i]_{E/\ell_\mathfrak{p}} R_1 = [i]_{E/\ell_\mathfrak{p}} R_2$, so that again $R_{j,\mathfrak{p}} \neq T_{k,\mathfrak{p}}$. We have seen:

Proposition 9.5. *The divisors $D_1^0 = D_1 - F_2$ and $D_2^0 = D_2 - F_2$ are disjoint.*

In the following section we will need the following computation:

Lemma 9.6. $(F_2, F_2) = -\sum_{\mathfrak{p}|2}(10 r_\mathfrak{p} + s_\mathfrak{p} + 11)\log \sharp\ell_\mathfrak{p}$.

Proof. We have $F_2 = \sum_\mathfrak{p} D_\mathfrak{p}$, and $(D_\mathfrak{p}, D_\mathfrak{p}) = -(10 r_\mathfrak{p} + s_\mathfrak{p} + 11)\log \sharp\ell_\mathfrak{p}$ follows from the equalities (where the factor $\log \sharp\ell_\mathfrak{p}$ is skipped):

$$D_\mathfrak{p} = (r_\mathfrak{p} + s_\mathfrak{p} + 1)X_{0,\mathfrak{p}} + s_\mathfrak{p} X_1 + (s_\mathfrak{p} + 4(r_\mathfrak{p}+1))X_{2,\mathfrak{p}} + r_\mathfrak{p} X_3 + E_\mathfrak{p},$$

$$E_\mathfrak{p} = \sum_{i=1}^{r_\mathfrak{p}}(s_\mathfrak{p} + i)A_{i,\mathfrak{p}} + \sum_{i=1}^{s_\mathfrak{p}}(r_\mathfrak{p} + i)C_{i,\mathfrak{p}} + \sum_{j=1}^{r_\mathfrak{p}}(s_\mathfrak{p} + 4(r_\mathfrak{p}+1) - 3j)B_{j,\mathfrak{p}},$$

$$\left(\sum_{i=1}^{r_\mathfrak{p}}(s+i)A_{i,\mathfrak{p}}\right)^2 = -(r_\mathfrak{p} + s_\mathfrak{p} + 1)^2 - s_\mathfrak{p}^2 + r_\mathfrak{p} + 1,$$

$$\left(\sum_{i=1}^{s_\mathfrak{p}}(r+i)C_{i,\mathfrak{p}}\right)^2 = -(r_\mathfrak{p} + s_\mathfrak{p} + 1)^2 - r_\mathfrak{p}^2 + s_\mathfrak{p} + 1,$$

$$\left(\sum_{j=1}^{r_{\mathfrak{p}}}(s_{\mathfrak{p}}+4(r_{\mathfrak{p}}+1)-3j)B_{j,\mathfrak{p}}\right)^2=-17r_{\mathfrak{p}}^2-10r_{\mathfrak{p}}s_{\mathfrak{p}}-25r_{\mathfrak{p}}-2s_{\mathfrak{p}}^2-10s_{\mathfrak{p}}-8,$$

$$\begin{aligned}(E_{\mathfrak{p}},E_{\mathfrak{p}})&=\left(\sum_{i=1}^{r_{\mathfrak{p}}}(s+i)A_{i,\mathfrak{p}}\right)^2+\left(\sum_{j=1}^{r_{\mathfrak{p}}}(s_{\mathfrak{p}}+4(r_{\mathfrak{p}}+1)-3j)B_{j,\mathfrak{p}}\right)^2\\&\quad+\left(\sum_{i=1}^{s_{\mathfrak{p}}}(r+i)C_{i,\mathfrak{p}}\right)^2\\&=-20r_{\mathfrak{p}}^2-14r_{\mathfrak{p}}s_{\mathfrak{p}}-28r_{\mathfrak{p}}-5s_{\mathfrak{p}}^2-13s_{\mathfrak{p}}-8,\end{aligned}$$

$$(X_{0,\mathfrak{p}},E_{\mathfrak{p}})=3(r_{\mathfrak{p}}+s_{\mathfrak{p}})+4,\qquad (X_{0,\mathfrak{p}},X_{0,\mathfrak{p}})=-3,$$
$$(X_{1,\mathfrak{p}},(r_{\mathfrak{p}}+s_{\mathfrak{p}}+1)X_{0,\mathfrak{p}}+E_{\mathfrak{p}})=s_{\mathfrak{p}}+1,\qquad (X_{1,\mathfrak{p}},X_{1,\mathfrak{p}})=-1,$$
$$(X_{2,\mathfrak{p}},(r_{\mathfrak{p}}+s_{\mathfrak{p}}+1)X_{0,\mathfrak{p}}+E_{\mathfrak{p}})=s_{\mathfrak{p}}+4r_{\mathfrak{p}}+1,\qquad (X_{2,\mathfrak{p}},X_{2,\mathfrak{p}})=-1,$$
$$(X_{3,\mathfrak{p}},(r_{\mathfrak{p}}+s_{\mathfrak{p}}+1)X_{0,\mathfrak{p}}+E_{\mathfrak{p}})=r_{\mathfrak{p}}+1,\qquad (X_{3,\mathfrak{p}},X_{3,\mathfrak{p}})=-1.$$

□

10. A formula for Arakelov self-intersection.

We will now give an explicit expression for the Arakelov self-intersection of the canonical sheaf of the arithmetic surfaces $\mathcal{V}_n$. Since this self-intersection is unaltered after a blow-up, we can compute it with the canonical divisors on $\mathcal{V}'_n$ which we have found in previous section.

First of all, we must extend the isomorphism (2) to an Arakelov isomorphism. The curvature of $\omega_{\mathcal{V}'_n/T_n}$ and $\mathcal{O}_{\mathcal{V}'_n}(D_j)$ being the same, it will be enough to add some vertical Arakelov components to the divisors D_1, D_2. Let us write:

$$\overline{\omega}_{\mathcal{V}'_n/T_n}\simeq\overline{\mathcal{O}}_{\mathcal{V}'_n}\left(D_j+\sum_{\sigma:L_n\to\mathbb{C}}r_{j,\sigma}C_{n,\sigma}\right).$$

We have taken $D_2=\varphi_{01}(D_1)$, and φ_{01} is defined over K_n, so that $r_{1,\sigma}=r_{2,\sigma}$ for every immersion $\sigma: L_n\to\mathbb{C}$, and the divisors D_1, D_2 are Arakelov equivalent. Moreover, this equality yields:

$$(\overline{\omega}_{\mathcal{V}'_n/T_n},\overline{\omega}_{\mathcal{V}'_n/T_n})_{\mathrm{Ar}}=(D_1,D_2)_{\mathrm{Ar}}+8\sum_{\sigma}r_{1,\sigma}.$$

We can determine this sum using the arithmetic adjunction formula (cf. [**Sz85**]), which in our case gives: $(\overline{\omega}_{\mathcal{V}'_n/T_n},\overline{\mathcal{O}}_{\mathcal{V}'_n}(M_j))_{\mathrm{Ar}}=-(M_j,M_j)_{\mathrm{Ar}}$. Combining this relation with the first equality we obtain

$$\sum_{\sigma}r_{1,\sigma}=-(M_j,M_j)_{\mathrm{Ar}}-(M_j,D_1)_{\mathrm{Ar}},$$

and summing up for all M_j's:

$$4\sum_{\sigma} r_{1,\sigma} = -\sum_{j=1}^{4}(M_j, M_j)_{\mathrm{Ar}} - (D_1, D_1)_{\mathrm{Ar}} + (F_2, D_1)_{\mathrm{Ar}}.$$

The self-intersection of the M_j's can be eliminated using the equality $(D_1^0, D_1^0)_{\mathrm{Ar}} = \sum_{j=1}^4 (M_j, M_j)_{\mathrm{Ar}} + 2\sum_{j<k}(M_j, M_k)_{\mathrm{Ar}}$:

$$4\sum_{\sigma} r_{1,\sigma} = -2(D_1, D_1)_{\mathrm{Ar}} + 3(D_1, F_2)_{\mathrm{Ar}} - (F_2, F_2)_{\mathrm{Ar}} + 2\sum_{j<k}(M_j, M_k)_{\mathrm{Ar}}.$$

If we now take into account that $(D_1, D_1)_{\mathrm{Ar}} = (D_1, D_2)_{\mathrm{Ar}}$ since the divisors D_1, D_2 are Arakelov equivalent, we obtain that:

$$\begin{aligned}(\overline{\omega}_{\mathcal{V}'_n/T_n}, \overline{\omega}_{\mathcal{V}'_n/T_n})_{\mathrm{Ar}} = -3(D_1, D_2)_{\mathrm{Ar}} + 4\sum_{j<k}(M_j, M_k)_{\mathrm{Ar}} \\ + 6(D_1, F_2)_{\mathrm{Ar}} - 2(F_2, F_2)_{\mathrm{Ar}}.\end{aligned}$$

We will now give more concrete expressions for the terms in the right of the previous equality. We must introduce some notations. For every $\sigma : L_n \to \mathbb{C}$, we denote by P_j^σ (resp. Q_k^σ) the points given by the sections M_j (resp. N_k) on the fibre $C_{n,\sigma}$ of $\mathcal{V}_n$ at σ.

Lemma 10.1. *Let G be the Green function of the Riemann surface given by the curve C_n. We have:*

$$(D_1, D_2)_{\mathrm{Ar}} = \sum_{\sigma}\sum_{j,k=1}^{4} \log G(P_j^\sigma, Q_k^\sigma) + 2(D_1^0, F_2)_{\mathrm{Ar}} + (F_2, F_2)_{\mathrm{Ar}}.$$

Proof. We have:

$$(D_1, D_2)_{\mathrm{Ar}} = (D_1^0, D_2^0)_{\mathrm{Ar}} + (D_1^0, F_2)_{\mathrm{Ar}} + (D_2^0, F_2)_{\mathrm{Ar}} + (F_2, F_2)_{\mathrm{Ar}}.$$

By Proposition 9.5, the first term equals $\sum_\sigma \sum_{j,k=1}^4 \log G(P_j^\sigma, Q_k^\sigma)$. The two intermediate terms are equal since $D_2^0 = \varphi_{01}(D_1^0)$, and F_2 is invariant through φ_{01}. □

Lemma 10.2.

$$\begin{aligned}\sum_{j<k}(M_j, M_k)_{\mathrm{Ar}} = \sum_{\sigma}\sum_{j<k} \log G(P_j^\sigma, P_k^\sigma) \\ + 2\sum_{\mathfrak{p}|2}(M_1, M_2)_{\mathfrak{p}} + 2\sum_{\mathfrak{p}|3}(M_1, M_3 + M_4)_{\mathfrak{p}}.\end{aligned}$$

Proof. We calculate the finite part of this intersection. If $\mathfrak{p}|2$, then

$$(M_1, M_3)_{\mathfrak{p}} = (M_1, M_4)_{\mathfrak{p}} = (M_2, M_3)_{\mathfrak{p}} = (M_2, M_4)_{\mathfrak{p}} = 0,$$

because the images of these pairs of points through the map ψ_2 are different. On the other hand, the pair of points M_1, M_2 are conjugated to the pair M_3, M_4 over $\mathbb{Q}$ by Lemma 9.1, and thus $\sum_{\mathfrak{p}|2}(M_1, M_2)_{\mathfrak{p}} = \sum_{\mathfrak{p}|2}(M_3, M_4)_{\mathfrak{p}}$.

We now look at an odd prime $\mathfrak{p}$ dividing $n(n-1)$. We know from Lemma 9.3 that two of the sections M_1, M_2, M_3, M_4 intersect the component $X_{1,\mathfrak{p}}$ of the fibre $\mathcal{V}_{n,\mathfrak{p}}$, and the other two intersect the component $X_{2,\mathfrak{p}}$. Moreover, M_1 and M_2 must intersect different components, since $M_2 = \varphi_{12}(M_1)$, $X_{2,\mathfrak{p}} = \varphi_{12}(X_{1,\mathfrak{p}})$. The same assertion is true for M_3, M_4. The M_j are contained in the smooth part of $\mathcal{V}_n$, so that we must have:

$$(M_1, M_2)_{\mathfrak{p}} = (M_3, M_4)_{\mathfrak{p}} = 0.$$

Now, $(M_1, M_3 + M_4)_{\mathfrak{p}} = (M_2, M_3 + M_4)_{\mathfrak{p}}$, again because $M_2 = \varphi_{12}(M_1)$, $M_4 = \varphi_{12}(M_3)$. Note that $\psi_2(P_1) = R_1$ is a 3-torsion point distinct from $\psi_2(P_3) = \psi_2(P_4) = R_2 = 2R_1$. Hence, if these sections do intersect at $\mathfrak{p}$, the points $R_{1,\mathfrak{p}}, R_{2,\mathfrak{p}}$ must specialize both to 0. This can only happen at those primes with residue characteristic 3. □

Putting all the above equalities together, and taking into account Lemma 9.6 we obtain:

Theorem 10.3. *The self-intersection of the Arakelov dualizing sheaf of the arithmetic surface $\mathcal{V}'_n$ can be expressed as:*

$$\begin{aligned}(\overline{\omega}_{\mathcal{V}'_n/T_n}, \overline{\omega}_{\mathcal{V}'_n/T_n})_{\mathrm{Ar}} = 4\sum_{\sigma}\sum_{j<k} &\log G(P_j^\sigma, P_k^\sigma) \\ - 3\sum_{\sigma}\sum_{j,k=1}^{4} &\log G(P_j^\sigma, Q_k^\sigma) \\ &+ 8\sum_{\mathfrak{p}|2}(M_1, M_2)_{\mathfrak{p}} + 8\sum_{\mathfrak{p}|3}(M_1, M_3 + M_4)_{\mathfrak{p}} \\ &- \sum_{\mathfrak{p}|2}(10r_{\mathfrak{p}} + s_{\mathfrak{p}} + 11)\log \sharp\ell_{\mathfrak{p}}.\end{aligned}$$

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QUOTIENTS OF NILALGEBRAS AND THEIR ASSOCIATED GROUPS

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We show that every finitely generated nilalgebra having nilalgebras of matrices is a homomorphic image of nilalgebras constructed by the Golod method (Golod, 1965 and 1969). By applying some elements of module theory to these results, we construct over any field non-residually finite nilalgebras and Golod groups with non-residually finite quotients. This solves Šunkov's problem (Kourovka Notebook, 1995, Problem 12.102). Also, we reduce Kaplansky's problem on the existence of a f.g. infinite p-group G such that the augmentation ideal $\omega K[G]$ over a nondenumerable field K is a nilideal (Kaplansky, 1957, Problem 9) to the study of the just-infinite quotients of Golod groups.

1. Introduction.

This paper deals with finitely generated (f.g.) infinite dimensional nilalgebras and their associated groups. Using Golod's algebras Anan'in and Puczyłowski constructed over fields of characteristic zero f.g. non-nilpotent nilalgebras which are not residually finite [**2**, **15**]. On the other hand, Rowen has proved their existence over every field [**16**]. Here we shall construct such examples over every field. This will enable us to solve in the negative Šunkov's problem [**11**, Problem 12.102] by constructing Golod groups with non-residually finite quotients. To this end we shall first start constructing Golod algebras as extensions of some nilalgebras. This is a completely different view from the classical one where Golod algebras are seen as homomorphic images. On the other hand the proofs of Theorems 2 and 3 are careful analysis of the Golod method. However, a great deal of information is extracted. For example, we prove that every f.g. nilalgebra over a nondenumerable field is a homomorphic image of a Golod algebra. As a consequence, Kaplansky's problem on the existence of a f.g. infinite p-group G such that the augmentation ideal $\omega K[G]$ over a nondenumerable field K is a nilideal [**10**, Problem 9] is reduced to the study of the just-infinite quotients of Golod groups. In the denumerable case we obtain some results, although because of the Köthe conjecture [**12**] the situation is quite complicated and we are far from understanding it.

Let K be any field and let $F^{(1)}$ be the free associative algebra of polynomials without constant terms in the non-commuting indeterminates $X_1, \dots, X_d$ ($d \geq 2$) over K. In this work an algebra means an associative algebra unless otherwise stated.

Lemma 1 ([6, 7]). *Let I be an ideal of $F^{(1)}$ generated by a family of homogeneous polynomials $f_1, f_2, \dots$ of non-decreasing degrees greater than or equal to 2. Let r_i be the number of polynomials of each degree $i \geq 2$ in the sequence $f_1, f_2, \dots$. If the coefficients of the series $\left(1 - dt + \sum_{i=2}^{+\infty} r_i t^i\right)^{-1}$ are positive, then the algebra $F^{(1)}/I$ is of infinite dimension. In particular this is true if for a fixed real ϵ, $0 < \epsilon < 1/2$, $r_i \leq \epsilon^2 (d - 2\epsilon)^{i-2}$, for every $i \geq 2$.*

A Golod algebra is a f.g. non-nilpotent nilalgebra which satisfies Lemma 1 and which is constructed by the Golod method as in [6, 7].

An algebra A over a field k is absolutely nil if for every extension field $K \supset k$, $A \bigotimes K$ is a nilalgebra [1, 1_c, p. 51].

We shall use the following characterization of absolutely nilalgebras:

Lemma 2 ([1, 3_c, p. 52]). *The algebra A is absolutely nil if for every finite set $g_1, \dots, g_n$ of elements of A, there exists an integer m such that for every partition $m = \mu_1 + \cdots + \mu_n$, $\mu_i \geq 0$, $\phi_{\mu_1 \dots \mu_n}(g_1, \dots, g_n) = \sum g_{i_1} \cdots g_{i_m} = 0$, where $\sum$ ranges over all the products which contain g_j, μ_j times for every j.*

The smallest such integer m is called the degree of absolute nillity of $g_1, \dots, g_n$. It is obvious that $\phi_{\mu_1 \dots \mu_n}(g_1, \dots, g_n)$ is a homogeneous polynomial of degree m in the subalgebra generated by $g_1, \dots, g_n$. $\phi_{\mu_1 \dots \mu_n}(g_1, \dots, g_n)$ is called a $\phi_{\mu_1, \mu_n}(g_1, \dots, g_n)$ homogeneous polynomial. When there is no ambiguity, we speak about the ϕ_{μ_1, μ_n} homogeneous polynomials (parts, components) where $\mu_1, \dots, \mu_n$ range over all the partitions of m.

It is well-known that every f.g. nilalgebra over a nondenumerable field and every locally nilpotent algebra are absolutely nil [1]. It is observed [1, p. 56] and is proved below (see Remark 2) that Golod algebras are examples of non-locally nilpotent absolutely nilalgebras.

2. Residually finite case.

Theorem 1. *Let $A = F^{(1)}/I$ be a nilalgebra with an absolutely nil ideal J/I such that J is a homogeneous ideal of $F^{(1)}$. Then A is a homomorphic image of a residually finite nilalgebra $B = F^{(1)}/T$ such that T is a homogeneous ideal.*

Proof. Let $g \in F^{(1)}$ and n be an integer such that $g^n \in J$. Then $g^n = \sum_i \mathcal{M}_i$, where $\mathcal{M}_j$ are homogeneous polynomials of J. Since J/I is an absolutely nilalgebra, there exists an integer $m = m(\mathcal{M}_{i_1}, \dots, \mathcal{M}_{i_k})$ such that all the homogeneous polynomials in the $\mathcal{M}_j$, $\phi_{\mu_1,\mu_k} = \sum \mathcal{M}_{j_1} \cdots \mathcal{M}_{j_m} \in I$. But every element $\mathcal{M}_j$ is homogeneous in $F^{(1)}$, so all the polynomials ϕ_{μ_1,μ_k} are homogeneous in $F^{(1)}$. From the fact that $(g^n)^m = \sum_{\mu_1+\cdots+\mu_k=m} \phi_{\mu_1,\mu_k}$ we see that $(g^n)^m$ is a sum of homogeneous elements of I. Let T be the ideal of $F^{(1)}$ generated by all the homogeneous polynomials ϕ_{μ_1,μ_k}, so constructed. It is obvious that $T \subset I$ is a homogeneous ideal and that $F^{(1)}/T$ is a residually finite nilalgebra.

In view of this theorem we ask the following natural question:

Question 1. Let A be an algebra as in the previous theorem. Is A absolutely nil?

Although this question seems to be difficult, one can observe that if J/I is an ideal of A of finite codimension then A is absolutely nil. This gives the following characterization of f.g. non-absolutely nilalgebras. Examples of this sort are the nilalgebras generated by 3 elements constructed recently by Smoktunowicz [**18**].

Corollary 1. *Let A be a f.g. non-absolutely nilalgebra. Then for every $n \geq 1$, A^n is a f.g. nilalgebra which is not absolutely nil.*

Theorem 2. *Let $A = F^{(1)}/I$ be a nilalgebra over a denumerable field such that I is a homogeneous ideal. Then A is a homomorphic image of a residually finite nilalgebra $B = F^{(1)}/J$ which satisfies Lemma 1.*

Proof. We will construct by induction a family of homogeneous polynomials $f_1, f_2, \dots$ which generate the ideal J.

We suppose that the base field K is denumerable. In this case $F^{(1)}$ is denumerable. Let us enumerate its elements as $\{y_1, y_2, \dots\}$. Choose an integer n greater than or equal to the index of nilpotency of $(y_1 + I)$. Then y_1^n is in I and since I is homogeneous, each of its homogeneous components $f_1, \dots, f_t$ (with $\deg f_j < \deg f_{j+1}$) is in I. Given any number k, there is no more than one f_i with degree k. So we have the set $\{f_1, \dots, f_t\}$ satisfying Lemma 1. In particular , there exit homogeneous polynomials $f_1, \dots, f_t$ with increasing degrees in I satisfying Lemma 1 such that y_1^n is in the ideal generated by $\{f_1, \dots, f_t\} \subseteq I$.

Suppose by induction, we have a Golod set $\{f_1, \dots, f_s\} \subseteq I$ such that $\deg f_i < \deg f_{i+1}$ and for each $i = 1, \dots, k$ there is an integer n_i with $y_i^{n_i}$ in the ideal generated by $\{f_1, \dots, f_s\}$. For y_{k+1} choose an integer m greater than both the index of nilpotency of $(y_{k+1} + I)$ and $\deg f_s$. Since A is nil and since I is a homogeneous ideal, we can write $(y_{k+1})^m$ in terms of its homogeneous components all of which are in I, and all of which have degree larger than $\deg f_s$. Label these components $f_{s+1}, \dots, f_r$. Then the set $\{f_1, \dots, f_s, f_{s+1}, \dots, f_r\} \subseteq I$ of homogeneous polynomials satisfies Lemma 1 such that for each $i = 1, \dots, k+1$, there is an integer n_i with $y_i^{n_i}$ in the ideal generated by $\{f_1, \dots, f_r\}$. Now, by the induction we have an infinite set of homogeneous polynomials $f_1, f_2, \dots$ in I satisfying Lemma 1, and which generates the ideal J, such that $F^{(1)}/J$ is a nilalgebra.

Theorem 3. *Let $A = F^{(1)}/I$ be an absolutely nilalgebra. Then A is a homomorphic image of a Golod algebra $B = F^{(1)}/J$.*

Proof. The proof is by induction on the degrees of general polynomials. Let $g_1 = c_1X_1 + \cdots + c_dX_d$ be a general polynomial of degree 1 in $F^{(1)}$ and choose an integer l greater than or equal to the degree of absolute nillity of $X_1 + I, \dots, X_d + I$. Since A is absolutely nil, by Lemma 2, for every partition $l = \mu_1 + \cdots + \mu_d$, $\mu_i \geq 0$, the $\phi_{\mu_1,\mu_d}(X_1, \dots, X_d)$ polynomials are in I. These polynomials are just the coefficients (homogeneous polynomials in $X_1, \dots, X_d$) of g_1^l when seen as a polynomial in the commuting unknowns $c_1, \dots, c_d$. Let us denote these $\phi_{\mu_1,\mu_d}(X_1, \dots, X_d)$ polynomials as $f_1, \dots, f_{l_1}$. Now, since the number r_i of polynomials of each degree i (in this case $i = l$) in $\{f_1, \dots, f_{l_1}\}$ does not exceed $(l + d - 1)^{d-1}$, for l big enough, $r_i \leq (l + d - 1)^{d-1} \leq \epsilon^2(d - 2\epsilon)^{i-2}$. Thus, the set $\{f_1, \dots, f_{l_1}\}$ satisfies Lemma 1.

Suppose that we have constructed in I a system of homogeneous polynomials $f_1, \dots, f_{l_k}$ satisfying Lemma 1 and that for every polynomial $y \in F^{(1)}$ of a degree not exceeding k there exists an integer $l' = l'(y)$ such that the homogeneous parts of $y^{l'}$ are in the ideal generated by $f_1, \dots, f_{l_k}$. Let

$$g_{k+1} = c_1^{(1)}X_1 + \cdots + c_d^{(1)}X_d + c_1^{(2)}X_1^2 + c_2^{(2)}X_1X_2 + \cdots + c_{d^2}^{(2)}X_d^2 + \cdots + c_{d^{k+1}}^{(k+1)}X_d^{k+1}$$

be a general polynomial of $F^{(1)}$ of degree $k+1$. Let n be an integer greater than $\max(\deg f_1, \dots, \deg f_{l_k}, m(X_1, \dots, X_1X_d, \dots, X_d^{k+1}))$, where, $m(X_1, \dots, X_d^{k+1})$ is the degree of absolute nillity of $X_1, \dots, X_1X_d, \dots, X_d^{k+1}$. By Lemma 2, for every partition $n = \mu_1 + \cdots + \mu_q$, $\mu_i \geq 0$, $q = d + \cdots + d^{k+1}$ the $\phi_{\mu_1,\mu_q}(X_1, \dots, X_d^{k+1})$ polynomials are in I. As in the case of g_1, by the choice of the integer n, the coefficients of g_{k+1}^n, seen as a polynomial in the commuting unknowns $c_1^{(1)}, \dots, c_{d^{k+1}}^{(k+1)}$, are the $\phi_{\mu_1,\mu_q}(X_1, \dots, X_d^{k+1}) \in I$. Let us denote them by $f_{l_k+1}, \dots, f_{l_{k+1}}$ and construct a new family of homogeneous polynomials $f_1, \dots, f_{l_k}, f_{l_k+1}, \dots, f_{l_{k+1}}$ satisfying Lemma 1. Indeed, the number r_i of polynomials of degree $i > \max(\deg f_1, \dots, \deg f_{l_k})$ does not exceed $(n + q - 1)^{q-1}$. For n big enough, we have $r_i \leq (n + q - 1)^{q-1} \leq \epsilon^2(d - 2\epsilon)^{i-2}$. For $i \leq \max(\deg f_1, \dots, \deg f_{l_k})$ this property is satisfied in the system $f_1, \dots, f_{l_k}$. So we have constructed a family of polynomials $f_1, \dots, f_{l_{k+1}}$ satisfying Lemma 1 and for every polynomial $z \in F^{(1)}$ of a degree not exceeding $k+1$ there exists an integer $n' = n'(z)$ such that the homogeneous parts of $z^{n'}$ are in the ideal generated by $f_1, \dots, f_{n_{k+1}}$. The union of all these families so constructed gives an infinite system of homogeneous polynomials $f_1, f_2, \dots$ which generate the ideal J. We have proved the theorem.

Remarks.

1. If A is such that specific elements generate a nilpotent (soluble, finite dimensional,...) subalgebra, then one can construct B with the same properties as A.
2. From the proof of Theorem 3, we see that the Golod algebras are absolutely nil. Therefore, Golod algebras have nilalgebras of matrices. This solves P.M. Cohn's question [**4**, p. 387 and Exercise 6^o, p. 395].

Having in mind that a f.g. nilalgebra over a nondenumerable field is absolutely nil [**1**], we obtain:

Corollary 2. *Every f.g. nilalgebra over a nondenumerable field is a homomorphic image of a Golod algebra.*

Let A be a Golod algebra generated by $\overline{X}_1, \dots, \overline{X}_d$ $(d \geq 2)$. The group generated by $1+\overline{X}_1, \dots, 1+\overline{X}_d$ is called the Golod group of A and the Lie algebra generated by $\overline{X}_1, \dots, \overline{X}_d$ is the Golod-Lie algebra.

Corollary 3. *For any integer $d \geq 2$, every d-generator group arising from an absolutely nilalgebra is a homomorphic image of a d-generator Golod group. In particular, so is every finite p-group, for every prime integer p.*

In [**10**, Problem 9], Kaplansky asked whether the augmentation ideal $\omega K[G]$ of a f.g. infinite p-group G could be a nilideal. A particular case is Passman's question on the use of Golod groups to solve this problem [**13**, p. 121 and Problem 18, p. 133], [**14**, p. 415]. The following result confirms Passman's observation and reduces Kaplansky's problem to the study of the quotients of Golod groups:

Corollary 4. *Let K be a nondenumerable field of characteristic $p > 0$. Then, there exists a f.g. infinite p-group $\overline{G}$ such that the augmentation ideal $\omega K[\overline{G}]$ is nil if and only if there exists a just-infinite homomorphic image G of a Golod p-group such that $\omega K[G]$ is nil.*

Proof. Let $\overline{G}$ be as in the corollary. Since it is f.g and infinite., it has a just-infinite homomorphic image G. Hence, the augmentation ideal $\omega K[G]$ is a quotient of $\omega\ K[\overline{G}]$ and so it is a nilalgebra over a nondenumerable field K. By Corollary 2, G and $\overline{G}$ are quotients of a Golod group. The converse is obvious.

On the other hand we point out that since non-absolutely nilalgebras cannot be quotients of Golod algebras, their associated groups have non-nil augmentation ideals. The only examples of this type are the nilalgebras generated by 3 elements constructed by Smoktunowicz [**18**]. The following result is analogous to the results obtained in the case of the 2-generated Grigorchuk groups [**5**], the 3-generated Gupta-Sidki groups [**17**] and the free Burnside groups [**9**]:

Corollary 5. *Let K be a nondenumerable field of characteristic $p > 0$. Let G be a f.g. p-group associated to a non-absolutely nilalgebra. Then the augmentation ideal $\omega K[G]$ is not nil. Moreover $\omega K[G]$ has a just-infinite primitive homomorphic image.*

Question 2. Could the group algebra in the preceeding Corollary contain a free associative algebra with two non-commuting indeterminates?

Corollary 6. *For any integer $d \geq 2$, every d-generator Lie algebra arising from an absolutely nilalgebra is a homomorphic image of a d-generator Golod-Lie algebra.*

3. Non-residually finite case.

We turn now to non-residually finite quotients of nilalgebras and their associated groups. We point out that a f.g. just-infinite nilalgebra or a f.g. just-infinite Jacobson radical ring is residually finite [**9**] and that some infinite dimensional quotients of Golod algebras are also Golod algebras (the same result holds for Golod groups and Golod-Lie algebras) [**8**, **19**]. A subset E of a ring A is T-nilpotent if for every sequence $g_1, g_2, \dots$ of elements of E, there exists an integer k with $g_1 g_2 \cdots g_k = 0$. It is obvious that T-nilpotency implies local nilpotency. In our investigations, a key role is played by the following generalization of Nakayama's lemma:

Lemma 3 ([**20**, §43.5, p. 386])**.** *Let A be an algebra. Then, $AM \neq M$ for every left A-module M, if and only if A is T-nilpotent.*

The existence of f.g. non-residually finite, infinite dimensional nilalgebras over every field was first proved in [**16**]. A simple observation yields a stronger result. Indeed, let $d \geq 2$ be an integer and suppose that for any d-generator nilalgebra A, any left A-module M satisfies $\cap A^i M = \langle 0 \rangle$. So, $AM \neq M$ and by Lemma 3, A is T-nilpotent. Thus every d-generator nilalgebra is nilpotent. This contradicts the Golod construction [**6**, **7**] and proves:

Proposition. *For every integer $d \geq 2$ and over any field, there exists a non-residually finite, non-nilpotent d-generator nilalgebra.*

Theorem 4. *Over any field, any f.g. non-nilpotent nilalgebra with involution is a homomorphic image of a f.g. non-residually finite nilalgebra.*

Proof. Let A be a f.g. non-nilpotent nilalgebra with involution. Since A is not locally finite, by Lemma 3 there exists a nondegenerate left A-module M such that $AM = M$. It is well-known that every left A-module can be considered as a right module over the opposite algebra A^o of A. But the fact that A has an involution yields $A \cong A^o$ and turns M to a nondegenerate (A, A)-bimodule such that $AM = MA = M$. Let m be a nondegenerate element of M and consider the submodule $N = \langle m \rangle$. Since A has an involution and N is nondegenerate, we have $AN = NA = N$. Denote by $\overline{A}$ the trivial extension of A by N,

$$\overline{A} = \{(a, n), a \in A, n \in N\}.$$

With the usual addition and the following multiplication:

$$(a, n)(a', n') = (aa', an' + na'), \quad a, a' \in A, \quad n, n' \in N,$$

$\overline{A}$ is a non-nilpotent nilalgebra such that $\overline{A}/I = A$, where I is the ideal $\langle (0, n), n \in N \rangle$. From the fact that $AN = NA = N$, it follows that I is in $\overline{A}^k$ for every integer k; thus $\overline{A}$ is not residually finite. Since A is f.g. and $N = \langle m \rangle$, $\overline{A}$ is f.g. Therefore, we proved the theorem.

Corollary 7. *Over every field, there exists a Golod algebra with non-residually finite quotients.*

Proof. Apply Theorems 1 and 2 or 3 to the non-residually finite nilalgebras of Theorem 4.

The following corollary solves in the negative Šunkov's problem [**11**, Problem 12.102]:

Corollary 8. *For every prime p (respectively $p = 0$), there exists Golod p-groups (respectively torsion free groups) with non-residually finite quotients.*

Proof. Let $\overline{A}$ be a non-residually finite homomorphic image of a Golod algebra B and denote by $Y_1, \ldots, Y_d$ its generators which are images of fixed generators of B. Since $\overline{A}$ is f.g., and $N = \langle m \rangle$ is a nondegenerate module satisfying $AN = NA = N$ (see the proof of Theorem 4), $1 + (0, m) \in \overline{G}$ where, $\overline{G} = \langle 1 + Y_1, \ldots, 1 + Y_d \rangle$. Thus the Golod group of B has $\overline{G}$ as a non-residually finite quotient.

We conclude with the following question which is related to Bergman's [**3**, Question 63]:

Question 3. Anan'in and Puczyłowski constructed over fields of characteristic zero, f.g. non-residually finite, non-nilpotent nilalgebras with non-radical tensor square [**2**, **15**]. Could we construct such examples in characteristic $p > 0$?

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SYMMETRIC SPACE VALUED MOMENT MAPS

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For a compact Lie group G, three examples of G-spaces which can serve as the target of a moment map are discussed. Abstracting the work of Alekseev, Meinrenken, and Malkin, we cast these theories into a unified framework.

Let G be a compact, connected Lie group, and M a manifold on which G acts. There are several natural G-spaces which can be considered as the target of a moment map originating from G. The first is the dual $\mathfrak{g}^*$ to the Lie algebra of G; we say that M is a *Hamiltonian G-space* if M has a G-invariant symplectic form ω and there exists an equivariant map $\Phi\colon M \to \mathfrak{g}^*$ such that

$$\iota(\xi_M)\omega = d\,\langle \Phi, \xi\rangle \tag{1}$$

for all $\xi \in \mathfrak{g}$. Φ is called the moment map [**5**], [**4**].

In [**3**], Alekseev, Meinrenken, and Malkin define a Hamiltonian theory in which the moment map has the group itself as target. Given an invariant inner product B on $\mathfrak{g}$, M is called *q-Hamiltonian* if there is an invariant two-form ω and an equivariant map $\Phi\colon M \to G$ (again called a moment map) such that

$$d\omega = \tfrac{1}{12}\Phi^* B(\theta, [\theta, \theta]), \tag{2a}$$

and for all $\xi \in \mathfrak{g}$,

$$\iota(\xi_M)\omega = \tfrac{1}{2}\Phi^* B(\theta + \overline{\theta}, \xi); \tag{2b}$$

$$\ker \omega_x = \left\{\xi_M(x) \mid \xi \in \ker\left(\mathrm{Ad}_{\Phi(x)} + 1\right)\right\}. \tag{2c}$$

Here θ and $\overline{\theta}$ are the left- and right-invariant Maurer-Cartan forms on G. This theory is more complicated, especially when G is nonabelian. For (2a) requires that ω may not be closed and (2b) requires that ω may not be nondegenerate.

If G is given the structure of a Poisson-Lie group, one can also consider the class of Poisson-Lie G-spaces [**11**], [**10**]. These have a symplectic form ω and equivariant map $\Phi\colon M \to G^*$ such that

$$\iota(\xi_M)\omega = \Phi^* \left\langle \overline{\theta}_{G^*}, \xi\right\rangle \tag{3}$$

for all ξ. Here $\overline{\theta}_{G^*}$, the right-invariant Maurer-Cartan form on G^*, takes values in $\mathfrak{g}^*$, and hence pairs with $\mathfrak{g}$. The complications here are that the individual maps $g\colon M \to M$ do not preserve the full structure of M, i.e., the Poisson structure, as they do in the two cases above. Rather, the action map itself $G \times M \to M$ is a Poisson map. However, in [1], Alekseev introduces another target: The canonical noncompact symmetric space Y within $G_{\mathbb{C}}$ transverse to G. Let $i\colon Y \to G_{\mathbb{C}}$ be the inclusion. The differential equations a moment map $\Phi\colon M \to Y$ must satisfy are

$$d\omega = \tfrac{1}{2}\Phi^* i^* \operatorname{Im} B\left(\overline{\theta}_{G_{\mathbb{C}}}, [\overline{\theta}_{G_{\mathbb{C}}}, \overline{\theta}_{G_{\mathbb{C}}}]\right); \tag{4a}$$

and for all ξ,

$$\iota(\xi_M)\omega = \tfrac{1}{2\sqrt{-1}}\Phi^* i^* B(\theta_{G_{\mathbb{C}}} + \overline{\theta}_{G_{\mathbb{C}}}, \xi); \tag{4b}$$

$$\ker \omega_x = 0. \tag{4c}$$

They are similar to (2a)-(2c). The spaces G^* and Y are equivariantly diffeomorphic, and Alekseev uses Y to construct a correspondence between ordinary Hamiltonian G-spaces and Poisson-Lie G-spaces. Both G^* and Y are equivariantly diffeomorphic to a slightly more natural space, $G_{\mathbb{C}}/G$.

Thus we have three moment map theories, or at least three natural G-spaces which serve as targets for moment maps: $\mathfrak{g}^*$ for the classical Hamiltonian theory; G for the q-Hamiltonian theory, and $G_{\mathbb{C}}/G$ representing the Poisson or Y-valued theory. In this note we bring these theories into a unified framework. Abstracting from [3], we show that given a symmetric pair (H, G), with a special pairing on the Lie algebra of H, we may construct an equivariantly-closed three-form on $P = H/G$ and a moment map theory. The three most obvious P which arise this way are $G \ltimes \mathfrak{g}^*/G = \mathfrak{g}^*$, $(G \times G)/G = G$, and $G_{\mathbb{C}}/G \cong Y$, and the moment map theories we will construct coincide with those which have already arisen in the literature. Furthermore, if (H, G) is one of these special symmetric pairs with H connected and G simply connected, H/G must decompose into a product of smaller such symmetric spaces each one of which is isomorphic to $\mathfrak{k}^*$, K, or $K_{\mathbb{C}}/K$ for a subgroup K of G.

These results were announced in [8]. Shortly thereafter, similar results were related to the author by Yvette Kosmann-Schwarzbach [2]. The author's preprint eventually developed into [9]. He would like to thank his advisors, Victor Guillemin and Shlomo Sternberg, as well as Eckhard Meinrenken and Chris Woodward for many useful discussions. The reviewer also deserves thanks for thoughtful and detailed feedback.

1. Definitions.

1.1. Moment space and moment map. There is a basic notion of a differential equation a moment map should obey, regardless of the target, as well as certain conditions of minimal degeneracy. Here we generalize these requirements.

Definition 1. A (possibly degenerate) *moment space* for G is a pair $(P, \widetilde{\chi})$, where P is a G-manifold and $\widetilde{\chi}$ is an equivariantly closed three-form on P.

The form $\widetilde{\chi}$ may be called the moment form. Since

$$\Omega^3_G(P) = \Omega^3(P)^G \oplus (\Omega^1(P) \otimes \mathfrak{g}^*)^G,$$

we can write $\widetilde{\chi}$ as $\chi + \tau$, where $\chi \in \Omega^3(P)^G$ is the invariant piece and $\tau \colon \mathfrak{g} \to \Omega^1(P)$ is the equivariant piece. For any G-manifold Q, the vector field generated by $\xi \in \mathfrak{g}$ will be denoted ξ_Q. The condition that $d_G\widetilde{\chi} = 0$ can be written as three equations:

$$d\chi = 0, \tag{5a}$$

and for all $\xi \in \mathfrak{g}$,

$$\iota(\xi_P)\chi = d\tau(\xi); \tag{5b}$$

$$\iota(\xi_P)\tau(\xi) = 0. \tag{5c}$$

Example 1. Let $\phi \colon \mathfrak{g} \to \mathfrak{g}^{**} \subset C^\infty(\mathfrak{g}^*)$ be the map $\phi(\xi)(\ell) = \langle \xi, \ell \rangle$. Then an equivariantly closed three-form on $\mathfrak{g}^*$ is $\tau_{\mathfrak{g}^*} = d_G\phi$, which has no invariant part. Written as a map $\mathfrak{g} \to \Omega^1(\mathfrak{g}^*)$, it takes the form

$$\tau_{\mathfrak{g}^*}(\xi)_\ell(\lambda) = \langle \lambda, \xi \rangle\,, \tag{6}$$

for each $\xi \in \mathfrak{g}$, $\ell \in \mathfrak{g}^*$, and $\lambda \in T_\ell\mathfrak{g}^* = \mathfrak{g}^*$.

Example 2. As explained in the introduction, the second example of a moment space is G itself. Let $\mathfrak{g}$ have an invariant, positive-definite inner product B. Then the form

$$\widetilde{\chi}_G(\xi) = \chi_G + \tau_G(\xi) \stackrel{\text{def}}{=} \tfrac{1}{12}B(\theta_G, [\theta_G, \theta_G]) + \tfrac{1}{2}B(\xi, \theta_G + \overline{\theta}_G) \tag{7}$$

is equivariantly closed. This is a consequence of the Cartan structure equations

$$d\theta = -\tfrac{1}{2}[\theta, \theta] \tag{8a}$$

$$d\overline{\theta} = \tfrac{1}{2}[\overline{\theta}, \overline{\theta}]. \tag{8b}$$

Example 3. There are two perspectives on the last example of moment space. The first connects with the Poisson-Lie G-spaces of Lu and Weinstein. Let T be a maximal torus for G, $\mathfrak{t}$ its Lie algebra, and $\mathfrak{a} = \sqrt{-1}\mathfrak{t} \subset \mathfrak{g}_{\mathbb{C}}$. $\mathfrak{a}$ is the Lie algebra of a subgroup $A \subset G_{\mathbb{C}}$. Let $\mathbf{n}$ be the sum of a set of positive root spaces. Then there is the Iwasawa decomposition of $\mathfrak{g}_{\mathbb{C}}$:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus \mathfrak{a} \oplus \mathbf{n};$$
$$G_{\mathbb{C}} = GAN.$$

Then the imaginary part of $B_{\mathbb{C}}$ restricts to a nondegenerate pairing between $\mathfrak{g}$ and $\mathfrak{a} \oplus \mathbf{n}$, and thus $\mathfrak{a} \oplus \mathbf{n} \cong \mathfrak{g}^*$. Call the group AN by G^*; then $G_C = G^*G$.

The groups G and G^* are both *Poisson-Lie groups*, and G^* the *dual Poisson-Lie group* to G. Left multiplication of G on $G_{\mathbb{C}}$ descends to an action of G on G^*, called the left dressing action.

We do not seek an equivariantly closed three-form on G^*, the main difficulty arising from the fact that the dressing action does not preserve the Poisson structure of G^*. However, one may also consider the space $G_{\mathbb{C}}/G$ as a subspace of $G_{\mathbb{C}}$. Set

$$Y = \left\{ h \in G_{\mathbb{C}} \middle| \overline{h} = h^{-1} \right\}.$$

Y is invariant under the adjoint action of G and equivariantly diffeomorphic to G^*, and $T_eY = \mathfrak{p}$. Let θ_Y, $\overline{\theta}_Y \in \Omega^1(Y, \mathfrak{p})$ be the restrictions of the Maurer-Cartan forms from $G_{\mathbb{C}}$ to Y. Then the form

$$\widetilde{\chi}_Y(\xi) = \chi_Y + \tau_Y(\xi) \overset{\text{def}}{=} \tfrac{1}{12} \operatorname{Im} B_{\mathbb{C}}(\overline{\theta}_Y, [\overline{\theta}_Y, \overline{\theta}_Y]) + \tfrac{1}{2\sqrt{-1}} B_{\mathbb{C}}(\xi, \overline{\theta}_Y + \theta_Y) \tag{9}$$

is real and equivariantly closed.

The equivariantly closed three-form allows us to define a moment map.

Definition 2. Let M be a G-manifold and P a moment space for G. M is called a *P-Hamiltonian* G-space if there exists an invariant two-form $\omega \in \Omega^2(M)^G$ and an equivariant map $\Phi\colon M \to P$ such that

$$d_G\omega = -\Phi^*\widetilde{\chi}. \tag{10}$$

The P-Hamiltonian G-space M will further be called *nondegenerate* if in addition

$$\ker \omega_x = \left\{ \xi_M(x) \middle| \xi \in \ker \tau_{\Phi(x)} \colon \mathfrak{g} \to T^*_{\Phi(x)}P \right\} \tag{11}$$

for all $x \in M$.

We may write (10) in terms of its components

$$d\omega = -\Phi^*\chi; \tag{12a}$$

$$\iota(\xi_M)\omega = \Phi^*\tau(\xi), \tag{12b}$$

for all $\xi \in \mathfrak{g}$. For $p \in P$, τ_p is defined to be the linear map $\mathfrak{g} \to T^*_pP$ which takes $\xi \in \mathfrak{g}$ to the evaluation of the one-form $\tau(\xi)$ at the point p. In light of (12b), we have that for $p \in P$, the fundamental vector fields of all Lie algebra vectors in the kernel of τ_p must annihilate ω. Thus (11) is a condition of minimal degeneracy.

Example 4. To revisit Example 1, the condition (12b) applied to $\widetilde{\chi}_{\mathfrak{g}^*} = \tau_{\mathfrak{g}^*}$ is precisely (1). Equations (12a) and (11) state that ω must be closed and nondegenerate, respectively.

Example 5. The conditions on G-valued moment maps are also clearly generalized by Definition 2 (in fact, one could say this example motivates the abstract theory). Alekseev-Meinrenken-Malkin show that the moduli space of flat G-connections on a Riemann surface with $r > 0$ boundary components (divided by the action of the restricted gauge group) has the structure of a Hamiltonian G^{r+2}-space with G^{r+2}-valued moment map.

Example 6. A G-space M with Poisson action is called a *Poisson-Lie G-space* if there is a G-invariant symplectic form ω and an equivariant map Φ such that for all $\xi \in \mathfrak{g}$,

$$\iota(\xi_M)\omega = 2\Phi^* \operatorname{Im}(\xi, \overline{\theta}_{G^*}). \tag{13}$$

Choosing instead to work with Y-valued moment maps, we may apply Definition 2 to $\widetilde{\chi}_Y$ and we get (4a)-(4c). Alekseev exhibits equivariant diffeomorphisms $G^* \cong Y$, and shows that the corresponding moment map theories are isomorphic.

For any moment space P, the most immediate candidates for P-Hamiltonian G-spaces are the orbits $\mathcal{O}$ of G. These have a natural inclusion map $i\colon \mathcal{O} \to P$. Indeed, $\widetilde{\chi}$ induces a two-form on each orbit $\mathcal{O}$. If $p \in \mathcal{O}$, $T_p\mathcal{O}$ is spanned by $\{\xi_P(p) \,|\, \xi \in \mathfrak{g}\}$, and we define

$$\omega_{\mathcal{O}}\left(\xi_P(p), \eta_P(p)\right) = \tau(\xi)_p\left(\eta_P(p)\right). \tag{14}$$

By (5c), this form is well-defined and alternating, and we immediately see that it satisfies (12b). In fact, $\omega_{\mathcal{O}}$ is characterized by the property that $\iota(\xi_M)\omega_{\mathcal{O}} = i^*\tau(\xi)$ for all $\xi \in \mathfrak{g}$. We claim $\omega_{\mathcal{O}}$ is G-invariant, and this is a consequence of the equivariance of τ. For, given $\xi, \eta \in \mathfrak{g}$ and $\mathfrak{g} \in G$,

$$\begin{aligned}
g^*\omega_{\mathcal{O}}\big|_p\left(\xi_P(p), \eta_P(p)\right) &= \omega_{\mathcal{O}}\big|_g\, p\left(g_* \cdot \xi_P(p), g_* \cdot \eta_P(p)\right)\\
&= \omega_{\mathcal{O}}\big|_g\, p\left((\mathrm{Ad}_g\, \xi)_P(gp), g_* \cdot \eta_P(p)\right)\\
&= \tau(\mathrm{Ad}_g\, \xi)_{gp}\left(g_* \cdot \eta_P(p)\right)\\
&= \tau(\xi)_p\left((g^{-1})_* \cdot g_* \cdot \eta_P(p)\right)\\
&= \omega_{\mathcal{O}}\big|_p\left(\xi_P(p), \eta_P(p)\right).
\end{aligned}$$

Therefore, by the relation

$$0 = \mathcal{L}_{\xi_{\mathcal{O}}}\omega_{\mathcal{O}} = d\iota(\xi_{\mathcal{O}}) + \iota(\xi_{\mathcal{O}})d\omega_{\mathcal{O}},$$

we must have that

$$\begin{aligned}
\iota(\xi_{\mathcal{O}})d\omega_{\mathcal{O}} &= -d\iota(\xi_{\mathcal{O}})\omega_{\mathcal{O}}\\
&= -di^*\tau(\xi)\\
&= i^*d\tau(\xi)\\
&= -i^*\iota(\xi_{\mathcal{O}})\chi,
\end{aligned}$$

which verifies the moment condition (12a). We have proved the following:

Proposition 1. *Let P be a moment space for G. Consider a G-orbit $\mathcal{O} \subseteq P$ with two-form $\omega_{\mathcal{O}}$ given by (14) and moment map given by inclusion. Then $(\mathcal{O}, \omega_{\mathcal{O}}, i\colon \mathcal{O} \to P)$ is a P-Hamiltonian G-space.*

We incorporate the minimal degeneracy along orbits into our definition.

Definition 3. A moment space P is called *nondegenerate* if all orbits $\mathcal{O} \subseteq P$ are nondegenerate P-Hamiltonian G-spaces with two-form given by (14) and moment map inclusion. This means that for all $p \in P$,

$$\ker \tau_p \cap \mathfrak{g}_p = \{0\}\,, \tag{15a}$$

and for all $\xi \in \mathfrak{g}$,

$$\tau(\xi)_p\big(\eta_p(p)\big) = 0 \ \forall \eta \in \mathfrak{g} \implies \tau(\xi)_p \equiv 0. \tag{15b}$$

Lemma 1. *Let M be a nondegenerate P-Hamiltonian G-space with two-form ω and moment map $\Phi\colon M \to P$. Let $x \in M$. Then:*

(a) *The map $\xi \mapsto \xi_M(x)$, restricted to $\ker \tau_x \to \ker \omega_x$, is an isomorphism.*
(b) *We have $\ker d\Phi|_x \cap \ker \omega|_x = \{0\}$.*

Proof. The first claim is obvious given (15a). For the second, let $v \in \ker d\Phi|_x \cap \ker \omega|_x$. Then $v = \xi_M(x)$ for some $\xi \in \ker \tau_{\Phi(x)}$. However, since

$$0 = d\Phi(v) = d\Phi(\xi_M(x)) = \xi_P(\Phi(x)),$$

(the last equality is by the equivariance of Φ), we must have that $\xi \in \mathfrak{g}_{\Phi(p)}$. Hence again by (15b), we have $\xi = 0$. □

Proposition 2. *Let M_1 and M_2 be nondegenerate P-Hamiltonian G-spaces and $F\colon M_1 \to M_2$ an equivariant map such that $F^*\omega_2 = \omega_1$ and $F^*\Phi_2 = \Phi_1$. Then F is an immersion.*

Proof. Since $F^*\omega_2 = \omega_1$, we have that $\ker dF|_x = \ker \omega_1|_x$. Also, since $F^*\Phi_2 = \Phi_1$, we have

$$\ker \omega_1|_x \cong \ker \omega_2|_{F(x)} \cong \ker \tau_{\Phi_1(x)}.$$

Thus $\ker dF|_x \cap \ker \omega_1|_x = \{0\}$, and therefore $dF|_x$ is injective. □

From this we can prove the "P-Hamiltonian Kostant Theorem."

Theorem 1. *Let M a transitive nondegenerate P-Hamiltonian G-space. Then the moment map $\Phi\colon M \to P$ is a covering map onto an orbit.*

Proof. For $x_0 \in M$, let $\mathcal{O}$ be the orbit of $\Phi(x_0) \in P$. Then since M is transitive, the image of Φ consists of $\mathcal{O}$ alone. That Φ is a submersion onto its image is clear since $T_p\mathcal{O} = \mathfrak{g}_P(p)$ and $\xi_P(p) = d\Phi(\xi_M(x))$ if $p = \Phi(x)$. Finally, we have that $\Phi^*\omega_{\mathcal{O}} = \omega$ by (12b) and (14), and applying the previous proposition, Φ is an immersion as well. □

Definition 4. Let M be a P-Hamiltonian G-space. Suppose that $o \in M$ is a G-fixed point. We define the *reduced space* of M at o to be $M_o = \Phi^{-1}(o)/G$.

M_o has a special two-form on it arising from that on M. To see this, put $Z = \Phi^{-1}(o)$, and let $i\colon Z \to M$, $\pi\colon Z \to M_o$ be the inclusion and projection. Denote the P-Hamiltonian two-form on M is by ω. It is G-invariant, and therefore so it is restriction to Z. Furthermore, for $\xi \in \mathfrak{g}$,

$$\begin{aligned}\iota(\xi_Z)i^*\omega &= i^*\iota(\xi_M)\omega \\ &= i^*\Phi^*\tau(\xi) \\ &= 0,\end{aligned}$$

since $\Phi \circ i$ is the constant map o. Thus $i^*\omega$ is G-basic; there exists $\omega_o \in \Omega^2(M_o)$ such that $\pi^*\omega_o = i^*\omega$. Notice also that since

$$\pi^* d\omega_o = d\pi^*\omega_o = di^*\omega = i^*d\omega = i^*\Phi^*\chi = 0,$$

we must have that $d\omega_o = 0$.

Theorem 2. *Let M be a nondegenerate P-Hamiltonian G-space. Suppose that* $\dim P = \dim G$. *Then ω_o is a symplectic form if and only o is a regular value of the moment map Φ.*

Proof. Let $z \in Z$. The map τ_o is injective by (15a) and therefore an isomorphism since $\dim P = \dim G$. Thus $\omega|_z$ is nondegenerate. Therefore, we have a commutative diagram

$$\begin{array}{ccc}\mathfrak{g} & \xrightarrow{\tau_o} & T_o^*P \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \Phi^*} \\ T_zM & \xrightarrow{\omega} & T_z^*M\end{array}$$

where the horizontal maps are both isomorphisms. It follows from the basic $(\ker T^*)^0 = \operatorname{im} T$ theorem of linear algebra that

$$(\ker d\Phi|_z)^\omega = \mathfrak{g}_M(z).$$

Now $T_zZ \subseteq \ker d\Phi|_z$, by the definition of Z as the inverse image of o. Thus we have

$$T_z(G\cdot z) = \mathfrak{g}_M(z) = (\ker d\Phi|_z)^\omega \subseteq (T_zZ)^{i^*\omega}.$$

The left-hand side of the above is the kernel of $\pi_*\colon T_zZ \to T_{\pi(z)}M_o$, and the right-hand side is the kernel of $i^*\omega_o$ at z. The two are equal (and ω_o is therefore nondegenerate) if and only equality holds in the last step. This is true if and only if $T_zZ = \ker d\Phi|_z$, i.e., if z is regular. □

2. Manin structure.

A symmetric pair over G consists of a Lie group $H \supset G$ and an involution σ of H such that $H^\sigma = G$. Let $\mathfrak{h}$ be the Lie algebra of H and s the derivative of σ at the identity. Then $(\mathfrak{h}, s)$ is a symmetric Lie algebra, and $\mathfrak{h}^\sigma = \mathfrak{g}$. $\mathfrak{h}$

has a canonical decomposition $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the -1 eigenspace of s. We have the commutation relations

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}; \quad [\mathfrak{g}, \mathfrak{p}] \subset \mathfrak{p}; \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{g}. \tag{16}$$

Let $P = H/G$ where, and $o = eG$. Then P is a symmetric space and $\mathfrak{p}$ is canonically identified with T_oP. See [6].

We will often take the involution to be understood and refer to the symmetric pair as (H, G). There are three important example of symmetric pairs over G.

Example 7. On $H_0 = G \ltimes \mathfrak{g}^*$, the involution σ_0 is the map $(g, \ell) \mapsto (g, -\ell)$. The corresponding symmetric Lie algebra is $\mathfrak{h}_0 = \mathfrak{g} \ltimes \mathfrak{g}^*$ with Lie bracket and involution

$$\begin{aligned} [(\xi, \lambda), (\eta, \mu)] &= \left([\xi, \eta], \operatorname{ad}^*_\xi \mu - \operatorname{ad}^*_\eta \lambda\right); \\ s_0(\xi, \lambda) &= (\xi, -\lambda). \end{aligned}$$

Example 8. On $H_+ = G \times G$, G is embedded as the diagonal. This subgroup is fixed by the involution $\sigma_+(g_1, g_2) = (g_2, g_1)$. The corresponding symmetric Lie algebra is $\mathfrak{h}_+ = \mathfrak{g} \times \mathfrak{g}$ with involution

$$\sigma_+(\xi_1, \xi_2) = (\xi_2, \xi_1)$$

fixing the diagonal subalgebra.

Example 9. Let G be simply connected as well, so that there is a complex, simply connected group $G_{\mathbb{C}}$ with G as its real form. There are the conjugation automorphisms of $G_{\mathbb{C}}$ and $\mathfrak{h}_- = \mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathfrak{C}$ singling out the real forms as their fixed point sets.

$$s_-(\xi + \sqrt{-1}\eta) = \xi - \sqrt{-1}\eta.$$

In order to consider the quotient spaces H/G as moment spaces for G, we need to pair elements of $\mathfrak{g}$ with elements of $\mathfrak{p}$. The following structure makes this possible:

Definition 5. Let $(\mathfrak{h}, s)$ be a symmetric Lie algebra. $\mathfrak{h}$ will be called a *Manin symmetric Lie algebra* if it admits a nondegenerate symmetric bilinear form q with respect to which s is skew-symmetric: I.e., for all $\zeta_1, \zeta_2 \in \mathfrak{h}$:

$$q(s\zeta_1, \zeta_2) = -q(\zeta_1, s\zeta_2). \tag{17}$$

The pairing q will be called a *Manin form* or *Manin pairing.* It is also assumed to be invariant with respect to the adjoint action of $\mathfrak{h}$ on itself: For all $\zeta_1, \zeta_2, \zeta_3 \in \mathfrak{h}$,

$$\begin{aligned} q(\operatorname{ad}_{\zeta_1} \zeta_2, \zeta_3) &= q(\zeta_2, \operatorname{ad}_{\zeta_1} \zeta_3), \text{ or} \\ q([\zeta_1, \zeta_2], \zeta_3) &= q(\zeta_1, [\zeta_2, \zeta_3]). \end{aligned} \tag{18}$$

Let (H, G) be a symmetric pair. H will be called a *Manin symmetric pair* if the associated symmetric Lie algebra $\mathfrak{h}$ admits a Manin form which is invariant with respect to the adjoint action of H on $\mathfrak{h}$. That is, in addition to (17) and (18), we must have, for all $\zeta_1, \zeta_2 \in \mathfrak{h}$ and $h \in H$,

$$q(\mathrm{Ad}_h \zeta_1, \zeta_2) = q(\zeta_1, \mathrm{Ad}_{h^{-1}} \zeta_2). \tag{19}$$

Proposition 3. *Let G be a compact, connected Lie group and $\mathfrak{g}$ its Lie algebra.*

(a) $\mathfrak{h}_0 = \mathfrak{g} \rtimes \mathfrak{g}^*$ *has a Manin pairing given by*

$$q_0\left((\xi_1, \lambda_1), (\xi_2, \lambda_2)\right) = \langle \xi_1, \lambda_2 \rangle + \langle \xi_2, \lambda_1 \rangle . \tag{20}$$

$(H_0 = G \rtimes \mathfrak{g}^*, G)$ is a Manin symmetric pair. The resulting symmetric space is isomorphic to $\mathfrak{g}^*$.

(b) *Let $\mathfrak{g}$ have an invariant inner product B. $\mathfrak{h}_+ = \mathfrak{g} \times \mathfrak{g}$ has a Manin pairing given by*

$$q_+\left((\xi_1, \eta_1), (\xi_2, \eta_2)\right) = \tfrac{1}{2}\left(B(\xi_1, \xi_2) - B(\eta_1, \eta_2)\right). \tag{21}$$

Since G is connected, $(H_+ = G \times G, \Delta(G))$ is a Manin symmetric pair. The resulting symmetric space is isomorphic to G.

(c) *Again assume $\mathfrak{g}$ has an inner product B. Then B extends to a $\mathbb{C}$-bilinear inner product on $\mathfrak{h}_- = \mathfrak{g} \otimes \mathbb{C}$. $\mathfrak{h}_-$ has a a Manin pairing given by*

$$q_-(\zeta_1, \zeta_2) = \mathrm{Im}\, B(\zeta_1, \zeta_2). \tag{22}$$

Proof. Clear. □

3. Construction of the moment form.

The purpose of this section is to show that given a Manin symmetric pair, we can construct a moment space. This space will in fact be the space of right cosets.

For this section (H, G) will be a Manin symmetric pair with involution σ and a Manin pairing q. The corresponding involution of $\mathfrak{h}$ will be denoted s.

3.1. The equivariant form. Let θ be the left-invariant Maurer-Cartan form on H taking values in $\mathfrak{h}$. Using s, we can decompose θ into its "$\mathfrak{g}$-part" and its "$\mathfrak{p}$-part," defining:

$$\gamma = \frac{1+s}{2}\theta; \qquad \pi = \frac{1-s}{2}\theta,$$

so $\gamma \in \Omega^1(H, \mathfrak{g})$ and $\pi \in \Omega^1(H, \mathfrak{p})$. Let $j \colon H \to P = H/G$ be the quotient map.

Proposition 4. *Define for $\xi \in \mathfrak{g}$ a one-form*

$$\beta(\xi)_h = q(\xi, \mathrm{Ad}_h \pi) \in \Omega^1(H). \tag{23}$$

Then:

(a) *$\beta(\xi)$ is basic with respect to the right action of G on H, so there is a unique one-form $\tau(\xi) \in \Omega^1(P)$ such that $j^*\tau(\xi) = \beta(\xi)$.*
(b) *The map $\xi \mapsto \beta(\xi)$ is equivariant with respect to the left action of G on H, so τ is an equivariant three-form on P.*
(c) *We have, for all $\xi \in \mathfrak{g}$,*

$$\iota(\xi_P)\tau(\xi) = 0,$$

where ξ_P is the vector field on P generated by the left action of G in the direction ξ.

Proof. For $h \in H$, let R_h and L_h denote left and right multiplication by h as diffeomorphisms of H. Since $R_g^*\theta = \operatorname{Ad}_{g^{-1}} \theta$ and $\sigma(g) = g$, it follows that $R_g^*\pi = \operatorname{Ad}_{g^{-1}} \pi$. Then

$$\begin{aligned}\left(R_g^*\beta(\xi)\right)_h &= q\left(\xi, R_g^* \operatorname{Ad}_h \pi\right) \\ &= q\left(\xi, \operatorname{Ad}_{hg} \operatorname{Ad}_{g^{-1}} \pi\right) \\ &= q\left(\xi, \operatorname{Ad}_h \pi\right) = \beta(\xi)_h,\end{aligned}$$

so $\beta(\xi)$ is right-invariant. Moreover, if $\eta_R(h) = (L_h)_*\eta$ is the fundamental vector field associated to the right action corresponding to η, then $\theta(\eta_R) = \eta$. Hence $\pi(\theta_R) = 0$ and

$$\beta(\xi)_h(\eta_R) = 0.$$

Thus $\beta(\xi)$ is also right-horizontal, hence right-basic. This proves the first claim of the proposition.

For the second, note that θ and hence π are left H-invariant, so

$$\begin{aligned}\left(L_{g^{-1}}^*\beta(\xi)\right)_h &= q(\xi, \operatorname{Ad}_{g^{-1}} \pi) \\ &= q(\operatorname{Ad}_g \xi, \operatorname{Ad}_h \pi) = \beta(\operatorname{Ad}_g \xi)_h.\end{aligned}$$

Finally, to prove the third claim, we will show that for $\xi \in \mathfrak{g}$,

$$\iota(\xi_L)\beta(\xi) = 0, \tag{24}$$

where ξ_L is the fundamental vector field on H associated to the left action. Indeed,

$$\begin{aligned}\beta(\xi_h)(\xi_L) &= q\left(\xi, \operatorname{Ad}_h \frac{\operatorname{Ad}_{h^{-1}} - \operatorname{Ad}_{\sigma(h^{-1})}}{2}\xi\right) \\ &= \tfrac{1}{2}q(\xi,\xi) - \tfrac{1}{2}\left(\operatorname{Ad}_{h-1}\xi, \operatorname{Ad}_{\sigma(h^{-1})}\xi\right) \\ &= 0.\end{aligned} \tag{25}$$

□

Remark. It is only in (25) that we used the full Ad_H-invariance of the pairing q. In fact, the first two claims of Proposition 4 can be proven with only a pairing between $\mathfrak{g}$ and $\mathfrak{p}$ which is Ad_G-invariant (note Ad_G preserves the decomposition $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{p}$). There is a unique extension of such a pairing to an s-skew pairing of the full Lie algebra, and to force the associated one-form to obey (24) is to require that

$$q(\mathrm{Ad}_h \xi, \eta) = q(\xi, \mathrm{Ad}_{h^{-1}} \eta),$$

for all $\xi, \eta \in \mathfrak{g}$, and for all h of the form $\sigma(k)k^{-1}$. Such h lie in a submanifold $V = \{h \in H \,|\, \sigma(h) = h^{-1}\}$, which is transverse to G at the identity of H. In fact $T_e V = \mathfrak{p}$.

3.2. The invariant form. Here we will extend $\tau \in \Omega^3_G(H/G)$ to an equivariantly closed three-form.

Proposition 5. *Define $\Xi \in \Omega^3(H)$ by*

$$\Xi = \tfrac{1}{3} q(\pi, [\pi, \pi]). \tag{26}$$

Then:

(a) *Ξ is right G-basic and left G-invariant. Hence there exists a unique $\chi \in \Omega^3(P)^G$ such that $\Xi = j^*\chi$.*

(b)
$$d\chi = 0. \tag{27}$$

(c) *For $\xi \in \mathfrak{g}$,*
$$\iota(\xi_P)\chi = d\tau(\xi). \tag{28}$$

Proof. Writing $\theta = \gamma + \pi$ as the decomposition of θ relative to that of $\mathfrak{h}$, we have

$$\begin{aligned} d\gamma &= -\tfrac{1}{2}\left([\gamma, \gamma] + [\pi, \pi]\right); \\ d\pi &= -[\gamma, \pi]. \end{aligned}$$

This is an immediate consequence of the bracket identities for a symmetric Lie algebra (16) and the Cartan structure equation (8a). The proposition reduces to a formal calculation.

(a) This is proved similarly to the analogous claim in Proposition 4.

(b) By the Jacobi identity

$$\big[\pi, [\pi, \pi]\big] = \big[\theta, [\theta, \theta]\big] = \big[\gamma, [\gamma, \gamma]\big] = 0.$$

Thus,

$$\begin{aligned} d\,\Xi &= \tfrac{1}{3} dq(\pi, [\pi, \pi]) \\ &= q(d\pi, [\pi, \pi]) \\ &= -q([\gamma, \pi], [\pi, \pi]) \\ &= -q\left(\gamma, \big[\pi, [\pi, \pi]\big]\right) = 0. \end{aligned}$$

So (27) is proved.

(c) Let $\xi \in \mathfrak{g}$. Then

$$\begin{aligned}\iota(\xi_L)\Xi &= \tfrac{1}{3}\iota(\xi_L)q(\pi,[\pi,\pi]) \\ &= q(\pi(\xi_L),[\pi,\pi]) \\ &= q(\operatorname{Ad}_{h^{-1}}\xi,[\pi,\pi]).\end{aligned}$$

On the other hand,

$$\begin{aligned}d\beta(\xi) &= dq(\xi,\operatorname{Ad}_h\pi) \\ &= q(\xi,\operatorname{Ad}_h\operatorname{ad}_\theta\pi) - q(\xi,\operatorname{Ad}_h[\pi,\gamma]) \\ &= q(\xi,\operatorname{Ad}_h[\gamma+\pi,\pi]) - q(\xi,\operatorname{Ad}_h[\pi,\gamma]) \\ &= q(\xi,\operatorname{Ad}_h[\pi,\pi]).\end{aligned}$$

Thus (28) is true as well.

□

As an immediate consequence, we have:

Theorem 3. *If (H,G,σ,q) is a Manin symmetric pair, the equivariant three-form $\widetilde{\chi} = \chi + \tau$ is equivariantly closed, thus giving H/G the structure of a moment space for G.* □

3.3. Nondegeneracy. Along with the equivariant condition (Definition 1), which we have just satisfied for an arbitrary Manin symmetric pair, there is the nondegeneracy (actually, minimal degeneracy) condition of Definition 3. Here we will use the nondegeneracy of the pairing to satisfy nondegeneracy of τ.

Proposition 6. *Let (H,G) be a Manin symmetric pair, and $\mathcal{O}$ an orbit of G in $P = H/G$. Then P with two-form given by (14) and moment map $i\colon \mathcal{O} \to P$ satisfies*

$$\ker\omega_p = \left\{\xi_P(p)\,\middle|\,\xi \in \ker\tau_p\colon \mathfrak{g} \to T_p^*P\right\}. \tag{29}$$

Hence $\mathcal{O}$ is a nondegenerate P-Hamiltonian G-space.

Thus:

Theorem 4. *Let (H,G,σ,q) be a Manin symmetric pair. Then $P = H/G$ is a nondegenerate moment space for G.*

Proof. What we are attempting to prove is

$$\iota(\xi_P)\omega_p = 0 \iff \begin{cases}\xi_P(p) = 0 & \text{or} \\ \xi \in \ker\tau_p.\end{cases} \tag{30}$$

Suppose that $\xi \in \ker\tau_p$, where $p = hG$. This means that

$$0 = q\left(\operatorname{Ad}_{h^{-1}}\xi, \frac{1-s}{2}\operatorname{Ad}_{h^{-1}}\eta\right) = -\tfrac{1}{2}q\left(\operatorname{Ad}_{h^{-1}}\xi, \operatorname{Ad}_{\sigma(h^{-1})}\eta\right) \tag{31}$$

for all $\eta \in \mathfrak{g}$. Since the q-orthogonal space to $\mathfrak{g}$ within $\mathfrak{h}$ is $\mathfrak{g}$ itself, we have that

$$\operatorname{Ad}_{\sigma(h)h^{-1}\xi} \in \mathfrak{g}. \tag{32}$$

Write $k = \sigma(h)h^{-1}$ and note that $\sigma(k) = k^{-1}$. Then by (32) we must have that

$$\operatorname{Ad}_{k^{-1}} \xi = \sigma(\operatorname{Ad}_k \xi) = \operatorname{Ad}_k \xi$$

and therefore $\xi = \operatorname{Ad}_k^2 \xi$ or

$$\xi \in \ker(1 - \operatorname{Ad}_k^2).$$

Now we have a direct sum decomposition

$$\ker(1 - \operatorname{Ad}_k^2) = \ker(1 - \operatorname{Ad}_k) \oplus \ker(1 + \operatorname{Ad}_k). \tag{33}$$

If ξ is in the first summand, we have $\operatorname{Ad}_{h^{-1}} \xi = \operatorname{Ad}_{\sigma(h^{-1})} \xi \in \mathfrak{g}$ and therefore $\xi_P(p) = 0$. On the other hand, if ξ is in the second summand we have

$$\operatorname{Ad}_{h^{-1}} \xi = -\operatorname{Ad}_{\sigma(h^{-1})} \xi \in \mathfrak{p}$$

and thus $\beta(\xi)_h = 0$. Therefore (30) is true. □

Proof of Theorem 4. The first summand in the right-hand side of (33) intersected with $\mathfrak{g}$ is $\ker \tau_p$, and the second summand intersected with $\mathfrak{g}$ is $\mathfrak{g}_p$. Hence $\ker \tau_p \cap \mathfrak{g}_p = \{0\}$. Since q is nondegenerate, $\mathfrak{g}$ and $\mathfrak{p}^*$ are isomorphic as vector spaces. Hence all nondegeneracy conditions are satisfied. □

4. Recovery of the original moment spaces.

Propositions 3, 4, and 5 give equivariantly closed three-forms on each of the symmetric spaces $\mathfrak{g}^*$, G, and $G_{\mathbb{C}}/G$. In this section we will how the forms we have constructed here coincide with those developed independently.

Consider first $H_0 = G \ltimes \mathfrak{g}^*$. The map $j_0 \colon G \ltimes \mathfrak{g}^* \to \mathfrak{g}^*$, given by projection onto the $\mathfrak{g}^*$ factor, is right G-invariant. Thus it gives a left G-equivariant diffeomorphism between H_0/G and $\mathfrak{g}^*$.

Proposition 7. *We have $j_0^* \tau_{\mathfrak{g}^*} = \beta_0$, where β_0 is the form given by applying Proposition 5 to the Manin form q_0 on $\mathfrak{h}_0 = \mathfrak{g} \ltimes \mathfrak{g}^*$.*

Proof. Let $\xi \in \mathfrak{g}$ and $h = (g, \ell) \in G \ltimes \mathfrak{g}^*$. Then a tangent vector to $h \in H_0$ can be written as $L_{(g,\ell)*}(\eta, \lambda)$ for some $(\eta, \lambda) \in \mathfrak{g} \ltimes \mathfrak{g}^*$. We have

$$\begin{aligned}(j_0^* \tau_{\mathfrak{g}^*}(\xi))_{(g,\ell)} \left(L_{(g,\ell)*}(\eta, \lambda)\right) &= \tau_{\mathfrak{g}^*}(\xi)_\ell \left(\left(j_0 \circ L_{(g,\ell)}\right)_* (\eta, \lambda)\right) \\ &= \tau_{\mathfrak{g}^*}(\xi)_\ell(\operatorname{Ad}_g^* \lambda) = \left\langle \xi, \operatorname{Ad}_g^* \lambda \right\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}\beta_0(\xi)_{(g,\ell)}\left(L_{(g,\ell)*}(\eta,\lambda)\right) &= q_0\left(\xi, \operatorname{Ad}_{(g,\ell)}\pi\left(L_{(g,\ell)*}(\eta,\lambda)\right)\right)\\ &= q_0\left(\xi, \operatorname{Ad}_{(g,\ell)}(0,\lambda)\right)\\ &= q_0\big((\xi,0),(0,\operatorname{Ad}_g^*\lambda)\big) = \left\langle \xi, \operatorname{Ad}_g^*\lambda\right\rangle.\end{aligned}$$

□

Let $H_+ = G\times G$ and define the map $j_+\colon H_+\to G$, $(g_1,g_2)\mapsto g_1g_2^{-1}$. Embed G into H_+ as the diagonal; it acts on H_+ on the left and the right. j_+ is then seen to be right G-invariant and thus a left G-equivariant diffeomorphism between H_+/G and G.

Proposition 8. *The map j_+ pulls back the Alekseev-Meinrenken-Malkin moment form (7) to $\Xi_+ + \beta_+$, the form constructed on H_+ from the Manin pairing q_+.*

Proof. We may write $\theta_{H_+} = \theta_G^1 + \theta_G^2$, etc. Then for each $(g_1,g_2)\in H_+$,

$$\begin{aligned} j_+^*\theta_G\big|_{(g_1,g_2)} &= \operatorname{Ad}_{g_2}(\theta_G^1-\theta_G^2);\\ j_+^*\overline{\theta}_G\big|_{(g_1,g_2)} &= \operatorname{Ad}_{g_1}(\theta_G^1-\theta_G^2).\end{aligned}$$

Therefore

$$j_+^*\chi_+ = j_+^*\tfrac{1}{12}B(\theta_G,[\theta_G,\theta_G]) = \tfrac{1}{12}B(\theta_G^1-\theta_G^2,[\theta_G^1-\theta_G^2,\theta_G^1-\theta_G^2]).$$

Now $\pi = \left(\frac{\theta_G^1-\theta_G^2}{2},\frac{\theta_G^1-\theta_G^2}{2}\right)$, so

$$\begin{aligned}\Xi_+ &= \tfrac{1}{3}q_+(\pi,[\pi,\pi])\\ &= \tfrac{1}{24}q_+\left((\theta_G^1-\theta_G^2,\theta_G^1-\theta_G^2),[(\theta_G^1-\theta_G^2,\theta_G^1-\theta_G^2),(\theta_G^1-\theta_G^2,\theta_G^1-\theta_G^2)]\right)\\ &= \tfrac{1}{24}q_+\left((\theta_G^1-\theta_G^2,\theta_G^1-\theta_G^2),\left([\theta_G^1-\theta_G^2,\theta_G^1-\theta_G^2],[\theta_G^1-\theta_G^2,\theta_G^1-\theta_G^2]\right)\right)\\ &= \tfrac{1}{12}B(\theta_G^1-\theta_G^2,[\theta_G^1-\theta_G^2,\theta_G^1-\theta_G^2]).\end{aligned}$$

Similarly,

$$\begin{aligned} J_+^*\tau(\xi) &= \tfrac{1}{2}B(\xi, j_+^*\theta_G+\overline{\theta}_G)\\ &= \tfrac{1}{2}B\left(\xi,(\operatorname{Ad}_{g_2}+\operatorname{Ad}_{g_1})(\theta_G^1-\theta_G^2)\right),\end{aligned}$$

while

$$\begin{aligned}\beta_+(\xi) &= q_+\left((\xi,\xi),\operatorname{Ad}_{(g_1,g_2)}\left(\frac{\theta_G^1-\theta_G^2}{2},\frac{\theta_G^1-\theta_G^2}{2}\right)\right)\\ &= q_+\left((\xi,\xi),\left(\operatorname{Ad}_{g_1}\frac{\theta_G^1-\theta_G^2}{2},\operatorname{Ad}_{g_2}\frac{\theta_G^1-\theta_G^2}{2}\right)\right)\\ &= \tfrac{1}{2}B(\xi,(\operatorname{Ad}_{g_1}+\operatorname{Ad}_{g_2})(\theta_G^1-\theta_G^2)).\end{aligned}$$

□

Let $j_-\colon G_{\mathbb{C}} \to G_{\mathbb{C}}$ be the map $h \mapsto h\overline{h}^{-}$. Then j_- takes values in Y, is right G-invariant, and descends to a left G-equivariant diffeomorphism of $G_{\mathbb{C}}/G$ with Y.

The factor of 2 appearing in (13) is not in its original definition; it is introduced in [**3**] to make the theorem connecting Y to G^* more clear. Up to that same factor of 2, we can connect our moment form on G_C/G to that on Y.

Proposition 9. *The map j_- pulls back the moment form $\widetilde{\chi}_Y$ to $\Xi_- + \beta_-$, the equivariantly closed three-form on $G_{\mathbb{C}}$ arising from applying Propositions 5 and 6 to $2q_-$.*

Proof. Let $\theta_{G_{\mathbb{C}}}$ be the left Maurer-Cartan form, and $\widetilde{\theta}_{G_{\mathbb{C}}}$ its complex conjugate. Then for all $h \in G_{\mathbb{C}}$,

$$\begin{aligned} j_-^*\theta_Y\big|_h &= \mathrm{Ad}_{\overline{h}}(\theta_{G_{\mathbb{C}}} - \widetilde{\theta}_{G_{\mathbb{C}}}) = 2\,\mathrm{Ad}_{\overline{h}}\,\pi; \\ j_-^*\overline{\theta}_Y\big|_h &= \mathrm{Ad}_{\overline{h}}(\theta_{G_{\mathbb{C}}} - \widetilde{\theta}_{G_{\mathbb{C}}}) = 2\,\mathrm{Ad}_h\,\pi. \end{aligned}$$

So

$$\begin{aligned} j_-^*\chi_Y &= \tfrac{1}{2}\,\mathrm{Im}\,B_{\mathbb{C}}\left(j_-^*\theta_Y, [j_-^*\theta_Y, j_-^*\theta_Y]\right) \\ &= \tfrac{2}{3}\,\mathrm{Im}\,B_{\mathbb{C}}(\pi, [\pi,\pi]) = \Xi_-. \end{aligned}$$

Likewise, we compute

$$\begin{aligned} j_-^*\tau_Y(\xi) &= \tfrac{1}{2\sqrt{-1}}B_{\mathbb{C}}(\xi, j_-^*\theta_Y + j_-^*\overline{\theta}_Y) \\ &= \tfrac{1}{2\sqrt{-1}}B_{\mathbb{C}}\left(\xi, (\mathrm{Ad}_h + \mathrm{Ad}_{\overline{h}})(\theta_{G_{\mathbb{C}}} - \widetilde{\theta}_{G_{\mathbb{C}}})\right). \end{aligned}$$

Note that $\mathrm{Ad}_h + \mathrm{Ad}_{\overline{h}}$ is real and $\theta_{G_{\mathbb{C}}} - \widetilde{\theta}_{G_{\mathbb{C}}}$ is imaginary, so the above is in fact real. One the other hand

$$\begin{aligned} \beta_-(\xi)_h &= q(\xi, \mathrm{Ad}_h\,\pi) = 2\,\mathrm{Im}\,B_{\mathbb{C}}(\xi, \mathrm{Ad}_h\,\pi) \\ &= \frac{1}{\sqrt{-1}}\left(B_C\left(\xi, \tfrac{1}{2}\,\mathrm{Ad}_h(\theta_{G_{\mathbb{C}}} - \widetilde{\theta}_{G_{\mathbb{C}}})\right) - \overline{B_C\left(\xi, \tfrac{1}{2}\,\mathrm{Ad}_h(\theta_{G_{\mathbb{C}}} - \widetilde{\theta}_{G_{\mathbb{C}}})\right)}\right) \\ &= \frac{1}{2\sqrt{-1}}B_{\mathbb{C}}\left(\xi, (\mathrm{Ad}_{\overline{h}} + \mathrm{Ad}_h)(\theta_{G_{\mathbb{C}}} - \widetilde{\theta}_{G_{\mathbb{C}}})\right). \end{aligned}$$

□

5. Decompositions.

We have shown how Manin symmetric pairs can give rise to moment spaces. We now show to extent to which the known examples of Manin symmetric pairs are the only ones.

If $G = G_1 \times G_2$ is a direct product of Lie groups, and P_1 and P_2 are moment spaces for G_1 and G_2, respectively, then $P_1 \times P_2$ with the equivariant form $\widetilde{\chi}_1 + \widetilde{\chi}_2$ is a moment space for G. Thus we have a way of "building up" moment spaces. It is natural to try to go the other way—i.e., to decompose. We start this process at the linear level, and integrate from there.

5.1. Structure of Manin symmetric pairs. We say that a symmetric pair (H, G) is *Riemannian* if G acts by isometries on H. We say that a symmetric Lie algebra $(\mathfrak{h}, s)$ is *orthogonal* if $\mathfrak{g} = \mathfrak{h}^s$ is compactly embedded in h and *effective* if $\mathfrak{g} \cap \mathfrak{h} = 0$. These last two conditions are satisfied whenever $(\mathfrak{h}, s)$ is the symmetric Lie algebra associated to a Riemannian symmetric pair (H, G).

Theorem 5. *let $(\mathfrak{h}, s)$ be an effective, orthogonal symmetric Lie algebra with Manin pairing q. Then there exists a unique canonical decomposition*

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_+ \oplus \mathfrak{h}_-; \qquad (\textit{direct sum of ideals}) \tag{34a}$$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-; \qquad (\textit{direct sum of ideals}) \tag{34b}$$

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-; \qquad (\textit{direct sum of subspaces}) \tag{34c}$$

such that, with the induced symmetric and Manin pairings given by restriction, we have

$$\begin{aligned} \mathfrak{h}_0 &= \mathfrak{g}_0 \oplus \mathfrak{p}_0 \cong \mathfrak{g}_0 \ltimes \mathfrak{g}_0^*; \\ \mathfrak{h}_+ &= \mathfrak{g}_+ \oplus \mathfrak{p}_+ \cong \mathfrak{g}_+ \times \mathfrak{g}_+; \\ \mathfrak{h}_- &= \mathfrak{g}_- \oplus \mathfrak{p}_- \cong \mathfrak{g}_- \otimes \mathbb{C}. \end{aligned}$$

These isomorphisms are in fact isometries with respect to q.

We will prove this in a series of lemmas. To begin, assume that $\mathfrak{g}$ is simple. Let κ be the negative of the Killing form on $\mathfrak{h}$. Then κ is positive-definite on $\mathfrak{g}$, $\mathrm{ad}_{\mathfrak{h}}$-invariant, and

$$\kappa(s\zeta_1, \zeta_2) = \kappa(\zeta_1, \zeta_2),$$

or,

$$\kappa(s\zeta_1, \zeta_2) = \kappa(\zeta_1, s\zeta_2),$$

for all $\zeta_1, \zeta_2 \in \mathfrak{h}$. Define $J \colon \mathfrak{h} \to \mathfrak{h}$ by

$$q(J\zeta_1, \zeta_2) = \kappa(\zeta_1, \zeta_2).$$

Lemma 2. (a) *The map J commutes with the adjoint action of $\mathfrak{h}$ on itself. That is, for all $\zeta \in \mathfrak{h}$,*

$$J \circ \mathrm{ad}_\zeta = \mathrm{ad}_\zeta \circ J;$$

or, for all ζ_1 and ζ_2,

$$J[\zeta_1, \zeta_2] = [J\zeta_1, \zeta_2].$$

(b) *The map J anticommutes with s: $J \circ s = -s \circ J$. So J takes $\mathfrak{g}$ into $\mathfrak{p}$ and vice versa.*

(c) *The map J is self-adjoint with respect to q.*

(d) *The restriction* $J|_{\mathfrak{g}}$ *is a vector space isomorphism* $\mathfrak{g} \cong \mathfrak{p}$.

Proof. The first two parts are straightforward. The third is a simple consequence of the symmetry of κ and of q. The last follows from the fact that κ is positive definite on $\mathfrak{g}$. □

It follows that J^2 is an endomorphism of $\mathfrak{g}$ as a $\mathfrak{g}$-module. By Lemma 2, Part (c), J^2 is self-adjoint. Therefore, $\mathfrak{g}$ has an orthonormal basis of eigenvectors with real eigenvalues. Since each eigenspace is an ideal of J^2, it follows by simplicity that $\mathfrak{g} = \mathfrak{g}_\lambda$ is a single eigenspace. Thus for all $\xi, \eta \in \mathfrak{g}$,

$$[J\xi, J\eta] = J^2[\xi, \eta] = \lambda[\xi, \eta]. \tag{35}$$

If $\lambda = 0$, then $\mathfrak{p}$ is an abelian ideal of $\mathfrak{h}$ dual by q to $\mathfrak{g}$, and hence $\mathfrak{h} \cong \mathfrak{g} \ltimes \mathfrak{g}^*$. Otherwise, the endomorphism $\frac{1}{\sqrt{|\lambda|}}J$ enjoys all the properties of Lemma 2, so we may assume that $|\lambda| = 1$. If $\lambda = 1$, the map

$$\begin{aligned} T_+ \colon \mathfrak{h} &\longrightarrow \mathfrak{g} \times \mathfrak{g}; \\ (\xi, J\eta) &\longmapsto \tfrac{1}{2}(\xi + \eta, \xi - \eta) \end{aligned}$$

is an isomorphism of $(\mathfrak{h}, s, q)$ onto $(\mathfrak{h}_+, s_+, q_+)$. On the other hand if $\lambda = -1$ the map

$$\begin{aligned} T_- \colon \mathfrak{h} &\longrightarrow \mathfrak{g} \otimes \mathbb{C}; \\ (\xi, J\eta) &\longmapsto \xi + \sqrt{-1}\eta \end{aligned}$$

is an isomorphism onto $(\mathfrak{h}_-, s_-, q_-)$. This concludes the proof of Theorem 5 in the case that $\mathfrak{g}$ is simple.

Now if $\mathfrak{h}$ is effective, then $\mathfrak{g}$ is at least semisimple. Therefore we have a decomposition

$$\mathfrak{g} = \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda$$

where Σ is the set of eigenvalues of J^2. The eigenspaces $\mathfrak{g}_\lambda$ are ideals of $\mathfrak{g}$. For each λ, let $\mathfrak{p}_\lambda = J\mathfrak{g}_\lambda$.

Lemma 3. *For each pair of eigenvalues (λ, μ), the following commutation relations hold:*

(a)

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \begin{cases} 0 & \textit{if } \lambda \neq \mu; \\ \mathfrak{g}_\lambda & \textit{if } \lambda = \mu. \end{cases}$$

(b)

$$[\mathfrak{g}_\lambda, \mathfrak{p}_\mu] \subseteq \begin{cases} 0 & \textit{if } \lambda \neq \mu; \\ \mathfrak{p}_\lambda & \textit{if } \lambda = \mu. \end{cases}$$

(c)

$$[\mathfrak{p}_\lambda, \mathfrak{p}_\mu] \subseteq \begin{cases} 0 & \textit{if } \lambda \neq \mu; \\ \mathfrak{g}_\lambda & \textit{if } \lambda = \mu. \end{cases}$$

Proof.

(a) Let $\xi \in \mathfrak{g}_\lambda$ and $\eta \in \mathfrak{g}_\mu$. Then since J^2 is a $\mathfrak{g}$-module homomorphism, we have that $[\xi, \eta] \in \mathfrak{g}_\lambda \cap \mathfrak{g}_\mu$.

(b) Given ξ and η as above, notice

$$[\xi, J\eta] = J[\xi, \eta] \in J[\mathfrak{g}_\lambda, \mathfrak{g}_\mu].$$

(c) Finally,

$$[J\xi, J\eta] = J^2[\xi, \eta] \in [\mathfrak{g}_\lambda, \mathfrak{g}_\mu].$$

□

This shows that each $\mathfrak{h}_\lambda = \mathfrak{g}_\lambda \oplus \mathfrak{p}_\lambda$ is an ideal of $\mathfrak{h}$. Each $\mathfrak{h}_\lambda$ is isomorphic to one of the three canonical types, and we can collect them by type. This proves Theorem 5.

Theorem 6. *Let (H, G) be a Manin symmetric pair, with G semisimple and H connected and simply connected. Then the moment space $P = H/G$ has a decomposition*

$$P = P_0 \times P_+ \times P_-$$

and G has a decomposition

$$G = G_0 \times G_+ \times G_-$$

such that P_0 is a moment space for G_0 isomorphic to $\mathfrak{g}_0^*$, P_+ is a moment space for G_+ isomorphic to G_+, and P_- is a moment space for G_- isomorphic to $(G_-)_{\mathbb{C}}/G_-$.

Proof. It follows from the homotopy exact sequence for the fibration $G \to H \to H/G$ that if H is simply connected and G is connected, then H/G is simply connected.

Since G is semisimple, $\mathfrak{h}$ is an effective, orthogonal, Manin symmetric Lie algebra. Therefore, we can decompose $\mathfrak{h}$ as in Theorem 5 into $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_+ \oplus \mathfrak{h}_-$. Let $H_0 \times H_+ \times H_-$ be the corresponding decomposition of H. Likewise $\mathfrak{g}$ decomposes and we can write $G = G_0 \times G_+ \times G_-$. Then

$$P = H/G = \frac{H_0 \times H_+ \times H_-}{G_0 \times G_+ \times G_-} = H_0/G_0 \times H_+/G_+ \times H_-/G_-.$$

□

5.2. Relaxing *H*-invariance. This shows that we have exhausted all possibilities of creating moment spaces from Manin symmetric pairs, once we allow suitable assumptions about semisimplicity and connectedness. In fact, we can relax one of the conditions of a Manin form, weakening a hypothesis in Theorem 5, and thus arriving at a stronger Theorem 6.

Let (H, G) be any symmetric pair such that H/G is a moment space. Let $\widetilde{\chi} = \chi + \tau$ be the equivariantly closed three-form. Then τ pulls back to a linear map $\beta\colon \mathfrak{g} \to \Omega^1(H)^{G^{\mathrm{op}}}$, where we use the op-superscript to denote the right action of G on H. Evaluating β at the identity of H gives a map $b\colon \mathfrak{g} \to \mathfrak{h}^*$. Notice that for $g \in G$, $\xi \in \mathfrak{g}$, and $\zeta \in \mathfrak{h}$,

$$b(\mathrm{Ad}_g, \xi, \zeta) = \beta(\mathrm{Ad}_g\, \xi)_e(\zeta).$$

By left-equivariance of β, we have

$$= \beta(\xi)_{g^{-1}}(L_{g^{-1}*}\zeta).$$

Because β is right-invariant, this is

$$\begin{aligned} &= \beta(\xi)_e(R_{g*}L_{g^{-1}*}\zeta) \\ &= b(\xi, \mathrm{Ad}_{g^{-1}}\, \zeta). \end{aligned}$$

Furthermore, again by right-invariance,

$$\begin{aligned} b(\xi, \eta) &= \beta(\xi)_e(\eta) \\ &= \langle \beta(\xi), \eta_R \rangle_e \\ &= 0. \end{aligned}$$

Since $\mathfrak{h}$ is symmetric and has a canonical decomposition, we can uniquely extend b to an inner product q on $\mathfrak{h}$ with respect to which $\mathfrak{g}$ and $\mathfrak{p}$ are dual isotropic subspaces and s is skew-symmetric. This form is not necessarily completely $\mathfrak{h}$-invariant, however, only $\mathfrak{g}$-invariant. Nevertheless, this suffices.

Theorem 7. *Let $(\mathfrak{h}, s, q)$ be an effective orthogonal symmetric Lie algebra, with $\mathfrak{g}$ semisimple and q a Manin pairing assumed to be only $\mathfrak{g}$-invariant. Then there is a canonical decomposition of $\mathfrak{h}$ as in Theorem 5.*

Then we can immediately, using techniques similar to Theorem 6, prove:

Theorem 8. *let (H, G) be a Riemannian symmetric pair, with G connected and H simply connected. Suppose that H/G is a moment space for G. Then there is a decomposition of H, H, and H/G as in Theorem 6.*

Proof of Theorem 7. Assume that $\mathfrak{g}$ is simple, and complexify $\mathfrak{h}$, $\mathfrak{g}$, s, and q. Then $J\colon \mathfrak{g} \to \mathfrak{p}$ can still by constructed by $q(J\xi, \eta) = \kappa(\xi, \eta)$. Define for $\xi, \eta \in \mathfrak{g}$,

$$\{\xi, \eta\} \stackrel{\text{def}}{=} [J\xi, J\eta].$$

Then clearly

$$[\xi_1, \{\xi_2, \xi_3\}] = \{[\xi_1, \xi_2], \xi_3\} + \{\xi_2, [\xi_1, \xi_3]\}.$$

Hence $\{\cdot, \cdot\}$ is a homomorphism of $\mathfrak{g}$-modules. For simple Lie algebras, however, all such homomorphisms are scalar multiples of the Lie bracket (see below). Thus there exists a complex number λ such that $\{\xi, \eta\} = \lambda[\xi, \eta]$ for all ξ and η. But since J is real, λ must be real, too, and we are in the same situation as in Theorem 5. $\square$

It remains to prove that

$$\mathrm{Hom}_{\mathfrak{g}} \left(\bigwedge^2 \mathfrak{g}, \mathfrak{g} \right) = \mathbb{C}[\cdot, \cdot].$$

Since $\mathfrak{g}$ is simple, it is enough to show that $\left(\bigwedge^2 \mathfrak{g} \right)_{\mathrm{ad}} = \mathfrak{g}$, where for any $\mathfrak{g}$ module M, M_{ad} denotes the ad-primary component of M. Though κ we may identify $\mathfrak{g}^* \cong \mathfrak{g}$; thus the algebra $\bigwedge \mathfrak{g}$ (on which $\mathfrak{g}$ acts preserving the grading) has a differential d which is also a $\mathfrak{g}$-module homomorphism. Then as shown by Kostant [**7**, Theorems D and E], $(\bigwedge \mathfrak{g})_{\mathrm{ad}} = A_{\mathrm{ad}} \otimes (\bigwedge \mathfrak{g})^{\mathfrak{g}}$, where A is the exterior subalgebra generated by the image of $d^1 \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$. By restricting to the degree two subspace, we see that $\left(\bigwedge^2 \mathfrak{g} \right)_{\mathrm{ad}}$ is the image of d^1. But since $H^1(\mathfrak{g}, \mathbb{C}) = 0$, d^1 is injective, and so the image of d^1 is isomorphic to $\mathfrak{g}$.

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LE SYSTÈME DIFFÉRENTIEL DE HÉNON–HEILES ET LES VARIÉTÉS DE PRYM

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On montre que la fibre F définie par l'intersection des invariants du système différentiel de Hénon–Heiles se complète en une surface abélienne $\widetilde{\mathbf{F}}$, par l'adjonction d'une surface de Riemann $\mathbf{\Gamma}$ lisse hyperelliptique de genre 3; laquelle est un revêtement double ramifié le long d'une courbe elliptique $\mathbf{\Gamma_0}$. Aussi $\widetilde{\mathbf{F}}$ peut être identifiée à la duale d'une variété de Prym $\widehat{\mathbf{Prym}(\mathbf{\Gamma}/\mathbf{\Gamma_0})}$ et le système se linéarise sur cette variété.

1. Position du problème.

Le système différentiel de Hénon-Heiles [7] s'écrit sous la forme

$$(1.1)\qquad \begin{cases} \dot{q}_1 = p_1, \\ \dot{q}_2 = p_2, \\ \dot{p}_1 = -Aq_1 - 2q_1q_2, \\ \dot{p}_2 = -Bq_2 - q_1^2 - \varepsilon q_2^2, \end{cases}$$

où A, B, ε sont des constantes et admet les invariants (intégrales premières) suivants:

(i) Pour $\varepsilon = 1$, on a

$$H_1 = \frac{1}{2}\left(p_1^2 + p_2^2\right) + q_1^2 q_2 + \frac{1}{3}q_2^3,$$
$$H_2 = p_1p_2 + \frac{1}{3}q_1^3 + q_1q_2^2.$$

(ii) Pour $\varepsilon = 6$, on a

(1.2)
$$H_1 = \frac{1}{2}\left(p_1^2 + p_2^2 + Aq_1^2 + Bq_2^2\right) + q_1^2q_2 + 6q_2^3,$$
$$H_2 = q_1^4 + 4q_1^2q_2^2 - 4p_1\left(p_1q_2 - p_2q_1\right) + 4Aq_1^2q_2 + (4A - B)\left(p_1^2 + Aq_1^2\right).$$

L'intégration des équations (1.1) dans le cas $\varepsilon = 1$, s'effectue au moyen d'intégrales elliptiques et ne pose pas de problèmes. Le cas $\varepsilon = 6$, est plus intéressant mais plus compliqué [4] et [5]. Lorsque $A = B = 0$, Adler et van Moerbeke [2] ont montré que ce cas est lié par une transformation birationnelle au problème de Kowalewski ainsi qu'au flot géodésique sur $SO(4)$ pour une métrique de Manakov. Le but de cette note est d'étudier géométriquement et d'une manière rigoureuse ce problème pour A et B quelconque. Dans tout ce qui va suivre, on pose $\varepsilon = 6$.

2. Complète intégrabilité algébrique.

Considérons un systéme hamiltonian complètement intégrable

$$X_H : \overset{\bullet}{x} = J\frac{\partial H}{\partial x}, \quad x \in \mathbb{R}^{2n},\ J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \tag{2.1}$$

où H est l'Hamiltonian et I est la matrice unité. Le système (2.1) possède n intégrales premières $H_1 = H, H_2, \ldots, H_n$ en involution et indépendantes. Pour presque tous les $c_i \in \mathbb{R}$, les variétés invariantes

$$\bigcap_{i=1}^{n} \left\{ x \in \mathbb{R}^{2n} : H_i\left(x\right) = c_i \right\},$$

sont compactes, connexes et par le théorème d'Arnold-Liouville [**3**] et [**18**], elles sont difféomorphes aux tores réels $\mathbb{R}^n/\textit{réseau}$ sur lesquels les flots $g_i^t\left(x\right)$ définies par les champs de vecteurs $X_{H_i}, 1 \leq i \leq n$, sont des mouvements rectilignes.

Soient $x \in \mathbb{C}^{2n}, t \in \mathbb{C}$ et $\Delta \subset \mathbb{C}^{2n}$ un ouvert de Zariski. Notons que l'application moment

$$\varphi : (H_1, ..., H_n) : \mathbb{C}^{2n} \rightarrow \mathbb{C}^n,$$

est submersive sur Δ. Soit

$$\begin{aligned} \Pi &= \varphi\left(\mathbb{C}^{2n} \setminus \Delta\right), \\ &= \left\{ c = (c_i) \in \mathbb{C}^n : \exists x \in \varphi^{-1}\left(c\right) \text{ avec } dH_1\left(x\right) \wedge \cdots \wedge dH_n\left(x\right) \ = 0 \right\}, \end{aligned}$$

le lieu critique de φ où $c = (c_i)$ est le point courant de $\mathbb{C}^{2n}$ et soit $\overline{\Pi}$ la fermeture de Zariski dans $\mathbb{C}^4$. Rappelons [**1**] et [**14**] que le système (2.1) est *algébriquement complètement intégrable* si pour $c \in \mathbb{C}^n \setminus \overline{\Pi}$, la fibre $\mathbf{F} = \varphi^{-1}\left(c\right)$ est la partie affine d'une variété abélienne (tore complexe algébrique $\widetilde{\mathbf{F}} \simeq \mathbb{C}^n/\textit{réseau}$), les flots $g_i^t\left(x\right), x \in \mathbf{F}$, $t \in \mathbb{C}$, définies par les champs de vecteurs $X_{H_i}, 1 \leq i \leq n$, sont des mouvements rectilignes sur $\widetilde{\mathbf{F}}$ et les coordonnées $x_i = x_i\left(t_1, \ldots, t_n\right)$ du problème sont des fonctions méromorphes de $\left(t_1, \ldots, t_n\right)$. En outre, si le flot hamiltonian (2.1) est algébriquement complètement intégrable, alors ce système admet des solutions sous la forme de séries de Laurent en t telles que chaque x_i explose pour au moins une valeur finie de t et les séries de Laurent de x_i admettent $n-1$ paramètres libres.

Le système (1.1) s'écrit sous la forme (2.1) avec $n = 2$. Plus précisément, on a

$$\overset{\bullet}{x} \equiv f(x) = J\frac{\partial H}{\partial x}, \tag{2.2}$$

avec $x = (q_1, q_2, p_1, p_2)$ et $H = H_1$ (1.2). Comme l'application polynomiale est continue pour la topologie de Zariski, l'ensemble

$$\left\{x \in \mathbb{C}^4 : \varphi(x) \in \mathbb{C}^4 \setminus \overline{\Pi}\right\},$$

est un ouvert de Zariski dans $\mathbb{C}^4$. On cherche à montrer que pour $c \in \mathbb{C}^4 \setminus \overline{\Pi}$, la fibre

$$\begin{aligned} \mathbf{F} &= \varphi^{-1}(c), \\ &= \bigcap_{i=1}^{2} \left\{x \in \mathbb{C}^4 : H_i(x) = c_i\right\}, \end{aligned} \tag{2.3}$$

forme la partie affine d'une surface abélienne et qu'en outre les flots définis par les champs de vecteurs hamiltoniens (engendrés par H_1 et H_2) sont des mouvements rectilignes sur cette surface abélienne. On procède comme suit: d'abord l'on montre l'existence de solutions $x = (q_1, q_2, p_1, p_2)$ du système (2.2) sous la forme de séries de Laurent

$$\begin{cases} q_1 = \frac{q_1^{(0)}}{t} + q_1^{(1)} + q_1^{(2)}t + q_1^{(3)}t^2 + \dots, & p_1 = \overset{\bullet}{q}_1, \\ q_2 = \frac{q_2^{(0)}}{t^2} + \frac{q_2^{(1)}}{t} + q_2^{(2)} + q_2^{(3)}t + q_2^{(4)}t^2 + \dots, & p_2 = \overset{\bullet}{q}_2, \end{cases} \tag{2.4}$$

dépendant de trois paramètres libres: α, β, γ. En substituant ces développements dans le système (2.2), on voit que les coefficients $x^{(0)}, x^{(1)}, \dots$, satisfont aux équations

$$x^{(0)} + f(x^{(0)}) = 0, \tag{2.5}$$

$$(L - kI)x^{(k)} = \text{polynôme en } x^{(0)}, x^{(1)}, \dots, x^{(k-1)},\ k \geq 1, \tag{2.6}$$

où L est la matrice jacobienne de (2.5). Les trois paramètres libres α, β et γ apparaissent respectivement dans l'équation (2.5), l'équation (2.6) pour $k = 1$ et l'équation (2.6) pour $k = 6$. L'étape suivante est fondamentale et consiste à considérer l'ensemble

$$\begin{aligned} \Gamma &= \text{fermeture des composantes continues de} \\ &\quad \left\{\text{séries de Laurent de } x(t) \text{ tels que: } H_1(x) = c_1 \text{ et } H_2(x) = c_2\right\}, \\ &= \bigcap_{i=1}^{2} \left\{\text{coefficient de } t^0 \text{ dans } H_i(x(t)) = c_i\right\}, \\ &= \text{deux relations polynomiales entre les variables } \alpha, \beta \text{ et } \gamma, \\ &= \text{une surface de Riemann hyperelliptique de genre 3 d'équation:} \end{aligned}$$

$$a_1\beta^2 + a_2\alpha^8 + a_3\alpha^6 + a_4\alpha^4 + a_5\alpha^2 + a_6 = 0, \tag{2.7}$$

où

$$a_1 = 36, \quad a_2 = \frac{7}{432}, \quad a_3 = \frac{5}{12}A - \frac{13}{216}B,$$
$$a_4 = \frac{671}{15120}B^2 + \frac{17}{7}A^2 - \frac{943}{1260}BA,$$
$$a_5 = \frac{2}{9}AB^2 - \frac{1}{2520}B^3 - \frac{10}{7}c_1 - \frac{13}{6}A^2B + 4A^3, \quad a_6 = -c_2.$$

Notons que l'application

$$\sigma : \Gamma \to \Gamma, (\alpha, \beta) \mapsto (-\alpha, \beta), \tag{2.8}$$

est une involution sur Γ et que cette dernière est un revêtement double

$$\Gamma \to \Gamma_0, (\alpha, \beta) \mapsto (\zeta, \beta), \tag{2.9}$$

ramifié en 4 points d'une courbe elliptique:

$$\Gamma_0 : a_1\beta^2 + a_2\zeta^4 + a_3\zeta^3 + a_4\zeta^2 + a_5\zeta + a_6 = 0. \tag{2.10}$$

Par conséquent, on a le:

Théorème 1. *Le système d'équations différentielles* (2.2) *admet une famille de solutions en séries de Laurent méromorphes* (2.4) *dépendant de trois paramètres libres. En outre, le diviseur* Γ (2.7) *des poles des fonctions* $x = (q_1, q_2, p_1, p_2)$ *est une surface de Riemann lisse hyperelliptique de genre* 3; *c'est un revêtement double ramifié en quatre points d'une courbe elliptique* Γ_0 (2.10).

On va procèder maintenant à la compactification de la fibre $\mathbf{F}$ (2.3) en une surface abélienne $\widetilde{\mathbf{F}}$. La méthode consiste à plonger $\mathbf{F}$ dans l'espace projectif complexe $\mathbb{P}^7(\mathbb{C})$ à l'aide des fonctions de $\mathbb{L}(2\Gamma)$. Ce sont des fonctions polynomiales $(1, f_1, \ldots, f_7)$ ayant au pire un pôle double de telle façon que:

$$\dim \mathbb{L}(2\Gamma) = \text{genre de } (2\Gamma) - 1 = 8.$$

Par ailleurs, on montre qu'il existe sur la surface $\widetilde{\mathbf{F}}$ deux différentielles holomorphes dt_1 et dt_2 telles que:

$$dt_1 \mid_\Gamma = \omega_1, \; dt_2 \mid_\Gamma = \omega_2,$$

où ω_1, ω_2 sont des différentielles holomorphes (voir Section 3, pour une expression explicite) sur la surface de Riemann Γ. En outre, l'espace des différentielles holomorphes sur Γ est

$$\left\{ f_i^{(0)}\omega_2, 1 \le i \le 7 \right\} \oplus \{\omega_1, \omega_2\},$$

où les $f_i^{(0)}$ sont les premiers coefficients des fonctions $f_i \in \mathbb{L}(2\Gamma)$ et le plongement de Γ dans $\mathbb{P}^7(\mathbb{C})$ est à deux différentielles holomorphes prés le plongement canonique:

$$(\alpha, \beta) \in \Gamma \mapsto [\omega_2, f_1^{(0)}\omega_2, \ldots, f_7^{(0)}\omega_2] \in \mathbb{P}^7(\mathbb{C}).$$

La suite consiste à montrer que les orbites du champ de vecteurs (2.2) passant à travers Γ forment une surface lisse S tout le long de Γ tel que: $S \setminus \Gamma \subseteq \mathbf{F}$. Alors, on prouve que $\widetilde{\mathbf{F}} = \mathbf{F} \cup S$ est une variété compacte (grâce au fait que les solutions issues des points de Γ pénètrent immédiatement dans la partie affine $\mathbf{F}$, plongée dans $\mathbb{P}^7(\mathbb{C})$ à l'aide des fonctions de $\mathbb{L}(2\Gamma)$) et est munie de deux champs de vecteurs réguliers, indépendants en chaque point et commutants. D'après le théorème d'Arnold-Liouville [**3**] et [**18**], la variété $\widetilde{\mathbf{F}}$ est un tore complexe et comme celui-ci possède un plongement projectif, alors $\widetilde{\mathbf{F}}$ est une surface abélienne. Par conséquent, on a le:

Théorème 2. *La fibre* $\mathbf{F}$ (2.3) *forme la partie affine d'une surface abélienne* $\widetilde{\mathbf{F}}$ *et le système* (2.2) *est algébriquement complètement intégrable.*

3. Surface abélienne en tant que variété de Prym.

Soit $(a_1, b_1, A, B, a_2, b_2)$ une base de cycles de Γ de telle façon que les indices d'intersection de cycles deux à deux s'écrivent: $AoB = 1$, $a_i o b_j = \delta_{ij}$ (symbole de Kroneker), $a_i o a_j = a_i o A = a_i o B = b_i o b_j = b_i o A = b_i o B = AoA = BoB = 0$ et qu'en outre: $\sigma(a_1) = a_2$, $\sigma(b_1) = b_2$, $\sigma(A) = -A$, $\sigma(B) = -B$ pour l'involution σ (2.8). Comme Γ est une surface de Riemann hyperelliptique de genre 3, alors les trois différentielles holomorphes sur Γ sont

$$\omega_0 = \frac{\alpha d\alpha}{\beta},\ \omega_1 = \frac{\alpha^2 d\alpha}{\beta},\ \omega_2 = \frac{d\alpha}{\beta},$$

et évidemment $\sigma^*(\omega_0) = \omega_0$, $\sigma^*(\omega_k) = -\omega_k$, $k = 1, 2$. Rappelons que l'involution σ échangeant les feuillets du revêtement double $\Gamma \to \Gamma_0$, identifie Γ_0 au quotient Γ/σ. Cette involution induit une involution $\sigma : Jac(\Gamma) \to Jac(\Gamma)$ et modulo un sous-groupe discret, la variété jacobienne $Jac(\Gamma)$ se décompose en deux parties : une partie paire à savoir Γ_0 et une partie impaire qui n'est autre que la variété de Prym $\mathrm{Prym}(\Gamma/\Gamma_0)$. Soit

$$\begin{pmatrix} \omega_0(A) & \omega_0(B) & \omega_0(a_1) & \omega_0(b_1) & \omega_0(a_2) & \omega_0(b_2) \\ \omega_1(A) & \omega_1(B) & \omega_1(a_1) & \omega_1(b_1) & \omega_1(a_2) & \omega_1(b_2) \\ \omega_2(A) & \omega_2(B) & \omega_2(a_1) & \omega_2(b_1) & \omega_2(a_2) & \omega_2(b_2) \end{pmatrix},$$

la matrice des périodes de $Jac(\Gamma)$ où $\omega_k(\circledast) = \int_{\circledast} \omega_k$, $k = 1, 2, 3$. Or

$$\begin{aligned}
\omega_0(A) &= \omega_0(B) = 0, \\
\omega_0(a_2) &= \omega_0(a_1), \\
\omega_0(b_2) &= \omega_0(b_1), \\
\omega_k(a_2) &= -\omega_k(a_1),\ \ k = 1, 2, \\
\omega_k(b_2) &= -\omega_k(b_1),\ \ k = 1, 2,
\end{aligned}$$

donc la matrice précédente s'écrit sous la forme

$$\begin{pmatrix} 0 & 0 & \omega_0(a_1) & \omega_0(b_1) & \omega_0(a_1) & \omega_0(b_1) \\ \omega_1(A) & \omega_1(B) & \omega_1(a_1) & \omega_1(b_1) & -\omega_1(a_1) & -\omega_1(b_1) \\ \omega_2(A) & \omega_2(B) & \omega_2(a_1) & \omega_2(b_1) & -\omega_2(a_1) & -\omega_2(b_1) \end{pmatrix}.$$

En effectuant des combinaisons linéaires simples sur les colonnes, on obtient les deux matrices suivantes:

$$\begin{pmatrix} 0 & 0 & \omega_0(a_1) & \omega_0(b_1) & 2\omega_0(a_1) & 2\omega_0(b_1) \\ \omega_1(A) & \omega_1(B) & \omega_1(a_1) & \omega_1(b_1) & 0 & 0 \\ \omega_2(A) & \omega_2(B) & \omega_2(a_1) & \omega_2(b_1) & 0 & 0 \end{pmatrix},$$

et

$$\begin{pmatrix} 0 & 0 & \omega_0(a_1) & \omega_0(b_1) & 0 & 0 \\ \omega_1(A) & \omega_1(B) & \omega_1(a_1) & \omega_1(b_1) & 2\omega_1(a_1) & 2\omega_1(b_1) \\ \omega_2(A) & \omega_2(B) & \omega_2(a_1) & \omega_2(b_1) & 2\omega_2(a_1) & 2\omega_2(b_1) \end{pmatrix}.$$

Notons que

$$\begin{pmatrix} 2\omega_0(a_1) & 2\omega_0(b_1) \end{pmatrix},$$

est la matrice des périodes de Γ_0, tandis que

$$\Omega = \begin{pmatrix} \omega_1(A) & \omega_1(B) & 2\omega_1(a_1) & 2\omega_1(b_1) \\ \omega_2(A) & \omega_2(B) & 2\omega_2(a_1) & 2\omega_2(b_1) \end{pmatrix},$$

est celle de $\mathrm{Prym}(\Gamma/\Gamma_0)$. Considérons l'application (uniformisante)

$$\widetilde{\mathbf{F}} \longrightarrow \mathbb{C}^2/L_\Lambda : p \mapsto \int_{p_0}^{p} \binom{dt_1}{dt_2},$$

où (dt_1, dt_2) est une base (considérée dans la Section 2) de différentielles holomorphes sur $\widetilde{\mathbf{F}}$ telles que: $dt_k \mid_\Gamma = \omega_k$, $k = 1, 2$,

$$L_\Lambda = \left\{ \sum_{k=1}^{4} n_k \binom{dt_1}{dt_2} (\nu_k) : n_k \in \mathbb{Z} \right\},$$

est le réseau associé à la matrice des périodes

$$\Lambda = \begin{pmatrix} dt_1(\nu_1) & dt_1(\nu_2) & dt_1(\nu_3) & dt_1(\nu_4) \\ dt_2(\nu_1) & dt_2(\nu_2) & dt_2(\nu_3) & dt_2(\nu_4) \end{pmatrix},$$

et $(\nu_1, \nu_2, \nu_3, \nu_4)$ une base de cycles dans le groupe d'homologie $H_1(\widetilde{\mathbf{F}}, \mathbb{Z})$. D'après le théorème de Lefschetz sur les sections hyperplanes [6], l'application $H_1(\Gamma, \mathbb{Z}) \rightarrow H_1(\widetilde{\mathbf{F}}, \mathbb{Z})$ induite par l'inclusion $\Gamma \hookrightarrow \widetilde{\mathbf{F}}$ est surjective et par conséquent on peut trouver quatre cycles $\nu_1, \nu_2, \nu_3, \nu_4$ sur la surface de Riemann Γ tels que:

$$\Lambda = \begin{pmatrix} \omega_1(\nu_1) & \omega_1(\nu_2) & \omega_1(\nu_3) & \omega_1(\nu_4) \\ \omega_2(\nu_1) & \omega_2(\nu_2) & \omega_2(\nu_3) & \omega_2(\nu_4) \end{pmatrix},$$

et

$$L_\Lambda = \left\{ \sum_{k=1}^{4} n_k \binom{\omega_1}{\omega_2} (\nu_k) : n_k \in \mathbb{Z} \right\}.$$

Ces cycles sont $\nu_1 = a_1, \nu_2 = b_1, \nu_3 = A, \nu_4 = B$ et ils engendrent $H_1(\widetilde{\mathbf{F}}, \mathbb{Z})$ de telle sorte que

$$\Lambda = \begin{pmatrix} \omega_1(a_1) & \omega_1(b_1) & \omega_1(A) & \omega_1(B) \\ \omega_2(a_1) & \omega_2(b_1) & \omega_2(A) & \omega_2(B) \end{pmatrix},$$

est une matrice de Riemann. On montre que $\Lambda = \Omega^\star$; la matrice des périodes de $\widehat{\mathrm{Prym}(\Gamma/\Gamma_0)}$ duale de $\mathrm{Prym}(\Gamma/\Gamma_0)$. Dès lors, les deux variétés abéliennes $\widetilde{\mathbf{F}}$ et $\widehat{\mathrm{Prym}(\Gamma/\Gamma_0)}$ sont analytiquement isomorphes au même tore complexe $\mathbb{C}^2/L_\Lambda$ et d'après le théorème de Chow, ces variétés sont algébriquement isomorphes. Par conséquent, on a le:

Théorème 3. *La surface abélienne* $\widetilde{\mathbf{F}}$ *qui complète la fibre* $\mathbf{F}$ (2.3) *peut être identifiée à la duale d'une variété de Prym* $\widehat{\mathrm{Prym}(\Gamma/\Gamma_0)}$ *du revêtement double* (2.9).

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CHARACTERIZATION OF THE SIMPLE $L^1(G)$-MODULES FOR EXPONENTIAL LIE GROUPS

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Let $\mathcal{G} = \exp \mathfrak{g}$ be a connected, simply connected, solvable exponential Lie group. Let $l \in \mathfrak{g}^*$ and let $\mathfrak{p}$ be an appropriate Pukanszky polarization for l in $\mathfrak{g}$. For every $\bar{p} = (p_1, \ldots, p_m) \in [1, \infty]^m$ we define a representation $\pi_{l,\mathfrak{p},\overline{p}}$ by induction on an $L^{\overline{p}}$-space, where the norm $\|\cdot\|_{\overline{p}}$ of this space is in fact obtained by successive L^{p_j}-norms, with distinct p_j's in different directions. These representations are topologically irreducible and their restrictions to the subspaces generated by the vectors of the form $\pi_{l,\mathfrak{p},\overline{p}}(f)\xi$ with $f \in L^1(\mathcal{G})$, $\pi_{l,\mathfrak{p},\overline{p}}(f)$ of finite rank and $\xi \in \mathfrak{H}_{l,\mathfrak{p},\overline{p}}$ are algebraically irreducible. All the simple $L^1(\mathcal{G})$-modules are of that form, up to equivalence. We show that these representations may in fact be characterized (up to equivalence) by the $\mathcal{G}$-orbits of couples (l, ν), where $l \in \mathfrak{g}^*$ and ν is a real linear form on $\mathfrak{g}(l)/\mathfrak{g}(l) \cap \mathfrak{n}$ satisfying a certain growth condition and where $\mathfrak{g}(l)$ is the stabilizer of l in $\mathfrak{g}$.

1. Introduction.

The aim of the present paper is to give an explicit description of the algebraically irreducible representations of $L^1(\mathcal{G})$, where $\mathcal{G}$ is a connected, simply connected, exponential, solvable Lie group. These representations have first been studied by D. Poguntke in 1983 ([**Po2**]). The method of Poguntke which has been adapted and used in ([**LuMo2**]), is an important ingredient in the present paper, as we shall see later with more details. But first we have to recall the following definitions: We say that $(T, \mathfrak{V})$ is a representation of $L^1(\mathcal{G})$, if $\mathfrak{V}$ is a vector space, $\mathcal{L}(\mathfrak{V})$ the space of all linear operators on $\mathfrak{V}$ and

$$T : L^1(\mathcal{G}) \to \mathcal{L}(\mathfrak{V})$$

an algebra homomorphism. Moreover $(T, \mathfrak{V})$ is said to be algebraically irreducible if $\mathfrak{V}$ has no nontrivial invariant subspaces for the action of $L^1(\mathcal{G})$ under T. In that case we also say that $\mathfrak{V}$ is a simple $L^1(\mathcal{G})$-module. If $\mathfrak{V}$ is a topological vector space, we require moreover the action of $L^1(\mathcal{G})$ on $\mathfrak{V}$ to be strongly continuous. In that case we say that $(T, \mathfrak{V})$ is a topologically irreducible representation of $L^1(\mathcal{G})$, if $\mathfrak{V}$ has no nontrivial closed invariant subspaces. As in the general theory, we can always assume for any representation $(T, \mathfrak{V})$ of $L^1(\mathcal{G})$ that $\mathfrak{V}$ is a Banach space and that the representation $(T, \mathfrak{V})$ is bounded (see [**BoDu**]).

Assume that $(T, \mathfrak{V})$ is a topologically irreducible representation on a Banach space and that there exists $f \in L^1(\mathcal{G})$ such that $T(f)$ is a nonzero operator of finite rank. Consider

$$\mathfrak{V}^0 = \text{span } \{T(f)\xi \mid f \in L^1(\mathcal{G}), T(f) \text{ of finite rank, } \xi \in \mathfrak{V}\}.$$

Then $\mathfrak{V}^0 \neq \{0\}$ and the restriction of T to $\mathfrak{V}^0$, $(T|_{\mathfrak{V}^0}, \mathfrak{V}^0)$, is a simple $L^1(\mathcal{G})$-module ([**Wa**]). We shall see that in our situation all the simple $L^1(\mathcal{G})$-modules are obtained in that way (up to equivalence) and we shall give a precise description of the representations $(T, \mathfrak{V})$ to consider.

The previous definitions and results may of course be given for an arbitrary Banach algebra $\mathcal{A}$ instead of $L^1(\mathcal{G})$. Moreover the representations of $L^1(\mathcal{G})$ may be considered as the integrated forms of bounded representations of the group $\mathcal{G}$. In fact, recall that $(T, \mathfrak{V})$ is said to be a representation of the group $\mathcal{G}$ if T is a group homomorphism of $\mathcal{G}$ into the general linear group of $\mathfrak{V}$. This representation is said to be bounded if $\sup_{x \in \mathcal{G}} \|T(x)\| < \infty$, where $\|T(x)\|$ is the operator norm of $T(x)$. For such a representation of $\mathcal{G}$, we get a representation of $L^1(\mathcal{G})$ by $T(f) = \int_{\mathcal{G}} f(x)T(x)dx, \forall f \in L^1(\mathcal{G})$.

A representation π of $\mathcal{G}$, resp. $L^1(\mathcal{G})$ on a Hilbert space $\mathfrak{H}_\pi$ is said to be unitary, if $\pi(x^{-1}) = \pi(x)^*$, resp. $\pi(f^*) = \pi(f)^*$ for all $x \in \mathcal{G}$, resp. $f \in L^1(\mathcal{G})$. Recall that the unitary topologically irreducible representations π of a solvable exponential Lie group $\mathcal{G} = \exp \mathfrak{g}$ may be described as induced representations. There exist $l \in \mathfrak{g}^*$ and a Pukanszky polarization $\mathfrak{p} \subset \mathfrak{g}$ at l such that $\pi = \text{ ind } {}_{\mathcal{P}}^{\mathcal{G}} \chi_l$ (up to unitary equivalence), where $\mathcal{P} = \exp \mathfrak{p}$ and $\chi_l(\exp X) = e^{-i\langle l, X\rangle}$ for all $X \in \mathfrak{p}$ ([**LeLu**]). The set of equivalence classes of topologically irreducible unitary representations of $\mathcal{G}$ is noted by $\hat{\mathcal{G}}$.

If $\mathcal{G}$ is a connected, simply connected nilpotent Lie group, then all the simple $L^1(\mathcal{G})$-modules are equivalent to a module of the form $(\pi|_{\mathfrak{H}_\pi^0}, \mathfrak{H}_\pi^0)$, where $\pi \in \hat{\mathcal{G}}$ and

$$\mathfrak{H}_\pi^0 = \text{ span } \{\pi(f)\xi \mid f \in L^1(\mathcal{G}), \pi(f) \text{ of finite rank, } \xi \in \mathfrak{H}_\pi\}.$$

The same remains true for $L^1(\mathcal{G}, \omega)$, where $\mathcal{G}$ is a connected, simply connected, nilpotent Lie group and ω is a polynomial weight on $\mathcal{G}$ ([**MiMo**]). In this paper these results are generalized in the following way: If $\mathcal{G}$ is a connected, simply connected, solvable exponential Lie group, we define representations $\pi_{l,\mathfrak{p},\bar{p}}$ by induction on $L^{\bar{p}}$-spaces, where $\bar{p} = (p_1, \ldots, p_m) \in [1, \infty]^m$ is a multi-index. The norm $\| \cdot \|_{\bar{p}}$ of such an $L^{\bar{p}}$-space is obtained by successive L^{p_j}-norms with distinct p_j's in different directions. To do this, we have to introduce a precise decomposition of the Lie algebra $\mathfrak{g}$ of the group $\mathcal{G}$. These representations are topologically irreducible and admit nontrivial operators of finite rank. Hence, if we write $\mathfrak{H}_{l,\mathfrak{p},\bar{p}}$ for the space of such a representation and

$$\mathfrak{H}^0_{l,\mathfrak{p},\bar{p}} = \text{ span } \{\pi_{l,\mathfrak{p},\bar{p}}(f)\xi \mid f \in L^1(\mathcal{G}), \pi_{l,\mathfrak{p},\bar{p}}(f) \text{ of finite rank, } \xi \in \mathfrak{H}_{l,\mathfrak{p},\bar{p}}\},$$

then $\left(\pi_{l,\mathfrak{p},\bar{p}}|_{\mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}}, \mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}\right) = \left(\pi^0_{l,\mathfrak{p},\bar{p}}, \mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}\right)$ is a simple $L^1(\mathcal{G})$-module. We show that all the simple $L^1(\mathcal{G})$-modules $(T, \mathfrak{V})$ are of this type (up to equivalence). To do this we rely on the work of Poguntke ([**Po1**], [**Po2**]). In his paper ([**Po2**]) Poguntke gives a first description of simple $L^1(\mathcal{G})$-modules. Let's notice first that a representation $(T, \mathfrak{V})$ of $L^1(\mathcal{G})$ defines unique representations of $\mathcal{G}$, of $\mathcal{N}$ (by restriction) and of $L^1(\mathcal{N})$, where $\mathcal{N} = \exp \mathfrak{n}$ and $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$. We shall write $\ker_{L^1(\mathcal{N})} T$ for the corresponding kernel in $L^1(\mathcal{N})$. This kernel is of the form $\ker(\mathcal{G} \cdot \tau)$, where $\tau \in \hat{\mathcal{N}}$ is the representation induced from a character χ_q defined by a linear form $q \in \mathfrak{n}^*$. Let $l \in \mathfrak{g}^*$ such that $l|_{\mathfrak{n}} = q$. The method of Poguntke ([**Po2**]) which has been adapted and used for the description of topologically irreducible representations in ([**LuMo2**]) consists in constructing an algebra of the type $L^1(\mathbb{R}^n, \omega)$, where ω is an exponential weight in general, uniquely determined by the given simple module $(T, \mathfrak{V})$ and where $\mathbb{R}^n \equiv \mathcal{G}(l)/\mathcal{G}(l) \cap \mathcal{N}$, with $\mathcal{G}(l) = \exp \mathfrak{g}(l)$ and $\mathfrak{g}(l)$ is the stabilizer of l in $\mathfrak{g}$. Then one shows that the simple $L^1(\mathcal{G})$-module $(T, \mathfrak{V})$ with given $\ker_{L^1(\mathcal{N})} T$ is completely characterized by a continuous character on $L^1(\mathbb{R}^n, \omega)$. Conversely every such character on $L^1(\mathbb{R}^n, \omega)$ leads to a unique simple $L^1(\mathcal{G})$-module (up to equivalence) with given $\ker_{L^1(\mathcal{N})} T$. In order to show that every simple $L^1(\mathcal{G})$-module is equivalent to a module of the form $\left(\pi_{l,\mathfrak{p},\bar{p}}|_{\mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}}, \mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}\right)$, it is then enough to show that every (continuous) character on $L^1(\mathbb{R}^n, \omega)$ is associated to such a representation. To do this we have to give an estimation of the weight ω using a method developed by Poguntke in ([**Po2**]). The equivalence classes of simple $L^1(\mathcal{G})$-modules are then completely characterized by the $\mathcal{G}$-orbits of the couple (l, ν), where $l \in \mathfrak{g}^*$ and ν is a real linear form on $\mathfrak{g}(l)/\mathfrak{g}(l) \cap \mathfrak{n}$ satisfying a certain growth condition.

2. Construction of special irreducible representations.

2.1. . For the rest of this paper $\mathcal{G} = \exp \mathfrak{g}$ will be a connected, simply connected, solvable exponential Lie group with Lie algebra $\mathfrak{g}$. The nil-radical of $\mathfrak{g}$ will be denoted by $\mathfrak{n}$ and $\mathcal{N} = \exp \mathfrak{n}$ will be the corresponding subgroup of $\mathcal{G}$. Take $l \in \mathfrak{g}^*$ and write $q = l|_{\mathfrak{n}} \in \mathfrak{n}^*$. We define the following stabilizers:

$$\begin{aligned}
\mathfrak{g}(l) &= \{X \in \mathfrak{g} \mid \langle l, [X, \mathfrak{g}]\rangle \equiv 0\},\\
\mathfrak{g}(q) &= \{X \in \mathfrak{g} \mid \langle q, [X, \mathfrak{n}]\rangle = \langle l, [X, \mathfrak{n}]\rangle \equiv 0\},\\
\mathfrak{n}(q) &= \{X \in \mathfrak{n} \mid \langle q, [X, \mathfrak{n}]\rangle = \langle l, [X, \mathfrak{n}]\rangle \equiv 0\} = \mathfrak{g}(q) \cap \mathfrak{n}.
\end{aligned}$$

Then we decompose the Lie algebra as follows:

$$\begin{aligned} \mathfrak{g}(l) + \mathfrak{n} &= \mathfrak{u} \oplus \mathfrak{n} & &\text{with } \mathfrak{u} \subset \mathfrak{g}(l) \subset \mathfrak{g}(q), \\ \mathfrak{g}(q) + \mathfrak{n} &= \mathfrak{w} \oplus (\mathfrak{g}(l) + \mathfrak{n}) = \mathfrak{w} \oplus \mathfrak{u} \oplus \mathfrak{n} & &\text{with } \mathfrak{w} \subset \mathfrak{g}(q), \\ \mathfrak{g} &= \mathfrak{v} \oplus (\mathfrak{g}(q) + \mathfrak{n}) = \mathfrak{v} \oplus \mathfrak{w} \oplus \mathfrak{u} \oplus \mathfrak{n}. \end{aligned}$$

2.2. . Now we choose $\mathfrak{Y} \subset \mathfrak{w} \subset \mathfrak{g}(q)$ a maximal l-isotropic subspace of $\mathfrak{w}$, i.e., a maximal subspace of $\mathfrak{w}$ such that $\langle l, [\mathfrak{Y}, \mathfrak{Y}]\rangle \equiv 0$. Then there exist a subspace $\mathfrak{X}$ in $\mathfrak{w}$ and bases $\{X_1, \dots, X_c\}$ of $\mathfrak{X}$, resp. $\{Y_1, \dots, Y_c\}$ of $\mathfrak{Y}$ such that $\mathfrak{w} = \mathfrak{X} \oplus \mathfrak{Y}$ with

$$\langle l, [X_i, X_j]\rangle = 0, \ \langle l, [Y_i, Y_j]\rangle = 0, \ \langle l, [X_i, Y_j]\rangle = \delta_{ij},$$

i.e., $\mathfrak{X}$ is a dual space of $\mathfrak{Y}$ with respect to l. This is possible because $\{Z \in \mathfrak{w} \mid \langle l, [Z, \mathfrak{w}]\rangle \equiv 0\} = \{0\}$. As a matter of fact, $\mathfrak{w} \oplus \mathfrak{n}(q)$ modulo $\ker(q|_{\mathfrak{n}(q)})$ is a Heisenberg algebra. We write $\mathcal{U} = \exp \mathfrak{u}$, $\mathcal{V} = \exp \mathfrak{v}$, $\mathcal{W} = \exp \mathfrak{w}$, $\mathcal{X} = \exp \mathfrak{X}$, $\mathcal{Y} = \exp \mathfrak{Y}$.

2.3. Polarizations. First let us choose $\mathfrak{p}^0$ a $\mathfrak{g}(q)$-invariant polarization of q in $\mathfrak{n}$ (for example a Vergne polarization). Then $\mathfrak{p} = \mathfrak{Y} \oplus \mathfrak{p}^0 \oplus \mathfrak{u}$ is a Pukanszky polarization of l in $\mathfrak{g}$. Moreover $\mathfrak{p}^0 = \mathfrak{p} \cap \mathfrak{n}$. For the rest of this paper we shall stick to these polarizations. We write $\mathcal{P}^0 = \exp \mathfrak{p}^0$, $\mathcal{P} = \exp \mathfrak{p}$.

2.4. Jordan-Hölder decomposition. Let

$$\mathfrak{n} = \mathfrak{n}_0 \supset \mathfrak{n}_1 \supset \cdots \supset \mathfrak{n}_k \supset \mathfrak{n}_{k+1} = \{0\}$$

be a Jordan-Hölder sequence for the action of $\mathfrak{g}(q) + \mathfrak{n}$ on $\mathfrak{n}$. Let

$$\begin{aligned} Y &= \{i \mid \mathfrak{p}^0 + \mathfrak{n}_i \neq \mathfrak{p}^0 + \mathfrak{n}_{i+1}, i = 0, \dots, k\} \\ &= \{i_j \mid 1 \leq j \leq m, 0 \leq i_1 \leq \cdots \leq i_m \leq k\}. \end{aligned}$$

We write $\mathfrak{p}_j = \mathfrak{p}^0 + \mathfrak{n}_{i_j}$, for $j = 1, \dots m$, and $\mathfrak{p}_{m+1} = \mathfrak{p}^0$. Obviously $\mathfrak{p}_1 = \mathfrak{n}$. For each $j \in \{1, \dots, m\}$ we choose a subspace $\mathfrak{v}_j \subset \mathfrak{n}_{i_j} \subset \mathfrak{p}_j$ such that $\mathfrak{v}_j \oplus \mathfrak{p}_{j+1} = \mathfrak{p}_j$. Then $\sum_{j=1}^{m}{}^{\oplus} \mathfrak{v}_j \oplus \mathfrak{p}^0 = \mathfrak{n}$ and

$$\begin{aligned} \Phi : \sum_{j=1}^{m}{}^{\oplus} \mathfrak{v}_j &\longrightarrow \mathcal{N}/\mathcal{P}^0 \\ V_1 + \cdots + V_m \equiv (V_1, \dots, V_m) &\longmapsto \exp\,(V_1) \dots \exp\,(V_m) \cdot \mathcal{P}^0 \end{aligned}$$

is a diffeomorphism.

2.5. Special representations. Let's write

$$\widetilde{\mathfrak{n}} = \sum_{j=1}^{m}{}^{\oplus} \mathfrak{v}_j, \ \mathcal{V}_j = \exp \mathfrak{v}_j \text{ and } \widetilde{\mathcal{N}} = \prod_{j=1}^{m} \mathcal{V}_j = \prod_{j=1}^{m} \exp \mathfrak{v}_j.$$

Consider the following decomposition of $\mathcal{G}$: $\mathcal{G} = \mathcal{V} \cdot \mathcal{X} \cdot \widetilde{\mathcal{N}} \cdot \mathcal{P}$. Take $\bar{p} = (p_1, \dots, p_m) \in [1, \infty]^m$. The representation space $L^{\bar{p}}(\mathcal{G}/\mathcal{P}, \chi_l)$ is then defined to be the completion, for the norm $\|\cdot\|_{\bar{p}}$ given below, of the space of all functions $\xi : \mathcal{V}\cdot\mathcal{X}\cdot\widetilde{\mathcal{N}}\cdot\mathcal{P}\to\mathbb{C}$ continuous with compact support mod $\mathcal{P}$, such that $\xi(x\cdot p)=\overline{\chi_l(p)}\xi(x), \forall x\in\mathcal{G}, \forall p\in\mathcal{P}$, and

$$\|\xi\|_{\bar{p}}=\Big(\int_{\mathcal{V}}\int_{\mathcal{X}}\Big(\Big(\int_{\mathcal{V}_1}\Big(\cdots\Big(\int_{\mathcal{V}_m}|\xi(vxv_1\dots v_m)|^{p_m}dv_m\Big)^{\frac{1}{p_m}} \\ \cdots\Big)^{p_1}dv_1\Big)^{\frac{1}{p_1}}\Big)^2 dxdv\Big)^{\frac{1}{2}}<\infty,$$

the different measures being the Lebesgue measures on $\mathfrak{v},\mathfrak{X},\mathfrak{v}_1,\dots,\mathfrak{v}_m$. If $p_j=\infty$, then $\big(\int_{\mathcal{V}_j}|\dots|^{p_j}dv_j\big)^{\frac{1}{p_j}}$ is replaced by the corresponding sup-norm. Let $L^{\bar{p}}(\mathcal{G}/\mathcal{P},\chi_l)=\mathfrak{H}_{l,\mathfrak{p},\bar{p}}$ be the space we get by completion. On this space we want to define a representation by isometric operators given essentially by left translation. This representation will be of the form

$$(\pi_{l,\mathfrak{p},\bar{p}}(s)\xi)(y)=\Delta^{-\frac{1}{\bar{p}}}(s)\xi(s^{-1}y),\quad\forall s,y\in\mathcal{G},$$

where the modular function $\Delta^{-\frac{1}{\bar{p}}}$ has to be defined in order to get isometric operators on $\mathfrak{H}_{l,\mathfrak{p},\bar{p}}$. It is easy to check that

$$\Delta^{\frac{1}{\bar{p}}}(v\cdot x\cdot n\cdot p)=e^{\sum_{j=1}^m\frac{1}{p_j}\mathrm{tr\ ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log p)}=e^{\sum_{j=1}^m\frac{1}{p_j}\mathrm{tr}\lambda_j(\log p)},$$

if we use the notation $\lambda_j(\cdot)=\mathrm{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\cdot)$. For $\bar{p}=\overline{2}=(2,\dots,2)$ we have

$$\Delta^{\frac{1}{2}}(s)=e^{\frac{1}{2}\mathrm{tr\ ad}_{\mathfrak{n}/\mathfrak{p}^0}(\log s)}=e^{\frac{1}{2}\mathrm{tr\ ad}_{\mathfrak{g}/\mathfrak{p}}(\log s)},$$

as $\mathfrak{n}/\mathfrak{p}^0=(\mathfrak{u}\oplus\mathfrak{Y}\oplus\mathfrak{n})/\mathfrak{p}$ and as $\mathrm{tr\ ad}_{\mathfrak{g}/(\mathfrak{u}\oplus\mathfrak{Y}\oplus\mathfrak{n})}=0$. The representation $\pi_l=\pi_{l,\mathfrak{p},\overline{2}}$ is the usual induced unitary representation $\mathrm{ind}_{\mathcal{P}}^{\mathcal{G}}(\chi_l,2)$. Notice that

$$\pi_{l,\mathfrak{p},\bar{p}}(s)=\Delta^{\frac{1}{2}-\frac{1}{\bar{p}}}(s)\pi_l(s)$$

on the dense subspace of all continuous functions of $L^{\bar{p}}(\mathcal{G}/\mathcal{P},\chi_l)$ with compact support in $\mathcal{G}/\mathcal{P}$, or, more generally, on the generalized Schwartz space $\mathcal{ES}(\mathcal{G}/\mathcal{P},\chi_l)$ (see (2.7.) for the precise definition of this space).

2.6. Remarks.

a) As $\mathcal{G}(l)\subset\mathcal{P}$ and as $\Delta^{\frac{1}{\bar{p}}}\equiv 1$ on $\mathcal{N}\cap\mathcal{G}(l)$, $\Delta^{\frac{1}{\bar{p}}}$ may be considered as a character on $\mathcal{G}(l)/\mathcal{G}(l)\cap\mathcal{N}\equiv\mathcal{G}/\mathcal{H}$ given by

$$\Delta^{\frac{1}{\bar{p}}}(\dot{s})=e^{\sum_{j=1}^m\frac{1}{p_j}\mathrm{\ tr\ ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log s)}$$

for all $\dot{s}\in\mathcal{G}(l)/\mathcal{G}(l)\cap\mathcal{N}$.

b) There is a relation between the Haar measures on $\mathcal{G}$, $\mathcal{P}$ and the measure on $\mathcal{G}/\mathcal{P}\equiv\mathcal{V}\cdot\mathcal{X}\cdot\widetilde{\mathcal{N}}=\mathcal{V}\cdot\mathcal{X}\cdot\prod_{j=1}^m\mathcal{V}_j$: If the Lie algebra $\mathfrak{g}$ is decomposed by

$$\mathfrak{g}=\mathfrak{v}\oplus\mathfrak{X}\oplus\mathfrak{Y}\oplus\mathfrak{u}\oplus(\widetilde{\mathfrak{n}}\oplus\mathfrak{p}^0),$$

we get a Haar measure on $\mathcal{G}$ by

$$\int_{\mathcal{G}} f(g)dg = \int_{\mathfrak{v}}\int_{\mathfrak{X}}\int_{\mathfrak{Y}}\int_{\mathfrak{u}}\int_{\mathfrak{v}_1}\cdots\int_{\mathfrak{v}_m}\int_{\mathcal{P}^0} f\big(\exp V\cdot\exp X\cdot\exp Y\cdot\exp U \cdot\exp V_1\ldots\exp V_m\cdot p^0\big)dp^0dV_m\ldots dV_1dUdYdXdV,$$

where we use the Haar measure on $\mathcal{P}^0$ and the Lebesgue measures on $\mathfrak{v},\mathfrak{X},\mathfrak{Y},\mathfrak{u},\mathfrak{v}_j$. The Haar measure on $\mathcal{P}$ is given by

$$\int_{\mathcal{P}} f(p)dp = \int_{\mathfrak{Y}}\int_{\mathfrak{u}}\int_{\mathcal{P}^0} f\big(\exp Y\cdot\exp U\cdot p^0\big)dp^0dUdY.$$

We check that

$$\int_{\mathcal{G}} f(g)dg = \int_{\mathcal{G}/\mathcal{P}}\int_{\mathcal{P}} f(gp)\Delta^{-1}(gp)dpd\dot{g}.$$

2.7. The $\boldsymbol{\mathcal{ES}}$-spaces. Let the polarizations be chosen as in (2.3.). Let $\mathfrak{B}_1 = \{A_1,\ldots,A_j\}$ be a coexponential basis for $\mathfrak{p}_0$ in $\mathfrak{n}$, which has for instance been chosen in the subspaces $\mathfrak{v}_j$. Let $\mathfrak{B}_2 = \{B_1,\ldots,B_k\}$ be a coexponential basis for $\mathfrak{n}+\mathfrak{p}$ in $\mathfrak{g}$. Then $\mathfrak{B}=\mathfrak{B}_1\cup\mathfrak{B}_2$ is a coexponential basis for $\mathfrak{p}$ in $\mathfrak{g}$. Given a function F on $\mathcal{G}/\mathcal{P}\times\mathcal{G}/\mathcal{P}$, we define a function $\widetilde{F}$ on $(\mathbb{R}^k\times\mathbb{R}^j)\times(\mathbb{R}^k\times\mathbb{R}^j)$ by

$$\begin{aligned}&\widetilde{F}\left(b_1,\ldots,b_k,a_1,\ldots a_j;b_1',\ldots,b_k',a_1',\ldots a_j'\right)\\ &= F\Big(\exp b_1B_1\ldots\exp b_kB_k\exp a_1A_1\ldots\exp a_jA_j;\\ &\qquad \exp b_1'B_1\ldots\exp b_k'B_k\exp a_1'A_1\ldots\exp a_j'A_j\Big).\end{aligned}$$

We proceed similarly for a function defined on $\mathcal{G}/\mathcal{P}$. This allows us to give the following definition:

Definition 2.7.1. a) The space $\mathcal{E}S(\mathcal{G}/\mathcal{P}\times\mathcal{G}/\mathcal{P},\chi_l)$ is the space of all C^∞-functions $F:\mathcal{G}\times\mathcal{G}\to\mathbb{C}$ such that:

(1) $F(xs,x's') = \overline{\chi_l(s)}\chi_l(s')F(x,x'),\quad \forall x,x'\in\mathcal{G},\forall s,s'\in\mathcal{P}.$

(2) $\|F\|_{\partial,\alpha,\alpha',R,R'}$
$$= \sup_{a,a'\in\mathbb{R}^j,b,b'\in\mathbb{R}^k}\left(e^{\alpha|b|}e^{\alpha'|b'|}|R(a)R'(a')\partial_a\partial_b\partial_{a'}\partial_{b'}\widetilde{F}(b,a;b',a')|\right)<\infty$$
for all $\alpha,\alpha'\geq 0$, for all polynomials R and R', for all derivation operators ∂, if $|b|$ and $|b'|$ denote the euclidean norm on $\mathbb{R}^k$.

(3) The same conditions as in (2) are required for all partial Fourier transforms of $\widetilde{F}$ in b and b'.

b) The space $\mathcal{E}S(\mathcal{G}/\mathcal{P},\chi_l)$ is defined similarly (see [**Lu**]).

Remark. The previous spaces are independent of the choice of the coexponential bases. They also contain real analytic functions which, therefore, may be extended to functions with complex variables ([**LeLu**]).

Let $\mathfrak{B}_3 = \{C_1, \dots, C_i\}$ be a coexponential basis for $\mathfrak{n}$ in $\mathfrak{g}$. We may choose the elements of $\mathfrak{B}_3$ in a nilpotent subalgebra $\mathfrak{Q}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{Q} + \mathfrak{n}$. Let $\mathfrak{B}_4 = \{D_1, \dots, D_g\}$ be a Jordan-Hölder basis for $\mathfrak{n}$. For a function f defined on $\mathcal{G}$, we define $\widetilde{f}$ on $\mathbb{R}^i \times \mathbb{R}^g$ as previously. We then define:

Definition 2.7.2. The space $\mathcal{ES}(\mathcal{G})$ is the space of all C^∞-functions $f : \mathcal{G} \to \mathbb{C}$ such that

$$\|f\|_{\partial,\alpha,R} = \sup_{c\in\mathbb{R}^i, d\in\mathbb{R}^g} \left(e^{\alpha|c|} \left| R(d)\partial_c\partial_d \widetilde{f}(c,d) \right| \right) < \infty$$

for every $\alpha \geq 0$, for every polynomial R, for all derivation operators ∂, if $|c|$ denotes the euclidean norm on $\mathbb{R}^i$.

Remarks.

a) The space $\mathcal{ES}(\mathcal{G})$ is independent of the choice of the bases. It is dense in $L^1(\mathcal{G})$ ([**Lu**]). Similarly for $\mathcal{ES}(\mathcal{G}/\mathcal{P}, \chi_l)$ and $L^{\bar{p}}(\mathcal{G}/\mathcal{P}, \chi_l)$.

b) The space $\mathcal{ES}(\mathcal{G}/\mathcal{P} \times \mathcal{G}/\mathcal{P}, \chi_l)$ is in the image of the map that sends every $f \in L^1(\mathcal{G})$ to the kernel function of the operator $\pi_l(f)$ ([**LeLu**], [**Lu**]). Similarly for $\pi_{l,\mathfrak{p},\bar{p}}(f)$ instead of π_l thanks to the following observation: For $f \in \mathcal{ES}(\mathcal{G}) \subset L^1(\mathcal{G})$, we have $\pi_{l,\mathfrak{p},\bar{p}}(f) = \pi_l(\Delta^{\frac{1}{2}-\frac{1}{\bar{p}}} \cdot f)$ and $\pi_l(f) = \pi_{l,\mathfrak{p},\bar{p}}(\Delta^{\frac{1}{\bar{p}}-\frac{1}{2}} \cdot f)$, where $\frac{1}{2} - \frac{1}{\bar{p}} = \left(\frac{1}{2} - \frac{1}{p_1}, \dots, \frac{1}{2} - \frac{1}{p_m}\right)$.

c) Put $\mathfrak{H}^0_{l,\mathfrak{p},\bar{p}} = \operatorname{span}\{\pi_{l,\mathfrak{p},\bar{p}}(f)\xi \mid \xi \in \mathfrak{H}_{l,\mathfrak{p},\bar{p}}, f \in L^1(\mathcal{G})$ such that $\pi_{l,\mathfrak{p},\bar{p}}(f)$ of finite rank$\}$. Hence $\mathcal{ES}(\mathcal{G}/\mathcal{P}, \chi_l) \subset \mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}$, by b).

As in ([**Wa**]) we can prove the following theorem, using c):

Theorem 2.7.3. *The representation* $\left(\pi_{l,\mathfrak{p},\bar{p}}, \mathfrak{H}_{l,\mathfrak{p},\bar{p}}\right)$ *is topologically irreducible and the sub-representation* $\left(\pi_{l,\mathfrak{p},\bar{p}}|_{\mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}}, \mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}\right) = \left(\pi^0_{l,\mathfrak{p},\bar{p}}, \mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}\right)$ *is algebraically irreducible.*

3. Analysis of an arbitrary simple $L^1(\mathcal{G})$-module.

3.1. . In this chapter we shall use the methods of Poguntke ([**Po1**], [**Po2**]) which have been used and modified in ([**LuMo2**]) in order to study the topologically irreducible representations. As a matter of fact most of the analysis of ([**LuMo2**]) remains true in the situation of simple $L^1(\mathcal{G})$-modules. Therefore we shall give no proofs in this chapter and just recall the main results of ([**LuMo2**]) and ([**Po2**]).

Proposition 3.2. *Let* $(T, \mathfrak{U})$ *be an algebraically irreducible representation of* $L^1(\mathcal{G})$. *Let's write* $\ker_{L^1(\mathcal{N})} T$ *for the kernel of the corresponding representation of* $L^1(\mathcal{N})$. *Then there exist* $\tau \in \hat{\mathcal{N}}$ *and* $q \in \mathfrak{n}^*$, $\mathfrak{p}_0$ *a polarization* *of q in $\mathfrak{n}$ and $\mathcal{P}_0 = \exp \mathfrak{p}_0$, such that*

$$\tau = \operatorname{ind}_{\mathcal{P}^0}^{\mathcal{N}} \chi_q \text{ and } \ker_{L^1(\mathcal{N})} T = \ker(\mathcal{G} \cdot \tau) = \bigcap_{g \in \mathcal{G}} \ker({}^g\tau).$$

The kernel $\ker_{L^1(\mathcal{N})} T$ is completely determined by the $\mathcal{G}$-orbit $\mathcal{G} \cdot \tau$.

3.3. Corresponding unitary representations. The aim of this section is to introduce the largest subgroup $\mathcal{H}$ on which it is possible, in a certain sense, to work with a unitary representation. Let $l \in \mathfrak{g}^*$ be such that $l|_{\mathfrak{n}} = q$. Using the same decompositions as in (2.1.), we define $\mathfrak{h} = \mathfrak{v} \oplus \mathfrak{w} \oplus \mathfrak{n}$, $\mathcal{H} = \exp \mathfrak{h}$, $r = l|_{\mathfrak{h}}$. Then $\mathfrak{p}^1 = \mathfrak{Y} \oplus \mathfrak{p}^0$ is a Pukanszky polarization of r in $\mathfrak{h}$. Moreover, $\mathfrak{p}^1 = \mathfrak{p} \cap \mathfrak{h}$. Let $\mathcal{P}^1 = \exp \mathfrak{p}^1$. As in (2.5.) we get a decomposition of $\mathcal{H}$ by writing $\mathcal{H} = \mathcal{V} \cdot \mathcal{X} \cdot \widetilde{\mathcal{N}} \cdot \mathcal{P}^1$. Imitating the definition of $\pi_{l,\mathfrak{p},\bar{p}}$, we similarly define representations $\gamma_{\bar{p}}$ of $\mathcal{H}$ and $L^1(\mathcal{H})$ on the representation space $\mathfrak{H}_{\gamma_{\bar{p}}} = L^{\bar{p}}(\mathcal{H}/\mathcal{P}^1, \chi_r)$. Notice that the corresponding character $\Delta^{\frac{1}{\bar{p}}}$ is the same as for $\pi_{l,\mathfrak{p},\bar{p}}$. For $\bar{p} = \overline{2} = (2, \dots, 2)$ we simply write $\gamma = \gamma_{\overline{2}} = \gamma_2$. For every extension l of r to $\mathfrak{g}$, the representation $\gamma_{\bar{p}}$ may be extended to a representation $\gamma_{l,\bar{p}}$ of $\mathcal{G}$ in the following way:

(1) $\mathfrak{H}_{\gamma_{l,\bar{p}}} = \mathfrak{H}_{\gamma_{\bar{p}}}$

(2) $\gamma_{l,\bar{p}}(h) = \gamma_{\bar{p}}(h), \quad \forall h \in \mathcal{H}$

(3) $\Big(\gamma_{l,\bar{p}}(t)\xi\Big)(x) = \Delta^{-\frac{1}{\bar{p}}}(t)\chi_l(t)\xi(t^{-1}xt), \quad \forall \xi \in \mathfrak{H}_{\gamma_{\bar{p}}}, \forall t \in \mathcal{U}, \forall x \in \mathcal{H}$

(4) $\gamma_{l,\bar{p}}(th) = \gamma_{l,\bar{p}}(t)\gamma_{l,\bar{p}}(h), \quad \forall t \in \mathcal{U}, \forall h \in \mathcal{H}.$

For $\bar{p} = \overline{2}$ we simply write γ_l instead of $\gamma_{l,\overline{2}}$. It is easy to check that $\gamma_{l,\bar{p}}$ is a well-defined representation that is equivalent to $\pi_{l,\mathfrak{p},\bar{p}}$. Hence $\gamma_{\bar{p}}$ may also be viewed as the restriction of $\pi_{l,\mathfrak{p},\bar{p}}$ to the subgroup $\mathcal{H}$. One may check that different extensions r and r' of $q \in \mathfrak{n}^*$ to $\mathfrak{h}$ give the same representation $\gamma_{\bar{p}}$ (up to equivalence), whereas different extensions l and l' of $r \in \mathfrak{h}^*$ to $\mathfrak{g}$ lead to representations $\gamma_{l,\bar{p}}$ and $\gamma_{l',\bar{p}}$ that differ by the unitary character $\chi_{l-l'}$ on $\mathcal{U}$. One defines of course the spaces $\mathcal{ES}(\mathcal{H})$, $\mathcal{ES}(\mathcal{H}/\mathcal{P}^1, \chi_r)$, $\mathcal{ES}(\mathcal{H}/\mathcal{P}^1 \times \mathcal{H}/\mathcal{P}^1, \chi_r)$ and one has the equivalent of (2.7.3.) for the representations $\gamma_{\bar{p}}$.

Take $\lambda \in \mathcal{ES}(\mathcal{H}/\mathcal{P}^1, \chi_r)$ such that $\langle \lambda, \lambda \rangle = 1$ and let $p_\lambda \in L^1(\mathcal{H})$ be an element such that the kernel of the operator $\gamma(p_\lambda)$ is the projector $P_{\lambda,\lambda}$, i.e., such that

$$\Big(\gamma(p_\lambda)\xi\Big)(x) = \int_{\mathcal{H}/\mathcal{P}^1} \lambda(x)\overline{\lambda(y)}\xi(y)d\dot{y}.$$

Put $p = p_\lambda \bmod \ker \gamma$. Then p is an idempotent element of $L^1(\mathcal{H})/\ker \gamma$. We have that

$$\ker \gamma = \Big(L^1(\mathcal{H}) * \ker(\mathcal{G} \cdot \tau)\Big)^{-L^1(\mathcal{H})} = \ker_{L^1(\mathcal{H})} T,$$

where $\ker_{L^1(\mathcal{H})} T$ stands for the kernel of the corresponding representation of $L^1(\mathcal{H})$ (obtained by $T|_{\mathcal{H}}$) and

$$\left(L^1(\mathcal{G}) * \ker\gamma\right)^{-L^1(\mathcal{G})} \subset \ker T.$$

In particular,

$$T(p_\lambda) \neq 0 \text{ and } \mathfrak{W} = T(p_\lambda)\mathfrak{U} \neq \{0\}.$$

3.4. Some quotient algebras. Thanks to the decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{h}$ with $\mathfrak{u} \subset \mathfrak{g}(l)$, we put $\mathcal{U} = \exp\mathfrak{u}$ and we may identify the sets $\mathcal{U}$ and $\mathcal{G}(l)/\mathcal{G}(l) \cap \mathcal{N}$. As in ([**Po2**]) and ([**LuMo2**]) we introduce generalized convolution and involution formulas in $L^1(\mathcal{U}, L^1(\mathcal{H})/\ker\gamma)$. It is then easy to check that the algebras $L^1(\mathcal{U}, L^1(\mathcal{H})/\ker\gamma)$ and $L^1(\mathcal{G})/(L^1(\mathcal{G}) * \ker\gamma)^{-L^1(\mathcal{G})}$ $= L^1(\mathcal{G})/(L^1(\mathcal{G}) * \ker_{L^1(\mathcal{N})} T)^{-L^1(\mathcal{G})}$ are isomorphic and isometric (see [**Po2**] and [**LuMo2**]). Notice that the latter algebra is completely determined by the initial representation $(T, \mathcal{U})$.

3.5. A special subalgebra. Take p_λ as in (3.3.). For any $f \in L^1(\mathcal{G})$, let's define $\widetilde{f} \in L^1(\mathcal{U}, L^1(\mathcal{H}))$ by $\widetilde{f}(u)(h) = f(u \cdot h)$ for almost all $u \in \mathcal{U}$ and almost all $h \in \mathcal{H}$. It is then easy to check that

$$(p_\lambda * f * p_\lambda)\widetilde{\ }(x) = p_\lambda^x *_{L^1(\mathcal{H})} \widetilde{f}(x) *_{L^1(\mathcal{H})} p_\lambda$$

for every $f \in L^1(\mathcal{G})$ and every $x \in \mathcal{G}$, where p_λ^x is the function of $L^1(\mathcal{H})$ obtained by the action of x on p_λ:

$$p_\lambda^x(y) = \Delta_{\mathcal{G}}(x) p_\lambda(xyx^{-1}), \quad \forall y \in \mathcal{H}.$$

We recall that $\pi = \pi_{l,\mathfrak{p},\overline{2}} = \operatorname{ind}_{\mathcal{P}}^{\mathcal{G}} \chi_l$, that $\gamma = \operatorname{ind}_{\mathcal{P}^1}^{\mathcal{H}} \chi_r$ and that the extension γ_l is equivalent to π. One has the following formulas:

$$\gamma(p_\lambda^x) = P_{\gamma_l(x)^*\lambda, \gamma_l(x)^*\lambda},$$
$$\gamma(p_\lambda^x * g * p_\lambda) = \langle \gamma(g)\lambda, \gamma_l(x)^*\lambda \rangle P_{\gamma_l(x)^*\lambda, \lambda},$$

for every $g \in L^1(\mathcal{H})$. By ([**LeLu**], [**Lu**]) there exists $v_{\lambda,l}(x) \in L^1(\mathcal{H})$ such that $\gamma(v_{\lambda,l}(x)) = P_{\gamma_l(x)^*\lambda,\lambda}$ and the map $x \to v_{\lambda,l}(x)$ from $\mathcal{G}$ to $L^1(\mathcal{H})$ is continuous. Hence, for every $g \in L^1(\mathcal{H})/\ker\gamma$ and every $x \in \mathcal{G}$, there is a constant $c(x,g) = \langle \gamma(g)\lambda, \gamma_l(x)^*\lambda \rangle$ such that

$$p_\lambda^x * g * p_\lambda = c(x,g) v_{\lambda,l}(x) \mod \ker\gamma.$$

Moreover

$$v_{\lambda,l}(x) = p_\lambda^x * v_{\lambda,l}(x) * p_\lambda \mod \ker\gamma.$$

Let's write

$$p = p_\lambda \mod \ker\gamma, \quad v_l(x) = v_{\lambda,l}(x) \mod \ker\gamma$$

in the quotient space $L^1(\mathcal{H})/\ker\gamma$. Then the space

$$p^x * (L^1(\mathcal{H})/\ker\gamma) * p = (p_\lambda^x * L^1(\mathcal{H}) * p_\lambda)/\ker\gamma$$

is one dimensional for every fixed $x \in \mathcal{G}$ and it has $v_l(x)$ as a basis.

On the other hand, it is easy to check that

$$\gamma(p_\lambda^x * p_\lambda) = \langle \gamma_l(x)\lambda, \lambda\rangle P_{\gamma_l(x)^*\lambda,\lambda} = \langle \gamma_l(x)\lambda, \lambda\rangle \gamma(v_{\lambda,l}(x)),$$

i.e., that

$$p_\lambda^x * p_\lambda = \langle \gamma_l(x)\lambda, \lambda\rangle v_{\lambda,l}(x) \quad \mod \ker \gamma.$$

If we apply the representation $\gamma_{\bar{p}}$ instead of γ, the formulas are more complicated. As $\ker_{L^1(\mathcal{H})} \pi_{l,\mathfrak{p},\bar{p}} = \ker_{L^1(\mathcal{H})} \gamma_{l,\bar{p}} = \ker \gamma$ (by (3.3.)),

$$\begin{aligned}\gamma_{l,\bar{p}}(v_{\lambda,l}(x)) &= \frac{1}{\langle \gamma_l(x)\lambda, \lambda\rangle}\gamma_{l,\bar{p}}(p_\lambda^x * p_\lambda) \\ &= \frac{1}{\langle \gamma_l(x)\lambda, \lambda\rangle}\gamma_{l,\bar{p}}(x^{-1})\gamma_{l,\bar{p}}(p_\lambda)\gamma_{l,\bar{p}}(x)\gamma_{l,\bar{p}}(p_\lambda).\end{aligned}$$

In order to compute the exact value of $\gamma_{l,\bar{p}}(v_{\lambda,l}(x))$, we have to introduce a more precise decomposition of the Lie algebra $\mathfrak{g}$ (see (5.)).

Definition of $v(x)$**.** The previous definition of $v_l(x)$ is the one used in ([**Po2**]) and ([**LuMo2**]). It depends on the extension l of q we have chosen. If l and l' are two different extensions such that $l|_{\mathfrak{h}} = l'|_{\mathfrak{h}} = r$, then γ_l and $\gamma_{l'}$ differ only by the unitary character $\chi_{l-l'}$ on $\mathcal{U}$. Hence, if the corresponding functions are named $v_{\lambda,l}$ and $v_{\lambda,l'}$, then

$$v_{\lambda,l'}(x) = \chi_{l-l'}(x)v_{\lambda,l}(x) \mod \ker\gamma, \forall x \in \mathcal{U}.$$

Let $l_0 \in \mathfrak{g}^*$ be a fixed extension of r. We have

$$v_{\lambda,l_0}(x) = \chi_{l-l_0}(x)v_{\lambda,l}(x) \mod \ker\gamma$$

and we define $v(x)$ to be $v_{l_0}(x) = v_{\lambda,l_0}(x) \mod \ker\gamma$.

Let's put $\omega(x) = \|v(x)\|_{L^1(\mathcal{H})/\ker\gamma}$. By ([**Po2**], [**LuMo2**]) the function ω is a symmetric weight function on $\mathcal{G}$, which is constant on the classes modulo $\mathcal{H}$. Notice that ω is independent of the choice of the fixed linear form l_0 used to define v. Moreover, ω may be considered as a function on $\mathcal{G}(l)/\mathcal{G}(l) \cap \mathcal{N} = \mathcal{G}(l_0)/\mathcal{G}(l_0) \cap \mathcal{N}$.

Recall that p_λ acts on $L^1(\mathcal{G})$ and $p = p_\lambda \mod \ker\gamma$ acts on $L^1(\mathcal{G})/(L^1(\mathcal{G})* \ker\gamma)^{-L^1(\mathcal{G})}$ by convolution. Moreover $f \mod (L^1(\mathcal{G})*\ker\gamma)^{-L^1(\mathcal{G})} \mapsto \widetilde{f} \mod \ker\gamma$ is an isometric isomorphism between $L^1(\mathcal{G})/(L^1(\mathcal{G}) * \ker\gamma)^{-L^1(\mathcal{G})}$ and $L^1(\mathcal{U}, L^1(\mathcal{H})/\ker\gamma)$. As

$$(p_\lambda * f * p_\lambda)\widetilde{\ }(x) = p_\lambda^x *_{L^1(\mathcal{H})} \widetilde{f}(x) *_{L^1(\mathcal{H})} p_\lambda$$

for every $f \in L^1(\mathcal{G})$ and every $x \in \mathcal{G}$, we may consider a similar action on $L^1(\mathcal{U}, L^1(\mathcal{H})/\ker\gamma)$ by

$$\begin{aligned}(p * \widetilde{f} * p)(x) &= p^x *_{L^1(\mathcal{H})/\ker\gamma} \widetilde{f}(x) *_{L^1(\mathcal{H})/\ker\gamma} p \in p^x * (L^1(\mathcal{H})/\ker\gamma) * p \\ &= \mathbb{C}\cdot v(x)\end{aligned}$$

for every $\widetilde{f} \in L^1(\mathcal{U}, L^1(\mathcal{H})/\ker\gamma)$. As a matter of fact,

$$\begin{aligned}(p * \widetilde{f} * p)(x) &= \langle \gamma(\widetilde{f}(x))\lambda, \gamma_l(x)^*\lambda\rangle \overline{\chi_l(x)}\chi_{l_0}(x) \cdot v(x) \mod \ker\gamma \\ &= h(x) \cdot v(x) \mod \ker\gamma,\end{aligned}$$

if we define the function $h : \mathcal{U} \to \mathbb{C}$ by $h(x) = \langle \gamma(\widetilde{f}(x))\lambda, \gamma_l(x)^*\lambda\rangle \overline{\chi_l(x)}\chi_{l_0}(x)$. Of course the same argument is valid for every function $f \in L^1(\mathcal{G})$ and every $x \in \mathcal{G}$, if we define $\widetilde{f}(x) \in L^1(\mathcal{H})/\ker\gamma$ by $\widetilde{f}(x)(h) = f(xh) \bmod \ker\gamma$. As shown in ([**LuMo2**]), the map $\Lambda : p * \widetilde{f} * p = h \cdot v \mapsto h$ is an isometric isomorphism from $p * L^1(\mathcal{U}, L^1(\mathcal{H})/\ker\gamma) * p$ onto $L^1(\mathcal{U}, \omega)$.

Remarks.

a) Notice that the function h given by

$$h(x) = \langle \gamma(\widetilde{f}(x))\lambda, \gamma_l(x)^*\lambda\rangle \overline{\chi_l(x)}\chi_{l_0}(x) = \langle \gamma(\widetilde{f}(x))\lambda, \gamma_{l_0}(x)^*\lambda\rangle$$

is independent of the choice of l such that $l|_{\mathfrak{h}} = r$ is fixed.

b) For a given f in $L^1(\mathcal{G})/(L^1(\mathcal{G}) * \ker\gamma)^{-L^1(\mathcal{G})}$, the function h defined by the previous formulas may be considered as a function on all of $\mathcal{G}(l)$. It is then constant on the classes of $\mathcal{G}(l)$ modulo $\mathcal{G}(l) \cap \mathcal{N}$. Hence we may consider h as a function in $L^1(\mathcal{G}(l)/\mathcal{G}(l) \cap \mathcal{N}, \omega)$, where $\mathcal{G}(l)$ just depends on $l|_{\mathfrak{n}} = q$. In particular, h is independent of the choice of the supplementary space $\mathfrak{u}$ in $\mathfrak{g}(l)$.

c) If we take another $l_0 \in \mathfrak{g}^*$ having the same restriction to $\mathfrak{h}$ and another $v(x) = v_{\lambda,l_0}(x) \mod \ker\gamma$, then the h functions are all multiplied by the same unitary character χ such that $\chi|_{\mathcal{H}} \equiv 1$.

d) Let's take $\lambda, \mu \in \mathcal{ES}(\mathcal{H}/\mathcal{P}^1, \chi_r)$ such that $\langle\lambda,\lambda\rangle = \langle\mu,\mu\rangle = 1$. If $p_\lambda, p_\mu \in L^1(\mathcal{H})$ are such that $\gamma(p_\lambda) = P_{\lambda,\lambda}$ and $\gamma(p_\mu) = P_{\mu,\mu}$, then the algebras

$$(p_\lambda \mod \ker\gamma) * L^1(\mathcal{U}, L^1(\mathcal{H})/\ker\gamma) * (p_\lambda \mod \ker\gamma)$$

and

$$(p_\mu \mod \ker\gamma) * L^1(\mathcal{U}, L^1(\mathcal{H})/\ker\gamma) * (p_\mu \mod \ker\gamma)$$

are $*$-isomorphic. The resulting weights are equivalent. In fact, take $s_{\lambda,\mu} \in L^1(\mathcal{H})$ and $s = s_{\lambda,\mu} \mod \ker\gamma$ such that $\gamma(s_{\lambda,\mu}) = P_{\lambda,\mu}$. Then the map

$$\begin{aligned}\Phi : (p_\lambda \bmod \ker\gamma) * \widetilde{f} * (p_\lambda \bmod \ker\gamma)& \\ &\mapsto s^* * ((p_\lambda \bmod \ker\gamma) * \widetilde{f} * (p_\lambda \bmod \ker\gamma)) * s\end{aligned}$$

is the corresponding $*$-isomorphism. Moreover the different $\lambda, \mu \in \mathcal{ES}(\mathcal{H}/\mathcal{P}^1, \chi_r)$ together with the corresponding $*$-isomorphism lead to the same function h, for given functions f and $\widetilde{f}$. The algebra $L^1(\mathcal{G}(l)/\mathcal{G}(l) \cap \mathcal{N}, \omega)$ is hence independent of the choice of λ.

e) The algebra $L^1(\mathcal{U},\omega) \equiv L^1(\mathcal{G}(l)/\mathcal{G}(l)\cap\mathcal{N},\omega)$ is abelian (see [**Po2**], [**LuMo2**]).

3.6. Relation with the simple $L^1(\mathcal{G})$-module. Let's recall that if $(T,\mathfrak{U})$ is a simple $L^1(\mathcal{G})$-module, there is a unique orbit $G\cdot\tau\subset\hat{\mathcal{N}}, \tau\in\hat{\mathcal{N}}$, such that $\ker_{L^1(\mathcal{N})} T = \ker(\mathcal{G}\cdot\tau)$. Then we construct $\mathcal{H}\subset\mathcal{G}$ and $\gamma\in\hat{\mathcal{H}}$ as explained previously. To characterize completely $(T,\mathfrak{U})$ with a given $\ker T$, it is of course enough to study the algebraically irreducible representations of

$$L^1(\mathcal{G})/\left(L^1(\mathcal{G})*\ker_{L^1(\mathcal{N})} T\right)^{-L^1(\mathcal{G})} = L^1(\mathcal{G})/(L^1(\mathcal{G})*\ker\gamma)^{-L^1(\mathcal{G})} \simeq L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma),$$

as $\left(L^1(\mathcal{G})*\ker_{L^1(\mathcal{N})} T\right)^{-L^1(\mathcal{G})}\subset\ker T$. By ([**Po2**], Theorem 1) these are determined by the simple $(p*L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)*p)$-modules. But $\mathcal{B} = p*L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)*p \simeq L^1(\mathbb{R}^n,\omega)$ is abelian and its simple modules coincide with the characters of $L^1(\mathbb{R}^n,\omega)$. Hence, if we put $\mathcal{A} = L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)$ and if $(S,\mathfrak{U})$ is a simple $\mathcal{A}$-module, this means that the subspace $\mathfrak{V} = S(p)\mathfrak{U}$ is one-dimensional. So there exists a character χ on $L^1(\mathbb{R}^n,\omega)\simeq p*L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)*p$ such that for every $v\in\mathfrak{V}$ and $f\in\mathcal{B}$ we have $S(f)v=\chi(f)v$. Hence the maximal modular left ideal M of $\mathcal{A}$ consisting of all f in $\mathcal{A}$ for which $S(f)v=0, v\in\mathfrak{V}$, is given by $M=\{f\in\mathcal{A}\mid \chi(p*\mathcal{A}*f*p)\equiv 0\}$. The given simple $L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)$-module is then isomorphic to $(L,\mathcal{A}/M)$ where L is the left multiplication on $\mathcal{A}/M$.

On the other hand, for a given $(T,\mathfrak{U})$, let $q\in\mathfrak{n}^*$ be as in (3.2.). We want to show that $(T,\mathfrak{U})$ is equivalent to $\pi^0_{l,\mathfrak{p},\bar{p}}$ for some $l,\mathfrak{p},\bar{p}$ such that $l|_{\mathfrak{n}}=q$. But $\ker_{L^1(\mathcal{N})} T = \ker(\mathcal{G}\cdot\tau) = \ker_{L^1(\mathcal{N})}\pi_{l,\mathfrak{p},\bar{p}}$ for every $l\in\mathfrak{g}^*$ such that $l|_{\mathfrak{n}}=q$, for every multi-index $\bar{p}$, τ being given by $\tau=\mathrm{ind}^{\mathcal{N}}_{\mathcal{P}^0}\chi_q$. Hence the algebraically irreducible representations $\left(\pi_{l,\mathfrak{p},\bar{p}}|_{\mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}},\mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}\right)$ give rise to the same algebra $L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)$ as $(T,\mathfrak{U})$ does (if we make the same choices for $\mathcal{H},\mathcal{U},\mathfrak{p},\dots$). To show that $(T,\mathfrak{U})$ is equivalent to such a $\left(\pi_{l,\mathfrak{p},\bar{p}}|_{\mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}},\mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}\right)$ with $l|_{\mathfrak{n}}=q$ it is therefore enough to show that the corresponding characters on $L^1(\mathbb{R}^n,\omega)$ coincide for some $\bar{p}$. To do this we first have to study the weight ω.

Example 3.7. Let $\gamma_l\equiv\pi_{l,\mathfrak{p},2}\in\hat{G}$ such that $\gamma_l|_{\mathcal{H}}=\gamma$ and consider the simple module $(\gamma_l|_{\mathfrak{H}^0_\gamma},\mathfrak{H}^0_\gamma)$. Let's compute the character of $L^1(\mathbb{R}^n,\omega)\equiv p*L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)*p$ associated to $\gamma_l|_{\mathfrak{H}^0_\gamma}$. Recall that this is done by considering the action of $p*L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)*p$ on $\gamma(p)\mathfrak{H}^0_\gamma=\gamma(p_\lambda)\mathfrak{H}^0_\gamma$. Take $h\cdot v\in p*L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)*p$ corresponding to $h\in L^1(\mathbb{R}^n,\omega)$. Then one checks that

$$\gamma_l(h\cdot v)(\gamma(p_\lambda)\xi)=\int_{\mathcal{U}}\int_{\mathcal{H}}h(t)v(t)(s)\gamma_l(t)\gamma(s)\gamma(p_\lambda)\xi ds dt=\hat{h}(l-l_0)\gamma(p_\lambda)\xi.$$

Hence the character of $L^1(\mathbb{R}^n,\omega)\equiv L^1(\mathcal{U},\omega)$ corresponding to $\gamma_l\equiv\pi_{l,\mathfrak{p},2}$ is χ_{l-l_0}. Similarly, we may compute $\gamma_{l,\bar{p}}(h\cdot v)$:

$$\gamma_{l,\bar{p}}(h\cdot v)(\gamma_{\bar{p}}(p_\lambda)\xi)=\int_{\mathcal{U}}h(t)\chi_{l-l_0}(t)\frac{1}{\langle\gamma_l(t)\lambda,\lambda\rangle}\gamma_{\bar{p}}(p_\lambda)\gamma_{l,\bar{p}}(t)\gamma_{\bar{p}}(p_\lambda)\xi dt,$$

by (3.5.). In order to conclude, we need to know $\gamma_{\bar{p}}(p_\lambda)$. This computation requires a more precise decomposition of the Lie algebra $\mathfrak{g}$ and will be done in (5.6.1.). We shall see that the character corresponding to $\gamma_{l,\bar{p}}\equiv\pi_{l,\mathfrak{p},\bar{p}}$ is $\chi_{l,\bar{p}}=\Delta^{\frac{1}{2}-\frac{1}{\bar{p}}}\cdot\chi_{l-l_0}$.

4. Characters of $L^1(\mathbb{R}^n,\omega)$.

4.1. . Let's fix $x=\exp X\in\mathcal{U}\subset\mathcal{G}(l)$ and let's study the growth of $\omega(\exp tX)$ for $t\in\mathbb{R}$ and X fixed. Take $\lambda,p_\lambda,v_\lambda,v$ as in (3.3.) and (3.5.). Recall that

$$\omega(\exp tX)=\|v(\exp tX)\|_{L^1(\mathcal{H})/\ker\gamma}$$

where $v(\exp tX)=v_{\lambda,l_0}(\exp tX)\bmod\ker\gamma$. Moreover let's choose for λ the Gaussian function. This is possible because different choices of λ give equivalent weights. Put $\sigma(g)=e^{\frac{1}{2}\sum_{j=1}^m|\operatorname{tr}\lambda_j(\log g)|}$ for $g\in\mathcal{G}(l)$, where $\lambda_j(\cdot)=\operatorname{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\cdot)$. Using a method developed by Poguntke ([**Po2**]), one checks that there are constants C and C' (depending on the choice of X but not on t) such that

$$\omega(\exp tX)\le C'\cdot(1+|t|)^C\cdot e^{\frac{|t|}{2}\sum_{j=1}^m|\operatorname{tr}\lambda_j(X)|}=C'\cdot(1+|t|)^C\cdot\sigma(\exp tX).$$

Proposition 4.2. *Let χ be a continuous character on $L^1(\mathcal{G}(l)/\mathcal{G}(l)\cap\mathcal{N},\omega)$ $\equiv L^1(\mathcal{U},\omega)\equiv L^1(\mathbb{R}^n,\omega)$. Then*

$$|\chi(\exp X)|\le\prod_{i=1}^m e^{\frac{1}{2}|\operatorname{tr}\lambda_i(X)|}=\sigma(\exp X)$$

for all $X\in\mathfrak{u}\equiv\mathfrak{g}(l)/\mathfrak{g}(l)\cap\mathfrak{n}$.

Proof. As χ is a continuous character on $L^1(\mathcal{U},\omega)$, $|\chi(\exp X)|\le\omega(\exp X)$, $X\in\mathcal{U}$. Let's write $\chi(\exp X)=e^{\rho(X)}$ with ρ a complex linear form on $\mathcal{U}\equiv\mathbb{R}^n$. Then $|\chi(\exp X)|=e^{\mathfrak{Re}\rho(X)}$, $\forall X\in\mathcal{U}$. Assume that there is $X_0\in\mathcal{U}$ such that $\mathfrak{Re}\rho(X_0)>0$ (otherwise, change X_0 to $-X_0$) and such that

$$|\chi(\exp X_0)|=e^{\mathfrak{Re}\rho(X_0)}=e^{|\mathfrak{Re}\rho(X_0)|}>\prod_{i=1}^m e^{\frac{1}{2}|\operatorname{tr}\lambda_i(X_0)|}.$$

Hence

$$\prod_{i=1}^{m} e^{\frac{|t|}{2}|\mathrm{tr}\,\lambda_i(X_0)|} < |\chi(\exp|t|X_0)|$$
$$\leq \omega(\exp|t|X_0) \leq C'(1+|t|)^C \prod_{i=1}^{m} e^{\frac{|t|}{2}|\mathrm{tr}\,\lambda_i(X_0)|}$$

and

$$1 < e^{|t|(|\mathfrak{Re}\rho(X_0)| - \frac{1}{2}\sum_{i=1}^{m}|\mathrm{tr}\,\lambda_i(X_0)|)} \leq C'(1+|t|)^C$$

for all $t \in \mathbb{R}^*$. As this is impossible, we have that

$$|\chi(\exp X)| \leq \prod_{i=1}^{m} e^{\frac{1}{2}|\mathrm{tr}\,\lambda_i(X)|} = \sigma(\exp X),$$

for all $X \in \mathcal{U}$. □

5. Characterization of all the simple modules.

We proceed now as written in (3.6.).

5.1. Identification of $\mathfrak{H}_{\gamma_{\bar{p}}} = L^{\bar{p}}(\mathcal{H}/\mathcal{P}^1, \chi_r)$ and $L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X})$.

a) We use the decompositions and notations introduced in (2.1.) to (2.5.) and in (3.3.). Recall in particular that $\mathfrak{h} = \mathfrak{v} \oplus \mathfrak{n} \oplus \mathfrak{w} = \mathfrak{v} \oplus \mathfrak{n} \oplus \mathfrak{Y} \oplus \mathfrak{X}$. Let's define $\mathfrak{k} = \mathfrak{v} \oplus \mathfrak{n}$. Hence $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{Y} \oplus \mathfrak{X}$. In order to get an isometry between $\mathfrak{H}_{\gamma_{\bar{p}}} = L^{\bar{p}}(\mathcal{H}/\mathcal{P}^1, \chi_r)$ and $L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X})$, let's define

$$\widetilde{\xi}(k, x) = \xi(k \cdot x), \quad \forall k \in \mathcal{K}, \forall x \in \mathcal{X}, \forall \xi \in \mathfrak{H}_{\gamma_{\bar{p}}},$$

and

$$(S_{\bar{p}}\xi)(k, x) = e^{-\sum_{j=1}^{m}\frac{1}{p_j}\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log x)}\widetilde{\xi}(k, x)$$
$$= e^{-\sum_{j=1}^{m}\frac{1}{p_j}\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log x)}\xi(k \cdot x).$$

Let's write $\delta^{-\frac{1}{\bar{p}}}(x) = e^{-\sum_{j=1}^{m}\frac{1}{p_j}\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log x)}$, $\forall x \in \mathcal{X}$. Then it is easy to see that the map $S_{\bar{p}} : \xi \mapsto \delta^{-\frac{1}{\bar{p}}} \cdot \widetilde{\xi}$ is an isometry between $L^{\bar{p}}(\mathcal{H}/\mathcal{P}^1, \chi_r)$ and $L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X})$ if the norm on $L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X})$ is given by

$$\|\widetilde{f}\| = \left(\int_{\mathcal{X}} \|\widetilde{f}(\cdot, x)\|_{\bar{p}}^2 dx\right)^{\frac{1}{2}}$$
$$= \left(\int_{\mathcal{X}}\int_{\mathcal{V}}\left(\left(\int_{\mathcal{V}_1}\left(\dots\left(\int_{\mathcal{V}_m}|\widetilde{f}(vv_1\dots v_m, x)|^{p_m}dv_m\right)^{\frac{1}{p_m}}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\dots\right)^{p_1}dv_1\right)^{\frac{1}{p_1}}\right)^2 dvdx\right)^{\frac{1}{2}}.$$

In particular, for $\xi \in \mathfrak{H}_{\gamma_{\bar{p}}}$,

$$
\begin{aligned}
(S_{\bar{p}}\xi)(k \cdot p_0, x) &= \delta^{-\frac{1}{\bar{p}}}(x)\xi(k \cdot p_0 \cdot x) \\
&= \delta^{-\frac{1}{\bar{p}}}(x)\overline{\chi_r(x^{-1}p_0x)}\xi(k \cdot x) = \overline{\chi_r(p_0)}(S_{\bar{p}}\xi)(k, x),
\end{aligned}
$$

because $x \in \mathcal{X} \subset \mathcal{G}(q), p_0 \in \mathcal{P}^0$, $\mathcal{P}^0$ is $\mathcal{G}(q)$-invariant and $\langle r, [\log x, \log p_0]\rangle = 0$.

b) Notice that $L^2(\mathcal{X})$ may be identified with $L^2(\mathfrak{X})$. In fact, if $x, x' \in \mathcal{X} \subset \mathcal{G}(q)$, then

$$x \cdot x' = q(x, x') \cdot \exp(\log x + \log x')$$

with $q(x, x') \in \mathcal{N}(q) \subset \mathcal{P}^0$ and $\chi_r(q(x, x')) = 1$. Hence

$$\xi(k \cdot x \cdot x') = \xi(k \cdot q(x, x') \cdot \exp(\log x + \log x')) = \widetilde{\xi}(k, \exp(\log x + \log x'))$$

and we may identify $L^2(\mathcal{X})$ with $L^2(\mathfrak{X})$ where $\mathcal{X} = \exp \mathfrak{X}$ as before. Moreover

$$(S_{\bar{p}}\xi)(k, x \cdot x') = e^{-\sum_{j=1}^{m} \frac{1}{p_j} \mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log x + \log x')} \cdot \widetilde{\xi}(k, \exp(\log x + \log x'))$$

and we may consider $S_{\bar{p}}\xi$ as a function on $(\mathcal{K}/\mathcal{P}^0) \times \mathfrak{X}$. Similarly we shall consider

$$\mathcal{E}S(\mathcal{X}) \equiv \mathcal{E}S(\mathfrak{X}) \subset L^2(\mathcal{X}) \equiv L^2(\mathfrak{X}),$$

an $\mathcal{E}S$-space with decay conditions as in (2.7.1.).

5.2. Equivalent representations. Let $\gamma_{\bar{p}}$ be the representation defined on $\mathfrak{H}_{\gamma_{\bar{p}}} = L^{\bar{p}}(\mathcal{H}/\mathcal{P}^1, \chi_r)$. If we define $\kappa_{\bar{p}}$ on $L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X})$ by

$$(\kappa_{\bar{p}}(h)(S_{\bar{p}}\xi))(k, x) = S_{\bar{p}}(\gamma_{\bar{p}}(h)\xi)(k, x) = \delta^{-\frac{1}{\bar{p}}}(x)\Delta^{-\frac{1}{\bar{p}}}(h)\xi(h^{-1}kx)$$

$\forall h \in \mathcal{H}$, then the representations $(\kappa_{\bar{p}}, L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X}))$ and $(\gamma_{\bar{p}}, L^{\bar{p}}(\mathcal{H}/\mathcal{P}^1, \chi_r))$ are equivalent. Similarly, we define $\kappa_{l,\bar{p}}$ on $L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X})$ by

$$
\begin{aligned}
(\kappa_{l,\bar{p}}(t)(S_{\bar{p}}\xi))(k, x) &= S_{\bar{p}}(\gamma_{l,\bar{p}}(t)\xi)(k, x) \\
&= \delta^{-\frac{1}{\bar{p}}}(x)\Delta^{-\frac{1}{\bar{p}}}(t)\chi_l(t)\xi(k^{t^{-1}} \cdot (t^{-1}xtx^{-1})x)
\end{aligned}
$$

for $t \in \mathcal{U} \subset \mathcal{G}(l)$ and $\kappa_{l,\bar{p}}(h) = \kappa_{\bar{p}}(h)$ for $h \in \mathcal{H}$. As $t^{-1}xtx^{-1} \in \mathcal{G}(q) \cap \mathcal{N} \subset \mathcal{P}^0$ and as $\chi_r(t^{-1}xtx^{-1}) = 1$, we have that

$$
\begin{aligned}
(\kappa_{l,\bar{p}}(t)(S_{\bar{p}}\xi))(k, x) &= \delta^{-\frac{1}{\bar{p}}}(x)\Delta^{-\frac{1}{\bar{p}}}(t)\chi_l(t)\widetilde{\xi}(k^{t^{-1}}, x) \\
&= \Delta^{-\frac{1}{\bar{p}}}(t)\chi_l(t)(S_{\bar{p}}\xi)(k^{t^{-1}}, x),
\end{aligned}
$$

i.e., $t \in \mathcal{G}(l)$ acts only on $\mathcal{K}$. The representations $(\kappa_{l,\bar{p}}, L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X}))$ and $(\gamma_{l,\bar{p}}, L^{\bar{p}}(\mathcal{H}/\mathcal{P}^1, \chi_r))$ are equivalent by construction.

5.3. Kernel of $\kappa_{\overline{p}}(f)$. The kernel of the operator $\kappa_{\overline{p}}(f)$, $f \in \mathcal{E}S(\mathcal{H}) \subset L^1(\mathcal{H})$, is given by the following computations:

$$
\begin{aligned}
&(\kappa_{\overline{p}}(f)(S_{\overline{p}}\xi))(k', x') \\
&= \delta^{-\frac{1}{\overline{p}}}(x') \int_{\mathcal{Y}} \int_{\mathcal{X}} \int_{\mathcal{K}} f((xy)k) \Delta^{-\frac{1}{\overline{p}}}(xyk) \xi(k^{-1}(xy)^{-1}k'x') dk dx dy \\
&= \delta^{-\frac{1}{\overline{p}}}(x') \int_{\mathcal{Y}} \int_{\mathcal{X}} \int_{\mathcal{K}} f((xy)((k')^{(xy)^{-1}})k^{-1}) \cdot \Delta^{-\frac{1}{\overline{p}}}(k'(xy)k^{-1}) \\
&\qquad \cdot \Delta_{\mathcal{K}}(k)^{-1} \xi(k(xy)^{-1}x') dk dx dy \\
&= \int_{\mathcal{Y}} \int_{\mathcal{X}} \int_{\mathcal{K}} f((xy)((k')^{(xy)^{-1}})k^{-1}) \Delta^{-\frac{1}{\overline{p}}}(k'(xy)k^{-1}) \\
&\qquad \cdot \Delta_{\mathcal{K}}(k)^{-1} \cdot e^{-i\langle r, \log(y)\rangle} e^{-i\langle r, [\log(y), \log(x^{-1}x')]\rangle} \\
&\qquad \cdot \delta^{-\frac{1}{\overline{p}}}(x)(S_{\overline{p}}\xi)(k, x^{-1}x') dk dx dy \\
&= \int_{\mathcal{Y}} \int_{\mathcal{X}} \int_{\mathcal{K}} f((x'x^{-1}y)((k')^{(x'x^{-1}y)^{-1}})k^{-1}) \Delta^{-\frac{1}{\overline{p}}}(k'(x'x^{-1}y)k^{-1}) \\
&\qquad \cdot \Delta_{\mathcal{K}}(k)^{-1} \cdot e^{-i\langle r, \log(y)\rangle} e^{-i\langle r, [\log(y), \log(x)]\rangle} \\
&\qquad \cdot \delta^{-\frac{1}{\overline{p}}}(x'x^{-1})(S_{\overline{p}}\xi)(k, x) dk dx dy \\
&= \int_{\mathcal{Y}} \int_{\mathcal{X}} \int_{\mathcal{K}/\mathcal{P}^0} \int_{\mathcal{P}^0} f(((x'x^{-1})y)((k')^{((x'x^{-1})y)^{-1}}) \cdot p_0 k^{-1}) \\
&\qquad \cdot \Delta_{\mathcal{K}}(k)^{-1} \cdot \Delta^{-\frac{1}{\overline{p}}}(y) \cdot e^{-i\langle r, \log(p_0)\rangle} \cdot e^{-i\langle r, \log(y)\rangle} \\
&\qquad \cdot e^{-i\langle r, [\log(y), \log(x)]\rangle} \delta^{-\frac{1}{\overline{p}}}(x'x^{-1})(S_{\overline{p}}\xi)(k, x) dp_0 d\dot{k} dx dy,
\end{aligned}
$$

as $\Delta \equiv 1$ on $\mathcal{K}$. Consider $\rho_{\overline{p}} = \mathrm{ind}_{\mathcal{P}^0}^{\mathcal{K}}(\chi_q, \overline{p})$ and $\rho_{\overline{2}} = \mathrm{ind}_{\mathcal{P}^0}^{\mathcal{K}}(\chi_q, 2)$. Let's write $f(x, y)(k) = f(x \cdot y \cdot k)$, $k \in \mathcal{K}$. Then the kernel of $\kappa_{\overline{p}}(f)$ may be written

$$
\begin{aligned}
&(f_{\kappa_{\overline{p}}})((k', x'), (k, x)) \\
&= \int_{\mathcal{Y}} f((x'x^{-1}), y)_{\rho_{\overline{p}}}((k')^{((x'x^{-1})\cdot y)^{-1}}, k) \\
&\qquad \cdot \Delta^{-\frac{1}{\overline{p}}}(y) \cdot e^{-i\langle r, \log(y)\rangle} \cdot e^{-i\langle r, [\log(y), \log(x)]\rangle} \delta^{-\frac{1}{\overline{p}}}(x'x^{-1}) dy,
\end{aligned}
$$

where the kernel of $\rho_{\overline{p}}(g), g \in L^1(\mathcal{K})$, is given by

$$
(\rho_{\overline{p}}(g))(k', k) = \int_{\mathcal{P}^0} g(k' p_0 k^{-1}) \cdot e^{-i\langle r, \log p_0\rangle} \Delta_{\mathcal{K}}(k)^{-1} dp_0.
$$

In particular, for $g \in \mathcal{E}S(\mathcal{K})$, $g_{\rho_{\overline{p}}}(k', k) = g_{\rho_{\overline{2}}}(k', k)$ for every multi-index $\overline{p}$. Hence the representations $\rho_{\overline{p}}$ and $\rho_{\overline{2}}$ are given by the same formulas (but act on different spaces).

5.4. Behavior of projectors. For $\bar{p} = (p_1, \ldots, p_m)$ we define $\bar{q} = (q_1, \ldots, q_m)$ such that $\frac{1}{\bar{q}} = 1 - \frac{1}{\bar{p}}$, which means that $\frac{1}{q_i} = 1 - \frac{1}{p_i}$ for each i. Take $\alpha, \beta \in \mathcal{ES}(\mathcal{H}/\mathcal{P}^1, \chi_r) \subset L^{\bar{p}}(\mathcal{H}/\mathcal{P}^1, \chi_r) \cap L^{\bar{q}}(\mathcal{H}/\mathcal{P}^1, \chi_r)$. One checks that $\langle \alpha, \beta \rangle = \langle S_{\bar{p}}\alpha, S_{\bar{q}}\beta \rangle$. Hence, if $\gamma_{\bar{p}}(f)$ is the projector $P_{\alpha,\beta}$, then $\kappa_{\bar{p}}(f)$ is the projector $P_{S_{\bar{p}}\alpha, S_{\bar{q}}\beta}$. Conversely, every projector $\kappa_{\bar{p}}(f) = P_{\alpha',\beta'}$, with $\alpha', \beta' \in \mathcal{ES}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}\mathcal{ES}(\mathcal{X})$, comes from a projector $\gamma_{\bar{p}}(f) = P_{\alpha,\beta}$ where $\alpha(vxv_1 \ldots v_m) = \delta^{-\frac{1}{\bar{p}}}(x^{-1})\alpha'(v \cdot (xv_1x^{-1}) \ldots (xv_mx^{-1}), x)$. Similarly for β and β'.

5.5. Choice of a particular p_λ. Let μ be an arbitrary function in $\mathcal{ES}(\mathcal{K}/\mathcal{P}^0, \chi_r)$ such that $\langle \mu, \mu \rangle = 1$. For every $s \in \mathcal{G}$ there is a function $g(s) \in \mathcal{ES}(\mathcal{K})$ such that the kernel of $\rho_{\bar{2}}(g(s))$ is given by $g(s)_{\rho_{\bar{2}}}(k', k) = \mu((k')^s)\overline{\mu(k)}$. Moreover, as $\Delta|_{\mathcal{P}^0} \equiv 1$, the kernels $g(s)_{\rho_{\bar{p}}}$ and $g(s)_{\rho_{\bar{2}}}$ coincide. Let's choose a real-valued analytic function $\nu \in \mathcal{ES}(\mathcal{X}) \equiv \mathcal{ES}(\mathfrak{X}) \subset L^2(\mathcal{X})$ such that $\langle \nu, \nu \rangle = 1$. Put

$$\alpha(x) = e^{\frac{1}{2}\sum_{j=1}^m \operatorname{tr} \operatorname{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log x)}\nu(x) = \delta^{\frac{1}{2}}(x)\nu(x)$$

and define $\lambda \in \mathcal{ES}(\mathcal{H}/\mathcal{P}^1, \chi_r)$ by the formulas

$$\begin{aligned}\lambda(vxv_1 \ldots v_m) &= \widetilde{\lambda}(v(xv_1x^{-1}) \ldots (xv_mx^{-1}), x) \\ &= \mu(v(xv_1x^{-1}) \ldots (xv_mx^{-1})) \cdot \alpha(x).\end{aligned}$$

One checks that

$$\langle \lambda, \lambda \rangle = \langle \mu, \mu \rangle \langle \nu, \nu \rangle = 1 = \langle S_{\bar{p}}\lambda, S_{\bar{q}}\lambda \rangle = \langle S_2\lambda, S_2\lambda \rangle,$$

where the last equalities are due to (5.4.). Moreover, for $k = vv_1 \ldots v_m \in \mathcal{K}$,

$$\begin{aligned}&(S_{\bar{2}}\lambda)(k, x) = \mu(k)\nu(x), \\ &(S_{\bar{p}}\lambda)(k, x) = e^{\sum_{j=1}^m (\frac{1}{2} - \frac{1}{p_j})\operatorname{tr} \operatorname{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log x)}\mu(k)\nu(x) = \delta^{\frac{1}{2} - \frac{1}{\bar{p}}}(x)\mu(k)\nu(x).\end{aligned}$$

In order to construct the function $p_\lambda \in L^1(\mathcal{H})$ that will give us the projectors associated to λ, let's put

$$a(x, y) = \int_{\mathcal{X}} \alpha(xu)\alpha(u)e^{i\langle r, [\log y, \log u]\rangle} \cdot e^{i\langle r, \log y\rangle} \cdot \Delta^{\frac{1}{2}}(y)du, \quad \forall x \in \mathcal{X}, \forall y \in \mathcal{Y},$$

and define $p_\lambda \in \mathcal{ES}(\mathcal{H}) \subset L^1(\mathcal{H})$ by $p_\lambda(xyk) = p_\lambda(x, y)(k) = a(x, y)g(xy)(k)$.

Proposition 5.5.1. *For every $\bar{p} \in [1, \infty]^m$, the operator $\gamma_{\bar{p}}(p_\lambda)$ is a rank one operator. This is in particular true for the operator $\gamma(p_\lambda) = \gamma_{\bar{2}}(p_\lambda)$.*

Proof. Using the previous computations and the observation (5.1.b) we get

$$\begin{aligned}
&p_\lambda(x'x^{-1}, y)_{\rho_{\overline{p}}}((k')^{(x'x^{-1}y)^{-1}}, k)\\
&= a(x'x^{-1}, y)\mu(k')\overline{\mu(k)}\\
&= \mu(k')\overline{\mu(k)} \int_{\mathcal{X}} \alpha(x'x^{-1}u)\alpha(u)\\
&\qquad \cdot e^{i\langle r,[\log y, \log u]\rangle} e^{i\langle r, \log y\rangle} \Delta^{\frac{1}{2}}(y)\delta^{-1}(u)du,
\end{aligned}$$

where $\delta^{-1}(u) = e^{-\sum_{j=1}^m \operatorname{tr} \operatorname{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log u)}$. Hence

$$\begin{aligned}
&(p_\lambda)_{\kappa_{\overline{p}}}((k', x'), (k, x))\\
&= \mu(k')\overline{\mu(k)} \int_{\mathcal{Y}}\int_{\mathcal{X}} \alpha(x'x^{-1}u)\alpha(u)\\
&\qquad \cdot e^{i\langle r,[\log y, \log u]\rangle} e^{-i\langle r,[\log y, \log x]\rangle} \Delta^{\frac{1}{2}-\frac{1}{\overline{p}}}(y)\delta^{-1}(u)\delta^{-\frac{1}{\overline{p}}}(x'x^{-1})du,\\
&= \mu(k')\overline{\mu(k)} \int_{\mathcal{Y}}\int_{\mathcal{X}} \nu(x'x^{-1}u)\nu(u)\\
&\qquad \cdot e^{i\langle r,[\log y, \log u]\rangle} e^{-i\langle r,[\log y, \log x]\rangle} \Delta^{\frac{1}{2}-\frac{1}{\overline{p}}}(y)\delta^{\frac{1}{2}-\frac{1}{\overline{p}}}(x'x^{-1})du.
\end{aligned}$$

In particular, for $\overline{p} = \overline{2}$ and $\kappa = \kappa_{\overline{2}}$, we have

$$\begin{aligned}
(p_\lambda)_\kappa((k', x'), (k, x)) &= \mu(k')\overline{\mu(k)} \int_{\mathcal{Y}}\int_{\mathcal{X}} \nu(x'x^{-1}u)\nu(u)\\
&\qquad \cdot e^{i\langle r,[\log y, \log u]\rangle} e^{-i\langle r,[\log y, \log x]\rangle} dudy\\
&= \mu(k')\overline{\mu(k)}\nu(x')\overline{\nu(x)}\\
&= (S_{\overline{2}}\lambda)(k', x')\overline{(S_{\overline{2}}\lambda)(k, x)},
\end{aligned}$$

as ν is in fact a real-valued function. Hence $\kappa(p_\lambda)$ is a projector, i.e.,

$$\kappa(p_\lambda) = P_{S_{\overline{2}}\lambda, S_{\overline{2}}\lambda} \text{ and } \gamma(p_\lambda) = P_{\lambda,\lambda}.$$

In order to characterize the kernel of $\kappa_{\overline{p}}(p_\lambda)$, let's recall that

$$\Delta^{(\frac{1}{2}-\frac{1}{\overline{p}})}(y) = e^{\sum_{j=1}^m \left(\frac{1}{2}-\frac{1}{p_j}\right)\operatorname{tr}\lambda_j(\log y)}$$

where $\sum_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{2}\right) \operatorname{tr}\lambda_j(\cdot)$ is a linear form on $\mathfrak{Y}$ and may hence be identified with an element of $\mathfrak{X}$, because of the duality between $\mathfrak{X}$ and $\mathfrak{Y}$ (see (2.2.)). Let's write $\frac{1}{\overline{p}} - \frac{1}{2}$ for this element of $\mathfrak{X}$, i.e.,

$$\left\langle r, \left[\frac{1}{\overline{p}} - \frac{1}{2}, \log y\right]\right\rangle = \sum_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{2}\right) \operatorname{tr}\lambda_j(\log y), \quad \forall y \in \mathcal{Y},$$

by definition. The function ν has been chosen analytic in $\mathcal{E}S(\mathcal{X}) \equiv \mathcal{E}S(\mathfrak{X})$. Hence it admits an extension to a complex-valued analytic function on $\mathfrak{X}_{\mathbb{C}}$ which we shall also denote by ν. We compute

$$\begin{aligned}
&(p_\lambda)_{\kappa_{\bar{p}}}((k',x'),(k,x)) \\
&= \mu(k')\overline{\mu(k)}\delta^{\frac{1}{2}-\frac{1}{\bar{p}}}(x'x^{-1})\int_{\mathcal{Y}}\int_{\mathcal{X}}\nu((x'x^{-1})\cdot u)\nu(u) \\
&\qquad \cdot e^{i\langle r,[\log y,\log u]\rangle}e^{-i\langle r,[\log y,\log x-i(\frac{1}{\bar{p}}-\frac{1}{2})]\rangle}du dy \\
&= \mu(k')\overline{\mu(k)}\delta^{-\frac{1}{\bar{p}}}(x')\delta^{\frac{1}{2}}(x')\nu\left(\log x'-i\left(\frac{1}{\bar{p}}-\frac{1}{2}\right)\right) \\
&\qquad \cdot \delta^{-\frac{1}{\bar{q}}}(x)\delta^{\frac{1}{2}}(x)\nu\left(\log x-i\left(\frac{1}{\bar{p}}-\frac{1}{2}\right)\right),
\end{aligned}$$

if we identify ν with a function on the complex vector space $\mathfrak{X}_{\mathbb{C}}$ and if $\frac{1}{\bar{q}}=1-\frac{1}{\bar{p}}$.

Let's define new functions $\nu_{\bar{p}} \in \mathcal{E}S(\mathcal{X}) \equiv \mathcal{E}S(\mathfrak{X})$, $\zeta_1,\zeta_2 \in L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X})$ and $\lambda_{\bar{p}},\lambda'_{\bar{p}} \in \mathcal{E}S(\mathcal{H}) \subset L^1(\mathcal{H})$ by

$$\nu_{\bar{p}}(x) = \delta^{\frac{1}{2}}(x)\nu\left(\log x - i\left(\frac{1}{\bar{p}}-\frac{1}{2}\right)\right), \quad \forall x \in \mathcal{X},$$

$$\begin{aligned}
(S_{\bar{p}}\lambda_{\bar{p}})(k,x) &= \delta^{-\frac{1}{\bar{p}}}(x)\mu(k)\nu_{\bar{p}}(x) = \zeta_1(k,x) \\
(S_{\bar{q}}\lambda'_{\bar{p}})(k,x) &= \delta^{-\frac{1}{\bar{q}}}(x)\mu(k)\overline{\nu_{\bar{p}}(x)} = \zeta_2(k,x),
\end{aligned}$$

i.e., $\lambda_{\bar{p}} = S_{\bar{p}}^{-1}(\zeta_1)$ and $\lambda'_{\bar{p}} = S_{\bar{q}}^{-1}(\zeta_2)$. Hence

$$(p_\lambda)_{\kappa_{\bar{p}}}((k',x'),(k,x)) = (S_{\bar{p}}\lambda_{\bar{p}})(k',x')\overline{(S_{\bar{q}}\lambda'_{\bar{p}})(k,x)},$$

i.e., $\kappa_{\bar{p}}(p_\lambda)$ is the projector $P_{S_{\bar{p}}\lambda_{\bar{p}},S_{\bar{q}}\lambda'_{\bar{p}}}$. So $\gamma_{\bar{p}}(p_\lambda)$ is also a projector, more precisely $\gamma_{\bar{p}}(p_\lambda) = P_{\lambda_{\bar{p}},\lambda'_{\bar{p}}}$. Both projectors are idempotent because

$$\begin{aligned}
\langle\lambda_{\bar{p}},\lambda'_{\bar{p}}\rangle &= \langle S_{\bar{p}}\lambda_{\bar{p}}, S_{\bar{q}}\lambda'_{\bar{p}}\rangle \\
&= \int_{\mathcal{X}}\int_{\mathcal{K}}\delta^{-\frac{1}{\bar{p}}}(x)\mu(k)\nu_{\bar{p}}(x)\delta^{-\frac{1}{\bar{q}}}(x)\overline{\mu(k)}\nu_{\bar{p}}(x)dkdx \\
&= \langle\mu,\mu\rangle\cdot\int_{\mathcal{X}}\delta^{-1}(x)(\nu_{\bar{p}}(x))^2dx \\
&= \int_{\mathcal{X}}\left(\nu\left(\log x - i\left(\frac{1}{\bar{p}}-\frac{1}{2}\right)\right)\right)^2 dx = 1,
\end{aligned}$$

as $\frac{1}{\bar{p}}+\frac{1}{\bar{q}}=1$, $\langle\mu,\mu\rangle=1$ and

$$\int_{\mathcal{X}}\left(\nu\left(\log x - i\left(\frac{1}{\bar{p}}-\frac{1}{2}\right)\right)\right)^2 dx = \int_{\mathcal{X}}(\nu(x))^2dx = \langle\nu,\nu\rangle = 1$$

by Cauchy's theorem. □

5.5.2. . The following computation will be important in the characterization of the character associated to $\gamma_{l,\bar{p}} \equiv \pi_{l,\mathfrak{p},\bar{p}}$:

$$\begin{aligned}
&\langle \gamma_{l,\bar{p}}(t)\lambda_{\bar{p}}, \lambda'_{\bar{p}}\rangle \\
&= \langle \kappa_{l,\bar{p}}(t)(S_{\bar{p}}\lambda_{\bar{p}}), S_{\bar{q}}\lambda'_{\bar{p}}\rangle \\
&= \Delta^{-\frac{1}{\bar{p}}}(t)\chi_l(t)\int_{\mathcal{X}}\int_{\mathcal{K}}(S_{\bar{p}}\lambda_{\bar{p}})(k^{t^{-1}},x)\overline{(S_{\bar{q}}\lambda'_{\bar{p}})(k,x)}dkdx \\
&= \Delta^{-\frac{1}{\bar{p}}}(t)\chi_l(t)\int_{\mathcal{X}}\int_{\mathcal{K}}\delta^{-1}(x)\mu(k^{t^{-1}})\nu_{\bar{p}}(x)\overline{\mu(k)}\nu_{\bar{p}}(x)dkdx \\
&= \Delta^{-\frac{1}{\bar{p}}}(t)\chi_l(t)\langle \mu^{t^{-1}}, \mu\rangle.
\end{aligned}$$

In particular, for $\bar{p} = \overline{2} = (2,\ldots,2)$, $\lambda_{\overline{2}} = \lambda'_{\overline{2}} = \lambda$ (as ν is real-valued) and

$$\frac{\langle \widetilde{\gamma}_{l,\bar{p}}(t)\lambda_{\bar{p}}, \lambda'_{\bar{p}}\rangle}{\langle \widetilde{\gamma}_l(t)\lambda, \lambda\rangle} = \Delta^{\frac{1}{2}-\frac{1}{\bar{p}}}(t).$$

5.6. Character of $L^1(\mathbb{R}^n,\omega)$ corresponding to $\gamma_{l,\bar{p}}$.

5.6.1. . Using the computations of (3.5.) and (3.7.) we get

$$\begin{aligned}
\gamma_{l,\bar{p}}(v_{\lambda,l}(t)) &= \frac{1}{\langle \gamma_l(t)\lambda, \lambda\rangle}\gamma_{l,\bar{p}}(t^{-1})P_{\lambda_{\bar{p}},\lambda'_{\bar{p}}}\gamma_{l,\bar{p}}(t)P_{\lambda_{\bar{p}}\lambda'_{\bar{p}}} \\
&= \Delta^{\frac{1}{2}-\frac{1}{\bar{p}}}(t)\gamma_{l,\bar{p}}(t^{-1})\gamma_{\bar{p}}(p_\lambda)
\end{aligned}$$

and

$$(\gamma_{l,\bar{p}}(h\cdot v))(\gamma_{\bar{p}}(p_\lambda)\xi) = \left(\int_{\mathcal{U}} h(t)\chi_{l-l_0}(t)\Delta^{\frac{1}{2}-\frac{1}{\bar{p}}}(t)dt\right)(\gamma_{\bar{p}}(p_\lambda)\xi).$$

Hence $\chi_{l,\bar{p}}(t) = \Delta^{\frac{1}{2}-\frac{1}{\bar{p}}}(t)\chi_{l-l_0}(t)$ is the character of $L^1(\mathcal{U},\omega) \equiv L^1(\mathcal{G}(l)/\mathcal{G}(l)\cap\mathcal{N},\omega)$ associated to the representation $\gamma_{l,\bar{p}}$.

Remark. If l and l' are two different extensions of $r \in \mathfrak{h}^*$ to $\mathfrak{g}$, then $\chi_{l',\bar{p}}(t) = \chi_{l'-l}(t)\cdot\chi_{l,\bar{p}}(t)$, $\forall t \in \mathcal{U} \subset \mathcal{G}(l)$. Hence, if l and l' are in the same $\mathcal{G}$-orbit, then $\chi_{l',\bar{p}}(t) = \chi_{l,\bar{p}}(t)$, $\quad \forall t \in \mathcal{U} \subset \mathcal{G}(l)$.

Corollary 5.6.2. *The weight ω satisfies the inequality*

$$\Delta^{(\frac{1}{2}-\frac{1}{\bar{p}})}(x) = e^{\sum_{j=1}^m\left(\frac{1}{2}-\frac{1}{p_j}\right)\mathrm{tr}\lambda_j(\log x)} \le \omega(x), \quad \forall x \in \mathcal{G}(l),$$

for all $\bar{p}$. Hence

$$\begin{aligned}
e^{\frac{|t|}{2}\sum_{j=1}^m|\mathrm{tr}\lambda_j(X)|} &\le \omega(\exp tX) \\
&\le C'(1+|t|)^C e^{\frac{|t|}{2}\sum_{j=1}^m|\mathrm{tr}\lambda_j(X)|}, \quad \forall t \in \mathbb{R}, \forall X \in \mathfrak{g}(l).
\end{aligned}$$

Proof. If $t \in \mathcal{G}(l) \cap \mathcal{N}$, then $\Delta^{(\frac{1}{2}-\frac{1}{\bar{p}})}(t) = 1$ and $\omega \geq 1$. For $t \in \mathcal{U}$, for every multi-index $\bar{p} \in [1,\infty]^m$, $\Delta^{(\frac{1}{2}-\frac{1}{\bar{p}})}(t) = |\chi_{l,\bar{p}}(t)| \leq \omega(t)$ as $\chi_{l,\bar{p}}$ is a continuous character on $L^1(\mathbb{R}^n,\omega) \equiv L^1(\mathcal{G}(l)/\mathcal{G}(l)\cap\mathcal{N},\omega)$ (see proof of (4.2.)). See (4.1.) for the last assertion. □

5.7. Characterization of an arbitrary simple $L^1(\mathcal{G})$-module.

Proposition 5.7.1. *Let S_1 and S_2 be the following subsets of $(\mathbb{R}^n)^* \equiv \mathfrak{u}^* \equiv (\mathfrak{g}(l)/\mathfrak{g}(l)\cap\mathfrak{n})^*$:*

$$S_1 = \left\{ \sum_{i=1}^m \left(\frac{1}{2}-\frac{1}{p_i}\right) \mathrm{tr}\lambda_i(\cdot) \mid 1 \leq p_i \leq \infty \right\}$$
$$= \left\{ \sum_{i=1}^m C_i \mathrm{tr}\lambda_i(\cdot) \mid |C_i| \leq \frac{1}{2}, C_i \in \mathbb{R} \right\}$$
$$S_2 = \left\{ \rho \in \mathfrak{u}^* \mid |\rho(X)| \leq \sum_{i=1}^m \frac{1}{2}|\mathrm{tr}\lambda_i(X)|,\ \forall X \in \mathfrak{u} \equiv \mathbb{R}^n \right\}.$$

Then $S_1 = S_2$.

Proof. Notice first that the linear form $\nu(\cdot) = \sum_{i=1}^m (\frac{1}{2}-\frac{1}{p_i})\mathrm{tr}\lambda_i(\cdot)$ of $\mathfrak{g}(l)$ is constant on the classes modulo $\mathfrak{g}(l)\cap\mathfrak{n}$ and may hence be considered as a linear form on $\mathfrak{g}(l)/\mathfrak{g}(l)\cap\mathfrak{n}$. The sets S_1 and S_2 are closed convex subsets of $(\mathbb{R}^n)^*$ such that $S_1 \subset S_2$. Assume there exists $\rho \in S_2 \backslash S_1$. By the Hahn-Banach theorem there is $X_0 \in \mathbb{R}^n \equiv \mathfrak{u}$ and $\alpha \in \mathbb{R}$ such that $s_1(X_0) < \alpha < \rho(X_0)$, $\forall s_1 \in S_1$. Let's then choose $s_1 \in S_1$ by $s_1(X) = \sum_{i=1}^m \frac{1}{2}\varepsilon_i \mathrm{tr}\lambda_i(X)$, $\forall X \in \mathfrak{u}$, where $\varepsilon_i = 1$ if $\mathrm{tr}\lambda_i(X_0) \geq 0$ and $\varepsilon_i = -1$ if $\mathrm{tr}\lambda_i(X_0) < 0$. Hence

$$\sum_{i=1}^m \frac{1}{2}|\mathrm{tr}\lambda_i(X_0)| = s_1(X_0) < \rho(X_0),$$

which contradicts the fact that $\rho \in S_2$. □

Corollary 5.7.2. *Let χ be a continuous character on $L^1(\mathbb{R}^n,\omega) \equiv L^1(\mathcal{U},\omega)$. Then there is a multi-index $\bar{p} = (p_1,\ldots,p_m)$ and $l' \in \mathfrak{g}^*$ with $l'|_{\mathfrak{h}} = l|_{\mathfrak{h}}$ such that*

$$|\chi(\exp X)| = e^{\sum_{i=1}^m \left(\frac{1}{2}-\frac{1}{p_i}\right)\mathrm{tr}\lambda_i(X)}, \quad \forall X \in \mathfrak{u}$$

and such that

$$\chi(\exp X) = \chi_{l'-l_0}(\exp X)\cdot e^{\sum_{i=1}^m \left(\frac{1}{2}-\frac{1}{p_i}\right)\mathrm{tr}\lambda_i(X)} = \chi_{l',\bar{p}}(\exp X), \quad \forall X \in \mathfrak{u},$$

i.e., every continuous character on $L^1(\mathcal{G}(l)/\mathcal{G}(l)\cap\mathcal{N},\omega)$ is of the form

$$\chi(\exp X) = \chi_{l'-l_0}(\exp X)e^{\nu(X)},$$

where $l' - l_0 \in \mathfrak{h}^\perp$ and $\nu \in (\mathfrak{g}(l)/\mathfrak{g}(l) \cap \mathfrak{n})^*$ such that

$$|\nu(X)| \leq \frac{1}{2} \sum_{j=1}^{m} |\mathrm{tr}\lambda_j(X)|.$$

Proof. We may write $\chi(\exp X) = e^{-i\rho_1(X)} \cdot e^{\rho_2(X)}$ with $\rho_1, \rho_2 \in (\mathbb{R}^n)^* \equiv \mathfrak{u}^* \equiv (\mathfrak{g}(l)/\mathfrak{g}(l) \cap \mathfrak{n})^*$. By (4.2.) and (5.6.1.) $\rho_2 \in S_2 = S_1$ and hence there is a multi-index $\bar{p} = (p_1, \ldots, p_m)$, $p_i \in [1, \infty]$ for all i, such that

$$|\chi(\exp X)| = e^{\rho_2(X)} = e^{\sum_{i=1}^m \left(\frac{1}{2} - \frac{1}{p_i}\right) \mathrm{tr}\lambda_i(X)}, \quad \forall X \in \mathfrak{u} \equiv (\mathfrak{g}(l)/\mathfrak{g}(l) \cap \mathfrak{n})^*.$$

We may then choose $l' \in \mathfrak{g}^*$ such that $l'|_{\mathfrak{h}} = l|_{\mathfrak{h}}$ and such that $l' - l_0 = \rho_1$ on $\mathfrak{u}$. □

Theorem 5.7.3.

a) *Let $(T, \mathfrak{U})$ be a simple $L^1(\mathcal{G})$-module. Then there exists $l \in \mathfrak{g}^*$, a polarization $\mathfrak{p}$ for l in $\mathfrak{g}$ and a multi-index $\bar{p} \in [1, \infty]^m$, such that $(T, \mathfrak{U})$ is equivalent to the simple module $\left(\pi^0_{l,\mathfrak{p},\bar{p}}, \mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}\right)$.*

b) *Let $\bar{p}$, $\bar{q} \in [1, \infty]^m$ be two multi-indices. Then $\left(\pi^0_{l,\mathfrak{p},\bar{q}}, \mathfrak{H}^0_{\pi_{l,\mathfrak{p},\bar{q}}}\right) \simeq \left(\pi^0_{l,\mathfrak{p},\bar{p}}, \mathfrak{H}^0_{\pi_{l,\mathfrak{p},\bar{p}}}\right)$ if and only if*

$$\sum_{i=1}^{m} \left(\frac{1}{2} - \frac{1}{q_i}\right) \mathrm{tr}\, \lambda_i(\cdot) = \sum_{i=1}^{m} \left(\frac{1}{2} - \frac{1}{p_i}\right) \mathrm{tr}\, \lambda_i(\cdot) = \nu(\cdot)$$

on $\mathfrak{u}$ and hence on $\mathfrak{g}(l)$, i.e., if the corresponding linear forms $\nu \in (\mathfrak{g}(l)/\mathfrak{g}(l) \cap \mathfrak{n})^*$ are the same.

Proof. By (3.6.) and (5.7.2.). □

5.7.4. Remarks.

a) One can show that up to equivalence the representations $\pi^0_{l,\mathfrak{p},\bar{p}}$ are independent of the choice of the polarization $\mathfrak{p}$.

b) Let's write $\widetilde{\mathcal{G}}$ for the space of the equivalence classes of simple $L^1(\mathcal{G})$-modules. Let's write $\widetilde{\mathfrak{g}^*}$ for the collection of all pairs (l, ν) with $l \in \mathfrak{g}^*$, $\nu \in (\mathfrak{g}(l)/\mathfrak{g}(l) \cap \mathfrak{n})^*$ such that $|\nu(X)| \leq \frac{1}{2} \sum_{j=1}^m |\mathrm{tr}\lambda_j(X)|, \forall X \in \mathfrak{g}(l)$. The group $\mathcal{G}$ acts on $\widetilde{\mathfrak{g}^*}$ by conjugation. Let $\widetilde{\mathfrak{g}^*}/\mathcal{G}$ be the set of all equivalence classes for this action.

We then get our final theorem:

Theorem 5.7.5. *There is a bijection between $\widetilde{\mathfrak{g}^*}/\mathcal{G}$ and $\widetilde{\mathcal{G}}$.*

6. Final remarks.

As it was already pointed out in the introduction, the algebraically simple $L^1(\mathcal{G})$-modules for a solvable exponential Lie group are essentially obtained in the same way as in the case of the nilpotent groups, except that one has to generalize the induced representations. This is no longer true for topologically irreducible representations, as it was shown in ([**LuMo2**]). Two major differences exist. Usually there are a lot of extensions of a topologically irreducible representation of the subalgebra $p * L^1(\mathcal{G}/\mathcal{H}, L^1(\mathcal{H})/\ker\gamma) * p$ to a topologically irreducible representation of the algebra $L^1(\mathcal{G}/\mathcal{H}, L^1(\mathcal{H})/\ker\gamma)$, whereas this extension is unique in the algebraic case. These different extensions are characterized by different extension norms. But the main difference arises from the irreducible representations of $L^1(\mathbb{R}^n, \omega)$. These representations coincide with the characters in the algebraic case. In the topological case there are a lot of irreducible inifinite dimensional representations of $L^1(\mathbb{R}^n, \omega)$ if the weight ω is exponential, which happens if and only if the group $\mathcal{G}$ is nonsymmetric. The corresponding representations of $L^1(\mathcal{G})$ are fundamentally different from induced representations. The construction of such representations is linked to the invariant subspace problem, as it was shown in ([**LuMo2**]).

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EXPLICIT REALIZATION OF THE DICKSON GROUPS $G_2(q)$ AS GALOIS GROUPS

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For any prime power q we determine a polynomial $f_q(X) \in \mathbf{F}_q(t,u)[X]$ whose Galois group over $\mathbf{F}_q(t,u)$ is the Dickson group $G_2(q)$. The construction uses a criterion and a method due to Matzat.

1. Introduction.

In this paper we are concerned with the construction of polynomials whose Galois groups are the exceptional simple Chevalley groups $G_2(q)$, q a prime power, first discovered by Dickson; see Theorems 4.1 and 4.3.

It was shown by Nori [**7**] that all semisimple simply-connected linear algebraic groups over $\mathbb{F}_q$ do occur as Galois groups of regular extension of regular function fields over $\mathbb{F}_q$, but his proof does not give an explicit equation or even a constructive method for obtaining such extensions. On the other hand, in a long series of papers Abhyankar has given families of polynomials for groups of classical types (see [**1**] and the references cited there). His ad hoc approach hasn't yet led to families with groups of exceptional type (but see [**2**] for a different construction of polynomials with Galois group the simple groups of Suzuki). Thus it seems natural to try to fill this gap. In his recent paper Matzat [**6**] describes an algorithmic approach which reduces the construction of generating polynomials for such extensions to certain group theoretic calculations.

More precisely, let $F := \mathbb{F}_q(\mathbf{t})$, with $\mathbf{t} = (t_1, \ldots, t_s)$ a set of indeterminates. We denote by $\phi_q : F \to F$, $x \mapsto x^q$, the Frobenius endomorphism. Let G be a reduced connected linear algebraic group defined over $\mathbb{F}_q$, with a faithful linear representation $\Gamma : G(F) \hookrightarrow \mathrm{GL}_n(F)$ in its defining characteristic, also defined over $\mathbb{F}_q$. We identify $G(F)$ with its image in $\mathrm{GL}_n(F)$. Fix an element $g \in G(F)$ and assume that $g \in \mathrm{GL}_n(R)$, where $R := \mathbb{F}_q[\mathbf{t}]$. Any specialization homomorphism $\psi : R \to \mathbb{F}_{q^a}$, $t_j \mapsto \psi(t_j)$, can be naturally extended to $\mathrm{GL}_n(R)$. We define

$$g_\psi := \psi(g) \cdot \psi(\phi_q(g)) \cdots \psi(\phi_q^{a-1}(g)) \in \mathrm{GL}_n(\mathbb{F}_q).$$

With these notations Matzat [**6**, Thm. 4.3 and 4.5] shows the following:

Theorem 1.1 (Matzat). *Let $G(F) \leq \mathrm{GL}_n(F)$ be a reduced connected linear algebraic group defined over $\mathbb{F}_q$. Let $g \in \mathrm{GL}_n(R)$ such that:*

(i) $g \in G(F)$,

(ii) *there exist specializations* $\psi_i : R \to \mathbb{F}_{q^{a_i}}$, $1 \leq i \leq k$, *such that no proper subgroup of* $G(\mathbb{F}_q) \leq \mathrm{GL}_n(\mathbb{F}_q)$ *contains conjugates of all the* g_{ψ_i}, $1 \leq i \leq k$.

Then $G(\mathbb{F}_q)$ *occurs as regular Galois group over* F, *and a generating polynomial* $f(\mathbf{t}, X) \in F[X]$ *for such a* $G(\mathbb{F}_q)$*-extension can be computed explicitly from the matrix* g.

Thus the strategy for the computation of a $G_2(q)$-polynomial will be the following: First construct a small faithful matrix representation of $G_2(F)$ in its defining characteristic. For this we use the well-known facts that $G_2(F)$ is a subgroup of an 8-dimensional orthogonal group over F, and that this 8-dimensional representation has a faithful irreducible constituent of dimension 6 for $G_2(F)$, if $\mathrm{char}(F) = 2$, respectively of dimension 7 if $\mathrm{char}(F) > 2$. Secondly, we need to find an element $g \in G_2(F)$ with the properties required in the Theorem. For this, we make use of the known lists of maximal subgroups of $G_2(q)$ by Cooperstein and Kleidman. (These results require the classification of finite simple groups, but only in a very weak form.) Finally, the corresponding polynomial has to be computed using a version of the Buchberger algorithm.

2. Identifying $G_2(F)$ inside the 8-dimensional orthogonal group.

We first introduce some notation. Let V be an 8-dimensional vector space over a field F of characteristic $p \geq 0$, with basis $e_1, \ldots, e_8$ and Q the quadratic form on V defined by

$$Q : V \to F, \qquad Q\left(\sum_{i=1}^{8} x_i e_i\right) = \sum_{i=1}^{4} x_i x_{9-i}.$$

We denote by $\mathrm{GO}_8(F)$ the group of isometries of Q, the full orthogonal group, and by $\mathrm{SO}_8(F)$ the connected component of the identity in $\mathrm{GO}_8(F)$, of index 2. Thus $\mathrm{SO}_8(F)$ is a simple split algebraic group over F of type D_4. The subgroup of upper triangular matrices of $\mathrm{GL}_8(F)$ contains a Borel subgroup B of $\mathrm{SO}_8(F)$. More precisely, the unipotent radical of B is generated by the root subgroups

$$X_i := \{x_i(t) \mid t \in F\}, \qquad i = 1, \ldots, 12,$$

where the $x_i(t)$ are defined as in Table 1. Here $E_{i,j}(t)$ denotes the matrix having 1's on the diagonal and one further nonzero entry t in position (i, j).

A maximal torus T in B is given by the set of diagonal matrices

$$T := \{t = \mathrm{diag}(t_1, t_2, t_3, t_4, t_4^{-1}, t_3^{-1}, t_2^{-1}, t_1^{-1}) \mid t_i \in F^\times\}.$$

The simple roots with respect to T are now α_i, $i = 1, \ldots, 4$, with $\alpha_i(t) = t_i/t_{i+1}$ for $i = 1, 2, 3$ and $\alpha_4(t) = t_3 t_4$. In Table 1 we have also recorded the

Table 1. Root subgroups of $\mathrm{SO}_8(F)$.

$x_1(t) = E_{1,2}(t) - E_{7,8}(t)$	1000		$x_7(t) = E_{2,5}(t) - E_{4,7}(t)$	0101
$x_2(t) = E_{2,3}(t) - E_{6,7}(t)$	0100		$x_8(t) = E_{1,4}(t) - E_{5,8}(t)$	1110
$x_3(t) = E_{3,4}(t) - E_{5,6}(t)$	0010		$x_9(t) = E_{2,6}(t) - E_{3,7}(t)$	0111
$x_4(t) = E_{3,5}(t) - E_{4,6}(t)$	0001		$x_{10}(t) = E_{1,5}(t) - E_{4,8}(t)$	1101
$x_5(t) = E_{1,3}(t) - E_{6,8}(t)$	1100		$x_{11}(t) = E_{1,6}(t) - E_{3,8}(t)$	1111
$x_6(t) = E_{2,4}(t) - E_{5,7}(t)$	0110		$x_{12}(t) = E_{1,7}(t) - E_{2,8}(t)$	1211

decomposition of the root corresponding to a root subgroup into the simple roots $\alpha_1, \ldots, \alpha_4$. Note that the simple root α_2 (with label `0100`) is the one belonging to the central node in the Dynkin diagram of type D_4.

The group $\mathrm{PSO}_8(F) := \mathrm{SO}_8(F)/Z(\mathrm{SO}_8(F))$ possesses an outer automorphism γ of order 3 induced by the graph automorphism of the Dynkin diagram D_4 which cyclically permutes the nodes $1, 3$ and 4 and fixes the middle node 2. The group $\mathrm{PSO}_8(F)^\gamma$ of fixed points in $\mathrm{PSO}_8(F)$ under γ is again a simple connected algebraic group over F, of type G_2. Note that γ does not stabilize the natural representation of $\mathrm{SO}_8(F)$. Nevertheless we can construct $G_2(F)$ as a preimage G of $\mathrm{PSO}_8(F)^\gamma$ in $\mathrm{SO}_8(F)$.

The Borel subgroup B of $\mathrm{SO}_8(F)$ contains a Borel subgroup of G. Its unipotent radical is the product of the subgroups

$$X_{i,j,k} := \{x_i(t)x_j(t)x_k(t) \mid t \in F\}$$

where $(i,j,k) \in \{(1,3,4), (5,6,7), (8,9,10)\}$, together with the root subgroups $X_i = \{x_i(t) \mid t \in F\}$ for $i \in \{2, 11, 12\}$ (see for example Carter [**3**, Prop. 13.6.3]). A maximal torus of G inside T consists of the elements

$$\{t = \mathrm{diag}(t_1, t_2, t_1t_2^{-1}, 1, 1, t_1^{-1}t_2, t_2^{-1}, t_1^{-1}) \mid t_i \in F^\times\}.$$

From this description we find that the simple roots for $G_2(F)$ are now α, β, with $\alpha(t) := t_1/t_2$ and $\beta(t) := t_2^2/t_1$, and with corresponding root subgroups $X_\alpha := X_{1,3,4}$, $X_\beta := X_2$ respectively.

An easy calculation with the generators of root subgroups given above now shows that G leaves invariant the hyperplane V_1 of V consisting of vectors with equal fourth and fifth coordinate, as well as the 1-dimensional subspace V_2 of V spanned by $e_4 - e_5$. Thus we obtain an induced action of G on V_1, respectively on V_1/V_2 when $\mathrm{char}(F) = 2$. This yields a faithful matrix representation $\Gamma : G_2(F) \hookrightarrow \mathrm{GL}_n(F)$ of $G_2(F)$, of dimension $n = 7$ when $\mathrm{char}(F) \neq 2$, respectively of dimension $n = 6$ when $\mathrm{char}(F) = 2$. It is well-known that the smallest possible degree of a faithful representation of $G_2(F)$ is 7, respectively 6 if $\mathrm{char}(F) = 2$, so our representation Γ is irreducible.

Remark 2.1. The matrices given in [**4**, p. 34] do not define a representation of $G_2(2^f)$. Indeed, the matrix for $h_a(t)$ does not have determinant 1, as it should have (since $G_2(2^f)$ is simple for $f > 1$). Its second diagonal entry should be t^{-1}. Conjugating $X_a(t)$ by $h_a(t')$ one sees that the middle off-diagonal entry of $X_a(t)$ should be t^2 instead of t. The commutator relations (see Carter [**3**, 12.4]; [**4**, (2.1)] contains misprints) then show that similarly in the matrices for $X_{a+b}(t)$ and $X_{2a+b}(t)$ the second nonzero off-diagonal entry t should be replaced by t^2. In this way one recovers the representation constructed above.

3. Finding a suitable element.

Let first $q = 2^f$ be even. Then an easy calculation shows that in our 6-dimensional representation $\Gamma : G_2(F) \to \mathrm{GL}_6(F)$ constructed above, we have

$$x_\alpha(t) = \begin{pmatrix} 1 & t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & t^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad x_\beta(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and the longest element of the Weyl group of $G_2(F)$ is represented by

$$w_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We choose $g := x_\alpha(t)x_\beta(u)w_0 \in G_2(F)$ and let

$$D := \Gamma(g) = \begin{pmatrix} 0 & 0 & 0 & tu & t & 1 \\ 0 & 0 & 0 & u & 1 & 0 \\ 0 & t^2u & t^2 & 1 & 0 & 0 \\ 0 & u & 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

Proposition 3.1. *Let q be even and D be defined as above. Then no proper subgroup of $G_2(q)$ contains conjugates of all specializations of D.*

Proof. We use the fact that all maximal subgroups of the finite groups $G_2(q)$ are known by Cooperstein [**4**]. For $q = 2$ specializations into $\mathbb{F}_8$ yield elements of orders 7 and 12, and no maximal subgroup of $G_2(2)$ contains elements of both orders. For $q = 4$ specializations into $\mathbb{F}_4$ yield elements of orders 13, 15 and 21. The only maximal subgroup of order divisible by $7 \cdot 13$ is $\mathrm{PSL}_2(13)$, but its order is not divisible by 5, so we are done again.

Now let $q \geq 8$. Let G be a subgroup of $G_2(q)$ containing conjugates of all specializations of D. Let $\alpha \in \mathbb{F}_{q^2}^\times$ of order $q+1$. Then the minimal polynomial of α over $\mathbb{F}_q$ has the form $X^2 + \mathrm{Tr}(\alpha)X + 1$, where $\mathrm{Tr}(\alpha) = \alpha + \alpha^q \in \mathbb{F}_q$. Thus any element of $\mathbb{F}_{q^2}^\times$ of order $q+1$ occurs as a root of a polynomial of the shape

$$X^2 + vX + 1, \qquad v \in \mathbb{F}_q.$$

Clearly, all elements of $\mathbb{F}_q^\times$ also occur as zeros of such a polynomial. Now for $v \in \mathbb{F}_q$ consider the specialization

$$\psi_v : \mathbb{F}_q[t,u] \to \mathbb{F}_q, \qquad t \mapsto 0,\ u \mapsto v.$$

Then the specialization $\psi_v(D)$ of D has characteristic polynomial

$$X^6 + (v^2+1)X^4 + (v^2+1)X^2 + 1 = (X+1)^2(X^2+vX+1)^2.$$

The 1-eigenspace of $\psi_v(D)$ only has dimension 1 for $v \neq 0$, so the order of $\psi_v(D)$ is divisible by 2. By our above considerations, we hence find elements of orders $2(q+1)$ and $2(q-1)$ as specializations of D. (This can also be seen as follows: If $t = 0$ then g specializes to

$$x_\beta(u)w_0 = x_\beta(u)(w_\beta w_\alpha)^3 = x_\beta(u)w_\beta \cdot w'$$

where $w' = w_\alpha w_\beta w_\alpha w_\beta w_\alpha$ has order 2, centralizes $x_\beta(u)w_\beta$, and $x_\beta(u)w_\beta$ represents the element

$$\begin{pmatrix} u & 1 \\ 1 & 0 \end{pmatrix}$$

in the subgroup $\langle X_\beta, X_{-\beta}\rangle \cong \mathrm{SL}_2(q)$.)

Next, consider the specialization

$$\psi'_v : \mathbb{F}_q[t,u] \to \mathbb{F}_q, \qquad t \mapsto v,\ u \mapsto 0.$$

Here, $\psi'_v(D)$ has characteristic polynomial

$$(X^2+vX+1)^2(X^2+v^2X+1).$$

By the argument above, this again yields elements of orders $2(q-1)$ and $2(q+1)$. But note that this time these elements never have an eigenvalue 1, nor have any of their powers of order larger than 2. Thus G contains subgroups of order $(q \pm 1)^2$. Theorem 2.3 in [4] shows that either $G \leq \mathrm{SL}_2(q) \times \mathrm{SL}_2(q)$ or $G = G_2(q)$.

Finally, consider the specialization

$$\psi''_v : \mathbb{F}_q[t,u] \to \mathbb{F}_q, \qquad t \mapsto v,\ u \mapsto 1.$$

The corresponding specialization of D has characteristic polynomial

$$(X^3+v^2X+1)(X^3+v^2X^2+1).$$

If X^3+v^2X+1 is reducible over $\mathbb{F}_q$, then it has a linear factor $X+a$, $a\in\mathbb{F}_q$, and $X^3+v^2X+1=(X+a)(X^2+aX+1/a)$. Clearly, the case $a=0$ is not possible, so for at least one of the q possibilities for $v\in\mathbb{F}_q$ the characteristic polynomial has an irreducible factor of degree 3. In this case, the specialization of D has order dividing q^2+q+1, but not $q-1$. Since $\mathrm{SL}_2(q)\times\mathrm{SL}_2(q)$ doesn't contain such elements, we have $G=G_2(q)$, as claimed. □

For odd $q=p^f$ we again choose $g:=x_\alpha(t)x_\beta(u)w_0\in G_2(F)$. With

$$x_\alpha(t)=\begin{pmatrix}1&t&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&t&-t^2&0&0\\0&0&0&1&-2t&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&-t\\0&0&0&0&0&0&1\end{pmatrix},$$

$$x_\beta(t)=\begin{pmatrix}1&0&0&0&0&0&0\\0&1&t&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&-t&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix},$$

and

$$w_0=\begin{pmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&-1&0\\0&0&0&0&1&0&0\\0&0&0&-1&0&0&0\\0&0&1&0&0&0&0\\0&-1&0&0&0&0&0\\1&0&0&0&0&0&0\end{pmatrix},$$

this gives

$$D:=\Gamma(g)=\begin{pmatrix}0&0&0&0&tu&-t&1\\0&0&0&0&u&-1&0\\0&-t^2u&-t^2&-t&1&0&0\\0&-2tu&-2t&-1&0&0&0\\0&u&1&0&0&0&0\\-t&-1&0&0&0&0&0\\1&0&0&0&0&0&0\end{pmatrix}\tag{2}$$

in this case. This matrix has separable characteristic polynomial

$$X^7 + (t^2+1)\,X^6 - (2t^2+u^2+3)\,X^5 - (t^4+3t^2+u^2+3)\,X^4 \\ + (t^4+3t^2+u^2+3)\,X^3 + (2t^2+u^2+3)\,X^2 - (t^2+1)\,X - 1.$$

We need the following result:

Lemma 3.2. *Let $q > 3$ be an odd prime power. Then there exists $v \in \mathbb{F}_q$ such that*

$$X^3 - (v^2+2)X - 1$$

is irreducible over $\mathbb{F}_q$.

Proof. Assume that $f := X^3 - (v^2+2)X - 1$ is reducible. Then f has a zero $a \in \mathbb{F}_q$, and $X^3 - (v^2+2)X - 1 = (X-a)(X^2+aX+a^{-1})$. These zeros are just the first coordinates of the $\mathbb{F}_q$-points on the elliptic curve E defined by $U^3 - (V^2+2)U - 1$. By the Weil bounds [8], E has at most $q+1+2\sqrt{q}$ points (u,v) over $\mathbb{F}_q$. Clearly, with (u,v) the point $(u,-v)$ also lies on E, hence there are at most $q/2+1+\sqrt{q}$ distinct values a which can occur as zeros of f.

Next, we estimate how often f splits completely into linear factors. This happens if in addition the discriminant $(a^3-4)/a$ of X^2+aX+a^{-1} is a square in $\mathbb{F}_q$. Thus we need to count points on the $\mathbb{F}_q$-curve C defined by the two equations

$$U^3 - (V^2+2)U - 1, \quad U^3 - W^2U - 4.$$

Subtracting these two equations we see that U lies in the function field $\mathbb{F}_q(V,W)$. Since both V,W have degree at most 2 over $\mathbb{F}_q(U)$, the curve C has genus at most 4. Moreover, the only singular point of C is the point with coordinates $(4,0,0)$ in characteristic 5. Again by the Weil bounds [8] this means that C has at least $q+1-2\cdot 4\sqrt{q}-6$ points over $\mathbb{F}_q$. For each such point, changing the sign of the V,W-coordinates again yields a point, hence there are at least $(q-5-8\sqrt{q})/4$ distinct $a \in \mathbb{F}_q$ for which f splits completely. Thus we obtain at most

$$q/2+1+\sqrt{q} - (q-5-8\sqrt{q})/4 = (q+9)/4 + 3\sqrt{q}$$

factorizations of f into a linear and a quadratic factor. The discriminant of f is a polynomial in v of degree 6, hence f is inseparable for at most six values of v. Apart from those, each completely splitting f accounts for three different values of a, so we obtain a total of at most

$$(q+9)/4 + 3\sqrt{q} + ((q-5-8\sqrt{q})/4 - 6)/3 + 6 = (2q+35)/6 + 7/3\sqrt{q}$$

reducible polynomials when v runs over $\mathbb{F}_q$. Hence there remain at least

$$(q+1)/2 - ((2q+35)/6 + 7/3\sqrt{q}) = (q-32)/6 - 7/3\sqrt{q}$$

irreducible polynomials. This is positive for $q \geq 257$. For the remaining prime powers $3 < q < 257$ a computer check shows that the assertion is also satisfied. (For $q = 5, 9$ there is just one irreducible polynomial of the required shape, for $q = 3$ there is none.)

Note that the counting of singular points and of inseparable f was very rough and a more detailed analysis would have reduced the bound considerably. □

Proposition 3.3. *Let q be odd and D be the matrix defined in (2). Then no proper subgroup of $G_2(q)$ contains conjugates of all specializations of D.*

Proof. Again all maximal subgroups of $G_2(q)$ are known by work of Kleidman [**5**]. For $q = 3$ specializations into $\mathbb{F}_9$ yield elements of orders 7, 9, 13. The only maximal subgroup of $G_2(3)$ of order divisible by $7 \cdot 13$ is $\mathrm{PSL}_2(13)$, but that has no elements of order 9. For $q = 5$, specialization into $\mathbb{F}_5$ yields element orders 7, 20 and 31, thus we are done again.

For $q \geq 7$ let G be a subgroup of $G_2(q)$ containing conjugates of all specializations of D. We again consider the specialization

$$\psi_v : \mathbb{F}_q[t, u] \to \mathbb{F}_q, \qquad t \mapsto 0,\ u \mapsto v.$$

Then the square of $\psi_v(D)$ has characteristic polynomial

$$(X - 1)^3(X^2 - (v^2 + 2)X + 1)^2.$$

This gives rise to elements of orders $q \pm 1$ in G. Similarly, the specialization

$$\psi'_v : \mathbb{F}_q[t, u] \to \mathbb{F}_q, \qquad t \mapsto v,\ u \mapsto 0,$$

yields the characteristic polynomial

$$(X - 1)(X^2 - (v^4 + 4v^2 + 2)X + 1)(X^2 - (v^2 + 2)X + 1)^2$$

for the image of D^2. So as in the previous proof we deduce that G must contain subgroups of orders $(q \pm 1)^2$. Theorem A in [**5**] shows that either G is contained in the central product $\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$, or $G = G_2(q)$. Finally, for the specialization

$$\psi''_v : \mathbb{F}_q[t, u] \to \mathbb{F}_q, \qquad t \mapsto v,\ u \mapsto 1,$$

we obtain the characteristic polynomial

$$(X - 1)(X^3 + (v^2 + 2)X^2 - 1)(X^3 - (v^2 + 2)X - 1)$$

for $\phi''_v(D)$. Since $q \geq 7$ is odd, Lemma 3.2 shows that there exists $v \in \mathbb{F}_q$ such that the degree 3 factors of this polynomial are irreducible over $\mathbb{F}_q$. But $\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$ does not contain such elements, hence we have $G = G_2(q)$. □

4. The polynomials.

It remains to determine generating polynomials for the $G_2(q)$-extensions whose existence is guaranteed by Theorem 1.1 in conjunction with Propositions 3.1 and 3.3.

Theorem 4.1. *Let $q = 2^f$ be a power of 2. Then the polynomial*

$$\begin{aligned} X^{q^6} &+ u^{e_2}t^{e_4}X^{q^5} + (u^{e_1}t^{e_1} + u^{e_3}t^{e_1} + t^{e_1} + t^{e_3} + 1)X^{q^4} \\ &+ u^{e_2}t^{e_4}(t^{q^3+q^2} + t^{q^3-q} + 1)X^{q^3} \\ &+ t^{e_1}(u^{e_1}t^{q^2-1} + u^{e_1}t^{q^2+q} + u^{e_1} + u^{e_3} + 1)X^{q^2} \\ &+ u^{e_2}t^{q^4+2q^2-q}X^q + u^{e_1}t^{q^4-1}X, \end{aligned}$$

with $e_1 := q^4 - q^2$, $e_2 := q^4 - q^3$, $e_3 := q^4 + q^3$, $e_4 := q^4 - q^3 + 2q^2$, has Galois group $G_2(q)$ over $\mathbb{F}_q(t, u)$.

Proof. In Proposition 3.1 we have shown that the assumptions of Matzat's Theorem 1.1 are satisfied for the matrix D defined in (1). According to Matzat [**6**, §1], a generating polynomial for a field extension with group $G_2(q)$ can now be obtained by solving the non-linear system of equations given by

$$\mathbf{y} = D\,\mathbf{y}^q,$$

where $\mathbf{y} = (y_1, \dots, y_6)^t$, for one of the variables. Solving for y_6 yields the equation displayed in the statement. □

By the Hilbert irreducibility theorem, there exist 1-parameter specializations of the polynomial in Theorem 4.1 with group $G_2(q)$.

Example 4.2. By arguments similar to those used in the proof of Proposition 3.1 it can be checked that the polynomial

$$\begin{aligned} X^{64} + t^{24}\,X^{32} + (t^{36} + t^{12} + 1)\,X^{16} + (t^{30} + t^{36} + t^{24})\,X^8 \\ + (t^{24} + t^{36} + t^{27} + t^{30} + t^{12})\,X^4 + t^{30}\,X^2 + t^{27}\,X \end{aligned}$$

obtained by setting $u = t$ has Galois group $G_2(2)$ over $\mathbb{F}_2(t)$.

Theorem 4.3. *Let $q = p^f$ be an odd prime power. Then the polynomial*

$$\begin{aligned} X^{q^7} &+ u^{e_1}t^{e_4}(t^{e_6} + 1)X^{q^6} - (t^{e_2}u^{e_3} + (t^{q^5+q^2} + t^{e_2})u^{e_2} + t^{e_3} + t^{e_2} + 1)X^{q^5} \\ &- u^{e_1}t^{e_4}(t^{e_5}(u^{q^4+q^3} + u^{e_5}) + (t^{e_6} + 1)(t^{q^4+q^3} + t^{e_5} + 1))X^{q^4} \\ &+ t^{e_2}(u^{e_3} + (t^{e_6} + 1)(t^{e_6} + t^{q^3-q} + 1)u^{e_2} + 1)X^{q^3} \\ &+ u^{e_1}t^{q^5+q^3-2q^2}(u^{q^4+q^3} + (t^{q^2+q} + t^{q^2-1} + 1)u^{e_5} + t^{e_6} + 1)X^{q^2} \\ &- u^{e_2}t^{q^5-q}(t^{e_6} + 1)X^q - u^{q^5-q^2}t^{q^5+q^3-q^2-1}X, \end{aligned}$$

where $e_1 := q^5 - q^4$, $e_2 := q^5 - q^3$, $e_3 := q^5 + q^4$, $e_4 := q^5 - q^4 + q^3 - q^2$,

$e_5 := q^4 - q^2$, $e_6 := q^3 + q^2$, has Galois group $G_2(q)$ over $\mathbb{F}_q(t, u)$.

The proof is as for the preceding theorem, starting this time from the matrix D given in (2), solving for y_7, and using Proposition 3.3.

Remark 4.4. The sporadic simple Janko groups J_1 and J_2 are subgroups of $G_2(11)$, respectively of $G_2(4)$. It would be nice to find Galois extensions for these groups in characteristic 11 respectively 2 by the above method, possibly as specializations of the polynomials in Theorems 4.1 and 4.3.

Remark 4.5. The next smallest simple exceptional group is the one of type F_4. Its smallest faithful representation has dimension 26, respectively 25 in characteristic 3. In principle, the methods of this paper should make it possible to produce an $F_4(q)$-polynomial.

Remark 4.6. The group $G_2(q)$, q odd, has q orbits on nonzero vectors in its 7-dimensional representation. Thus, the polynomial $f_q(t, u, X)$ in Theorem 4.3 has q factors, of degrees roughly q^6, and a linear factor. On the other hand, any specialization of f_q has factors of degree at most $q^2 + q + 1$, the maximal element order in $G_2(q)$. Thus, f_q seems a good candidate for testing factorization algorithms. Using Maple we have not been able to find the factorization of f_q for $q = 3$.

Similarly, for q even $G_2(q)$ has a single orbit on the nonzero vectors of the 6-dimensional module. Hence $f_q(t, u, X)$ in Theorem 4.1 is irreducible apart from the trivial linear factor. Again Maple was not able to confirm this for $q = 4$.

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POINCARÉ–EINSTEIN METRICS AND THE SCHOUTEN TENSOR

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We examine the space of conformally compact metrics g on the interior of a compact manifold with boundary which have the property that the k^{th} elementary symmetric function of the Schouten tensor A_g is constant. When $k = 1$ this is equivalent to the familiar Yamabe problem, and the corresponding metrics are complete with constant negative scalar curvature. We show for every k that the deformation theory for this problem is unobstructed, so in particular the set of conformal classes containing a solution of any one of these equations is open in the space of all conformal classes. We then observe that the common intersection of these solution spaces coincides with the space of conformally compact Einstein metrics, and hence this space is a finite intersection of closed analytic submanifolds.

Let $\overline{M}^{n+1}$ be a smooth compact manifold with boundary. A metric g defined on its interior is said to be *conformally compact* if there is a nonnegative defining function ρ for $\partial\overline{M}$ (i.e., $\rho = 0$ only on $\partial\overline{M}$ and $d\rho \neq 0$ there) such that $\overline{g} = \rho^2 g$ is a nondegenerate metric on $\overline{M}$. The precise regularity of ρ and $\overline{g}$ is somewhat peripheral and shall be discussed later. Such a metric is automatically complete. Metrics which are conformally compact and also Einstein are of great current interest in (some parts of) the physics community, since they serve as the basis of the AdS/CFT correspondence [**24**], and they are also quite interesting as geometric objects. Since they are natural generalizations of the hyperbolic metric on the ball B^{n+1}, as well as the complete constant negative Gauss curvature metrics on hyperbolic Riemann surfaces – which exist in particular on the interiors of arbitrary smooth surfaces with boundary – and which are often called Poincaré metrics [**19**], we say that a metric which is both conformally compact and Einstein is *Poincaré-Einstein* (or P-E for short). Until recently, beyond a handful of examples, the only general existence result concerning the existence of P-E metrics was the local perturbation theory of Graham and Lee [**11**], which gives an infinite dimensional family of such metrics in a neighborhood of the hyperbolic metric on the ball, parametrized by conformal classes on the boundary sphere near to the standard one. Recently many new existence results have been obtained, including further perturbation results by Biquard [7] and Lee [13], and Anderson has some important global existence results in dimension four [3]. Many interesting geometric and topological properties of these metrics have also been found [10], [23], [1] and [2]; this last paper also surveys a number of intriguing examples of P-E metrics.

A common thread through the analytic approaches to the construction of these metrics is the possible existence of an L^2 obstruction, or more simply a finite dimensional cokernel of the (suitably gauged) linearization of the Einstein equations around a solution. For any P-E metric where this obstruction is trivial, the implicit function theorem readily implies that the moduli space $\mathcal{E}$ of P-E metrics is (locally) a Banach manifold, parametrized by conformal classes of metrics on $\partial\overline{M}$. (Actually, the smoothness of $\mathcal{E}$ is true in generality [3], but this geometric parametrization breaks down.) Unfortunately, the only known geometric criteria ensuring the vanishing of this obstruction are strong global ones [13].

One purpose of this note is to introduce some new ideas into this picture which may help elucidate the structure of this moduli space. We consider a related family of conformally compact metrics which satisfy certain *scalar* nonlinear equations, including and generalizing the familiar Yamabe equation, which we introduce below. These are sometimes called the σ_k-Yamabe equations, $k = 1, \cdots, n+1$. The hyperbolic metric on the ball, or indeed an arbitrary P-E metric on any manifold with boundary satisfies each of these equations, and conversely, in this particular (conformally compact) setting, metrics which satisfy every one of these scalar problems are also P-E. The punchline is that, in the conformally compact case, the deformation theory for the σ_k-Yamabe equations is always unobstructed! This fact seems to have been unappreciated, except for the case $k = 1$. The full implications of this statement in the conformally compact case for the moduli space of P-E metrics is not completely evident at this point, but this relationship seems quite likely to be of some value. Furthermore, the deformation theory for these σ_k-Yamabe metrics is new, and also of some interest.

To define these equations, recall the Schouten tensor A_g, defined for any metric g on a manifold of dimension $n+1$ by the formula

$$A_g := \frac{1}{n-1}\left(\mathrm{Ric} - \frac{R}{2n} g\right); \tag{1}$$

here $\mathrm{Ric} := \mathrm{Ric}_g$ and $R := R_g$ are the Ricci tensor and scalar curvature function for g. This tensor occupies a prominent position in conformal geometry because it transforms quite nicely under conformal changes of metric. In fact, if $\widetilde{g} := e^{2u}g$, then

$$A_{\widetilde{g}} = A_g - Ddu + du \otimes du - \frac{1}{2}|du|^2 g. \tag{2}$$

We refer to [6] for a derivation of this formula. For later reference, note that in terms of any local coordinate system w,

$$|du|^2\, g := \sum_{i,j,k,\ell} g^{k\ell}\partial_{w_k}u\,\partial_{w_\ell}u\, g_{ij}\, dw^i\, dw^j,$$
$$du \otimes du := \sum_{i,j} \partial_{w_i}u\,\partial_{w_j}u\, dw^i\ dw^j,$$

and

$$D\,du := \sum_{i,j}\left(\partial^2_{w_iw_j}u - \sum_k \Gamma^k_{ij}\partial_{w_k}u\right) dw^i dw^j.$$

We have (somewhat inconsistently) used raised indices in the differentials (i.e., dw^i, etc.) in accord with the standard summation convention.

Definition 1. The metric g is a σ_k-Yamabe metric if $\sigma_k(A_g)$, the k^{th} elementary symmetric function of the eigenvalues of A_g computed with respect to g, is constant.

The problem of finding σ_k-Yamabe metrics is usually posed as a problem in conformal geometry: Starting with an arbitrary metric g and given $\beta \in \mathbb{R}$, the σ_k-Yamabe problem consists in finding a new metric $\widetilde{g} = e^{2u}g$, in the conformal class of g, such that $\sigma_k(A_{\widetilde{g}}) = (-1)^k\beta$. (In the main case of interest here, the eigenvalues of A_g are all negative, and so the constant β is positive; this explains our choice of sign.) This way we reduce the problem to finding a solution u to some scalar nonlinear partial differential equation. Notice that when $k = 1$, $\sigma_1(A_g) = R/2n$, and so $\widetilde{g}$ is a σ_1-Yamabe metric if and only if its scalar curvature is constant. In this case the equation for u becomes

$$\Delta_g u + \frac{n-1}{2}\,|\nabla_g u|^2 - \frac{R}{2n} = \beta\, e^{2u}.$$

Defining v by $v^{4/(n-1)} = e^{2u}$ (and keeping in mind that $\dim M = n+1$), the equation for v assumes the familiar form

$$\Delta_g v - \frac{R(n-1)}{4n}\, v = -\frac{n-1}{2}\beta\, v^{\frac{n+3}{n-1}},$$

and the existence theory when M is compact is complete and by now well-known [14]. However, when $k > 1$, the equation for u is fully nonlinear and the existence theory is much less well understood. Recent significant progress has been made by Chang-Gursky-Yang [8] when $k = 2$, and also by Viaclovsky [22], but much remains to be understood. In particular, in contrast with the ordinary Yamabe problem, for $k > 1$ the σ_k-Yamabe problem seems to be somewhat more tractable for positively curved metrics: a crucial a priori $\mathcal{C}^2$ estimate is missing in the case where all eigenvalues of A_g are negative [**22**].

We now write out the σ_k-Yamabe equations (within a conformal class) more explicitly. Fixing g and using (2), we see that

$$\widetilde{g} = e^{2u}\, g \qquad \text{satisfies} \qquad \sigma_k(A_{\widetilde{g}}) = (-1)^k \beta \tag{3}$$

provided

$$\mathcal{F}_k(g,u,\beta) := \sigma_k\left(D\, du - du \otimes du + \frac{1}{2}|du|^2 g - A_g\right) - \beta e^{2ku} = 0. \tag{4}$$

The symmetric function of the eigenvalues of $A_{\widetilde{g}}$ here is computed with respect to g rather than $\widetilde{g}$, which accounts for the exponential factor; the sign on the final term comes from taking σ_k of $-A_{\widetilde{g}}$. For any constant β, we define

$$\Sigma_k(\beta) := \left\{\widetilde{g} = e^{2u}\, g : \quad \mathcal{F}_k(g,u,\beta) = 0\right\}, \tag{5}$$

which is some subset within the space of all metrics on M.

As already indicated, the main result here involves the perturbation theory for solutions of $\mathcal{F}_k(g,u,\beta) = 0$, or equivalently, the structure of the sets $\Sigma_k(\beta)$, in the case where M^{n+1} is a manifold with boundary and all metrics are conformally compact. In this case, we will fix a defining function ρ for $\partial\overline{M}$ and write any conformally compact metric g on M as $g = \rho^{-2}\,\bar{g}$ where $\bar{g}$ is a metric on $\overline{M}$.

Since conformally compact metrics have asymptotically negative (in fact, isotropic) sectional curvatures, $\Sigma_k(\beta)$ is nonempty only when $\beta > 0$. Indeed, a brief calculation shows that when $g = \rho^{-2}\overline{g}$, then near any point of the boundary (where $\rho = 0$),

$$A_g = -\frac{1}{2}|d\rho|^2_{\overline{g}}\, g + \mathcal{O}(\rho^{-1}). \tag{6}$$

Notice that although g only determines ρ and $\overline{g}$ up to a conformal factor (i.e., g is also equal to $(a\rho)^{-2}(a^2\overline{g})$ for any $a \in \mathcal{C}(\overline{M})$), the function $|d\rho|^2_{\overline{g}}$ is well-defined at $\rho = 0$, regardless of this choice. Also, since ρ is a defining function for $\partial\overline{M}$, this quantity is by definition strictly positive at the boundary. We conclude that for any conformally compact metric g,

$$\sigma_k(A_g) = \left(-\frac{1}{2}|d\rho|^2_{\overline{g}}\right)^k \binom{n+1}{k} + \mathcal{O}(\rho), \tag{7}$$

near $\partial\overline{M}$. If $\sigma_k(A_g)$ is constant on M, then necessarily $|d\rho|^2_{\overline{g}}$ is constant along the boundary, and so, multiplying g by a constant, we may as well assume that

$$|d\rho|^2_{\bar{g}} \equiv 1 \qquad \text{when} \quad \rho = 0.$$

In this case, the limit of $\sigma_k(A_g)$ at any point of $\partial\overline{M}$ equals the particular constant $(-1)^k\beta_k^0$ which corresponds to the hyperbolic metric g_0 on B^{n+1}, namely

$$\beta_k^0 := 2^{-k}\binom{n+1}{k}. \tag{8}$$

Our main result gives a rich class of conformally compact σ_k-Yamabe metrics on the manifold M, granting the existence of at least one such metric. In particular, it states that the deformation theory for this problem is always unobstructed whenever $\beta > 0$. More precisely, we have:

Theorem 1. *Let $\overline{M}$ be a compact smooth manifold with boundary and ρ a fixed defining function for $\partial\overline{M}$. For any metric $\bar{g}$ on $\overline{M}$, denote by $[\bar{g}]$ its conformal class. Suppose that $\sigma_k(A_{\rho^{-2}\overline{g}}) = (-1)^k\beta_k^0$. Then there is a $\mathcal{C}^{2,\alpha}$-neighborhood $\mathcal{U}$ of $[\bar{g}]$ in the space of conformal classes on $\overline{M}$ such that every conformal class $[\overline{g}']$ in this neighborhood contains a unique metric $\overline{g}'_u = e^{2u}\overline{g}'$ with*

$$\sigma_k(A_{\rho^{-2}\,\overline{g}'_u}) = (-1)^k\beta_k^0,$$

which is near to $\bar{g}$; the set of these solution metrics fills out an (open piece of an) analytic Banach submanifold, with respect to an appropriate Banach topology.

As noted above, the analogue of this theorem holds also when M is compact without boundary, and the proof is similar but even more straightforward. For the record, we state this result too:

Theorem 2. *Fix $\beta > 0$. Let g be a metric on the compact manifold M and $[g]$ its conformal class. Suppose that $A_g \in \Gamma_k^-$ (see §1) and $\sigma_k(A_g) = (-1)^k\beta$. Then there is a neighborhood $\mathcal{U}$ of $[g]$ in the space of conformal classes on M such that every conformal class $[g']$ in this neighborhood contains a unique metric $g'_u = e^{2u}g'$ with*

$$\sigma_k(A_{g'_u}) = (-1)^k\beta,$$

which is near to g; the set of these solution metrics fills out an (open piece of an) analytic Banach submanifold, with respect to an appropriate Banach topology.

Let us return to conformally compact metrics, and connect Theorem 1 with the first theme discussed in the introduction. To begin with, notice that a metric g is Einstein if and only if:

$$\mathrm{Ric}_g = \frac{R}{n+1}g. \tag{9}$$

It is well-known that if this is the case then the scalar curvature R is constant. Now (9) is equivalent to either of the two conditions:

$$A_g = \frac{R}{2n(n+1)} g, \qquad \text{or}$$

$$\sigma_k(A_g) = \left(\frac{R}{2n(n+1)}\right)^k \binom{n+1}{k} \quad \text{for} \quad k = 1, \dots, n+1.$$

Hence, Poincaré-Einstein metrics are also σ_k-Yamabe metrics for every $k = 1, \cdots, n+1$ and, if the scalar curvature is normalized so that $R = -n(n+1)$, the constants $\sigma_k(A_g)$ must equal the constants $(-1)^k \beta_k^0$ for hyperbolic space. In particular, the moduli space $\mathcal{E}$ of P-E metrics is equal to the intersection of the $\Sigma_k(\beta_k^0)$ over all k. This gives:

Corollary 1. *The moduli space $\mathcal{E}$ of conformally compact Poincaré-Einstein metrics is a finite intersection of Banach submanifolds, and is closed in the space of conformally compact metrics on M.*

The first statement in this corollary follows directly from the preceding discussion and Theorem 1, while the second statement follows from the fact that the space of σ_1-Yamabe metrics with scalar curvature equal to a fixed negative constant is closed.

In some sense, Corollary 1 shows that the somewhat less tractable space $\mathcal{E}$ is a finite intersection of submanifolds Σ_j, each of which is an *analytic* submanifold, but more importantly, each of which has an unobstructed deformation theory. This amounts to some sort of figurative 'factorization' of the Einstein equations into $n+1$ scalar (albeit fully nonlinear) equations.

The plan for the rest of this paper is as follows: §1 reviews the structure of the functionals $\mathcal{F}_k$ and their linearizations $\mathcal{L}_k$, and this is followed in §2 by a discussion of the function spaces and of the mapping properties of the $\mathcal{L}_k$ on these spaces. The deformation theory for the σ_k-Yamabe equations and the proof of Theorems 1 and 2 is the topic of §3. Finally, §4 contains a list of some interesting open questions raised by the results here.

1. The functionals $\mathcal{F}_k$.

Let us fix a conformally compact metric g_0, which we may as well take to be smooth, i.e., $g_0 = \rho^{-2}\overline{g}_0$, where both ρ and $\overline{g}_0$ are $\mathcal{C}^\infty$ on $\overline{M}$. Fix also a constant $\beta > 0$. Recall that the metric $g = e^{2u} g_0$ is in $\Sigma_k(\beta)$, and so has $\sigma_k(A_g) = (-1)^k \beta$, provided

$$\mathcal{F}(g_0, u, \beta) = \sigma_k\left(Ddu - du \otimes du + \frac{1}{2}|du|^2 g_0 - A_{g_0}\right) - \beta\, e^{2ku} = 0.$$

In this section we recall some facts about the ellipticity of this operator and the structure of its linearization. These facts are taken from [22], and we refer there for all proofs and further discussion.

To approach the issue of ellipticity, first consider the k^{th} elementary symmetric function σ_k as a function on vectors $\lambda = (\lambda_1, \cdots, \lambda_{n+1}) \in \mathbb{R}^{n+1}$. Let Γ_k^+ denote the connected component of the open set $\{\lambda : \sigma_k(\lambda) > 0\}$ containing the positive orthant $\{\lambda : \lambda_j > 0 \, \forall \, j\}$. These are all convex cones with vertices at the origin and

$$\{\lambda : \lambda_j > 0 \; \forall j\} = \Gamma_{n+1}^+ \subset \Gamma_n^+ \subset \cdots \subset \Gamma_1^+ = \{\lambda : \sigma_1(\lambda) > 0\}.$$

Also, let $\Gamma_k^- = -\Gamma_k^+$. A real symmetric matrix is said to lie in $\Gamma_k^\pm$ if its eigenvalues lie in the corresponding set.

We may equally well consider symmetric two-tensors and their eigenvalues relative to a metric g, and so we shall transfer to this setting, which is more natural in terms of the geometric notation.

Proposition 1. *If $A_g \in \Gamma_k^-$, then $u \to \mathcal{F}_k(g, u, \beta)$ is elliptic at any solution of $\mathcal{F}_k(g, u, \beta) = 0$.*

The proof of this in [22] (see also [21]) relies on the computation of the linearization of $\mathcal{F}_k$ in the direction of the conformal factor u. The neatest formulation of this requires a definition from linear algebra. For any real, symmetric matrix B, and any $q = 0, \cdots, n+1$, define the q^{th} Newton transform of B as the new real, symmetric matrix

$$T_q(B) = \sigma_q(B)I - \sigma_{q-1}(B)B + \cdots + (-1)^q B^q.$$

Of course, $T_{n+1}(B) = 0$. If B is a symmetric two-tensor, then $T_q(B)$ is defined as a symmetric two-tensor in the obvious way. Now suppose that $B = B(\varepsilon)$ depends smoothly on a parameter ε, and write $B'(0) = \dot{B}$. It is proved in [20] that

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \sigma_k(B(\varepsilon)) = \operatorname{Tr}(\dot{B} T_{k-1}(B)). \tag{10}$$

(For symmetric two-tensors, this trace is just the g-inner product of $\dot{B}$ with $T_{k-1}(B)$.)

We apply this to the Schouten tensors associated to the family of metrics $g(\varepsilon) = e^{2\varepsilon\phi} g$ where $g \in \Sigma_k(\beta_k)$. We may use the metric g to identify symmetric 2-tensors with $(n+1) \times (n+1)$ matrices, or completely equivalently, compute traces of such tensors using the metric and regard the trace of the product of matrices on the right side of (10) as the g-inner product of tensors. We have

$$B(\varepsilon) = \left(-A_g + \varepsilon \, D \, d\phi + \varepsilon^2 \left(\frac{1}{2} |d\phi|^2 - d\phi \otimes d\phi \right) \right),$$

so that $B(0) = B = -A_g$ and $\dot{B}(0) = \dot{B} = Dd\phi$. This gives the formula

$$\mathcal{L}_k\phi := D\mathcal{F}_k\big|_{g,0}(0,\phi) = \langle T_{k-1}(-A_g), Dd\phi\rangle_g - 2k\beta_k\phi. \tag{11}$$

The proof of Proposition 1 in general (i.e., when g is not necessarily a solution itself and when the linearization is computed at some solution $u \neq 0$) relies on the (nonobvious) fact that $T_{k-1}(B)$ is positive definite when $B \in \Gamma_k^+$. We refer to [**22**] (or [**21**]) for further details.

Let us compute $\mathcal{L}_k$ more explicitly when g is hyperbolic, or in fact, when g is an arbitrary Poincaré-Einstein metric. We shall always normalize the metric so that the scalar curvature is given by $R = -n(n+1)$, in which case the Einstein condition becomes $\mathrm{Ric} = -ng$ and $\sigma_k(A_g) = (-1)^k\,\beta_k^0$. By (9), if g is P-E then $A_g = -\frac{1}{2}\,g$, and so

$$T_{k-1}(-A_g) = 2^{1-k}T_{k-1}(g) = 2^{1-k}\sum_{j=0}^{k-1}(-1)^j\binom{n+1}{k-1-j}\,g.$$

One may check by induction that this sum has a closed form expression, and this leads to the identity

$$T_{k-1}(-A_g) = c_{k,n}\,g, \qquad \text{where} \qquad c_{k,n} = 2^{1-k}\binom{n}{k}.$$

The key observation here is that $c_{k,n} > 0$. Altogether, we obtain the formula

$$\mathcal{L}_k\phi = c_{k,n}\,\Delta\phi - 2k\,\beta_k^0\,\phi, \tag{12}$$

which holds whenever g is Poincaré-Einstein with $\mathrm{Ric} = -ng$.

If $g \in \Sigma_k(\beta)$ is a more general solution (i.e., not necessarily P-E), then $\mathcal{L}_k$ is more complicated. However, certain properties remain valid. A direct calculation yields:

Proposition 2. *Suppose $g \in \Sigma_k(\beta)$, $\beta > 0$, and let $\mathcal{L}_k$ denote the linearization of $\mathcal{F}_k$ at $u = 0$. Then*

$$\mathcal{L}_k\phi = c_{k,n}\Delta_g\phi - 2k\beta\phi + \rho^3E\phi, \tag{13}$$

where E is a second order operator with bounded coefficients on $\overline{M}$ (smooth if ρ and $\overline{g} = \rho^2 g$ are smooth), and without constant term.

Note especially one of the key point in this result that the operator E contains no constant term.

We remark that (13) may also be obtained from general principles involving the theory of uniformly degenerate operators [**15**] and [**16**]. Since some of the main results of this theory will be invoked later anyway, we digress briefly to explain this setup. Choose coordinates $(x, y) := (x, y_1, \ldots, y_n)$,

$x \geq 0$ near a point of the boundary of $\overline{M}$. A second order operator L is said to be uniformly degenerate if it may be expressed in the form

$$L = \sum_{j+|\alpha|\leq 2} a_{j,\alpha}(x,y)(x\,\partial_x)^j(x\,\partial_y)^\alpha. \tag{14}$$

The coefficients may be scalar or matrix-valued, and although we usually assume they are smooth, it is easy to extend most of the main conclusions of this theory when they are polyhomogeneous, or of some finite regularity. Operators of this type arise naturally in geometry, and in particular all of the natural geometric operators associated to a conformally compact metric are uniformly degenerate. Note that the error term $\rho^3 E$ in (13) is actually of the form $\rho E'$ where E' is some second order uniformly degenerate operator without constant term.

The 'uniformly degenerate symbol' of this operator is elliptic provided

$$\sigma(L)(x,y;\xi,\eta) := \sum_{j+|\alpha|=2} a_{j,\alpha}(x,y)\xi^j\eta^\alpha \neq 0 \qquad \text{when} \quad (\xi,\eta)\neq 0.$$

(For systems, we require $\sigma(L)$ to be invertible as a matrix when $(\xi,\eta)\neq 0$.) We also define the associated *normal operator*

$$N(L) := \sum_{j+|\alpha|\leq 2} a_{j,\alpha}(0,y)(s\partial_s)^j(s\partial_v)^\alpha.$$

The boundary variable y enters only as a parameter, while the 'active' variables (s,v) in this expression may be regarded as formal, but in fact are naturally identified with linear coordinates on the inward pointing half-tangent space $T^+_{(0,y)}M$. In particular:

Proposition 3. *If g is a smooth conformally compact metric (normalized so that $|d\rho|^2_{\bar{g}} = 1$ at $\partial\overline{M}$), then its Laplace-Beltrami operator Δ_g is an elliptic uniformly degenerate operator with normal operator*

$$N(\Delta_g) = \Delta_{\mathbb{H}^{n+1}} := (s\,\partial_s)^2 + s^2\,\Delta_{\mathbb{R}^n} - n\,s\,\partial_s. \tag{15}$$

Furthermore, if $g\in\Sigma_k(\beta^0_k)$, then the linearization $\mathcal{L}_k$ of $\mathcal{F}_k$ at $u=0$ is also elliptic and uniformly degenerate, with normal operator

$$\mathcal{L}^0_k := N(\mathcal{L}_k) = c_{k,n}((s\,\partial_s)^2 + s^2\,\Delta_{\mathbb{R}^n} - n\,s\,\partial_s) - 2k\,\beta^0_k. \tag{16}$$

As we explain in the next section, the operator $\mathcal{L}_k$ is Fredholm on various natural function spaces. This specializes a criterion which is applicable to other more general uniformly degenerate operators L, namely that L is Fredholm if and only if two separate ellipticity conditions hold: First, the symbol $\sigma(L)$ should be invertible, and in addition, the normal operator $N(L)$ must be invertible on certain weighted L^2 spaces.

2. Function spaces and mapping properties.

Let $\mathcal{L}_k$ be the linearization considered in the last section. We shall now describe some of its mapping properties. As indicated above, these properties also hold for more general elliptic, uniformly degenerate operators L.

We first review one particular scale of function spaces which is convenient in the present setting, and then state the mapping properties on them enjoyed by $\mathcal{L}_k$. The material here is taken from [**15**], to which we refer for further discussion and proofs.

Fix a reference (smooth) conformally compact metric $g_0 = \rho^{-2}\overline{g}_0$; also, choose a smooth boundary coordinate chart (x, y) as in the previous section, and recall the basic vector fields $x\,\partial_x$ and $x\,\partial_{y_j}$, $j = 1, \ldots, n$. Since x is a smooth nonvanishing multiple of ρ near $\partial\overline{M}$, these vector fields are all of uniformly bounded lengths with respect to g_0, and are also uniformly independent as $x \searrow 0$. There are two equivalent ways to define the Hölder space $\Lambda^{\ell,\alpha}(M)$, $\ell \in \mathbb{N}$, $\alpha \in (0,1)$. In either case, it suffices to work in a boundary coordinate chart. The first is to set

$$\Lambda^{0,\alpha}(M) := \left\{ u : \sup \frac{|u(x,y) - u(x',y')|(x+x')^\alpha}{|x-x'|^\alpha + |y-y'|^\alpha} \right\},$$

where the supremum is taken first over all points $w = (x,y)$, $w' = (x',y')$, $w \neq w'$, which lie in some coordinate cube B centered at a point $w_0 = (x_0, y_0)$ of sidelength $\frac{1}{2}x_0$, and then over all such cubes. The other is to let B denote a ball of unit radius with respect to the metric g_0 centered at w_0, and to replace the quotient in this definition by

$$\frac{|u(x,y) - u(x',y')|}{\mathrm{dist}_{g_0}(w,w')^\alpha}$$

and then take the supremum over all $w \neq w' \in B$, and then over all such balls B.

This latter definition is more geometric, while the former definition clearly implies the scale invariance of these spaces, namely that if $u(w)$ is defined (and, say, compactly supported) in one of these coordinate charts and if we define $u_\varepsilon(w) = u(w/\varepsilon)$, then the associated norms of u and u_ε are the same.

We shall also use a few other closely related spaces:

- For $\ell \in \mathbb{N}$ and $\alpha \in (0,1)$, let

$$\Lambda^{\ell,\alpha}(M) := \left\{ u : (x\,\partial_x)^j (x\,\partial_y)^\beta u \in \Lambda^{0,\alpha}(M)\ \forall\ j + |\beta| \leq \ell \right\}.$$

- For $\gamma \in \mathbb{R}$, $\ell \in \mathbb{N}$ and $\alpha \in (0,1)$, let

$$\rho^\gamma \Lambda^{\ell,\alpha}(M) := \left\{ u : u = \rho^\gamma \widetilde{u}, \quad \text{where} \quad \widetilde{u} \in \Lambda^{\ell,\alpha}(M) \right\}.$$

Thus the first of these are the natural higher order Hölder spaces associated to the geometry of g_0, or equivalently, to differentiations with respect to the vector fields $x\,\partial_x$ and $x\,\partial_y$. The second of these spaces are the usual weighted analogues. The corresponding norms are $||\cdot||_{\ell,\alpha}$ and $||\cdot||_{\ell,\alpha,\gamma}$, respectively.

We could equally easily have defined L^2- and L^p-based Sobolev spaces, corresponding to differentiations with respect to the vector fields $x\,\partial_x$ and $x\,\partial_y$. The mapping properties we state below all have direct analogues for these spaces. However, as usual, Hölder spaces are perhaps the simplest to deal with for nonlinear PDE.

Now let us turn to the mapping properties of $\mathcal{L}_k$ in the case where the conformally compact metric g at which $\mathcal{L}_k$ is computed satisfies $\sigma_k(A_g) = (-1)^k\,\beta_0^k$. First of all, it follows immediately from the definitions that

$$\mathcal{L}_k : \rho^\gamma\,\Lambda^{\ell+2,\alpha}(M) \longrightarrow \rho^\gamma\,\Lambda^{\ell,\alpha}(M) \tag{17}$$

is a bounded mapping for any $\gamma \in \mathbb{R}$ and $0 \le \ell$. However, this map is not well-behaved for many values of the weight parameter γ. There are two ways this may occur. First if γ is sufficiently large positive, then it is not hard to see that (17) has an infinite dimensional cokernel, while dually, if γ is sufficiently large negative, then (17) has an infinite dimensional nullspace. Although we do not use it here, less trivial is the fact that in either of these two cases the mapping is semi-Fredholm (i.e., has closed range and either the kernel or cokernel are finite-dimensional).

However, for certain values of γ the range of this mapping may not be closed. This is determined by a consideration of the indicial roots of $\mathcal{L}_k$. We say that γ is an indicial root of $\mathcal{L}_k$ if $\mathcal{L}_k(\rho^\gamma) = O(\rho^{\gamma+1})$ (note that because of the uniform degeneracy of $\mathcal{L}_k$, $\mathcal{L}_k(\rho^\gamma) = O(\rho^\gamma)$ is true for any value of γ). Thus γ is an indicial root only if some special cancellation occurs. It is clear that the indicial roots of $\mathcal{L}_k$ agree with those of its normal operator $\mathcal{L}_k^0$, and then (16) shows that γ is an indicial root if and only if

$$c_{k,n}(\gamma^2 - n\gamma) - 2k\beta_k^0 = 0,$$

or in other words $\gamma \in \{\gamma_\pm\}$ where

$$\gamma_\pm := \frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{2k\beta_k^0}{c_{k,n}}}.$$

In particular

$$\gamma_- < 0 < n < \gamma_+$$

since $\beta_k, c_{k,n} > 0$.

The relevance of these indicial roots to the mapping properties of (17) is that when γ is equal to one of these two values, then (17) does not have closed range. At heart, this stems from the fact that the equation

$$\mathcal{L}_k^0 u = s^{\gamma_\pm},$$

has solution $u = cs^{\gamma_\pm}(\log s)$ for some constant c, i.e., the inhomogeneous term is in the appropriate weighted Hölder space but the solution u just misses being in this space.

Despite these cautions, we have the following basic result:

Mapping properties: *If $\gamma_- < \gamma < \gamma_+$, then the mapping (17) is Fredholm of index zero.*

The main result of [**15**] is a considerably more general theorem of this sort for more general elliptic uniformly degenerate differential operators. There are two special features of $\mathcal{L}_k$ which enter into the precise form of the statement here. First, there is a nontrivial interval (γ_-, γ_+) between the two indicial roots $\gamma_\pm$, allowing for the possibility of a 'Fredholm range'. Second, the Fredholm index is zero for γ in this interval ultimately because $\mathcal{L}_k$ is self-adjoint on $L^2(dV_g)$.

We claim that the mapping (17) is actually invertible when γ is in this Fredholm range. By the result just stated, this claim will be proved if we show that when $\gamma \in (\gamma^-, \gamma^+)$ the nullspace of $\mathcal{L}_k$ is trivial. We do this now. The basic observation is that the constant term in $\mathcal{L}_k$ is negative. If $\gamma > 0$, then any $\phi \in \rho^\gamma \Lambda^{\ell+2,\alpha}$ vanishes at $\partial \overline{M}$, and thus if $\mathcal{L}_k \phi = 0$, the maximum principle implies that $\phi = 0$, as desired. To prove the claim for every $\gamma \in (\gamma_-, \gamma_+)$, we first compute that for any two values $\gamma', \gamma'' \in (\gamma_-, \gamma_+)$ and constants $c', c'' > 0$ the function

$$w_{c',c''} := c'\, \rho^{\gamma'} + c''\, \rho^{\gamma''}$$

satisfies $\mathcal{L}_k w_{c,c'} < 0$ in some small collar neighbourhood $\overline{M}_\tau$ of the boundary, where $0 \leq \rho \leq \tau$. Now fix $\gamma'' \in (\gamma, \gamma_+)$ and choose c'' so that $|\phi| \leq c''\, \rho^{\gamma''}$ on the set where $\rho = \tau$. Fixing $\gamma' \in (\gamma_-, \gamma)$ and $c' > 0$, then for any $\varepsilon > 0$, the functions $v_\pm := w_{c',c''} \pm \phi$ satisfy $\mathcal{L}_k v_\pm < 0$ in $\overline{M}_\tau$ and $v_\pm \geq 0$ both at the inner boundary $\rho = \tau$ and also at near the outer boundary where $\rho = 0$. By the maximum principle again, $v_\pm > 0$ in $\overline{M}_\tau$. Letting $c' \searrow 0$, we conclude that $|\phi| \leq c''\, \rho^{\gamma''}$. Since we may choose $\gamma'' > 0$, this implies that ϕ is bounded in M and vanishes at $\partial \overline{M}$. This reduces us to the previous case.

This reasoning shows that, if $\gamma \in (\gamma_-, \gamma_+)$ and $\phi \in \rho^\gamma \Lambda^{\ell+2,\alpha}$ is a solution of $\mathcal{L}_k \phi = 0$, then $\phi \in \rho^{\gamma'} \Lambda^{\ell+2,\alpha}$ for any $\gamma' < \gamma_+$. However, on account of the following basic result from [**15**], a much sharper result is true.

Regularity of solutions: *If $\gamma_- < \gamma < \gamma_+$ and $\phi \in \rho^\gamma \Lambda^{\ell+2,\alpha}$ is a solution of $\mathcal{L}_k \phi = f$ where f vanishes to all orders at $\partial\overline{M}$, then as $x \to 0$,*

$$\phi(x,y) \sim \sum_{j=0}^{\infty} \phi_j(y)\, x^{\gamma_+ + j}, \qquad \textit{with} \qquad \phi_j(y) \in \mathcal{C}^\infty(\partial\overline{M}),$$

in particular $\phi \in \rho^{\gamma_+}\mathcal{C}^\infty(\overline{M})$.

3. Perturbation theory in $\boldsymbol{\Sigma_k}$.

We now proceed to the main deformation result. Let $\overline{M}$ be a smooth compact, $n+1$ dimensional manifold with boundary. We fix a smooth defining function ρ for $\partial\overline{M}$. For any $\ell \in \mathbb{N}$ and any $\alpha \in (0,1)$ we define

$$\mathfrak{M}^{\ell,\alpha}(\overline{M}) := \Big\{ \bar{g} \in \mathcal{C}^{\ell,\alpha}(\overline{M}; S^2(\overline{M})) : \quad |d\rho|^2_{\bar{g}} = 1 \quad \text{on} \quad \partial\overline{M} \Big\}.$$

Having set things up carefully, the proof of Theorem 1 is almost immediate. Let $g \in \Sigma_k(\beta_k^0)$ and consider the mapping

$$\mathcal{H} : \mathfrak{M}^{2,\alpha}(\overline{M}) \times \rho^\gamma\, \Lambda^{2,\alpha}(M) \longrightarrow \rho^\gamma\, \Lambda^{0,\alpha} \tag{18}$$

defined by

$$\mathcal{H}(h,u) := \mathcal{F}_k(\rho^{-2}h, u, \beta_k^0).$$

Near g, the set $\Sigma_k(\beta_k^0)$ is identified with the zero set of $\mathcal{H}$. In particular, $(h,u) = (g,0) \in \mathcal{H}^{-1}(0)$.

To find all other nearby solutions, we shall apply the implicit function theorem, very much in the spirit of the closely related papers [**18**] and [**11**]. Thus we must check two things:

(i) The mapping $\mathcal{H}$ in (18) is a smooth mapping between the corresponding Banach spaces.
(ii) The linear map $u \longrightarrow D\mathcal{H}|_{g,0}\,(0,u)$ is surjective between these spaces.

The first of these is straightforward from the definitions and (2) and (7), provided we choose the weight parameter $\gamma \in (0,1)$. As for the other, recall that the restriction of this Fréchet derivative to tangent vectors of the form $(0,u)$ corresponds to the operator $\mathcal{L}_k$. We have already checked that this is surjective provided we choose the weight parameter $\gamma \in (\gamma_-, \gamma_+)$. But since $(0,1) \subset (\gamma_-, \gamma_+)$, these restrictions on γ are not inconsistent. Thus fixing $\gamma \in (0,1)$, we obtain a smooth mapping

$$\Phi : \mathfrak{M}^{2,\alpha}(\overline{M}) \longrightarrow \rho^\gamma \Lambda^{2,\alpha}(M)$$

with $\Phi(g) = 0$ and such that

$$\mathcal{H}(h, \Phi(h)) \equiv 0.$$

Furthermore, all solutions of $\mathcal{H}(h,u)$ in a sufficiently small neighborhood of $(g,0)$ are of this form. This concludes the proof of Theorem 1.

We omit the proof of Theorem 2 because it is nearly identical; indeed, the only difference is that standard elliptic theory replaces the Fredholm theory for uniformly degenerate operators we have quoted.

4. Open questions and further directions.

We conclude this note by raising a few other problems and questions related to the results and methods here.

a) Because of the difficulty in obtaining a $\mathcal{C}^2$ estimate for the σ_k-Yamabe problem when $k > 1$, it is worth wondering whether it might be worthwhile to pose a weaker version of this problem, at least for conformally compact metrics on manifolds with boundary: Namely, given a conformal class $[h_0]$ on $\partial\overline{M}$, is it possible to extend this conformal class to at least some conformal class $[\overline{g}]$ on the interior such that the σ_k-Yamabe problem is solvable in $[\overline{g}]$? Probably there are infinitely many such extensions, as is the case when $k = 1$, but the added flexibility in this formulation may be of some use.

b) We have shown in Theorem 1 that $\Sigma_k(\beta_k^0)$ is a Banach submanifold in a neighborhood of g, and furthermore that it may be regarded as a graph over the space of conformal classes, or at least those conformal classes near to g. For $k = 1$, every conformal class on $\overline{M}$ contains a unique representative lying in $\Sigma_1(\beta_1^0)$, and thus $\Sigma_1(\beta_1^0)$ is a graph globally over the space $\mathfrak{C}$ of all conformal classes. It is not known whether this remains true when $k > 1$, and thus we define

$$\mathfrak{C}_k = \left\{\mathfrak{c} \in \mathfrak{C} : \mathfrak{c} \text{ contains at least one } g \in \Sigma_k(\beta_k^0)\right\}. \tag{19}$$

Note that $\mathfrak{C}_1 = \mathfrak{C}$, and Theorem 1 shows that $\mathfrak{C}_k$ is open in $\mathfrak{C}$ for every k.

It seems central to understand whether $\mathfrak{C}_k = \mathfrak{C}$, or in other words, whether every conformal class on M contains a conformally compact σ_k-Yamabe metric. Related to this is the observation that we do not know whether each of the submanifolds Σ_k itself is closed; this depends ultimately on whether some version of this $\mathcal{C}^2$ estimate holds.

It also seems interesting to ask for which ℓ-tuple $J = \{j_1, \dots, j_\ell\} \subset \{1, \dots, n+1\}$ is the set

$$\Sigma_J := \Sigma_{j_1}(\beta_{j_1}^0) \cap \cdots \cap \Sigma_{j_\ell}(\beta_{j_\ell}^0),$$

closed. Notice that if $1 \in J$, then this is certainly true because the $\mathcal{C}^2$ estimate for the conformal factor is routine for the scalar curvature equation.

c) The regularity of the metrics $g \in \Sigma_k(\beta_k^0)$ is an interesting question. When $k = 1$ this is resolved in [**16**], cf. also [**19**]: If g is a smooth conformally compact metric, then the conformal factor u corresponding to the unique solution $\widetilde{g} = e^{2u}g \in [g]$ has a polyhomogeneous expansion. Presumably a similar result holds for all k. Note that unless $\gamma_+ \in \mathbb{N}$, this expansion will involve nonintegral powers of ρ; this should not be viewed negatively, since functions with expansions of this form may be manipulated just as easily as smooth functions.

d) The σ_k-Yamabe problem considered here extends naturally to the more general setting of the singular σ_k-Yamabe problem: Given a smooth metric g_0 on a compact manifold M and a closed subset $\Lambda \subset M$, when is it possible to find a conformally related metric $g = e^{2u}g_0$ which is both complete on $M \subset \Lambda$ and a σ_k-Yamabe metric? When $k = 1$ it is known that the dimension of Λ is intimately related to the sign of the imposed scalar curvature of the solution, and very good existence results are known when Λ is a submanifold [**5**] and [**17**]. What is the correct statement, and to what extent is this true when $k > 1$? There are a number of interesting analytic problems of this nature, and we shall return to this soon.

e) In general (not just in the conformally compact setting), $\mathcal{E}$ sits inside the finite intersection $\cap\Sigma_k$. Does it appear here as a finite codimensional analytic set, and if so, is this related to some sort of Kuranishi reduction for the perturbation theory for $\mathcal{E}$?

f) If g is a conformally compact metric on M and if $\sigma_k(A_g) = (-1)^k\beta$ and $\sigma_{k'}(A_g) = (-1)^{k'}\beta'$ are constant, then necessarily

$$\left(\frac{\beta}{\beta_k^0}\right)^{1/k} = \left(\frac{\beta'}{\beta_{k'}^0}\right)^{1/k'}. \tag{20}$$

This follows at once from (6) and (7).

For any $(n+1)$-tuple of numbers $\beta \equiv (\beta_1, \ldots, \beta_{n+1})$, define

$$\Sigma(\beta) := \bigcap_{k=1}^{n+1} \Sigma_k(\beta_k).$$

By (20) it is clear that in the class of conformally compact metrics, $\Sigma(\beta) = \emptyset$ unless $\beta_k = (-\lambda)^k \beta_k^0$ for some $\lambda > 0$ and for all k.

By contrast, on any compact manifold without boundary, it may happen that for some other $(n+1)$-tuple β of positive numbers, the set $\Sigma(\beta)$ is nonempty. If $g \in \Sigma(\beta)$, then Ric_g has constant eigenvalues. In particular, metrics with $\nabla\mathrm{Ric} = 0$ are in $\Sigma(\beta)$ for some β. However, the reverse inclusion may not be true and it appears that very little is known about metrics with Ricci tensor having constant eigenvalues, but cf. [**4**].

The (presumably) smaller class of metrics with parallel Ricci tensor is more tractable, but it does not seem to be known if the eigenvalues can be constant without the Ricci tensor being parallel. Examples would be very welcome. Also, in any setting (compact or conformally compact or ...) it seems to be a very basic problem in Riemannian geometry to ask what are the possible $(n+1)$-tuples $(\beta_1, \ldots, \beta_{n+1})$ for which $\Sigma_1(\beta_1) \cap \cdots \cap \Sigma_{n+1}(\beta_{n+1})$ is nonempty?

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ON ANTIPODES ON A MANIFOLD ENDOWED WITH A GENERIC RIEMANNIAN METRIC

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We prove that a generic point of a C^r manifold endowed with a generic Riemannian structure has an unique antipode (i.e., farthest point). Furthermore, in the case of 2-dimensional manifolds, such a point is joined to its antipode by at most three minimizing geodesics ($r \geq 2$).

1. Introduction.

In a compact metric space (X, d), we call an *antipode* of a point $p \in X$, any point which realizes the maximum of the distance from x. Some conjectures about antipodes were formulated by H. Steinhauss [**2**]. As an example: Is the sphere the only surface on which each point admits a single antipode and such that the antipodal function is an involution? This question was recently solved by C. Vîlcu who exhibits some counterexamples [**6**]. Now, it is quite natural to investigate the generic case.

In the case of a convex surface S, Tudor Zamfirescu proves in [**7**] that a generic point of S has an unique antipode, and is joined to it by exactly three segments (A *segment*, or a *minimizing geodesic*, is simply a path whose length equals the distance between its extremities).

The aim of this paper is to give an analogous result in the frame of Riemannian geometry. We obtain that the generic uniqueness holds for a C^r manifold (of any dimension) endowed with a generic Riemannian metric (Theorem 1), and the fact that at most three geodesics go from a generic point to its antipode holds for such 2-dimensional manifolds (Theorem 2) ($r \geq 2$).

The first result is optimal, in sense that it fails if you remove any occurrence of "generic". The projective plane with constant curvature is obviously a counterexample to the attempt at deletion of the second occurrence. Concerning the first one, we only need to notice that any sufficiency "long" space admits points with more than one antipode. The antipodes, in such a space, are localized near its "extremities". Any point near enough one extremity has all its antipodes near the other one. By semicontinuity of the antipodal function (see Lemma 5), there exists a point which should have (at least) one antipode near each extremity.

However, in the case of convex surfaces [8, Theorem 4], as in the case of manifolds homeomorphic to the 2-dimensional sphere [5], a stronger result exists.

2. Baire categories notion of genericity.

For elementary results about Baire categories, we refer to any topology book. We simply recall that a subset of topological space is said to be of *first category* or *meager* if it is included in a countable union of closed sets with empty interior. A *Baire space* is a topological space where meager subsets have empty interior. Baire's theorem states that complete metric spaces are Baire spaces. In a Baire space, we say that a *generic* point satisfies a property, if all points in a *residual* subset (i.e., a subset whose complement is meager) satisfy this property.

Now we need to endow the set G^r of all C^r Riemannian structures on a given manifold M, with a topology which makes it a Baire space. This can be done in several slightly different ways (see [1] for an another construction).

We fix $g_0 \in G^r$. On one hand, g_0 provides a norm $\| \, \|_x$, on each fibre over x of the vector bundle

$$B_n = \overbrace{TM^* \otimes \cdots \otimes TM^*}^{n \text{ times}} \otimes TM^* \odot TM^*,$$

where $\odot$ denotes the symmetric tensor product. As M is compact, we can define a norm $\| \, \|_n$ on the set $\Gamma^r(B_n)$ of C^rsections of B_n by

$$\|\cdot\|_n = \sup_{x \in M} \|\cdot\|_x \, .$$

On the other hand, g_0 supplies its Levi-Cività covariant derivation ∇, from $\Gamma^r(B_n)$ to $\Gamma^{r-1}(B_{n+1})$. Put $\nabla^p = \nabla \circ \cdots \circ \nabla$, and define for $g \in G'^r \stackrel{\text{def}}{=} \Gamma^r(B_0)$

$$\|g\|_{C^r} = \max_{p=0,\ldots,r} \|\nabla^p g\|_p \, .$$

It is obvious that this real valued map is a norm on G'^r. Moreover G'^r endowed with this norm is a Banach space. For C^∞ manifolds, we define the metric

$$d^\infty(g, g') = \sum_{r=0}^{\infty} 2^{-r} \min\left(1, \|g - g'\|_{C^r}\right).$$

It is well-known that (G'^∞, d^∞) is a complete metric space whose topology equals the one defined by the norm family $(\| \, \|_{C^r})_{r \geq 0}$. Moreover, the topology induced by $\| \, \|_{C^r}$ does not depend on $g_0 \in G^r$.

The set G^r of C^r Riemannian structures can be defined by

$$G^r = \left\{ g \in G'^r \, | \forall x \in TM, g(x,x) > 0 \Leftrightarrow x \neq 0 \right\}.$$

It is obvious that G^r is open in G'^r, and so, is a Baire space.

3. Some continuity or upper semicontinuity results.

3.1. Some notation. As a matter of geometric notation, we define for $g \in G^r$, $L^g(\gamma)$ as the g-length of a curve γ, $\mathfrak{A}^g_x$ as the set of g-antipodes of a point $x \in M$, $\mathfrak{D}^g_x$ as the distance between x and its g-antipodes. The g-unit tangent bundle over M will be denoted by $T^{1g}M$. We also denote by D the set of all metrics on M which induce its topology. D is endowed by the metric δ defined by

$$\delta\left(d, d'\right) = \max_{(x,y)\in M^2} \left|d\left(x, y\right) - d'\left(x, y\right)\right|.$$

For $g \in G^r$, we denote by $d^g \in D$ the metric corresponding to the Riemannian structure g. At last, we define Σ^g_{xy} as the set of g-segments from x to y, and A^g_2 as the set of points of M which admit at least two g-antipodes.

We also need more abstract notations. For any metric space (X, d) we define $\mathcal{H}(X)$ as the set of all nonempty compact subsets of X, which is endowed with the well-known Hausdorff metric, denoted by the same symbol d:

$$d\left(K_1, K_2\right) = \max\left(d^{\subset}\left(K_1, K_2\right), d^{\subset}\left(K_2, K_1\right)\right),$$
$$d^{\subset}\left(K_1, K_2\right) = \max_{x\in K_1}\min_{y\in K_2} d(x, y).$$

It is well-known that $\mathcal{H}(X)$ is compact whenever X is compact. An $\mathcal{H}(X)$-valued function is said to be upper semicontinuous, if it is continuous for the topology induced by $d^{\subset}$. Note that $d^{\subset}$ satisfies the triangle inequality, and $d^{\subset}\left(K_1, K_2\right)$ vanishes if and only if $K_1 \subset K_2$.

Given a subset P of X, and a positive real number ρ, we denote by $P + \rho$ the union of all open balls of radius ρ centered at the elements of P.

We define $\mathcal{HG}$ as the set of compact metric spaces up to isometries. Given two spaces $X, Y \in \mathcal{HG}$, we say that X is included in Y, and write $X \subset Y$, if there exists an isometric injective map from X into Y. This defines a partial order on $\mathcal{HG}$. Now we put for (X, d_X), $(Y, d_Y) \in \mathcal{HG}$

$$d^{\subset}_{HG}\left(X, Y\right) = \inf_{(Z,d_z),\phi,\psi} d^{\subset}_Z(\phi(X), \psi(Y))$$
$$d_{HG}\left(X, Y\right) = \max\left(d^{\subset}_{HG}\left(X, Y\right), d^{\subset}_{HG}\left(Y, X\right)\right),$$

where the infimum is taken over all metric spaces (Z, d_Z), and all isometric injective maps, ϕ, from X to Z, and, ψ, from Y to Z. d_{HG} is nothing but the well-known Hausdorff-Gromov metric on $\mathcal{HG}$ [**3**]. A $\mathcal{HG}$-valued function is said to be upper semicontinuous if it is continuous for the topology induced by $d^{\subset}_{HG}$. It is easy to see that $d^{\subset}_{HG}$ satisfies the triangle inequality, moreover we have:

Lemma 1. *Let X,$Y \in \mathcal{HG}$. We have $d^{\subset}_{HG}(X,Y) = 0$ if and only if $X \subset Y$.*

Proof. It is clear that $X \subset Y$ implies $d^{\subset}_{HG}(X,Y) = 0$. Conversely, if $d^{\subset}_{HG}(X,Y) = 0$, then there exist metric spaces (Z_n, d_n), and injective isometries $\phi_n : X \to Z_n$ and $\psi_n : Y \to Z_n$ such that $d^{\subset}(\phi_n(X), \psi_n(Y)) < \frac{1}{n}$. Let Z be a coproduct of all compact metric spaces $Z'_n \stackrel{\text{def}}{=} \phi_n(X) \cup \psi_n(Y)$, $n \in \mathbb{N}$. We define on Z an equivalence relation R by zRz' if and only if both z and z' are the image of the same point $y \in Y$ by ψ_n for some $n \in \mathbb{N}$. Put $\psi = s \circ i_n \circ \psi_n$, with $i_n : Z'_n \to Z$ the canonical injection, and $s : Z \to Z/R$ the canonical surjection. Z/R is a metric space with the metric d_Z defined by

$$d_{Z/R}(z_n, z_m) = \inf_{y \in Y} \left(d_n(z_n, \psi(y)) + d_m(z_m, \psi(y))\right),$$
$$\text{for } z_i \in Z_i,\ i = n \text{ or } m,\ n \neq m$$
$$d_{Z/R}\left(z_n, z'_n\right) = d_n\left(z_n, z'_n\right), \text{ for } z_n, z'_n \in Z_n.$$

We claim that Z/R is compact. Let $(z_p) = \left(s\left(z'_p\right)\right)$ be a sequence of Z/R. z'_p belongs to $i_{n_p}\left(Z'_{n_p}\right)$ for some integer n_p, and then, there exists a point $y_p \in \psi(Y)$ such that $d_{Z/R}(z_p, y_p) \leq \frac{1}{n_p}$. If the sequence (n_p) is bounded by an integer N, then we can select from (z_p) a converging subsequence, by compactness of Z'_1, Z'_2, ... , Z'_N, else, we can assume by selecting suitable subsequences, that $\lim_p \frac{1}{n_p} = 0$. As Y is compact, we can select from (y_p), and so from (z_p) too, a converging subsequence. This proves the claim. Now, as $s \circ i_n \circ \phi_n$ are isometries (hence, form an equicontinuous family), we can extract a subsequence converging to an isometry $\phi : X \to \psi(Y)$. This completes the proof. □

Given a compact metric space X, we denote by $\mathfrak{j}_X$ the canonical map from $\mathcal{H}(X)$ to $\mathcal{HG}$. Of course, $\mathfrak{j}_X$ is upper semicontinuous and order preserving.

Let $X \in \mathcal{HG}$ and $P \subset \mathcal{HG}$, we will denote by $d^{\subset}_{HG}(X,P)$ the infimum $\inf\limits_{Y \in P} d^{\subset}_{HG}(X,Y)$. As $d^{\subset}_{HG}$ is not a metric, the following lemma is not obvious.

Lemma 2. *Let P be a compact subset of $\mathcal{HG}$, take $X \in \mathcal{HG}$. Then $d^{\subset}_{HG}(X, P) = 0$ if and only if there exists a compact metric space $Y \in P$ such that $X \subset Y$.*

Proof. Given an integer n, there exists a metric space $Y_n \in P$ such that $d^{\subset}_{HG}(X, Y_n) < \frac{1}{n}$. Let Y be the limit of a converging subsequence of (Y_n). We have $d^{\subset}_{HG}(X,Y) \leq d^{\subset}_{HG}(X,Y_n) + d_{HG}(Y_n, Y)$. As the right-hand side tends to zero, left-hand side must vanish. □

We have to consider a special subset $\mathcal{T} \subset \mathcal{HG}$, which is the set of those metric spaces, included in the unit circle (i.e., $\mathbb{R}/2\pi\mathbb{Z}$), whose cardinality is at most 3. It is clear that $\mathcal{T}$ is compact.

3.2. Useful lemmas. This miscellany is nothing but the list of lemmas we need.

Lemma 3. *The map from G^r to D which associates to g the corresponding metric d^g is locally Lipschitz continuous with respect to the metric δ and the norm $\| \; \|_{C^0}$.*

Proof. We denote by $N(g)$ the real number $\sup_{x \in T^{1g_0}M} \left| \frac{1}{g(x,x)} \right|$. Consider two Riemannian structures g and $g' = g + h$. Let x, y be two points of M, such that $\delta(d^g, d^{g'}) = d^{g'}(x,y) - d^g(x,y)$ (you may exchange g and g' if necessary). Let σ be a g-segment from x to y. We have

$$\begin{aligned}
d^{g\prime}(x,y) &\leq L^{g'}(\sigma) \\
&= \int \sqrt{g+h}\,(\dot\sigma(t), \dot\sigma(t))\, dt \\
&\leq \int \sqrt{g}\,(\dot\sigma(t), \dot\sigma(t))\, dt + \frac{1}{2} \int \frac{|h|}{\sqrt{g}}\,(\dot\sigma(t), \dot\sigma(t))\, dt \\
&\leq L^g(\sigma) + \frac{1}{2} \|h\|_{C^0} N(g) \int \sqrt{g}\,(\dot\sigma(t), \dot\sigma(t))\, dt \\
&\leq d^g(x,y) + \frac{1}{2} \|h\|_{C^0} N(g) L^g(\sigma) \\
&\leq d^g(x,y) + \frac{1}{2} \|h\|_{C^0} N(g) \operatorname{diam}(M,g).
\end{aligned}$$

Hence

$$\begin{aligned}
\delta(d^g, d^{g'}) &= d^{g\prime}(x,y) - d^g(x,y) \\
&\leq \frac{1}{2} \|h\|_{C^0} N(g) \operatorname{diam}(M,g).
\end{aligned}$$

□

Lemma 4. *The map $\mathfrak{D}$ from $G^r \times M$ to $\mathbb{R}$ is locally Lipschitz continuous with respect to both variables.*

Proof. Let x, x' be in M, g, g' be in G^r, and take a g-antipode y of x.

$$\begin{aligned}
\mathfrak{D}^g_x &= d^g(x,y) \\
&\leq d^{g'}(x',y) + d^g(x,x') + \delta(d^g, d^{g'}) \\
&\leq \mathfrak{D}^{g'}_{x'} + d^g(x,x') + \delta(d^g, d^{g'}).
\end{aligned}$$

Of course, the same holds when you exchange (x,g) and (x',g'). Lemma 3 completes the proof. □

Lemma 5. *The map $\mathfrak{A}$ from $G^r \times M$ to $\mathcal{H}(M)$ is upper semicontinuous.*

Proof. Suppose the result fails. There exists a real number $\varepsilon > 0$, such that for each integer n, there exists a point x_n and a Riemannian structure g_n such that

$$\|g - g_n\|_{C^r} < \frac{1}{n} \tag{1}$$
$$d^g(x, x_n) < \frac{1}{n}$$
$$\mathfrak{A}_{x_n}^{g_n} \not\subset A_x^g + \varepsilon.$$

The formula (1) implies the existence of a sequence (y_n) of g_n-antipodes of x_n such that $y_n \notin \mathfrak{A}_x^g + \varepsilon$. Now, select a converging subsequence from y_n, and denote by y its limit. On one hand, as $\mathfrak{A}_x^g + \varepsilon$ is open, $y \notin \mathfrak{A}_x^g$. On the other hand, by Lemmas 4 and 3, the identity $\mathfrak{D}_{x_n}^{g_n} = d^{g_n}(y_n, x_n)$ leads to $\mathfrak{D}_x^g = d^g(y, x)$, and we obtain a contradiction. □

Lemma 6. *The map*

$$G^r \times M \times M \to \mathcal{H}(\mathcal{H}(M))$$
$$(g, x, y) \mapsto \Sigma_{xy}^g$$

is upper semicontinuous.

Proof. Suppose the result fails. There exists $\varepsilon > 0$, and three sequences (g_n), (x_n) and (y_n), converging respectively to $g \in G^r$, $x \in M$, and $y \in M$, such that a g_n-segment σ_n from x_n to y_n satisfying $\min\{d^g(\sigma_n, \sigma') | \sigma' \in \Sigma_{xy}^g\} > \varepsilon$ exists. We can select from σ_n a converging subsequence which tends to a g-segment σ from x to y, and a contradiction is found. □

Lemma 7. *Let (X, d) be a metric space, K be a subset of $\mathcal{HG}$, and $F : X \to \mathcal{HG}$ be an upper semicontinuous function. The map from X to $\mathbb{R}$,*

$$x \mapsto d_{HG}^{\subset}(F(x), K)$$

is upper semicontinuous.

Proof. Choose $\varepsilon > 0$, $x \in X$, and put $\delta = d_{HG}^{\subset}(F(x), K)$. Choose $\chi \in K$ such that $d_{HG}^{\subset}(F(x), \chi) < \delta + \frac{\varepsilon}{2}$. There exists a compact metric space (Z_1, d) and two isometric injective maps $g_1 : F(x) \to Z_1$ and $h_1 : \chi \to Z_1$ such that

$$g_1(F(x)) \subset h_1(\chi) + \left(\frac{\varepsilon}{2} + \delta\right).$$

By upper semicontinuity of F, there exists a real number $\eta > 0$, such that for all points y of the open ball $\{x\} + \eta$ we have

$$d_{HG}^{\subset}(F(y), F(x)) < \frac{\varepsilon}{2}.$$

Hence, there exists a metric space (Z_2, d) and two isometric injective maps $g_2 : F(x) \to Z_2$ and $f_2 : F(y) \to Z_2$ such that

$$f_2(F(y)) \subset g_2(F(x)) + \frac{\varepsilon}{2}.$$

Let Z be a coproduct of Z_1 and Z_2 where $g_1(F(x))$ and $g_2(F(x))$ have been identified. We obtain three injective isometric maps $f : F(y) \to Z$, $g : F(x) \to Z$ and $h : \chi \to Z$ such that

$$g\left(F\left(x\right)\right) \subset h(\chi) + \left(\frac{\varepsilon}{2} + \delta\right)$$
$$f(F(y)) \subset g\left(F\left(x\right)\right) + \frac{\varepsilon}{2}.$$

It follows that

$$f(F(y)) \subset h(\chi) + (\varepsilon + \delta)\,.$$

We have proved that for all $y \in \{x\} + \eta$,

$$\begin{aligned} d^{\subset}_{HG}(F(y), K) &\leq d^{\subset}_{HG}(F(y), \chi) \\ &\leq d^{\subset}_{HG}(F(x), K) + \varepsilon. \end{aligned}$$

□

Lemma 8. *Let (X, d) and (Y, d) be metric spaces. Let $F : X \to \mathcal{H}\,(Y)$ be an upper semicontinuous function, and $G : \mathcal{H}\,(Y) \to \mathcal{HG}$ be an order preserving (for order $\subset$) and upper semicontinuous function. Then $G \circ F$ is upper semicontinuous.*

Proof. Choose $\varepsilon > 0$ and $x \in X$. The upper semicontinuity of G implies the existence of a real number $\varepsilon' > 0$ such that for all K in $\mathcal{H}\,(Y)$,

$$d\left(K, F(x)\right) \leq \varepsilon' \Longrightarrow d^{\subset}_{HG}(G(K), G \circ F(x)) < \varepsilon. \tag{2}$$

By upper semicontinuity of F, there exists a real number $\eta > 0$ such that for all $y \in \{x\} + \eta$

$$F\left(y\right) \subset F\left(x\right) + \varepsilon'. \tag{3}$$

Put $K_0 = F\left(x\right) \cup F\left(y\right)$. As G preserves order, we have

$$G \circ F(y) \subset G\left(K_0\right). \tag{4}$$

As by (3) $d(K_0, F(x)) \leq \varepsilon'$, (2) leads to

$$d^{\subset}_{HG}\left(G\left(K_0\right), G \circ F(x)\right) < \varepsilon.$$

Hence, there exists a metric space (Z, d) and two injective isometric maps $f : G \circ F(x) \to Z$ and $g : G\left(K_0\right) \to Z$ such that

$$g \circ G\left(K_0\right) \subset f \circ G \circ F\left(x\right) + \varepsilon.$$

This formula and (4) lead to

$$g \circ G \circ F(y) \subset f \circ G \circ F\left(x\right) + \varepsilon.$$

Hence $d^{\subset}_{HG}\left(G \circ F(y), G \circ F\left(x\right)\right) \leq \varepsilon$ for all $y \in \{x\} + \eta$, and the lemma is proved. □

4. Comeback to geometry.

4.1. Generic uniqueness of antipodes. Now, we can enunciate and prove the theorems described in the introduction.

Theorem 1. *Let M be a C^r compact manifold ($r = 2, \dots, \infty$) endowed with a generic Riemannian structure of G^r. A generic point of M admits an unique antipode.*

For proving this, we need the following:

Lemma 9. *Let n be a positive integer. The interior of*

$$\mathcal{U}_x(n) \stackrel{\text{def}}{=} \{g \in G^r \,|\, \operatorname{diam}(\mathfrak{A}_x^g) \geq 1/n\}$$

is empty.

Proof. Let g be in $\mathcal{U}_x(n)$, ρ be an integer less or equal to r, and ε be a positive real number. We shall exhibit a Riemannian structure $g' \notin \mathcal{U}_x(n)$ such that $\|g - g'\|_{C^\rho} < \varepsilon$. We denote by λ the g-injectivity radius at point x. Take y, a g-antipode of x. Let $\Sigma = \Sigma_{xy}^g$ be the set of segments from x to y, with their arc length parameter. We define $S = \{\sigma(\lambda/2) |\, \sigma \in \Sigma\}$. Let Φ be the g-exponential mapping at x. As Φ is continuous on T_xM there exists a real number α such that for all tangent vectors u, v in the ball $\{0\} + 2\mathfrak{D}_x^g$ of T_xM

$$\|u - v\| < \alpha \Longrightarrow d^g(\Phi(u), \Phi(v)) < \frac{1}{12n},$$

where the norm $\|\ \|$ is the g-norm. As, restricted to $\{0\} + \frac{2\lambda}{3}$, Φ has a well-defined and continuous inverse, we can find a positive real number η, such that

$$\forall z, z' \in \{x\} + \frac{2\lambda}{3},\ d^g(z, z') < \eta \Longrightarrow \left\|\Phi^{-1}(z') - \Phi^{-1}(z)\right\| < \frac{\lambda\alpha}{2\mathfrak{D}_x^g} \stackrel{\text{def}}{=} \beta.$$

Choose a positive C^ρ function ϕ such that $V \stackrel{\text{def}}{=} \{x | \phi(x) > 0\}$ satisfies $S \subset V \subset S + \eta$. Now put $g' = g(1 + \iota\phi)$, with ι a positive small real number such that:

(i) $\|g - g'\|_{C^\rho} < \varepsilon$

(ii) $\delta(d^g, d^{g'}) < \min(\alpha, \frac{1}{4n})$.

Consider a g'-segment σ' from x to y. We have the following inequalities:

$$\mathfrak{D}_x^{g'} \geq d^{g'}(x, y) = L^{g'}(\sigma') \geq L^g(\sigma') \geq d^g(x, y) = \mathfrak{D}_x^g.$$

Moreover, either σ' passes across V, and $L^{g'}(\sigma') > L^g(\sigma')$, or σ' is not a g-segment, and $L^g(\sigma') > d^g(x, y)$. In both cases, we have

$$\mathfrak{D}_x^{g'} > \mathfrak{D}_x^g. \tag{5}$$

Let y' be in $\mathfrak{A}_x^{g'}$ and σ be a g-segment from x to y'. If $\sigma \cap V = \emptyset$, we would have

$$\mathfrak{D}_x^g \geq d^g(x, y') = L^g(\sigma) = L^{g'}(\sigma) \geq d^{g'}(x, y') = \mathfrak{D}_x^{g'},$$

which is in contradiction with (5). Hence $\sigma \cap V \neq \emptyset$, and there exists a g-segment $\sigma_0 \in \Sigma$ and a real number τ such that $d^g(\sigma_0(\frac{\lambda}{2}), \sigma(\tau)) < \eta$. Let $u_0, u \in T_xM$ be two unit vectors such that $\Phi(tu) = \sigma(t)$ and $\sigma_0(t) = \Phi(tu_0)$. We have $\left|\frac{\lambda}{2} - \tau\right| \leq \left\|\frac{\lambda}{2}u_0 - \tau u\right\| < \beta$. Hence

$$\left\|\mathfrak{D}_x^g u_0 - 2\frac{\mathfrak{D}_x^g}{\lambda}\tau u\right\| < \alpha$$

$$\left\|\mathfrak{D}_x^g u - 2\frac{\mathfrak{D}_x^g}{\lambda}\tau u\right\| < \alpha,$$

and then

$$d^g\left(\sigma_0(\mathfrak{D}_x^g), \sigma\left(\frac{2\mathfrak{D}_x^g}{\lambda}\tau\right)\right) < \frac{1}{12n} \tag{6}$$

$$d^g\left(\sigma(\mathfrak{D}_x^g), \sigma\left(\frac{2\mathfrak{D}_x^g}{\lambda}\tau\right)\right) < \frac{1}{12n}. \tag{7}$$

On the other hand, by Hypothesis (ii), we have

$$d^g(x, y') \leq \mathfrak{D}_x^g \leq \mathfrak{D}_x^{g'} = d^{g'}(x, y') < d^g(x, y') + \alpha,$$

hence $|d^g(x, y') - \mathfrak{D}_x^g| < \alpha$ and

$$d^g\left(\sigma\left(\mathfrak{D}_x^g\right), \sigma\left(d^g\left(x, y'\right)\right)\right) < \frac{1}{12n}. \tag{8}$$

As $y = \sigma_0\left(\mathfrak{D}_x^g\right)$ and $y' = \sigma\left(d^g\left(x, y'\right)\right)$, (6), (7) and (8) lead to $d^g(y, y') < \frac{1}{4n}$, which becomes together with Hypothesis (ii) $d^{g'}(y, y') < \frac{1}{2n}$. This holds for each g'-antipode y' of x, hence $\operatorname{diam}^{g'}\left(\mathfrak{A}_x^{g'}\right) < \frac{1}{n}$, and finally $g' \notin \mathcal{U}_x(n)$. $\square$

Proof of Theorem 1. Put $\mathcal{U}^g(n) = \{x \in M \,|\operatorname{diam}(\mathfrak{A}_x^g) \geq 1/n\}$. Lemma 5 implies that $\mathcal{U}^g(n)$ and $\mathcal{U}_x(n)$ are closed subsets of M and G^r respectively. Let S be a countable dense subset of M. We have

$$\begin{aligned}
\{g \in G^r \,|A_2^g \text{ is not meager}\} &= \left\{g \in G^r \,\middle|\, \exists n \in \mathbb{N}, \overbrace{\mathcal{U}^g(n)}^{\circ} \neq \emptyset\right\} \\
&= \bigcup_{n\in\mathbb{N}} \left\{g \in G^r \,\middle|\, \overbrace{\mathcal{U}^g(n)}^{\circ} \neq \emptyset\right\} \\
&\subset \bigcup_{n\in\mathbb{N}} \bigcup_{x\in S} \{g \in G^r \,|x \in \mathcal{U}^g(n)\} \\
&= \bigcup_{n\in\mathbb{N}} \bigcup_{x\in S} \mathcal{U}_x(n).
\end{aligned}$$

Hence, by Lemma 9, $\{g \in G^r | A_2^g \text{ is not meager}\}$ is meager. $\square$

4.2. Generic number of segments from a point to its antipode. Given a C^r manifold M, we denote by $\mathcal{S} \subset G^r \times M$ the set of all ordered pairs (g,x) such that $\mathfrak{A}_x^g$ contains a single point. Let $a : \mathcal{S} \to M$ be the continuous function which associates to (g,x) the only g-antipode of x. We define $\mathcal{S}^g = M \backslash A_2^g = \{x \in M | \, (g,x) \in \mathcal{S}\}$ and $\mathcal{S}_x = \{g \in G^r | \, (g,x) \in \mathcal{S}\}$.

In order to prove the result concerning surfaces, we have to use the following:

Lemma 10. *Let (M,g) be a C^r Riemannian 2-dimensional manifold ($r \geq 2$). Let x be a point of M, and y be a g-antipode of x. Denote by $\vec{\Sigma} \subset T_y^{1g}M$ the set of unit vectors tangent to a segment from x to y.*

Then $\vec{\Sigma}$ cannot be included in any open half-plane of T_yM. This implies that either $\vec{\Sigma}$ has cardinality at least three, or $\vec{\Sigma} = \{-u, u\}$ for a suitable vector $u \in T_y^{1g}M$.

Proof. Assume that $\vec{\Sigma}$ is included in some open half-plane, and let $\gamma : [0, \varepsilon] \to M$ be a arclength parameterized arc, starting at y, and directed by the bisector $u \in T_y^{1g}M$ of the other half-plane. For each integer n, there exists a minimizing geodesic σ_n from x to $\gamma(1/n)$. By selecting a subsequence, we can assume that σ_n is tending to a segment σ from x to y. Let v be the unit tangent vector at y to σ. Of course $g(u,v) < 0$. By a variant of the first variation formula of arclength $L^g(\sigma_n) = L^g(\sigma) - \frac{1}{n} g(u,v) + o(\frac{1}{n})$, hence y_n is farther from x than y for n large enough, and we obtain a contradiction. $\square$

Now, we can enunciate the following:

Theorem 2. *Let M be C^r 2-dimensional manifold ($r = 2, 3, \ldots, \infty$), endowed with a generic Riemannian structure. A generic point of M is joined to its only antipode by at most three segments.*

We need three lemmas.

Lemma 11. *We denote by $\vec{\Sigma}_{xy}^g$ the set of the g-unit tangent vectors at y, to segments from x to y. Assume $r \geq 2$. The map ξ from $\mathcal{S}$ to $\mathcal{HG}$ defined by $\xi(g,x) = \mathfrak{j}_{T^{1g}M} \left(\vec{\Sigma}_{xa_x^g}^g \right)$, is upper semicontinuous.*

Proof. It is obvious, by continuity of a, and Lemma 6, that $(g,x) \longmapsto \Sigma_{xa_x^g}^g$ is upper semicontinous. Now, with hypothesis $r \geq 2$, the convergence of a sequence of geodesics $(\sigma_n : I \to M)_n$ to σ with respect to the Hausdorff metric, implies, for a suitable parametrization, the uniform convergence of the derivatives $(\dot{\sigma}_n : I \to TM)$ to $\dot{\sigma}$. Hence, the map

$$\tau : \Sigma_{xa_x^g}^g \mapsto \vec{\Sigma}_{xa_x^g}^g,$$

from a suitable subset of $\mathcal{H}(\mathcal{H}(M))$ to $\mathcal{H}(TM)$, is continuous. As τ preserves order, we obtain by virtue of Lemma 8 that $\mathfrak{j}_{TM}\circ\tau$ is an (order preserving) upper semicontinuous map. Applying once again the same lemma gives the desired result. $\square$

Lemma 12. *The map μ from $\mathcal{S}$ to $\mathbb{R}$, defined by*

$$\mu_x^g = d_{HG}^{\subset}\left(\mathfrak{j}_{TM}\left(\vec{\Sigma}^g_{xa_x^g}\right), \mathcal{T}\right)$$

is upper semicontinuous.

Proof. This is a consequence of Lemmas 11 and 7. $\square$

Denote by $\mathcal{V}(n)$ the set $\left\{(g,x)\in\mathcal{S}|\mu_x^g\geq\frac{1}{n}\right\}$, and put

$$\mathcal{V}^g(n) = \{x\in M|\,(g,x)\in\mathcal{V}(n)\}$$
$$\mathcal{V}_x(n) = \{g\in G^r|\,(g,x)\in\mathcal{V}(n)\}\,.$$

By Lemma 12, $\mathcal{V}_x(n)$ and $\mathcal{V}^g(n)$ are closed subsets of $\mathcal{S}_x$ and $\mathcal{S}^g$ respectively, moreover we can prove the following:

Lemma 13. *Assume $r\geq 2$. $\mathcal{V}_x(n)$ has empty interior in $\mathcal{S}_x$.*

Proof. Fix $x\in M$, $n\in\mathbb{N}$, $g\in\mathcal{V}_x(n)$ and put $y=a_x^g$. If σ is a curve going from x to y, we denote by $\vec{\sigma}\in T_y^{1g}M$, the g-unit tangent vector to σ. The g-distance in $T_y^{1g}M$ (i.e., $\arccos g(\cdot,\cdot)$) is denoted by $(\cdot,\cdot)$.

Let σ_0 be a g-segment from x to y, and choose a positive function $\phi: M\to\mathbb{R}$ such that $V\overset{\text{def}}{=}\{x\in M|\phi(x)>0\}$ satisfies:

(i) Each g-segment from x to y passing across V satisfies $(\vec{\sigma},\vec{\sigma}_0)<\frac{1}{4n}$.
(ii) There exists a real number $\varepsilon>0$, such that all segments from x to y satisfying $(\vec{\sigma},\vec{\sigma}_0)<\varepsilon$ pass across V.

We define a sequence of Riemannian structures $g_p=\left(1+\frac{\phi}{p}\right)g$. We will discuss two cases.

Case 1. For p large enough, $y\in\mathfrak{A}_x^{g_p}$. By Lemma 10, there exists a g-segment σ from x to y which does not pass across V. We have on one hand

$$\mathfrak{D}_x^g = L^g(\sigma) = L^{g_p}(\sigma)\geq d^{g_p}(x,y) = \mathfrak{D}_x^{g_p}. \tag{9}$$

On the other hand, as $g_p\geq g$, we have $\mathfrak{D}^g\leq\mathfrak{D}^{g_p}$. Hence $\mathfrak{D}_x^g=\mathfrak{D}_x^{g_p}$.

Let σ^p be a g_p-segment from x to y. Suppose that σ^p passes across V, then

$$\mathfrak{D}_x^{g_p} = L^{g_p}(\sigma^p) > L^g(\sigma^p)\geq d^g(x,y) = \mathfrak{D}_x^g,$$

which is in contradiction with (9). Hence σ^p cannot pass across V,

$$L^g(\sigma^p) = L^{g_p}(\sigma^p) = \mathfrak{D}_x^{g_p} = \mathfrak{D}_x^g,$$

and, by Hypothesis (ii), σ_p is a g-segment such that $(\vec{\sigma}_0, \vec{\sigma}^p) \geq \varepsilon$. Now consider the process consisting on choosing a g-segment σ_0, and replacing g by g_p. By repeating this process finitely many times, we would obtain a situation where no segment from x to y will exist. This is obviously impossible, so after finitely many steps, we obtain a Case 2.

Case 2. We can select a subsequence of $(g_p)_p$ such that $y \notin \mathfrak{A}_x^{g_p}$. For each integer p, take a sequence $(g_{p,q})_q$ of Riemannian structures of $\mathcal{S}_x$, converging to g_p (this is possible because Lemmas 5 and 9 involve that $\mathcal{S}_x$ is dense in G^r). Let $y_{p,q}$ be the only $g_{p,q}$-antipode of x. By selecting suitable subsequences, we assume that each sequence $(y_{p,q})_q$ is converging to a g_p-antipode, say y_p. We also can assume that y_p tends to y. Let σ^p be a g_p-segment from x to y_p which does not pass across V. By Lemma 6, each cluster point σ of $(\sigma^p)_p$ is g-segment from x to y, which does not pass across V. It follows that σ^p and σ are g-geodesics. By a variant of the first variation formula of arclength

$$\begin{aligned} \mathfrak{D}_x^{g_p} - \mathfrak{D}_x^g &= L^g(\sigma^p) - L^g(\sigma) \\ &= -g(\vec{\sigma}, \overrightarrow{yy_p}) + o(\overrightarrow{yy_p}), \end{aligned} \tag{10}$$

where $\overrightarrow{yy_p} \in T_yM$ is the tangent vector such that $y_p = \operatorname{Exp}_y(\overrightarrow{yy_p})$. We denote by τ_p the g-norm of $\overrightarrow{yy_p}$, and put $u_p = \frac{\overrightarrow{yy_p}}{\tau_p}$. As $T_y^{1g}M$ is compact, we can assume (otherwise select a subsequence) that (u_p) is converging to a unit vector u. Equation (10) leads to

$$\Phi \overset{\text{def}}{=} \lim_{p\to\infty} \frac{\mathfrak{D}_x^{g_p} - \mathfrak{D}_x^g}{\tau_p} = -g(\vec{\sigma}, u).$$

As Φ does not depend on (σ_p), there are at most two possible values for $\vec{\sigma}$, say $\vec{\sigma}_1$ and $\vec{\sigma}_2$. By Hypothesis (i), we have

$$\overline{\lim_{p\to\infty}} d^{\subset}\left(\vec{\Sigma}_{xy_p}^{g_p}, \{\vec{\sigma}_0, \vec{\sigma}_1, \vec{\sigma}_2\}\right) \leq \frac{1}{4n}. \tag{11}$$

Hence, for p large enough, we have

$$d^{\subset}\left(\vec{\Sigma}_{xy_p}^{g_p}, \{\vec{\sigma}_0, \vec{\sigma}_1, \vec{\sigma}_2\}\right) \leq \frac{1}{3n}. \tag{12}$$

In order to conclude, we claim that for q large enough, we have

$$\mu_x^{g_{p,q}} \leq d^{\subset}\left(\vec{\Sigma}_{xy_{p,q}}^{g_{p,q}}, \{\vec{\sigma}_0, \vec{\sigma}_1, \vec{\sigma}_2\}\right) \leq \frac{1}{2n}.$$

Suppose not, there would exist a sequence $(\sigma^{p,q})_q$of $g_{p,q}$-segments from x to $y_{p,q}$ such that $(\overrightarrow{\sigma_{p,q}}, \vec{\sigma}_i) > \frac{1}{2n}$, $0 \leq i \leq 2$. A converging subsequence must tend to a g_p-segment σ_p from x to y_p such that $(\overrightarrow{\sigma_p}, \vec{\sigma}_i) \geq \frac{1}{2n}$, which is in contradiction with (12). $\square$

Proof of Theorem 2. Take $(g,x) \in \mathcal{S}$, x is joined to its only antipode by at most three segments if and only if $\vec{\Sigma}^g_{xa^g_x}$ has cardinality at most 3, that is, by Lemma 2, $\mu^g_x = 0$. Let T^g be the subset of (M,g) of those points which are joined to their only antipodes by at least four segments. We shall prove that $G^r_1 \overset{\text{def}}{=} \{g \in G^r | A^g_2 \cup T^g \text{ not meager}\}$ is meager. As

$$G^r_1 \subset \{g \in G^r | A^g_2 \ \text{ not meager}\} \cup \{g \in G^r | T^g \ \text{ not meager}\},$$

we only need to prove that $G^r_2 \overset{\text{def}}{=} \{g \in G^r | T^g \text{ not meager}\}$ is meager. As $T^g = \bigcup\limits_{n \geq 1} \mathcal{V}^g(n)$ and $\mathcal{V}^g(n)$ is closed in $\mathcal{S}^g$, we obtain

$$\begin{aligned} G^r_2 &\subset \left\{ g \in G^r \,\middle|\, \exists n \geq 1, \overset{\circ}{\overline{\mathcal{V}^g(n)}} \neq \emptyset \right\} \\ &= \bigcup_{n \geq 1} \left\{ g \in G^r \,\middle|\, \overset{\circ}{\overline{\mathcal{V}^g(n)}} \neq \emptyset \right\} \\ &\subset \bigcup_{n \geq 1} \left\{ g \in G^r \,\middle|\, \overset{\circ}{\overbrace{\mathcal{V}^g(n) \cup A^g_2}} \neq \emptyset \right\}. \end{aligned}$$

Choosing a dense countable subset S of M,

$$\begin{aligned} G^r_2 &\subset \bigcup_{n \geq 1} \bigcup_{x \in S} \{g \in G^r \, | x \in \mathcal{V}^g(n) \cup A^g_2\} \\ &\subset \bigcup_{n \geq 1} \bigcup_{x \in S} \{g \in G^r \, | x \in \mathcal{V}^g(n)\} \cup \bigcup_{x \in S} \{g \in G^r \, | x \in A^g_2\} \\ &= \bigcup_{n \geq 1} \bigcup_{x \in S} \mathcal{V}_x(n) \cup \bigcup_{x \in S} G^r \backslash \mathcal{S}_x. \end{aligned}$$

By Lemmas 5 and 9, $G^r \backslash \mathcal{S}_x$ is meager in G^r; by Lemmas 12 and 13, $\bigcup\limits_{n \geq 1} \bigcup\limits_{x \in S} \mathcal{V}_x(n)$ is meager in $\mathcal{S}_x$ and consequently in G^r. □

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