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Several very interesting results connecting the theory of abelian ideals of Borel subalgebras, some ideas of D. Peterson relating the previous theory to the combinatorics of affine Weyl groups, and the theory of discrete series are stated in a recent paper (Kostant, 1998) by B. Kostant.

In this paper we provide proofs for most of Kostant's results extending them to ad-nilpotent ideals and develop one direction of Kostant's investigation, the compatible discrete series.

Introduction.

This paper arises from the attempt to understand in detail a recent paper [**Ko2**] by B. Kostant, which can be regarded as an extended research announcement of several very interesting results connecting (at least) three topics: The theory of abelian ideals of Borel subalgebras (which originated from a much earlier paper by Kostant, [**Ko1**]), some ideas of D. Peterson relating the previous theory to the combinatorics of affine Weyl groups, and, finally, the theory of discrete series.

In this paper we provide proofs for most of Kostant's results and we develop one direction of Kostant's investigation, the compatible discrete series. The paper naturally divides into three parts that we now describe.

Let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra, G the corresponding connected simply connected Lie group, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ be a fixed Borel subalgebra (with $\mathfrak{h}$ Cartan and $\mathfrak{n}$ nilradical), Δ the root system of $(\mathfrak{g}, \mathfrak{h})$, Δ^+ the positive system induced by the choice of $\mathfrak{b}$, W, $\widehat{W}$, $\widetilde{W}$ the finite, affine and extended affine Weyl groups of Δ respectively (for more details on notation see the list at the end of the introduction). Let $\mathcal{I}$ denote the set of ad-nilpotent ideals of $\mathfrak{b}$, i.e., the ideals of $\mathfrak{b}$ consisting of ad-nilpotent elements.

The first section of the paper is devoted to build up bijections

$$\widetilde{Z} \leftrightarrow \mathcal{W} \times P^\vee/Q^\vee \leftrightarrow \mathcal{I} \times \mathrm{Cent}\,(G)$$

where $\widetilde{Z}$ is the set of points in $P^\vee$ of the simplex $\{\sigma \in \mathfrak{h}^*_{\mathbb{R}} \mid (\sigma, \alpha_i) \leq 1 \text{ for each } i \in \{1, \dots, n\} \text{ and } (\sigma, \theta) \geq -2\}$ and $\mathcal{W}$ is a suitable subset of $\widehat{W}$ in bijection with $\mathcal{I}$.

The previous results generalize to the ad-nilpotent case the work of Kostant and Peterson for the abelian ideals of $\mathfrak{b}$; their results can be easily obtained replacing $\widetilde{Z}$ with $\widetilde{Z}_{\rm ab}=\{\sigma\in P^\vee\mid(\sigma,\beta)\in\{0,-1,1,-2\}\,\forall\,\beta\in\Delta^+\}$.

Let us explain in more detail the meaning of the previous bijections. To any point $z\in\widetilde{Z}$ we can associate:

(1) An ad-nilpotent ideal $\mathfrak{i}_z$;
(2) an element $w_z\in\widetilde{W}$.

We define a natural action of $\mathrm{Cent}\,(G)$ on $\widetilde{Z}$; $\mathfrak{i}_z$ turns out to be constant on the orbits of this action. It is well-known that $\widetilde{W}$ is a $\mathrm{Cent}\,(G)$-extension of $\widehat{W}$. In fact we have $\widetilde{W}=\widehat{W}\rtimes\Omega$, where Ω is the subgroup, isomorphic to $\mathrm{Cent}\,(G)$, of the elements in $\widetilde{W}$ which stabilize the fundamental alcove. Then $z\mapsto w_z$ maps each $\mathrm{Cent}\,(G)$-orbit of $\widetilde{Z}$ onto a left coset of $\widetilde{W}/\Omega$. Moreover, if $\widehat{w}_z$ is the unique element in $(w_z\Omega)\cap\widehat{W}$, then $\mathfrak{i}_z\mapsto\widehat{w}_z$ realizes a bijection $\mathcal{I}\to\mathcal{W}$.

Restricting to $z\in\widetilde{Z}_{\rm ab}$, if we set v_z^{-1} to be the W-component of $w_z\in\widetilde{W}=W\ltimes T(P^\vee)$, we have that $\mathfrak{i}_z$ is the sum of the root spaces $\mathfrak{g}_\alpha$ for all $\alpha\in-v_z^{-1}(\Delta_z^{-2})\cup v_z^{-1}(\Delta_z^1)$, where by definition $\Delta_z^i=\{\alpha\in\Delta^+\mid(\alpha,z)=i\}$.

The connection of these results with representation theory is the main theme of Section 2. Set $X=\overline{C}_2\cap P^\vee$ and $\mathrm{dom}:\widetilde{Z}_{\rm ab}\to X$ to be the map defined by $\mathrm{dom}\,(z)=v_z^{-1}(z)$. Given $\tau\in X$, then $\Theta_\tau=\mathrm{Ad}(\exp(\sqrt{-1}\pi\tau))$ is a Cartan involution for G and the corresponding Cartan decomposition $\mathfrak{g}=\mathfrak{k}_\tau\oplus\mathfrak{p}_\tau$ has the property that $\mathfrak{k}_\tau$ is an equal rank symmetric Lie subalgebra of G. Given a Cartan involution Θ and a Θ-stable Borel subalgebra $\mathfrak{b}'$, we introduce, after Kostant, a notion of compatibility of $\mathfrak{b}'$ with Θ and we prove that $\mathfrak{b}'$ is compatible with Θ if and only if the pair $(\Theta,\mathfrak{b}')$ is conjugate under G to a pair $(\Theta_\tau,\mathfrak{b})$ with $\tau\in X$.

Now, for $\tau\in X$, set $\widetilde{Z}_\tau=\mathrm{dom}^{-1}(\tau)\cap\widetilde{Z}_{\rm ab}$; we provide proofs for two important results of [**Ko2**]. First, $\widetilde{Z}_\tau$ is in canonical bijection with $W_\tau\backslash W$, W_τ being the Weyl group of $(\mathfrak{k}_\tau,\mathfrak{h})$. Moreover we consider

$$\widetilde{Z}_\tau^{\rm cmpt}=\{z\in\widetilde{Z}_\tau\mid v_z^{-1}(\mathfrak{b})\text{ is compatible with }\Theta_\tau\}$$

and we prove that it has $|\mathrm{Cent}\,(G)|$ elements by exhibiting a bijection with a suitable copy of $\mathrm{Cent}\,(G)$ inside $\widetilde{W}$.

To introduce the connection with discrete series, fix again $\tau\in X$ and set G_τ to be the real form of G corresponding to Θ_τ. Set K to be a maximal compact subgroup of G_τ. If μ is a regular integral weight denote by π_μ the corresponding discrete series representation for G_τ in Harish Chandra parametrization.

The previous results imply that, if λ is regular and dominant, then the map $z\mapsto\pi_{v_z^{-1}(\lambda)}$ defines a bijection between $\widetilde{Z}_\tau$ and the discrete series having infinitesimal character χ_λ. What is more important, the minimal K-type of $\pi_{v_z^{-1}(\lambda)}$ and the L^2-cohomological degree on G_τ/T in which $\pi_{v_z^{-1}(\lambda)}$ appears, can be expressed in terms of combinatorial data related to $\mathfrak{i}_z$.

We can also single out the discrete series $\pi_{v_z^{-1}(\lambda)}$ that correspond to $z \in \widetilde{Z}_\tau^{\mathrm{cmpt}}$. These are called *compatible* discrete series: We note that they are exactly the small discrete series of Gross and Wallach ([**GW**]). We then go on to show that the K-spectrum of the compatible discrete series can be computed fairly explicitly. The relevant results in this direction are given in Theorem 2.8 and the discussion thereafter.

Section 3 is devoted to the proofs of the results of [**Ko2**] that did not fit in our previous discussion, namely, we develop Kostant's theory of special and nilradical abelian ideals. The main outcome is that the structure theory for an abelian ideal $\mathfrak{i}_z$ can be given a symmetric space significance in terms of $\tau = \mathrm{dom}\,(z)$ and of the associate decomposition $\mathfrak{g} = \mathfrak{k}_\tau \oplus \mathfrak{p}_\tau$. More precisely, if $\mathfrak{i}_z = \left(\bigoplus_{\alpha \in -v_z^{-1}(\Delta_z^{-2})} \mathfrak{g}_\alpha\right) \oplus \left(\bigoplus_{\alpha \in v_z^{-1}(\Delta_z^{1})} \mathfrak{g}_\alpha\right)$, then the two summands in the r.h.s. of the previous expression correspond to $\mathfrak{i}_z \cap \mathfrak{k}_\tau$, $\mathfrak{i}_z \cap \mathfrak{p}_\tau$, respectively. Other features of this decomposition are the following: On one hand $\mathfrak{i}_z \cap \mathfrak{k}_\tau$ is again an abelian ideal of $\mathfrak{b}$, and the abelian ideals arising in this way (called *special*) can be abstractly characterized. On the other hand, $\mathfrak{i}_z \cap \mathfrak{p}_\tau$ is an abelian $\mathfrak{b} \cap \mathfrak{k}_\tau$-submodule of $\mathfrak{b} \cap \mathfrak{p}_\tau$ and the map $z \mapsto \mathfrak{i}_z \cap \mathfrak{p}_\tau$ sets up a bijection between $\widetilde{Z}_\tau$ and the set of such submodules; this map extends to a bijection between $\mathrm{dom}^{-1}(\tau) \cap \widetilde{Z}$ and all the $\mathfrak{b} \cap \mathfrak{k}_\tau$-submodules of $\mathfrak{b} \cap \mathfrak{p}_\tau$. Finally, we give a criterion to decide whether a special abelian ideal is the nilradical of a parabolic subalgebra of $\mathfrak{g}$ and we determine explicitly all such ideals. All these results are obtained combining the results of Section 1 with elementary combinatorics of root systems.

Notation.

$\mathfrak{g}$	finite-dimensional complex simple Lie algebra,
G	connected simply connected semisimple Lie group with Lie algebra $\mathfrak{g}$,
$\mathrm{Cent}\,(G)$	center of G,
T	maximal compact torus of G, with Lie algebra $\mathfrak{t}$, hence $\mathfrak{h} = \mathfrak{t} \oplus \sqrt{-1}\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$,
$\mathfrak{h}_\mathbb{R} = \sqrt{-1}\mathfrak{t}$	
$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$	Borel subalgebra of $\mathfrak{g}$, with Cartan component $\mathfrak{h}$ and nilradical $\mathfrak{n}$,

$\Delta \subset \mathfrak{h}_{\mathbb{R}}^*$	finite (irreducible) root system of $\mathfrak{g}$ with positive system Δ^+,
$\mathfrak{g}_\alpha$	root space relative to $\alpha \in \Delta$,
$\Pi = \{\alpha_1, \dots, \alpha_n\}$	simple roots of Δ^+,
$Q = \sum\limits_{i=1}^n \mathbb{Z}\alpha_i$, $Q^\vee = \sum\limits_{i=1}^n \mathbb{Z}\alpha_i^\vee$	root and coroot lattices,
$P = \sum\limits_{i=1}^n \mathbb{Z}\omega_i$, $P^\vee = \sum\limits_{i=1}^n \mathbb{Z}\omega_i^\vee$	weight and coweight lattices,
$\theta = \sum\limits_{i=1}^n m_i\alpha_i$	highest root of Δ w.r.t. Π,
$J = \{i \mid 1 \leq i \leq n,\ m_i = 1\}$	
$o_i = \omega_i^\vee / m_i$, $1 \leq i \leq n$	nonzero vertices of the fundamental alcove,
$\rho = \omega_1 + \cdots + \omega_n$	sum of fundamental weights,
$\rho^\vee = \omega_1^\vee + \cdots + \omega_n^\vee$	sum of fundamental coweights,
h	Coxeter number of Δ,
δ	fundamental imaginary root,
$\widehat{\Delta} = \Delta + \mathbb{Z}\delta$	affine root system associated to Δ,
$\widehat{\Delta}^+ = (\Delta^+ + \mathbb{N}\delta) \cup (\Delta^- + \mathbb{N}^+\delta)$	positive system for $\widehat{\Delta}$,
W	finite Weyl group,
$T(L) \subset Aff(\mathfrak{h}_{\mathbb{R}}^*)$	group of translations corresponding to the lattice $L \subset \mathfrak{h}_{\mathbb{R}}^*$,
$\widehat{W} = W \ltimes T(Q^\vee)$	affine Weyl group,
$\widehat{W}_i = W \ltimes T(iQ^\vee)$	$i \in \mathbb{N}, i \geq 1$, $(\widehat{W}_1 \equiv \widehat{W})$,
$\widetilde{W} = W \ltimes T(P^\vee)$	extended affine Weyl group,
$\mathcal{I} = \{\mathfrak{i} \mid \mathfrak{i}$ ideal of $\mathfrak{b}$, $\mathfrak{i} \subset \mathfrak{n}\}$	ad-nilpotent ideals of $\mathfrak{b}$,
$\mathcal{I}_{\text{ab}}$	abelian ideals of $\mathfrak{b}$.

We refer to [**CP**] for a brief description of $\widehat{\Delta}$ and $\widehat{W}$, though we do not completely follow the notation we used there. In particular, in [**CP**, Section 1] we described in great detail the relationship between $\widehat{W}$ viewed as the Weyl group of $\widehat{\Delta}$ and its affine representation on $\mathfrak{h}_{\mathbb{R}}^*$. By virtue of that description, we do not distinguish here between $\widehat{W}$ and its affine representation. We note that $\widehat{W}_i$ can be viewed as the affine Weyl group of $\frac{1}{i}\Delta$.

We will denote by $(\ ,\ )$ at the same time the Killing form on $\mathfrak{h}_{\mathbb{R}}$, the induced form on $\mathfrak{h}_{\mathbb{R}}^*$ and the natural pairing $\mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}^* \to \mathbb{R}$. We set:

$H_{\alpha,k} = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid (x,\alpha) = k\},\ \alpha \in \Delta^+,\ k \in \mathbb{Z}$	affine reflection hyperplane of $\widehat{W}$,
$C_\infty = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid (x,\alpha_i) > 0\,\forall\, i = 1,\dots,n\}$	the fundamental chamber of W,
$C_i = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid (x,\alpha) > 0\,\forall\, \alpha \in \Pi,\ (x,\theta) < i\}$	the fundamental alcove of $\widehat{W}_i$.

Recall that Δ can be given a partial order defined by $\alpha < \beta$ if $\beta - \alpha$ is a sum of positive roots. We will denote by V_α the principal dual order ideal generated by α, namely:

$$V_\alpha = \{\beta \in \Delta^+ \mid \beta \geq \alpha\}.$$

Moreover, for $\mathfrak{i} \in \mathcal{I}$ we define $\Phi_{\mathfrak{i}} = \{\alpha \in \Delta^+ \mid \mathfrak{g}_\alpha \subset \mathfrak{i}\}$, and for $\Phi \subset \Delta^+$ we set $\mathfrak{i}(\Phi) = \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, so that $\mathfrak{i} \mapsto \Phi_{\mathfrak{i}},\ \Phi \mapsto \mathfrak{i}(\Phi)$ are mutually inverse maps between $\mathcal{I}$ and the set of dual order ideals of the poset $(\Delta^+, \leq)$. We shall use the notation $\mathfrak{i}(\Phi)$ even when Φ is any subset of Δ^+ (so $\mathfrak{i}(\Phi)$ is not necessarily an ideal).

We set $\mathfrak{i}(\emptyset)$ to be the zero ideal.

1. Connections between ad-nilpotent ideals, the affine Weyl group and the center of G.

In this section we will explain how ad-nilpotent ideals can be naturally parametrized by a certain subset $\mathcal{W}$ of $\widehat{W}$, and in turn, by the points in $Q^\vee$ of a suitable simplex $D \subset \mathfrak{h}_{\mathbb{R}}^*$; on this simplex there is a natural action of $\mathrm{Cent}\,(G)$, which should be regarded as embedded into $\widetilde{W}$. We will describe the relationships of this action with the ad-nilpotent ideals. These constructions specialize nicely to the case of abelian ideals. The idea of relating abelian ideals with elements in $\widehat{W}$ appears in [**Ko2**] (where it is attributed to D. Peterson); the generalization to the ad-nilpotent case can be found in [**CP**]. The role of the center has been pointed out in [**Ko2**] (in the abelian case).

First we explain how to view $\mathcal{I}$ as a subset of $\widehat{W}$.

Let $\mathfrak{i} = \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, $\Phi \subset \Delta^+$, be an ad-nilpotent ideal. Set

$$L_{\mathfrak{i}} = \bigcup_{k\geq 1} \left(-\Phi^k + k\delta\right) \subset \widehat{\Delta}^+,$$

where $\Phi^1 = \Phi$ and $\Phi^k = (\Phi^{k-1} + \Phi) \cap \Delta$, $k \geq 2$. Note that, since $\mathfrak{i}$ is nilpotent, $L_{\mathfrak{i}}$ is a finite set. For $w \in \widehat{W}$ set

$$N(w) = \left\{\alpha \in \widehat{\Delta}^+ \mid w^{-1}(\alpha) < 0\right\}.$$

Then:

Proposition A ([**CP**, Theorem 2.6])**.** *There exists a unique $w_{\mathfrak{i}} \in \widehat{W}$ such that $L_{\mathfrak{i}} = N(w_{\mathfrak{i}})$.*

Hence we have an injective map $f : \mathcal{I} \to \widehat{W}$, $f(\mathfrak{i}) = w_{\mathfrak{i}}$. We set

$$\mathcal{W} = f(\mathcal{I}), \quad \text{and} \quad \mathcal{W}_{\text{ab}} = f(\mathcal{I}_{\text{ab}}).$$

$\mathcal{W}$ and $\mathcal{W}_{\text{ab}}$ are characterized inside $\widehat{W}$ by the following properties:

Proposition B ([**CP**, Theorem 2.9, Proposition 2.12])**.**

(1) *We have*

$$\mathcal{W}_{\text{ab}} = \{w \in \widehat{W} \mid w(C_1) \subset C_2\}.$$

(2) *Assume $w \in \widehat{W}$ and $w = t_\tau v$, with $\tau \in Q^\vee$, $v \in W$. Then $w \in \mathcal{W}$ if and only if the following conditions hold:*
  (i) $w(C_1) \subset C_\infty$;
  (ii) $(v^{-1}(\tau), \alpha_i) \leq 1$ *for each* $i \in \{1, \ldots, n\}$ *and* $(v^{-1}(\tau), \theta) \geq -2$.

Corollary (D. Peterson)**.** $|\mathcal{I}_{\text{ab}}| = 2^n$.

For $\mathfrak{i} \in \mathcal{I}$ and $\alpha \in \Delta^+$ we have that $\alpha \in \Phi_{\mathfrak{i}}$ if and only if $-\alpha + \delta \in N(w_{\mathfrak{i}})$. As shown in [**CP**, Section 1], we have that $-\alpha + \delta \in N(w_{\mathfrak{i}})$ if and only $H_{\alpha,1}$ separates C_1 and $w_{\mathfrak{i}}(C_1)$. We shall need the following result [**IM**, §1.9]:

Lemma C. *Let $w \in \widetilde{W}$, $w = t_\tau v$, $\tau \in P^\vee$, $v \in W$. Set $\Delta_\tau^i = \{\alpha \in \Delta^+ \mid (\alpha, \tau) = i\}$. Then $H_{\alpha,1}$ separates C_1 and $w(C_1)$ if and only if $\alpha \in \left(\bigcup\limits_{i>1} \Delta_\tau^i\right) \cup (\Delta_\tau^1 \setminus N(v))$.*

We shall also need the following classical results, which can be extracted from [**IM**, §1], on the embedding of $\text{Cent}\,(G)$ into W. Let $\Delta(j)$ denote the root subsystem of Δ generated by $\Pi \setminus \{\alpha_j\}$ and by w_0^j the longest (w.r.t. $\Pi \setminus \{\alpha_j\}$) element of the corresponding parabolic subgroup of W. Let w_0 be the longest element of W with respect to Π, and

$$\Omega = \left\{t_{\omega_j^\vee} w_0^j w_0 \mid j \in J\right\} \cup \{1\} \subset \widetilde{W}.$$

Proposition D.

(1) *Ω is a group. Precisely, Ω is the subgroup of all elements $w \in \widetilde{W}$ such that $w(\overline{C}_1) = \overline{C}_1$, hence it is isomorphic to $P^\vee/Q^\vee$. Composing this isomorphism with the one induced by the exponential map we also obtain an isomorphism $\Omega \to \text{Cent}\,(G)$.*

(2) *The projection* $t_{\omega_j^\vee} w_0^j w_0 \mapsto w_0^j w_0$ *from* Ω *to* W *is injective, hence it embeds* $\mathrm{Cent}\,(G)$ *into* W.

Remark. It is clear that, for any $1 \le j \le n$, $N(w_0^j w_0)$ equals $\Delta^+ \setminus \Delta(j)^+$. In particular, for $j \in J$ we have $N(w_0^j w_0) = \{\alpha \in \Delta^+ \mid (\alpha, \omega_j^\vee) = 1\}$.

The last tool which will be relevant in what follows is the encoding of ad-nilpotent ideals by means of lattice points of a certain simplex D in $\mathfrak{h}_\mathbb{R}^*$. The definition of this simplex is motivated by Proposition B; set

$$D = \{\sigma \in \mathfrak{h}_\mathbb{R}^* \mid (\sigma, \alpha_i) \le 1 \text{ for each } i \in \{1, \dots, n\} \text{ and } (\sigma, \theta) \ge -2\}$$

and define

$$\begin{aligned}
\widetilde{Z} &= D \cap P^\vee, \\
Z &= D \cap Q^\vee, \\
\widetilde{Z}_{\mathrm{ab}} &= \left\{\sigma \in P^\vee \mid (\sigma, \beta) \in \{0, -1, 1, -2\} \, \forall \beta \in \Delta^+\right\}, \\
Z_{\mathrm{ab}} &= \left\{\sigma \in Q^\vee \mid (\sigma, \beta) \in \{0, -1, 1, -2\} \, \forall \beta \in \Delta^+\right\}.
\end{aligned}$$

It is easily seen that $D = t_{\rho^\vee} w_0(\overline{C}_{h+1}) = \rho^\vee - \overline{C}_{h+1}$, h being the Coxeter number of Δ. Note that $t_{\rho^\vee} w_0 \in \widetilde{W}$; on the other hand in [**CP2**, Lemma 1] it is proved that there exists an element $\widetilde{w} \in \widehat{W}$ such that $D = \widetilde{w}\,(\overline{C}_{h+1})$. Set, for $r \ge 1$

$$\Omega_r = \left\{t_{r\omega_j^\vee} w_0^j w_0 \mid j \in J\right\} \cup \{1\};$$

then Ω_r is the subgroup of all elements $w \in \widetilde{W}$ such that $w(\overline{C}_r) = \overline{C}_r$. It follows that $\widetilde{w}\,\Omega_{h+1}\,\widetilde{w}^{-1}$ is the subgroup of all elements $w \in \widetilde{W}$ such that $w(D) = D$.

We set $\Sigma = \left\{t_{-\omega_j^\vee} w_0^j w_0 \mid j \in J\right\} \cup \{1\}$. Clearly, Σ is a subgroup of $\widetilde{W}$ isomorphic to Ω. We have the following result:

Proposition 1.1.

(1) $\Sigma = \left\{w \in \widetilde{W} \mid w(D) = D\right\}$, *so that* $\Sigma = \widetilde{w}\,\Omega_{h+1}\,\widetilde{w}^{-1}$. *In particular* Σ *acts on* $\widetilde{Z}$.

(2) *For any* $z \in \widetilde{Z}$ *the orbit of* z *under the action of* Σ *is a set of representatives of* $P^\vee/Q^\vee$. *In particular* Σ *acts freely on* $\widetilde{Z}$.

(3) *The action of* Σ *on* $\widetilde{Z}$ *preserves* $\widetilde{Z}_{\mathrm{ab}}$.

Proof. (1) It suffices to prove that any element in Σ preserves D. Take $x \in D$ and consider $t_{-\omega_j^\vee} w_0^j w_0(x)$, $j \in J$. For $i = 1, \dots, n$, we have $(t_{-\omega_j^\vee} w_0^j w_0(x), \alpha_i) = (x, w_0 w_0^j(\alpha_i)) - (\omega_j^\vee, \alpha_i)$. Now if $\alpha_i \in \Delta(j)$ we have $(\omega_j^\vee, \alpha_i) = 0$ and that $w_0 w_0^j(\alpha_i)$ is a simple root (see [**IM**]). Since $x \in D$, we get $(x, w_0 w_0^j(\alpha_i)) \leq 1$ as desired. If $\alpha_i \notin \Delta(j)$, then $\alpha_i = \alpha_j$, so that $(\omega_j^\vee, \alpha_i) = 1$ and $w_0 w_0^j(\alpha_i) = -\theta$; therefore $(t_{-\omega_j^\vee} w_0^j w_0(x), \alpha_i) = -(x,\theta) - 1 \leq 2 - 1 = 1$. Finally, we have

$$(t_{-\omega_j^\vee} w_0^j w_0(x), \theta) = (x, w_0 w_0^j(\theta)) - (\omega_j^\vee, \theta) = (x, w_0(\alpha_j)) - 1 \geq -1 - 1 = -2.$$

The fact that Σ also preserves $\widetilde{Z}$ is immediate.

(2) Let $z \in \widetilde{Z}$. For any root α, denote by s_α the reflection associated to α; we have $s_\alpha(z) = z - (z, \alpha^\vee)\alpha = z - (z,\alpha)\alpha^\vee$ and, since $z \in P^\vee$, $(z,\alpha)\alpha^\vee \in Q^\vee$. It follows that, for any $v \in W$, z and $v(z)$ differ by an element in $Q^\vee$. Therefore, for any $j \in J$, z and $t_{-\omega_j^\vee} w_0^j w_0(z) = -\omega_j^\vee + w_0^j w_0(z)$ are distinct $\mod Q^\vee$, since $\omega_j^\vee \notin Q^\vee$. The claim follows directly.

(3) If $x \in \widetilde{Z}_{\text{ab}}$, we have

$$\begin{aligned}
&(t_{-\omega_i^\vee} w_0^i w_0(x), \beta)\\
&= (x, w_0 w_0^i(\beta)) - (\omega_i^\vee, \beta)\\
&= \begin{cases} (x, w_0 w_0^i(\beta)) \in \{0,1,-1,-2\} & \text{if } \beta \in \Delta(i)\\ (x, w_0 w_0^i(\beta)) - 1 \in \{0,1,-1,2\} - 1 = \{0,1,-1,-2\} & \text{otherwise.} \end{cases}
\end{aligned}$$

Hence $t_{-\omega_i^\vee} w_0^i w_0(x) \in \widetilde{Z}_{\text{ab}}$, as desired. □

Lemma 1.2. $W \cdot \overline{C}_k = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid -k \leq (x,\beta) \leq k \ \forall \beta \in \Delta^+\}$.

Proof. Consider $x \in W \cdot \overline{C}_k$; then $x = \sum_{i=1}^n \lambda_i w(o_i)$ for some $w \in W$ and $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i \leq k$. Set $\beta_i = w(\alpha_i)$ for $1 \leq i \leq n$; then $\beta_1, \dots, \beta_n$ is another basis for Δ, hence for any $\beta \in \Delta^+$ we have $\beta = \sum_{i=1}^n \mu_i \beta_i$ with $|\mu_i| \leq m_i$ and $\mu_i \geq 0 \ \forall i$ or $\mu_i \leq 0 \ \forall i$. Finally we have $|(x,\beta)| = \left|\sum_{i=1}^n \lambda_i \frac{\mu_i}{m_i}\right| \leq \sum_{i=1}^n \lambda_i \leq k$. □

Proposition 1.3. *For any $z \in \widetilde{Z}$ we have $z + \overline{C}_1 \subset W \cdot \overline{C}_h$. Moreover $z \in \widetilde{Z}_{\text{ab}}$ if and only if $z + \overline{C}_1 \subset W \cdot \overline{C}_2$. In particular, for any $z \in \widetilde{Z}$ there exists a unique $v \in W$ such that $v(z + \overline{C}_1) \subset \overline{C}_h$ and, for such a v, $v(z + \overline{C}_1) \subset \overline{C}_2$ if and only $z \in \widetilde{Z}_{\text{ab}}$.*

Proof. Let $z \in \widetilde{Z}$. We may write $z = \rho^\vee - \sum_{i=1}^n \lambda_i o_i$, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i \leq h+1$. For any positive root $\beta = \sum_{i=1}^n \mu_i \alpha_i$ $(0 \leq \mu_i \leq m_i)$ we have

$$(z,\beta) = \sum_{i=1}^n \mu_i - \sum_{i=1}^n \lambda_i \frac{\mu_i}{m_i} \leq \sum_{i=1}^n m_i = h - 1.$$

On the other hand, since at least one of the μ_i is positive, $(z,\beta) \geq 1 - \sum_{i=1}^n \lambda_i \geq 1 - (h+1) = -h$.

Now for any $y \in \overline{C}_1$ and any $\beta \in \Delta^+$ we have $0 \leq (y, \beta) \leq 1$, whence $-h \leq (z + y, \beta) \leq h$. By Lemma 1.2 we get $z + \overline{C}_1 \subset W \cdot \overline{C}_h$.

We see directly that $z \in \widetilde{Z}_{\text{ab}}$ if and only if $-2 \leq (z + y, \beta) \leq 2$, or equivalently, by Lemma 1.2, if and only if $z + \overline{C}_1 \subset W \cdot \overline{C}_2$. $\square$

Definition. Let $z \in \widetilde{Z}$ and $v \in W$ be (the unique element of W) such that $v(z + \overline{C}_1) \subset \overline{C}_h$. We define the map

$$\widetilde{F} : \widetilde{Z} \to \widetilde{W}, \quad z \mapsto t_{v(z)}v, \qquad \text{and we set} \qquad v_z = v^{-1}.$$

There exists a unique $w \in \widehat{W}$ such that $v(z + \overline{C}_1) = w(\overline{C}_1)$, thus we also have a map

$$F : \widetilde{Z} \to \widehat{W}, \quad z \mapsto w.$$

(In the Introduction $\widetilde{F}(z)$, $F(z)$ have been called w_z, $\widehat{w}_z$, respectively.)

By Proposition A we can thus associate to z an ad-nilpotent ideal

$$\mathfrak{i}_z = f^{-1}(F(z)).$$

Proposition 1.4.

(a) *$F(z) = F(z')$ $(z, z' \in \widetilde{Z})$ if and only if $z = \psi(z')$ for $\psi \in \Sigma$.*

(b) *F is a surjection $\widetilde{Z} \twoheadrightarrow \mathcal{W}$ and $F_{|Z} : Z \to \mathcal{W}$ is a bijection, with inverse map $t_\tau v \mapsto v^{-1}(\tau)$.*

(c) *Set $H(z) = (F(z), z \bmod Q^\vee)$. Then H is bijective, hence the same holds for the composite map:*

$$\widetilde{Z} \xrightarrow{\quad H \quad} \mathcal{W} \times P^\vee/Q^\vee \xrightarrow{(f^{-1}, \exp)} \mathcal{I} \times \text{Cent}\,(G).$$

(d) *Restricting F to $\widetilde{Z}_{\text{ab}}$ induces a surjection $\widetilde{Z}_{\text{ab}} \twoheadrightarrow \mathcal{W}_{\text{ab}}$, and as above bijections $Z_{\text{ab}} \leftrightarrow \mathcal{W}_{\text{ab}}$, $\widetilde{Z}_{\text{ab}} \leftrightarrow \mathcal{I}_{\text{ab}} \times \text{Cent}\,(G)$.*

Proof. (a) Assume that $F(z) = F(z')$ for $z, z' \in \widetilde{Z}$ and set $v = v_z^{-1}$, $u = v_{z'}^{-1}$. Then $v(z + C_1) = u(z' + C_1)$, so that $C_1 = v^{-1}u(z') - z + v^{-1}u(C_1)$ and $t_{[v^{-1}u(z')-z]}v^{-1}u \in \Omega$. By Proposition D, (2), we have either $v = u$ and in turn $z = z'$ (and in this case we are done), or there exists $j \in J$ such that $v^{-1}u = w_0^j w_0$; moreover

$$(*) \qquad z = w_0^j w_0(z') - \omega_j^\vee$$

which means $z = \psi(z')$ with $\psi = t_{-\omega_j^\vee} w_0^j w_0 \in \Sigma$. Viceversa, if $z = \psi(z')$, $1 \neq \psi \in \Sigma$, relation $(*)$ implies $z + \overline{C}_1 = w_0^j w_0(z' + \overline{C}_1)$, hence $F(z) = F(z')$.

(b) In [**CP2**, Prop. 3] it is proved that $F_{|Z} : Z \to \mathcal{W}$ is a bijection, with inverse map $t_\tau v \mapsto v^{-1}(\tau)$: By (a) we obtain that F maps the whole $\widetilde{Z}$ onto $\mathcal{W}$.

(c) This statement is a direct consequence of (b) and Proposition 1.1, (2).

(d) This follows from Propositions B, (1), Proposition 1.3, and from Part (c) of this proposition. □

Proposition 1.5. *If $z \in \widetilde{Z}$, then $N(v_z) = \{\alpha \in \Delta^+ \mid (\alpha, z) < 0\}$.*

Proof. We have to prove that the element $v \in W$ such that $v(z + C_1) \subset C_\infty$ is defined by the condition $N(v^{-1}) = \{\alpha \in \Delta^+ \mid (\alpha, z) < 0\}$. It suffices to verify that the element u defined by $N(u^{-1}) = \{\alpha \in \Delta^+ \mid (\alpha, z) < 0\}$ is such that $(u(z + e), \beta) > 0 \; \forall \beta \in \Delta^+, \; \forall e \in C_1$. We can then conclude that $u = v$ by uniqueness.

First remark that $N(u) = -uN(u^{-1})$; then suppose $\beta \in N(u)$, or $\beta = -u(\gamma)$, $\gamma \in N(u^{-1})$; by hypothesis $(z, \gamma) \leq -1$, hence we have $(u(z+e), \beta) = -(z + e, \gamma) \geq 1 - (e, \gamma) > 0$. It remains to consider the case $\beta \notin N(u)$; in that case $\beta = u(\gamma)$, $\gamma \notin N(u^{-1})$ and in particular $(z, \gamma) \geq 0$, so that $(u(z + e), \beta) = (z + e, \gamma) = (z, \gamma) + (e, \gamma) \geq (e, \gamma) > 0$ as desired. □

Corollary 1.6. *If $z \in \widetilde{Z}_{\mathrm{ab}}$, then $N(v_z) = \Delta_z^{-2} \cup \Delta_z^{-1}$.*

Proposition 1.7. *Suppose $z \in \widetilde{Z}_{\mathrm{ab}}$. Then*

$$\mathfrak{i}_z = -v_z^{-1}(\Delta_z^{-2}) \cup v_z^{-1}(\Delta_z^1).$$

Proof. Let $w = \widetilde{F}(z)$, $w = t_\tau v$; thus $v_z = v^{-1}$ and $z = v_z(\tau)$. By Lemma C we obtain that $\Phi_{\mathfrak{i}_z} = \Delta_\tau^2 \cup (\Delta_\tau^1 \setminus N(v))$. Let $\alpha \in \Delta_\tau^2$. Since $(v_z(\alpha), z) = (\alpha, \tau) = 2$ and $\Delta_z^2 = \emptyset$, we obtain $v_z(\alpha) < 0$, thus $-v_z(\alpha) \in \Delta_z^{-2}$, or equivalently $\alpha \in -v_z^{-1}(\Delta_z^{-2})$. Conversely, if $\beta \in \Delta_z^{-2}$, then $(\tau, v_z^{-1}(\beta)) = -2$, whence $-v_z^{-1}(\beta) \in \Delta_\tau^2$. Therefore $\Delta_\tau^2 = -v_z^{-1}(\Delta_z^{-2})$. Similarly, we see that if $\alpha \in \Delta_\tau^1 \setminus N(v)$, then $v_z(\alpha) \in \Delta_z^1$, and, conversely, if $\beta \in \Delta_z^1$, then $v_z^{-1}(\beta)$ belongs to $\Delta_\tau^1 \setminus N(v)$. This concludes the proof. □

Remark. Corollary 1.6 and Proposition 1.7 show that our constructions coincide for points in $\widetilde{Z}_{\mathrm{ab}}$ with those performed by Kostant in [**Ko2**], taking into account that Kostant's Z is our $-\widetilde{Z}_{\mathrm{ab}}$. In particular, we have provided proofs for the results of Sections 2 and 3 and for Proposition 5.2 and Theorem 5.3 of that paper.

2. Compatible Borel subalgebras and compatible discrete series.

A symmetric Lie subalgebra of $\mathfrak{g}$ is a subalgebra $\mathfrak{k}$ that is the fixed point set of an involutory automorphism Θ of $\mathfrak{g}$. If $\mathfrak{k}$ is a symmetric Lie subalgebra, we set $\mathfrak{p}$ to denote the -1 eigenspace of Θ. It follows that we have a decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

that is usually referred to as a Cartan decomposition of $\mathfrak{g}$. An equal rank symmetric Lie subalgebra is a symmetric Lie subalgebra that contains a Cartan subalgebra of $\mathfrak{g}$. A procedure to construct equal rank symmetric Lie subalgebras is the following: If $\tau \in P^\vee$, set $\Theta_\tau = \mathrm{Ad}(\exp(\sqrt{-1}\pi\tau))$. Then Θ_τ is an involutory automorphism of $\mathfrak{g}$. If we set

$$\mathfrak{g} = \mathfrak{k}_\tau \oplus \mathfrak{p}_\tau$$

to be the corresponding Cartan decomposition, then $\mathfrak{k}_\tau$ is an equal rank symmetric Lie subalgebra.

On the other hand, if $\mathfrak{k}$ is an equal rank symmetric Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{h}'$ is a Cartan subalgebra contained in $\mathfrak{k}$, then, by [**Hel**], Ch. IX, Proposition 5.3, there is $x \in \mathfrak{g}$ such that $\Theta = \mathrm{Ad}(\exp(x))$. Moreover, by Theorem 5.15 of Ch. X of [**Hel**], there is an automorphism ϕ of $\mathfrak{g}$ such that $\phi\Theta\phi^{-1}$ is an automorphism Θ' of type $(s_0, \dots, s_r; k)$ for $\mathfrak{h}$, r being the rank of the subalgebra of Θ'-fixed points. Since $\mathfrak{k}$ is an equal rank symmetric subalgebra, we have $r = n$. Since $\Theta' = \mathrm{Ad}(\exp(\phi(x)))$, it follows from Theorem 5.16 (i), Ch. X of [**Hel**] that $k = 1$, i.e., Θ' fixes $\mathfrak{h}$ pointwise.

Set $X = \overline{C}_2 \cap P^\vee$. If we set $\tau = \sum_i s_i\omega_i^\vee$, then $\tau \in X$. Indeed, since Θ' is an involution, then, according to Theorem 5.15 of [**Hel**] again,

$$2 = s_0 + \sum_{i=1}^{n} m_i s_i$$

(recall that m_i is the coefficient of α_i in the highest root).

Note that $\Theta' = \Theta_\tau$. In fact, calculating the action on the root vectors X_{α_i}, $1 \le i \le n$, $X_{-\theta}$ (which are Lie algebra generators for $\mathfrak{g}$), we have:

$$\begin{aligned}\Theta_\tau(X_{\alpha_i}) &= (-1)^{s_i} X_{\alpha_i} = \Theta'(X_{\alpha_i})\\ \Theta_\tau(X_{-\theta}) &= (-1)^{s_0-2} X_{-\theta} = (-1)^{s_0} X_{-\theta} = \Theta'(X_{-\theta}).\end{aligned}$$

Finally we observe that, by Theorem 5.4 of Ch. IX of [**Hel**], we can write $\phi = \nu\mathrm{Ad}(g)$, with ν an automorphism of $\mathfrak{g}$ leaving $\mathfrak{h}$ and C_∞ invariant. Then

$$\nu^{-1}\Theta_\tau\nu = \Theta_{\nu^{-1}(\tau)} = \mathrm{Ad}(g)\Theta\mathrm{Ad}(g^{-1}).$$

Since $\nu^{-1}(\tau) \in X$, we have proved a weaker form of Proposition 4.1 of [**Ko2**], namely:

Theorem 2.1. *If $\mathfrak{k}$ is an equal rank symmetric Lie subalgebra of $\mathfrak{g}$ then there is $\tau \in X$ such that $\mathfrak{k}_\tau = \mathrm{Ad}(g)\mathfrak{k}$ for some $g \in G$.*

We now turn our attention to the Θ-stable Borel subalgebras. Suppose that $\mathfrak{k}$ is an equal rank symmetric subalgebra of $\mathfrak{g}$ and let Θ be the corresponding involution.

We let K_τ be the subgroup of G corresponding to $\mathfrak{k}_\tau$.

Theorem 1 of [**Ma**] gives the following characterization of Θ-stable Borel subalgebras:

Theorem 2.2. *If $\mathfrak{b}'$ is a Θ-stable Borel subalgebra of $\mathfrak{g}$, then there exist $g \in G$, $w \in W$, and $\tau \in X$ such that $\mathrm{Ad}(g)\mathfrak{k} = \mathfrak{k}_\tau$ and $\mathrm{Ad}(g)\mathfrak{b}' = w\mathfrak{b}$.*

Proof. By Theorem 2.1, we can find $g' \in G$ such that

$$\mathrm{Ad}(g')\mathfrak{k} = \mathfrak{k}_\tau \text{ and } \mathrm{Ad}(g')\Theta\mathrm{Ad}(g'^{-1}) = \Theta_\tau,$$

so $\mathrm{Ad}(g')\mathfrak{b}'$ is Θ_τ-stable. By Theorem 1 of [**Ma**], we can find an element $k \in K_\tau$ such that $\mathfrak{b}'' = \mathrm{Ad}(kg')\mathfrak{b}'$ contains a Θ_τ-stable Cartan subalgebra $\mathfrak{h}'$. Set Δ' to be the set of roots of $(\mathfrak{g}, \mathfrak{h}')$ and let Δ'^+ denote the positive system in Δ' defined by $\mathfrak{b}''$. Since $\mathfrak{b}''$ and $\mathfrak{h}'$ are Θ_τ-stable, it follows that the map $\alpha \mapsto \alpha \circ \Theta_\tau$ defines an automorphism of the Dynkin diagram. Since Θ_τ is of inner type we conclude that Θ_τ fixes pointwise $\mathfrak{h}'$, i.e., $\mathfrak{h}' \subset \mathfrak{k}_\tau$. Hence there is $k' \in K_\tau$ such that $\mathrm{Ad}(k')\mathfrak{h}' = \mathfrak{h}$. Set $g = k'kg'$; then $\mathrm{Ad}(g)\mathfrak{k} = \mathfrak{k}_\tau$ and $\mathrm{Ad}(g)\mathfrak{b}'$ is a Borel subalgebra containing $\mathfrak{h}$. Then $\mathrm{Ad}(g)\mathfrak{b}'$ defines a positive system in Δ, therefore there is an element $w \in W$ such that $\mathrm{Ad}(g)\mathfrak{b}' = w\mathfrak{b}$. □

Definition. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition, with Cartan involution Θ. We will say that a Borel subalgebra $\mathfrak{b}'$ is compatible with Θ (or with $\mathfrak{k}$) if it is Θ-stable and

$$[[\mathfrak{b}'_\mathfrak{p}, \mathfrak{b}'_\mathfrak{p}], \mathfrak{b}'_\mathfrak{p}] = 0,$$

where $\mathfrak{b}'_\mathfrak{p} = \mathfrak{b}' \cap \mathfrak{p}$.

Clearly $\mathfrak{b}$ is compatible with $\mathfrak{k}_\tau$ for any $\tau \in X$. Conversely we have the following theorem:

Theorem 2.3 ([**Ko2**, Theorem 4.3])**.** *Let $\mathfrak{k}$ be an equal rank symmetric subalgebra such that $\mathfrak{h} \subset \mathfrak{k}$. If $\mathfrak{k}$ is compatible with $\mathfrak{b}$ then $\mathfrak{k} = \mathfrak{k}_\tau$ for some $\tau \in X$. Moreover if $\mathfrak{k}'$ is any equal rank symmetric subalgebra, then a Borel subalgebra $\mathfrak{b}'$ is compatible with $\mathfrak{k}'$ if and only if there exist $g \in G$ and $\tau \in X$ such that* $\mathrm{Ad}(g)\mathfrak{k}' = \mathfrak{k}_\tau$ *and* $\mathrm{Ad}(g)\mathfrak{b}' = \mathfrak{b}$.

Proof. Suppose that $\mathfrak{k}$ is compatible with $\mathfrak{b}$. Since $\mathfrak{h} \subset \mathfrak{k}$, by Exercise C.3 of Ch. IX of [**Hel**], we have that $\Theta = \mathrm{Ad}(\sqrt{-1}\pi h)$ with $h \in P^\vee$. Clearly we can choose $h = \sum \epsilon_i \omega_i^\vee$ with $\epsilon_i \in \{0, 1\}$. We claim that, since $\mathfrak{b}$ is compatible, then $\sum m_i\epsilon_i \le 2$, i.e., $h \in X$.

Indeed write

$$\mathfrak{b}_0 = \mathfrak{h} \oplus \sum_{\alpha(h)=0} \mathfrak{g}_\alpha, \quad \mathfrak{b}_1 = \sum_{\alpha(h)=1} \mathfrak{g}_\alpha.$$

Clearly

$$[\mathfrak{b}_0 + \mathfrak{b}_1, \mathfrak{b}_0 + \mathfrak{b}_1] \subset \mathfrak{b}_0 + \mathfrak{b}_1 + [\mathfrak{b}_1, \mathfrak{b}_1]$$

hence, since $\mathfrak{b}$ is compatible,

$$[[\mathfrak{b}_0 + \mathfrak{b}_1, \mathfrak{b}_0 + \mathfrak{b}_1], \mathfrak{b}_0 + \mathfrak{b}_1] \subset \mathfrak{b}_0 + \mathfrak{b}_1 + [\mathfrak{b}_1, \mathfrak{b}_1].$$

Since $\mathfrak{b}_0 + \mathfrak{b}_1$ generates $\mathfrak{b}$ we have that $\mathfrak{b} = \mathfrak{b}_0 + \mathfrak{b}_1 + [\mathfrak{b}_1, \mathfrak{b}_1]$, thus, in particular, the root vector X_θ belongs to either $\mathfrak{b}_0$, or $\mathfrak{b}_1$, or $[\mathfrak{b}_1, \mathfrak{b}_1]$.

If $X_\theta \in \mathfrak{b}_0$ then $\theta(h) = 0$, if $X_\theta \in \mathfrak{b}_1$ then $\theta(h) = 1$, and, if $X_\theta \in [\mathfrak{b}_1, \mathfrak{b}_1]$, then $\theta(h) = 2$. Since $\theta(h) = \sum \epsilon_i m_i$ the first result follows.

For the second assertion we use Theorem 2.2 to deduce that there is $g \in G$ such that $\mathrm{Ad}(g)\mathfrak{b}' = \mathfrak{b}$ and $\mathrm{Ad}(g)\mathfrak{k}' = \mathfrak{k}_{w\tau'}$ for some $\tau' \in X$ and $w \in W$. By the first part of the proof we obtain $\mathfrak{k}_{w\tau'} = \mathfrak{k}_\tau$ for some $\tau \in X$. $\square$

Given $z \in \mathfrak{h}_\mathbb{R}$, let v_z be the unique element of W such that

$$N(v_z) = \{\alpha \in \Delta^+ \mid (\alpha, z) < 0\}.$$

This definition coincides for $z \in \widetilde{Z}$ with the one already introduced (see Proposition 1.5).

It is easy to check that, if $z \in \mathfrak{h}_\mathbb{R}$, then $v_z^{-1}(z)$ is dominant, hence we can define

$$\mathrm{dom} : \mathfrak{h}_\mathbb{R} \to \overline{C}_\infty \quad z \mapsto v_z^{-1}(z).$$

If $z \in \widetilde{Z}$ and $\widetilde{F}(z) = t_\tau v$, then $\mathrm{dom}\,(z) = \tau$. Clearly $\mathrm{dom}\,(\widetilde{Z}_{\mathrm{ab}}) = X$, in fact $w_0\tau \in \widetilde{Z}_{\mathrm{ab}}$ for all $\tau \in X$.

If $\tau \in X$, set $\widetilde{Z}_\tau = \mathrm{dom}^{-1}(\tau) \cap \widetilde{Z}_{\mathrm{ab}}$. Let W_τ be the Weyl group of $(\mathfrak{k}_\tau, \mathfrak{h})$ and denote by n_τ the index of W_τ in W. The following result affords a proof of [**Ko2**, Theorem 4.5]:

Theorem 2.4. *The map $z \mapsto W_\tau v_z^{-1}$ is a bijection between $\widetilde{Z}_\tau$ and $W_\tau \backslash W$. In particular, for any $\tau \in X$, one has*

$$|\widetilde{Z}_\tau| = n_\tau.$$

Proof. We note that, if $z \in \widetilde{Z}_\tau$ then

$$\begin{aligned}
\Delta^2_\tau &= -v_z^{-1}(\Delta_z^{-2}),\\
\Delta^0_\tau &= v_z^{-1}(\Delta_z^{0}),\\
\Delta^1_\tau &= v_z^{-1}(\Delta_z^{1}) \cup -v_z^{-1}(\Delta_z^{-1}).
\end{aligned}$$

The root system for $(\mathfrak{k}_\tau, \mathfrak{h})$ is $\Delta_\tau = \pm\Delta^0_\tau \cup \pm\Delta^2_\tau$. By our formulas above $v_z^{-1}(\Delta^+)$ contains $-\Delta^2_\tau \cup \Delta^0_\tau$. This implies that $W_\tau v_z^{-1} = W_\tau v_{z'}^{-1}$, $z, z' \in \widetilde{Z}_\tau$, if and only if $z = z'$: Indeed if $v_z^{-1} = w' v_{z'}^{-1}$ for some $w' \in W_\tau$ then

$$-\Delta^2_\tau \cup \Delta^0_\tau \subset v_z^{-1}(\Delta^+) = w' v_{z'}^{-1}(\Delta^+) \supset w'(-\Delta^2_\tau \cup \Delta^0_\tau).$$

Since $-\Delta^2_\tau \cup \Delta^0_\tau$ is a positive system for Δ_τ, it follows that $w' = 1$. Hence $v_z = v_{z'}$ and in turn $z = v_z(\tau) = v_{z'}(\tau) = z'$.

We now verify that $W_\tau w = W_\tau v_z^{-1}$ for some $z \in \widetilde{Z}_\tau$. Let $w' \in W_\tau$ be the unique element such that $w'(w(\Delta^+) \cap \Delta_\tau) = -\Delta^2_\tau \cup \Delta^0_\tau$ so that we have that $w'w(\Delta^+) \supset -\Delta^2_\tau \cup \Delta^0_\tau$.

Set $z = (w'w)^{-1}(\tau)$. Let us verify that $z \in \widetilde{Z}_{\mathrm{ab}}$: If $\alpha \in \Delta^+$ then $\alpha(z) = w'w\alpha(\tau) \in \{\pm 2, \pm 1, 0\}$, so it is enough to verify that Δ^2_τ is disjoint from $w'w(\Delta^+)$, but this is obvious since $w'w(\Delta^+)$ contains $-\Delta^2_\tau$.

We are left with showing that $W_\tau w = W_\tau v_z^{-1}$. Since z is in the W-orbit of τ, then $v_z^{-1}(z) = \tau = w'w(z)$, hence $w'wv_z \in \mathrm{Stab}_W(\tau) \subset W_\tau$; therefore $W_\tau w = W_\tau v_z^{-1}$. $\square$

We are going to prove a result which implies [**Ko2**, Theorem 5.5]. As shown in the proof of Theorem 2.4, $\Delta_\tau = \pm\Delta_\tau^0 \cup \pm\Delta_\tau^2$ is the set of roots for $\mathfrak{k}_\tau$ and $\Delta_\tau^+ = -\Delta_\tau^2 \cup \Delta_\tau^0$ is a positive system for Δ_τ. We let Π_τ denote the set of simple roots for Δ_τ corresponding to Δ_τ^+.

We consider $\widehat{W}_2 = W \ltimes T(2Q^\vee)$: $\widehat{W}_2$ can be viewed as the affine Weyl group of $\frac{1}{2}\Delta$, so that C_2 is its fundamental alcove relative to $\frac{1}{2}\Pi$. In the following lemma we describe Π_τ and see that W_τ is strictly related with the stabilizer $\mathrm{Stab}_{\widehat{W}_2}(\tau)$ of τ in $\widehat{W}_2$.

Lemma 2.5.

(1) *We have* $\mathrm{Stab}_{\widehat{W}_2}(\tau) \subset W_\tau \ltimes T(2Q^\vee)$. *Moreover,* $\mathrm{Stab}_{\widehat{W}_2}(\tau) \cong W_\tau$ *via the canonical projection of* $\widehat{W}$ *onto* W.

(2) *Let* Π_τ *denote the set of simple roots for* Δ_τ *relative to* Δ_τ^+. *If* $(\tau,\theta) < 2$, *then* $\Pi_\tau = \Pi \cap \tau^\perp$; *if* $(\tau,\theta) = 2$, *then* $\Pi_\tau = (\Pi \cap \tau^\perp) \cup \{-\theta\}$.

Proof. We start with some general facts. Consider any point $\nu \in \overline{C}_1$. It is well-known (see [**Bou**, V, 3.3]) that the stabilizer $\mathrm{Stab}_{\widehat{W}}(\nu)$ of ν in $\widehat{W}$ is the parabolic subgroup generated by the simple reflections which fix ν. Set $\Delta(\nu) = \{\alpha \in \Delta \mid (\alpha,\nu) \in \mathbb{Z}\}$, $\widehat{\Delta}_\nu = \{\alpha - (\alpha,\nu)\delta \mid \alpha \in \Delta(\nu)\}$, and $\widehat{\Pi}_\nu = \widehat{\Delta}_\nu \cap \widehat{\Pi}$. It is clear that both $\Delta(\nu)$ and $\widehat{\Delta}_\nu$ are root subsystems of Δ and $\widehat{\Delta}$, respectively; moreover, $\widehat{\Delta}_\nu$ is isomorphic to $\Delta(\nu)$, via the natural projection on $\mathfrak{h}_\mathbb{R}^*$. Since $\nu \in \overline{C}_1$, we have that $(\alpha,\nu) \in \{0,-1,1\}$ for each $\alpha \in \Delta(\nu)$, hence we easily obtain that $\widehat{\Delta}_\nu$ is the standard parabolic subsystem generated by $\widehat{\Pi}_\nu$. Since the simple reflections that fix ν are exactly the reflections corresponding to $\widehat{\Pi}_\nu$, we have that $\mathrm{Stab}_{\widehat{W}}(\nu)$ is the Weyl group of $\widehat{\Delta}_\nu$. This implies in particular that the canonical projection of $\mathrm{Stab}_{\widehat{W}}(\nu)$ on W provides an isomorphism of $\mathrm{Stab}_{\widehat{W}}(\nu)$ with the Weyl group of $\Delta(\nu)$. Moreover, if we project $\widehat{\Pi}_\nu$ on Δ, we obtain a basis for $\Delta(\nu)$. Thus we have that: If $(\nu,\theta) < 1$, then $\Pi \cap \nu^\perp$ is a basis of $\Delta(\nu)$; if $(\nu,\theta) = 1$, then $(\Pi \cap \nu^\perp) \cup \{-\theta\}$ is a basis of $\Delta(\nu)$.

If we apply the above remarks to $\widehat{W}_2$ in place of $\widehat{W}$, C_2 in place of C_1, and $\tau \in X$ in place of ν, we obtain that $\mathrm{Stab}_{\widehat{W}_2}(\tau)$ is a parabolic subgroup of $\widehat{W}_2$ which projects isomorphically onto W_τ via the canonical projection of $\widehat{W}_2$ on W. In particular, we have $\mathrm{Stab}_{\widehat{W}_2}(\tau) \subset W_\tau \ltimes T(2Q^\vee)$. Moreover we have that if $(\tau,\theta) < 2$, then $\Pi \cap \tau^\perp$ is a basis of Δ_τ; if $(\tau,\theta) = 2$, then a basis of Δ_τ is given by $\{-\theta\} \cup (\Pi \cap \tau^\perp)$. $\square$

Recall that $\Omega_2 = \left\{ t_{2\omega_j^\vee} w_0^j w_0 \mid j \in J \right\} \cup \{1\}$ and define, for $z \in \widetilde{Z}$,

$$\mathfrak{b}_z = v_z^{-1}\mathfrak{b}.$$

Theorem 2.6. *Suppose that $\tau \notin \{2\omega_i^\vee \mid i \in J\} \cup \{0\}$. Set*
$$\widetilde{Z}_\tau^{\mathrm{cmpt}} = \left\{ z \in \widetilde{Z}_\tau \mid \mathfrak{b}_z \text{ is compatible with } \mathfrak{k}_\tau \right\}.$$
Then there is a canonical bijection between $\widetilde{Z}_\tau^{\mathrm{cmpt}}$ and Ω_2. In particular
$$|\mathrm{Cent}\,(G)| = \left|\widetilde{Z}_\tau^{\mathrm{cmpt}}\right|.$$

Proof. If $s \in \Omega_2$ write $s = t_\nu v$ with $v \in W$ and $\nu \in 2P^\vee$. By Theorem 2.4 there is a unique $z_s \in \widetilde{Z}_\tau$ such that $W_\tau v_{z_s}^{-1} = W_\tau v$. Notice that $z_s \in \widetilde{Z}_\tau^{\mathrm{cmpt}}$: Indeed, if $\tau' = s^{-1}(\tau)$, we have that $\tau' \in X$, hence $\mathfrak{b}$ is compatible with $\mathfrak{k}_{\tau'} = \mathfrak{k}_{s^{-1}(\tau)} = \mathfrak{k}_{v^{-1}(\tau)}$. In turn $v\mathfrak{b}$ is compatible with $\mathfrak{k}_\tau$, but $v_{z_s}^{-1} = w'v$ with $w' \in W_\tau$, hence $\mathfrak{b}_{z_s} = w'v\mathfrak{b}$ is compatible with $w'\mathfrak{k}_\tau = \mathfrak{k}_\tau$. We can thus define a map $B : \Omega_2 \to \widetilde{Z}_\tau^{\mathrm{cmpt}}$ by setting $B(s) = z_s$.

Let us show that B is injective. We need to show that, if $s = t_\nu v$ and $s' = t_{\nu'} v'$ are such that $W_\tau v = W_\tau v'$, then $v = v'$. Since Ω_2 is a group, it is enough to check that, if $s = t_\nu v$ is such that $v \in W_\tau$, then $v = 1$. The assumption that $v \in W_\tau$ implies that $v(\Delta_\tau) = \Delta_\tau$ and, by [**IM**, Prop. 1.26, (ii)], $v(\Pi \cup \{-\theta\}) = \Pi \cup \{-\theta\}$. Since $\tau \notin \{2\omega_i^\vee \mid i \in J\} \cup \{0\}$, we see that $\Pi_\tau = \Delta_\tau \cap (\Pi \cup \{-\theta\})$, hence $v(\Pi_\tau) = \Pi_\tau$ and $v = 1$.

It remains to show that B is surjective. Fix $z \in \widetilde{Z}_\tau^{\mathrm{cmpt}}$: Since $v_z^{-1}\mathfrak{b}$ is compatible with $\mathfrak{k}_\tau$ it follows that $\mathfrak{b}$ is compatible with $\mathfrak{k}_{v_z(\tau)} = \mathfrak{k}_z$. By Theorem 2.3, we deduce that there is $\tau' \in X$ such that $\mathfrak{k}_z = \mathfrak{k}_{\tau'}$, hence $z = \tau' + \nu$ with $\nu \in 2P^\vee$, or, equivalently,
$$\tau = v_z^{-1}(\tau' + \nu) = t_{\nu'} v_z^{-1}(\tau),$$
where $\nu' = v_z^{-1}(\nu)$.

Since $\overline{C}_2$ is a fundamental domain for $\widehat{W}_2$, there is a unique element $\widetilde{u} \in \widehat{W}_2$ such that $\widetilde{u} t_{\nu'} v_z^{-1}(\overline{C}_2) = \overline{C}_2$. Set $s = \widetilde{u} t_{\nu'} v_z^{-1}$: Clearly $s \in \Omega_2$. Since $\tau \in t_{\nu'} v_z^{-1}(\overline{C}_2) \cap \overline{C}_2$, we find that $\widetilde{u}\tau = \tau$, therefore, by the previous lemma we have $\widetilde{u} \in W_\tau \ltimes T(2Q^\vee)$. Thus $B(s) = z$ and we are done. ☐

Fix again $\tau \in X$ and set σ to denote a conjugation in $\mathfrak{g}$ corresponding to a compact real form $\mathfrak{g}_u$ such that $\Theta_\tau \sigma = \sigma \Theta_\tau$ and set $\sigma_\tau = \Theta_\tau \sigma$. Then σ_τ is a conjugation of $\mathfrak{g}$ defining a real form $\mathfrak{g}_\tau$. We set G_τ to be the subgroup of G corresponding to $\mathfrak{g}_\tau$ and K to be the subgroup of G_τ corresponding to $\mathfrak{g}_\tau \cap \mathfrak{k}_\tau$.

Set $P_{\mathrm{reg}} = \{\lambda \in P \mid (\lambda, \alpha) \neq 0 \text{ for all } \alpha \in \Delta\}$. Let $\widehat{G}_\tau^{\mathrm{disc}} \subset \widehat{G}_\tau$ denote the set of equivalence classes of discrete series for G_τ. If $\lambda \in P_{\mathrm{reg}}$ we let $\pi_\lambda \in \widehat{G}_\tau^{\mathrm{disc}}$ denote the equivalence class corresponding to the parameter λ in Harish Chandra parametrization. We recall that given $\lambda, \mu \in P_{\mathrm{reg}}$ then $\pi_\lambda = \pi_\mu$ if and only if there is $w \in W_\tau$ such that $w\lambda = \mu$.

If $\lambda \in P_{\mathrm{reg}}$ then we set $\Delta_\lambda^+ = \{\alpha \in \Delta \mid (\lambda, \alpha) > 0\}$ and $\mathfrak{b}_\lambda = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_\lambda^+} \mathfrak{g}_\alpha$ to be the corresponding Borel subalgebra. Given $z \in \widetilde{Z}_\tau$, we set $P_{\mathrm{reg}}(z) =$

$\{\lambda \in P_{\text{reg}} \mid \mathfrak{b}_\lambda = \mathfrak{b}_z\}$ and $\rho_z = v_z^{-1}\rho$. Set also $\widehat{G}_\tau^{\text{disc}}(z) = \{\pi_\lambda \mid \lambda \in P_{\text{reg}}(z)\}$. As observed in [**Ko2**], Theorem 2.4 implies that the sets $\widehat{G}_\tau^{\text{disc}}(z)$ give a partition of the set of equivalence classes of discrete series for G_τ, namely

$$\widehat{G}_\tau^{\text{disc}} = \bigcup_{z \in Z_\tau} \widehat{G}_\tau^{\text{disc}}(z).$$

With the notation of §1, if $z \in \widetilde{Z}_{\text{ab}}$, recall that $\mathfrak{i}_z \in \mathcal{I}_{\text{ab}}$ denotes $f^{-1}(F(z))$ and $\Phi_z \equiv \Phi_{\mathfrak{i}_z} = \{\alpha \in \Delta^+ \mid \mathfrak{g}_\alpha \subset \mathfrak{i}_z\}$. Set also $\Phi_z^2 = -v_z^{-1}(\Delta_z^{-2})$, $\Phi_z^1 = v_z^{-1}(\Delta_z^1)$. By Proposition 1.7 we know that $\Phi_z = \Phi_z^2 \cup \Phi_z^1$.

The relation between abelian ideals and discrete series stated above is not as good as one might expect, for example the correspondence depends on the choice of τ even when different choices for τ give rise to isomorphic real forms of G. Nevertheless Kostant observed in [**Ko2**] that there are relations between the elements of $\widetilde{Z}_\tau$ and the structure of the corresponding discrete series. The first occurrence of such a relation involves the realization of the discrete series via L^2-cohomology: One can give to the principal bundle G_τ/T a complex structure by declaring that $\mathfrak{n}$ is the space of antiholomorphic tangent vectors; if $\lambda \in P_{\text{reg}}(z)$ then set $\mathcal{L}_{\lambda-\rho}$ to be the holomorphic line bundle on G_τ/T associated to the character $\exp(\lambda - \rho)$ of T. According to [**S2**], the L^2-cohomology groups $H^p(\mathcal{L}_{\lambda-\rho})$ vanish except in one degree $k(\lambda)$ and $H^{k(\lambda)}(\mathcal{L}_{\lambda-\rho})$ gives a realization of π_λ. It turns out that the decomposition $\Phi_z = \Phi_z^2 \cup \Phi_z^1$ implies, applying Theorem 1.5 of [**S2**], that, if $\lambda \in P_{\text{reg}}(z)$, then $k(\lambda) = \dim(\mathfrak{i}_z)$ (cf. [**Ko2**, Theorem 6.5]).

Another instance, discussed in [**Ko2**], where there is a connection between abelian ideals and discrete series involves the K-structure of π_λ. Recall that, if $\lambda \in P_{\text{reg}}(z)$, then $\Delta_\tau^+ = -\Delta_\tau^2 \cup \Delta_\tau^0$ is a positive system for Δ_τ contained in $\Delta_\lambda^+ = v_z^{-1}\Delta^+$. Set $\rho_c = \frac{1}{2}\sum_{\alpha \in \Delta_\tau^+} \alpha$ and $\rho_n = \rho_z - \rho_c$. It is well-known that the minimal K-type of π_λ has highest weight with respect to Δ_τ^+ given by the formula

$$\mu_\lambda = \lambda + \rho_n - \rho_c.$$

We are going to prove, using this formula, that if $\lambda \in P_{\text{reg}}(z)$ then the highest weight μ_λ of the minimal K-type of π_λ with respect to Δ_τ^+ is

$$\mu_\lambda = \lambda - \rho_z + 2\langle \mathfrak{i}_z \rangle - \frac{1}{2}\widetilde{\tau}.$$

Notation is as follows: For $h \in \mathfrak{h}_\mathbb{R}$ we set $\widetilde{h} = \sum_{\alpha \in \Delta}(\alpha, h)\alpha$. and for $\mathfrak{i} \in \mathcal{I}$ we put $\langle \mathfrak{i} \rangle = \sum\limits_{\alpha \in \Phi_\mathfrak{i}} \alpha$. A special case of the previous formula is

$$\mu_{\rho_z} = 2\langle \mathfrak{i}_z \rangle - \frac{1}{2}\widetilde{\tau}.$$

To prove the formula, just compute:

$$\begin{aligned}
\mu_\lambda &= \lambda + \rho_n - \rho_c \\
&= \lambda - \rho_z + 2\rho_n \\
&= \lambda - \rho_z + \sum_{\alpha\in\Delta_z^1} v_z^{-1}\alpha + \sum_{\alpha\in\Delta_z^{-1}} v_z^{-1}\alpha \\
&= \lambda - \rho_z + \sum_{\alpha\in\Phi_z^1} \alpha - \sum_{\alpha\in\Delta_\tau^1\setminus\Phi_z^1} \alpha \\
&= \lambda - \rho_z + 2\sum_{\alpha\in\Phi_z} \alpha - \sum_{\alpha\in\Phi_z^1} \alpha - \sum_{\alpha\in\Delta_\tau^1\setminus\Phi_z^1} \alpha - 2\sum_{\alpha\in\Phi_z^2} \alpha \\
&= \lambda - \rho_z + 2\langle \mathfrak{i}_z\rangle - \sum_{\alpha\in\Delta_\tau^1} \alpha - \sum_{\alpha\in\Delta_\tau^2} 2\alpha \\
&= \lambda - \rho_z + 2\langle \mathfrak{i}_z\rangle - \sum_{\alpha\in\Delta^+} \alpha(\tau)\alpha \\
&= \lambda - \rho_z + 2\langle \mathfrak{i}_z\rangle - \frac{1}{2}\widetilde{\tau}.
\end{aligned}$$

The previous calculation proves Theorem 6.6 of [**Ko2**].

We now turn our attention to the discrete series that correspond to elements of $\widetilde{Z}_\tau^{\mathrm{cmpt}}$.

Definition. A discrete series π_λ is said compatible if $\lambda \in P_{\mathrm{reg}}(z)$ with $z \in \widetilde{Z}_\tau^{\mathrm{cmpt}}$.

By Theorem 2.6, the number of compatible discrete series is $|P^\vee/Q^\vee|$. In particular it is independent from the particular real form G_τ.

Following §8 of Ch. XI of [**KV**], we recall how a representative of π_λ is constructed: Set

$$\mathfrak{n}_\lambda = [\mathfrak{b}_\lambda, \mathfrak{b}_\lambda],$$

$S = \dim \mathfrak{n}_\lambda \cap \mathfrak{k}$, and $\rho(\mathfrak{n}_\lambda) = \frac{1}{2}\sum_{\alpha\in\Delta_\lambda^+}\alpha$. If $\mu \in P$ we denote by $\mathbb{C}_\mu$ the one dimensional representation of $\mathfrak{b}_\lambda$ defined by setting $(h+n)\cdot c = \lambda(h)c$.

Then, according to Theorem 11.178 of [**KV**], the $(\mathfrak{g}, K)$-module of a representative of π_λ is

$$V^\lambda_{(\mathfrak{g},K)} = \left({}^u\mathcal{R}^{(\mathfrak{g},K)}_{(\mathfrak{b}_\lambda,T)}\right)^S (\mathbb{C}_{\lambda+\rho(\mathfrak{n}_\lambda)}).$$

Recall that $({}^u\mathcal{R}^{(\mathfrak{g},K)}_{(\mathfrak{b}_\lambda,T)})^j = (\Gamma^{(\mathfrak{g},K)}_{(\mathfrak{g},T)})^j \circ \mathrm{pro}^{(\mathfrak{g},T)}_{(\mathfrak{b}_\lambda,T)}$, where $(\Gamma^{(\mathfrak{g},K)}_{(\mathfrak{g},T)})^j$ is the j-th derived functor of the Zuckerman functor, while $\mathrm{pro}^{(\mathfrak{g},T)}_{(\mathfrak{b}_\lambda,T)}$ is the ordinary algebraic induction functor: $\mathrm{pro}^{(\mathfrak{g},T)}_{(\mathfrak{b}_\lambda,T)}(Z) = \mathrm{Hom}_{\mathfrak{b}_\lambda}(U(\mathfrak{g}), Z)_T$.

In what follows we adopt the following notation: If $\mathfrak{a}$ is any subspace of $\mathfrak{g}$ that is stable for the action of $\mathfrak{h}$, we let $\Delta(\mathfrak{a})$ be the set of roots occurring in $\mathfrak{a}$, in other words $\alpha \in \Delta(\mathfrak{a})$ if and only if $\mathfrak{g}_\alpha \subset \mathfrak{a}$.

Set Π_λ to be the set of simple roots for Δ^+_λ and $\Pi^c_\lambda \subset \Pi_\lambda$ to be the set of simple compact roots. Let

$$\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{u} \tag{1}$$

be the corresponding parabolic; i.e., $\Delta(\mathfrak{m})$ is the subsystem of Δ generated by Π^c_λ and $\Delta(\mathfrak{u}) = \Delta^+_\lambda \setminus (\Delta(\mathfrak{m}) \cap \Delta^+_\lambda)$.

Set $\mathfrak{b}_\mathfrak{m} = \mathfrak{b}_\lambda \cap \mathfrak{m}$, $\Delta^+(\mathfrak{m}) = \Delta(\mathfrak{m}) \cap \Delta^+_\lambda$, $\rho_\mathfrak{m} = \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{m})} \alpha$, M the subgroup of K corresponding to $\mathfrak{m} \cap \mathfrak{g}_\tau$, and

$$\lambda_1 = \lambda + \rho(\mathfrak{n}_\lambda) - \rho_\mathfrak{m}. \tag{2}$$

Then λ_1 is dominant and regular for $\Delta^+(\mathfrak{m})$ and, according to Corollary 4.160 of [**KV**],

$$\left({}^u\mathcal{R}^{(\mathfrak{m},M)}_{(\mathfrak{b}_\mathfrak{m},T)}\right)^q (\mathbb{C}_{\lambda+\rho(\mathfrak{n}_\lambda)}) = \begin{cases} 0 & \text{if } q \neq \dim \mathfrak{b}_\mathfrak{m} \cap \mathfrak{u} \\ V^{\lambda_1}_{(\mathfrak{m},M)} & \text{if } q = \dim \mathfrak{b}_\mathfrak{m} \cap \mathfrak{u}. \end{cases} \tag{3}$$

Let $s = \dim(\mathfrak{u} \cap \mathfrak{k})$. Combining (3) with Corollary 11.86 of [**KV**], we find that

$$V^\lambda_{(\mathfrak{g},K)} = \left({}^u\mathcal{R}^{(\mathfrak{g},K)}_{(\mathfrak{q},M)}\right)^s \left(V^{\lambda_1}_{(\mathfrak{m},M)}\right). \tag{4}$$

Fix $z \in \widetilde{Z}^{\mathrm{cmpt}}_\tau$. We are going to compute the K-spectrum of the compatible discrete series, that is the restriction to K of $V^\lambda_{(\mathfrak{g},K)}$ when $\lambda \in P_{\mathrm{reg}}(z)$. Set $S(\mathfrak{u} \cap \mathfrak{p})$ to denote the symmetric algebra of $\mathfrak{u} \cap \mathfrak{p}$.

Lemma 2.7. *Suppose that $z \in \widetilde{Z}^{\mathrm{cmpt}}_\tau$ and $\lambda \in P_{\mathrm{reg}}(z)$. If μ is $\Delta^+(\mathfrak{m})$-dominant and there is $n \in \mathbb{N}$ such that*

$$\dim \operatorname{Hom}_M \left(V^{\mu+\rho_\mathfrak{m}}_{(\mathfrak{m},M)}, S^n(\mathfrak{u} \cap \mathfrak{p}) \otimes V^{\mu_\lambda+\rho_\mathfrak{m}}_{(\mathfrak{m},M)}\right) \neq 0$$

then μ is dominant for Δ^+_τ.

Proof. Since $\mathfrak{b}_z$ is compatible with $\mathfrak{k}_\tau$, then $\mathfrak{b}$ is compatible with $\mathfrak{k}_z$, hence, by Theorem 2.3, $\mathfrak{k}_z = \mathfrak{k}_{\tau'}$ for some $\tau' \in X$. As in the proof of Theorem 2.3, we can write

$$\mathfrak{b} = \mathfrak{b}_0 + \mathfrak{b}_1 + [\mathfrak{b}_1, \mathfrak{b}_1],$$

where $\mathfrak{b}_0 = \mathfrak{h} \oplus \sum_{\substack{\alpha\in\Delta^+\\ \alpha(\tau')=0}} \mathfrak{g}_\alpha$ and $\mathfrak{b}_1 = \sum_{\alpha(\tau')=1} \mathfrak{g}_\alpha$. It is then clear that $\mathfrak{m} \cap \mathfrak{b}_z \supset v_z^{-1}\mathfrak{b}_0$ and $v_z^{-1}\mathfrak{b}_1 = \mathfrak{u} \cap \mathfrak{p}$. This implies that $v_z^{-1}[\mathfrak{b}_1, \mathfrak{b}_1] \subset \mathfrak{u} \cap \mathfrak{k}$. Since $\mathfrak{b}_z = v_z^{-1}\mathfrak{b}_0 + v_z^{-1}\mathfrak{b}_1 + v_z^{-1}[\mathfrak{b}_1, \mathfrak{b}_1]$, we deduce that $\mathfrak{u} \cap \mathfrak{k} = v_z^{-1}[\mathfrak{b}_1, \mathfrak{b}_1]$ and $\mathfrak{m} \cap \mathfrak{b}_z = v_z^{-1}\mathfrak{b}_0$. Hence $[\mathfrak{u} \cap \mathfrak{p}, \mathfrak{u} \cap \mathfrak{k}] = 0$ and $\mathfrak{u} \cap \mathfrak{k}$ is abelian. In particular, we have that $(\alpha, \beta) \geq 0$ whenever $X_\alpha \in \mathfrak{u} \cap \mathfrak{p}$ and $X_\beta \in \mathfrak{u} \cap \mathfrak{k}$.

Since μ_λ is dominant for Δ^+_τ, our result follows easily as in Lemma 3.1 of [**EPWW**]. $\square$

Remark. What we actually observe in Lemma 2.7 is the fact that the compatible discrete series of [**Ko2**] are exactly the small discrete series of [**GW**]. In [**EPWW**] it is also shown that Lemma 2.7 implies that one can compute the full K-spectrum of the discrete series. In the next result we prove this fact using directly Blattner's formula.

Theorem 2.8. *If $z \in \widetilde{Z}_\tau^{\mathrm{cmpt}}$, $\lambda \in P_{\mathrm{reg}}(z)$ and μ is a Δ_τ^+-dominant weight, then*

$$\dim \operatorname{Hom}_K \left(V^{\mu+\rho_c}_{(\mathfrak{k},K)}, V^{\lambda}_{(\mathfrak{g},K)} \right) = \sum_{n=0}^{+\infty} \dim \operatorname{Hom}_M \left(V^{\mu+\rho_{\mathfrak{m}}}_{(\mathfrak{m},M)}, S^n(\mathfrak{u} \cap \mathfrak{p}) \otimes V^{\mu_\lambda+\rho_{\mathfrak{m}}}_{(\mathfrak{m},M)} \right). \tag{5}$$

Proof. Write

$$m_\mu = \dim \operatorname{Hom}_K \left(V^{\mu+\rho_c}_{(\mathfrak{k},K)}, V^{\lambda}_{(\mathfrak{g},K)} \right).$$

As in (11.73) of [**KV**], we write

$$\left({}^u\mathcal{R}^{(\mathfrak{g},K)}_{(\mathfrak{q},M)} \right)^j (Z) = \mathcal{R}^j (Z \otimes (\Lambda^{\mathrm{top}} \mathfrak{u})^*),$$

so, by (4),

$$\mathcal{R}^q \left(V^{\lambda_1}_{(\mathfrak{m},M)} \otimes (\Lambda^{\mathrm{top}} \mathfrak{u})^* \right) = \begin{cases} 0 & \text{if } q \neq s \\ V^{\lambda}_{(\mathfrak{g},K)} & \text{if } q = s. \end{cases}$$

Applying Theorem 5.64 of [**KV**] we find that

$$m_\mu = \sum_{j=0}^{s} (-1)^{s-j} \sum_{n=0}^{\infty} \dim \operatorname{Hom}_M \left(H_j \left(\mathfrak{u} \cap \mathfrak{k}, V^{\mu+\rho_c}_{(\mathfrak{k},K)} \right), S^n(\mathfrak{u} \cap \mathfrak{p}) \otimes V^{\lambda_1}_{(\mathfrak{m},M)} \right).$$

By Corollary 3.8 of [**KV**], as M-modules,

$$H_j \left(\mathfrak{u} \cap \mathfrak{k}, V^{\mu+\rho_c}_{(\mathfrak{k},K)} \right) \simeq H^{s-j} \left(\mathfrak{u} \cap \mathfrak{k}, V^{\mu+\rho_c}_{(\mathfrak{k},K)} \otimes \Lambda^{\mathrm{top}} \mathfrak{u} \cap \mathfrak{k} \right).$$

Since the action of $\mathfrak{u} \cap \mathfrak{k}$ on $\Lambda^{\mathrm{top}} \mathfrak{u} \cap \mathfrak{k}$ is trivial we can write

$$H_j \left(\mathfrak{u} \cap \mathfrak{k}, V^{\mu+\rho_c}_{(\mathfrak{k},K)} \right) \simeq H^{s-j} \left(\mathfrak{u} \cap \mathfrak{k}, V^{\mu+\rho_c}_{(\mathfrak{k},K)} \right) \otimes \Lambda^{\mathrm{top}} \mathfrak{u} \cap \mathfrak{k},$$

hence, using the fact that $V^{\lambda_1}_{(\mathfrak{m},M)} \otimes (\Lambda^{\mathrm{top}} \mathfrak{u} \cap \mathfrak{k})^* = V^{\mu_\lambda+\rho_{\mathfrak{m}}}_{(\mathfrak{m},M)}$, we find that

$$m_\mu = \sum_{j=0}^{s} (-1)^{s-j} \sum_{n=0}^{\infty} \dim \operatorname{Hom}_M \left(H^{s-j} \left(\mathfrak{u} \cap \mathfrak{k}, V^{\mu+\rho_c}_{(\mathfrak{k},K)} \right), S^n(\mathfrak{u} \cap \mathfrak{p}) \otimes V^{\mu_\lambda+\rho_{\mathfrak{m}}}_{(\mathfrak{m},M)} \right).$$

Set $W^1 = \{w \in W_\tau \mid N(w) \subset \Delta(\mathfrak{u} \cap \mathfrak{k})\}$. Applying Kostant's Theorem we find that

$$m_\mu = \sum_{j=0}^{s} (-1)^{s-j} \sum_{n=0}^{\infty} \sum_{\substack{w \in W^1 \\ \ell(w)=s-j}} \dim \operatorname{Hom}_M \left(V_{(\mathfrak{m},M)}^{w(\mu+\rho_c)-\rho_c+\rho_\mathfrak{m}}, S^n(\mathfrak{u} \cap \mathfrak{p}) \otimes V_{(\mathfrak{m},M)}^{\mu_\lambda+\rho_\mathfrak{m}} \right).$$

By Lemma 2.7, if

$$\dim \operatorname{Hom}_M \left(V_{(\mathfrak{m},M)}^{w(\mu+\rho_c)-\rho_c+\rho_\mathfrak{m}}, S^n(\mathfrak{u} \cap \mathfrak{p}) \otimes V_{(\mathfrak{m},M)}^{\mu_\lambda+\rho_\mathfrak{m}} \right) \neq 0,$$

then $w(\mu + \rho_c) - \rho_c$ is dominant for Δ_τ^+, hence $w(\mu + \rho_c)$ is dominant and regular. Since $\mu + \rho_c$ is dominant and regular, we find that $w = 1$ and the above sum reduces to (5). $\square$

Theorem 2.8 says that to compute the K-spectrum of $V_{(\mathfrak{g},K)}^\lambda$ it is enough to compute the M-spectrum of $S(\mathfrak{u} \cap \mathfrak{p})$. The point is the fact that the M-spectrum of $S(\mathfrak{u} \cap \mathfrak{p})$ is somewhat easy to compute. Indeed suppose that $\lambda \in P_{\mathrm{reg}}(z)$ with $z \in \widetilde{Z}_\tau^{\mathrm{cmpt}}$ and let τ' be as in the proof of Lemma 2.7. We classify τ' according to Proposition 3.5 below. If τ' is of Type 4 then $\mathfrak{m} = \mathfrak{k}$ and $\mathfrak{u} = \mathfrak{u} \cap \mathfrak{p}$. This means that $V_{(\mathfrak{g},K)}^\lambda$ is an holomorphic discrete series and the K-structure of $S(\mathfrak{u})$ is very well-known.

If τ' is of Type 2, then $V_{(\mathfrak{g},K)}^\lambda$ is a Borel-de Siebenthal discrete series and the M-spectrum of $S(\mathfrak{u} \cap \mathfrak{p})$ is given in [**M-F**] for the classical cases and F_4.

We now discuss the case of τ' of Type 3, that is $\tau' = \omega_i^\vee + \omega_j^\vee$ with $i \neq j$ and $i, j \in J$. Set

$$\Delta_i = \{\alpha \in \Delta^+ \mid (\omega_i^\vee, \alpha) = 1,\ (\omega_j^\vee, \alpha) = 0\},$$
$$\Delta_j = \{\alpha \in \Delta^+ \mid (\omega_j^\vee, \alpha) = 1,\ (\omega_i^\vee, \alpha) = 0\},$$

and $\mathfrak{u}_i$ (resp. $\mathfrak{u}_j$) be the subspace of $\mathfrak{u} \cap \mathfrak{p}$ such that $\Delta(\mathfrak{u}_i) = v_z^{-1}\Delta_i$ (resp. $\Delta(\mathfrak{u}_j) = v_z^{-1}\Delta_j$). Then, as a M-module, $\mathfrak{u} \cap \mathfrak{p} = \mathfrak{u}_i \oplus \mathfrak{u}_j$, hence

$$S^n(\mathfrak{u} \cap \mathfrak{p}) = \sum_{h+k=n} S^h(\mathfrak{u}_i) \otimes S^k(\mathfrak{u}_j).$$

Set $\mathfrak{u}_i^- = \sum_{\alpha \in v_z^{-1}\Delta_i} \mathfrak{g}_{-\alpha}$, $\mathfrak{u}_j^- = \sum_{\alpha \in v_z^{-1}\Delta_j} \mathfrak{g}_{-\alpha}$, and

$$\mathfrak{g}_i = \mathfrak{u}_i^- \oplus \mathfrak{m} \oplus \mathfrak{u}_i \quad \mathfrak{g}_j = \mathfrak{u}_j^- \oplus \mathfrak{m} \oplus \mathfrak{u}_j.$$

We notice that $\mathfrak{u}_i$ and $\mathfrak{u}_j$ are abelian in $\mathfrak{g}_i$ and $\mathfrak{g}_j$ respectively, thus we can apply [**S1**] to compute the M-spectrum of $S(\mathfrak{u}_i)$ and $S(\mathfrak{u}_j)$.

As an example we compute the M-spectrum of $S(\mathfrak{u}\cap\mathfrak{p})$ for G of type E_6 and $\tau' = \omega_1^\vee + \omega_6^\vee$. We use the notations of [**Bou**]. In this case

$$\Delta_1 = \left\{ \frac{1}{2}(\epsilon_8 - \epsilon_7 - \epsilon_6 - \epsilon_5) + \frac{1}{2}\sum_{i=1}^{4}(-1)^{\nu(i)}\epsilon_i \text{ with } \sum_{i=1}^{4}\nu(i) \text{ odd} \right\},$$

$$\Delta_6 = \{\pm\epsilon_i + \epsilon_5 \mid 1 \le i \le 4\}.$$

If we set $\Delta_0 = \{\pm\epsilon_i \pm \epsilon_j \mid 2 \le i < j \le 4\}$ and $\mathfrak{m}' = \mathfrak{h} \oplus \sum_{\alpha\in\Delta_0}\mathfrak{g}_\alpha$, then $\mathfrak{m} = v_z^{-1}\mathfrak{m}'$.

We set

$$\begin{aligned}
\eta_1 &= \frac{1}{2}(\epsilon_8 - \epsilon_7 - \epsilon_6 - \epsilon_5)\\
\eta_2 &= \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)\\
\eta_3 &= \frac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4)\\
\eta_4 &= \frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4)\\
\eta_5 &= \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4),
\end{aligned}$$

then we see that $\{v_z^{-1}(\eta_i \pm \eta_j) \mid 1 \le i < j \le 5\}$ is the set of positive roots for $(\mathfrak{g}_1, \mathfrak{h})$ contained in $v_z^{-1}\Delta^+$ and that $v_z^{-1}(\eta_1 - \eta_2)$ is the unique simple noncompact root. Applying the results of [**S1**], we see that the highest weights of the M-types of $S(\mathfrak{u}_1)$ are given by $\mu(h_1,h_2) = v_z^{-1}(h_1(\eta_1+\eta_2) + h_2(\eta_1 - \eta_2))$, with $h_1 \ge h_2$.

Similarly the positive system for $(\mathfrak{g}_6, \mathfrak{h})$ contained in $v_z^{-1}\Delta^+$ is $\{v_z^{-1}(\pm\epsilon_i + \epsilon_j) \mid 1 \le i < j \le 5\}$ and $v_z^{-1}(\epsilon_5 - \epsilon_4)$ is the unique noncompact simple root. It follows that the highest weights of the M-types of $S(\mathfrak{u}_6)$ are given by $\nu(k_1,k_2) = v_z^{-1}(k_1(\epsilon_4+\epsilon_5) + k_2(\epsilon_5 - \epsilon_4))$, with $k_1 \ge k_2$. Putting all together we obtain that

$$S^n(\mathfrak{u}\cap\mathfrak{p}) = \sum_{\substack{h_1+h_2+k_1+k_2=n\\ h_1\ge h_2,\ k_1\ge k_2}} F_M(\mu(h_1,h_2)) \otimes F_M(\nu(k_1,k_2)),$$

where $F_M(\mu) = V_{(\mathfrak{m},M)}^{\mu+\rho_{\mathfrak{m}}}$ denotes the M-type of highest weight μ.

3. Nilradical and special abelian ideals.

If $\tau \in \mathfrak{h}_\mathbb{R}$ we set $\mathfrak{q}_\tau = \mathfrak{m}_\tau \oplus \mathfrak{n}_\tau$, where $\mathfrak{m}_\tau = \mathfrak{h} \oplus \sum\limits_{(\alpha,\tau)=0}\mathfrak{g}_\alpha$ is a Levi subalgebra and $\mathfrak{n}_\tau = \sum\limits_{\alpha\in\Delta^+:(\alpha,\tau)>0}\mathfrak{g}_\alpha$ is the nilradical of $\mathfrak{q}_\tau$.

Remark. If $\mathfrak{i}$ is abelian, then $\mathfrak{n}_{\langle\mathfrak{i}\rangle}$ is an ad-nilpotent ideal of $\mathfrak{b}$; moreover $\mathfrak{i} \subset \mathfrak{n}_{\langle\mathfrak{i}\rangle}$. The first assertion follows from the fact that $\langle\mathfrak{i}\rangle$ is a dominant weight (cf. [**Ko1**, Prop. 6]). For the second claim, remark that, since $\alpha+\beta$ is not a root for any $\alpha, \beta \in \Phi_{\mathfrak{i}}$, we have $(\alpha,\beta) \geq 0$; therefore, for $\alpha \in \Phi_{\mathfrak{i}}$, we have $(\alpha, \langle\mathfrak{i}\rangle) = (\alpha,\alpha) + \sum\limits_{\beta\in\Phi_{\mathfrak{i}}\setminus\{\alpha\}} (\alpha,\beta) > 0$.

In the following lemma we recall two well-known facts about roots:

Lemma 3.1.

(1) *If $\alpha, \beta \in \Delta^+$ and $\alpha < \beta$, then there exists $\beta_1, \dots, \beta_h \in \Delta^+$ such that $\alpha + \sum\limits_{i=1}^{j} \beta_i \in \Delta^+$ for $j = 1, \dots, h$, and $\alpha + \sum\limits_{i=1}^{h} \beta_i = \beta$.*

(2) *If $\alpha+\beta, \gamma, \alpha+\beta+\gamma \in \Delta$, then at least one of $\alpha+\gamma$, $\beta+\gamma$ is a root.*

Proof. To prove (1), fix $\beta \in \Delta^+$. We will show that for each $\alpha \in \Delta^+$ such that $\alpha < \beta$, there exist $\beta_1, \dots, \beta_h \in \Delta^+$ such that

$$\beta = \alpha + \beta_1 + \cdots + \beta_h$$

and $\alpha + \beta_1 + \cdots + \beta_i \in \Delta^+$ for all $i = 1, \dots, h$. We shall prove the statement by induction on $ht(\beta-\alpha)$. If $ht(\beta-\alpha) = 1$ the result is obvious. Suppose that $ht(\beta-\alpha) > 1$. If $(\alpha,\beta) > 0$, then $\beta-\alpha$ is a root and we are done, so we assume $(\alpha,\beta) \leq 0$. By assumption, there exist $\eta_1, \dots, \eta_k \in \Delta^+$ such that $\beta = \alpha+\eta_1+\cdots+\eta_k$ and we obtain that, for at least one of the η_i, $(\alpha,\eta_i) < 0$. We may assume $(\alpha,\eta_1) < 0$, so that $\alpha+\eta_1 \in \Delta^+$. If $\alpha+\eta_1 = \beta$ we are done, otherwise $\alpha+\eta_1 < \beta$ and we can apply the induction hypothesis.

Assertion (2) is an immediate consequence of the Jacobi identity. □

Recall that $X = \overline{C}_2 \cap P^\vee$. Let $\tau \in X$ and set $\mathfrak{i}_{(\tau)} = \bigoplus\limits_{\alpha\in\Delta^2_\tau} \mathfrak{g}_\alpha$. Then $\mathfrak{i}_{(\tau)}$ is an abelian ideal of $\mathfrak{b}$ that should not be confused with $\mathfrak{i}_\tau$ in case $\tau \in X \cap \widetilde{Z}_{\text{ab}}$.

Lemma 3.2. *If $\tau \in X$ and $\Delta^2_\tau \neq \emptyset$ then:*

(1) *for all $\alpha \in \Delta^1_\tau$ there exists $\beta \in \Delta^1_\tau$ such that $\alpha+\beta \in \Delta^2_\tau$.*

(2) $\mathfrak{n}_{\langle\mathfrak{i}_{(\tau)}\rangle} = \mathfrak{n}_\tau = \sum\limits_{\alpha\in\Delta^1_\tau\cup\Delta^2_\tau} \mathfrak{g}_\alpha$.

Proof. (1) Assume $\alpha \in \Delta^1_\tau$. Consider $\Delta(\alpha) = \{\beta \in \Delta^2_\tau \mid \beta > \alpha\}$. Since $(\tau,\theta) = 2$ we have $\alpha \neq \theta$, hence $\Delta(\alpha) \neq \emptyset$. Pick β minimal in $\Delta(\alpha)$.

By Lemma 3.1 (1), we can find $\beta_1, \dots, \beta_k \in \Delta^+$ such that $\gamma_j = \alpha + \sum\limits_{i=1}^{j} \beta_i$ is a root for $j = 1, \dots, k$ and $\gamma_k = \beta$. Choose among such sequences $\beta_1, \dots, \beta_k$ one such that k is minimal. By the choice of β, $(\beta_k, \tau) = 1$. We shall prove the claim by showing that $k = 1$. If $k > 1$, we have $\gamma_k = \gamma_{k-2} + \beta_{k-1} + \beta_k$, where we set $\gamma_0 = \alpha$. By Lemma 3.1 (2), either $\gamma_{k-2} + \beta_k \in \Delta^+$ (hence $\gamma_{k-2} + \beta_k \in \Delta^2_\tau$), but this is not possible by the minimality of β, or $\beta_{k-1} + \beta_k \in \Delta^+$, and indeed $\beta_{k-1} + \beta_k \in \Delta^1_\tau$. But in this latter case, setting

$\beta'_{k-1} = \beta_{k-1} + \beta_k$, we see that $\beta = \alpha + \beta_1 + \cdots + \beta_{k-2} + \beta'_{k-1}$ contradicting the minimality of k.

(2) Set $\mathfrak{i} = \mathfrak{i}_{(\tau)}$. We first prove the inclusion $\Delta^0_\tau \subset \Delta^0_{\langle\mathfrak{i}\rangle}$. If $\alpha \in \Delta^0_\tau$ and $\beta \in \Delta^2_\tau$, then clearly $s_\alpha(\beta) \in \Delta^2_\tau$. This implies that $s_\alpha\left(\sum_{\beta\in\Delta^2_\tau} \beta\right) = \sum_{\beta\in\Delta^2_\tau} \beta$, hence that $\left(\alpha, \sum_{\beta\in\Delta^2_\tau} \beta\right) = 0$, or $(\alpha, \langle\mathfrak{i}\rangle) = 0$. It remains to prove the reverse inclusion: We assume that $(\alpha, \tau) > 0$ and prove that $(\alpha, \langle\mathfrak{i}\rangle) > 0$. If $\alpha \in \Delta^2_\tau$, then by definition $\alpha \in \Phi_{\mathfrak{i}}$, hence, by the above Remark, $(\alpha, \langle\mathfrak{i}\rangle) > 0$. Thus it suffices to prove that if $\alpha \in \Delta^1_\tau$, then $(\alpha, \langle\mathfrak{i}\rangle) > 0$. If $\alpha \in \Delta^1_\tau$ and $\gamma \in \Delta^2_\tau$ then necessarily $(\alpha, \gamma) \geq 0$: Otherwise $\alpha + \gamma$ is a root and $(\alpha + \gamma, \tau) = 1 + 2 = 3$, a contradiction. It remains to prove that for at least one $\gamma \in \Delta^2_\tau$ we have $(\alpha, \gamma) > 0$. By (1) we can find $\beta \in \Delta^1_\tau$ such that $\alpha + \beta \in \Delta^2_\tau$; we shall prove that then $(\alpha, \alpha + \beta) > 0$. Assume by contradiction $(\alpha, \alpha + \beta) \leq 0$. Then in particular $(\alpha, \beta) < 0$, so that $(s_\alpha(\beta), \tau) = (\beta - (\beta, \alpha^\vee)\alpha, \tau) \geq 2$. It would follow that $(\beta - (\beta, \alpha^\vee)\alpha, \tau) = 2$, hence that $(\beta, \alpha^\vee) = -1$ and therefore that $(\alpha^\vee, \alpha + \beta) = 1$. This is impossible since $(\alpha^\vee, \alpha + \beta)$ differs from $(\alpha, \alpha + \beta)$ by a positive factor. □

Theorem 3.3 ([**Ko2**, Theorem 4.4]). *Given $\mathfrak{i} \in \mathcal{I}_{\mathrm{ab}}$, the following conditions are equivalent:*

(1) $[\mathfrak{n}_{\langle\mathfrak{i}\rangle}, \mathfrak{n}_{\langle\mathfrak{i}\rangle}] \subset \mathfrak{i} = \operatorname{cent} \mathfrak{n}_{\langle\mathfrak{i}\rangle}$.
(2) $\Phi_{\mathfrak{i}} = \Delta^2_\tau$ *for some* $\tau \in X$.

Proof. ($1 \Rightarrow 2$) We already remarked, as a general fact, that $\mathfrak{i} \subset \mathfrak{n}_{\langle\mathfrak{i}\rangle}$. We assume that Condition (1) holds and for $\alpha \in \Delta^+$ we define

$$\tau(\alpha) = \begin{cases} 2 & \text{if} \quad \mathfrak{g}_\alpha \subset \mathfrak{i} \\ 1 & \text{if} \quad \mathfrak{g}_\alpha \subset \mathfrak{n}_{\langle\mathfrak{i}\rangle} \setminus \mathfrak{i} \\ 0 & \text{if} \quad \mathfrak{g}_\alpha \not\subset \mathfrak{n}_{\langle\mathfrak{i}\rangle}. \end{cases}$$

We shall prove that if $\alpha, \beta, \alpha + \beta \in \Delta^+$, then $\tau(\alpha + \beta) = \tau(\alpha) + \tau(\beta)$. We first verify that if $\mathfrak{g}_\alpha \not\subset \mathfrak{n}_{\langle\mathfrak{i}\rangle}$, $\mathfrak{g}_\beta \subset \mathfrak{n}_{\langle\mathfrak{i}\rangle}$, and $\alpha + \beta \in \Delta^+$, then $\mathfrak{g}_{\alpha+\beta} \subset \mathfrak{n}_{\langle\mathfrak{i}\rangle}$, and $\mathfrak{g}_{\alpha+\beta} \subset \mathfrak{i}$ if and only if $\mathfrak{g}_\beta \subset \mathfrak{i}$. The first condition is immediate since $\mathfrak{n}_{\langle\mathfrak{i}\rangle}$ is an ideal of $\mathfrak{q}_{\langle\mathfrak{i}\rangle}$. Since $\mathfrak{i} = \operatorname{cent} \mathfrak{n}_{\langle\mathfrak{i}\rangle}$, we have in particular that $\mathfrak{i}$ is an ideal of $\mathfrak{q}_{\langle\mathfrak{i}\rangle}$, too; hence, if $\mathfrak{g}_\beta \subset \mathfrak{i}$, $\mathfrak{g}_{\alpha+\beta} \subset \mathfrak{i}$, too. Similarly, if $\mathfrak{g}_\beta \subset \mathfrak{n}_{\langle\mathfrak{i}\rangle} \setminus \mathfrak{i}$, then $\mathfrak{g}_{\alpha+\beta} \subset \mathfrak{n}_{\langle\mathfrak{i}\rangle} \setminus \mathfrak{i}$, otherwise we should obtain that $\mathfrak{g}_\beta = [\mathfrak{g}_{-\alpha}, \mathfrak{g}_{\alpha+\beta}] \subset \mathfrak{i}$. The next case to consider is when $\mathfrak{g}_\alpha, \mathfrak{g}_\beta$ are both contained in $\mathfrak{n}_{\langle\mathfrak{i}\rangle} \setminus \mathfrak{i}$; if $\alpha + \beta \in \Delta$, the first equality in (1) implies $\mathfrak{g}_{\alpha+\beta} \subset \mathfrak{i}$, as desired. Finally, if $\mathfrak{g}_\alpha \subset \mathfrak{i} = \operatorname{cent} \mathfrak{n}_{\langle\mathfrak{i}\rangle}$ and $\mathfrak{g}_\beta \subset \mathfrak{n}_{\langle\mathfrak{i}\rangle}$ we have that $\alpha + \beta \notin \Delta$, so we can conclude that τ is additive and less or equal than 2 on Δ^+. This fact implies that τ can be extended to a linear functional on $\mathfrak{h}^*_{\mathbb{R}}$ which clearly corresponds to an element in X; if we still denote by τ this element, we obtain by definition that $\Phi_{\mathfrak{i}} = \Delta^2_\tau$.

$(2 \Rightarrow 1)$ By Lemma 3.2, (2) we have $\mathfrak{n}_{\langle \mathfrak{i}\rangle} = \bigoplus\limits_{\alpha\in\Delta^1_\tau\cup\Delta^2_\tau} \mathfrak{g}_\alpha$. Hence it is clear that $[\mathfrak{n}_{\langle \mathfrak{i}\rangle}, \mathfrak{n}_{\langle \mathfrak{i}\rangle}] \subset \mathfrak{i}$; it is also clear that $\mathfrak{i} \subset \operatorname{cent}\mathfrak{n}_{\langle \mathfrak{i}\rangle}$. Item (1) of Lemma 3.2 shows that no element in $\bigoplus\limits_{\alpha\in\Delta^1_\tau} \mathfrak{g}_\alpha$ belongs to $\operatorname{cent}\mathfrak{n}_{\langle \mathfrak{i}\rangle}$, so indeed $\mathfrak{i} = \operatorname{cent}\mathfrak{n}_{\langle \mathfrak{i}\rangle}$. □

Definition. An abelian ideal of $\mathfrak{b}$ is said special if it satisfies one of the conditions of Theorem 3.3. An abelian ideal is nilradical if it is the nilradical of a parabolic subalgebra of $\mathfrak{g}$.

Now we determine the structure of both nilradical and special abelian ideals. In particular we provide a proof of Theorem 4.9 of [**Ko2**].

Set $\mathcal{M} = \{\omega_i^\vee \mid i \in J\} \cup \{0\}$. It is well-known that $\mathcal{M}$ is a set of representatives for $P^\vee/Q^\vee$.

Proposition 3.4. *An abelian ideal $\mathfrak{i}$ of $\mathfrak{b}$ is nilradical if and only if there exists $\omega \in \mathcal{M}$ such that $\mathfrak{i} = \mathfrak{n}_\omega$. In particular nilradical abelian ideals are in bijection with* Cent (G)*.*

Proof. The only nontrivial thing to prove is the fact that

$$0 \neq \mathfrak{n}_\tau = \bigoplus_{\alpha\in\Delta^+:(\alpha,\tau)>0} \mathfrak{g}_\alpha$$

is abelian only if we can choose $\tau = \omega_i^\vee$ for some $i \in J$.

Let $\alpha \in \Delta^+$ be such that $(\alpha,\tau) > 0$ and such that $\alpha = \alpha' + \alpha''$ with $\alpha', \alpha'' \in \Delta^+$. If $\mathfrak{n}_\tau$ is abelian, (α',τ) and (α'',τ) cannot be both nonzero. Suppose that $(\alpha',\tau) = 0$: Then we have that $(\alpha'',\tau) > 0$. Repeating this argument we obtain that $\alpha = \alpha_i + \gamma$ where γ is a sum of positive roots β such that $(\beta,\tau) = 0$ and α_i is a simple root such that $(\alpha_i,\tau) > 0$. Apply now this observation to the highest root θ: Then $\theta = \alpha_i + \gamma$. This implies that $\tau = k\omega_i^\vee$ with $i \in J$, hence $\mathfrak{n}_\tau = \mathfrak{n}_{\omega_i^\vee}$. □

Recall that $\theta = \sum_{i=1}^n m_i\alpha_i$.

Proposition 3.5. *Suppose $\tau \in X$. Then we have the following five possibilities:*

(1) $\tau = 2\omega_i^\vee$, $i \in J$;
(2) $\tau = \omega_i^\vee$, $m_i = 2$;
(3) $\tau = \omega_i^\vee + \omega_j^\vee$, $i, j \in J$;
(4) $\tau = \omega_i^\vee$, $i \in J$;
(5) $\tau = 0$.

Then $\Delta^2_\tau \neq \emptyset$, whence $\mathfrak{i}_{(\tau)} \neq 0$, exactly in Cases (1), (2), (3).

In Case (1) the special abelian ideal $\mathfrak{i}_{(\tau)}$ *is nilradical (and viceversa); in Cases (2) and (3)* $\mathfrak{i}_{(\tau)}$ *is not nilradical and relations* $\mathfrak{i}_{(\tau)} \subsetneq \mathfrak{n}_\tau$, $\mathfrak{i}_{(\tau)} = [\mathfrak{n}_\tau, \mathfrak{n}_\tau]$ *hold.*

Proof. Since $\tau \in \overline{C}_2$ we have $\tau = \sum_{i=1}^{n} \varepsilon_i o_i$ with $\varepsilon_i \geq 0$ and $0 < \sum_{i=1}^{n} \varepsilon_i \leq 2$ where $o_i = \omega_i^\vee / m_i$, $i = 1, \dots, n$. Since, moreover, $\tau \in P^\vee = \bigoplus_{i=1}^{n} \mathbb{Z}\omega_i^\vee$, the only possibilities for τ are the five ones listed above. It is clear that $\Delta_\tau^2 = \emptyset$ if and only if we are in Cases (4) or (5). Suppose that $\tau = 2\omega_i^\vee$, $i \in J$: Then $\Delta_\tau^1 = \emptyset$ and Part (2) of Lemma 3.2 implies that $\mathfrak{i}_{(\tau)} = \mathfrak{n}_{\langle \mathfrak{i}_{(\tau)} \rangle}$, hence $\mathfrak{i}_{(\tau)}$ is nilradical. It is clear that in Cases (2) and (3) $\Delta_\tau^1 \neq \emptyset$, whence, by Lemma 3.2, (2), $\mathfrak{i}_{(\tau)} \subsetneq \mathfrak{n}_{\langle \mathfrak{i}_{(\tau)} \rangle}$; moreover, by Theorem 3.3, $\mathfrak{i}_{(\tau)} \supset [\mathfrak{n}_{\langle \mathfrak{i}_{(\tau)} \rangle}, \mathfrak{n}_{\langle \mathfrak{i}_{(\tau)} \rangle}]$. So we are left with proving the reverse inclusion. It suffices to prove that any $\beta \in \Delta_\tau^2$ splits as the sum of two roots in Δ_τ^1. We note that in Cases (2) and (3) if $\beta \in \Delta_\tau^2$ then β is not simple, so $\beta = \beta_1 + \beta_2$ with $\beta_1, \beta_2 \in \Delta^+$, $\beta_1, \beta_2 < \beta$. If β_1 and β_2 belong to Δ_τ^1 we are done; otherwise one of them, say for example β_1, belongs to Δ_τ^2. By induction on the height of β we can assume that there exist $\gamma_1, \gamma_2 \in \Delta_\tau^1$ such that $\beta_1 = \gamma_1 + \gamma_2$. Then by Lemma 3.1, (2) at least one among $\gamma_1 + \beta_2$, $\gamma_2 + \beta_2$ is a root which clearly belongs to Δ_τ^1 and we get our claim. From the previous proposition it follows that $\mathfrak{i}_{(\tau)}$ is not nilradical. $\square$

Putting together the results of Lemma 3.2 and Theorem 3.3 we obtain the following result:

Corollary 3.6. *The special abelian ideals of* $\mathfrak{b}$ *are exactly the* $\mathfrak{i}_{(\tau)}$, *with* $\tau \in X$. *For any* $\tau \in X$ *we have* $[\mathfrak{n}_\tau, \mathfrak{n}_\tau] \subset \mathfrak{i}_{(\tau)} = \operatorname{cent} \mathfrak{n}_\tau$.

Remark. Lemma 3.2 allows easily to classify and enumerate the special abelian ideals by looking at the coefficients of the highest root. Indeed, if $\mathfrak{i}$ is a nonzero special abelian ideal and $\mathfrak{i} = \mathfrak{i}_{(\tau)} = \mathfrak{i}_{(\tau')}$, then Part (2) of Lemma 3.2 says that $\mathfrak{n}_\tau = \mathfrak{n}_{\tau'}$, so $\Delta_\tau^0 = \Delta_{\tau'}^0$. Since we are assuming that $\Delta_\tau^2 = \Delta_{\tau'}^2$, we can conclude that $\Delta_\tau^1 = \Delta_{\tau'}^1$ also. This implies that $\tau = \tau'$. With reference to the five possibilities for τ listed Proposition 3.5, the list of the $\tau \in X$ such that $\mathfrak{i}_{(\tau)} \neq 0$ is given in Table 1; we label Dynkin diagrams as in [**Bou**].

For $\alpha \in \Delta^+$ we set $V_\alpha = \{\beta \in \Delta^+ \mid \beta \geq \alpha\}$. Let $\mathfrak{i}_{(\tau)}$, $\tau \in X$, be a special abelian ideal. Then if $\tau = 2\omega_i^\vee$, $i \in J$, or $\tau = \omega_i^\vee$, $m_i = 2$, we have $\mathfrak{n}_\tau = \bigoplus_{\beta \in V_{\alpha_i}} \mathfrak{g}_\beta$, whereas if $\tau = \omega_i^\vee + \omega_j^\vee$, $i, j \in J$ then $\mathfrak{n}_\tau = \bigoplus_{\beta \in V_{\alpha_i} \cup V_{\alpha_j}} \mathfrak{g}_\beta$.

Table 1.

Δ	τ of Type 1	τ of Type 2	τ of Type 3
A_n	$2\omega_1^\vee,\dots,2\omega_n^\vee$		$\omega_i^\vee+\omega_j^\vee,\ 1\le i<j\le n$
B_n	$2\omega_1^\vee$	$\omega_2^\vee,\dots,\omega_n^\vee$	
C_n	$2\omega_n^\vee$	$\omega_1^\vee,\dots,\omega_{n-1}^\vee$	
D_n	$2\omega_1^\vee,2\omega_{n-1}^\vee,2\omega_n^\vee$	$\omega_2^\vee,\dots,\omega_{n-2}^\vee$	$\omega_1^\vee+\omega_{n-1}^\vee,\ \omega_1^\vee+\omega_n^\vee,\ \omega_{n-1}^\vee+\omega_n^\vee$
E_6	$2\omega_1^\vee,2\omega_6^\vee$	$\omega_2^\vee,\omega_3^\vee,\omega_5^\vee$	$\omega_1^\vee+\omega_6^\vee$
E_7	$2\omega_7^\vee$	$\omega_1^\vee,\omega_2^\vee,\omega_6^\vee$	
E_8		$\omega_1^\vee,\omega_8^\vee$	
F_4		$\omega_1^\vee,\omega_4^\vee$	
G_2		$\omega_2^\vee$	

We want to restate the previous results in terms of the machinery introduced in Section 1. If $\tau\in X$, set $\mathfrak{k}=\mathfrak{k}_\tau$ and $\mathfrak{p}=\mathfrak{p}_\tau$. Set also $\mathfrak{b}_\mathfrak{k}=\mathfrak{b}\cap\mathfrak{k}$, $\mathfrak{b}_\mathfrak{p}=\mathfrak{b}\cap\mathfrak{p}$ and $\mathfrak{i}(\Phi)=\bigoplus\limits_{\alpha\in\Phi}\mathfrak{g}_\alpha$, for $\Phi\subset\Delta^+$. Recall that $\Phi_z^1=v_z^{-1}(\Delta_z^1)$, $\Phi_z^2=-v_z^{-1}(\Delta_z^{-2})$, so that $\mathfrak{i}_z=\mathfrak{i}(\Phi_z^1)\oplus\mathfrak{i}(\Phi_z^2)$. Moreover we have $\mathfrak{b}_\mathfrak{k}=\mathfrak{h}\oplus\left(\bigoplus\limits_{\alpha\in\Delta_\tau^0\cup\Delta_\tau^2}\mathfrak{g}_\alpha\right)$, $\mathfrak{b}_\mathfrak{p}=\bigoplus\limits_{\alpha\in\Delta_\tau^1}\mathfrak{g}_\alpha$ and $\Delta_\tau^2=-v_z^{-1}(\Delta_z^{-2})$, $\Delta_\tau^0=v_z^{-1}(\Delta_z^0)$, $\Delta_\tau^1=v_z^{-1}(\Delta_z^1)\cup -v_z^{-1}(\Delta_z^{-1})$.

With this notation we have (see [**Ko2**, Prop. 4.7, Theorem 4.9]):

Proposition 3.7. *Let $z\in\widetilde{Z}_{\mathrm{ab}}$ and set $\tau=\mathrm{dom}\,(z)$.*

(1) *The following relations are equivalent:*
 i) $\tau=2\omega$ *for some* $\omega\in\mathcal{M}$*;*
 ii) $z=-2\omega$ *for some* $\omega\in\mathcal{M}$*;*
 iii) $n_\tau=1$*;*
 iv) $\mathfrak{k}_\tau=\mathfrak{g}$.

(2) $\mathfrak{i}(\Phi_z^2)$ *is a special abelian ideal.*

(3) $\mathfrak{i}(\Phi_z^2)$ *is nilradical if and only if* $\tau=2\omega$ *for some* $\omega\in\mathcal{M}$.

(4) *If* $\mathfrak{i}(\Phi_z^2)$ *is not nilradical then* $\mathfrak{i}(\Phi_z^2)=[\mathfrak{b}_\mathfrak{p},\mathfrak{b}_\mathfrak{p}]$.

(5) $\mathfrak{i}(\Phi_z^1)=\mathfrak{b}_z\cap\mathfrak{b}_\mathfrak{p}$.

(6) $\mathfrak{i}(\Phi_z^1)=\mathfrak{i}_z\cap\mathfrak{p}$ *and* $\mathfrak{i}(\Phi_z^2)=\mathfrak{i}_z\cap\mathfrak{k}$.

Proof. i) $\Rightarrow$ ii): If $\alpha \in \Delta^+$ then $\alpha(z) = v_z^{-1}(\alpha)(\tau) \in \{0, -2\}$, hence $-z/2$ is minuscule.

ii) $\Rightarrow$ iii): By the proof of 2.4, $\Delta_\tau = \Delta$, hence $n_\tau = 1$.

iii) $\Rightarrow$ iv): $W_\tau = W$ implies that $\Delta = \Delta_\tau$ and in turn that $\mathfrak{k}_\tau = \mathfrak{g}$.

iv) $\Rightarrow$ i): If $\Delta_\tau = \Delta$ then, if $\alpha \in \Delta^+$, $(\alpha, \tau) \in \{0, 2\}$ hence $\tau/2$ is minuscule. At this point Part (1) is completely proved. The other parts follow immediately combining the relations listed just before the statement of this proposition with Lemma 3.2 and Proposition 3.4. $\square$

Proposition 3.8. *Let $\tau \in X$, $\mathfrak{s}$ be an $\mathfrak{h}$-submodule of $\mathfrak{b}_\mathfrak{p}$, $\Delta(\mathfrak{s}) = \{\alpha \mid \mathfrak{g}_\alpha \subset \mathfrak{s}\}$, and set $\mathfrak{i} = \mathfrak{i}(\Delta^2_\tau) \oplus \mathfrak{s}$. Then the following facts are equivalent:*

(1) *$\mathfrak{s}$ is a $\mathfrak{b}_\mathfrak{k}$-submodule of $\mathfrak{b}_\mathfrak{p}$;*
(2) *$\mathfrak{i}$ is an* ad*-nilpotent ideal of $\mathfrak{b}$ included in $\mathfrak{n}_\tau$;*
(3) *$\mathfrak{i}$ is an* ad*-nilpotent ideal of $\mathfrak{b}$ and $(\beta, \tau) > 0$ for all β in $\Delta(\mathfrak{s})$.*

*The same equivalences hold if we replace "*ad*-nilpotent" with "abelian" in (2) and (3) and consider abelian submodules in (1).*

Proof. The equivalence of (2) and (3) is immediate from the definitions. We prove the equivalence of (1) and (2). By the definition of $\mathfrak{n}_\tau$ and $\mathfrak{b}_\mathfrak{p}$ we have that in any case $\mathfrak{i} \subset \mathfrak{n}_\tau$. By assumption $[\mathfrak{h}, \mathfrak{s}] \subset \mathfrak{s}$; since $\mathfrak{s} \subset \mathfrak{i}(\Delta^1_\tau)$, we have $[\mathfrak{i}(\Delta^2_\tau), \mathfrak{s}] = 0$. Moreover, $[\mathfrak{i}(\Delta^1_\tau), \mathfrak{s}] \subset \mathfrak{i}(\Delta^2_\tau)$, thus we obtain that $\mathfrak{i}$ is an (ad-nilpotent) ideal of $\mathfrak{b}$ if and only if $[\mathfrak{i}(\Delta^\tau_0), \mathfrak{s}] \subset \mathfrak{s}$. But by the definition of $\mathfrak{b}_\mathfrak{k}$, we also obtain that $\mathfrak{s}$ is a $\mathfrak{b}_\mathfrak{k}$-module if and only $[\mathfrak{i}(\Delta^\tau_0), \mathfrak{s}] \subset \mathfrak{s}$, so we get our claim. It is clear that all the above arguments still work when restricting to abelian objects. $\square$

Proposition 3.9. *Let $\mathfrak{i} \in \mathcal{I}$, $\tau \in X$, and assume $\mathfrak{i} = \mathfrak{i}(\Delta^2_\tau) \oplus \mathfrak{s}$, with $\mathfrak{s} \subset \mathfrak{i}(\Delta^1_\tau)$. Then there exists $z \in \widetilde{Z}$ such that $\mathfrak{i} = \mathfrak{i}_z$, $\tau = \mathrm{dom}\,(z)$, and $\mathfrak{s} = \mathfrak{i}(v_z^{-1}(\Delta^1_z))$.*

Conversely, if $\mathfrak{i} = \mathfrak{i}_z$ and $\mathrm{dom}\,(z) \in X$, we have $\mathfrak{i} = \mathfrak{i}\left(\Delta^2_{\mathrm{dom}\,(z)}\right) \oplus \mathfrak{s}$, with $\mathfrak{s} \subset \mathfrak{i}\left(\Delta^1_{\mathrm{dom}\,(z)}\right)$.

Proof. Set $\Phi = \Delta(\mathfrak{s})$, $\Phi^2 = (\Phi + \Phi) \cap \Delta^+$, and and $\Phi' = (\Delta^2_\tau \setminus \Phi^2) \cup (\Delta^1_\tau \setminus \Phi)$. We first prove that there exists $v \in W$ such that $\Phi' = N(v)$. For this it suffices to prove that Φ' and its complement in Δ^+ are closed. Since $\tau \in \overline{C}_2 \cap P^\vee$, to prove that Φ' is closed it suffices to show that if $\alpha, \beta \in \Delta^1_\tau$ and $\alpha + \beta \in \Phi^2$ then either α or β belongs to Φ. Set $\gamma = \alpha + \beta$; since $\gamma \in \Phi^2$, there exist $\xi, \eta \in \Phi$ such that $\gamma = \xi + \eta$. Now from the expansion

$$0 < (\gamma, \gamma) = (\alpha, \xi) + (\beta, \xi) + (\alpha, \eta) + (\beta, \eta)$$

we deduce that one of the summands in the right-hand side of the previous relation is positive. Since the difference of two roots having positive scalar product is a root, and since $\eta - \beta = \alpha - \xi$ and $\eta - \alpha = \beta - \xi$, we have that either $\eta - \beta \in \Delta$ or $\eta - \alpha \in \Delta$. It suffices to consider the case $\eta - \alpha \in \Delta$.

Suppose $\alpha - \eta \in \Delta^+$ and remark that $\alpha - \eta \in \Delta^0_\tau$; since $\mathfrak{s}$ is a $\mathfrak{b}_\mathfrak{k}$-module, we have $\alpha = (\alpha - \eta) + \eta \in \Phi$ as desired. If instead $\eta - \alpha \in \Delta^+$, we note that $\eta - \alpha = \beta - \xi \in \Delta$ and as above we deduce that $\beta = (\beta - \xi) + \xi \in \Phi$.

Next we consider $\Delta^+ \setminus \Phi' = \Delta^0_\tau \cup \Phi \cup \Phi^2$. Obviously, Δ^0_τ and $\Phi \cup \Phi^2$ are closed. If $\alpha \in \Delta^0_\tau, \beta \in \Phi$ and $\alpha + \beta \in \Delta$ we have that $\alpha + \beta \in \Phi$ since $\mathfrak{s}$ is a $\mathfrak{b}_\mathfrak{k}$-module. Finally, if $\alpha \in \Delta^0_\tau, \beta \in \Phi^2, \beta = \xi + \eta$, $\xi, \eta \in \Phi$, and $\alpha + \beta = \alpha + \xi + \eta \in \Delta^+$, we have by Lemma 3.1, (2), that either $\alpha + \xi$ or $\alpha + \eta$ is a root and hence belongs to Φ, so that $\alpha + \beta \in \Phi^2$.

So there exists $v \in W$ such that $\Phi' = N(v)$. Set $z = v^{-1}(\tau)$. Then $N(v_z) = \{\alpha \in \Delta^+ \mid (\alpha, v^{-1}(\tau)) < 0\} = \{\alpha \in \Delta^+ \mid (v(\alpha), \tau) < 0\} = \{\alpha \in \Delta^+ \mid v(\alpha) < 0\} = N(v^{-1})$, hence $v_z = v^{-1}$ and therefore $\mathrm{dom}\,(z) = \tau$.

In order to conclude we have to prove that $\mathfrak{i} = \mathfrak{i}_z$. We have $\widetilde{F}(z) = t_\tau v$, hence, by Lemma C, $\mathfrak{i}_z = \mathfrak{i}(\Delta^2_\tau) \oplus \mathfrak{i}(\Delta^1_\tau \setminus N(v))$. Since $\Delta(\mathfrak{s}) \subset \Delta^1_\tau$, and $N(v) \cap \Delta^1_\tau = \Delta^1_\tau \setminus \Delta(\mathfrak{s})$, we have $\Delta^1_\tau \setminus N(v) = \Delta(\mathfrak{s})$, whence $\mathfrak{i} = \mathfrak{i}_z$. Finally by a direct check we obtain that $\Delta^1_\tau \setminus N(v) = v(\Delta^1_z)$, so $\mathfrak{s} = \mathfrak{i}(v_z^{-1}(\Delta^1_z))$.

The converse statement follows directly from Lemma C. $\square$

Recall that $\mathrm{dom} : \mathfrak{h}_\mathbb{R} \to C_\infty$ is defined by $z \mapsto v_z^{-1}(z)$. As already observed $\mathrm{dom}\,(\widetilde{Z}_{\mathrm{ab}}) = X = \overline{C}_2 \cap P^\vee$, whereas, by Proposition 1.3, we have $\mathrm{dom}\,(\widetilde{Z}) \subset \overline{C}_h \cap P^\vee$. We recall that for $\tau \in X$ we set $\widetilde{Z}_\tau = \mathrm{dom}^{-1}(\tau) \cap \widetilde{Z}_{\mathrm{ab}}$; we also set $\widehat{Z}_\tau = \mathrm{dom}^{-1}(\tau) \cap \widetilde{Z}$. Fix $\tau \in X$ and let $\mathcal{S}_\tau$ denote the set of $\mathfrak{b}_\mathfrak{k}$-submodules of $\mathfrak{b}_\mathfrak{p}$.

Proposition 3.10. *The map $z \mapsto \mathfrak{i}(v_z^{-1}(\Delta^1_z))$ establishes a bijection between $\widehat{Z}_\tau$ and $\mathcal{S}_\tau$. Moreover, this map restricts to a bijection between $\widetilde{Z}_\tau$ and the abelian subspaces in $\mathcal{S}_\tau$. In particular, the number of abelian $\mathfrak{b}_\mathfrak{k}$-submodules of $\mathfrak{b}_\mathfrak{p}$ is n_τ.*

Proof. Since $\tau + Q^\vee = z + Q^\vee$ for any $z \in \widehat{Z}_\tau$, the map $z \mapsto \mathfrak{i}_z$ is injective on $\widehat{Z}_\tau$ (see Prop. 1.4, (c)). For any $z \in \widehat{Z}_\tau$ we have $\widetilde{F}(z) = t_\tau v_z^{-1}$, hence, by Lemma C, $\Phi_{\mathfrak{i}_z} = \Delta^2_\tau \cup (\Delta^1_\tau \setminus N(v_z^{-1}))$. As in the proof of 3.9, we have $\Delta^1_\tau \setminus N(v_z^{-1}) = v_z^{-1}(\Delta^1_z)$, so that $\mathfrak{i}_z = \mathfrak{i}(\Delta^2_\tau) \oplus \mathfrak{i}(v_z^{-1}(\Delta^1_z))$. In particular $v_z^{-1}(\Delta^1_z)$ determines $\mathfrak{i}_z$, and therefore $z \mapsto \mathfrak{i}(v_z^{-1}(\Delta^1_z))$ is one to one. Moreover, by Proposition 3.8, we obtain that $\mathfrak{i}(v_z^{-1}(\Delta^1_z))$ is a $\mathfrak{b}_\mathfrak{k}$-submodule of $\mathfrak{b}_\mathfrak{p}$, so $z \mapsto \mathfrak{i}(v_z^{-1}(\Delta^1_z))$ is a one to one map from $\widehat{Z}_\tau$ to $\mathcal{S}_\tau$. It remains to prove that this map is onto. Take any $\mathfrak{s} \in \mathcal{S}_\tau$; then, by Proposition 3.8, $\Delta^2_\tau \oplus \mathfrak{s}$ is an ad-nilpotent ideal of $\mathfrak{b}$ included in $\mathfrak{n}_\tau$. Thus by Lemma 3.9 there exists $z \in \widetilde{Z}$ such that $\mathrm{dom}\,(z) = \tau$ and $\mathfrak{s} = \mathfrak{i}(v_z^{-1}(\Delta^1_z))$. So we have proved that $z \mapsto \mathfrak{i}(v_z^{-1}(\Delta^1_z))$ is a bijection between $\widehat{Z}_\tau$ and $\mathcal{S}_\tau$. The final assertions are clear. $\square$

Recall now the $\mathrm{Cent}\,(G)$ action on $\widetilde{Z}$ introduced in Proposition 1.1, explicitly given by the action of the subgroup Σ of $\widetilde{W}$. Let $\Sigma \cdot z$ denote the orbit of $z \in \widetilde{Z}$ under this action. As before we denote by $\mathcal{M}$ the set $\{\omega_i^\vee \mid i \in J\} \cup \{0\}$ of representatives for $P^\vee/Q^\vee$ and for any $\omega \in \mathcal{M}$ we set $V_\omega = \{\alpha \in \Delta^+ \mid (\alpha, \omega) > 0\}$. So $V_{\omega_i^\vee} = V_{\alpha_i}$ and $V_0 = \emptyset$. For $\mathfrak{i} \in \mathcal{I}_{\text{ab}}$ we set

$$C_\mathfrak{i} = \{\omega \in \mathcal{M} \mid \Phi_\mathfrak{i} \subset V_\omega\},$$
$$C'_\mathfrak{i} = \mathrm{dom}\,(\Sigma \cdot z) \cap \mathcal{M},$$

where z is any element of $\widetilde{Z}_{\text{ab}}$ such that $\mathfrak{i}_z = \mathfrak{i}$.

Corollary 3.11 ([**Ko2**, Theorem 5.1])**.** *Let $\mathfrak{i} \in \mathcal{I}_{\text{ab}}$ and let $z \in \widetilde{Z}_{\text{ab}}$ be such that $\mathfrak{i} = \mathfrak{i}_z$. Then there are exactly $|\mathrm{Cent}\,(G)| - |C_\mathfrak{i}|$ decompositions*

$$\mathfrak{i} = \mathfrak{i}(\Delta^2_\tau) + \mathfrak{s},$$

with $\mathfrak{i}(\Delta^2_\tau)$ nonzero special abelian ideal and $\mathfrak{s} \subset \mathfrak{i}(\Delta^1_\tau)$.

Proof. By Proposition 1.4, for any $z' \in \widetilde{Z}$ we have $\mathfrak{i}_{z'} = \mathfrak{i}_z$ if and only if $z' \in \Sigma \cdot z$. Hence, by Proposition 3.9, we have $\mathfrak{i} = \mathfrak{i}(\Delta^2_\tau) + \mathfrak{s}$, with $\tau \in X$ and $\mathfrak{s} \subset \Delta^1_\tau$, if and only if $\tau = \mathrm{dom}\,(z')$ for some $z' \in \Sigma \cdot z$. By the remark following 3.6, the map $\tau \to \Delta^2_\tau$ is injective. Therefore we have only to prove that $\Delta^2_{\mathrm{dom}\,(z')} \neq \emptyset$ for exactly $|\mathrm{Cent}\,(G)| - |C_\mathfrak{i}|$ elements $z' \in \Sigma \cdot z$. We first notice that the restriction of dom to a Σ-orbit is injective. This follows from the fact that distinct elements in $\Sigma \cdot z$ are distinct mod-$Q^\vee$, as clear from the definition of Σ, while $z' + Q^\vee = \mathrm{dom}\,(z') + Q^\vee$ for each $z' \in \Sigma \cdot z$. In particular we have $|\mathrm{dom}\,(\Sigma \cdot z)| = |\mathrm{Cent}\,(G)|$. Now it suffices to prove that $\Delta^2_{\mathrm{dom}\,(z')} = 0$ for exactly $|C_\mathfrak{i}|$ elements $z' \in \Sigma \cdot z$. It is clear that if $\tau \in X$, we have $\Delta^2_\tau = \emptyset$ if and only if $\tau \in \mathcal{M}$, therefore if we prove that $C'_\mathfrak{i} = C_\mathfrak{i}$ we are done. Assume first $\omega \in C'_\mathfrak{i}$. Then $\omega = \mathrm{dom}\,(z')$ for some z' such that $\mathfrak{i} = \mathfrak{i}_{z'}$. Since $\widetilde{F}(z') = t_\omega v_{z'}^{-1}$ using Lemma C we obtain $\Phi_\mathfrak{i} \subset V_\omega$. Conversely, assume $\omega \in C_\mathfrak{i}$. Then, clearly, $\Delta^2_\omega = \emptyset$ and $\mathfrak{i} \subset \mathfrak{i}(\Delta^1_\omega) = \mathfrak{i}(V_\omega)$. Therefore, by Proposition 3.9, there exists $z' \in \Sigma \cdot z$ such that $\mathrm{dom}\,(z') = \omega$: This concludes the proof. $\square$

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