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**Let $R$ and $S$ be arbitrary associative rings. A left $R$-module ${}_RW$ is said to be cotilting if the class of modules cogenerated by ${}_RW$ coincides with the class of modules for which the functor $\mathrm{Ext}^1_R(-,W)$ vanishes. In this paper we characterize the cotilting modules which are pure-injective. The two notions seem to be strictly connected: Indeed all the examples of cotilting modules known in the literature are pure-injective. We observe that if ${}_RW_S$ is a *pure-injective cotilting bimodule*, both $R$ and $S$ are semiregular rings and we give a characterization of the reflexive modules in terms of a suitable "linear compactness" notion.**

## Introduction.

Cotilting modules first appeared as vector space duals of tilting modules over finite dimensional algebras [**12**, IV, 7.8]. Recently they have been introduced [**5**] in the framework of modules over arbitrary associative rings, acquiring a proper independent role. The cotilting modules generalize the notion of injective cogenerator: They are injectives with respect to short exact sequences of modules cogenerated by them.

For arbitrary rings $R$ and $S$, a Morita duality between left $R$-modules and right $S$-modules is given by the contravariant Hom functors associated to a *Morita bimodule*, i.e., a faithfully balanced bimodule ${}_RW_S$ with ${}_RW$ and $W_S$ both injective cogenerators. One of the major component in the theory of Morita dualities is Müller's theorem [**13**] which states that the reflexive modules are precisely the linearly compact modules. If ${}_RW_S$ is a Morita bimodule, both $R$ and $S$ are semiperfect rings [**16**, Theorem 2.7].

For arbitrary rings $R$ and $S$, a cotilting duality between left $R$-modules and right $S$-modules is given by the contravariant Hom functors and the contravariant Ext functors associated to a cotilting bimodule, i.e., a faithfully balanced bimodule ${}_RW_S$ with both ${}_RW$ and $W_S$ cotilting modules (see [**4**]).

All known examples of cotilting modules are *pure-injective*. In this paper we characterize the pure-injective cotilting modules. We observe that if ${}_RW_S$ is a *pure-injective cotilting bimodule*, both $R$ and $S$ are semiregular rings and we give a characterization of the reflexive modules in terms of a suitable "linear compactness" notion.

## 1. About the pure-injectivity of a cotilting module.

Let $R$ be an associative ring with $1 \neq 0$. We denote by $R$-Mod the category of left unitary$R$-modules and their homomorphisms. Given a left $R$-module $W$, we consider the following classes:

- $\operatorname{Cogen} W$ denotes the class of all left $R$-modules *cogenerated* by ${}_RW$, that is all $M$ in $R$-Mod such that there exist a cardinal $\lambda$ and a monomorphism $M \hookrightarrow W^\lambda$;
- ${}^\perp W$ denotes the class of all left $R$-modules $M$ such that $\operatorname{Ext}^1_R(M, W) = 0$.

A left $R$-module ${}_RW$ is said to be *cotilting* [**5**] if $\operatorname{Cogen} {}_RW = {}^\perp W$. The cotilting modules generalize injective cogenerators: Clearly ${}_RW$ is an injective cogenerator if and only if both the classes $\operatorname{Cogen} {}_RW$ and ${}^\perp W$ coincide with the whole category of left $R$-modules. A short exact sequence

$$0 \to K \to L \to M \to 0$$

is said to be *pure* if any morphism $P \to M$, with $P$ finitely presented, lifts to a morphism $P \to L$.

**Definition 1.1.** A module ${}_RW$ is *pure-injective* if it is injective with respect to any pure exact sequence.

All known examples of cotilting modules are pure-injective. It naturally arises the question how the two notions are related.

**Proposition 1.2.** *Let* ${}_RW$ *be a cotilting module. If the class* $\operatorname{Cogen} W$ *is closed under direct limits, then* $W$ *is pure-injective.*

*Proof.* Let us show that for any pure exact sequence $0 \to A \to B \to C \to 0$ and for any map $f : A \to W$ there exists a map $g$ making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \overset{*}{\hookrightarrow} & B & \longrightarrow & C & \longrightarrow & 0. \\ & & \downarrow f & \swarrow g & & & & & \\ & & W & & & & & & \end{array}$$

Replacing $A \overset{*}{\hookrightarrow} B$ by $A/\operatorname{Rej}_W A \overset{*}{\hookrightarrow} B/\operatorname{Rej}_W A$,we can assume that $\operatorname{Rej}_W A = 0$. The pure exact sequence $0 \to A \to B \to C \to 0$ is a direct limit of split exact sequences $0 \to A \to B_i \to C_i \to 0$ with $C_i$ finitely presented (cf. 34.2 [**14**]). For each index $i$ we have the commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & \operatorname{Rej}_W B_i & \xrightarrow{\cong} & \operatorname{Rej}_W C_i & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B_i & \longrightarrow & C_i & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B_i/\operatorname{Rej}_W B_i & \longrightarrow & C_i/\operatorname{Rej}_W C_i & \longrightarrow & 0.
\end{array}$$

Applying the direct limit functor we get

$$\begin{array}{ccccccccc}
 & & & & \varinjlim \operatorname{Rej}_W B_i & \xrightarrow{\cong} & \varinjlim \operatorname{Rej}_W C_i & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & \varinjlim(B_i/\operatorname{Rej}_W B_i) & \longrightarrow & \varinjlim(C_i/\operatorname{Rej}_W C_i) & \longrightarrow & 0.
\end{array}$$

Since $\varinjlim \operatorname{Rej}_W B_i$ is in the kernel of $\operatorname{Hom}(-,W)$ and $\varinjlim(B_i/\operatorname{Rej}_W B_i)$ belongs to $\operatorname{Cogen} W$ by the assumption, we infer that $\varinjlim(B_i/\operatorname{Rej}_W B_i) \cong B/\operatorname{Rej}_W B$. Also $\varinjlim(C_i/\operatorname{Rej}_W C_i)$ belongs to $\operatorname{Cogen} W = {}^{\perp}W$. So $f$ can be extended to a morphism $g' : B/\operatorname{Rej}_W B \to W$; the composition of $g'$ with the canonical projection $B \to B/\operatorname{Rej}_W B$ yields the desired map $g$. □

In the cotilting case, since $\operatorname{Cogen} W = {}^{\perp}W$, the hypothesized closure under direct limits of the class of modules cogenerated by $W$ is suggested by the following proposition:

**Proposition 1.3.** *If ${}_RW$ is a pure-injective module, then ${}^{\perp}_{R}W$ is closed under direct limits.*

*Proof.* Consider a direct system $\{M_i : i \in I\}$ in ${}^{\perp}W$. The canonical exact sequence

$$0 \to K \to \oplus_{i\in I} M_i \to \varinjlim_{i\in I} M_i \to 0$$

is pure (cf. [**14**, 33.9, (2)]). Applying $\operatorname{Hom}_R(-,W)$ we get the long exact sequence

$$\cdots \to \operatorname{Hom}(\oplus_{i\in I} M_i, W) \xrightarrow{f} \operatorname{Hom}(K,W) \to \operatorname{Ext}^1_R(\varinjlim_{i\in I} M_i, W) \to$$
$$\to \operatorname{Ext}^1_R(\oplus_{i\in I} M_i, W) = 0.$$

Since $W$ is pure-injective, $f$ is surjective; so $\varinjlim_{i\in I} M_i$ belongs to ${}^{\perp}W$. □

This result has been used in [**9**, Lemma 9] to prove that, if $\mathcal{C}$ is a class of pure-injective modules, every module $M$ which has a ${}^{\perp}\mathcal{C}$-precover has a

${}^{\perp}\mathcal{C}$-cover (see [**15**] for an extensive introduction to theory of (pre)covers and (pre)envelopes of modules, including various recent results).

**Corollary 1.4.** *If ${}_RW$ is a cotilting module, then $W$ is pure-injective if and only if* $\operatorname{Cogen} W$ *is closed under direct limits. In such a case any module has a* $\operatorname{Cogen} W$*-cover.*

*Proof.* The first claim follows by Propositions 1.2 and 1.3. The second one follows by [**1**, Corollary 2.6]. □

**Open Problem 1.5.** Are all cotilting (bi)modules pure-injective?[1]

It is well-known that the endomorphism rings of a Morita bimodule are semiperfect; indeed a Morita bimodule is injective and finitely cogenerated on both sides (see [**16**, Theorem 2.7, Proposition 1.19]). We are able to give an analogous result for cotilting bimodules, assuming the closure under direct limits of the classes cogenerated by them. We recall that a ring $R$ is said to be *semiregular* if $R/J(R)$ is regular and the idempotents lift over the Jacobson radical $J(R)$.

**Proposition 1.6.** *Let ${}_RW_S$ be a pure-injective cotilting bimodule, i.e., faithfully balanced and cotilting and pure-injective on both sides. Then both $R$ and $S$ are semiregular rings.*

*Proof.* The notions of pure-injective and algebraically compact module coincide (cf. [**14**, 34.4]). Then, by [**17**, Theorem 9], $R$ and $S$ are both semiregular. □

**Remark 1.7.** Observe that if the ring $R$ is regular, the classes of pure-injective and of injective $R$-modules coincide [**14**, 37.6]. Therefore pure-injective cotilting bimodules which are not Morita bimodules "live in the space between semiregular and regular rings".

## 2. Characterizing the reflexive modules.

Let ${}_RW_S$ be an arbitrary bimodule. In the sequel we denote by $\Delta$ the functors $\operatorname{Hom}_?(-, W)$, and by $\Gamma$ the functors $\operatorname{Ext}^1_?(-, W)$, where ? stands for $R$ or $S$. We denote by $\Delta^2$ both the compositions $\operatorname{Hom}_R(\operatorname{Hom}_S(-, W), W)$ and $\operatorname{Hom}_S(\operatorname{Hom}_R(-, W), W)$. Given a (left $R$, or right $S$)-module $M$, we denote by $\delta_M$ the canonical homomorphism $M \to \Delta^2(M)$ defined by $m \mapsto [f \mapsto f(m)]$. A module $M$ is called *reflexive* (resp. *torsionless*) if $\delta_M$ is an isomorphism (resp. monomorphism).

Clearly a torsionless module $M$ is reflexive if and only if the evaluation map $\delta_M$ is surjective. Endowed $M$ and $\Delta^2 M$ with any topology, the surjectivity of $\delta_M$ can be tested in a topological way asking for $\operatorname{Im}\delta_M$ to be both dense and closed in $\Delta^2 M$. As the approach of Müller [**13**] to the classical

[1] Recently Silvana Bazzoni proved that any cotilting module is pure-injective [**2**].

case of Morita dualities suggests, we introduce topological tools in order to characterize the reflexive modules.

Let us endow $\Delta^2 M$ with the *finite topology* $\varphi$: The linear topology for which the family of submodules $V(F) = \{\alpha \in \Delta^2 M : \alpha(F) = 0\}$, where $F$ is a finite subset of $\Delta M$, is a base for the filter of neighbourhoods of zero.

Let us endow $W$ with the discrete topology. Given any torsionless module $M$, we associate with each subset $A$ of $\Delta M$ the *weak topology* with respect to morphisms in $A$, denoted by $\tau_A$. By definition $\tau_A$ is the coarsest topology on $M$ such that all morphisms in $A$ are continuous: It is a linear topology with a base for its filter of neighbourhoods of zero formed by finite intersections of kernels of morphisms in $A$. In the sequel the topology $\tau_{\Delta M}$ will be shortly denoted by $\tau$. Note that $\tau$ is the maximum element of the set of linear topologies $\{\tau_A : A \subseteq \Delta M\}$ partially ordered by inclusion. Let $L_\sigma$ be a linearly topologized module. Denote by $H$ the $\sigma$-closure of zero in $L$. Note that $H$ is equal to the intersection of all neighbourhoods of zero and, since $\sigma$ is a linear topology, $\sigma$ is Hausdorff if and only if $H = 0$. A $\sigma$-*Cauchy net* in $L$ is a family $X_\lambda$, $\lambda \in \Lambda$, indexed by the upwards directed partially ordered set $\Lambda$, such that for every neighbourhood $U$ of zero there exists an upper subset $\Lambda'$ of $\Lambda$ with $x_\lambda - x_{\lambda'} \in U$ for every $\lambda$, $\lambda' \in \Lambda'$. The topology $\sigma$ is complete, i.e., any $\sigma$-Cauchy net in $L$ converges in $L$, if and only if the topological quotient $L/H$ is complete (see [**3**, Chap. 3, §2]). Note that a closed submodule of a complete module is complete. The completion of $L/H$ is called the Hausdorff completion of $L$: Denoted by $\mathcal{J} = \{J_\lambda : \lambda \in \Lambda\}$ a base for the filter of neighbourhoods of zero in $L_\sigma$ consisting of open submodules, it coincides with the inverse limit $\varprojlim L/J_\lambda$ (see [**10**, Proposition 13.7]).

**Proposition 2.1.** *Let $M$ be a torsionless module.*

(i) *The topologies $\tau$ on $M$ and $\varphi$ on $\Delta^2 M$ are Hausdorff.*
(ii) *$\delta_M : M_\tau \to \Delta^2 M_\varphi$ is a topological embedding.*
(iii) *The topology $\varphi$ on $\Delta^2 M$ is complete.*

*Proof.* (i) Since $M$ is cogenerated by $W$, there exists a set $X$ and the following maps:

$$M \overset{i}{\hookrightarrow} W^X \overset{\pi_x}{\to} W.$$

Clearly $\{0\}$ is the intersection of $\operatorname{Ker}(\pi_x \circ i)$, $x \in X$. Since any $\operatorname{Ker}(\pi_x \circ i)$ is $\tau$-open and hence $\tau$-closed, $\tau$ is Hausdorff. Let us consider the topology $\varphi$. The open submodules $V(F) = \{\alpha \in \Delta^2 M : \alpha(F) = 0\}$, with $F$ finite subset of $\Delta M$, have intersection zero: Hence $\varphi$ is Hausdorff.

(ii) It follows from the fact that for any finite subset $F$ of $\Delta M$ we have

$$V(F) \cap \delta_M(M) = \{\delta_M(m) : f(m) = 0 \ \forall\ f \in F\} = \delta_M(\cap_{f \in F} \operatorname{Ker} f).$$

(iii) The topological module $\Delta^2 M_\varphi$ is a closed submodule of $W^{\Delta M}$ endowed with the product of the discrete topologies. Since the product of complete topologies is also complete, we can conclude. $\square$

**Corollary 2.2.** *Each reflexive module $M$ endowed with the topology $\tau$ is complete.*

*Proof.* If $M$ is reflexive, $\delta_M$ is an isomorphism. Therefore, by Proposition 2.1, (ii), it is a topological isomorphism. We conclude by Proposition 2.1, (iii). □

**Proposition 2.3.** *For a torsionless module $M$ the following statements are equivalent:*

(i) *$M$ is reflexive.*
(ii) *$\tau$ is a complete topology and $\delta_M(M)$ is dense in $\Delta^2 M_\varphi$.*

*Proof.* (i) ⇒ (ii): It follows by Corollary 2.2.

(ii) ⇒ (i): Since $\tau$ is complete, by Proposition 2.1 $\delta_M(M)$ is a complete, and hence closed, topological submodule of $\Delta^2 M_\varphi$. Being $\delta_M(M)$ dense in $\Delta^2 M_\varphi$ and $\varphi$ an Hausdorff topology, $\delta_M$ is an isomorphism. □

As suggested by Müller [**13**] in the case of Morita dualities, we look for a suitable notion of compactness for a module $M$ in order to guarantee the completeness of the topology $\tau$ on $M$.

Note that $\delta_M(M)$ is dense in $\Delta^2 M_\varphi$ if and only if for each $\alpha$ in $\Delta^2 M$ and $f_1, \dots, f_n$ in $\Delta M$ there exists $m$ in $M$ such that $\alpha(f_i) = f_i(m)$ for each $i = 1, \dots, n$. Following [**11**], a module $M$ satisfying the above property will be called *$W$-dense*.

**Definition 2.4.** Let $M$ be a torsionless left $R$-module. A submodule $K$ of $M$ is called *$W$-closed* if $M/K$ is torsionless (see [**11**, §2]). A linear topology on $M$ is said to be a *$W$-topology* if it has a basis of neighbourhoods of zero consisting of $W$-closed submodules.

Each $W$-closed submodule $K$ of $M$ is closed in $M_\tau$. Indeed, for a suitable set $X$, there exist the following maps:

$$M \xrightarrow{\pi} M/K \overset{i}{\hookrightarrow} W^X \xrightarrow{\pi_x} W.$$

Then, $K = \bigcap_{x \in X} \operatorname{Ker}(\pi_x \circ i \circ \pi)$ is closed, since it is an intersection of open and hence closed submodules. The converse is not true in general.

**Example 2.5.** Let $R$ denote the $k$-algebra given by the quiver $1 \to 2 \to 3$. It is easy to verify that ${}_R R_R$ is a cotilting bimodule. Consider the projective $R$-module $P(2)$. The topology $\tau_{\Delta P(2)}$ is discrete since $P(2)$ embeds in ${}_R R$. Therefore each submodule of $P(2)$, in particular the simple module $S(3)$, is closed. Nevertheless $S(3)$ is not a $R$-closed submodule of $P(2)$, since $P(2)/S(3) \cong S(2)$ is not cogenerated by ${}_R R$.

**Definition 2.6.** A left $R$-module $M$ is said to be *$W$-linearly compact* (see [**11**, §3]), briefly $W$-lc, (resp. *$HW$-linearly compact*, briefly $HW$-lc) if it is complete in any $W$-topology (resp. in any Hausdorff $W$-topology).

Analogously to the usual linear compactness, a left $R$-module $M$ is $W$-linear compact if and only if any finitely solvable system of congruences $x \equiv x_\lambda \mod M_\lambda$, where $\{M_\lambda : \lambda \in \Lambda\}$ is a downwards directed collection of $W$-closed submodules of $M$, is solvable. Similarly a module $M$ is $HW$-linearly compact if and only if it satisfies the previous condition restricted to downwards directed collections of $W$-closed submodules of $M$ with intersection equal to zero.

**Proposition 2.7.** *Let ${}_RM$ be a torsionless left $R$-module. If $M$ is $HW$-linearly compact, then $M$ is complete in the topology $\tau$.*

*Proof.* Since the intersection of the kernels of a finite number of elements of $\Delta M$ is a $W$-closed submodule of $M$, $\tau$ is a $W$-topology. Since $M$ is torsionless, by Proposition 2.1 the topology $\tau$ is Hausdorff. Since $M$ is $HW$-lc, $\tau$ is complete. □

**Corollary 2.8.** *Let ${}_RM$ be a torsionless left $R$-module. If $M$ is $HW$-linearly compact and $W$-dense, then $M$ is reflexive.*

*Proof.* It follows by Propositions 2.3 and 2.7. □

We can obtain a more precise result for cotilting bimodules.

**Theorem 2.9.** *Let ${}_RW_S$ be a cotilting bimodule. For a torsionless left $R$-module $M$ the following statements are equivalent:*

(a) *$M$ is $HW$-linearly compact and $W$-dense.*
(b) *$M$ is reflexive and $\tau$ is the unique Hausdorff topology among those induced by subsets of $\Delta M$.*

*Proof.* (a ⇒ b): By Corollary 2.8 we only have to prove that $\tau$ is the unique Hausdorff topology induced by subsets of $\Delta M$. Since $\tau$ is the maximum element in $\{\tau_A : A \subseteq \Delta M\}$, it is sufficient to prove that if $\tau_A$ is Hausdorff, then $\tau$ is coarser than $\tau_A$ and hence $\tau_A = \tau$.

Let $F$ be a finite subset of $A$. We denote by $f_F : M \to W^F$ the diagonal morphism. Let $M_F := \bigcap_{f\in F} \operatorname{Ker} f = \operatorname{Ker} f_F$ and $N_F := M/M_F$. By [4, Proposition 5] both the left $R$-modules $M_F$ and $N_F$ are reflexive. We call $\pi_F$ the induced map $M \to N_F$. Since $\{M_F : F \subseteq A,\ F \text{ finite}\}$ is a base for the filter of $\tau_A$-neighbourhoods of zero consisting of open submodules, $\varprojlim N_F$ is the Hausdorff completion of $M$ endowed with the topology $\tau_A$.

But, since $\tau_A$ is an Hausdorff $W$-topology and, by hypothesis, $M$ is $HW$-lc, $\tau_A$ is complete. Thus $M \cong \varprojlim N_F$.

Applying the functors $\Delta$ and $\varinjlim$ to the exact sequences

$$0 \to M_F \to M \overset{\pi_F}{\to} N_F \to 0$$

we get the exact sequence of right $S$-modules

$$(*) \qquad 0 \longrightarrow \varinjlim \Delta N_F \xrightarrow{\varinjlim \Delta(\pi_F)} \Delta M \longrightarrow \varinjlim \Delta M_F \longrightarrow 0$$

Now $\Delta(\varinjlim \Delta(\pi_F)) \cong \varprojlim \Delta^2(\pi_F) \cong \varprojlim \pi_F$ is an isomorphism. Then, from the exact sequence

$$0 \to \Delta \varinjlim \Delta M_F \longrightarrow \Delta^2 M \xrightarrow{\Delta(\varinjlim \Delta(\pi_F))} \Delta \varinjlim \Delta N_F \longrightarrow \Gamma \varinjlim \Delta M_F \to 0,$$

we get $\varinjlim \Delta M_F$ belongs to $\operatorname{Ker} \Delta \cap \operatorname{Ker} \Gamma = 0$. Hence $\Delta M \cong \varinjlim \Delta N_F$.

Let now $g$ be in $\Delta M$. Since $g$ belongs to $\Delta(\pi_F)(\Delta(N_F))$ for some finite subset $F$ of $A$, there exists a morphism $h : N_F \to W$ such that $g = h \circ \pi_F$. Then since $\operatorname{Ker} g \supseteq \operatorname{Ker} \pi_F = \bigcap_{f \in F} \operatorname{Ker} f$, $\operatorname{Ker} g$ is $\tau_A$-open. Therefore $\tau$ is coarser than $\tau_A$.

(b $\Rightarrow$ a): We only have to prove that $M$ is $HW$-lc. Let $\sigma$ be a Hausdorff $W$-topology on $M$. By definition $\sigma$ has a basis $\mathcal{B}$ for the filter of neighbourhoods of zero consisting of $W$-closed submodules; since $\sigma$ is Hausdorff, the intersection of elements in $\mathcal{B}$ is equal to zero. Observe that any element $V$ of $\mathcal{B}$ is the intersection of the kernels of a (not necessarily finite) subset $A_V$ of $\Delta M$. Let $A$ the union $\cup_{V \in \mathcal{B}} A_V$. If $f$ belongs to $A_V$, $\operatorname{Ker} f$ contains $V$ and hence it is $\sigma$-open. Therefore the topology $\tau_A$ is coarser than $\sigma$. Since

$$\cap_{f \in A} \operatorname{Ker} f = \cap_{V \in \mathcal{B}} V = \{0\},$$

0 is a closed subset of $M_{\tau_A}$, i.e., $\tau_A$ is Hausdorff. By hypothesis $\tau_A = \tau$ and, since $M$ is reflexive, by Proposition 2.3 $\tau_A$ is complete. By [**3**, Proposition III.3.10], also the topology $\sigma$ is complete. ☐

**Lemma 2.10.** *Let ${}_R W_S$ be a cotilting bimodule with* $\operatorname{Cogen} W_S$ *closed under direct limits. Let $M$ be a reflexive left $R$-module. Then $\tau$ is the unique Hausdorff topology among those induced by subsets of $\Delta M$.*

*Proof.* We can follow the first two paragraphs of the proof of Theorem 2.9, (a $\Rightarrow$ b). Applying the functors $\Delta$ and $\varinjlim$ to the exact sequence

$$0 \to M_F \to M \stackrel{\pi_F}{\to} N_F \to 0$$

we get the exact sequence

$$0 \to \varinjlim \Delta N_F \xrightarrow{\varinjlim \Delta(\pi_F)} \Delta M \longrightarrow \varinjlim \Delta M_F \to 0$$

of right $S$-modules. Observe that $\varinjlim \Delta\, M_F$ again belongs to $\operatorname{Ker} \Gamma = \operatorname{Cogen} W$ since, by hypothesis, $\operatorname{Cogen} W$ is closed under direct limits. Moreover

$$\Delta \varinjlim \Delta M_F \cong \varprojlim \Delta^2 M_F \cong \varprojlim M_F = \cap_{f \in A} \operatorname{Ker} f = 0.$$

Therefore, since $\operatorname{Ker} \Delta \cap \operatorname{Ker} \Gamma = 0$, we get $\varinjlim \Delta M_F = 0$. Hence $\Delta M \cong \varinjlim \Delta N_F$. We can thus conclude following the last paragraph of the proof of Theorem 2.9, (a $\Rightarrow$ b). ☐

Thus we obtain the characterization of reflexive modules for pure-injective cotilting bimodules.

**Theorem 2.11.** *Let ${}_RW_S$ be a pure-injective cotilting bimodule. For a torsionless module $M$ the following are equivalent:*

(a) *$M$ is reflexive.*
(b) *$M$ is $HW$-linearly compact and $W$-dense.*
(c) *$M$ is $W$-linearly compact and $W$-dense.*

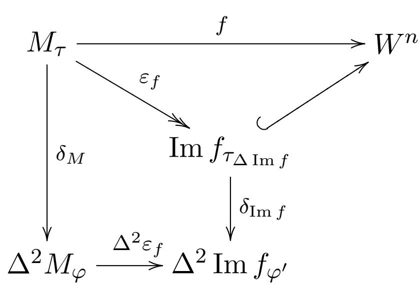
*Proof.* (c $\Rightarrow$ b $\Rightarrow$ a): They follow by Definition 2.6 and by Theorem 2.9.

(a $\Rightarrow$ c): Trivially $M$ is $W$-dense. Let $\sigma$ be a $W$-topology on $M$ and let $H \leq M$ be the $\tau$-closure of zero. Since $H$ is a $W$-closed submodule of $M$, $M/H$ is reflexive (see [**4**, Proposition 5]). By Theorem 2.9 and Lemma 2.10 $M/H$ is $HW$-lc and hence complete endowed with the quotient topology of $\tau$. Therefore $M_\tau$ is complete. □

In [**13**] Müller proved that if ${}_RW_S$ is a Morita bimodule, a module $M$ is reflexive if and only if it is linearly compact in the discrete topology if and only if it is complete in any Hausdorff linear topology. In such a case $W$ cogenerates the whole category of modules, hence any submodule is $W$-closed. Therefore the notions of $W$-linear compactness, of $HW$-linear compactness and of linear compactness in the discrete topology coincide. In our setting a density condition comes out. Let us better investigate its role.

**Proposition 2.12.** *Let ${}_RW_S$ be a bimodule such that $\operatorname{Cogen} W_S \subseteq \operatorname{Ker}\Gamma$. A left $R$-module $M$ is $W$-dense if and only if $\operatorname{Im}(f)$ is a reflexive left $R$-module for every $f \in \operatorname{Hom}_R(M, W^n)$, ($n \in \mathbb{N}$).*

*Proof.* Let us consider for each $f$ in $\operatorname{Hom}(M, W^n)$, $n \in \mathbb{N}$, the following commutative diagram of linearly topologized modules and continuous morphisms:




where $\varphi'$ is the finite topology on $\Delta^2 \operatorname{Im} f$. Since $\operatorname{Im} f \leq W^n$, $\tau_{\Delta \operatorname{Im} f}$ is the discrete topology; in particular it is complete.

Suppose that $M$ is $W$-dense. Applying $\Delta$ to the exact sequence $0 \to \operatorname{Ker} f \to M \xrightarrow{\varepsilon_f} \operatorname{Im} f \to 0$, we get the exact sequence $0 \to \Delta \operatorname{Im} f \xrightarrow{\Delta(\varepsilon_f)} \Delta M \to C \to 0$ with $C$ in $\operatorname{Cogen} W_S$. Since $\operatorname{Cogen} W_S \subseteq \operatorname{Ker}\Gamma$, $\Delta^2(\varepsilon_f)$ is an epimorphism. Therefore, the $W$-density of $M$ implies the $W$-density of $\operatorname{Im} f$. By Proposition 2.3, $\operatorname{Im} f$ is reflexive.

Conversely, let $f_1, \ldots, f_n$ be in $\Delta M$. We denote by $f : M \to W^n$ their diagonal morphism, $m \mapsto (f_1(m), \ldots, f_n(m))$, and by $\mu_f \circ \varepsilon_f$ the usual factorization of $f$ through $\operatorname{Im} f$. Since $\operatorname{Im} f$ is reflexive, for each $\alpha$ in $\Delta^2 M$ there exists $m_\alpha$ in $M$ such that

$$\Delta^2(\varepsilon_f)(\alpha) = \delta_{\operatorname{Im} f}(f(m_\alpha)).$$

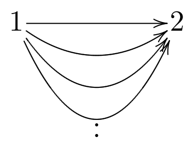
In particular, denoted by $p_i : \operatorname{Im} f \to W$ the $i$-th projection, we have

$$(\Delta^2(\varepsilon_f)(\alpha))(p_i) = \alpha(\Delta(\varepsilon_f))(p_i) = \alpha(p_i \circ \varepsilon_f) = \alpha(f_i)$$
$$(\delta_{\operatorname{Im} f}(f(m_\alpha))(p_i) = p_i(f(m_\alpha)) = f_i(m_\alpha);$$

therefore $\alpha$ and $\delta_M(m_\alpha)$ coincide on $f_1, \ldots, f_n$. Therefore $M$ is $W$-dense. □

If ${}_RW_S$ is a Morita bimodule, then the class of reflexive modules contains $W$ and it is closed under submodules and finite direct sums. Therefore the $W$-density condition is always satisfied: Any module $M$ is $W$-dense. This is not the case for cotilting bimodules.

**Example 2.13.** Let $k$ be an algebraically closed field. Denote by $A$ the *generalized Kronecker algebra* of dimension $\aleph_0$ over $k$ given by the quiver




with a countable set of arrows from 1 to 2, i.e., the ring of lower triangular matrices

$$\begin{pmatrix} k & 0 \\ V & k \end{pmatrix} = \left\{ \begin{pmatrix} a & 0 \\ v & b \end{pmatrix} : a, b \in k, v \in V \right\}$$

where $V$ is a $k$-vector space of dimension $\aleph_0$ (see [**7**, **8**]). Then, by [**8**, Lemma 2.2], $A$ is a hereditary, coherent and perfect ring. It is easily verified that ${}_AA_A$ is a cotilting bimodule and $\operatorname{Cogen}(A)$ consists of projective modules, while $\operatorname{Ker}(\Delta)$ contains exactly the modules without projective direct summands. The reflexive modules coincide with the finitely generated projective modules [**8**, Lemma 2.3]. Denote by $e_1$, $e_2$ the primitive idempotents, i.e., $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Let $P_i = Ae_i$ and $Q_i = e_iA$, $i = 1, 2$. The socle $S$ of $P_1$ is isomorphic to $P_2^{(\aleph_0)}$; therefore it is a not reflexive submodule of $A$. By Proposition 2.12, $S$ is not $A$-dense.

In [**6**, Definition 1.2] Colpi and Fuller introduced the *$W$-torsionless linear compactness*, a generalization of the notion of linear compactness with respect to the torsion theories associated to a cotilting bimodule ${}_RW_S$. They prove that if a module is $W$-torsionless linearly compact, then it is reflexive, i.e., (see Proposition 2.3) $\tau$ is a complete topology and $M$ is $W$-dense.

Our notion of $W$-linear compactness is strong enough to assure the completeness, but to obtain the $\Delta$-reflexivity we need to assume explicitly the $W$-density. Assuming ${}_RW$ and $W_S$ pure-injective (as in all examples known in the literature), these two notions together completely characterize the classes of reflexive modules. The notion of $W$-torsionless linear compactness is too strong to characterize the classes of reflexive modules in the general case; this happens if and only if the classes of reflexive left $R$- and right $S$- modules are closed under submodules [**6**, Corollary 1.9]. Observe that in this case, by Proposition 2.12, any module is $W$-dense. Adding the hypotheses of both the contexts we get:

**Corollary 2.14.** *Let ${}_RW_S$ be a pure-injective cotilting bimodule. Assume the classes of reflexive left $R$- and right $S$- modules being closed under submodules. For a module $M$, the following statements are equivalent:*

(a) *$M$ is reflexive;*
(b) *$M$ is $W$-torsionless linearly compact;*
(c) *$M$ is $W$-linearly compact.*

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Dipartimento di Matematica Pura ed Applicata
Università di Padova
via Belzoni 7
I-35131 Padova
Italy
*E-mail address*: fmantese@math.unipd.it

Dipartimento di Matematica Pura ed Applicata
Università di Padova
via Belzoni 7
I-35131 Padova
Italy
*E-mail address*: tonolo@math.unipd.it

Katedra algebry MFF UK
Sokolovská 83
186 75 Prague 8
Czech Republic
*E-mail address*: ruzicka@karlin.mff.cuni.cz