COTITLING VERSUS PURE-INJECTIVE MODULES
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Let $R$ and $S$ be arbitrary associative rings. A left $R$-module $\mathcal{R}W$ is said to be cotilting if the class of modules cogenerated by $\mathcal{R}W$ coincides with the class of modules for which the functor $\text{Ext}_R^1(\cdot, W)$ vanishes. In this paper we characterize the cotilting modules which are pure-injective. The two notions seem to be strictly connected: Indeed all the examples of cotilting modules known in the literature are pure-injective. We observe that if $\mathcal{R}W_S$ is a pure-injective cotilting bimodule, both $R$ and $S$ are semiperfect rings and we give a characterization of the reflexive modules in terms of a suitable “linear compactness” notion.

Introduction.

Cotilting modules first appeared as vector space duals of tilting modules over finite dimensional algebras [12, IV, 7.8]. Recently they have been introduced [5] in the framework of modules over arbitrary associative rings, acquiring a proper independent role. The cotilting modules generalize the notion of injective cogenerator: They are injectives with respect to short exact sequences of modules cogenerated by them.

For arbitrary rings $R$ and $S$, a Morita duality between left $R$-modules and right $S$-modules is given by the contravariant Hom functors associated to a Morita bimodule, i.e., a faithfully balanced bimodule $\mathcal{R}W_S$ with both $\mathcal{R}W$ and $W_S$ injective cogenerators. One of the major components in the theory of Morita dualities is Müller’s theorem [13] which states that the reflexive modules are precisely the linearly compact modules. If $\mathcal{R}W_S$ is a Morita bimodule, both $R$ and $S$ are semiperfect rings [16, Theorem 2.7].

For arbitrary rings $R$ and $S$, a cotilting duality between left $R$-modules and right $S$-modules is given by the contravariant Hom functors and the contravariant Ext functors associated to a cotilting bimodule, i.e., a faithfully balanced bimodule $\mathcal{R}W_S$ with both $\mathcal{R}W$ and $W_S$ cotilting modules (see [4]).

All known examples of cotilting modules are pure-injective. In this paper we characterize the pure-injective cotilting modules. We observe that if $\mathcal{R}W_S$ is a pure-injective cotilting bimodule, both $R$ and $S$ are semiperfect
rings and we give a characterization of the reflexive modules in terms of a suitable “linear compactness” notion.

1. About the pure-injectivity of a cotilting module.

Let $R$ be an associative ring with $1 \neq 0$. We denote by $R$-Mod the category of left unitary $R$-modules and their homomorphisms. Given a left $R$-module $W$, we consider the following classes:

- $\text{Cogen}_W$ denotes the class of all left $R$-modules cogenerated by $R^\lambda W$, that is all $M$ in $R$-Mod such that there exist a cardinal $\lambda$ and a monomorphism $M \rightarrowtail W^\lambda$;
- $\perp W$ denotes the class of all left $R$-modules $M$ such that $\text{Ext}_R^1(M, W) = 0$.

A left $R$-module $R^\lambda W$ is said to be cotilting [5] if $\text{Cogen}_R W = \perp W$. The cotilting modules generalize injective cogenerators: Clearly $R^\lambda W$ is an injective cogenerator if and only if both the classes $\text{Cogen}_R W$ and $\perp W$ coincide with the whole category of left $R$-modules. A short exact sequence

$$0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$$

is said to be pure if any morphism $P \rightarrow M$, with $P$ finitely presented, lifts to a morphism $P \rightarrow L$.

**Definition 1.1.** A module $R^\lambda W$ is pure-injective if it is injective with respect to any pure exact sequence.

All known examples of cotilting modules are pure-injective. It naturally arises the question how the two notions are related.

**Proposition 1.2.** Let $R^\lambda W$ be a cotilting module. If the class $\text{Cogen}_W$ is closed under direct limits, then $W$ is pure-injective.

**Proof.** Let us show that for any pure exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ and for any map $f : A \rightarrow W$ there exists a map $g$ making the following diagram commute:

$$
\begin{array}{c}
0 \rightarrow \begin{array}{c}
A \rightarrow B \rightarrow C \rightarrow 0.
\end{array} \\
\downarrow f \\
W
\end{array}
$$

Replacing $A \rightarrow B$ by $A/\text{Rej}_W A \rightarrow B/\text{Rej}_W A$, we can assume that $\text{Rej}_W A = 0$. The pure exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a direct limit of split exact sequences $0 \rightarrow A \rightarrow B_i \rightarrow C_i \rightarrow 0$ with $C_i$ finitely presented (cf. 34.2 [14]). For each index $i$ we have the commutative diagram with exact rows...
Applying the direct limit functor we get

\[ \text{lim} \rightarrow \text{Rej}_W B_i \cong \text{lim} \rightarrow \text{Rej}_W C_i \]

\[ \begin{array}{ccc}
0 & \rightarrow & A \\
& \downarrow & \\
& B_i & \rightarrow & C_i & \rightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & A & \rightarrow & B_i/\text{Rej}_W B_i & \rightarrow & C_i/\text{Rej}_W C_i & \rightarrow & 0.
\end{array} \]

Since \( \text{lim} \rightarrow \text{Rej}_W B_i \) is in the kernel of \( \text{Hom}(-, W) \) and \( \text{lim} \rightarrow (B_i/\text{Rej}_W B_i) \) belongs to \( \text{Cogen}_W \) by the assumption, we infer that \( \text{lim} \rightarrow (B_i/\text{Rej}_W B_i) \cong B/\text{Rej}_W B \). Also \( \text{lim} \rightarrow (C_i/\text{Rej}_W C_i) \) belongs to \( \text{Cogen}_W \cong \perp W \). So \( f \) can be extended to a morphism \( g' : B/\text{Rej}_W B \rightarrow W \); the composition of \( g' \) with the canonical projection \( B \rightarrow B/\text{Rej}_W B \) yields the desired map \( g \). \( \Box \)

In the cotilting case, since \( \text{Cogen}_W = \perp W \), the hypothesized closure under direct limits of the class of modules cogenerated by \( W \) is suggested by the following proposition:

**Proposition 1.3.** If \( RW \) is a pure-injective module, then \( \perp RW \) is closed under direct limits.

**Proof.** Consider a direct system \( \{M_i : i \in I\} \) in \( \perp W \). The canonical exact sequence

\[ 0 \rightarrow K \rightarrow \oplus_{i \in I} M_i \rightarrow \text{lim}_{i \in I} M_i \rightarrow 0 \]

is pure (cf. [14, 33.9, (2)]). Applying \( \text{Hom}_R(-, W) \) we get the long exact sequence

\[ \cdots \rightarrow \text{Hom}(\oplus_{i \in I} M_i, W) \xrightarrow{f} \text{Hom}(K, W) \rightarrow \text{Ext}^1_R(\text{lim}_{i \in I} M_i, W) \rightarrow \]

\[ \rightarrow \text{Ext}^1_R(\oplus_{i \in I} M_i, W) = 0. \]

Since \( W \) is pure-injective, \( f \) is surjective; so \( \text{lim}_{i \in I} M_i \) belongs to \( \perp W \). \( \Box \)

This result has been used in [9, Lemma 9] to prove that, if \( \mathcal{C} \) is a class of pure-injective modules, every module \( M \) which has a \( \perp \mathcal{C} \)-precover has a
Corollary 1.4. If $R W$ is a cotilting module, then $W$ is pure-injective if and only if $\text{Cogen} W$ is closed under direct limits. In such a case any module has a $\text{Cogen} W$-cover.

Proof. The first claim follows by Propositions 1.2 and 1.3. The second one follows by [1, Corollary 2.6].

Open Problem 1.5. Are all cotilting (bi)modules pure-injective?

It is well-known that the endomorphism rings of a Morita bimodule are semiperfect; indeed a Morita bimodule is injective and finitely cogenerated on both sides (see [16, Theorem 2.7, Proposition 1.19]). We are able to give an analogous result for cotilting bimodules, assuming the closure under direct limits of the classes cogenerated by them. We recall that a ring $R$ is said to be semiregular if $R/J(R)$ is regular and the idempotents lift over the Jacobson radical $J(R)$.

Proposition 1.6. Let $R W S$ be a pure-injective cotilting bimodule, i.e., faithfully balanced and cotilting and pure-injective on both sides. Then both $R$ and $S$ are semiregular rings.

Proof. The notions of pure-injective and algebraically compact module coincide (cf. [14, 34.4]). Then, by [17, Theorem 9], $R$ and $S$ are both semiregular. 

Remark 1.7. Observe that if the ring $R$ is regular, the classes of pure-injective and of injective $R$-modules coincide [14, 37.6]. Therefore pure-injective cotilting bimodules which are not Morita bimodules “live in the space between semiregular and regular rings”.

2. Characterizing the reflexive modules.

Let $R W S$ be an arbitrary bimodule. In the sequel we denote by $\Delta$ the functors $\text{Hom}_?(-, W)$, and by $\Gamma$ the functors $\text{Ext}_?^1(-, W)$, where $?$ stands for $R$ or $S$. We denote by $\Delta^2$ both the compositions $\text{Hom}_R(\text{Hom}_S(-, W), W)$ and $\text{Hom}_S(\text{Hom}_R(-, W), W)$. Given a (left $R$, or right $S$)-module $M$, we denote by $\delta_M$ the canonical homomorphism $M \to \Delta^2(M)$ defined by $m \mapsto [f \mapsto f(m)]$. A module $M$ is called reflexive (resp. torsionless) if $\delta_M$ is an isomorphism (resp. monomorphism).

Clearly a torsionless module $M$ is reflexive if and only if the evaluation map $\delta_M$ is surjective. Endowed $M$ and $\Delta^2 M$ with any topology, the surjectivity of $\delta_M$ can be tested in a topological way asking for $\text{Im} \delta_M$ to be both dense and closed in $\Delta^2 M$. As the approach of Müller [13] to the classical

\footnote{Recently Silvana Bazzoni proved that any cotilting module is pure-injective [2].}
case of Morita dualities suggests, we introduce topological tools in order to characterize the reflexive modules.

Let us endow $\Delta^2 M$ with the finite topology $\varphi$: The linear topology for which the family of submodules $V(F) = \{\alpha \in \Delta^2 M : \alpha(F) = 0\}$, where $F$ is a finite subset of $\Delta M$, is a base for the filter of neighbourhoods of zero.

Let us endow $W$ with the discrete topology. Given any torsionless module $M$, we associate with each subset $A$ of $\Delta M$ the weak topology with respect to morphisms in $A$, denoted by $\tau_A$. By definition $\tau_A$ is the coarsest topology on $M$ such that all morphisms in $A$ are continuous: It is a linear topology with a base for its filter of neighbourhoods of zero formed by finite intersections of kernels of morphisms in $A$. In the sequel the topology $\tau_{\Delta M}$ will be shortly denoted by $\tau$. Note that $\tau$ is the maximum element of the set of linear topologies $\{\tau_A : A \subseteq \Delta M\}$ partially ordered by inclusion. Let $L_\sigma$ be a linearly topologized module. Denote by $H$ the $\sigma$-closure of zero in $L$. Note that $H$ is equal to the intersection of all neighbourhoods of zero and, since $\sigma$ is a linear topology, $\sigma$ is Hausdorff if and only if $H = 0$. A $\sigma$-Cauchy net in $L$ is a family $X_\lambda$, $\lambda \in \Lambda$, indexed by the upwards directed partially ordered set $\Lambda$, such that for every neighbourhood $U$ of zero there exists an upper subset $\Lambda'$ of $\Lambda$ with $x_\lambda - x_\lambda' \in U$ for every $\lambda, \lambda' \in \Lambda'$. The topology $\sigma$ is complete, i.e., any $\sigma$-Cauchy net in $L$ converges in $L$, if and only if the topological quotient $L/H$ is complete (see [3, Chap. 3, §2]). Note that a closed submodule of a complete module is complete. The completion of $L/H$ is called the Hausdorff completion of $L$: Denoted by $J = \{J_\lambda : \lambda \in \Lambda\}$ a base for the filter of neighbourhoods of zero in $L_\sigma$ consisting of open submodules, it coincides with the inverse limit $\varprojlim L/J_\lambda$ (see [10, Proposition 13.7]).

**Proposition 2.1.** Let $M$ be a torsionless module.

(i) The topologies $\tau$ on $M$ and $\varphi$ on $\Delta^2 M$ are Hausdorff.

(ii) $\delta_M : M_\tau \to \Delta^2 M_\varphi$ is a topological embedding.

(iii) The topology $\varphi$ on $\Delta^2 M$ is complete.

*Proof.* (i) Since $M$ is cogenerated by $W$, there exists a set $X$ and the following maps:

$$M \xrightarrow{i} W^X \xrightarrow{\pi_x} W.$$  

Clearly $\{0\}$ is the intersection of $\text{Ker}(\pi_x \circ i)$, $x \in X$. Since any $\text{Ker}(\pi_x \circ i)$ is $\tau$-open and hence $\tau$-closed, $\tau$ is Hausdorff. Let us consider the topology $\varphi$. The open submodules $V(F) = \{\alpha \in \Delta^2 M : \alpha(F) = 0\}$, with $F$ finite subset of $\Delta M$, have intersection zero: Hence $\varphi$ is Hausdorff.

(ii) It follows from the fact that for any finite subset $F$ of $\Delta M$ we have

$$V(F) \cap \delta_M(M) = \{\delta_M(m) : f(m) = 0 \forall f \in F\} = \delta_M(\cap_{f \in F} \text{Ker} f).$$

(iii) The topological module $\Delta^2 M_\varphi$ is a closed submodule of $W^{\Delta M}$ endowed with the product of the discrete topologies. Since the product of complete topologies is also complete, we can conclude. $\square$
Corollary 2.2. Each reflexive module $M$ endowed with the topology $\tau$ is complete.

Proof. If $M$ is reflexive, $\delta_M$ is an isomorphism. Therefore, by Proposition 2.1, (ii), it is a topological isomorphism. We conclude by Proposition 2.1, (iii).

Proposition 2.3. For a torsionless module $M$ the following statements are equivalent:

(i) $M$ is reflexive.

(ii) $\tau$ is a complete topology and $\delta_M(M)$ is dense in $\Delta^2 M\varphi$.

Proof. (i) $\Rightarrow$ (ii): It follows by Corollary 2.2.

(ii) $\Rightarrow$ (i): Since $\tau$ is complete, by Proposition 2.1 $\delta_M(M)$ is a complete, and hence closed, topological submodule of $\Delta^2 M\varphi$. Being $\delta_M(M)$ dense in $\Delta^2 M\varphi$ and $\varphi$ an Hausdorff topology, $\delta_M$ is an isomorphism.

As suggested by Müller [13] in the case of Morita dualities, we look for a suitable notion of compactness for a module $M$ in order to guarantee the completeness of the topology $\tau$ on $M$.

Note that $\delta_M(M)$ is dense in $\Delta^2 M\varphi$ if and only if for each $\alpha$ in $\Delta^2 M$ and $f_1, \ldots, f_n$ in $\Delta M$ there exists $m$ in $M$ such that $\alpha(f_i) = f_i(m)$ for each $i = 1, \ldots, n$. Following [11], a module $M$ satisfying the above property will be called $W$-dense.

Definition 2.4. Let $M$ be a torsionless left $R$-module. A submodule $K$ of $M$ is called $W$-closed if $M/K$ is torsionless (see [11, §2]). A linear topology on $M$ is said to be a $W$-topology if it has a basis of neighbourhoods of zero consisting of $W$-closed submodules.

Each $W$-closed submodule $K$ of $M$ is closed in $M\tau$. Indeed, for a suitable set $X$, there exist the following maps:

$$M \xrightarrow{\pi} M/K \xrightarrow{i} W^{-}\xrightarrow{\pi} W.$$

Then, $K = \bigcap_{x \in X} \ker(\pi_x \circ i \circ \pi)$ is closed, since it is an intersection of open and hence closed submodules. The converse is not true in general.

Example 2.5. Let $R$ denote the $k$-algebra given by the quiver $1 \to 2 \to 3$. It is easy to verify that $R_R$ is a cotilting bimodule. Consider the projective $R$-module $P(2)$. The topology $\tau_{\Delta P(2)}$ is discrete since $P(2)$ embeds in $R_R$. Therefore each submodule of $P(2)$, in particular the simple module $S(3)$, is closed. Nevertheless $S(3)$ is not a $R$-closed submodule of $P(2)$, since $P(2)/S(3) \cong S(3)$ is not cogenerated by $R_R$.

Definition 2.6. A left $R$-module $M$ is said to be $W$-linearly compact (see [11, §3]), briefly $W$-lc, (resp. $HW$-linearly compact, briefly $HW$-lc) if it is complete in any $W$-topology (resp. in any Hausdorff $W$-topology).
Analogously to the usual linear compactness, a left $R$-module $M$ is $W$-linear compact if and only if any finitely solvable system of congruences $x \equiv x_\lambda \mod M_\lambda$, where $\{M_\lambda : \lambda \in \Lambda\}$ is a downwards directed collection of $W$-closed submodules of $M$, is solvable. Similarly a module $M$ is $HW$-linearly compact if and only if it satisfies the previous condition restricted to downwards directed collections of $W$-closed submodules of $M$ with intersection equal to zero.

**Proposition 2.7.** Let $\mathrm{RM}$ be a torsionless left $R$-module. If $M$ is $HW$-linearly compact, then $M$ is complete in the topology $\tau$.

**Proof.** Since the intersection of the kernels of a finite number of elements of $\Delta M$ is a $W$-closed submodule of $M$, $\tau$ is a $W$-topology. Since $M$ is torsionless, by Proposition 2.1 the topology $\tau$ is Hausdorff. Since $M$ is $HW$-lc, $\tau$ is complete. \qed

**Corollary 2.8.** Let $\mathrm{RM}$ be a torsionless left $R$-module. If $M$ is $HW$-linearly compact and $W$-dense, then $M$ is reflexive.

**Proof.** It follows by Propositions 2.3 and 2.7. \qed

We can obtain a more precise result for cotilting bimodules.

**Theorem 2.9.** Let $\mathrm{RWS}$ be a cotilting bimodule. For a torsionless left $R$-module $M$ the following statements are equivalent:

(a) $M$ is $HW$-linearly compact and $W$-dense.

(b) $M$ is reflexive and $\tau$ is the unique Hausdorff topology among those induced by subsets of $\Delta M$.

**Proof.** (a $\Rightarrow$ b): By Corollary 2.8 we only have to prove that $\tau$ is the unique Hausdorff topology induced by subsets of $\Delta M$. Since $\tau$ is the maximum element in $\{\tau_A : A \subseteq \Delta M\}$, it is sufficient to prove that if $\tau_A$ is Hausdorff, then $\tau$ is coarser than $\tau_A$ and hence $\tau_A = \tau$.

Let $F$ be a finite subset of $A$. We denote by $f_F : M \to WF$ the diagonal morphism. Let $M_F := \bigcap_{f \in F} \ker f = \ker f_F$ and $N_F := M/M_F$. By [4, Proposition 5] both the left $R$-modules $M_F$ and $N_F$ are reflexive. We call $\pi_F$ the induced map $M \to N_F$. Since $\{M_F : F \subseteq A, F \text{ finite}\}$ is a base for the filter of $\tau_A$-neighbourhoods of zero consisting of open submodules, $\lim N_F$ is the Hausdorff completion of $M$ endowed with the topology $\tau_A$.

But, since $\tau_A$ is an Hausdorff $W$-topology and, by hypothesis, $M$ is $HW$-lc, $\tau_A$ is complete. Thus $M \cong \lim N_F$.

Applying the functors $\Delta$ and $\lim$ to the exact sequences

$$0 \to M_F \to M \xrightarrow{\pi_F} N_F \to 0$$

we get the exact sequence of right $S$-modules

$$
\begin{align*}
0 \to \lim \Delta N_F & \xrightarrow{\lim \Delta(\pi_F)} \Delta M \xrightarrow{\lim \Delta M_F} 0
\end{align*}
$$

(*)
Now $\Delta(\lim \Delta(\pi_F)) \cong \lim \Delta^2(\pi_F) \cong \lim \pi_F$ is an isomorphism. Then, from the exact sequence

$$0 \to \Delta \lim \Delta M \to \Delta^2 M \to \Delta \lim \Delta N \to \Gamma \lim \Delta M \to 0,$$

we get $\lim \Delta M$ belongs to $\text{Ker } \Delta \cap \text{Ker } \Gamma = 0$. Hence $\Delta \cong \lim \Delta N$.

Let now $g$ be in $\Delta M$. Since $g$ belongs to $\Delta(\pi_F)(\Delta(N_F))$ for some finite subset $F$ of $A$, there exists a morphism $h : N_F \to W$ such that $g = h \circ \pi_F$.

Then since $\text{Ker } g \supseteq \text{Ker } \pi_F = \bigcap_{f \in F} \text{Ker } f$, $\text{Ker } g$ is $\tau_A$-open. Therefore $\tau$ is coarser than $\tau_A$.

$(b \Rightarrow a)$: We only have to prove that $M$ is $HW$-lc. Let $\sigma$ be a Hausdorff $W$-topology on $M$. By definition $\sigma$ has a basis $B$ for the filter of neighbourhoods of zero consisting of $W$-closed submodules; since $\sigma$ is Hausdorff, the intersection of elements in $B$ is equal to zero. Observe that any element $V$ of $B$ is the intersection of the kernels of a (not necessarily finite) subset $A_V$ of $\Delta M$. Let $A$ the union $\cup_{V \in B} A_V$. If $f$ belongs to $A_V$, $\text{Ker } f$ contains $V$ and hence it is $\sigma$-open. Therefore the topology $\tau_A$ is coarser than $\sigma$. Since $\bigcap_{f \in A} \text{Ker } f = \bigcap_{V \in B} V = \{0\}$,

$0$ is a closed subset of $M_{\tau_A}$, i.e., $\tau_A$ is Hausdorff. By hypothesis $\tau_A = \tau$ and, since $M$ is reflexive, by Proposition 2.3 $\tau_A$ is complete. By [3, Proposition III.3.10], also the topology $\sigma$ is complete. □

**Lemma 2.10.** Let $R W_S$ be a cotilting bimodule with $\text{Cogen } W_S$ closed under direct limits. Let $M$ be a reflexive left $R$-module. Then $\tau$ is the unique Hausdorff topology among those induced by subsets of $\Delta M$.

**Proof.** We can follow the first two paragraphs of the proof of Theorem 2.9, $(a \Rightarrow b)$. Applying the functors $\Delta$ and $\lim$ to the exact sequence

$$0 \to M_F \to M \xrightarrow{\pi_F} N_F \to 0$$

we get the exact sequence

$$0 \to \lim \Delta N_F \xrightarrow{\Delta(\pi_F)} \Delta M \xrightarrow{\lim \Delta M} \lim \Delta N_F \to 0$$

of right $S$-modules. Observe that $\lim \Delta M_F$ again belongs to $\text{Ker } \Gamma = \text{Cogen } W$ since, by hypothesis, $\text{Cogen } W$ is closed under direct limits. Moreover

$$\Delta \lim \Delta M_F \cong \lim \Delta^2 M_F \cong \lim \Delta M_F = \bigcap_{f \in A} \text{Ker } f = 0.$$  

Therefore, since $\text{Ker } \Delta \cap \text{Ker } \Gamma = 0$, we get $\lim \Delta M_F = 0$. Hence $\Delta M \cong \lim \Delta N_F$. We can thus conclude following the last paragraph of the proof of Theorem 2.9, $(a \Rightarrow b)$. □

Thus we obtain the characterization of reflexive modules for pure-injective cotilting bimodules.
**Theorem 2.11.** Let $R_W S$ be a pure-injective cotilting bimodule. For a torsionless module $M$ the following are equivalent:

(a) $M$ is reflexive.
(b) $M$ is $HW$-linearly compact and $W$-dense.
(c) $M$ is $W$-linearly compact and $W$-dense.

**Proof.** (c $\Rightarrow$ b $\Rightarrow$ a): They follow by Definition 2.6 and by Theorem 2.9.

(a $\Rightarrow$ c): Trivially $M$ is $W$-dense. Let $\sigma$ be a $W$-topology on $M$ and let $H \leq M$ be the $\tau$-closure of zero. Since $H$ is a $W$-closed submodule of $M$, $M/H$ is reflexive (see [4, Proposition 5]). By Theorem 2.9 and Lemma 2.10 $M/H$ is $HW$-lc and hence complete endowed with the quotient topology of $\tau$. Therefore $M_{\tau}$ is complete. □

In [13] Müller proved that if $R_W S$ is a Morita bimodule, a module $M$ is reflexive if and only if it is linearly compact in the discrete topology if and only if it is complete in any Hausdorff linear topology. In such a case $W$ cogenerates the whole category of modules, hence any submodule is $W$-closed. Therefore the notions of $W$-linear compactness, of $HW$-linear compactness and of linear compactness in the discrete topology coincide. In our setting a density condition comes out. Let us better investigate its role.

**Proposition 2.12.** Let $R_W S$ be a bimodule such that $\text{Cogen}_W S \subseteq \text{Ker} \Gamma$. A left $R$-module $M$ is $W$-dense if and only if $\text{Im}(f)$ is a reflexive left $R$-module for every $f \in \text{Hom}_R(M, W^n)$, $(n \in \mathbb{N})$.

**Proof.** Let us consider for each $f$ in $\text{Hom}(M, W^n)$, $n \in \mathbb{N}$, the following commutative diagram of linearly topologized modules and continuous morphisms:

$$
\begin{array}{ccc}
M_{\tau} & \xrightarrow{f} & W^n \\
\downarrow \delta_M & & \downarrow \delta_{\text{Im} f} \\
\text{Im} \epsilon_f \tau_{\Delta \text{Im} f} & \xrightarrow{\delta_{\text{Im} f}} & \Delta^2 \text{Im} f_{\omega'}
\end{array}
$$

where $\varphi'$ is the finite topology on $\Delta^2 \text{Im} f$. Since $\text{Im} f \leq W^n$, $\tau_{\Delta \text{Im} f}$ is the discrete topology; in particular it is complete.

Suppose that $M$ is $W$-dense. Applying $\Delta$ to the exact sequence $0 \rightarrow \text{Ker} f \rightarrow M \xrightarrow{\epsilon_f} \text{Im} f \rightarrow 0$, we get the exact sequence $0 \rightarrow \Delta \text{Im} f \xrightarrow{\Delta(\epsilon_f)} \Delta M \rightarrow C \rightarrow 0$ with $C$ in $\text{Cogen} W S$. Since $\text{Cogen} W S \subseteq \text{Ker} \Gamma$, $\Delta^2(\epsilon_f)$ is an epimorphism. Therefore, the $W$-density of $M$ implies the $W$-density of $\text{Im} f$. By Proposition 2.3, $\text{Im} f$ is reflexive.
Conversely, let \( f_1, \ldots, f_n \) be in \( \Delta M \). We denote by \( f : M \to W^n \) their diagonal morphism, \( m \mapsto (f_1(m), \ldots, f_n(m)) \), and by \( \mu_f \circ \varepsilon_f \) the usual factorization of \( f \) through \( \text{Im} f \). Since \( \text{Im} f \) is reflexive, for each \( \alpha \) in \( \Delta^2 M \) there exists \( m_\alpha \) in \( M \) such that

\[
\Delta^2(\varepsilon_f)(\alpha) = \delta_{\text{Im} f}(f(m_\alpha)).
\]

In particular, denoted by \( p_i : \text{Im} f \to W \) the \( i \)-th projection, we have

\[
(\Delta^2(\varepsilon_f)(\alpha))(p_i) = \alpha(\Delta(\varepsilon_f))(p_i) = \alpha(p_i \circ \varepsilon_f) = \alpha(f_i)
\]

\[
(\delta_{\text{Im} f}(f(m_\alpha))(p_i) = p_i(f(m_\alpha)) = f_i(m_\alpha);
\]

therefore \( \alpha \) and \( \delta_M(m_\alpha) \) coincide on \( f_1, \ldots, f_n \). Therefore \( M \) is \( W \)-dense.

\[\square\]

If \( R W S \) is a Morita bimodule, then the class of reflexive modules contains \( W \) and it is closed under submodules and finite direct sums. Therefore the \( W \)-density condition is always satisfied: Any module \( M \) is \( W \)-dense. This is not the case for cotilting bimodules.

**Example 2.13.** Let \( k \) be an algebraically closed field. Denote by \( A \) the generalized Kronecker algebra of dimension \( \aleph_0 \) over \( k \) given by the quiver

\[
\begin{array}{c}
1 \\
\vdots \\
2
\end{array}
\]

with a countable set of arrows from 1 to 2, i.e., the ring of lower triangular matrices

\[
\begin{pmatrix}
  k & 0 \\
  V & k
\end{pmatrix} = \left\{ \begin{pmatrix}
  a & 0 \\
  v & b
\end{pmatrix} : a, b \in k, v \in V \right\}
\]

where \( V \) is a \( k \)-vector space of dimension \( \aleph_0 \) (see [7, 8]). Then, by [8, Lemma 2.2], \( A \) is a hereditary, coherent and perfect ring. It is easily verified that \( AA \) is a cotilting bimodule and \( \text{Cogen}(A) \) consists of projective modules, while \( \text{Ker}(\Delta) \) contains exactly the modules without projective direct summands. The reflexive modules coincide with the finitely generated projective modules [8, Lemma 2.3]. Denote by \( e_1, e_2 \) the primitive idempotents, i.e.,

\[
e_1 = \begin{pmatrix}
  1 & 0 \\
  0 & 0
\end{pmatrix}, \quad e_2 = \begin{pmatrix}
  0 & 0 \\
  0 & 1
\end{pmatrix}.
\]

Let \( P_i = Ae_i \) and \( Q_i = e_i A, i = 1, 2 \). The socle \( S \) of \( P_1 \) is isomorphic to \( P_2^{(\aleph_0)} \); therefore it is a not reflexive submodule of \( A \). By Proposition 2.12, \( S \) is not \( A \)-dense.

In [6, Definition 1.2] Colpi and Fuller introduced the \( W \)-torsionless linear compactness, a generalization of the notion of linear compactness with respect to the torsion theories associated to a cotilting bimodule \( R W S \). They prove that if a module is \( W \)-torsionless linearly compact, then it is reflexive, i.e., (see Proposition 2.3) \( \tau \) is a complete topology and \( M \) is \( W \)-dense.
Our notion of \( W \)-linear compactness is strong enough to assure the completeness, but to obtain the \( \Delta \)-reflexivity we need to assume explicitly the \( W \)-density. Assuming \( RW \) and \( WS \) pure-injective (as in all examples known in the literature), these two notions together completely characterize the classes of reflexive modules. The notion of \( W \)-torsionless linear compactness is too strong to characterize the classes of reflexive modules in the general case; this happens if and only if the classes of reflexive left \( R \)- and right \( S \)-modules are closed under submodules [6, Corollary 1.9]. Observe that in this case, by Proposition 2.12, any module is \( W \)-dense. Adding the hypotheses of both the contexts we get:

**Corollary 2.14.** Let \( RW_S \) be a pure-injective cotilting bimodule. Assume the classes of reflexive left \( R \)- and right \( S \)-modules being closed under submodules. For a module \( M \), the following statements are equivalent:

(a) \( M \) is reflexive;
(b) \( M \) is \( W \)-torsionless linearly compact;
(c) \( M \) is \( W \)-linearly compact.

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