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Several very interesting results connecting the theory of abelian ideals of Borel subalgebras, some ideas of D. Peterson relating the previous theory to the combinatorics of affine Weyl groups, and the theory of discrete series are stated in a recent paper (Kostant, 1998) by B. Kostant.

In this paper we provide proofs for most of Kostant’s results extending them to ad-nilpotent ideals and develop one direction of Kostant’s investigation, the compatible discrete series.

Introduction.

This paper arises from the attempt to understand in detail a recent paper [Ko2] by B. Kostant, which can be regarded as an extended research announcement of several very interesting results connecting (at least) three topics: The theory of abelian ideals of Borel subalgebras (which originated from a much earlier paper by Kostant, [Ko1]), some ideas of D. Peterson relating the previous theory to the combinatorics of affine Weyl groups, and, finally, the theory of discrete series.

In this paper we provide proofs for most of Kostant’s results and we develop one direction of Kostant’s investigation, the compatible discrete series. The paper naturally divides into three parts that we now describe.

Let \( \mathfrak{g} \) be a simple finite dimensional complex Lie algebra, \( G \) the corresponding connected simply connected Lie group, \( \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \) be a fixed Borel subalgebra (with \( \mathfrak{h} \) Cartan and \( \mathfrak{n} \) nilradical), \( \Delta \) the root system of \((\mathfrak{g}, \mathfrak{h})\), \( \Delta^+ \) the positive system induced by the choice of \( \mathfrak{b} \), \( W, \hat{W}, \tilde{W} \) the finite, affine and extended affine Weyl groups of \( \Delta \) respectively (for more details on notation see the list at the end of the introduction). Let \( \mathcal{I} \) denote the set of ad-nilpotent ideals of \( \mathfrak{b} \), i.e., the ideals of \( \mathfrak{b} \) consisting of ad-nilpotent elements.

The first section of the paper is devoted to build up bijections

\[ \tilde{Z} \leftrightarrow W \times P^\vee / Q^\vee \leftrightarrow \mathcal{I} \times \text{Cent} \,(G) \]

where \( \tilde{Z} \) is the set of points in \( P^\vee \) of the simplex \( \{ \sigma \in \mathfrak{h}^*_R \mid (\sigma, \alpha_i) \leq 1 \text{ for each } i \in \{1, \ldots, n\} \text{ and } (\sigma, \theta) \geq -2 \} \) and \( W \) is a suitable subset of \( \tilde{W} \) in bijection with \( \mathcal{I} \).
The previous results generalize to the ad-nilpotent case the work of Kostant and Peterson for the abelian ideals of \( h \); their results can be easily obtained replacing \( Z \) with \( \tilde{Z} = \{ \sigma \in P^\vee \mid (\sigma, \beta) \in \{0, -1, 1, -2\} \forall \beta \in \Delta^+ \} \).

Let us explain in more detail the meaning of the previous bijections. To any point \( z \in \tilde{Z} \) we can associate:

1. An ad-nilpotent ideal \( i_z \);
2. an element \( w_z \in \tilde{W} \).

We define a natural action of \( \text{Cent}(G) \) on \( \tilde{Z} \); \( i_z \) turns out to be constant on the orbits of this action. It is well-known that \( \tilde{W} \) is a \( \text{Cent}(G) \)-extension of \( \tilde{W} \). In fact we have \( \tilde{W} = \tilde{W} \rtimes \Omega \), where \( \Omega \) is the subgroup, isomorphic to \( \text{Cent}(G) \), of the elements in \( \tilde{W} \) which stabilize the fundamental alcove. Then \( z \mapsto w_z \) maps each \( \text{Cent}(G) \)-orbit of \( \tilde{Z} \) onto a left coset of \( \tilde{W}/\Omega \).

Moreover, if \( \tilde{w}_z \) is the unique element in \( (w_z\Omega) \cap \tilde{W} \), then \( i_z \mapsto \tilde{w}_z \) realizes a bijection \( \tilde{I} \to \tilde{W} \).

Restricting to \( z \in \tilde{Z}_{ab} \), if we set \( v_z^{-1} \) to be the \( W \)-component of \( w_z \in \tilde{W} = W \rtimes T(P^\vee) \), we have that \( i_z \) is the sum of the root spaces \( g_\alpha \) for all \( \alpha \in -v_z^{-1}(\Delta_z^2) \cup v_z^{-1}(\Delta_z^1) \), where by definition \( \Delta_z^i = \{ \alpha \in \Delta^+ \mid (\alpha, z) = i \} \).

The connection of these results with representation theory is the main theme of Section 2. Set \( X = \mathcal{C}_2 \cap P^\vee \) and \( \text{dom} : \tilde{Z}_{ab} \to X \) to be the map defined by \( \text{dom} (z) = v_z^{-1}(z) \). Given \( \tau \in X \), then \( \Theta_\tau = \text{Ad}(\exp(\sqrt{-1}\pi \tau)) \) is a Cartan involution for \( G \) and the corresponding Cartan decomposition \( g = \mathfrak{k}_\tau \oplus \mathfrak{p}_\tau \) has the property that \( \mathfrak{k}_\tau \) is an equal rank symmetric Lie subalgebra of \( G \). Given a Cartan involution \( \Theta \) and a \( \Theta \)-stable Borel subalgebra \( \mathfrak{b}' \), we introduce, after Kostant, a notion of compatibility of \( \mathfrak{b}' \) with \( \Theta \) and we prove that \( \mathfrak{b}' \) is compatible with \( \Theta \) if and only if the pair \( (\Theta, \mathfrak{b}') \) is conjugate under \( G \) to a pair \( (\Theta_\tau, \mathfrak{b}) \) with \( \tau \in X \).

Now, for \( \tau \in X \), set \( \tilde{Z}_\tau = \text{dom}^{-1}(\tau) \cap \tilde{Z}_{ab} \); we provide proofs for two important results of [Ko2]. First, \( \tilde{Z}_\tau \) is in canonical bijection with \( W_\tau \setminus \tilde{W} \), \( W_\tau \) being the Weyl group of \( (\mathfrak{t}_\tau, \mathfrak{h}) \). Moreover we consider

\[ \tilde{Z}_\tau^{\text{cmpt}} = \{ z \in \tilde{Z}_\tau \mid v_z^{-1}(\mathfrak{b}) \text{ is compatible with } \Theta_\tau \} \]

and we prove that it has \(|\text{Cent}(G)|\) elements by exhibiting a bijection with a suitable copy of \( \text{Cent}(G) \) inside \( \tilde{W} \).

To introduce the connection with discrete series, fix again \( \tau \in X \) and set \( G_\tau \) to be the real form of \( G \) corresponding to \( \Theta_\tau \). Set \( K \) to be a maximal compact subgroup of \( G_\tau \). If \( \mu \) is a regular integral weight denote by \( \pi_\mu \) the corresponding discrete series representation for \( G_\tau \) in Harish Chandra parametrization.

The previous results imply that, if \( \lambda \) is regular and dominant, then the map \( z \mapsto \pi_{v_z^{-1}(\lambda)} \) defines a bijection between \( \tilde{Z}_\tau \) and the discrete series
having infinitesimal character $\chi_\lambda$. What is more important, the minimal $K$-type of $\pi_{v_z^{-1}(\lambda)}$ and the $L^2$-cohomological degree on $G_\tau/T$ in which $\pi_{v_z^{-1}(\lambda)}$ appears, can be expressed in terms of combinatorial data related to $i_z$.

We can also single out the discrete series $\pi_{v_z^{-1}(\lambda)}$ that correspond to $z \in \tilde{Z}_\tau^{\text{cmpt}}$. These are called *compatible* discrete series: We note that they are exactly the small discrete series of Gross and Wallach ([GW]). We then go on to show that the $K$-spectrum of the compatible discrete series can be computed fairly explicitly. The relevant results in this direction are given in Theorem 2.8 and the discussion thereafter.

Section 3 is devoted to the proofs of the results of [Ko2] that did not fit in our previous discussion, namely, we develop Kostant’s theory of special and nilradical abelian ideals. The main outcome is that the structure theory for an abelian ideal $i_z$ can be given a symmetric space significance in terms of $\tau = \text{dom} (z)$ and of the associate decomposition $\mathfrak{g} = \mathfrak{k}_\tau \oplus \mathfrak{p}_\tau$. More precisely, if $i_z = \left( \bigoplus_{\alpha \in -v_z^{-1}(\Delta^-)} \mathfrak{g}_\alpha \right) \oplus \left( \bigoplus_{\alpha \in v_z^{-1}(\Delta^+_1)} \mathfrak{g}_\alpha \right)$, then the two summands in the r.h.s. of the previous expression correspond to $i_z \cap \mathfrak{k}_\tau$, $i_z \cap \mathfrak{p}_\tau$, respectively. Other features of this decomposition are the following: On one hand $i_z \cap \mathfrak{k}_\tau$ is again an abelian ideal of $\mathfrak{b}$, and the abelian ideals arising in this way (called *special*) can be abstractly characterized. On the other hand, $i_z \cap \mathfrak{p}_\tau$ is an abelian $\mathfrak{b} \cap \mathfrak{k}_\tau$-submodule of $\mathfrak{b} \cap \mathfrak{p}_\tau$ and the map $z \mapsto i_z \cap \mathfrak{p}_\tau$ sets up a bijection between $\tilde{Z}_\tau$ and the set of such submodules; this map extends to a bijection between $\text{dom}^{-1}(\tau) \cap \tilde{Z}$ and all the $\mathfrak{b} \cap \mathfrak{k}_\tau$-submodules of $\mathfrak{b} \cap \mathfrak{p}_\tau$. Finally, we give a criterion to decide whether a special abelian ideal is the nilradical of a parabolic subalgebra of $\mathfrak{g}$ and we determine explicitly all such ideals. All these results are obtained combining the results of Section 1 with elementary combinatorics of root systems.

**Notation.**

- $\mathfrak{g}$: finite-dimensional complex simple Lie algebra,
- $G$: connected simply connected semisimple Lie group with Lie algebra $\mathfrak{g}$,
- $\text{Cent} (G)$: center of $G$,
- $T$: maximal compact torus of $G$, with Lie algebra $\mathfrak{t}$, hence $\mathfrak{h} = \mathfrak{t} \oplus \sqrt{-1}\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$,
- $\mathfrak{h}_\mathbb{R} = \sqrt{-1}\mathfrak{t}$
- $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$: Borel subalgebra of $\mathfrak{g}$, with Cartan component $\mathfrak{h}$ and nilradical $\mathfrak{n}$,
\[\Delta \subset h^*_R\] finite (irreducible) root system of \(\mathfrak{g}\) with positive system \(\Delta^+\),

\[\mathfrak{g}_\alpha\] root space relative to \(\alpha \in \Delta\),

\[\Pi = \{\alpha_1, \ldots, \alpha_n\}\] simple roots of \(\Delta^+\),

\[Q = \sum_{i=1}^n \mathbb{Z}\alpha_i, \quad Q^\vee = \sum_{i=1}^n \mathbb{Z}\alpha_i^\vee\] root and coroot lattices,

\[P = \sum_{i=1}^n \mathbb{Z}\omega_i, \quad P^\vee = \sum_{i=1}^n \mathbb{Z}\omega_i^\vee\] weight and coweight lattices,

\[\theta = \sum_{i=1}^n m_i\alpha_i\] highest root of \(\Delta\) w.r.t. \(\Pi\),

\[J = \{i \mid 1 \leq i \leq n, \ m_i = 1\}\] nonzero vertices of the fundamental alcove,

\[o_i = \omega_i^\vee/m_i, \ 1 \leq i \leq n\] sum of fundamental weights,

\[\rho = \omega_1 + \cdots + \omega_n\] sum of fundamental coweights,

\[h\] Coxeter number of \(\Delta\),

\[\delta\] fundamental imaginary root,

\[\hat{\Delta} = \Delta + \mathbb{Z}\delta\] affine root system associated to \(\Delta\),

\[\hat{\Delta}^+ = (\Delta^+ + \mathbb{N}\delta) \cup (\Delta^- + \mathbb{N}^+\delta)\] positive system for \(\hat{\Delta}\),

\(W\) finite Weyl group,

\(T(L) \subset Aff(h^*_R)\) group of translations corresponding to the lattice \(L \subset h^*_R\),

\(\hat{W} = W \ltimes T(Q^\vee)\) affine Weyl group,

\(\hat{W}_i = W \ltimes T(iQ^\vee)\) \(i \in \mathbb{N}, i \geq 1, \ (\hat{W}_1 \equiv \hat{W})\),

\(\hat{W} = W \ltimes T(P^\vee)\) extended affine Weyl group,

\(\mathcal{I} = \{i \mid \text{ideal of } \mathfrak{b}, \ i \subset \mathfrak{n}\}\) ad-nilpotent ideals of \(\mathfrak{b}\),

\(\mathcal{I}_{ab}\) abelian ideals of \(\mathfrak{b}\).

We refer to [CP] for a brief description of \(\hat{\Delta}\) and \(\hat{W}\), though we do not completely follow the notation we used there. In particular, in [CP, Section 1] we described in great detail the relationship between \(\hat{W}\) viewed as the Weyl group of \(\hat{\Delta}\) and its affine representation on \(h^*_R\). By virtue of that description, we do not distinguish here between \(\hat{W}\) and its affine representation. We note that \(\hat{W}_i\) can be viewed as the affine Weyl group of \(\frac{1}{i}\hat{\Delta}\).
We will denote by \((\ , \ )\) at the same time the Killing form on \(h_{\mathbb{R}}\), the induced form on \(h^*_{\mathbb{R}}\) and the natural pairing \(h_{\mathbb{R}} \times h^*_{\mathbb{R}} \rightarrow \mathbb{R}\). We set:

\[H_{\alpha,k} = \{x \in h^*_{\mathbb{R}} \mid (x, \alpha) = k\}, \quad \alpha \in \Delta^+, \ k \in \mathbb{Z}\]

affine reflection

\[C_\infty = \{x \in h^*_{\mathbb{R}} \mid (x, \alpha_i) > 0 \ \forall \ i = 1, \ldots, n\}\]

the fundamental chamber of \(W\),

\[C_i = \{x \in h^*_{\mathbb{R}} \mid (x, \alpha) > 0 \ \forall \ \alpha \in \Pi, \ (x, \theta) < i\}\]

the fundamental alcove of \(\widehat{W}_i\).

Recall that \(\Delta\) can be given a partial order defined by \(\alpha < \beta\) if \(\beta - \alpha\) is a sum of positive roots. We will denote by \(V_\alpha\) the principal dual order ideal generated by \(\alpha\), namely:

\[V_\alpha = \{\beta \in \Delta^+ \mid \beta \geq \alpha\}\]

Moreover, for \(i \in I\) we define \(\Phi_i = \{\alpha \in \Delta^+ \mid g_\alpha \subset i\}\), and for \(\Phi \subset \Delta^+\) we set \(i(\Phi) = \bigoplus_{\alpha \in \Phi} g_\alpha\), so that \(i \mapsto \Phi_i, \ \Phi \mapsto i(\Phi)\) are mutually inverse maps between \(I\) and the set of dual order ideals of the poset \((\Delta^+, \leq)\). We shall use the notation \(i(\Phi)\) even when \(\Phi\) is any subset of \(\Delta^+\) (so \(i(\Phi)\) is not necessarily an ideal).

We set \(i(\emptyset)\) to be the zero ideal.

1. Connections between ad-nilpotent ideals, the affine Weyl group and the center of \(G\).

In this section we will explain how ad-nilpotent ideals can be naturally parametrized by a certain subset \(W\) of \(\widehat{W}\), and in turn, by the points in \(Q^\vee\) of a suitable simplex \(D \subset h^*_\mathbb{R}\): on this simplex there is a natural action of \(\text{Cent}(G)\), which should be regarded as embedded into \(\widehat{W}\). We will describe the relationships of this action with the ad-nilpotent ideals. These constructions specialize nicely to the case of abelian ideals. The idea of relating abelian ideals with elements in \(\widehat{W}\) appears in [Ko2] (where it is attributed to D. Peterson); the generalization to the ad-nilpotent case can be found in [CP]. The role of the center has been pointed out in [Ko2] (in the abelian case).

First we explain how to view \(I\) as a subset of \(\widehat{W}\).
Let \(i = \bigoplus_{\alpha \in \Phi} g_\alpha, \ \Phi \subset \Delta^+\), be an ad-nilpotent ideal. Set

\[L_i = \bigcup_{k \geq 1} (-\Phi^k + k\delta) \subset \widehat{\Delta}^+\]
where $\Phi^1 = \Phi$ and $\Phi^k = (\Phi^{k-1} + \Phi) \cap \Delta$, $k \geq 2$. Note that, since $i$ is nilpotent, $L_i$ is a finite set. For $w \in \widehat{W}$ set
\[ N(w) = \left\{ \alpha \in \widehat{\Delta^+} \mid w^{-1}(\alpha) < 0 \right\}. \]

Then:

**Proposition A** ([CP, Theorem 2.6]). There exists a unique $w_i \in \widehat{W}$ such that $L_i = N(w_i)$.

Hence we have an injective map $f : \mathcal{I} \rightarrow \widehat{W}$, $f(i) = w_i$. We set
\[ \mathcal{W} = f(\mathcal{I}), \quad \text{and} \quad \mathcal{W}_{ab} = f(\mathcal{I}_{ab}). \]

$\mathcal{W}$ and $\mathcal{W}_{ab}$ are characterized inside $\widehat{W}$ by the following properties:

**Proposition B** ([CP, Theorem 2.9, Proposition 2.12]).

1. We have
\[ \mathcal{W}_{ab} = \{ w \in \widehat{W} \mid w(C_1) \subset C_2 \}. \]
2. Assume $w \in \widehat{W}$ and $w = t_\tau v$, with $\tau \in Q^\vee$, $v \in W$. Then $w \in \mathcal{W}$ if and only if the following conditions hold:
   (i) $w(C_1) \subset C_\infty$;
   (ii) $(v^{-1}(\tau), \alpha_i) \leq 1$ for each $i \in \{1, \ldots, n\}$ and $(v^{-1}(\tau), \theta) \geq -2$.

**Corollary** (D. Peterson). $|\mathcal{I}_{ab}| = 2^n$.

For $i \in \mathcal{I}$ and $\alpha \in \Delta^+$ we have that $\alpha \in \Phi_i$ if and only if $-\alpha + \delta \in N(w_i)$. As shown in [CP, Section 1], we have that $-\alpha + \delta \in N(w_i)$ if and only $H_{\alpha,1}$ separates $C_1$ and $w_i(C_1)$. We shall need the following result [IM, §1.9]:

**Lemma C.** Let $w \in \widehat{W}$, $w = t_\tau v$, $\tau \in P^\vee$, $v \in W$. Set $\Delta^+_i = \{ \alpha \in \Delta^+ \mid (\alpha, \tau) = i \}$. Then $H_{\alpha,1}$ separates $C_1$ and $w(C_1)$ if and only if $\alpha \in \left( \bigcup_{i>1} \Delta^+_i \right) \cup (\Delta^1_{\tau} \setminus N(v))$.

We shall also need the following classical results, which can be extracted from [IM, §1], on the embedding of $\text{Cent}(G)$ into $W$. Let $\Delta(j)$ denote the root subsystem of $\Delta$ generated by $\Pi \setminus \{\alpha_j\}$ and by $w_0$ the longest (w.r.t. $\Pi \setminus \{\alpha_j\}$) element of the corresponding parabolic subgroup of $W$. Let $w_0$ be the longest element of $W$ with respect to $\Pi$, and
\[ \Omega = \left\{ t_{\omega, j} w_0^j w_0 \mid j \in J \right\} \cup \{1\} \subset \widehat{W}. \]

**Proposition D.**

1. $\Omega$ is a group. Precisely, $\Omega$ is the subgroup of all elements $w \in \widehat{W}$ such that $w(C_1) = C_1$, hence it is isomorphic to $P^\vee/Q^\vee$. Composing this isomorphism with the one induced by the exponential map we also obtain an isomorphism $\Omega \rightarrow \text{Cent}(G)$. 

Proposition 1.1. \(\Omega\) isomorphic to \(\Omega\). We have the following result:

Remark. It is clear that, for any \(1 \leq j \leq n\), \(N(w_0^j w_0)\) equals \(\Delta^+ \setminus \Delta(j)^+\). In particular, for \(j \in J\) we have \(N(w_0^j w_0) = \{\alpha \in \Delta^+ \mid (\alpha, \omega_j^\vee) = 1\}\).

The last tool which will be relevant in what follows is the encoding of ad-nilpotent ideals by means of lattice points of a certain simplex \(D\) in \(h_\mathbb{R}^\ast\). The definition of this simplex is motivated by Proposition B; set

\[\det = \{\sigma \in h_\mathbb{R}^\ast \mid (\sigma, \alpha_i) \leq 1 \text{ for each } i \in \{1, \ldots, n\} \text{ and } (\sigma, \theta) \geq -2\}\]

and define

\[\tilde{Z} = D \cap P^\vee,\]
\[Z = D \cap Q^\vee,\]
\[\tilde{Z}_{ab} = \{\sigma \in P^\vee \mid (\sigma, \beta) \in \{0, -1, 1, -2\} \forall \beta \in \Delta^+\},\]
\[Z_{ab} = \{\sigma \in Q^\vee \mid (\sigma, \beta) \in \{0, -1, 1, -2\} \forall \beta \in \Delta^+\} .\]

It is easily seen that \(D = t_{\rho^\vee} w_0(\mathcal{C}_{h+1}) = \rho^\vee - \mathcal{C}_{h+1}, h\) being the Coxeter number of \(\Delta\). Note that \(t_{\rho^\vee} w_0 \in \tilde{W}\); on the other hand in [CP2, Lemma 1] it is proved that there exists an element \(\tilde{w} \in \tilde{W}\) such that \(D = \tilde{w} (\mathcal{C}_{h+1})\). Set, for \(r \geq 1\)

\[\Omega_r = \{t_{\rho^\vee} w_0^j \mid j \in J\} \cup \{1\};\]

then \(\Omega_r\) is the subgroup of all elements \(w \in \tilde{W}\) such that \(w(\mathcal{C}_r) = \mathcal{C}_r\). It follows that \(\tilde{w} \Omega_{h+1} \tilde{w}^{-1}\) is the subgroup of all elements \(w \in \tilde{W}\) such that \(w(D) = D\).

We set \(\Sigma = \{t_{-\rho^\vee} w_0^j w_0 \mid j \in J\} \cup \{1\}\). Clearly, \(\Sigma\) is a subgroup of \(\tilde{W}\) isomorphic to \(\Omega\). We have the following result:

Proposition 1.1.

(1) \(\Sigma = \{w \in \tilde{W} \mid w(D) = D\}\), so that \(\Sigma = \tilde{w} \Omega_{h+1} \tilde{w}^{-1}\). In particular \(\Sigma\) acts on \(\tilde{Z}\).

(2) For any \(z \in \tilde{Z}\) the orbit of \(z\) under the action of \(\Sigma\) is a set of representatives of \(P^\vee/Q^\vee\). In particular \(\Sigma\) acts freely on \(\tilde{Z}\).

(3) The action of \(\Sigma\) on \(\tilde{Z}\) preserves \(\tilde{Z}_{ab}\).

Proof. (1) It suffices to prove that any element in \(\Sigma\) preserves \(D\). Take \(x \in D\) and consider \(t_{-\rho^\vee} w_0^j w_0(x), j \in J\). For \(i = 1, \ldots, n\), we have \((t_{-\rho^\vee} w_0^j w_0(x), \alpha_i) = (x, w_0^j w_0(\alpha_i)) - (\omega_j^\vee, \alpha_i)\). Now if \(\alpha_i \in \Delta(j)\) we have \((\omega_j^\vee, \alpha_i) = 0\) and that \(w_0^j w_0(\alpha_i)\) is a simple root (see [IM]). Since \(x \in D\),
we get \((x, w_0 w_0^j(\alpha_i)) \leq 1\) as desired. If \(\alpha_i \not\in \Delta(j)\), then \(\alpha_i = \alpha_j\), so that \((\omega^\vee, \alpha_i) = 1\) and \(w_0 w_0^j(\alpha_i) = -\theta\); therefore \((t_{-\omega^\vee} w_0^j w_0(x), \alpha_i) = -(x, \theta) \leq -2 - 1 = 1\). Finally, we have

\[
(t_{-\omega^\vee} w_0^j w_0(x), \theta) = (x, w_0 w_0^j(\theta)) - (\omega^\vee, \theta) = (x, w_0(\alpha_j)) - 1 \geq -1 - 1 = -2.
\]

The fact that \(\Sigma\) also preserves \(\tilde{Z}\) is immediate.

(2) Let \(z \in \tilde{Z}\). For any root \(\alpha\), denote by \(s_\alpha\) the reflection associated to \(\alpha\); we have \(s_\alpha(z) = z - (z, \alpha^\vee)\alpha = z - (z, \alpha)\alpha^\vee\) and, since \(z \in P^\vee\), \((z, \alpha)\alpha^\vee \in Q^\vee\). It follows that, for any \(\nu \in W\), \(z\) and \(v(z)\) differ by an element in \(Q^\vee\). Therefore, for any \(j \in J\), \(z\) and \(t_{-\omega^\vee} w_0^j w_0(\nu) = -\omega^\vee + w_0^j w_0(\nu)\) are distinct \(\text{mod } Q^\vee\), since \(\omega^\vee \not\in Q^\vee\). The claim follows directly.

(3) If \(x \in \tilde{Z}_{ab}\), we have

\[
(t_{-\omega^\vee} w_0^j w_0(x), \beta)
= (x, w_0 w_0^j(\beta)) - (\omega^\vee, \beta)
= \begin{cases}
(x, w_0 w_0^j(\beta)) \in \{0, 1, -1, -2\} & \text{if } \beta \in \Delta(i) \\
(x, w_0 w_0^j(\beta)) - 1 \in \{0, 1, -1, 2\} - 1 = \{0, 1, -1, -2\} & \text{otherwise}.
\end{cases}
\]

Hence \(t_{-\omega^\vee} w_0^i w_0(x) \in \tilde{Z}_{ab}\), as desired. \(\square\)

Lemma 1.2. \(W \cdot \overline{C}_k = \{ x \in h^*_R \mid -k \leq (x, \beta) \leq k \forall \beta \in \Delta^+ \}\).

Proof. Consider \(x \in W \cdot \overline{C}_k\); then \(x = \sum_{i=1}^n \lambda_i w(\alpha_i)\) for some \(w \in W\) and \(\lambda_i \geq 0\), \(\sum_{i=1}^n \lambda_i \leq k\). Set \(\beta_i = w(\alpha_i)\) for \(1 \leq i \leq n\); then \(\beta_1, \ldots, \beta_n\) is another basis for \(\Delta\), hence for any \(\beta \in \Delta^+\) we have \(\beta = \sum_{i=1}^n \mu_i \beta_i\) with \(|\mu_i| \leq m_i\) and \(\mu_i \geq 0 \forall i\) or \(\mu_i \leq 0 \forall i\). Finally we have \(|(x, \beta)| = \sum_{i=1}^n \lambda_i \frac{\mu_i}{m_i} |\leq \sum_{i=1}^n \lambda_i | \leq k\). \(\square\)

Proposition 1.3. For any \(z \in \tilde{Z}\) we have \(z + \overline{C}_1 \subset W \cdot \overline{C}_h\). Moreover \(z \in \tilde{Z}_{ab}\) if and only if \(z + \overline{C}_1 \subset W \cdot \overline{C}_2\). In particular, for any \(z \in \tilde{Z}\) there exists a unique \(\nu \in W\) such that \(\nu(z + \overline{C}_1) \subset \overline{C}_h\) and, for such a \(\nu\), \(\nu(z + \overline{C}_1) \subset \overline{C}_2\) if and only \(z \in \tilde{Z}_{ab}\).

Proof. Let \(z \in \tilde{Z}\). We may write \(z = \rho^\vee - \sum_{i=1}^n \lambda_i \alpha_i\), \(\lambda_i \geq 0\), \(\sum_{i=1}^n \lambda_i \leq h+1\). For any positive root \(\beta = \sum_{i=1}^n \mu_i \alpha_i\) \((0 \leq \mu_i \leq m_i)\) we have

\[
(z, \beta) = \sum_{i=1}^n \mu_i - \sum_{i=1}^n \lambda_i \frac{\mu_i}{m_i} \leq \sum_{i=1}^n m_i = h - 1.
\]

On the other hand, since at least one of the \(\mu_i\) is positive, \((z, \beta) \geq 1 - \sum_{i=1}^n \lambda_i \geq 1 - (h + 1) = -h\).
Now for any $y \in C_1$ and any $\beta \in \Delta^+$ we have $0 \leq (y, \beta) \leq 1$, whence $-h \leq (z + y, \beta) \leq h$. By Lemma 1.2 we get $z + C_1 \subset W \cdot C_h$.

We see directly that $z \in \tilde{Z}$ if and only if $-2 \leq (z + y, \beta) \leq 2$, or equivalently, by Lemma 1.2, if and only if $z + C_1 \subset W \cdot C_2$. \hfill \square

**Definition.** Let $z \in \tilde{Z}$ and $v \in W$ be (the unique element of $W$) such that $v(z + C_1) \subset C_h$. We define the map

$$
\tilde{F} : \tilde{Z} \rightarrow \tilde{W}, \quad z \mapsto t_{v(z)} v,
$$

and we set $v_z = v^{-1}$.

There exists a unique $w \in \tilde{W}$ such that $v(z + C_1) = w(C_1)$, thus we also have a map

$$
F : \tilde{Z} \rightarrow \tilde{W}, \quad z \mapsto w.
$$

(In the Introduction $\tilde{F}(z), F(z)$ have been called $w_z, \tilde{w}_z$, respectively.)

By Proposition A we can thus associate to $z$ an ad-nilpotent ideal

$$
i_z = f^{-1}(F(z)).
$$

**Proposition 1.4.**

(a) $F(z) = F(z')$ $(z, z' \in \tilde{Z})$ if and only if $z = \psi(z')$ for $\psi \in \Sigma$.

(b) $F$ is a surjection $\tilde{Z} \rightarrow W$ and $F|_Z : Z \rightarrow W$ is a bijection, with inverse map $t_{\tau} v \mapsto v^{-1}(\tau)$.

(c) Set $H(z) = (F(z), z \mod Q^\vee)$. Then $H$ is bijective, hence the same holds for the composite map:

$$
\tilde{Z} \xrightarrow{H} W \times P^\vee/Q^\vee \xrightarrow{(f^{-1}, \exp)} I \times \text{Cent}(G).
$$

(d) Restricting $F$ to $Z_{ab}$ induces a surjection $Z_{ab} \rightarrow W_{ab}$, and as above bijections $Z_{ab} \leftrightarrow W_{ab}, \tilde{Z}_{ab} \leftrightarrow I_{ab} \times \text{Cent}(G)$.

**Proof.** (a) Assume that $F(z) = F(z')$ for $z, z' \in \tilde{Z}$ and set $v = v_z^{-1}, u = v_{z'}^{-1}$. Then $v(z + C_1) = u(z' + C_1)$, so that $C_1 = v^{-1}u(z') - z + v^{-1}u(C_1)$ and $t_{v^{-1}u(z') - z} v^{-1}u \in \Omega$. By Proposition D, (2), we have either $v = u$ and in turn $z = z'$ (and in this case we are done), or there exists $j \in J$ such that $v^{-1}u = w_0^j w_0$; moreover

$$(*) \quad z = w_0^j w_0 (z') - \omega_j^j$$

which means $z = \psi(z')$ with $\psi = t_{-\omega_j^j} w_0^j w_0 \in \Sigma$. Viceversa, if $z = \psi(z')$, $1 \neq \psi \in \Sigma$, relation $(*)$ implies $z + C_1 = w_0^j w_0 (z' + C_1)$, hence $F(z) = F(z')$.

(b) In [CP2, Prop. 3] it is proved that $F|_Z : Z \rightarrow W$ is a bijection, with inverse map $t_{\tau} v \mapsto v^{-1}(\tau)$. By (a) we obtain that $F$ maps the whole $\tilde{Z}$ onto $W$.

(c) This statement is a direct consequence of (b) and Proposition 1.1, (2).
(d) This follows from Propositions B, (1), Proposition 1.3, and from Part (c) of this proposition. □

**Proposition 1.5.** If $z \in \tilde{Z}$, then $N(v_z) = \{\alpha \in \Delta^+ \mid (\alpha, z) < 0\}$.

*Proof.* We have to prove that the element $v \in W$ such that $v(z + C_1) \subset C_\infty$ is defined by the condition $N(v^{-1}) = \{\alpha \in \Delta^+ \mid (\alpha, z) < 0\}$. It suffices to verify that the element $u$ defined by $N(u^{-1}) = \{\alpha \in \Delta^+ \mid (\alpha, z) < 0\}$ is such that $(u(z + e), \beta) > 0 \forall \beta \in \Delta^+$, $\forall e \in C_1$. We can then conclude that $u = v$ by uniqueness.

First remark that $N(u) = -uN(u^{-1})$; then suppose $\beta \in N(u)$, or $\beta = -u(\gamma), \gamma \in N(u^{-1})$; by hypothesis $(z, \gamma) \leq -1$, hence we have $(u(z + e), \beta) = -(z + e, \gamma) \geq 1 - (e, \gamma) > 0$. It remains to consider the case $\beta \notin N(u)$; in that case $\beta = u(\gamma), \gamma \notin N(u^{-1})$ and in particular $(z, \gamma) \geq 0$, so that $(u(z + e), \beta) = (z + e, \gamma) = (z, \gamma) + (e, \gamma) \geq (e, \gamma) > 0$ as desired. □

**Corollary 1.6.** If $z \in \tilde{Z}_{ab}$, then $N(v_z) = \Delta_z^{-2} \cup \Delta_z^{-1}$.

**Proposition 1.7.** Suppose $z \in \tilde{Z}_{ab}$. Then

$$i_z = -v_z^{-1}(\Delta_z^{-2}) \cup v_z^{-1}(\Delta_z^1).$$

*Proof.* Let $w = \tilde{F}(z), w = t_\tau v$; thus $v_z = v^{-1}$ and $z = v_z(\tau)$. By Lemma C we obtain that $\Phi_{i_z} = \Delta_z^2 \cup (\Delta_z^1 \setminus N(v))$. Let $\alpha \in \Delta_z^2$. Since $(v_z(\alpha), z) = (\alpha, \tau) = 2$ and $\Delta_z^2 = 0$, we obtain $v_z(\alpha) < 0$, thus $-v_z(\alpha) \in \Delta_z^{-2}$, or equivalently $\alpha \in -v_z^{-1}(\Delta_z^{-2})$. Conversely, if $\beta \in \Delta_z^{-2}$, then $(\tau, v_z^{-1}(\beta)) = -2$, whence $-v_z^{-1}(\beta) \in \Delta_z^2$. Therefore $\Delta_z^2 = -v_z^{-1}(\Delta_z^{-2})$. Similarly, we see that if $\alpha \in \Delta_z^1 \setminus N(v)$, then $v_z(\alpha) \in \Delta_z^1$; and, conversely, if $\beta \in \Delta_z^1$, then $v_z^{-1}(\beta) \in 1 \setminus N(v)$. This concludes the proof. □

**Remark.** Corollary 1.6 and Proposition 1.7 show that our constructions coincide for points in $\tilde{Z}_{ab}$ with those performed by Kostant in [Ko2], taking into account that Kostant’s $Z$ is our $-\tilde{Z}_{ab}$. In particular, we have provided proofs for the results of Sections 2 and 3 and for Proposition 5.2 and Theorem 5.3 of that paper.

2. Compatible Borel subalgebras and compatible discrete series.

A symmetric Lie subalgebra of $\mathfrak{g}$ is a subalgebra $\mathfrak{k}$ that is the fixed point set of an involutory automorphism $\Theta$ of $\mathfrak{g}$. If $\mathfrak{k}$ is a symmetric Lie subalgebra, we set $\mathfrak{p}$ to denote the $-1$ eigenspace of $\Theta$. It follows that we have a decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

that is usually referred to as a Cartan decomposition of $\mathfrak{g}$. An equal rank symmetric Lie subalgebra is a symmetric Lie subalgebra that contains a Cartan subalgebra of $\mathfrak{g}$. A procedure to construct equal rank symmetric Lie
subalgebras is the following: If \( \tau \in \mathcal{P} \), set \( \Theta_\tau = \text{Ad}(\exp(\sqrt{-1} \pi \tau)) \). Then \( \Theta_\tau \) is an involutory automorphism of \( \mathfrak{g} \). If we set
\[
\mathfrak{g} = \mathfrak{k}_\tau \oplus \mathfrak{p}_\tau
\]
to be the corresponding Cartan decomposition, then \( \mathfrak{k}_\tau \) is an equal rank symmetric Lie subalgebra.

On the other hand, if \( \mathfrak{k} \) is an equal rank symmetric Lie subalgebra of \( \mathfrak{g} \) and \( \mathfrak{h}' \) is a Cartan subalgebra contained in \( \mathfrak{k} \), then, by \cite{Hel}, Ch. IX, Proposition 5.3, there is \( x \in \mathfrak{g} \) such that \( \Theta = \text{Ad}(\exp(x)) \). Moreover, by Theorem 5.15 of Ch. X of \cite{Hel}, there is an automorphism \( \phi' \) of type \( (s_0, \ldots, s_r; k) \) for \( \mathfrak{h} \), \( r \) being the rank of the subalgebra of \( \Theta' \)-fixed points. Since \( \mathfrak{k} \) is an equal rank symmetric subalgebra, we have \( r = n \). Since \( \Theta' = \text{Ad}(\exp(\phi'(x))) \), it follows from Theorem 5.16 (i), Ch. X of \cite{Hel} that \( k = 1 \), i.e., \( \Theta' \) fixes \( \mathfrak{h} \) pointwise.

Set \( \mathfrak{X} = \mathfrak{C}_2 \cap \mathcal{P} \). If we set \( \tau = \sum_i s_i \omega_i' \), then \( \tau \in \mathfrak{X} \). Indeed, since \( \Theta' \) is an involution, then, according to Theorem 5.15 of \cite{Hel} again,
\[
2 = s_0 + \sum_{i=1}^n m_i s_i
\]
(recall that \( m_i \) is the coefficient of \( \alpha_i \) in the highest root).

Note that \( \Theta' = \Theta_\tau \). In fact, calculating the action on the root vectors \( X_{\alpha_i}, 1 \leq i \leq n \), \( X_{-\theta} \) (which are Lie algebra generators for \( \mathfrak{g} \)), we have:
\[
\Theta_\tau(X_{\alpha_i}) = (-1)^{s_i} X_{\alpha_i} = \Theta'(X_{\alpha_i}) \\
\Theta_\tau(X_{-\theta}) = (-1)^{s_0-2} X_{-\theta} = (-1)^{s_0} X_{-\theta} = \Theta'(X_{-\theta})
\]

Finally we observe that, by Theorem 5.4 of Ch. IX of \cite{Hel}, we can write \( \phi = \nu \text{Ad}(g) \), with \( \nu \) an automorphism of \( \mathfrak{g} \) leaving \( \mathfrak{h} \) and \( C_\infty \) invariant. Then
\[
\nu^{-1} \Theta_\tau \nu = \Theta_{\nu^{-1}(\tau)} = \text{Ad}(g) \Theta \text{Ad}(g^{-1})
\]
Since \( \nu^{-1}(\tau) \in \mathfrak{X} \), we have proved a weaker form of Proposition 4.1 of \cite{Ko2}, namely:

**Theorem 2.1.** If \( \mathfrak{k} \) is an equal rank symmetric Lie subalgebra of \( \mathfrak{g} \) then there is \( \tau \in \mathfrak{X} \) such that \( \mathfrak{k}_\tau = \text{Ad}(g) \mathfrak{k} \) for some \( g \in G \).

We now turn our attention to the Θ-stable Borel subalgebras. Suppose that \( \mathfrak{k} \) is an equal rank symmetric subalgebra of \( \mathfrak{g} \) and let \( \Theta \) be the corresponding involution.

We let \( K_\tau \) be the subgroup of \( G \) corresponding to \( \mathfrak{k}_\tau \).

Theorem 1 of \cite{Ma} gives the following characterization of Θ-stable Borel subalgebras:

**Theorem 2.2.** If \( \mathfrak{b}' \) is a Θ-stable Borel subalgebra of \( \mathfrak{g} \), then there exist \( g \in G, w \in W \), and \( \tau \in \mathfrak{X} \) such that \( \text{Ad}(g) \mathfrak{k} = \mathfrak{k}_\tau \) and \( \text{Ad}(g) \mathfrak{b}' = w \mathfrak{b} \).
Proof. By Theorem 2.1, we can find \( g' \in G \) such that
\[
\text{Ad}(g')\xi = \xi \text{ and } \text{Ad}(g')\Theta \text{Ad}(g'^{-1}) = \Theta,
\]
so \( \text{Ad}(g')b' \) is \( \Theta \)-stable. By Theorem 1 of [Ma], we can find an element \( k \in K_\tau \) such that \( b'' = \text{Ad}(kg')b' \) contains a \( \Theta \)-stable Cartan subalgebra \( h' \).

Set \( \Delta' \) to be the set of roots of \( (g, h') \) and let \( \Delta'^+ \) denote the positive system in \( \Delta' \) defined by \( b'' \). Since \( b'' \) and \( h' \) are \( \Theta \)-stable, it follows that the map \( \alpha \mapsto \alpha \circ \Theta \) defines an automorphism of the Dynkin diagram. Since \( \Theta_\tau \) is of inner type we conclude that \( \Theta_\tau \) fixes pointwise \( h' \), i.e., \( h' \subseteq \xi_\tau \). Hence there is \( k' \in K_\tau \) such that \( \text{Ad}(k')h' = h \). Set \( g = k'kg' \); then \( \text{Ad}(g)\xi = \xi_\tau \) and \( \text{Ad}(g)b' \) is a Borel subalgebra containing \( h \). Then \( \text{Ad}(g)b' \) defines a positive system in \( \Delta \), therefore there is an element \( w \in W \) such that \( \text{Ad}(g)b' = wb' \).

\( \square \)

Definition. Let \( g = \xi \oplus p \) be a Cartan decomposition, with Cartan involution \( \Theta \). We will say that a Borel subalgebra \( b' \) is compatible with \( \Theta \) (or with \( \xi \)) if it is \( \Theta \)-stable and
\[
[[b'_p, b'_p], b'_p] = 0,
\]
where \( b'_p = b' \cap p \).

Clearly \( b \) is compatible with \( \xi_\tau \) for any \( \tau \in X \). Conversely we have the following theorem:

Theorem 2.3 ([Ko2, Theorem 4.3]). Let \( \xi \) be an equal rank symmetric subalgebra such that \( h \subseteq \xi \). If \( \xi \) is compatible with \( b \) then \( \xi = \xi_\tau \) for some \( \tau \in X \). Moreover if \( \xi' \) is any equal rank symmetric subalgebra, then a Borel subalgebra \( b' \) is compatible with \( \xi' \) if and only if there exist \( g \in G \) and \( \tau \in X \) such that \( \text{Ad}(g)\xi' = \xi_\tau \) and \( \text{Ad}(g)b' = b \).

Proof. Suppose that \( \xi \) is compatible with \( b \). Since \( h \subseteq \xi \), by Exercise C.3 of Ch. IX of [Hel], we have that \( \Theta = \text{Ad}(\sqrt{-1} \pi h) \) with \( h \in P' \). Clearly we can choose \( h = \sum \epsilon_i \omega_i \) with \( \epsilon_i \in \{0, 1\} \). We claim that, since \( b \) is compatible, then \( \sum m_i \epsilon_i \leq 2 \), i.e., \( h \in X \).

Indeed write
\[
b_0 = h \oplus \sum_{\alpha(h)=0} g_\alpha, \quad b_1 = \sum_{\alpha(h)=1} g_\alpha.
\]

Clearly
\[
[b_0 + b_1, b_0 + b_1] \subseteq b_0 + b_1 + [b_1, b_1]
\]
hence, since \( b \) is compatible,
\[
[[b_0 + b_1, b_0 + b_1], b_0 + b_1] \subseteq b_0 + b_1 + [b_1, b_1].
\]
Since \( b_0 + b_1 \) generates \( b \) we have that \( b = b_0 + b_1 + [b_1, b_1] \), thus, in particular, the root vector \( X_\theta \) belongs to either \( b_0 \), or \( b_1 \), or \( [b_1, b_1] \).

If \( X_\theta \in b_0 \) then \( \theta(h) = 0 \), if \( X_\theta \in b_1 \) then \( \theta(h) = 1 \), and, if \( X_\theta \in [b_1, b_1] \), then \( \theta(h) = 2 \). Since \( \theta(h) = \sum \epsilon_i m_i \) the first result follows.
Theorem 2.4. The map \( z \mapsto W_\tau v_z^{-1} \) is a bijection between \( \tilde{Z}_\tau \) and \( W_\tau \setminus W \). In particular, for any \( \tau \in X \), one has
\[
|\tilde{Z}_\tau| = n_\tau.
\]

Proof. We note that, if \( z \in \tilde{Z}_\tau \) then
\[
\begin{align*}
\Delta_\tau^2 & = -v_z^{-1}(\Delta_z^2), \\
\Delta_\tau^0 & = v_z^{-1}(\Delta_z^0), \\
\Delta_\tau^1 & = v_z^{-1}(\Delta_z^1) \cup -v_z^{-1}(\Delta_z^{-1}).
\end{align*}
\]

The root system for \((\mathfrak{t}_\tau, \mathfrak{h})\) is \( \Delta_\tau = \pm \Delta_\tau^0 \cup \pm \Delta_\tau^2 \). By our formulas above \( v_z^{-1}(\Delta^+) \) contains \( -\Delta_\tau^2 \cup \Delta_\tau^0 \). This implies that \( W_\tau v_z^{-1} = W_\tau v_{z'}^{-1} \), \( z, z' \in \tilde{Z}_\tau \), if and only if \( z = z' \): Indeed if \( v_z^{-1} = w'v_z^{-1} \) for some \( w' \in W_\tau \) then
\[
-\Delta_\tau^2 \cup \Delta_\tau^0 \subseteq v_z^{-1}(\Delta^+) = w'v_z^{-1}(\Delta^+) \supset w'(-\Delta_\tau^2 \cup \Delta_\tau^0).
\]
Since \( -\Delta_\tau^2 \cup \Delta_\tau^0 \) is a positive system for \( \Delta_\tau \), it follows that \( w' = 1 \). Hence \( v_z = v_{z'} \) and in turn \( z = v_z(\tau) = v_{z'}(\tau) = z' \).

We now verify that \( W_\tau w = W_\tau v_z^{-1} \) for some \( z \in \tilde{Z}_\tau \). Let \( w' \in W_\tau \) be the unique element such that \( w'(w(\Delta^+) \cap \Delta_\tau) = -\Delta_\tau^2 \cup \Delta_\tau^0 \) so that we have that \( w'w(\Delta^+) \supset -\Delta_\tau^2 \cup \Delta_\tau^0 \).

Set \( z = (w'w)^{-1}(\tau) \). Let us verify that \( z \in \tilde{Z}_{ab} \): If \( \alpha \in \Delta^+ \) then \( \alpha(z) = w'w\alpha(\tau) \in \{ \pm 2, \pm 1, 0 \} \), so it is enough to verify that \( \Delta_\tau^2 \) is disjoint from \( w'w(\Delta^+) \), but this is obvious since \( w'w(\Delta^+) \) contains \( -\Delta_\tau^2 \).

We are left with showing that \( W_\tau w = W_\tau v_z^{-1} \). Since \( z \) is in the \( W \)-orbit of \( \tau \), then \( v_z^{-1}(z) = \tau = w'w(z) \), hence \( w'wv_z \in \text{Stab}_W(\tau) \subset W_\tau \); therefore \( W_\tau w = W_\tau v_z^{-1} \). \( \square \)
We are going to prove a result which implies [Ko2, Theorem 5.5]. As shown in the proof of Theorem 2.4, $\Delta_\tau = \pm \Delta_\tau^0 \cup \pm \Delta_\tau^2$ is the set of roots for $\mathfrak{t}_\tau$ and $\Delta_\tau^± = -\Delta_\tau^2 \cup \Delta_\tau^0$ is a positive system for $\Delta_\tau$. We let $\Pi_\tau$ denote the set of simple roots for $\Delta_\tau$ corresponding to $\Delta_\tau^\perp$.

We consider $\hat{W}_2 = W \ltimes T(2Q^\vee)$: $\hat{W}_2$ can be viewed as the affine Weyl group of $\frac{1}{2}\Delta$, so that $C_2$ is its fundamental alcove relative to $\frac{1}{2}\Pi$. In the following lemma we describe $\Pi_\tau$ and see that $W_\tau$ is strictly related with the stabilizer $\text{Stab}_{\hat{W}_2}(\tau)$ of $\tau$ in $\hat{W}_2$.

**Lemma 2.5.**

1. We have $\text{Stab}_{\hat{W}_2}(\tau) \subset W_\tau \ltimes T(2Q^\vee)$. Moreover, $\text{Stab}_{\hat{W}_2}(\tau) \cong W_\tau$ via the canonical projection of $\hat{W}$ onto $W$.

2. Let $\Pi_\tau$ denote the set of simple roots for $\Delta_\tau$ relative to $\Delta_\tau^\perp$. If $(\tau, \theta) < 2$, then $\Pi_\tau = \Pi \cap \tau^\perp$; if $(\tau, \theta) = 2$, then $\Pi_\tau = (\Pi \cap \tau^\perp) \cup \{-\theta\}$.

**Proof.** We start with some general facts. Consider any point $\nu \in C_1$. It is well-known (see [Bou, V, 3.3]) that the stabilizer $\text{Stab}_{\hat{W}}(\nu)$ of $\nu$ in $\hat{W}$ is the parabolic subgroup generated by the simple reflections which fix $\nu$. Set $\Delta(\nu) = \{\alpha \in \Delta \mid (\alpha, \nu) \in \mathbb{Z}\}$, $\Delta_\nu = \{\alpha - (\alpha, \nu)\delta \mid \alpha \in \Delta(\nu)\}$, and $\tilde{\Pi}_\nu = \tilde{\Delta}_\nu \cap \tilde{\Pi}$. It is clear that both $\Delta(\nu)$ and $\Delta_\nu$ are root subsystems of $\Delta$ and $\tilde{\Delta}$, respectively; moreover, $\tilde{\Delta}_\nu$ is isomorphic to $\Delta(\nu)$, via the natural projection on $\mathfrak{b}^+_\Delta$. Since $\nu \in C_1$, we have that $(\alpha, \nu) \in \{0, -1, 1\}$ for each $\alpha \in \Delta(\nu)$, hence we easily obtain that $\tilde{\Delta}_\nu$ is the standard parabolic subsystem generated by $\tilde{\Pi}_\nu$. Since the simple reflections that fix $\nu$ are exactly the reflections corresponding to $\tilde{\Pi}_\nu$, we have that $\text{Stab}_{\hat{W}}(\nu)$ is the Weyl group of $\tilde{\Delta}_\nu$. This implies in particular that the canonical projection of $\text{Stab}_{\hat{W}}(\nu)$ on $W$ provides an isomorphism of $\text{Stab}_{\hat{W}}(\nu)$ with the Weyl group of $\Delta(\nu)$. Moreover, if we project $\tilde{\Pi}_\nu$ on $\Delta$, we obtain a basis for $\Delta(\nu)$. Thus we have that: If $(\nu, \theta) < 1$, then $\Pi \cap \nu^\perp$ is a basis of $\Delta(\nu)$; if $(\nu, \theta) = 1$, then $(\Pi \cap \nu^\perp) \cup \{-\theta\}$ is a basis of $\Delta(\nu)$.

If we apply the above remarks to $\hat{W}_2$ in place of $\hat{W}$, $C_2$ in place of $C_1$, and $\tau \in \widetilde{X}$ in place of $\nu$, we obtain that $\text{Stab}_{\hat{W}_2}(\tau)$ is a parabolic subgroup of $\hat{W}_2$ which projects isomorphically onto $W_\tau$ via the canonical projection of $\hat{W}_2$ on $W$. In particular, we have $\text{Stab}_{\hat{W}_2}(\tau) \subset W_\tau \ltimes T(2Q^\vee)$. Moreover we have that if $(\tau, \theta) < 2$, then $\Pi \cap \tau^\perp$ is a basis of $\Delta_\tau$; if $(\tau, \theta) = 2$, then a basis of $\Delta_\tau$ is given by $\{-\theta\} \cup (\Pi \cap \tau^\perp)$.

Recall that $\Omega_2 = \left\{t_{2\nu_j} w_0^j w_0 \mid j \in J\right\} \cup \{1\}$ and define, for $z \in \tilde{Z}$,

$$b_z = v_z^{-1} b.$$
Theorem 2.6. Suppose that $\tau \notin \{2\omega_i^\vee \mid i \in J\} \cup \{0\}$. Set

$$\widetilde{Z}_\tau^{\text{cmpt}} = \{z \in \widetilde{Z}_\tau \mid b_z \text{ is compatible with } t_\tau\}.$$ 

Then there is a canonical bijection between $\widetilde{Z}_\tau^{\text{cmpt}}$ and $\Omega_2$. In particular

$$|\text{Cent}(G)| = |\widetilde{Z}_\tau^{\text{cmpt}}|.$$

Proof. If $s \in \Omega_2$ write $s = t_\nu v$ with $v \in W$ and $\nu \in 2P^\vee$. By Theorem 2.4 there is a unique $z_s \in \widetilde{Z}_\tau$ such that $W_\tau v z_s^{-1} = W_\tau v$. Notice that $z_s \in \widetilde{Z}_\tau^{\text{cmpt}}$. Indeed, if $\tau' = s^{-1}(\tau)$, we have that $\tau' \in X$, hence $b$ is compatible with $t_{\tau'} = t_{s^{-1}(\tau)} = t_{v^{-1}(\tau)}$. In turn $v b$ is compatible with $t_\tau$, but $v z_s^{-1} = w' v$ with $w' \in W_\tau$, hence $b z_s = w' v b$ is compatible with $w' t_\tau = t_\tau$. We can thus define a map $B : \Omega_2 \rightarrow \widetilde{Z}_\tau^{\text{cmpt}}$ by setting $B(s) = z_s$.

Let us show that $B$ is injective. We need to show that, if $s = t_\nu v$ and $s' = t_{\nu'} v'$ are such that $W_\tau v = W_\tau v'$, then $v = v'$. Since $\Omega_2$ is a group, it is enough to check that, if $s = t_\nu v$ is such that $v \in W_\tau$, then $v = 1$. The assumption that $v \in W_\tau$ implies that $v(\Delta_\tau) = \Delta_\tau$ and, by [IM, Prop. 1.26, (ii)], $\nu(\Pi \cup \{-\theta\}) = \Pi \cup \{-\theta\}$. Since $\tau \notin \{2\omega_i^\vee \mid i \in J\} \cup \{0\}$, we see that $\Pi_\tau = \Delta_\tau \cap (\Pi \cup \{-\theta\})$, hence $v(\Pi_\tau) = \Pi_\tau$ and $v = 1$.

It remains to show that $B$ is surjective. Fix $z \in \widetilde{Z}_\tau^{\text{cmpt}}$: Since $v z_s^{-1} b$ is compatible with $t_\tau$, it follows that $b$ is compatible with $t_z = t_{v z_s^{-1}}$. By Theorem 2.3, we deduce that there is $\tau' \in X$ such that $t_z = t_{\tau'}$, hence $z = \tau' + \nu$ with $\nu \in 2P^\vee$, or, equivalently,

$$\tau = v z_s^{-1}(\tau' + \nu) = t_{\nu'} v z_s^{-1}(\tau),$$

where $\nu' = v z_s^{-1}(\nu)$.

Since $\widetilde{C}_2$ is a fundamental domain for $\widetilde{W}_2$, there is a unique element $\tilde{u} \in \widetilde{W}_2$ such that $\tilde{u} t_{\nu'} v z_s^{-1}(\tau) = \widetilde{C}_2$. Set $s = \tilde{u} t_{\nu'} v z_s^{-1}: Clearly s \in \Omega_2$. Since $\tau \in t_{\nu'} v z_s^{-1}(\widetilde{C}_2) \cap \widetilde{C}_2$, we find that $\tilde{u} \tau = \tau$, therefore, by the previous lemma we have $u \in W_\tau \times T(2Q^\vee)$. Thus $B(s) = z$ and we are done. \(\Box\)

Fix again $\tau \in X$ and set $\sigma$ to denote a conjugation in $g$ corresponding to a compact real form $g_u$ such that $\Theta_\tau \sigma = \sigma \Theta_\tau$ and set $\sigma_\tau = \Theta_\tau \sigma$. Then $\sigma_\tau$ is a conjugation of $g$ defining a real form $g_\tau$. We set $G_\tau$ to be the subgroup of $G$ corresponding to $g_\tau$ and $K$ to be the subgroup of $G_\tau$ corresponding to $g_\tau \cap k_\tau$.

Set $P_{\text{reg}} = \{\lambda \in P \mid (\lambda, \alpha) \neq 0 \text{ for all } \alpha \in \Delta\}$. Let $\hat{G}_{\tau}^{\text{disc}} \subset \hat{G}_{\tau}$ denote the set of equivalence classes of discrete series for $G_\tau$. If $\lambda \in P_{\text{reg}}$ we let $\pi_{\lambda} \in \hat{G}_{\tau}^{\text{disc}}$ denote the equivalence class corresponding to the parameter $\lambda$ in Harish-Chandra parametrization. We recall that given $\lambda, \mu \in P_{\text{reg}}$ then $\pi_{\lambda} = \pi_{\mu}$ if and only if there is $w \in W_\tau$ such that $w(\lambda) = \mu$.

If $\lambda \in P_{\text{reg}}$ then we set $\Delta_\lambda^+ = \{\alpha \in \Delta \mid (\lambda, \alpha) > 0\}$ and $b_\lambda = h \oplus \sum_{\alpha \in \Delta_\lambda^+} g_\alpha$ to be the corresponding Borel subalgebra. Given $z \in \widetilde{Z}_\tau$, we set $P_{\text{reg}}(z) = \{\xi \in \hat{G}_{\tau}^{\text{disc}} \mid \hat{Z}_\tau \xi \subset \hat{Z}_\tau z\}$. If $G_{\text{reg}}(z) = \{\xi \in \hat{G}_{\tau}^{\text{disc}} \mid \hat{Z}_\tau \xi \subset \hat{Z}_\tau z\}$.
\{ \lambda \in P_{reg} \mid b_\lambda = b_z \} \) and \( \rho_z = v_z^{-1}\rho \). Set also \( \hat{G}_\tau^{disc}(z) = \{ \pi_\lambda \mid \lambda \in P_{reg}(z) \} \).

As observed in [Ko2], Theorem 2.4 implies that the sets \( \hat{G}_\tau^{disc}(z) \) give a partition of the set of equivalence classes of discrete series for \( G_\tau \), namely
\[
\hat{G}_\tau^{disc} = \bigcup_{z \in \hat{Z}_\tau} \hat{G}_\tau^{disc}(z).
\]

With the notation of \( \S 1 \), if \( z \in \hat{Z}_{ab} \), recall that \( i_z \in \mathcal{I}_{ab} \) denotes \( f^{-1}(F(z)) \) and \( \Phi_z \equiv \Phi_{i_z} = \{ \alpha \in \Delta^+ \mid g_\alpha \subset i_z \} \). Set also \( \Phi_z^2 = -v_z^{-1}(\Delta_z)^2 \), \( \Phi_z^1 = v_z^{-1}(\Delta_z^1) \). By Proposition 1.7 we know that \( \Phi_z = \Phi_z^2 \cup \Phi_z^1 \).

The relation between abelian ideals and discrete series stated above is not as good as one might expect, for example the correspondence depends on the choice of \( \tau \) even when different choices for \( \tau \) give rise to isomorphic real forms of \( G \). Nevertheless Kostant observed in [Ko2] that there are relations between the elements of \( \hat{Z}_\tau \) and the structure of the corresponding discrete series.

The first occurrence of such a relation involves the realization of the discrete series via \( L^2 \)-cohomology: One can give to the principal bundle \( G_\tau/T \) a complex structure by declaring that \( n \) is the space of antiholomorphic tangent vectors; if \( \lambda \in P_{reg}(z) \) then set \( \mathcal{L}_{\lambda - \rho} \) to be the holomorphic line bundle on \( G_\tau/T \) associated to the character \( \exp(\lambda - \rho) \) of \( T \). According to [S2], the \( L^2 \)-cohomology groups \( H^p(\mathcal{L}_{\lambda - \rho}) \) vanish except in one degree \( k(\lambda) \) and \( H^{k(\lambda)}(\mathcal{L}_{\lambda - \rho}) \) gives a realization of \( \pi_\lambda \). It turns out that the decomposition \( \Phi_z = \Phi_z^2 \cup \Phi_z^1 \) implies, applying Theorem 1.5 of [S2], that, if \( \lambda \in P_{reg}(z) \), then \( k(\lambda) = \dim(i_z) \) (cf. [Ko2, Theorem 6.5]).

Another instance, discussed in [Ko2], where there is a connection between abelian ideals and discrete series involves the \( K \)-structure of \( \pi_\lambda \). Recall that, if \( \lambda \in P_{reg}(z) \), then \( \Delta_\tau^+ = -\Delta_z^2 \cup \Delta_z^0 \) is a positive system for \( \Delta_z \) contained in \( \Delta_z^1 = v_z^{-1}\Delta_z^+ \). Set \( \rho_c = \frac{1}{2} \sum_{\alpha \in \Delta_z^+} \alpha \) and \( \rho_n = \rho_z - \rho_c \). It is well-known that the minimal \( K \)-type of \( \pi_\lambda \) has highest weight with respect to \( \Delta_\tau^+ \) given by the formula
\[
\mu_\lambda = \lambda + \rho_n - \rho_c.
\]

We are going to prove, using this formula, that if \( \lambda \in P_{reg}(z) \) then the highest weight \( \mu_\lambda \) of the minimal \( K \)-type of \( \pi_\lambda \) with respect to \( \Delta_\tau^+ \) is
\[
\mu_\lambda = \lambda - \rho_z + 2(i_z) - \frac{1}{2} \tilde{\tau}.
\]

Notation is as follows: For \( h \in h_\mathbb{R} \) we set \( \tilde{h} = \sum_{\alpha \in \Delta} (\alpha, h) \alpha \) and for \( i \in \mathcal{I} \) we put \( (i) = \sum_{\alpha \in \Phi_i} \alpha \). A special case of the previous formula is
\[
\mu_{\rho_z} = 2(i_z) - \frac{1}{2} \tilde{\tau}.
\]
To prove the formula, just compute:

\[
\mu_\lambda = \lambda + \rho_n - \rho_c \\
= \lambda - \rho_z + 2\rho_n \\
= \lambda - \rho_z + \sum_{\alpha \in \Delta_z} v_z^{-1} \alpha + \sum_{\alpha \in \Delta_z^{-1}} v_z^{-1} \alpha \\
= \lambda - \rho_z + \sum_{\alpha \in \Phi_z} \alpha - \sum_{\alpha \in \Delta_z^1 \setminus \Phi_z^1} \alpha - 2 \sum_{\alpha \in \Phi_z^2} \alpha \\
= \lambda - \rho_z + 2\langle i_z \rangle - \sum_{\alpha \in \Delta_z} \alpha - \sum_{\alpha \in \Delta_z^2} 2\alpha \\
= \lambda - \rho_z + 2\langle i_z \rangle - \sum_{\alpha \in \Delta^+} \alpha(\tau) \alpha \\
= \lambda - \rho_z + 2\langle i_z \rangle - \frac{1}{2} \tilde{\tau}.
\]

The previous calculation proves Theorem 6.6 of [Ko2].

We now turn our attention to the discrete series that correspond to elements of \( \tilde{Z}_{\tau}^{\text{cmpt}} \).

**Definition.** A discrete series \( \pi_\lambda \) is said compatible if \( \lambda \in P_{\text{reg}}(z) \) with \( z \in \tilde{Z}_{\tau}^{\text{cmpt}} \).

By Theorem 2.6, the number of compatible discrete series is \( |P^\vee/Q^\vee| \). In particular it is independent from the particular real form \( G_\tau \).

Following §8 of Ch. XI of [KV], we recall how a representative of \( \pi_\lambda \) is constructed: Set

\[
\mathfrak{n}_\lambda = [\mathfrak{b}_\lambda, \mathfrak{b}_\lambda],
\]

\( S = \dim \mathfrak{n}_\lambda \cap \mathfrak{k} \), and \( \rho(\mathfrak{n}_\lambda) = \frac{1}{2} \sum_{\alpha \in \Delta_z^1} \alpha \). If \( \mu \in P \) we denote by \( \mathbb{C}_\mu \) the one dimensional representation of \( \mathfrak{b}_\lambda \) defined by setting \( (h + n) \cdot c = \lambda(h)c \).

Then, according to Theorem 11.178 of [KV], the \((\mathfrak{g}, K)\)-module of a representative of \( \pi_\lambda \) is

\[
V^\lambda_{(\mathfrak{g}, K)} = \left( u^R(\mathfrak{g}, K) \right)^{\mathfrak{g}_\lambda(T)} \left( \mathbb{C}_{\lambda + \rho(\mathfrak{n}_\lambda)} \right).
\]

Recall that \( u^R(\mathfrak{g}, K) \) is the derived functor of the Zuckerman functor, while \( \pi_{(\mathfrak{g}, K)} \) is the ordinary algebraic induction functor: \( \pi_{(\mathfrak{g}, K)}(Z) = \text{Hom}_{\mathfrak{g}_\lambda}(U(\mathfrak{g}, Z)) \).
In what follows we adopt the following notation: If \( a \) is any subspace of \( \mathfrak{g} \) that is stable for the action of \( \mathfrak{h} \), we let \( \Delta(a) \) be the set of roots occurring in \( a \), in other words \( \alpha \in \Delta(a) \) if and only if \( \mathfrak{g}_\alpha \subset a \).

Set \( \Pi_\lambda \) to be the set of simple roots for \( \Delta_\lambda^+ \) and \( \Pi_\lambda^+ \subset \Pi_\lambda \) to be the set of simple compact roots. Let

\[
q = m \oplus u
\]

be the corresponding parabolic; i.e., \( \Delta(m) \) is the subsystem of \( \Delta \) generated by \( \Pi_\lambda^+ \) and \( \Delta(u) = \Delta_\lambda^+ \setminus (\Delta(m) \cap \Delta_\lambda^+) \).

Set \( \mathfrak{b}_m = \mathfrak{b}_\lambda \cap m \), \( \Delta^+(m) = \Delta(m) \cap \Delta_\lambda^+ \), \( \rho_m = \frac{1}{2} \sum_{\alpha \in \Delta^+(m)} \alpha \), \( M \) the subgroup of \( K \) corresponding to \( m \cap \mathfrak{g}_\tau \), and

\[
(2) \quad \lambda_1 = \lambda + \rho(n_\lambda) - \rho_m.
\]

Then \( \lambda_1 \) is dominant and regular for \( \Delta^+(m) \) and, according to Corollary 4.160 of [KV],

\[
(3) \quad \begin{cases} 
0 & \text{if } q \neq \dim \mathfrak{b}_m \cap u \\
V_{(m,M)}^{\lambda_1} & \text{if } q = \dim \mathfrak{b}_m \cap u.
\end{cases}
\]

Let \( s = \dim (u \cap t) \). Combining (3) with Corollary 11.86 of [KV], we find that

\[
(4) \quad V_{(\mathfrak{g},K)}^\lambda = \left( u\mathcal{R}_{(\mathfrak{g},M)}^{(m,M)} \right)^s \left( V_{(m,M)}^{\lambda_1} \right).
\]

Fix \( z \in \widetilde{Z}_r^{\text{comp}} \). We are going to compute the \( K \)-spectrum of the compatible discrete series, that is the restriction to \( K \) of \( V_{(\mathfrak{g},K)}^\lambda \) when \( \lambda \in P_{\text{reg}}(z) \). Set \( S(u \cap p) \) to denote the symmetric algebra of \( u \cap p \).

**Lemma 2.7.** Suppose that \( z \in \widetilde{Z}_r^{\text{comp}} \) and \( \lambda \in P_{\text{reg}}(z) \). If \( \mu \) is \( \Delta^+(m) \)-dominant and there is \( n \in \mathbb{N} \) such that

\[
\dim \text{Hom}_M \left( V_{(m,M)}^{\mu_1 + \rho_m}, S^n(u \cap p) \otimes V_{(m,M)}^{\mu_1 + \rho_m} \right) \neq 0
\]

then \( \mu \) is dominant for \( \Delta_\lambda^+ \).

**Proof.** Since \( \mathfrak{b}_z \) is compatible with \( \mathfrak{t}_r \), then \( \mathfrak{b} \) is compatible with \( \mathfrak{t}_z \), hence, by Theorem 2.3, \( \mathfrak{t}_z = \mathfrak{t}_r' \) for some \( r' \in X \). As in the proof of Theorem 2.3, we can write

\[
\mathfrak{b} = \mathfrak{b}_0 + \mathfrak{b}_1 + [\mathfrak{b}_1, \mathfrak{b}_1],
\]

where \( \mathfrak{b}_0 = \mathfrak{h} \oplus \sum_{\alpha \in \Delta^+_{(r')}} \mathfrak{g}_\alpha \) and \( \mathfrak{b}_1 = \sum_{\alpha((r')=1} \mathfrak{g}_\alpha \). It is then clear that \( \mathfrak{m} \cap \mathfrak{b}_z \supset v_z^{-1} \mathfrak{b}_0 \) and \( v_z^{-1} \mathfrak{b}_1 = \mathfrak{u} \cap \mathfrak{p} \). This implies that \( v_z^{-1}[\mathfrak{b}_1, \mathfrak{b}_1] \subset \mathfrak{u} \cap \mathfrak{t} \).

Since \( \mathfrak{b}_z = v_z^{-1} \mathfrak{b}_0 + v_z^{-1} \mathfrak{b}_1 + v_z^{-1}[\mathfrak{b}_1, \mathfrak{b}_1] \), we deduce that \( \mathfrak{u} \cap \mathfrak{t} = v_z^{-1}[\mathfrak{b}_1, \mathfrak{b}_1] \) and \( \mathfrak{m} \cap \mathfrak{b}_z = v_z^{-1} \mathfrak{b}_0 \). Hence \( [\mathfrak{u} \cap \mathfrak{p}, \mathfrak{u} \cap \mathfrak{t}] = 0 \) and \( \mathfrak{u} \cap \mathfrak{t} \) is abelian. In particular, we have that \( (\alpha, \beta) \geq 0 \) whenever \( X_\alpha \in \mathfrak{u} \cap \mathfrak{p} \) and \( X_\beta \in \mathfrak{u} \cap \mathfrak{t} \).

Since \( \mu_1 \) is dominant for \( \Delta_\lambda^+ \), our result follows easily as in Lemma 3.1 of [EPWW]. \( \square \)
Remark. What we actually observe in Lemma 2.7 is the fact that the compatible discrete series of \([Ko2]\) are exactly the small discrete series of \([GW]\). In \([EPWW]\) it is also shown that Lemma 2.7 implies that one can compute the full \(K\)-spectrum of the discrete series. In the next result we prove this fact using directly Blattner’s formula.

**Theorem 2.8.** If \(z \in \tilde{Z}_\tau^{\text{cmpt}}\), \(\lambda \in P_{\text{reg}}(z)\) and \(\mu \) is a \(\Delta^+_\tau\)-dominant weight, then

\[
\dim \text{Hom}_K \left( V_{\mu + \rho_c}^{\lambda, K}, V_{\nu, K}^{\lambda} \right) = \sum_{n=0}^{+\infty} \dim \text{Hom}_M \left( V_{\mu + \rho_m}^{\lambda, (m,M)}, S^n(u \cap p) \otimes V_{\lambda, (m,M)}^{\mu + \rho_m} \right).
\]

**Proof.** Write

\[ m_\mu = \dim \text{Hom}_K \left( V_{\mu + \rho_c}^{\lambda, K}, V_{\nu, K}^{\lambda} \right). \]

As in (11.73) of \([KV]\), we write

\[
(u^{(g,K)})^j (Z) = \mathcal{R}^j (Z \otimes (\Lambda^{\text{top}} u)^*),
\]

so, by (4),

\[
\mathcal{R}^q \left( V_{(m,M)}^{\lambda_1} \otimes (\Lambda^{\text{top}} u)^* \right) = \begin{cases} 0 & \text{if } q \neq s \\ V_{(g,K)}^{\lambda} & \text{if } q = s. \end{cases}
\]

Applying Theorem 5.64 of \([KV]\) we find that

\[
m_\mu = \sum_{j=0}^{s} (-1)^{s-j} \sum_{n=0}^{\infty} \dim \text{Hom}_M \left( H_j \left( u \cap \mathfrak{k}, V_{\mu + \rho_c}^{\lambda, (\mathfrak{k},K)} \right), S^n(u \cap p) \otimes V_{(m,M)}^{\lambda_1} \right).
\]

By Corollary 3.8 of \([KV]\), as \(M\)-modules,

\[
H_j \left( u \cap \mathfrak{k}, V_{\mu + \rho_c}^{\lambda, (\mathfrak{k},K)} \right) \simeq H^{s-j} \left( u \cap \mathfrak{k}, V_{\mu + \rho_c}^{\lambda, (\mathfrak{k},K)} \otimes \Lambda^{\text{top}} u \cap \mathfrak{k} \right).
\]

Since the action of \(u \cap \mathfrak{k}\) on \(\Lambda^{\text{top}} u \cap \mathfrak{k}\) is trivial we can write

\[
H_j \left( u \cap \mathfrak{k}, V_{\mu + \rho_c}^{\lambda, (\mathfrak{k},K)} \right) \simeq H^{s-j} \left( u \cap \mathfrak{k}, V_{\mu + \rho_c}^{\lambda, (\mathfrak{k},K)} \otimes \Lambda^{\text{top}} u \cap \mathfrak{k} \right),
\]

hence, using the fact that \(V_{(m,M)}^{\lambda_1} \otimes (\Lambda^{\text{top}} u \cap \mathfrak{k})^* = V_{(m,M)}^{\mu + \rho_m}\), we find that

\[
m_\mu = \sum_{j=0}^{s} (-1)^{s-j} \sum_{n=0}^{\infty} \dim \text{Hom}_M \left( H^{s-j} \left( u \cap \mathfrak{k}, V_{\mu + \rho_c}^{\lambda, (\mathfrak{k},K)} \right), S^n(u \cap p) \otimes V_{(m,M)}^{\mu + \rho_m} \right).
\]
Set \( W^1 = \{ w \in W_\tau \mid N(w) \subset \Delta(u \cap \mathfrak{t}) \} \). Applying Kostant’s Theorem we find that

\[
m_{\mu} = \sum_{j=0}^{s} (-1)^{s-j} \sum_{n=0}^{\infty} \dim \text{Hom}_M \left( V^w(\mu+\rho_c-\rho_c+\rho_m, S^n(u \cap p) \otimes V^{\mu+\rho_m}_{(m,M)} \right).
\]

By Lemma 2.7, if

\[
\dim \text{Hom}_M \left( V^w(\mu+\rho_c-\rho_c+\rho_m, S^n(u \cap p) \otimes V^{\mu+\rho_m}_{(m,M)} \right) \neq 0,
\]

then \( w(\mu+\rho_c-\rho_c \) is dominant for \( \Delta^+_s \), hence \( w(\mu+\rho_c \) is dominant and regular. Since \( \mu+\rho_c \) is dominant and regular, we find that \( w = 1 \) and the above sum reduces to (5).

\[\square\]

Theorem 2.8 says that to compute the \( K \)-spectrum of \( V^\lambda_{(\mathfrak{g},K)} \) it is enough to compute the \( M \)-spectrum of \( S(u \cap p) \). The point is the fact that the \( M \)-spectrum of \( S(u \cap p) \) is somewhat easy to compute. Indeed suppose that \( \lambda \in P^{\text{reg}}(z) \) with \( z \in \mathbb{Z}^{\text{cmax}} \) and let \( \tau' \) be as in the proof of Lemma 2.7. We classify \( \tau' \) according to Proposition 3.5 below. If \( \tau' \) is of Type 4 then \( m = \mathfrak{t} \) and \( u = u \cap p \). This means that \( V^\lambda_{(\mathfrak{g},K)} \) is an holomorphic discrete series and the \( K \)-structure of \( S(u) \) is very well-known.

If \( \tau' \) is of Type 2, then \( V^\lambda_{(\mathfrak{g},K)} \) is a Borel-de Siebenthal discrete series and the \( M \)-spectrum of \( S(u \cap p) \) is given in [M-F] for the classical cases and \( F_4 \).

We now discuss the case of \( \tau' \) of Type 3, that is \( \tau' = \omega^\vee_i + \omega^\vee_j \) with \( i \neq j \) and \( i, j \in J \). Set

\[
\Delta_i = \{ \alpha \in \Delta^+ \mid (\omega^\vee_i, \alpha) = 1, (\omega^\vee_j, \alpha) = 0 \},
\]

\[
\Delta_j = \{ \alpha \in \Delta^+ \mid (\omega^\vee_j, \alpha) = 1, (\omega^\vee_i, \alpha) = 0 \},
\]

and \( u_i \) (resp. \( u_j \)) be the subspace of \( u \cap p \) such that \( \Delta(u_i) = v^{-1}_z \Delta_i \) (resp. \( \Delta(u_j) = v^{-1}_z \Delta_j \)). Then, as a \( M \)-module, \( u \cap p = u_i \oplus u_j \), hence

\[
S^n(u \cap p) = \sum_{h+k=n} S^h(u_i) \otimes S^k(u_j).
\]

Set \( u_i^- = \sum_{\alpha \in v^{-1}_z \Delta_i} \mathfrak{g}(-\alpha), \) \( u_j^- = \sum_{\alpha \in v^{-1}_z \Delta_j} \mathfrak{g}(-\alpha), \) and

\[
\mathfrak{g}_i = u_i^- \oplus \mathfrak{m} \oplus u_i \quad \mathfrak{g}_j = u_j^- \oplus \mathfrak{m} \oplus u_j.
\]

We notice that \( u_i \) and \( u_j \) are abelian in \( \mathfrak{g}_i \) and \( \mathfrak{g}_j \) respectively, thus we can apply [S1] to compute the \( M \)-spectrum of \( S(u_i) \) and \( S(u_j) \).
As an example we compute the $M$-spectrum of $S(u \cap p)$ for $G$ of type $E_6$ and $\tau' = \omega_1^\nu + \omega_6^\nu$. We use the notations of \cite{Bou}. In this case

$$\Delta_1 = \left\{ \frac{1}{2}(\epsilon_8 - \epsilon_7 - \epsilon_6 - \epsilon_5) + \frac{1}{2} \sum_{i=1}^{4} (-1)^{\nu(i)} \epsilon_i \middle| \sum_{i=1}^{4} \nu(i) \text{ odd} \right\},$$

$$\Delta_6 = \{ \pm \epsilon_i + \epsilon_5 \mid 1 \leq i \leq 4 \}. $$

If we set $\Delta_0 = \{ \pm \epsilon_i \pm \epsilon_j \mid 2 \leq i < j \leq 4 \}$ and $m' = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_0} \mathfrak{g}_\alpha$, then $m = v_z^{-1}m'$.

We set

$$\eta_1 = \frac{1}{2}(\epsilon_8 - \epsilon_7 - \epsilon_6 - \epsilon_5),$$

$$\eta_2 = \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4),$$

$$\eta_3 = \frac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4),$$

$$\eta_4 = \frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4),$$

$$\eta_5 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4),$$

then we see that $\{v_z^{-1}(\eta_i \pm \eta_j) \mid 1 \leq i < j \leq 5\}$ is the set of positive roots for $(\mathfrak{g}_1, \mathfrak{h})$ contained in $v_z^{-1}\Delta^+$ and that $v_z^{-1}(\eta_1 - \eta_2)$ is the unique simple noncompact root. Applying the results of \cite{S1}, we see that the highest weights of the $M$-types of $S(u_1)$ are given by $\mu(h_1, h_2) = v_z^{-1}(h_1(\eta_1 + \eta_2) + h_2(\eta_1 - \eta_2))$, with $h_1 \geq h_2$.

Similarly the positive system for $(\mathfrak{g}_6, \mathfrak{h})$ contained in $v_z^{-1}\Delta^+$ is $\{v_z^{-1}(\pm \epsilon_i + \epsilon_j) \mid 1 \leq i < j \leq 5\}$ and $v_z^{-1}(\epsilon_5 - \epsilon_4)$ is the unique noncompact simple root. It follows that the highest weights of the $M$-types of $S(u_6)$ are given by $\nu(k_1, k_2) = v_z^{-1}(k_1(\epsilon_4 + \epsilon_5) + k_2(\epsilon_5 - \epsilon_4))$, with $k_1 \geq k_2$. Putting all together we obtain that

$$S^\mu(u \cap p) = \sum_{\substack{h_1 + h_2 + k_1 + k_2 = n \\h_1 \geq h_2, \ k_1 \geq k_2}} F_M(\mu(h_1, h_2)) \otimes F_M(\nu(k_1, k_2)),$$

where $F_M(\mu) = V^{\mu + \rho_M}_{(m,M)}$ denotes the $M$-type of highest weight $\mu$.

3. Nilradical and special abelian ideals.

If $\tau \in \mathfrak{h}_R$ we set $q_{\tau} = m_\tau + n_\tau$, where $m_\tau = \mathfrak{h} \oplus \sum_{(\alpha, \tau) = 0} \mathfrak{g}_\alpha$ is a Levi subalgebra and $n_\tau = \sum_{\alpha \in \Delta^+; \langle \alpha, \tau \rangle > 0} \mathfrak{g}_\alpha$ is the nilradical of $q_\tau$.

**Remark.** If $i$ is abelian, then $n_{(i)}$ is an ad-nilpotent ideal of $\mathfrak{b}$; moreover $i \subseteq n_{(i)}$. The first assertion follows from the fact that $\langle i \rangle$ is a dominant
weight (cf. [Ko1, Prop. 6]). For the second claim, remark that, since \( \alpha + \beta \) is not a root for any \( \alpha, \beta \in \Phi_1 \), we have \( (\alpha, \beta) \geq 0 \); therefore, for \( \alpha \in \Phi_1 \), we have \( (\alpha, (i)) = (\alpha, \alpha) + \sum_{\beta \in \Phi_1 \setminus \{\alpha\}} (\alpha, \beta) > 0 \).

In the following lemma we recall two well-known facts about roots:

**Lemma 3.1.**

1. If \( \alpha, \beta \in \Delta^+ \) and \( \alpha < \beta \), then there exists \( \beta_1, \ldots, \beta_h \in \Delta^+ \) such that 
   \[
   \alpha + \sum_{i=1}^{j} \beta_i \in \Delta^+ \text{ for } j = 1, \ldots, h, \text{ and } \alpha + \sum_{i=1}^{h} \beta_i = \beta.
   \]
2. If \( \alpha + \beta, \gamma, \alpha + \beta + \gamma \in \Delta \), then at least one of \( \alpha + \gamma, \beta + \gamma \) is a root.

**Proof.** To prove (1), fix \( \beta \in \Delta^+ \). We will show that for each \( \alpha \in \Delta^+ \) such that \( \alpha < \beta \), there exist \( \beta_1, \ldots, \beta_h \in \Delta^+ \) such that 
   \[
   \beta = \alpha + \beta_1 + \cdots + \beta_h
   \]
and \( \alpha + \beta_1 + \cdots + \beta_i \in \Delta^+ \) for all \( i = 1, \ldots, h \). We shall prove the statement by induction on \( \text{ht}(\beta - \alpha) \). If \( \text{ht}(\beta - \alpha) = 1 \) the result is obvious. Suppose that \( \text{ht}(\beta - \alpha) > 1 \). If \( (\alpha, \beta) > 0 \), then \( \beta - \alpha \) is a root and we are done, so we assume \( (\alpha, \beta) \leq 0 \). By assumption, there exist \( \eta_1, \ldots, \eta_k \in \Delta^+ \) such that 
   \[
   \beta = \alpha + \eta_1 + \cdots + \eta_k
   \]
and we obtain that, for at least one of the \( \eta_i \), \( (\alpha, \eta_i) < 0 \). We may assume \( (\alpha, \eta_1) < 0 \), so that \( \alpha + \eta_1 \in \Delta^+ \). If \( \alpha + \eta_1 = \beta \) we are done, otherwise \( \alpha + \eta_1 < \beta \) and we can apply the induction hypothesis.

Assertion (2) is an immediate consequence of the Jacobi identity. \( \square \)

Recall that \( X = \overline{\mathbb{Q}}_2 \cap P^\vee \). Let \( \tau \in X \) and set \( i_{(\tau)} = \bigoplus_{\alpha \in \Delta^+_2} g_{\alpha} \). Then \( i_{(\tau)} \) is an abelian ideal of \( \mathfrak{b} \) that should not be confused with \( i_{\tau} \) in case \( \tau \in X \cap \overline{Z}_{ab} \).

**Lemma 3.2.** If \( \tau \in X \) and \( \Delta_2^2 \neq \emptyset \) then:

1. for all \( \alpha \in \Delta_1^1 \), there exists \( \beta \in \Delta_2^1 \) such that \( \alpha + \beta \in \Delta_2^2 \).
2. \( n_{i_{(\tau)}} = n_\tau = \sum_{\alpha \in \Delta_1^1 \cup \Delta_2^2} g_{\alpha} \).

**Proof.** (1) Assume \( \alpha \in \Delta_1^1 \). Consider \( \Delta(\alpha) = \{ \beta \in \Delta_2^2 \mid (\beta, \alpha) = 1 \} \). Since \( (\tau, \theta) = 2 \) we have \( \alpha \neq \theta \), hence \( \Delta(\alpha) \neq \emptyset \). Pick \( \beta \) minimal in \( \Delta(\alpha) \).

By Lemma 3.1 (1), we can find \( \beta_1, \ldots, \beta_k \in \Delta^+ \) such that \( \gamma_j = \alpha + \sum_{i=1}^{j} \beta_i \) is a root for \( j = 1, \ldots, k \) and \( \gamma_k = \beta \). Choose among such sequences \( \beta_1, \ldots, \beta_k \) one such that \( k \) is minimal. By the choice of \( \beta \), \( (\beta_k, \tau) = 1 \). We shall prove the claim by showing that \( k = 1 \). If \( k > 1 \), we have \( \gamma_k = \gamma_{k-2} + \beta_{k-1} + \beta_k \), where we set \( \gamma_0 = \alpha \). By Lemma 3.1 (2), either \( \gamma_{k-2} + \beta_k \in \Delta^+ \) (hence \( \gamma_{k-2} + \beta_k \in \Delta_2^2 \)), but this is not possible by the minimality of \( \beta \), or \( \beta_{k-1} + \beta_k \in \Delta^+ \), and indeed \( \beta_{k-1} + \beta_k \in \Delta_2^1 \). But in this latter case, setting
\( \beta'_{k-1} = \beta_{k-1} + \beta_k \), we see that \( \beta = \alpha + \beta_1 + \cdots + \beta_{k-2} + \beta'_{k-1} \) contradicting the minimality of \( k \).

(2) Set \( i = i(\tau) \). We first prove the inclusion \( \Delta^0_\tau \subset \Delta^0_{(i)} \). If \( \alpha \in \Delta^0_\tau \) and \( \beta \in \Delta^2_\tau \), then clearly \( s_\alpha(\beta) \in \Delta^2_\tau \). This implies that \( s_\alpha \left( \sum_{\beta \in \Delta^2_\tau} \beta \right) = \sum_{\beta \in \Delta^2_\tau} \beta \), hence that \( \left( \alpha, \sum_{\beta \in \Delta^2_\tau} \beta \right) = 0 \), or \( \langle \alpha, \langle i \rangle \rangle = 0 \). It remains to prove the reverse inclusion: We assume that \( \langle \alpha, \tau \rangle > 0 \) and prove that \( \langle \alpha, \langle i \rangle \rangle > 0 \). If \( \alpha \in \Delta^2_\tau \), then by definition \( \alpha \in \Phi_1 \), hence, by the above Remark, \( \langle \alpha, \langle i \rangle \rangle > 0 \). Thus it suffices to prove that if \( \alpha \in \Delta^2_1 \), then \( \langle \alpha, \langle i \rangle \rangle > 0 \). If \( \alpha \in \Delta^2_1 \) and \( \gamma \in \Delta^2_\tau \) then necessarily \( \langle \alpha, \gamma \rangle \geq 0 \): Otherwise \( \alpha + \gamma \) is a root and \( \langle \alpha + \gamma, \tau \rangle = 1 + 2 = 3 \), a contradiction. It remains to prove that for at least one \( \gamma \in \Delta^2_\tau \) we have \( \langle \alpha, \gamma \rangle > 0 \). By (1) we can find \( \beta \in \Delta^2_1 \) such that \( \alpha + \beta \in \Delta^2_\tau \); we shall prove that then \( \langle \alpha, \alpha + \beta \rangle > 0 \). Assume by contradiction \( \langle \alpha, \alpha + \beta \rangle \leq 0 \). Then in particular \( \langle \alpha, \beta \rangle < 0 \), so that \( \langle s_\alpha(\beta), \tau \rangle = \langle \beta - (\beta, \alpha')^2, \alpha, \tau \rangle \geq 2 \). It would follow that \( \langle \beta - (\beta, \alpha')^2, \alpha, \tau \rangle = 2 \), hence that \( \langle \beta, \alpha' \rangle = -1 \) and therefore that \( \langle \alpha', \alpha + \beta \rangle = 1 \). This is impossible since \( \langle \alpha', \alpha + \beta \rangle \) differs from \( \langle \alpha, \alpha + \beta \rangle \) by a positive factor. \( \square \)

**Theorem 3.3 ([Ko2, Theorem 4.4])**. Given \( i \in I_{ab} \), the following conditions are equivalent:

1. \( \lbrack n_{(i)}, n_{(i)} \rbrack \subset i = \text{cent } n_{(i)} \).
2. \( \Phi_1 = \Delta^2_\tau \) for some \( \tau \in X \).

**Proof.** (1 \( \Rightarrow \) 2) We already remarked, as a general fact, that \( i \subset n_{(i)} \). We assume that Condition (1) holds and for \( \alpha \in \Delta^+ \) we define

\[
\tau(\alpha) = \begin{cases} 
2 & \text{if } g_\alpha \subset i \\
1 & \text{if } g_\alpha \subset n_{(i)} \setminus i \\
0 & \text{if } g_\alpha \not\subset n_{(i)}. 
\end{cases}
\]

We shall prove that if \( \alpha, \beta, \alpha + \beta \in \Delta^+ \), then \( \tau(\alpha + \beta) = \tau(\alpha) + \tau(\beta) \). We first verify that if \( g_\alpha \not\subset n_{(i)} \), \( g_\beta \subset n_{(i)} \), and \( \alpha + \beta \in \Delta^+ \), then \( g_{\alpha + \beta} \subset n_{(i)} \), and \( g_{\alpha + \beta} \subset i \) if and only if \( g_\beta \subset i \). The first condition is immediate since \( n_{(i)} \) is an ideal of \( q_{(i)} \). Since \( i = \text{cent } n_{(i)} \), we have in particular that \( i \) is an ideal of \( q_{(i)} \), too; hence, if \( g_\beta \subset i \), \( g_{\alpha + \beta} \subset i \), too. Similarly, if \( g_\beta \subset n_{(i)} \setminus i \), then \( g_{\alpha + \beta} \subset n_{(i)} \setminus i \), otherwise we should obtain that \( g_\beta = [g_{-\alpha}, g_{\alpha + \beta}] \subset i \). The next case to consider is when \( g_\alpha, g_\beta \) are both contained in \( n_{(i)} \setminus i \); if \( \alpha + \beta \in \Delta \), the first equality in (1) implies \( g_{\alpha + \beta} \subset i \), as desired. Finally, if \( g_\alpha \subset i = \text{cent } n_{(i)} \) and \( g_\beta \subset n_{(i)} \) we have that \( \alpha + \beta \not\in \Delta \), so we can conclude that \( \tau \) is additive and less or equal than 2 on \( \Delta^+ \). This fact implies that \( \tau \) can be extended to a linear functional on \( h_{kR}^* \) which clearly corresponds to
an element in $X$; if we still denote by $\tau$ this element, we obtain by definition that $\Phi_i = \Delta^2_\tau$.

(2 $\Rightarrow$ 1) By Lemma 3.2, (2) we have $n_{(i)} = \bigoplus_{\alpha \in \Delta^1_\tau \cup \Delta^2_\tau} g_\alpha$. Hence it is clear that $[n_{(i)}, n_{(i)}] \subset i$; it is also clear that $i \subset \text{cent } n_{(i)}$. Item (1) of Lemma 3.2 shows that no element in $\bigoplus_{\alpha \in \Delta^1_\tau} g_\alpha$ belongs to $\text{cent } n_{(i)}$, so indeed $i = \text{cent } n_{(i)}$. □

**Definition.** An abelian ideal of $b$ is said special if it satisfies one of the conditions of Theorem 3.3. An abelian ideal is nilradical if it is the nilradical of a parabolic subalgebra of $g$.

Now we determine the structure of both nilradical and special abelian ideals. In particular we provide a proof of Theorem 4.9 of [Ko2].

Set $M = \{\omega \lor i | i \in J\} \cup \{0\}$. It is well-known that $M$ is a set of representatives for $P^\lor/Q^\lor$.

**Proposition 3.4.** An abelian ideal $i$ of $b$ is nilradical if and only if there exists $\omega \in M$ such that $i = n_\omega$. In particular nilradical abelian ideals are in bijection with $\text{Cent } (G)$.

**Proof.** The only nontrivial thing to prove is the fact that

$$0 \neq n_\tau = \bigoplus_{\alpha \in \Delta^+ : (\alpha, \tau) > 0} g_\alpha$$

is abelian only if we can choose $\tau = \omega_i^\lor$ for some $i \in J$.

Let $\alpha \in \Delta^+$ be such that $(\alpha, \tau) > 0$ and such that $\alpha = \alpha' + \alpha''$ with $\alpha', \alpha'' \in \Delta^+$. If $n_\tau$ is abelian, $(\alpha', \tau)$ and $(\alpha'', \tau)$ cannot be both nonzero. Suppose that $(\alpha', \tau) = 0$: Then we have that $(\alpha'', \tau) > 0$. Repeating this argument we obtain that $\alpha = \alpha_i + \gamma$ where $\gamma$ is a sum of positive roots $\beta$ such that $(\beta, \tau) = 0$ and $\alpha_i$ is a simple root such that $(\alpha_i, \tau) > 0$. Apply now this observation to the highest root $\theta$: Then $\theta = \alpha_i + \gamma$. This implies that $\tau = k\omega_i^\lor$ with $i \in J$, hence $n_\tau = n_{\omega_i^\lor}$. □

Recall that $\theta = \sum_{i=1}^n m_i \alpha_i$.

**Proposition 3.5.** Suppose $\tau \in X$. Then we have the following five possibilities:

1. $\tau = 2\omega_i^\lor$, $i \in J$;
2. $\tau = \omega_i^\lor$, $m_i = 2$;
3. $\tau = \omega_i^\lor + \omega_j^\lor$, $i, j \in J$;
4. $\tau = \omega_i^\lor$, $i \in J$;
5. $\tau = 0$.

Then $\Delta^2_\tau \neq \emptyset$, whence $i(\tau) \neq 0$, exactly in Cases (1), (2), (3).
In Case (1) the special abelian ideal \(i_{(\tau)}\) is nilradical (and viceversa); in Cases (2) and (3) \(i_{(\tau)}\) is not nilradical and relations \(i_{(\tau)} \subset n_{\tau}, i_{(\tau)} = [n_{\tau}, n_{\tau}]\) hold.

Proof. Since \(\tau \in \mathcal{C}_2\) we have \(\tau = \sum_{i=1}^{n} \varepsilon_i o_i\) with \(\varepsilon_i \geq 0\) and \(0 < \sum_{i=1}^{n} \varepsilon_i \leq 2\) where \(o_i = \omega_i^{\gamma}/m_i\), \(i = 1, \ldots, n\). Since, moreover, \(\tau \in P^\vee = \bigoplus_{i=1}^{n} \mathbb{Z}\omega_i^{\gamma}\), the only possibilities for \(\tau\) are the five listed above. It is clear that \(\Delta_{\tau}^2 = \emptyset\) if and only if we are in Cases (4) or (5). Suppose that \(\tau = 2\omega_i^{\gamma}, i \in J\): Then \(\Delta_{\tau}^1 = \emptyset\) and Part (2) of Lemma 3.2 implies that \(i_{(\tau)} = n_{(i_{(\tau)})}\), hence \(i_{(\tau)}\) is nilradical. It is clear that in Cases (2) and (3) \(\Delta_{\tau}^1 \neq \emptyset\), whence, by Lemma 3.2, (2), \(i_{(\tau)} \subset n_{(i_{(\tau)})}\); moreover, by Theorem 3.3, \(i_{(\tau)} \supset [n_{(i_{(\tau)})}, n_{(i_{(\tau)})}]\). So we are left with proving the reverse inclusion. It suffices to prove that any \(\beta \in \Delta_{\tau}^2\) splits as the sum of two roots in \(\Delta_{\tau}^1\). We note that in Cases (2) and (3) if \(\beta \in \Delta_{\tau}^2\) then \(\beta\) is not simple, so \(\beta = \beta_1 + \beta_2\) with \(\beta_1, \beta_2 \in \Delta_{\tau}^1\), \(\beta_1, \beta_2 < \beta\). If \(\beta_1\) and \(\beta_2\) belong to \(\Delta_{\tau}^1\) we are done; otherwise one of them, say for example \(\beta_1\), belongs to \(\Delta_{\tau}^2\). By induction on the height of \(\beta\) we can assume that there exist \(\gamma_1, \gamma_2 \in \Delta_{\tau}^1\) such that \(\beta_1 = \gamma_1 + \gamma_2\). Then by Lemma 3.1, (2) at least one among \(\gamma_1 + \beta_2, \gamma_2 + \beta_2\) is a root which clearly belongs to \(\Delta_{\tau}^1\) and we get our claim. From the previous proposition it follows that \(i_{(\tau)}\) is not nilradical.

Putting together the results of Lemma 3.2 and Theorem 3.3 we obtain the following result:

Corollary 3.6. The special abelian ideals of \(b\) are exactly the \(i_{(\tau)}\), with \(\tau \in X\). For any \(\tau \in X\) we have \([n_{\tau}, n_{\tau}] \subset i_{(\tau)} = \text{cent } n_{\tau}\).

Remark. Lemma 3.2 allows easily to classify and enumerate the special abelian ideals by looking at the coefficients of the highest root. Indeed, if \(i\) is a nonzero special abelian ideal and \(i = i_{(\tau)} = i_{(\tau')}\), then Part (2) of Lemma 3.2 says that \(n_{\tau} = n_{\tau'}\), so \(\Delta_{\tau}^0 = \Delta_{\tau'}^0\). Since we are assuming that \(\Delta_{\tau}^2 = \Delta_{\tau'}^2\), we can conclude that \(\Delta_{\tau}^1 = \Delta_{\tau'}^1\) also. This implies that \(\tau = \tau'\). With reference to the five possibilities for \(\tau\) listed Proposition 3.5, the list of the \(\tau \in X\) such that \(i_{(\tau)} \neq 0\) is given in Table 1; we label Dynkin diagrams as in \([\text{Bou}]\).

For \(\alpha \in \Delta^+\) we set \(V_{\alpha} = \{\beta \in \Delta^+ \mid \beta \geq \alpha\}\). Let \(i_{(\tau)}\), \(\tau \in X\), be a special abelian ideal. Then if \(\tau = 2\omega_i^{\gamma}, i \in J\), or \(\tau = \omega_j^{\gamma}, m_i = 2\), we have \(n_{\tau} = \bigoplus_{\beta \in V_{\alpha}} g_{\beta}\), whereas if \(\tau = \omega_i^{\gamma} + \omega_j^{\gamma}, i, j \in J\) then \(n_{\tau} = \bigoplus_{\beta \in V_{\alpha} \cup V_{\beta}} g_{\beta}\).
Table 1.

<table>
<thead>
<tr>
<th>Δ</th>
<th>τ of Type 1</th>
<th>τ of Type 2</th>
<th>τ of Type 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$2\omega_1^\lor, \ldots, 2\omega_n^\lor$</td>
<td>$\omega_i^\lor + \omega_j^\lor$, $1 \leq i &lt; j \leq n$</td>
<td></td>
</tr>
<tr>
<td>$B_n$</td>
<td>$2\omega_1^\lor$</td>
<td>$\omega_2^\lor, \ldots, \omega_n^\lor$</td>
<td></td>
</tr>
<tr>
<td>$C_n$</td>
<td>$2\omega_n^\lor$</td>
<td>$\omega_1^\lor, \ldots, \omega_{n-1}^\lor$</td>
<td></td>
</tr>
<tr>
<td>$D_n$</td>
<td>$2\omega_1^\lor, 2\omega_{n-1}^\lor, 2\omega_n^\lor$</td>
<td>$\omega_1^\lor + \omega_{n-1}^\lor, \omega_1^\lor + \omega_n^\lor, \omega_{n-1}^\lor + \omega_n^\lor$</td>
<td></td>
</tr>
<tr>
<td>$E_6$</td>
<td>$2\omega_1^\lor, 2\omega_6^\lor$</td>
<td>$\omega_2^\lor, \omega_3^\lor, \omega_5^\lor$</td>
<td>$\omega_1^\lor + \omega_6^\lor$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$2\omega_7^\lor$</td>
<td>$\omega_1^\lor, \omega_2^\lor, \omega_6^\lor$</td>
<td></td>
</tr>
<tr>
<td>$E_8$</td>
<td></td>
<td>$\omega_1^\lor, \omega_8^\lor$</td>
<td></td>
</tr>
<tr>
<td>$F_4$</td>
<td></td>
<td>$\omega_1^\lor, \omega_4^\lor$</td>
<td></td>
</tr>
<tr>
<td>$G_2$</td>
<td></td>
<td>$\omega_2^\lor$</td>
<td></td>
</tr>
</tbody>
</table>

We want to restate the previous results in terms of the machinery introduced in Section 1. If $\tau \in X$, set $\mathfrak{k} = \mathfrak{k}_\tau$ and $\mathfrak{p} = \mathfrak{p}_\tau$. Set also $\mathfrak{b}_\tau = \mathfrak{b} \cap \mathfrak{k}$, $\mathfrak{b}_\mathfrak{p} = \mathfrak{b} \cap \mathfrak{p}$ and $i(\Phi) = \bigoplus_{\alpha \in \Phi} g_\alpha$, for $\Phi \subset \Delta^+$. Recall that $\Phi_z^1 = v_z^{-1}(\Delta_z^1)$, $\Phi_z^2 = -v_z^{-1}(\Delta_z^2)$, so that $i_z = i(\Phi_z^1) \oplus i(\Phi_z^2)$. Moreover we have $\mathfrak{b}_\mathfrak{p} = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in \Delta^\lor \cup \Delta^\lor_z} g_\alpha \right)$, $\mathfrak{b}_\mathfrak{p} = \bigoplus_{\alpha \in \Delta^\lor} g_\alpha$ and $\Delta^2_z = -v_z^{-1}(\Delta_z^2)$, $\Delta^0_z = v_z^{-1}(\Delta_z^0)$, $\Delta^1_z = v_z^{-1}(\Delta_z^1) \cup -v_z^{-1}(\Delta_z^1)$.

With this notation we have (see [Ko2, Prop. 4.7, Theorem 4.9]):

**Proposition 3.7.** Let $z \in \tilde{Z}_{ab}$ and set $\tau = \text{dom}(z)$.

1. The following relations are equivalent:
   i) $\tau = 2\omega$ for some $\omega \in \mathcal{M}$;
   ii) $z = -2\omega$ for some $\omega \in \mathcal{M}$;
   iii) $n_\tau = 1$;
   iv) $\mathfrak{k}_\tau = \mathfrak{g}$.
2. $i(\Phi_z^2)$ is a special abelian ideal.
3. $i(\Phi_z^2)$ is nilradical if and only if $\tau = 2\omega$ for some $\omega \in \mathcal{M}$.
4. If $i(\Phi_z^2)$ is not nilradical then $i(\Phi_z^2) = [\mathfrak{b}_\mathfrak{p}, \mathfrak{b}_\mathfrak{p}]$.
5. $i(\Phi_z^1) = \mathfrak{b}_z \cap \mathfrak{b}_\mathfrak{p}$.
6. $i(\Phi_z^1) = i_z \cap \mathfrak{p}$ and $i(\Phi_z^2) = i_z \cap \mathfrak{k}$. 
Proof. \( i \Rightarrow ii \): If \( \alpha \in \Delta^+ \) then \( \alpha(z) = v_{z}^{-1}(\alpha)(\tau) \in \{0, -2\} \), hence \(-z/2\) is minuscule.

\( ii \Rightarrow iii \): By the proof of 2.4, \( \Delta_\tau = \Delta \), hence \( n_\tau = 1 \).

\( iii \Rightarrow iv \): \( W_\tau = W \) implies that \( \Delta = \Delta_\tau \) and in turn that \( \tau_\tau = \mathfrak{g} \).

\( iv \Rightarrow i \): If \( \Delta_\tau = \Delta \) then, if \( \alpha \in \Delta^+ \), \( (\alpha, \tau) \in \{0, 2\} \) hence \( \tau/2 \) is minuscule. At this point Part (1) is completely proved. The other parts follow immediately combining the relations listed just before the statement of this proposition with Lemma 3.2 and Proposition 3.4. \( \square \)

**Proposition 3.8.** Let \( \tau \in X \), \( s \) be an \( \mathfrak{h} \)-submodule of \( \mathfrak{b}_0 \), \( \Delta(s) = \{ \alpha \mid \mathfrak{g}_\alpha \subset s \} \), and set \( i = i(\Delta^2) \oplus s \). Then the following facts are equivalent:

1. \( s \) is a \( \mathfrak{b}_1 \)-submodule of \( \mathfrak{b}_0 \);
2. \( i \) is an ad-nilpotent ideal of \( \mathfrak{b} \) included in \( n_\tau \);
3. \( i \) is an ad-nilpotent ideal of \( \mathfrak{b} \) and \( (\beta, \tau) > 0 \) for all \( \beta \) in \( \Delta(s) \).

The same equivalences hold if we replace “ad-nilpotent” with “abelian” in (2) and (3) and consider abelian submodules in (1).

Proof. The equivalence of (2) and (3) is immediate from the definitions. We prove the equivalence of (1) and (2). By the definition of \( n_\tau \) and \( \mathfrak{b}_0 \) we have that in any case \( i \subset n_\tau \). By assumption \( [\mathfrak{h}, s] \subset s \); since \( s \subset i(\Delta^1) \), we have \([i(\Delta^2), s] = 0 \). Moreover, \([i(\Delta^1), s] \subset i(\Delta^2) \), thus we obtain that \( i \) is an (ad-nilpotent) ideal of \( \mathfrak{b} \) if and only if \([i(\Delta^2), s] \subset s \). But by the definition of \( \mathfrak{b}_0 \), we also obtain that \( s \) is a \( \mathfrak{b}_1 \)-module if and only if \([i(\Delta^2), s] \subset s \), so we get our claim. It is clear that all the above arguments still work when restricting to abelian objects. \( \square \)

**Proposition 3.9.** Let \( i \in I \), \( \tau \in X \), and assume \( i = i(\Delta^2) \oplus s \), with \( s \subset i(\Delta^1) \). Then there exists \( z \in \bar{Z} \) such that \( i = i_z \), \( \tau = \text{dom}(z) \), and \( s = i(v_{z}^{-1}(\Delta^1)) \).

Conversely, if \( i = i_z \) and \( \text{dom}(z) \in X \), we have \( i = i(\Delta^2_{\text{dom}(z)}) \oplus s \), with \( s \subset i(\Delta^1_{\text{dom}(z)}) \).

Proof. Set \( \Phi = \Delta(s) \), \( \Phi^2 = (\Phi + \Phi) \cap \Delta^+ \), and and \( \Phi' = (\Delta^2 \setminus \Phi^2) \cup (\Delta^1 \setminus \Phi) \).

We first prove that there exists \( v \in W \) such that \( \Phi' = N(v) \). For this it suffices to prove that \( \Phi' \) and its complement in \( \Delta^+ \) are closed. Since \( \tau \in \mathcal{O}_2 \cap P^v \), to prove that \( \Phi' \) is closed it suffices to show that if \( \alpha, \beta \in \Delta^1 \) and \( \alpha + \beta \in \Phi^2 \) then either \( \alpha \) or \( \beta \) belongs to \( \Phi \). Set \( \gamma = \alpha + \beta \); since \( \gamma \in \Phi^2 \), there exist \( \xi, \eta \in \Phi \) such that \( \gamma = \xi + \eta \). Now from the expansion \[ 0 < (\gamma, \gamma) = (\alpha, \xi) + (\beta, \xi) + (\alpha, \eta) + (\beta, \eta) \]
we deduce that one of the summands in the right-hand side of the previous relation is positive. Since the difference of two roots having positive scalar product is a root, and since \( \eta - \beta = \alpha - \xi \) and \( \eta - \alpha = \beta - \xi \), we have that either \( \eta - \beta \in \Delta \) or \( \eta - \alpha \in \Delta \). It suffices to consider the case \( \eta - \alpha \in \Delta \).
Suppose $\alpha - \eta \in \Delta^+$ and remark that $\alpha - \eta \in \Delta^+_0$; since $\mathfrak{s}$ is a $\mathfrak{b}_\mathfrak{f}$-module, we have $\alpha = (\alpha - \eta) + \eta \in \Phi$ as desired. If instead $\eta - \alpha \in \Delta^+$, we note that $\eta - \alpha = \beta - \xi \in \Delta$ and as above we deduce that $\beta = (\beta - \xi) + \xi \in \Phi$.

Next we consider $\Delta^+ \setminus \Phi' = \Delta^+_0 \cup \Phi \cup \Phi^2$. Obviously, $\Delta^+_0$ and $\Phi \cup \Phi^2$ are closed. If $\alpha \in \Delta^+_0, \beta \in \Phi$ and $\alpha + \beta \in \Delta$ we have that $\alpha + \beta \in \Phi$ since $\mathfrak{s}$ is a $\mathfrak{b}_\mathfrak{f}$-module. Finally, if $\alpha \in \Delta^+_0, \beta \in \Phi^2, \beta = \xi + \eta \in \Phi$, and $\alpha + \beta = \alpha + \xi + \eta \in \Delta^+$, we have by Lemma 3.1, (2), that either $\alpha + \xi$ or $\alpha + \eta$ is a root and hence belongs to $\Phi$, so that $\alpha + \beta \in \Phi^2$.

So there exists $v \in W$ such that $\Phi' = N(v)$. Set $z = v^{-1}(\tau)$. Then $N(v_z) = \{\alpha \in \Delta^+ \mid (\alpha, v^{-1}(\tau)) < 0\} = \{\alpha \in \Delta^+ \mid (v(\alpha), \tau) < 0\} = \{\alpha \in \Delta^+ \mid v(\alpha) < 0\} = N(v^{-1})$, hence $v_z = v^{-1}$ and therefore $\text{dom}(z) = \tau$.

In order to conclude we have to prove that $i = i_z$. We have $\widetilde{F}(z) = t_\tau v$, hence, by Lemma C, $i_z = i(\Delta^+_2) \oplus i(\Delta^+_1 \setminus N(v_z))$. Since $\Phi(\mathfrak{s}) \subset \Delta^+_1$, and $N(v) \cap \Delta^+_1 = \Delta^+_1 \setminus N(\mathfrak{s})$, we have $\Delta^+_1 \setminus N(v) = \Delta(\mathfrak{s})$, whence $i = i_z$. Finally by a direct check we obtain that $\Delta^+_1 \setminus N(v) = v(\Delta^+_1)$, so $\mathfrak{s} = i(v^{-1}_z(\Delta^+_1))$.

The converse statement follows directly from Lemma C.

Recall that $\text{dom} : \mathfrak{h}_\mathfrak{R} \rightarrow C_\infty$ is defined by $z \mapsto v^{-1}_z(\Delta)$. As already observed $\text{dom}(\tilde{Z}_{ab}) = X = C_2 \cap P^\vee$, whereas, by Proposition 1.3, we have $\text{dom}(\tilde{Z}) \subset C_h \cap P^\vee$. We recall that for $\tau \in X$ we set $\tilde{Z}_\tau = \text{dom}^{-1}(\tau) \cap \tilde{Z}_{ab}$; we also set $\tilde{Z}_\tau = \text{dom}^{-1}(\tau) \cap \tilde{Z}$. Fix $\tau \in X$ and let $\mathcal{S}_\tau$ denote the set of $\mathfrak{b}_\mathfrak{f}$-submodules of $\mathfrak{b}_\mathfrak{p}$.

**Proposition 3.10.** The map $z \mapsto i(v^{-1}_z(\Delta^+_1))$ establishes a bijection between $\tilde{Z}_\tau$ and $\mathcal{S}_\tau$. Moreover, this map restricts to a bijection between $\tilde{Z}_\tau$ and the abelian subspaces in $\mathcal{S}_\tau$. In particular, the number of abelian $\mathfrak{b}_\mathfrak{f}$-submodules of $\mathfrak{b}_\mathfrak{p}$ is $n_\tau$.

**Proof.** Since $\tau + Q^\vee = \tau + Q^\vee$ for any $z \in \tilde{Z}_\tau$, the map $z \mapsto i_z$ is injective on $\tilde{Z}_\tau$ (see Prop. 1.4, (e)). For any $z \in \tilde{Z}_\tau$ we have $\tilde{F}(z) = t_\tau v^{-1}_z$, hence, by Lemma C, $\Phi_{i_z} = \Delta^2 \cup (\Delta^+_1 \setminus N(v^{-1}))$. As in the proof of 3.9, we have $\Delta^+_1 \setminus N(v^{-1}) = v^{-1}_z(\Delta^+_1)$, so that $i_z = i(\Delta^+_2) \oplus i(v^{-1}_z(\Delta^+_1))$. In particular $v^{-1}_z(\Delta^+_1)$ determines $i_z$, and therefore $z \mapsto i(v^{-1}_z(\Delta^+_1))$ is one to one. Moreover, by Proposition 3.8, we obtain that $i(v^{-1}_z(\Delta^+_1))$ is a $\mathfrak{b}_\mathfrak{f}$-submodule of $\mathfrak{b}_\mathfrak{p}$, so $z \mapsto i(v^{-1}_z(\Delta^+_1))$ is a one to one map from $\tilde{Z}_\tau$ to $\mathcal{S}_\tau$. It remains to prove that this map is onto. Take any $\mathfrak{z} \in \mathcal{S}_\tau$; then, by Proposition 3.8, $\Delta^2 \oplus \mathfrak{z}$ is an ad-nilpotent ideal of $\mathfrak{b}$ included in $n_\tau$. Thus by Lemma 3.9 there exists $z \in \tilde{Z}$ such that dom $(z) = \tau$ and $\mathfrak{z} = i(v^{-1}_z(\Delta^+_1))$. So we have proved that $z \mapsto i(v^{-1}_z(\Delta^+_1))$ is a bijection between $\tilde{Z}_\tau$ and $\mathcal{S}_\tau$. The final assertions are clear.

Recall now the Cent $(G)$ action on $\tilde{Z}$ introduced in Proposition 1.1, explicitly given by the action of the subgroup $\Sigma$ of $W$. Let $\Sigma \cdot z$ denote
the orbit of \( z \in \overline{Z} \) under this action. As before we denote by \( M \) the set \\
\( \{ \omega^\vee \mid i \in J \} \cup \{ 0 \} \) of representatives for \( P^\vee / Q^\vee \) and for any \( \omega \in M \) we set \\
\( V_\omega = \{ \alpha \in \Delta^+ \mid (\alpha, \omega) > 0 \} \). So \( V_\omega^\vee = V_\alpha \) and \( V_0 = \emptyset \). For \( i \in I_{ab} \) we set \\
\( C_i = \{ \omega \in M \mid \Phi_i \subset V_\omega \}, \)
\( C'_i = \text{dom} (\Sigma \cdot z) \cap M, \)

where \( z \) is any element of \( \overline{Z}_{ab} \) such that \( i_z = i \).

**Corollary 3.11 ([Ko2, Theorem 5.1]).** Let \( i \in I_{ab} \) and let \( z \in \overline{Z}_{ab} \) be such 
that \( i = i_z \). Then there are exactly \( |\text{Cent} (G)| - |C_i| \) decompositions 
\( i = i(\Delta^2) + s, \)

with \( i(\Delta^2) \) nonzero special abelian ideal and \( s \subset i(\Delta^1). \)

**Proof.** By Proposition 1.4, for any \( z' \in \overline{Z} \) we have \( i_{z'} = i_z \) if and only if 
\( z' \in \Sigma \cdot z \). Hence, by Proposition 3.9, we have \( i = i(\Delta^2) + s, \) with \( \tau \in X \) and \( s \subset \Delta^1 \), if and only if \( \tau = \text{dom} (z') \) for some \( z' \in \Sigma \cdot z \). By the remark 
following 3.6, the map \( \tau \to \Delta^2 \) is injective. Therefore we have only to prove 
that \( \Delta^2_{\text{dom}(z')} \neq \emptyset \) for exactly \( |\text{Cent} (G)| - |C_i| \) elements \( z' \in \Sigma \cdot z \). We first 
notice that the restriction of dom to a \( \Sigma \)-orbit is injective. This follows 
from the fact that distinct elements in \( \Sigma \cdot z \) are distinct mod-\( Q^\vee \), as clear 
from the definition of \( \Sigma \), while \( z' + Q^\vee = \text{dom} (z') + Q^\vee \) for each \( z' \in \Sigma \cdot z \). 
In particular we have \( |\text{dom} (\Sigma \cdot z)| = |\text{Cent} (G)| \). Now it suffices to prove 
that \( \Delta^2_{\text{dom}(z')} = \emptyset \) if and only if \( \tau \in \mathcal{M} \), therefore if we prove that 
\( C'_i = C_i \) we are done. Assume first \( \omega \in C'_i \). Then \( \omega = \text{dom} (z') \) for some \( z' \) 
such that \( i = i_{z'} \). Since \( \tilde{F}(z') = t_\omega v_\omega^{-1} \) using Lemma C we obtain \( \Phi_i \subset V_\omega \). 
Conversely, assume \( \omega \in C_i \). Then, clearly, \( \Delta^2 = \emptyset \) and \( i \subset i(\Delta^1) = i(V_\omega). \) 
Therefore, by Proposition 3.9, there exists \( z' \in \Sigma \cdot z \) such that \( \text{dom} (z') = \omega \). 
This concludes the proof. \( \square \)

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THE MASS OF ASYMPTOTICALLY HYPERBOLIC Riemannian Manifolds

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We present a set of global invariants, called “mass integrals”, which can be defined for a large class of asymptotically hyperbolic Riemannian manifolds. When the “boundary at infinity” has spherical topology one single invariant is obtained, called the mass; we show positivity thereof. We apply the definition to conformally compactifiable manifolds, and show that the mass is completion-independent. We also prove the result, closely related to the problem at hand, that conformal completions of conformally compactifiable manifolds are unique.

Introduction.

Let \((M, b)\) be a smooth \(n\)-dimensional Riemannian manifold, \(n \geq 2\) and let \(N_b\) denote the set of functions \(V\) on \(M\) such that
\[
\Delta_b V + \lambda V = 0, \\
\hat{D}_i \hat{D}_j V = V (\text{Ric}(b)_{ij} - \lambda b_{ij}),
\]
for some constant \(\lambda < 0\). Here \(\text{Ric}(b)_{ij}\) denotes the Ricci tensor of the metric \(b\), \(\hat{D}\) the Levi-Civita connection of \(b\), and \(\Delta_b := \hat{D}^k \hat{D}_k\) is the Laplacian of \(b\). Rescaling \(b\) if necessary, we can without loss of generality assume that
\[
\lambda = -n \quad \text{so that} \quad R_b = b^{ij} \text{Ric}(b)_{ij} = -n(n - 1).
\]
\((M, b)\) will be called static if
\[N_b \neq \{0\}.
\]
This terminology is motivated by the fact that for every \(V \in N_b\) the Lorentzian metrics defined on \(\mathbb{R} \times (M \setminus \{V = 0\})\) by the formula
\[
\gamma = -V^2 dt^2 + b
\]
are static solution of the Einstein equations, \(\text{Ric}(\gamma) = \lambda \gamma\) (and the Riemannian metrics \(V^2 dt^2 + b\) are actually Einstein as well).

The object of this work is to present a set of global invariants, constructed using \(N_b\), for metrics which are asymptotic to a class of static metrics. The model case of interest is the hyperbolic metric, which is static in our sense: The set \(N_b\) is then linearly isomorphic to \(\mathbb{R}^{n+1}\), however, other classes of
metrics will also be allowed in our framework. The invariants introduced here stem from a Hamiltonian analysis of general relativity, and part of the work here is a transcription to a Riemannian setting of the Lorentzian analysis in [16]. Related definitions have been given recently by Wang [39] (with a spherical conformal infinity) and Zhang [41] (in dimension 3, again with spherical asymptotic geometry), under considerably more restrictive asymptotic and global conditions. Wang’s definition of mass for asymptotically hyperbolic manifolds [39] coincides with ours, when his much stronger asymptotic decay conditions are satisfied (note, however, that his proof of the geometric character of the mass is incomplete, as he ignores the possibility of existence of inequivalent conformal completions). Moreover, the hypothesis of [39] and [41] that \((M, g)\) has compact interior is replaced by that of completeness; this strengthening of the positivity theorem is essential when the associated Lorentzian space-time contains “degenerate” event horizons.

This work is organised as follows: In Section 1 we review a few static metrics, and discuss their properties relevant to the work here. In Section 2 we define the “mass integrals”, we make precise the classes of metrics considered, and we show how to obtain global invariants out of the mass integrals. We show that our boundary conditions are sharp, in the sense that their weakening leads to mass integrals which do not provide geometric invariants. It is conceivable that the strengthening of some of our conditions could allow the weakening of some other ones, leading to geometric invariants for other classes of manifolds; we expect that such a mechanism occurs for the Trautman-Bondi mass of asymptotically hyperboloidal manifolds. In several cases of interest one obtains a single invariant, which we call the mass of \((M, g)\), but more invariants are possible depending upon the topology of the “boundary at infinity” \(\partial M\) of \(M\) — this is determined by the number of invariants of the action of the group of isometries of \(b\) on \(N_b\), see Section 3 for details. Regardless of this issue, we emphasise that we only consider the “global charges” of [16] related to Killing vectors which are normal to the level sets of \(t\) in the space-time metric (0.3): The remaining space-time invariants of [16], associated, \(e.g.,\) to “rotations” of \(M\), involve the extrinsic curvature of the initial data hypersurface and are of no concern in the purely Riemannian setting here. In Section 4 we prove positivity of the mass so obtained for metrics asymptotic to the hyperbolic one. As a corollary of the positivity results we obtain a new uniqueness result for anti-de Sitter space-time, Theorem 4.3. We also consider there the case of manifolds with a compact inner boundary. In Section 5 we show how to define mass for a class of conformally compactifiable manifolds. The question of geometric invariance of the mass is then closely related to the question of uniqueness of conformal completions; in Section 6 we prove that such completions are indeed unique.
1. The reference metrics.

Throughout this paper we will assume that the manifold contains a region $M_{\text{ext}} \subset M$ together with a diffeomorphism
\[(1.1) \quad \Phi^{-1} : M_{\text{ext}} \to [R, \infty) \times N,\]
where $N$ is a compact boundaryless manifold, such that the reference metric $b$ on $M_{\text{ext}}$ takes the form
\[(1.2) \quad \Phi^* b = \frac{dr^2}{r^2 + k} + r^2 \tilde{h} =: b_0,\]
with $\tilde{h}$ — a Riemannian metric on $N$ with scalar curvature $\tilde{R}_{\tilde{h}}$ and the constant $k$ equal to
\[(1.3) \quad \tilde{R}_{\tilde{h}} = \begin{cases} (n-1)(n-2)k, & k \in \{0, \pm 1\}, \text{ if } n > 2, \\ 0, & k = 1, \text{ if } n = 2, \end{cases}\]
(recall that the dimension of $N$ is $(n-1)$); here $r$ is a coordinate running along the $[R, \infty)$ factor of $[R, \infty) \times N$. There is some freedom in the choice of $k$ in (1.2) when $n = 2$, associated with the range of the angular variable $\varphi$ on $N = S^1$ (see the discussion in Remark 3.1 below) and we make the choice $k = 1$, as it corresponds to the usual form of the two-dimensional hyperbolic space.

When $(N, \tilde{h})$ is the unit round $(n-1)$-dimensional sphere $(S^{n-1}, g_{S^{n-1}})$, then $b$ is the hyperbolic metric. Equations (1.2) and (1.3) imply that the scalar curvature $R_b$ of the metric $b$ is constant:
\[R_b = -n(n-1).\]
Moreover the metric $b$ will be Einstein if and only if $\tilde{h}$ is. We emphasise that for all our purposes we only need $b$ on $M_{\text{ext}}$, and we continue $b$ in an arbitrary way to $M \setminus M_{\text{ext}}$ whenever required.

The cases of main interest seem to be those where $\tilde{h}$ is a space form — it then follows from the results in [16, Appendix B] that we have
\[(1.4) \quad k = 0, -1 \implies \mathcal{N}_{b_0} = \text{Vect}\{\sqrt{r^2 + k}\},\]
\[(1.5) \quad k = 1, (N, \tilde{h}) = (S^{n-1}, g_{S^{n-1}}) \implies \mathcal{N}_{b_0} = \text{Vect}\{V(\mu)\}_{\mu=0,\ldots,n},\]
\[(1.6) \quad V(0) = \sqrt{r^2 + k}, \quad V(i) = x^i,\]
with the usual identification of $[R, \infty) \times S^{n-1}$ with a subset of $\mathbb{R}^n$ in (1.6). However, we shall not assume that $\tilde{h}$ is a space-form, or that Equations (1.4)-(1.6) hold unless explicitly stated.

For the purposes of Section 5 we note the following conformal representation of the metrics (1.2): One replaces the coordinate $r$ by a coordinate $x$
defined as
\[ x = \frac{2}{r + \sqrt{r^2 + k}} \iff r = \frac{1 - kx^2/4}{x}, \]
which brings \( b \) into the form
\[ b = x^{-2} \left( dx^2 + (1 - kx^2/4)^2 \tilde{h} \right) =: x^{-2} \tilde{b}, \]
with \( \tilde{b} \) — a metric smooth up to boundary on \( \{ x \in [0, x_R] \times N \} \), for a suitable \( x_R \).

2. The mass integrals.

Let \( g \) and \( b \) be two Riemannian metrics on a manifold \( M \), and let \( V \) be any function there. We set
\[ e_{ij} := g_{ij} - b_{ij} \]
(the reader is warned that the tensor field \( e \) here is not a direct Riemannian counterpart of the one in [16]; the latter makes appeal to the contravariant and not the covariant representation of the metric tensor). As before we denote by \( \tilde{D} \) the Levi-Civita connection of \( b \), and we use the symbol \( R_f \) to denote the scalar curvature of any metric \( f \). The basic identity from which our mass integrals arise is the following:
\[ \sqrt{\det g} V(R_g - R_b) = \partial_i \left( U^i(V) \right) + \sqrt{\det g} (\rho + Q), \]
where
\[ U^i(V) := 2\sqrt{\det g} \left( Vg^{ik}g^{jl}D_jD_k + D^i[Vg^{il}k_e k_l] \right), \]
\[ \rho := (-VRic(b))_{ij} + \tilde{D}_i\tilde{D}_jV - \Delta_b Vb_{ij})g^{ik}g^{jl}e_{kl}, \]
\[ Q := V(g^{ij} - b^{ij} + g^{ik}g^{jl}e_{kl})Ric(b)_{ij} + Q'. \]
Brackets over a symbol denote anti-symmetrisation, with an appropriate numerical factor (1/2 in the case of two indices). Here \( Q' \) denotes an expression which is bilinear in \( e_{ij} \) and \( \tilde{D}_k e_{ij} \), linear in \( V, dV \) and Hess\( V \), with coefficients which are constants in an ON frame for \( b \). The idea behind this calculation is to collect all terms in \( R_g \) that contain second derivatives of the metric in \( \partial_i U^i \); in what remains one collects in \( \rho \) the terms which are linear in \( e_{ij} \), while the remaining terms are collected in \( Q \); one should note that the first term at the right-hand-side of (2.5) does indeed not contain any terms linear in \( e_{ij} \) when Taylor expanded at \( g_{ij} = b_{ij} \). The mass integrals will be flux integrals — understood as a limiting process — over the “boundary at infinity of \( M \)” of the vector density \( U^i \).
In general relativity a normalising factor $1/16\pi$, arising from physical considerations, is usually thrown in into the definition of $U^i$. From a geometric point of view this seems purposeful when the boundary at infinity is a round two dimensional sphere; however, for other topologies and dimensions, this choice of factor does not seem very useful, and for this reason we do not include it in the definition.

We note that the linearisation of the mass integrands $U^i$ coincides with the linearisation of the charge integrands of Equation 16 evaluated for the Lorentzian metrics $4b = -V^2dt^2 + b$, $4g = -V^2dt^2 + g$, with $X = \partial_t$, on the hypersurface $t = 0$; however the integrands do not seem to be identical. Nevertheless, under the conditions of Theorem 2.3 the resulting numbers coincide, because under the asymptotic conditions of Theorem 2.3 only the linearised terms matter.

The convergence of the mass integrals requires appropriate boundary conditions, which are defined using the following orthonormal frame $\{f_i\}_{i=1,n}$ on $M_{\text{ext}}$:

$$\Phi^{-1}f_i = r^{-1}\epsilon_i, \quad i = 1, \ldots, n-1, \quad \Phi^{-1}f_n = \sqrt{r^2 + k} \partial_r,$$

where the $\epsilon_i$'s form an orthonormal frame for the metric $\bar{h}$. We moreover set

$$g_{ij} := g(f_i, f_j).$$

Asymptotic decay conditions 2.1. We shall require:

$$\int_{M_{\text{ext}}} \left( \sum_{i,j} |g_{ij} - \delta_{ij}|^2 + \sum_{i,j,k} |f_k(g_{ij})|^2 \right) r \circ \Phi \, d\mu_g < \infty,$$

$$\int_{M_{\text{ext}}} |R_g - R_b| r \circ \Phi \, d\mu_g < \infty,$$

$$\exists \, C > 0 \text{ such that } C^{-1}b(X, X) \leq g(X, X) \leq Cb(X, X).$$

For the $V$'s of Equations (1.4) or (1.6) we have

$$V = O(r), \quad \sqrt{b^\#(dV, dV)} = O(r),$$

where $b^\#$ is the metric on $T^*M$ associated to $b$, and this behavior will be assumed in what follows:

Proposition 2.2. Let the reference metric $b$ on $M_{\text{ext}}$ be of the form (1.2), suppose that $V$ satisfies (2.10), and assume that $\Phi$ is such that Equations (2.8)-(2.9) hold. Then for all $V \in \mathcal{N}_{b_0}$ the limits

$$H_\Phi(V) := \lim_{R \to \infty} \int_{r=R} U^i(V \circ \Phi^{-1})dS_i$$

exist, and are finite.
The integrals (2.11) will be referred to as the mass integrals.

**Proof.** For any $R_1, R_2$ we have
\begin{equation}
\int_{r=R_1} U^i dS_i = \int_{r=R_2} U^i dS_i + \int_{[R_1,R_2] \times N} \partial_i U^i d^n x,
\end{equation}
and the result follows from (2.2)-(2.5). \hfill \Box

Under the conditions of Proposition 2.2, the integrals (2.11) define a linear map from $N_{b_0}$ to $\mathbb{R}$. Now, each map $\Phi$ used in (1.2) defines in general a different background metric $b$ on $M_{\text{ext}}$, so that the maps $H_\Phi$ are potentially dependent upon $\Phi$. (It should be clear that, given a fixed $\tilde{h}$, (2.11) does not depend upon the choice of the frame $\epsilon_i$ in (2.6).) It turns out that this dependence can be controlled, as follows:

**Theorem 2.3.** Consider two maps $\Phi_a$, $a = 1, 2$, satisfying (2.8) together with
\begin{equation}
\sum_{i,j} |g_{ij} - \delta_{ij}| + \sum_{i,j,k} |f_k(g_{ij})| = \begin{cases} o(r^{-n/2}), & \text{if } n > 2, \\ O(r^{-1-\epsilon}), & \text{if } n = 2, \text{for some } \epsilon > 0. \end{cases}
\end{equation}
Then there exists an isometry $A$ of $b_0$, defined perhaps only for $r$ large enough, such that
\begin{equation}
H_{\Phi_2}(V) = H_{\Phi_1}(V \circ A^{-1}).
\end{equation}

**Proof.** The arguments of the proof of Theorem 2.3 follow closely those given at the beginning of Section 4 and in Section 2 of [16], we need, however, to adapt some of the necessary ingredients to our different setup here. The conclusion of Proposition 5.2 below, which holds for all manifolds $(N, \tilde{h})$ considered here, enables us to use Theorem 3.3 (2) of [16]: If $\Phi_1$ and $\Phi_2$ are two maps as above satisfying the decay assumptions (2.8)-(2.9) and (2.13) with respect to (isometric) reference metrics $b_1 := (\Phi_1^{-1})^* b_0$ and $b_2 := (\Phi_2^{-1})^* b_0$, then there exists an isometry $A$ of the background metric $b_0$, defined perhaps only for $r$ large enough, as made clear in Proposition 5.2, such that
\[ \Phi_2 - \Phi_1 \circ A = o(r^{-n/2}). \]
One also has a similar — when appropriately formulated in terms of $b$-orthonormal frames, as in [16] — decay of first two derivatives.

It follows directly from the definition of $H_\Phi$ that
\[ H_{\Phi_1 \circ A}(V) = H_{\Phi_1}(V \circ A^{-1}). \]
In order to establish (2.14) it remains to show that
\begin{equation}
H_{\Phi_1 \circ A}(V) = H_{\Phi_2}(V).
\end{equation}
Now, Corollary 3.5 of [16] shows that $\Phi_1 \circ A$ has the same decay properties as $\Phi_1$, so that — replacing $\Phi_1$ by $\Phi_1 \circ A$ — to prove (2.15) it remains to consider two maps $\Phi_1^{-1} = (r_1, v_1^A)$ and $\Phi_2^{-1} = (r_2, v_2^A)$ (where $v^A$ denote abstract local coordinates on $N$) satisfying

$$r_2 = r_1 + o\left(r_1^{1-\frac{n}{2}}\right),$$

$$v_2^A = v_1^A + o\left(r_1^{-\frac{n}{2}}\right),$$

together with elements $V_1 := V \circ \Phi_1^{-1}$ of $\mathcal{N}_{b_1}$ and $V_2 := V \circ \Phi_2^{-1}$ of $\mathcal{N}_{b_2}$ having the same expression in the first or the second system of coordinates. Local coordinates $v^A$ might not be defined on the whole of $M$; we shall remove this problem by embedding the manifold $N$ in $\mathbb{R}^{2(n-1)}$, so that local coordinates are turned into global coordinates. This has no effect in the sequel of the proof but enables us to consider a well-defined vector field

$$\zeta = (r_2 - r_1) \frac{\partial}{\partial r_1} + \sum_A (v_2^A - v_1^A) \frac{\partial}{\partial v_1^A},$$

defined only along $M$, and tangent to $M$. The decay estimates above imply that $\zeta = o(r^{-n/2})$ in the reference metric $b_1$; by Theorem 3.3 (2) of [16] the same holds for its first two $\hat{D}$-derivatives. Elementary calculations show then that

$$b_2 = b_1 + \mathcal{L}_\zeta b_1 + o(r^{-n}), \quad V_2 = V_1 + \hat{D}_i V \zeta^i + o(r^{1-n}),$$

together with their first derivatives. Hence, to leading order in powers of $r \approx r_1$, everything behaves as if we were considering a first order variation of metrics through the action of the flow of the vector field $\zeta$.

We shall now show that $H_{\Phi_1}(V) = H_{\Phi_2}(V)$. For the purpose of the calculations that follow it will be easier to replace the local integrand $U$ by the following one:

$$U_i = \sqrt{\det b} \left(-V \hat{D}_j g^{ij} + Vb^{ij}b_{kl}\hat{D}_j g^{kl} + 2 \hat{D}^i[Vb^j]k e_{jk}\right),$$

which yields the same limit at infinity when integrated on an element $V$ of $\mathcal{N}_b$ on larger and larger spheres (strictly speaking, we should not denote them by the same letter $U$, since they are different vector densities which give identical results only after an integration process; we shall however do so since expression (2.18) will only be used in the course of the current proof; we emphasise that the definition (2.3) is used in all other places in the paper).

We now compute the variation of $U$ when passing from the asymptotic map $\Phi_1$ (with reference metric $b_1$ and function $V_1$) to the second map $\Phi_2$ (with reference metric $b_2$ and function $V_2$). From Equation (2.17), we deduce

$$U_2^i - U_1^i = \delta U + o(r^{1-n}),$$
where $\delta U^i$ is obtained by linearisation in $\zeta$ at $g = b$ and will be computed below, while the remainder terms decay sufficiently fast so that they do not contribute when integrated at infinity against either $b_1$ or $b_2$. It remains to show that $\delta U^i$ does not contribute either when integrated at infinity. In Equation (2.18), the only terms that contribute \textit{a priori} to the variation of $U$ are the following: $b^{ij}$, $b_{kl}$, $\sqrt{\det b}$, $\hat{D}$ and $V$, but the decay estimates (2.17) show at first glance that only the variation of $\hat{D}$ will contribute to the first-order term $\delta \bar{U}$. We now compute it using Formulae 1.174 of [8]. In all what follows, we denote $b = b_1$ and $V = V_1$. Then,

$$
\delta U^i = \sqrt{\det b} \left( -V \hat{D}_k \hat{D}^k \zeta^i + V \hat{D}_k \hat{D}^i \zeta^k - 2V \text{Ric}(b)^i_k \zeta^k \right) + \sqrt{\det b} \left( -2(\hat{D}^i V) \hat{D}_k \zeta^k + (\hat{D}_k V) \hat{D}^i \zeta^k + (\hat{D}^k V) \hat{D}_k \zeta^i \right).
$$

Fortunately, this will appear to be the sum of a divergence term plus lower order terms. The first step is to use the following elementary facts:

$$
\begin{align*}
\hat{D}_k \hat{D}^k \zeta^i &= -\hat{D}_k (V \hat{D}^k \zeta^i) + (\hat{D}_k V) \hat{D}^i \zeta^k, \\
V \hat{D}_k \hat{D}^i \zeta^k &= \hat{D}_k (V \hat{D}^i \zeta^k) - (\hat{D}_k V) \hat{D}^i \zeta^k,
\end{align*}
$$

which yield

$$
\delta U^i = 2\sqrt{\det b} \left( \left( \hat{D}_k V \right) \hat{D}^k \zeta^i - \left( \hat{D}^i V \right) \hat{D}_k \zeta^k - V \text{Ric}(b)^i_k \zeta^k \right) + \text{divergence term}.
$$

Each of the first two terms in the right-hand side may be transformed with

$$
\begin{align*}
(\hat{D}_k V) \hat{D}^k \zeta^i &= \hat{D}^k (\zeta^i \hat{D}^k V) - (\hat{D}^k \hat{D}_k V) \zeta^i, \\
-(\hat{D}^i V) \hat{D}_k \zeta^k &= -\hat{D}_k (\zeta^k \hat{D}^i V) + (\hat{D}_k \hat{D}^i V) \zeta^k,
\end{align*}
$$

and one may also use that $V$ is an element of $N_0$ to conclude that

$$
U_2^i - U_1^i = \text{divergence term} + o(r^{1-n}).
$$

This establishes the covariance of the mass functional. \hfill \Box

\textbf{Remark 2.4.} For the purpose of explicit calculations we note that under (2.13) the mass integral $H_\Phi(V_{(0)}) = H_\Phi(\sqrt{r^2 + \bar{k}})$ can be written as:

$$
(2.25)
$$

$$
H_\Phi(V_{(0)}) = \lim_{R \to \infty} (R^2 + k) \times \\
\int_{\{r = R\}} \left( -\sum_{i=1}^{n-1} \left\{ \frac{\partial e_{ii}}{\partial r} + \frac{ke_{ii}}{r(r^2 + k)} \right\} + \frac{(n-1)e_{nn}}{r} \right) d^{n-1} \mu_h,
$$

assuming that the right-hand-side of (2.25) converges. Here $d^{n-1} \mu_h$ is the Riemannian measure associated with the metric $h$ induced on the level sets of the function $r$. 

Remark 2.5. Conditions (2.13) are sharp, in the following sense: Let \( g \) be the standard hyperbolic metric, thus in a coordinate system \((\bar{r}, \bar{v}^A)\), where the \( \bar{v}^A \)'s are local coordinates on \( S^{n-1} \), we have

\[
g = \frac{d\bar{r}^2}{\bar{r}^2 + k} + \bar{v}^2 \bar{h}.
\]

Let, for sufficiently large \( r \), \( \Phi^{-1}_\gamma(r, \bar{v}^A) = (\bar{r}(r, \bar{v}^A), \bar{v}^B(r, \bar{v}^A)) \) be given by the formula

\[
\bar{r} = r + \gamma r^{1-n/2}, \quad \bar{v}^A = v^A,
\]

where \( \gamma \) is a constant. Then

\[
H_{\Phi, \gamma}(\sqrt{\bar{r}^2 + k}) \text{ does depend upon } \gamma: \text{ In order to see that, consider any background metric of the form}
\]

\[
b = a^2(r) dr^2 + r^2 \bar{h},
\]

and let \( g \) satisfy

\[
g = g_{mn}(r, v^A) a^2(r) dr^2 + c(r, v^A) r^2 \bar{h},
\]

for some differentiable functions \( g_{mn} \) and \( c \). One finds

\[
U^i dS_i|_{r=\text{const}} := U^i \partial_i [dr \wedge dv^1 \wedge \cdots \wedge dv^{n-1}]|_{r=\text{const}}
\]

\[
= \frac{(n-1)c^{(n-3)/2} r^{n-2}}{a(r) \sqrt{g_{mn}}} \left\{ V \left( g_{mn} - 1 - r \frac{\partial c}{\partial r} \right) + \left( r \frac{\partial V}{\partial r} - V \right) (c - 1) \right\} \sqrt{\det \bar{h}_{AB} \, dv^1 \wedge \cdots \wedge dv^{n-1}}.
\]

Applying this formula to the above \( g \) and \( b \) one obtains

\[
H_{\Phi, \gamma}(\sqrt{\bar{r}^2 + k}) = \frac{1}{4} (n + 8)n(n - 1) \gamma^2 \text{Vol}_{g_{n-1}}(S^{n-1}).
\]

One can also check that the numerical value of the linearised expression (2.25) reproduces the right-hand-side of (2.30) for the metrics at hand, thus is again not invariant under (2.27).

3. The mass.

In the asymptotically flat case the mass is a single number which one assigns to each end of \( M \) ([4] and [15]); it is then natural to enquire whether there are some geometrically defined numbers one can extract out of the family of maps \( H_\Phi \). This will depend upon the structure of \( N_{b_0} \) and we shall give here a few important examples. Throughout this section we assume that \( N_{b_0} \neq \emptyset \).

A. The simplest case is that of the manifold \((N, \bar{h})\) of (1.1)-(1.2) having a strictly negative Ricci tensor, with scalar curvature \( R_{\bar{h}} = -(n - 1)(n - 2) \),
so that \( n \geq 3 \) and \( k = -1 \) in (1.2). Similarly to the space forms discussed in Section 1, \( \mathcal{N}_{b_0} \) is then \([16, \text{Appendix B}]\) one dimensional:

\[
(3.1) \quad V \in \mathcal{N}_{b_0} \iff V = \lambda V_0, \quad \lambda \in \mathbb{R}, \quad V_0 := \sqrt{r^2 + k}.
\]

The coordinate system of (1.2) is uniquely defined, so is the function \( V_0 \); the number

\[
(3.2) \quad m := H_{\Phi}(V_0),
\]

calculated using any \( \Phi \) satisfying the conditions of Theorem 2.3, provides the desired, \( \Phi \)-independent definition of mass relative to \( b_0 \), whenever (2.6) and (2.13) hold.

**B.** Consider, next, the case of a flat \((N, \tilde{h})\) with \( n \geq 3 \), so that \( k = 0 \) in (1.2). Equation (3.1) holds again; however, the coordinate \( r \) is not anymore uniquely defined by \( b \), since (1.2) is invariant under the rescalings

\[
r \to ar, \quad \tilde{h} \to a^{-2}\tilde{h}, \quad a \in \mathbb{R}^*.
\]

This freedom can be gotten rid of by requiring, e.g.,

\[
\text{Vol}_{\tilde{h}}(N) = 1;
\]

the number \( m \) obtained then from (3.2), with \( V_0 \) as in (3.1), provides the desired invariant.

**C.** The case \( k = +1 \) requires more work. Consider first the case where \((N, \tilde{h}) = (S^{n-1}, g_{S^{n-1}})\), so that the reference metric is the hyperbolic metric; it is convenient to start with a discussion of \( \mathcal{N}_b \) in two models of the hyperbolic space: In the ball model, we consider the ball \( \mathbb{B} = \{ x \in \mathbb{R}^n, |x| < 1 \} \) endowed with the metric \( b = \omega^{-2}\delta \), where

\[
(3.3) \quad \omega = \frac{1}{2}(1 - |x|^2),
\]

and \( \delta \) is the flat Euclidean metric. From (1.5)-(1.6) one finds that the set \( \mathcal{N}_b \) defined in (0.1)-(0.2) is the \((n + 1)\)-dimensional vector space spanned by the following basis of functions:

\[
(3.4) \quad V_0 = \frac{1 + |x|^2}{1 - |x|^2}, \quad V_i = \frac{2x^i}{1 - |x|^2},
\]

where \( x^i \) is any of the Cartesian coordinates on the flat ball. In geodesic coordinates around an arbitrary point in the hyperbolic space, the hyperbolic metric is \( b = dr^2 + \sinh^2(r)g_{S^{n-1}} \) and the above orthonormal basis of \( \mathcal{N}_b \) may be rewritten as \( V_0 = \cosh(r) \) and \( V_i = \sinh(r)n^i \), where \( n^i \) is the restriction of \( x^i \) to the unit sphere centred at the pole.

The space \( \mathcal{N}_b \) is naturally endowed with a Minkowski metric \( \eta \), with signature \((+, -, \cdots, -)\), issued from the action of the group of isometries \( O^+(n, 1) \) of the hyperbolic metric (cf., e.g., \([16, \text{Appendix B}]\)). The basis given above
is then orthonormal with respect to this metric, with the vector \( V(0) \) being timelike, i.e., \( \eta(V(0), V(0)) > 0 \). We define the time orientation of \( N_b \) using this basis — by definition a timelike vector \( X^{(\mu)}V(\mu) \) is future directed if \( X^{(0)} > 0 \), similarly for covectors; this finds its roots in a Hamiltonian analysis in the associated Lorentzian space-time. Assuming that there exists a map \( \Phi \) for which the convergence conditions of Proposition 2.2 are satisfied, we set

\[
p(\mu) := H_\Phi(V(\mu)).
\]

Under isometries of \( b \) the \( V(\mu) \)'s are reshuffled amongst each other under the usual covariant version of the defining representation of the Lorentz group \( O^+(n, 1) \). It follows that the number

\[
m^2 := \left| (p(0))^2 - \sum_{i=1}^{n} (p(i))^2 \right|
\]

is a geometric invariant, which provides the desired notion of mass for a spherical asymptotic geometry. The nature of the action of \( O^+(n, 1) \) on \( N_b \) shows that the only invariants which can be extracted out of the \( H_\Phi \)'s are \( m^2 \) together with the causal character of \( p(\mu) \) and its future/past pointing nature if relevant. Under natural geometric conditions \( p(\mu) \) is timelike future pointing or vanishing, see Section 4 below. For timelike \( p(\mu) \)'s it appears natural to choose the sign of \( m \) to coincide with that of \( p(0) \), and this is the choice we shall make.

Suppose, finally, that the manifold \( (N, \tilde{h}) \) is the quotient of the unit round sphere \( (S^{n-1}, g_{S^{n-1}}) \) by a subgroup \( \Gamma \) of its group of isometries. For generic \( \Gamma \)'s one expects the conformal isometry group of \( (N, \tilde{h}) \) to be trivial, in which case all the integrals \( p(\mu) \) defined by Equation (3.5) define invariants. In any case, for nontrivial \( \Gamma \)'s conformal isometries of \( (N, \tilde{h}) \) are isometries (for compact Einstein manifolds which are not round spheres the group of conformal isometries coincides with the group of isometries; this follows immediately, e.g., from what is said in [29]; P.T.C. is grateful to A. Zeghib and C. Frances for useful comments concerning the structure of the group of conformal isometries of quotients of spheres), and, in addition to \( m, p(0) \) becomes then a geometric invariant. Further invariants may occur depending upon the details of the action of the group of isometries of \( (N, \tilde{h}) \) on \( N_{b_0} \).

Remark 3.1. Our results also apply in dimension \( n = 2 \). This might seem somewhat surprising at first sight, because there is no direct useful equivalent of asymptotic flatness and of the associated notion of mass given by an ADM-type integral in dimension 2: When the scalar curvature is in \( L^1 \), the appropriate analogue of mass is the deficit angle, as made precise by the Shiohama theorem [37]. For the metrics considered here the Shiohama theorem does not apply; however, the metrics we study can be thought of as
having a minus infinite deficit angle, consistently with a naively understood version of the Shiohama theorem — the ratio of the length of distance circles to the distance from any compact set tends to infinity as the distance does. Examples of metrics on $M_{\text{ext}}$ which satisfy our asymptotic conditions and have a well-defined nontrivial mass — with respect to a background given by the 2-dimensional hyperbolic metric — are provided by the Riemannian counterpart of the generalised $(2+1)$-dimensional Kottler metrics,

$$
g = \frac{dr^2}{r^2 - \eta} + r^2 d\varphi^2, \quad \varphi \in [0, 2\pi] \text{ mod } 2\pi,$$

for some constant $\eta \in \mathbb{R}$; $\eta = -1$ corresponds to the standard hyperbolic metric (as pointed out by Bañados, Teitelboim and Zanelli [3], for positive $\eta$ the associated static Lorentzian space-times with $V = \sqrt{r^2 - \eta}$ can be extended to space-times containing a black-hole region). The metrics (3.7) have constant Gauss curvature equal to minus one for all $\eta \in \mathbb{R}$, so the integral condition on $R_g$ in (2.8) holds with

$$b = \frac{dr^2}{r^2 + 1} + r^2 d\varphi^2;$$

the remaining conditions arising from (2.8), as well as (2.9) and (2.13), are easily checked. Applying formula (2.25) one obtains

$$m = p_{(0)} = 2\pi(1 + \eta)$$

(the remaining $p_{(\mu)}$'s are zero by symmetry considerations). For strictly negative $\eta$ there is a sense in which $m$ is related to a deficit angle, as follows: A coordinate transformation $r \to \lambda r$, $\varphi \to \varphi/\lambda$, with $\lambda^2 = -\eta$ brings the metric (3.7) to the standard hyperbolic space form,

$$
g = \frac{dr^2}{r^2 + 1} + r^2 d\varphi^2, \quad \varphi \in [0, 2\pi/\lambda] \text{ mod } 2\pi/\lambda,$$

except for the changed range of variation of the angular variable $\varphi$; that range will coincide with the standard one if and only if the mass vanishes. For metrics asymptotic to (3.7) the geometric invariance of the mass should follow directly from this deficit angle character. This suggests strongly that some methods specific to dimension 2, perhaps in the spirit of the Shiohama theorem (cf. also [32]), could provide simpler proofs of geometric invariance and positivity when $n = 2$; we have not investigated this issue any further.


In this section we consider metrics asymptotic to the standard hyperbolic metric $b$ of constant negative curvature $-1$; by this we mean that the Riemannian manifold $(N, \hat{h})$ is the unit round sphere $(S^{n-1}, g_{S^{n-1}})$. We wish
to show that the usual positivity theorem holds under the weak asymptotic hypotheses considered in the previous sections.

**Theorem 4.1.** Let \((M, g)\) be a complete boundaryless spin manifold with a \(C^2\) metric, and with scalar curvature satisfying
\[
R_g \geq -n(n - 1),
\]
and suppose that the asymptotic conditions (2.8) and (2.13) hold with \((N, \tilde{h}) = (\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})\). Then the covector \(p(\mu)\) defined by Equation (3.5) is timelike future directed or zero (in particular \(p(0) \geq 0\)). Moreover, it vanishes if and only if \((M, g)\) is isometrically diffeomorphic to the hyperbolic space.

**Remark 4.2.**
1. The \(C^2\) differentiability of the metric can be replaced by a weighted \(W^{2,p}\) Sobolev condition.
2. As already pointed out, we say that a linear functional \(p\) on \(N_b\) is causal (resp. timelike) and future-directed if it can be written as \((p(0), \ldots, p(n))\) in any orthonormal and future-oriented basis \((V(0), \ldots, V(n))\) with
\[
(p(0))^2 - \sum_{i=1}^{n} (p(i))^2 \geq 0 \quad \text{and} \quad p(0) \geq 0 \quad (\text{resp.} > 0).
\]
This is the obvious equivalent of the corresponding definition for vectors; note, however, that with our signature \((- , + , \ldots , +)\) future directed vectors are not mapped to future directed covectors by the isomorphism of \(TM\) with \(T^*M\) associated with the metric.

We emphasise that the Lorentz vector character of \(p(\mu)\) is not related to the tangent space of some point of \(M\), or of some “abstract asymptotic point” (“the tangent space at \(i^\infty\) — this last interpretation can be given to energy-momentum in the asymptotically Euclidean context), but arises from the fact that the adjoint action of the isometry group of the (standard) hyperbolic space, on the subspace of its Lie algebra singled out by Equations (0.1)-(0.2), is that of the defining representation of the Lorentz group on \(\mathbb{R}^{n+1}\).

3. Condition (2.8b) is actually not necessary for positivity, in the following sense: Under the remaining conditions of Theorem 4.1, the argument of the proof of Proposition 2.2 shows that \(p(0) = \infty\) whenever (2.8b) does not hold.

4. Such a theorem cannot be obtained in a more general setting. For instance, in the asymptotically Euclidean context, it is well-known that positivity statements may fail if the metric is asymptotic to some \(\mathbb{Z}_2\)-quotient of the Euclidean space [30]. In the asymptotically hyperbolic setting, Horowitz and Myers [25] have constructed an infinite family of metrics with ends asymptotic to a cuspidal hyperbolic metric (the topology of the end is \(\mathbb{R} \times \mathbb{T}^2\)), and with masses as negative as desired. If the topology of the end is the product of a half-line with a negatively curved Riemann surface, the mass
may also be negative when minimal interior boundaries are allowed, and it is expected that the infimum of the possible masses is achieved only for the Kottler black-hole metrics \([25, 28, 38, 12]\) and \([17]\).

5. The \(C^2\) differentiability of the metric can be replaced by weighted Sobolev-type conditions; this is, however, of no concern to us here.

As a corollary of Theorem 4.1, together with \([17, \text{Theorem I.3}]\) and the remarks at the end of Section V of \([17]\) one has (see \([17, \text{Corollary I.4}]\); compare \([9]\)):

**Theorem 4.3.** Let \(V\) be a strictly positive function on a three dimensional manifold \(M\) such that the metric
\[
\gamma := -V^2 dt^2 + g,
\]
is a static solution of the vacuum Einstein equations with strictly negative cosmological constant on the space-time \(\mathcal{M} := \mathbb{R} \times M\). If:

(i) \((M, g)\) is \(C^3\) compactifiable in the sense of Section 5 below, and if
(ii) the conformal boundary at infinity of \(M\) is \(S^2\), with \(V^{-2}g\) extending by continuity to the unit round metric on \(S^2\),

then \((M, g)\) is the hyperbolic space, so that \((\mathcal{M}, \gamma)\) is the anti-de Sitter space-time.

**Preliminaries to the proof.** The Proof of Theorem 4.1 will follow the Gibbons-Hawking-Horowitz-Perry variation \([21]\) of the classical Witten argument for the positivity of mass \([40]\) (cf. also \([34, 1]\) and the remarks done in \([18]\)), and relies on the existence on the hyperbolic space of a wealth of distinguished spinor fields, called *imaginary Killing spinors*. These are solutions \(\varphi\) of the differential equation
\[
\hat{D}_X^b \varphi = D_X^b \varphi + \frac{i}{2} c_b(X) \varphi = 0,
\]
where we denote by \(c_b(X) \varphi\) the Clifford action of a vector \(X\) on a spinor \(\varphi\) with respect to the metric \(b\). On hyperbolic space there is a set of maximal dimension of imaginary Killing spinors, which trivialise the spinor bundle. They can be described explicitly in the following manner \([6]\): One may choose the standard basis \(\{\partial_i\}\) of the flat space as reference frame, thus inducing an isomorphism between the spinor frame bundle of \((\mathbb{B}, e)\) and \(\mathbb{B} \times \text{Spin}(n)\). This can be transferred to the hyperbolic space through the usual conformal covariance (of “weight zero”) of spinor bundles \([20]\). In this trivialisation, the Killing spinors of \(b\) are then the spinor fields \(\varphi_u\) given by
\[
\varphi_u(x) = \omega^{-\frac{1}{2}} (1 - ic_3(x)) u
\]
where \(u\) is any nonzero constant spinor on the flat ball \(\mathbb{B}\), and \(\omega\) is the conformal factor of the hyperbolic metric defined in \((3.3)\).
Following the terminology due to H. Baum, Th. Friedrich and I. Kath \[19, 7\] and \[6\], the spinor \(\varphi_u\) is said to be of Type I (resp. of Type II) if

\[
\|u\|_\delta^4 + \sum_{i=1}^{n} \langle c_\delta(\partial_i)u, u \rangle_\delta^2 \text{ is zero (resp. positive). (4.4)}
\]

Type I spinors are actually sufficient for our purposes, we shall describe these ones only. For any imaginary Killing spinor, the function

\[
V_u = \langle \varphi_u, \varphi_u \rangle_b
\]

is always an element of \(\mathcal{N}_b\). If \(\varphi_u\) is moreover of Type I, then there is a set of \(n\) constants \((a_i) \in \mathbb{S}^{n-1} \subset \mathbb{R}^n\) and a constant \(\lambda > 0\), such that

\[
V_u = \lambda (V_0 - \sum_i a_i V_i) \text{: Indeed, an explicit computation from Equation (4.3) above shows that, in the ball model,}
\]

\[
V_u(x) = \|u\|_\delta^2 \frac{1 + |x|^2}{1 - |x|^2} + i \sum_{j=1}^{n} \langle c_\delta(\partial_j)u, u \rangle_\delta \frac{2 x_j}{1 - |x|^2},
\]

which is clearly future directed, and the Type I condition (4.4) yields that \(V_u\) is isotropic in \(\mathcal{N}_b\). This shows in particular that Killing spinors of Type I always exist on the hyperbolic space in any dimension. Further, e.g., as a result of covariance under isometries, any future directed isotropic combination \(V_0 - \sum_i a_i V_i\), \((a_i) \in \mathbb{S}^{n-1}\), can be obtained as a \(V_u\) for some Type I Killing spinor \((i.e., \text{for some constant spinor } u \text{ on the flat ball})\) in any dimension. For later use we also note that

\[
dV_u(X) = i \langle c_b(X)\varphi_u, \varphi_u \rangle_b.
\]

**Proof of Theorem 4.1.** Let \(A\) be the symmetric endomorphism defined over \(M_{ext}\) by \(g(A^i, A^j) = b(\cdot, \cdot)\), which we will take to be of the form

\[
A = I - \frac{1}{2} e + \{\text{quadratic and higher order in } e\}
\]

if \(e\) is small enough; by this we mean that \(A^i_j = \delta^i_j - \frac{1}{2} b^{ik} e_{kj} + \text{a second order Taylor expansion error term. One may use } A\text{ as an isomorphism between the orthonormal frame bundles of } b \text{ and } g \text{ and any lift of it as an isomorphism between their spinor frame bundles. This enables, as in [1]} \text{ (compare [10]), to transfer the spin connection } D^b \text{ of } (M_{ext}, b) \text{ on the spinor bundle of } (M_{ext}, g)\); for notational convenience, the new connection will be denoted by \(\hat{D}^b\). Note that this has the effect that the Clifford action \(c_b(X)\) of a vector \(X\) is transformed into the Clifford action \(c_g(AX)\) of \(AX\). As a consequence, the transferred spinors, still denoted by \(\varphi_u\), are now solutions of

\[
\hat{D}^b_X \varphi_u = D^b_X \varphi_u + \frac{i}{2} c_g(AX) \varphi_u = 0.
\]
We now denote by $D$ the spinor connection associated to the Levi-Civita connection of the metric $g$ and define the modified connection on spinors

$$\hat{D}_X = D_X + i\frac{c_g(X)}{2}.$$ 

For any $\varphi_u$ we set

$$\Phi_u = \chi\varphi_u + \psi_u,$$

where $\chi$ is a cut-off function that vanishes outside of $M_{\text{ext}}$ and is equal to 1 for $r$ large enough. Suppose, first, that $\psi_u$ is compactly supported, hence vanishes for $r \geq R$ for some $R$ on $M_{\text{ext}}$. We apply the standard Schrödinger-Lichnerowicz ([33] and [36]) formula relating the rough Laplacian of the modified connection $\hat{D}$ to the Dirac Laplacian $\hat{D}^*\hat{D}$ [1], where

$$\hat{D}\Phi_u = D\Phi_u - ni\frac{c_g(X)}{2}\Phi_u,$$

with $\hat{D}$ being the usual Dirac operator associated with the metric $g$. Letting $S_R = \{r = R\} \subset M_{\text{ext}}$ one obtains

$$\int_{M \setminus \{r \geq R\}} \|\hat{D}\Phi_u\|^2_g + \frac{1}{4} (R_g + n(n-1)) \|\Phi_u\|^2_g - \|\hat{D}\Phi_u\|^2_g$$

(4.7)

$$= \int_{S_R} B_{A\nu}(\Phi_u)$$

$$= \int_{S_R} B_{A\nu}(\varphi_u),$$

where $\nu$ is the outer $b$-unit normal to $S_r$, so that $A\nu$ is its outer $g$-unit normal, and $B_{A\nu}(\varphi_u)$ is the boundary integrand, explicitly defined by

$$B_Y(\rho) = \langle \hat{D}_Y\rho + c_g(Y)\hat{D}\rho, \rho \rangle_g$$

(4.8)

for any spinor $\rho$ and vector $Y$.

Assume that $(M, g)$ is not the hyperbolic space, otherwise there is nothing to prove. Let $H$ be the usual Hilbertian completion of the space of compactly supported smooth spinors $\psi$ on $M$ with respect to the norm defined as

$$\|\psi\|^2_H := \int_M \left(\|\hat{D}\psi\|^2_g + \frac{1}{4} (R_g + n(n-1)) \|\psi\|^2_g\right) \, d\mu_g.$$ 

(4.9)

We wish to show that for any $\Phi_u = \chi\varphi_u + \psi_u$, with $\psi_u \in H$, we will have

$$\int_M \|\hat{D}\Phi_u\|^2_g + \frac{1}{4} (R_g + n(n-1)) \|\Phi_u\|^2_g - \|\hat{D}\Phi_u\|^2_g = \lim_{R \to \infty} \int_{S_R} B_{A\nu}(\varphi_u).$$

(4.10)

We start by showing that $H$ can be identified with a space $\mathcal{H}$ of $H^1_{\text{loc}}$ spinor fields on $M$, with the norm $\| \cdot \|_H$ still given by (4.10) (after identification) for all $\psi \in H$. First, it is not too difficult to show [5] that in dimension
larger than or equal to three there exists a strictly positive $L^\infty_{\text{loc}}$ function $w$ on $M$ such that for all $H^1_{\text{loc}}$ spinor fields $\psi$ with compact support we have

$$\int_M \|\psi\|_g^2 w \, d\mu_g \leq \int_M \|	ilde{D}\psi\|_g^2 d\mu_g.$$ \hfill (4.12)

The function $w$ can be chosen to be constant in the asymptotically hyperbolic end. In dimension two one can also prove (4.12) if one assumes further that there are no imaginary Killing spinors. This is sufficient for our purposes because, if there exists a Killing spinor then, by [6], we are in hyperbolic space, where there is nothing to prove. So one might as well suppose that there are no such spinors.

Let $\mathcal{H}$ be the space of measurable spinor fields on $M$ such that

$$\|\psi\|_{\mathcal{H}}^2 := \int_M \|\psi\|_g^2 \left( w + \frac{1}{4}(R_g + n(n-1)) \right) \, d\mu_g + \int_M \|	ilde{D}\psi\|_g^2 d\mu_g < \infty$$ \hfill (4.13)

where $\tilde{D}\psi$ is understood in the distributional sense. Define $\hat{\mathcal{H}} \subset \mathcal{H}$ as the completion of $C^\infty_c$ in $\mathcal{H}$, with respect to the $\| \cdot \|_{\mathcal{H}}$ norm. It is then easy to verify the following:

**Proposition 4.4.** The inequality (4.12) remains true for all $\psi \in \hat{\mathcal{H}}$.

**Proof.** Both sides of (4.12) are continuous on $(\mathcal{H}, \| \cdot \|_{\mathcal{H}})$. \hfill \Box

**Proposition 4.5.** If $(M, g)$ is complete then $\mathcal{H} = \hat{\mathcal{H}}$.

**Proof.** If $\phi \in \mathcal{H}$ then the sequence $\chi_i \phi$ converges to $\phi$ in $(\mathcal{H}, \| \cdot \|_{\mathcal{H}})$, where $\chi_i(p) = \chi(d_{p_0}(p)/i)$, where $d_{p_0}$ is the distance to some chosen point $p_0 \in M$, while $\chi : \mathbb{R} \to [0, 1]$ is a smooth function such that $\chi|_{[0,1]} = 1$, $\chi|_{[2,\infty)} = 0$. Smoothing $\chi_i \phi$ using the usual convolution operator yields the result. \hfill \Box

**Proposition 4.6.** If $(M, g)$ is complete then there is a natural continuous bijection between $(\mathcal{H}, \| \cdot \|_{\mathcal{H}})$ and $(H, \| \cdot \|_{\mathcal{H}})$ which is the identity on $C^1_c$: in particular, elements of $H$ can be identified with spinor fields on $M$ which are in $\mathcal{H}$.

**Proof.** By Proposition 4.5 both spaces are Hilbert spaces containing $C^1_c$ as a dense subspace, with the norms being equivalent when restricted to $C^1_c$ by Proposition 4.4. \hfill \Box

Let $F(\psi)$ denote the left-hand side of Equation (4.11) with $\Phi_u = \chi \varphi_u + \psi$ there, let $\psi_i \in C^1_c$ converge to $\psi$ in $H$, we have
\[ F(\psi) - F(\psi_i) = \|\psi\|_H^2 - \|\psi_i\|_H^2 + 2 \int_M \langle \hat{D}(\chi \varphi_u), \hat{D}(\psi - \psi_i) \rangle - 2 \int_M \langle \hat{D}(\chi \varphi_u), \hat{D}(\psi - \psi_i) \rangle + \frac{1}{2} \int_M (R_g + n(n-1)) \langle \chi \varphi_u, \psi - \psi_i \rangle. \]

It should be clear from the fact that \( \hat{D}(\chi \varphi_u) \in L^2(M) \) that all the terms above converge to zero as \( i \) tends to infinity, except perhaps for the last one (recall that we are only assuming that \( 0 \leq (R_g + n(n-1))|V| \in L^1(M_{ext}) \)); the convergence of that last term can be justified as follows:

\[
\left| \int_M (R_g + n(n-1)) \langle \chi \varphi_u, \psi - \psi_i \rangle \right| \\
\leq \left( \int_M (R_g + n(n-1)) \|\chi \varphi_u\|_g^2 \right)^{1/2} \left( \int_M (R_g + n(n-1)) \|\psi - \psi_i\|_g^2 \right)^{1/2} \\
\leq \|\chi \varphi_u\|_H \|\psi - \psi_i\|_H.
\]

(Here we have applied the Cauchy-Schwarz inequality associated with the positive quadratic form occurring in the left-hand-side above.) Now, \( F(\psi_i) = F(0) \), and we have shown that Equation (4.11) holds for all \( \psi_u \in H \), as claimed.

To obtain positivity of the left-hand-side of Equation (4.11) we seek a \( \Phi_u \) such that

\[ \hat{D} \Phi_u = 0 \iff \hat{D} \psi_u = -\hat{D}(\chi \varphi_u). \]

We now use the fact that \( \varphi_u \) solves (4.6), hence

\[ \hat{D}_X \varphi_u = \left( D_X - D^b_X \right) \varphi_u - \frac{i}{2} c_g(AX - X) \varphi_u. \]

Now, in any \( g \)-orthonormal frame \( \{f_\alpha\}_{\alpha=1,...,n} \), if \( \omega \) denotes the connection 1-forms of either \( D \) or \( D^b \) (with the obvious notations), one has:

\[ D_X - D^b_X = \frac{1}{4} \sum_{\alpha,\beta=1}^n \left( \omega_{\alpha\beta}(X) - \omega_{\alpha\beta}^b(X) \right) c_g(f_\alpha) c_g(f_\beta). \]

The calculations of [1, Section 2.2] lead then to

\[ |\hat{D} \varphi_u| \leq C \left( |\hat{D} A|_b + |A - id|_b \right) |\varphi_u| \implies \hat{D}(\chi \varphi_u) \in L^2(M,d\mu_g). \]

Equation (4.17) and arguments known in principle [1, 35, 14] and [23] (cf. also [5]) show that there exists \( \psi_u \) in \( H \) such that Equation (4.14) holds.

\[ \hat{D} \psi_u = -\hat{D}(\chi \varphi_u). \]
holds. It then remains to show that the integral at the right-hand-side of Equation (4.11) is related to the map $H_\Phi$. In order to do this, we complement $\nu$ into a direct $b$-orthonormal basis $\{\nu, e_i\}_{i=1,\ldots,n-1}$. Seen as sitting in $M$, $\{A\nu, Ae_i\}_{i=1,\ldots,n-1}$ is a direct $g$-orthonormal basis on $S_r$. One easily finds

$$B_{A\nu}(\varphi_u) = \sum_{i=1}^{n-1} \langle c_g(A\nu)c_g(Ae_i)\tilde{D}_{Ae_i}\varphi_u, \varphi_u \rangle_g.$$  

From (4.15) and (4.16), we obtain

$$B_{A\nu}(\varphi_u) = \frac{1}{4} \sum_{i=1}^{n-1} \sum_{\alpha,\beta=1}^{n} (\omega_{\alpha\beta}(Ae_i)$$

$$- \omega_{\alpha\beta}^b(Ae_i) \rangle \langle c_g(A\nu)c_g(Ae_i)c_g(f_\alpha)c_g(f_\beta)\varphi_u, \varphi_u \rangle$$

$$+ \frac{i}{2} \sum_{i=1}^{n-1} \langle c_g(A\nu)c_g(Ae_i)c_g(A(Ae_i) - Ae_i)\varphi_u, \varphi_u \rangle$$

$$= -\frac{1}{2} \sum_{i,j=1}^{n-1} (\omega_{ij0}(Ae_i) - \omega_{ij0}^b(Ae_i)) \langle c_g(Ae_i)c_g(Ae_j)\varphi_u, \varphi_u \rangle$$

$$+ \frac{1}{4} \sum_{i,j,k=1}^{n-1} (\omega_{ijk}(Ae_i)$$

$$- \omega_{ijk}^b(Ae_i)) \langle c_g(A\nu)c_g(Ae_i)c_g(Ae_j)c_g(Ae_k)\varphi_u, \varphi_u \rangle$$

$$+ \frac{i}{2} \sum_{i=1}^{n-1} \langle c_g(A\nu)c_g(Ae_i)c_g(A(Ae_i) - Ae_i)\varphi_u, \varphi_u \rangle,$$

where the subscript $\cdot_0$ in the last formula denotes the basis element $A\nu$. These formulae correct Equation (34), page 20, in [1]: In the second line of that equation the multiplicative factor $1/4$ should be changed to $1/8$; this minor mistake carries over to all the equations that follow.

One may now use [1, Formulae (2-3)] to compute the difference between the connection 1-forms of $D$ and $D^b$ (or, equivalently, the connection 1-form of $D^b$) with respect to the covariant derivative $D^b \cdot A$. Following again Andersson and Dahl’s argument [1], and noting that all the imaginary-valued terms have to cancel out because the left-hand-side of (4.8) is real, one eventually gets:
Using the (spin) isomorphism $A$, this can immediately be rewritten as:

\begin{align}
B_{Ab}(\varphi_u) &= \frac{1}{2} \sum_{i=1}^{n-1} \left( g((D_{Ab}^b A) e_i, Ae_i) - g((D_{Ae}^b A) e_i, Ae_i) \right) \langle \varphi_u, \varphi_u \rangle_g \\
&\quad + \frac{1}{4} \sum_{i,j,k \text{ distinct}} g((D_{e_i}^b A) e_j, Ae_k) \langle c_b(A
\nu)c_b(Ae_i)c_g(Ae_j)c_g(Ae_k)\varphi_u, \varphi_u \rangle_g \\
&\quad + \frac{i}{2} \sum_{i=1}^{n-1} \langle c_b(A\nu)c_g(Ae_i)c_g(A(Ae_i) - Ae_i) \varphi_u, \varphi_u \rangle_g.
\end{align}

Using the (spin) isomorphism $A$, this can immediately be rewritten as:

\begin{align}
B_{Ab}(\varphi_u) &= \frac{1}{2} \sum_{i=1}^{n-1} \left( b(A^{-1}(D_{Ab}^b A)e_i, e_i) - b(A^{-1}(D_{Ae}^b A)\nu, e_i) \right) \langle \varphi_u, \varphi_u \rangle_b \\
&\quad + \frac{1}{4} \sum_{i,j,k \text{ distinct}} b(A^{-1}(D_{e_i}^b A)e_j, e_k) \langle c_b(\nu)c_b(e_i)c_b(e_j)c_b(e_k)\varphi_u, \varphi_u \rangle_b \\
&\quad + \frac{i}{2} \sum_{i=1}^{n-1} \langle c_b(\nu)c_b(e_i)c_b(Ae_i - e_i) \varphi_u, \varphi_u \rangle_b,
\end{align}

where all computations take now place on the spinor bundle of the reference hyperbolic metric. Taking into account the relationship between the squared norm of $\varphi_u$ and $V_u$, we now recall that our asymptotic conditions imply that, in the last formula, any quadratic term in $A - I$ and $D_{bA}^b A$ (or, equivalently, in $e = g - b$), when integrated on $S_r$, has limit value zero as $r$ goes to infinity. One may then eliminate a large number of occurrences of the map $A$ from the above formula. We will use below the notation $U \simeq V$ to mean that $V$ is the only term that contributes when integrating $U$ on larger and larger spheres. Equivalently,

$$U \simeq V \implies \lim_{r \to \infty} \int_{S_r} U = \lim_{r \to \infty} \int_{S_r} V.$$ 

Then, one obtains in our case:
(4.21)

\[
B_{Av}(\varphi_u) \approx \frac{1}{2} \sum_{i=1}^{n-1} \left( b((D^b_{e_i} A)e_i, e_i) - b((D^b_{e_i} A)\nu, e_i) \right) \langle \varphi_u, \varphi_u \rangle_b + \frac{1}{4} \sum_{i,j,k \text{ distinct}} b((D^b_{e_i} A)e_j, e_k) \langle c_b(\nu)c_b(e_i)c_b(e_j)c_b(e_k)\varphi_u, \varphi_u \rangle_b + \frac{i}{2} \sum_{i=1}^{n-1} (c_b(\nu)c_b(e_i)c_b(e_i - e_i) \varphi_u, \varphi_u \rangle_b.
\]

Furthermore, the last term in the previous formula is easily computed and it remains:

\[
B_{Av}(\varphi_u) \approx \frac{1}{2} \sum_{i=1}^{n-1} \left( b((D^b_{e_i} A)e_i, e_i) - b((D^b_{e_i} A)\nu, e_i) \right) \langle \varphi_u, \varphi_u \rangle_b + \frac{1}{4} \sum_{i,j,k \text{ distinct}} b((D^b_{e_i} A)e_j, e_k) \langle c_b(\nu)c_b(e_i)c_b(e_j)c_b(e_k)\varphi_u, \varphi_u \rangle_b - \frac{i}{2} (\text{tr}(A - I) \langle c_b(\nu)\varphi_u, \varphi_u \rangle_b - \langle c_b((A - I)\nu)\varphi_u, \varphi_u \rangle_b).
\]

The second term in this last formula contributes as zero, since \( A \) (hence \( D^b A \)) is symmetric and \( e_i \cdot e_j \cdot \) is antisymmetric. We then get:

\[
\lim_{r \to \infty} \int_{S_r} B_{Av}(\varphi_u) = \lim_{r \to \infty} \int_{S_r} \frac{1}{2} \sum_{i=1}^{n-1} \left( b((D^b_{e_i} A)e_i, e_i) - b((D^b_{e_i} A)\nu, e_i) \right) \langle \varphi_u, \varphi_u \rangle_b - \frac{i}{2} (\text{tr}(A - I) \langle c_b(\nu)\varphi_u, \varphi_u \rangle_b - \langle c_b((A - I)\nu)\varphi_u, \varphi_u \rangle_b).
\]

To compare with our previous formulae for mass integrals, we now replace \( A \) by \( I - \frac{1}{4} e \) (as remainder terms in a Taylor expansion contribute as zero in the limit) and relate the norms \( \langle \varphi_u, \varphi_u \rangle_b \) and \( \langle c_b(\nu)\varphi_u, \varphi_u \rangle_b \) to \( V_u \) using \( |\varphi_u|^2_b = V_u \) and \( i\langle c_b(X)\varphi_u, \varphi_u \rangle_b = dV_u(X) \) to obtain in the limit:

\[
- \frac{1}{4} \lim_{r \to \infty} \int_{S_r} V_u \left( d(\text{tr}_b e)(\nu) - \sum_{i=1}^{n-1} D^b_{e_i} e(\nu, e_i) \right) - (\text{tr}_b e) dV_u(\nu) + dV_u(e(\nu)),
\]

or, equivalently,

\[
(4.22) \quad \lim_{r \to \infty} \int_{S_r} B_{Av}(\varphi_u) = \frac{1}{4} \lim_{r \to \infty} \int_{S_r} U^b \nu_t.
\]

As the left-hand side in the Schrödinger-Lichnerowicz formula (4.11) is nonnegative, this implies that the mass (seen as a linear functional on \( \mathcal{N}_{b0} \)) is nonnegative on any future-directed null vector in \( \mathcal{N}_{b0} \). Standard considerations in Lorentzian geometry yield that this linear functional is causal.
and future directed (notice that we use here only the existence of imaginary Killing spinors of Type I, which is valid in any dimension). It remains to show that it is either timelike, or vanishes.

Suppose, then, that the mass $H_\Phi$ (still seen as a linear functional on $N_{b_0}$) is isotropic, then there exists a nonzero element $W$ of the light cone in $N_{b_0}$ such that it evaluates against $W$ as zero. Up to rescaling, $W$ can be written as $V(0) - \sum a_i V(i)$ and we already noticed there exists a constant spinor $u$ on $(\mathbb{B}, e)$ such that $V_u = W$. The Lichnerowicz-Schrödinger formula above has then a vanishing contribution at infinity. This implies the associated spinor $\Phi_u$ is a Killing spinor, and H. Baum’s work shows that $(M, g)$ is isometric to the hyperbolic space ([1] and [6]). The mass functional $H_\Phi$ is then zero and this ends the Proof of Theorem 4.1.

\textit{Manifolds with boundary.} When $(M, g)$ has a compact boundary one expects that the correct statement is the Penrose inequality ([11, 26] and [27]), which seems to lie outside of the scope of the Witten-type argument given above. Recall, however, that this last argument does lead to a positivity statement ([21, 22] and [23]) when compact boundaries occur:

\textbf{Theorem 4.7.} Let $(M, g)$ be a complete spin manifold with a $C^2$ metric, with a compact nonempty boundary of mean curvature

$$\Theta \leq n - 1,$$

and with scalar curvature satisfying

$$R_g \geq -n(n - 1).$$

If the asymptotic conditions (2.8) and (2.13) hold with $(N, \tilde{h}) = (\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$, then the covector $p(\mu)$ defined by Equation (3.5) is timelike future directed.

\textit{Proof.} When $\partial M$ is nonempty, a supplementary boundary integral over $\partial M$, given by

$$\int_{\partial M} B_{A\nu}(\Phi_u) = \int_{\partial M} \left\langle D_{\partial M} \Phi_u + \frac{1}{2}(\Theta - (n - 1)icg(n)) \Phi_u, \Phi_u \right\rangle,$$

appears in Equations (4.8) and (4.11), where $\Theta$ is the inwards oriented mean extrinsic curvature of $\partial M$, while $D_{\partial M}$ is a boundary Dirac operator. It is defined as

$$D_{\partial M} = c_g(n) \sum_{i=1}^{n-1} c_g(e_i) D_{e_i},$$

where $\mathcal{D}$ is the spin connection intrinsic to $\partial M$, explicitly defined on spinors fields on $M$ restricted to $\partial M$ as

$$\mathcal{D}_X = D_X - \frac{1}{2}c_g(n)c_g(B(X)),$$

where $B$ is the shape operator of the boundary.
Following [21] (compare [23]) we impose the following boundary condition on the spinors $\Phi_u$:

\begin{equation}
\Phi_u = \varepsilon \Phi_u, \tag{4.25}
\end{equation}

where $\varepsilon$ is a hermitian involution on spinors given by

\begin{equation}
\varepsilon = icg(\nu) \quad \text{as in [24].} \tag{4.26}
\end{equation}

It is proved in this paper that this leads to a self-adjoint elliptic problem for the Dirac operator which can be solved. Positivity of the mass is obtained through the same argument as before, the boundary contribution (4.23) having the correct sign since $\varepsilon \mathcal{D}_{\partial M} = -\mathcal{D}_{\partial M} \varepsilon$ so that

\begin{equation}
\langle \Phi_u, \mathcal{D}_{\partial M} \Phi_u \rangle = \langle \Phi_u, \mathcal{D}_{\partial M} \varepsilon \Phi_u \rangle = -\langle \Phi_u, \varepsilon \mathcal{D}_{\partial M} \Phi_u \rangle = -\langle \varepsilon \Phi_u, \mathcal{D}_{\partial M} \Phi_u \rangle = -\langle \Phi_u, \mathcal{D}_{\partial M} \Phi_u \rangle. \tag{4.27}
\end{equation}

As a result, $\langle \Phi_u, \mathcal{D}_{\partial M} \Phi_u \rangle$ vanishes and it remains

\begin{equation}
\int_{\partial M} B_{\mu
u}(\Phi_u) = \int_{\partial M} \left\langle \frac{1}{2} (\Theta - (n-1)) \Phi_u, \Phi_u \right\rangle \tag{4.28}
\end{equation}

for the boundary contribution. This proves as above that the covector $p(\mu)$ defined by Equation (3.5) is timelike future directed, or lightlike future directed, or vanishing. Let us show that those last two possibilities cannot occur: Clearly, $p(\mu)$ can be lightlike or vanish if and only if $M$ carries an imaginary Killing spinor $\Phi_u$ satisfying the boundary condition (4.25) at $\partial M$. Further, Equation (4.28) implies that $\Theta$ is identically equal to $(n-1)$ — otherwise, the imaginary Killing spinor field $\Phi_u$ would be zero on an open set on the boundary, a situation which is forbidden by the uniqueness property of solutions of ordinary differential equations. Moreover it is a classical fact [6] that existence of an imaginary Killing spinor implies that $(M,g)$ is Einstein, of scalar curvature $-n(n-1)$.

Hence we have the following situation: A noncompact Einstein manifold, looking like the hyperbolic space at infinity and with a compact inner boundary of constant mean curvature $n-1$. Choose any very large sphere-like compact submanifold $S$ in the asymptotically hyperbolic end of $M$ and consider the part of $M$ located inside $S$. It is a compact Einstein $n$-manifold with boundary having two components, one of constant mean curvature $\Theta_{\partial M} = n-1$ and the other one having (not necessarily constant) mean curvature $\Theta_S$ close to that of a sphere in the hyperbolic space by (2.13), hence which can be taken so that $\Theta_S > n-1$ at each point of $S$ (here, both mean curvatures are computed with respect to the normal unit vector pointing towards infinity).
Let now \( p \) be in \( \partial M \) the closest point to \( S \) and \( \gamma \) be a minimising geodesic from \( S \) to \( p \), starting from a point \( q \) in \( S \). Let \( \ell \) be the distance from \( \partial M \) to \( S \), i.e., the distance from \( p \) to \( q \). We now consider the family \( \partial M_\delta \) of submanifolds obtained by pushing the boundary \( \partial M \) a distance \( \delta \) through its normal exponential map towards \( S \), and the analogously defined submanifolds \( S_\eta \) obtained from \( S \) by pushing it a distance \( \eta \) towards \( \partial M \).

For \( \delta > 0 \) small enough, the submanifold \( \partial M_\delta \) is still smooth. Moreover, the contact point \( r \) of \( \partial M_\delta \) and \( \gamma \) is necessarily the closest point of \( \partial M_\delta \) to \( S \) (and is at distance \( \ell - \delta \)). As \( \gamma \) is minimising, the distance function to \( S \) is smooth in an open neighborhood of \( \gamma \setminus \{p\} \), hence the submanifold \( S_{\ell-\delta} \) is smooth around \( r \), contained in the (closure of the) unbounded part of \( M \) delimited by \( \partial M_\delta \) and is necessarily tangent at \( r \) to \( \partial M_\delta \).

It remains to show that this leads to a contradiction. This follows from classical comparison geometry: The usual Riccati equation for the normalised mean curvature \( H = \frac{\Theta}{n-1} \) reads [13]:

\[
H' \leq -H^2 - \frac{\text{Ric}(\gamma', \gamma')}{n-1}
\]

where \( H \) stands either for the outwards normalised mean curvature of the family \( \{\partial M_\delta\} \) or for the outwards normalised mean curvature of the family \( \{S_\eta\} \) and \( ' \) denotes differentiation with respect to either \( \delta \) or \( -\eta \). As \((M,g)\) is Einstein, this translates in our context as:

\[
H' \leq 1 - H^2
\]

and one gets by standard arguments that

\[
H_{\partial M_\delta} \leq 1 \quad \text{and} \quad H_{S_\eta} > 1 \quad \text{for any} \quad \eta > 0, \delta > 0.
\]

At the point \( r \), this contradicts the comparison principle for the mean curvature equation, which ends the proof. \( \square \)

**Remark 4.8.** This result is of special interest in general relativity, where the condition on the mean curvature (\( \Theta \leq n-1 \)) has the following interpretation: Let \( \alpha \in \mathbb{R} \) be a constant satisfying \( |\alpha| \leq 1 \), then our Riemannian manifold can be thought of as arising from a spacelike slice \( i(M) \) in a vacuum space-time with cosmological constant \( \Lambda := -(1-\alpha^2)n(n-1) \leq 0 \), such that \( i(M) \) has extrinsic curvature \( K_{ij} = \alpha g_{ij} \). The condition \( R_g \geq -n(n-1) \) is then equivalent to requiring positive energy density on \( i(M) \), while the condition \( \Theta \leq -\alpha(n-1) \) is equivalent to the statement that \( \partial M \) is an outer-future-trapped, or marginally outer-future-trapped, compact hypersurface in \( i(M) \). Under suitable global conditions existence of such surfaces implies existence of a black hole region in the associated space-time. Similarly the condition \( \Theta \leq \alpha(n-1) \) is associated with outer-past-trapped surfaces, and leads to existence of white hole regions. A significant consequence of the above result is then that the trapped surface situation is far away from...
the case of vanishing mass. This gives mathematical support (disjoint from all known physical reasons) to the idea that some statement analogous to the Penrose inequality ([11] and [27]) should hold in the asymptotically hyperbolic case as well.

**Remark 4.9.** In the special (and interesting for physics) case $\alpha = 0$ (i.e., $\Theta \leq 0$), there is a shorter way to prove that mass cannot vanish: In the proof of Theorem 4.7, one may take for $\varepsilon$ any self-adjoint involution satisfying $c_g(n)\varepsilon = -\varepsilon c_g(n)$ and $\varepsilon \mathcal{D}_{\partial M} = -\mathcal{D}_{\partial M} \varepsilon$ (such an $\varepsilon$ will certainly exist if our Riemannian manifold is isometrically embedded as a Riemannian slice in a Lorentzian $(n+1)$-dimensional manifold, or more generally, if the spinor bundle carries a representation of the Clifford algebra of the Lorentzian metric $\gamma = -e_0 \otimes e_0 + g$; in any of those cases one sets $\varepsilon = \gamma(e_0)\gamma(n)$ — note however that, in the rest of the proof as well as in the other parts of the paper, our discussion will stay purely Riemannian, as opposed to [21, 23] and [34]). A calculation identical to (4.27) shows that the boundary integral will have the right sign and the proof goes through without modifications, implying that the covector $p(\mu)$ is timelike future directed, or lightlike future directed, or vanishing. Assuming one of the last two conclusions, the equality case in the Lichnerowicz-Weitzenböck formula yields again existence of an imaginary Killing spinor. When restricted to the boundary, this spinor field would then be an eigenspinor of the formally self-adjoint boundary Dirac operator $\mathcal{D}_{\partial M}$ for a purely imaginary eigenvalue, which is certainly impossible on a compact manifold.

### 5. The mass of conformally compactifiable asymptotically hyperbolic ends.

The metric $g$ of a Riemannian manifold $(M, g)$ will be said to be $C^k$ compactifiable if there exists a compact Riemannian manifold with boundary $(\overline{M} \approx M \cup \partial_\infty M \cup \partial M, \tilde{g})$, where $\partial \overline{M} = \partial M \cup \partial_\infty M$ is the metric boundary of $(\overline{M}, \tilde{g})$, with $\partial M$ — the metric boundary of $(M, g)$, together with a diffeomorphism

$$\psi : \text{int}\overline{M} \to M$$

such that

$$\psi^* g = \Omega^{-2}\tilde{g},$$

where $\Omega$ is a defining function for $\partial_\infty M$ (i.e., $\Omega \geq 0$, $\{\Omega = 0\} = \partial_\infty M$, and $d\Omega$ is nowhere vanishing on $\partial_\infty M$), with $\tilde{g}$ — a metric which is $C^k$ up-to-boundary on $\overline{M}$. The triple $(\overline{M}, \tilde{g}, \Omega)$ will then be called a $C^k$ conformal completion of $(M, g)$. Clearly the definition allows $M$ to have a usual compact boundary. $(M, g)$ will be said to have a conformally compactifiable end $M_{\text{ext}}$ if $M$ contains an open submanifold $M_{\text{ext}}$ (of the same dimension that
$M$ such that $(M_{\text{ext}}, g|_{M_{\text{ext}}})$ is conformally compactifiable, with a connected conformal boundary $\partial_\infty M_{\text{ext}}$.

In the remainder of this section we shall assume for simplicity that the conformally rescaled metric $\tilde{g}$ is smooth up to boundary; it should be clear how the conditions here can be adapted to a weighted Hölder or Sobolev setting to allow lower differentiability compactifications consistent with the requirements of Theorem 2.3.

It is easily seen, using the transformation properties of the Riemann tensor under conformal transformations (cf., e.g., [29]) that for smoothly compactifiable metrics all the sectional curvatures $\kappa$ of $g$ satisfy
\begin{equation}
\left(\kappa + |d\Omega|_g^2\right)(p) \to_{p\to\partial_\infty M} 0,
\end{equation}
where $|\cdot|_k$ denotes the norm of a tensor with respect to a metric $k$; recall that $g^#$ is the metric on $T^*M$ associated to $g$.

Now, Equation (5.1) determines only the conformal class $[\tilde{g}]$ of $\tilde{g}$. Without loss of generality we can restrict the representative $\tilde{g}$ of $[\tilde{g}]$ so that the metric $h_0$ induced by $\tilde{g}$ on $\partial_\infty M$ has constant scalar curvature normalised as in (1.3), and this restriction will be made in what follows.

A compactifiable metric will be called *asymptotically hyperbolic* in an end $M_{\text{ext}}$ if
\begin{equation}
\forall p \in \partial_\infty M_{\text{ext}} \quad |d\Omega|_g^2(p) = 1.
\end{equation}
In what follows we restrict our considerations to a single end $M_{\text{ext}}$, replacing $M$ by $M_{\text{ext}}$ we will assume that $M = M_{\text{ext}}$, so that $\partial_\infty M = \partial_\infty M_{\text{ext}}$. Whenever (5.3) holds on $\partial_\infty M$, a preferred representative of $[\tilde{g}]$ in a neighborhood of $\partial_\infty M$ can be chosen by requiring that
\begin{equation}
|d\Omega|_g^2 = 1.
\end{equation}
Using $x := \Omega$ as the first coordinate, a coordinate system can be constructed (in some perhaps smaller neighborhood of $\partial_\infty M$) in which $g$ takes the form
\begin{equation}
g = x^{-2} (dx^2 + h_x),
\end{equation}
where $h_x$ is an $x$-dependent family of metrics on $N := \partial_\infty M$. We define the reference metric $b$ as
\begin{equation}
b := x^{-2} (dx^2 + (1 - kx^2/4)^2 h_0)
\end{equation}
(compare Equation (1.8)). To make contact with Section 3, we assume that $b$ is one of the metrics considered there. If $r$ is defined by Equation (1.7), then the asymptotic conditions of Proposition 2.2 and of Theorem 2.3 will hold if and only if
\begin{equation}
0 \leq i \leq \lfloor n/2 \rfloor \quad \partial^i_x (h_x - (1 - kx^2/4)^2 h_0) \bigg|_{x=0} = 0
\end{equation}
\[ \iff \quad h_x = (1 - kx^2/4)^2 h_0 + o(x^{\lfloor n/2 \rfloor}), \]
where \([\alpha]\) denotes the integer value of \(\alpha\), and if
\[
R_g + n(n - 1) = O(x^{n-1}).
\]
For instance, in the physically significant case \(n = 3\), Equation (5.6) is equivalent to the requirement that the second fundamental form of \(\partial_\infty M\) vanishes in the conformal gauge (5.4).

Under Equations (5.6)-(5.7), given some compactification \((\mathcal{M}_1, g_1, \Omega_1)\) of an end \((M_{\text{ext}}, g)\) of \(M\), we use the background (5.5) to define its mass, whenever the resulting background is one of those discussed in Section 3. (As already pointed out, when \((\partial_\infty M, h_0) = (\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})\) this definition coincides with that of [39].) Consider a second compactification \((\mathcal{M}_2, g_2, \Omega_2)\) of \((M, g)\) satisfying the above requirements; it is far from clear that the resulting mass will be the same. This turns out to be the case:

\textbf{Theorem 5.1.} Suppose that \((M, g)\) contains a conformally compactifiable end \(M_{\text{ext}}\) such that \((\partial_\infty M_{\text{ext}}, g|_{\partial_\infty M_{\text{ext}}})\) is one of the manifolds considered in Section 3 (that is, the scalar curvature of \(\tilde{h}\) is as in (1.3) and either \(\tilde{h}\) has strictly negative Ricci curvature, or is flat, or is the round sphere, or a quotient thereof). Assume, moreover, that Equations (5.6)-(5.7) hold. Then the mass of \(M_{\text{ext}}\), as defined above, is independent of the compactification of \(M_{\text{ext}}\) chosen for its calculation.

\textbf{Proof.} As already pointed out above, we can modify the \(g_a\)’s and \(\Omega_a\)’s so that
\[
|d\Omega_a|_{g_a} = 1
\]
in a neighborhood of \(\partial_\infty M_a\). Using the \(\Omega_a\)’s as the first coordinate, in neighborhoods of respective boundaries we can write the metrics \(g_a\) as
\[
g_a = d\Omega_a^2 + h^a,
\]
where \(h^a\) is the metric induced by \(g_a\) on the level sets of \(\Omega_a\). We introduce radial coordinates \(r_a\) as in (1.7),
\[
r_a = \frac{1 - k\Omega_a^2/4}{\Omega_a},
\]
so that
\[
g = \frac{dr_1^2}{r_1^2 + k} + r_1^2 h_{AB}^1 dv_1^A dv_1^B = \frac{dr_2^2}{r_2^2 + k} + r_2^2 h_{AB}^2 dv_2^A dv_2^B,
\]
where \(k\) is defined by Equation (1.3) using the boundary metric arising out from \(g_1\), and we have denoted by \((r_a, v_a^A)\), \(a = 1, 2\), the corresponding local coordinates near \(\partial_\infty M_a\). It follows from [16, Theorem 3.3] that the map
\[
(r_1, v_1^A) \rightarrow (r_2, v_2^A)
\]
extends by continuity to a differentiable map from $\overline{M}_1$ to $\overline{M}_2$. Equivalently, $\phi^{-1}_1 \circ \phi_2$ extends by continuity to a continuous map from $\overline{M}_1$ to $\overline{M}_2$. Further, point 1 of [16, Theorem 3.3] shows that the limit
\[
\lim_{r_1 \to \infty} v_2^A(r_1, v_1^A)
\]
exists, and defines a $C^\infty$ conformal diffeomorphism $\Psi$ from $(\partial_\infty M_1, h^1|_{\partial_\infty M_1})$ to $(\partial_\infty M_2, h^2|_{\partial_\infty M_2})$:
\[
\Psi^* h^2|_{\partial_\infty M_2} = e^\psi h^1|_{\partial_\infty M_1}.
\]
Replacing $g_1$ by $e^\psi g_1$ and $\Omega_1$ by $\Omega_1 e^{\psi/2}$, where, by an abuse of notation, we use the same symbol $e^\psi$ to denote the extension of $e^\psi$ from $\partial_\infty M_1$ to $\overline{M}_1$ such that
\[
|d(\Omega_1 e^{\psi/2})(e^\psi g)| = 1,
\]
we obtain Equation (5.9) with $\psi = 0$, hence $h^1|_{\partial_\infty M_1}$ is isometric to $h^2|_{\partial_\infty M_2}$. As a result, Theorem 2.3 together with the discussion of Section 3 establishes Theorem 5.1.

We note that the argument just given also proves the following:

**Proposition 5.2.** Consider $(M_{\text{ext}}, b)$ with a metric of the form (1.2). Then for every conformal isometry $\Psi$ of $(N, \tilde{h})$ there exists $R_\ast > 0$ and a $b$-isometric map $\Phi : [R_\ast, \infty) \times N \to [R, \infty) \times N$, such that
\[
\lim_{r \to \infty} \Phi(r, \cdot) = \Psi(\cdot).
\]

### 6. Uniqueness of conformal completions.

It should be clear that the invariance of the mass is related to the question of uniqueness of conformal compactifications. There are several issues to address here: $g_1$ is conformal to an appropriate pull-back of $g_2$ on the interior of $\overline{M}_1$, but the relative conformal factor could blow up as one approaches the boundary of $M_1$. Even if the relative conformal factor remains uniformly bounded both from above and below, it is not clear whether or not it extends differentiably — or even just continuously — to the boundary. Let us show that things behave as expected, so that conformal completions are conformally diffeomorphic in the sense of manifolds with boundary; notations are the same as in the previous section.

**Theorem 6.1.** Let $(M, g)$ be a Riemannian manifold endowed with two $C^\infty$-conformal compactifications $(\overline{M}_1, g_1, \Omega_1)$ and $(\overline{M}_2, g_2, \Omega_2)$. Then
\[
\phi^{-1}_1 \circ \phi_2 : \text{int} M_2 \to \text{int} M_1
\]
extends by continuity to a $C^\infty$ conformal up-to-boundary diffeomorphism from $(\overline{M}_2, g_2)$ to $(\overline{M}_1, g_1)$, in particular $\overline{M}_1$ and $\overline{M}_2$ are diffeomorphic as manifolds with boundary.
Remark 6.2. Results about completions of finite differentiability can be obtained in a similar way, by chasing the order of differentiability through various steps of the arguments below.

Proof. Let \( \varphi_2 \) be defined on \( \text{int} M_1 \) by the equation
\[
\varphi_2 := \frac{\Omega_2 \circ \phi_2^{-1} \circ \phi_1}{\Omega_1} > 0;
\]
Equation (5.8) gives
\[
1 = \left| d\Omega_2 \right|^2_{g_2^\#}
= \left| \varphi_2 d\Omega_1 + \Omega_1 d\varphi_2 \right|^2_{g_2^\#}
= \varphi_2^2 \left| d\Omega_1 \right|^2_{g_2^\#} + 2\varphi_2 \Omega_1 g_2^\# (d\varphi_2, d\Omega_1) + \Omega_1^2 \left| d\varphi_2 \right|^2_{g_2^\#}
= \left| d\Omega_1 \right|^2_{g_1^\#} + 2\varphi_2^{-1} \Omega_1 g_1^\# (d\varphi_2, d\Omega_1) + \varphi_2^{-2} \Omega_1^2 \left| d\varphi_2 \right|^2_{g_1^\#},
\]
hence
\[
2\Omega_1 g_1^\# (d(\ln \varphi_2), d\Omega_1) = -\Omega_1^2 \left| d(\ln \varphi_2) \right|^2_{g_1^\#} \leq 0.
\]
We can identify a neighborhood of \( \partial M_1 \) with \( \partial M_1 \times [0, x_0] \) using the flow of \( g_1^\# (d\Omega_1, \cdot) \). Equation (6.2) shows that \( \ln \varphi_2 \) is monotonously increasing along the integral curves of the vector field \( g_1^\# (d\Omega_1, \cdot) \) when \( \Omega_1 \) decreases, so that there exists a constant \( C_2 := \inf_{\partial M_1 \times \{x_0\}} \varphi_2 \) such that on \( \partial M_1 \times [0, x_0] \) we have
\[
\varphi_2 \geq C_2 > -\infty.
\]
Applying the same argument, with \( g_1 \) and \( g_2 \) interchanged, to
\[
\varphi_1 := \frac{\Omega_1 \circ \phi_1^{-1} \circ \phi_2}{\Omega_2} = \frac{1}{\varphi_2} \circ \phi_1^{-1} \circ \phi_1 : \text{int} M_2 \to \mathbb{R},
\]
shows that on \( \partial M_1 \times [0, x_0] \) it holds
\[
\varphi_1 \geq C_1 > -\infty.
\]
Equations (6.3) and (6.4) clearly imply that the \( \varphi_a \)'s are uniformly bounded and uniformly bounded away from zero.
Set
\[
\phi_{12} := \phi_1^{-1} \circ \phi_2, \quad \phi_{21} := \phi_2^{-1} \circ \phi_1.
\]
Let $\sigma_{ga}$ denote the distance function associated with the metric $g_a$. For $p, q \in \text{int} M_1$ we have

\begin{equation}
\sigma_{g_2}(\phi_{21}(p), \phi_{21}(q)) = \inf_{\Gamma} \int_{\Gamma} \sqrt{g_2 \left( \frac{d\Gamma}{ds}, \frac{d\Gamma}{ds} \right)}
\geq \inf_{\Gamma} C \int_{\Gamma} \sqrt{\left( \phi_{12}^* g_1 \right) \left( \frac{d\Gamma}{ds}, \frac{d\Gamma}{ds} \right)}
= C \inf_{\phi_{12}(\Gamma)} \int_{\phi_{12}(\Gamma)} \sqrt{g_1 \left( \frac{d\phi_{12}(\Gamma)}{ds}, \frac{d\phi_{12}(\Gamma)}{ds} \right)}
= C \sigma_{g_1}(p, q).
\end{equation}

This, together with an identical calculation with $g_1$ and $g_2$ interchanged shows that $\phi_{12}$ and $\phi_{21}$ are uniformly Lipschitz continuous.

Clearly, $\overline{M}_2$ is the metric completion of the manifold $M$ with respect to the metric $(\phi_{a}^{-1})^* g_a$; similarly for $\overline{M}_1$. An identical calculation shows that the metrics $(\phi_{a}^{-1})^* g_a$, $a = 1, 2$ define uniformly equivalent distance functions. But completions obtained using equivalent distances are homeomorphic; it follows that $\overline{M}_1$ is homeomorphic to $\overline{M}_2$, in particular $\partial M_1$ is homeomorphic to $\partial M_2$. In fact, by definition we have

\begin{equation}
\phi_{21} \circ \phi_{12} = \text{id}_{\overline{M}_2}, \quad \phi_{12} \circ \phi_{21} = \text{id}_{\overline{M}_1}.
\end{equation}

Since $\phi_{12}$ and $\phi_{21}$ are continuous, they have an extension by continuity to the metric completed spaces; we will use the same symbol to denote those extensions. It is then easily seen that (6.6) with $M_a$ replaced by $\overline{M}_a$ holds for the extensions, so that the extensions do directly provide the desired homeomorphism. Equation (6.5), together with its equivalent with $g_1$ interchanged with $g_2$, further show that the extensions $\phi_{21}$ and $\phi_{12}$ are uniformly Lipschitz continuous on $\overline{M}_1$ and $\overline{M}_2$. Obviously

\[ \phi_{21} : \partial M_1 \to \partial M_2, \quad \phi_{12} : \partial M_2 \to \partial M_1, \]

with $\phi_{21}|_{\partial M_1}$ and $\phi_{12}|_{\partial M_2}$ being homeomorphisms inverse to each other by the completed spaces equivalent of (6.6). We have:

**Lemma 6.3.** The map $\phi_{21}$ is $C^1$ up-to-boundary.

**Proof.** We can conformally rescale $g$ so that (5.3) holds with $\tilde{g} = g_1$ and $\tilde{\Omega} = \Omega_1$; as (5.3) is conformally invariant, (5.3) will also hold with $\tilde{g} = g_2$ and $\tilde{\Omega} = \Omega_2$. Introducing coordinates $r_a$ as in the proof of Theorem 5.1 with, say $k = 0$, we can apply Theorem 3.3 of [16] to obtain the desired conclusion. \qed
Returning to the proof of Theorem 6.1, Lemma 6.3 shows that for all \( p \in \partial M_1 \) the maps \( (\phi_{21})_*(p) \) are similarities with nonzero ratio (in general depending upon \( p \)). Differentiability of \( \phi_{21} \) further implies that \( \phi_{21} \) is ACL\(^n\), as defined in [31]. We can then use a deep result of Lelong-Ferrand [31, Theorem A] to conclude that \( \phi_2|_{\partial M_1} \) and \( \phi_{21}|_{\partial M_1} \) are smooth. Now, \( u := (\varphi_2)^{2\frac{n-2}{n}} \) solves the Yamabe equation,

\[
\Delta_{g_1} u - \frac{n - 2}{4(n - 1)} R_{g_1} u = (R_{g_2} \circ \phi_{21}) u^{(n+2)/(n-2)}.
\]

Here, as before, \( R_{g_a} \) denotes the curvature scalar of the metric \( g_a \). The right-hand-side of this equation is in \( L^\infty(\overline{M}_1) \), and standard results on the Dirichlet problem imply that \( u \) — and hence \( \varphi_2 \) — is uniformly \( C^1 \) on \( \overline{M}_1 \). Now, \( \phi_{21} \) is an isometry between \( g_1 \) and \( \varphi_2^2 g_2 \) which implies that, in local coordinates, \( \phi_{21} \) satisfies on \( M_1 \) the over-determined set of equations

\[
\frac{\partial^2 \phi_{21}}{\partial x^\ell \partial x^m} = \Gamma^k_{\ell m}(x) \frac{\partial \phi_{21}}{\partial x^k} - \Gamma^i_{rs}(x) \frac{\partial \phi_{21}^s}{\partial x^\ell} \frac{\partial \phi_{21}^r}{\partial x^m},
\]

where \( \Gamma^i_{rs} \) are, in local coordinates, the Christoffel symbols of the Riemannian metric \( (\varphi_2 \circ \phi_{12})^2 g_2 \). The right-hand-side of this set of equations extends by continuity to a continuous function on \( \overline{M}_1 \), which shows that \( \phi_{21} \) is uniformly \( C^2 \) on \( M_1 \). It follows that the right-hand-side of Equation (6.7) is uniformly \( C^1 \) on \( \overline{M}_1 \), hence \( \varphi_2 \) is uniformly \( C^2 \) on \( \overline{M}_1 \). An inductive repetition of this argument establishes our claims.

\[\square\]

**Remark 6.4.** It would be of interest to find a proof of Lemma 6.3 which is more in the spirit of conformal geometry than the methods of [16].

**Remark 6.5.** Recall that there exists a conformally invariant version of the Abbott-Deser mass, due to Ashtekar and Magnon [2], in a Lorentzian spacetime setting. We expect this expression to have a Riemannian counterpart which is also conformally invariant, with the numerical value thereof identical to that of the expression we propose. If that is the case, Theorem 5.1 is actually a straightforward corollary of Theorem 6.1; in particular one would not need to invoke the rather messy calculations of Theorem 2.3, which are implicitly used in the proof of Theorem 5.1.

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RIBAUCOUR TRANSFORMATIONS FOR CONSTANT MEAN CURVATURE AND LINEAR WEINGARTEN SURFACES

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We provide a method to obtain linear Weingarten surfaces from a given such surface, by imposing a one parameter algebraic condition on a Ribaucour transformation. Our main result extends classical results for surfaces of constant Gaussian or mean curvature. By applying the theory to the cylinder, we obtain a two-parameter family of complete linear Weingarten surfaces (hyperbolic, elliptic and tubular), asymptotically close to the cylinder, which have constant mean curvature when one of the parameters vanishes. The family contains \( n \)-bubble Weingarten surfaces which are 1-periodic, have genus zero and two ends of geometric index \( m \), where \( n/m \) is an irreducible rational number. Their total curvature vanishes, while the total absolute curvature is \( 8\pi n \). We also apply the method to obtain families of complete constant mean curvature surfaces, associated to the Delaunay surfaces, which are 1-periodic for special values of the parameter.

Introduction.

In the last two decades a great activity in research has been devoted to surfaces of constant mean curvature (cmc) surfaces. After the first example of a non-totally umbilical compact cmc surface immersed in \( \mathbb{R}^3 \) found by Wente [W1], a series of papers by Meeks [M], Korevaar, Kusner and Solomon [KKS], Pinkal and Sterling [PS], Kapouleas [K], Karcher [Ka], Abresch [A], Walter [Wa] gave important contributions to the theory and to the construction of examples of complete immersed cmc surfaces.

Some of the results proved for cmc surfaces were also extended by Rosenberg and Sa Earp [RS], and by Brito and Sa Earp [BS] to the so called special Weingarten immersed surfaces, and in particular to surfaces whose mean curvature \( H \) and Gaussian curvature \( K \) satisfy a linear relation \( aH + K = b \) where \( a \geq 0, b > 0 \).

In this paper, we consider a method of constructing linear Weingarten surfaces based on Ribaucour transformations. A linear Weingarten surface of \( \mathbb{R}^3 \) is a surface whose Gaussian curvature \( K \) and mean curvature \( H \) satisfy a linear relation \( \alpha + \beta H + \gamma K = 0 \), where \( \alpha, \beta, \gamma \in \mathbb{R} \). Such a surface is said
to be hyperbolic (resp. elliptic) when $\Delta := \beta^2 - 4\alpha\gamma < 0$ (resp. $\Delta > 0$). The relation $\Delta = 0$ characterizes the tubular surfaces. In particular, surfaces of constant negative Gaussian curvature are hyperbolic, while surfaces of constant mean curvature (including minimal) and constant positive curvature are elliptic. Ribaucour transformations for constant Gaussian curvature and constant mean curvature surfaces, were considered at the beginning of last century (see Bianchi [Bi]) and they were recently applied for the first time to obtain minimal surfaces [CFT2]. We should mention that Bäcklund transformations for hyperbolic linear Weingarten surfaces were considered by [Bi] and generalized to higher dimensions by [Bu].

Our main result in this paper extends Ribaucour transformations to linear Weingarten surfaces and also provides a unified version for the classical results. As an application of the theory, we obtain an interesting, two-parameter family of complete linear Weingarten surfaces associated to the cylinder. This family shows that the Ribaucour transformation under consideration is not necessarily a Darboux transformation and it provides an unexpected result. Namely, the existence of complete hyperbolic linear Weingarten surfaces immersed in $\mathbb{R}^3$. Hilbert’s theorem shows that there are no complete surfaces of constant negative curvature immersed in $\mathbb{R}^3$. Although such surfaces and hyperbolic linear Weingarten surfaces correspond to solutions of the sine-Gordon equation, the family of examples associated to the cylinder shows that there exist infinitely many complete hyperbolic linear Weingarten surfaces immersed in $\mathbb{R}^3$. The results of this paper were announced in [T].

We point out that although linear Weingarten surfaces are locally parallel to minimal surfaces or surfaces of constant Gaussian curvature, with same $\Delta$, the parallelism procedure cannot be applied to extend the method of Ribaucour transformation to linear Weingarten surfaces, since it is not a global construction and in general it produces singularities. This is reinforced by the existence of complete hyperbolic linear Weingarten surfaces, which obviously cannot be parallel to any complete surface of constant negative Gaussian curvature.

The paper is organized as follows: In Section 1, we consider Ribaucour transformations for surfaces and we provide an algebraic condition for such a transformation to relate two linear Weingarten surfaces. We show that the linear system of differential equations corresponding to the Ribaucour transformation with the additional algebraic condition is integrable on linear Weingarten surfaces. As a consequence of the theory we get the corresponding results for $H$-cmc surfaces. In this case, for a given such surface, the parameter must satisfy $c(c - 2H) > 0$.

In Section 2, by considering the cylinder as a linear Weingarten surface satisfying $-1/2 + H + \gamma K = 0$, we obtain a two-parameter family of immersions $\tilde{X}_{c\gamma}$, of $\mathbb{R}^2$ into euclidean space $\mathbb{R}^3$, which are linear Weingarten
surfaces. Not all of them are complete. We characterize the complete ones in terms of the pair of real numbers \((c, \gamma)\). This family of complete surfaces contains hyperbolic, elliptic and tubular linear Weingarten surfaces which are all asymptotically close to the cylinder. One family of lines of curvature is planar while the other one is spherical (which may degenerate into planar). We describe the symmetries of these surfaces, which are quite distinct whenever \(c < 0\) and \(c > 0\). For generic values of the parameters, the immersions are not periodic. However, for special values of \((c, \gamma)\), namely for each \(c < 0\) and \(\sqrt{1 - c(2\gamma + 1)} = n/m\) an irreducible rational number, we get an \(n\)-bubble surface \(\tilde{X}_{c\gamma}\). This is an immersed cylinder into \(\mathbb{R}^3\), with two ends of geometric index \(m\) and \(n\) isolated points of maximum and of minimum for the Gaussian curvature. We show that its total curvature vanishes while its total absolute curvature is \(8\pi n\). Moreover, we prove that the ends are embedded if and only if \(m = 1\) and in this case, they are cylindrical ends. If \(c > 0\) or \(c < 0\) and \(\sqrt{1 - c(2\gamma + 1)}\) is not a rational number, then \(\tilde{X}_{c\gamma}\) is a complete immersion of \(\mathbb{R}^2\) into \(\mathbb{R}^3\), not periodic in any variable and it has an infinite number of isolated critical points for the Gaussian curvature.

In Section 3, by restricting the constant \(\gamma = 0\) in the previous section, we get a one parameter family of complete 1/2-cmc immersions \(\tilde{X}_c\) from \(\mathbb{R}^2\) into \(\mathbb{R}^3\). These surfaces are not periodic for generic values of \(c\). However, special values of \(c\), produce 1-periodic cmc \(n\)-bubble surfaces. These immersed cmc cylinders were first described by Sievert [S] for \(n = 2\) (see also [PS]), and their existence was proved later in [G-B] and [SW]. We also show that these surfaces are of finite type one (as defined in [PS]).

In Section 4, we obtain families of cmc surfaces associated to the Delaunay surface by Ribaucour transformations. By restricting the range of the parameter \(c\) conveniently we obtain families of complete \(H\)-cmc surfaces. We describe their symmetries and as in the case of the surfaces associated to the cylinder, for special values of \(c\) we get a family of 1-periodic surfaces.

1. Ribaucour transformation for linear Weingarten surfaces.

In this section, we first recall the theory of Ribaucour transformation for surfaces. For the proofs and more details see [Bi] and [CFT1]. We then prove that, imposing a one-parameter algebraic condition on a Ribaucour transformation, we have a correspondence between linear Weingarten surfaces. We show that starting with such a surface the system of equations is integrable and provides a family of new Weingarten surfaces. As a consequence of the theory we get the corresponding results for cmc surfaces.

Let \(M\) be an orientable surface of \(\mathbb{R}^3\) without umbilic points. We denote by \(N\) its Gauss map. We say that \(\tilde{M}\) is associated by a Ribaucour transformation to \(M\), if and only if, there exists a differentiable function \(h\) defined on \(M\) and a diffeomorphism \(\psi : M \rightarrow \tilde{M}\) such that: \(p + h(p)N(p) = \)
\[ \psi(p) + h(p)\tilde{N}(\psi(p)), \] for all \( p \in M \), where \( \tilde{N} \) is the normal map of \( \tilde{M} \); the subset \( p + h(p)\tilde{N}(p) \), \( p \in M \), is an 2-dimensional submanifold and \( \psi \) preserves lines of curvature.

We say that \( \tilde{M} \) is locally associated by a Ribaucour transformation to \( M \) if for all \( \tilde{p} \in \tilde{M} \) there exists a neighborhood of \( \tilde{p} \) in \( \tilde{M} \) which is associated by a Ribaucour transformation to an open subset of \( M \). Similarly, one may consider the corresponding definitions for parametrized surfaces.

The following results give a characterization of Ribaucour transformations.

**Theorem 1.1.** Let \( M \) be an orientable surface of \( \mathbb{R}^3 \), without umbilic points and \( N \) its Gauss map. Let \( e_i, 1 \leq i \leq 2 \) be orthonormal principal directions, \( \lambda^i \) the corresponding principal curvatures, i.e., \( dN(e_i) = \lambda_i e_i \). A surface \( \tilde{M} \) is locally associated to \( M \) by a Ribaucour transformation, if and only if, there exist parametrizations \( \tilde{X}: U \subset \mathbb{R}^2 \rightarrow \tilde{M} \) and \( X: U \subset \mathbb{R}^2 \rightarrow M \) and a differentiable function \( h: U \rightarrow \mathbb{R} \) such that

\[ 1 + h \lambda^i \neq 0, \]

\[ \tilde{X} = X + h(N - \tilde{N}), \] 

where \( \tilde{N} \) is a unit vector field normal to \( \tilde{X}(U) \) given by

\[ \tilde{N} = \frac{1}{\Delta + 1} \left( \sum_{i=1}^{n} 2Z^i e_i + (\Delta - 1)N \right), \]

\[ Z^i = \frac{dh(e_i)}{1 + h \lambda^i}, \quad \Delta = \sum_{i=1}^{n} (Z^i)^2 \]

and \( h \) is a generic solution of the differential equation

\[ dZ^j(e_i) + Z^i \omega_{ij}(e_i) - Z^i Z^j \lambda^i = 0, \quad 1 \leq i \neq j \leq 2, \]

where \( \omega_{ij} \) are the connection forms of the frame \( e_i \).

**Proof.** Let \( \tilde{N} \) be a unit vector field given by

\[ \tilde{N} = \sum_{i=1}^{2} b^i e_i + b^3 N, \quad \text{where} \quad \sum_{i=1}^{2} (b^i)^2 + (b^3)^2 = 1. \]

We introduce the following notation:

\[ d\tilde{N}(e_i) = \sum_{k=1}^{2} L^k_i e_k + L^3_i N, \]

where for \( 1 \leq i, k \leq 2 \)

\[ L^k_i = db^k(e_i) + \sum_j b^j \omega_{jk}(e_i) + b^3 \lambda^i \delta_{ik}, \quad L^3_i = db^3(e_i) - b^i \lambda^i. \]
We will later show that the following relations hold:

\[ \dot{b}^i = Z^i(1 - b^3). \]  

(8)

In this case, from (5) we get

\[ b^3 = \frac{\Delta - 1}{\Delta + 1}. \]

(9)

We will now prove the theorem. Assume that \( \tilde{M} \) is locally associated to \( M \) by a Ribaucour transformation. Then by definition there exist local parametrizations \( X \) of \( M \), \( \tilde{X} \) of \( \tilde{M} \) and a function \( h \) defined on \( U \subset \mathbb{R}^2 \) such that \( \tilde{X} + h \tilde{N} = X + h N \), where \( \tilde{N} \) is a unit vector field normal to \( \tilde{M} \), which may be considered as in (5). Since

\[ d\tilde{X} = dX + dh(N - \tilde{N}) + h(dN - d\tilde{N}) \]

it follows from the relations \( dX = \sum \omega_j e_j \) and \( dN(e_i) = \lambda^i e_i \) that

\[ d\tilde{X}(e_i) = (1 + h\lambda^i) e_i + dh(e_i)(N - \tilde{N}) - hd\tilde{N}(e_i). \]

(10)

Hence, \( \langle d\tilde{X}(e_i), \tilde{N} \rangle = 0 \) implies

\[ (1 + h\lambda^i) b^i + dh(e_i)(b^3 - 1) = 0, \quad i = 1, 2. \]

(12)

Since \( X + hN \) is two dimensional, it follows that \( 1 + h\lambda^i \neq 0 \) for all \( i \). Therefore we conclude from (12) that the relations (8) hold and hence \( b^3 \) is given by (9). \( d\tilde{X}(e_i) \) are orthogonal principal directions, i.e.,

\[ \langle d\tilde{X}(e_i), d\tilde{X}(e_j) \rangle = \langle d\tilde{N}(e_j), d\tilde{X}(e_i) \rangle = \langle d\tilde{N}(e_j), d\tilde{N}(e_i) \rangle = 0 \quad \text{for} \quad i \neq j. \]

Hence, using (6) and Equation (11), we get

\[ \langle d\tilde{N}(e_i), d\tilde{X}(e_j) \rangle = L^j_i(1 + h\lambda^j) + L^3_i dh(e_j) = 0, \quad \text{for} \quad i \neq j. \]

It follows from (7) and the last equality, that \( h \) satisfies Equation (4).

Conversely, assume \( h \) is a solution of (4) such that \( 1 + h\lambda^i \neq 0, \forall i \), then we define \( Z^i \) and \( \Delta \) by (3), \( b^i \) and \( b^3 \) by (8) and (9). It follows from (7) and (4) that

\[ L^k_i + Z^k L^3_i = 0, \quad i \neq k, \]

(13)

and

\[ Z^i L^i_3 + \left( (Z^i)^2 - \frac{\Delta + 1}{2} \right) L^3_i = 0. \]

(14)

We consider \( \tilde{N} \) and \( \tilde{X} \) as in (5) and (1) respectively. We need to show that \( \tilde{X} \) is associated to \( X \) by a Ribaucour transformation. We first observe that \( \tilde{N} \) is a unit vector field. In fact, \( \sum (b^i)^2 + (b^3)^2 = (1 - b^3)^2 \Delta + (b^3)^2 = 1 \), since \( b^3 \) is given by (9). We next verify that \( \tilde{N} \) is normal to \( \tilde{X} \). From the definition of
\( \tilde{X} \), we have that \( d\tilde{X}(e_i) \) is given by (11). Hence, using the fact that \( |N| = 1 \), we conclude that \( \langle d\tilde{X}(e_i), \tilde{N} \rangle = (1 + h\lambda^i)Z^i + dh(e_i) (b^3 - 1) = 0 \).

Using (6), Equation (14) and the definition of \( \Delta \), one proves that \( \langle d\tilde{N}(e_1), d\tilde{N}(e_2) \rangle = 0 \). Therefore, it follows from Equations (10) and (13), that for \( i \neq j \)

\[ \langle d\tilde{N}(e_i), d\tilde{X}(e_j) \rangle = L^j_i (1 + h\lambda^j) + L^j_i dh(e_j) = 0. \]

Finally, we prove that \( d\tilde{X}(e_1) \) and \( d\tilde{X}(e_2) \) are orthogonal. In fact,

\[ \langle d\tilde{X}(e_1), d\tilde{X}(e_2) \rangle = dh(e_2) [- (1 + h\lambda^j) b^j + dh(e_1) (1 - b^3)] = 0, \]

where the last equality follows from the definition of \( b^j \) and Equation (13).

Moreover, for generic \( h \), \( \tilde{X} \) is an immersed surface. Introducing the eigenvalues \( \lambda^i \), \( d\tilde{N}(e_i) = \tilde{\lambda}^i d\tilde{X}(e_i) \), it follows from (11) that

\[ (1 + h\tilde{\lambda}^i) d\tilde{X}(e_i) = (1 + h\lambda^i) e_i + dh(e_i) (N - \tilde{N}). \]

Hence we conclude from (5) and (8) that

\[ |1 + h\tilde{\lambda}^i||d\tilde{X}(e_i)| = |1 + h\lambda^i|. \]

One can linearize Equation (4) as we will show in Proposition 1.2. We first observe that from the proof given above and from (6), we have \( \langle d\tilde{N}(e_i), N \rangle = L^3_i = \tilde{\lambda}^i (d\tilde{X}(e_i), N) \). Therefore, whenever \( dh(e_i) \neq 0 \), using (11), (5) and (6) we obtain the principal curvatures given by

\begin{equation}
\tilde{\lambda}^i = \frac{L^3_i}{dh(e_i)(1 - b^3) - hL^3_i}.
\end{equation}

**Proposition 1.2.** If \( h \) is a solution of (4) which does not vanish on a simply connected domain, then \( h = \Omega / W \) where \( \Omega \) and a non-vanishing function \( W \) satisfy

\begin{equation}
d\Omega_i(e_j) = \Omega_j \omega_{ij}(e_j), \quad \text{for } i \neq j,
\end{equation}

\begin{equation}
d\Omega = \sum_{i=1}^2 \Omega_i \omega_i,
\end{equation}

\begin{equation}
dW = -\sum_{i=1}^2 \Omega_i \lambda^i \omega_i.
\end{equation}

Conversely, suppose (16)-(18) are satisfied, such that \( W(W + \Omega \lambda^3) \neq 0 \), then \( h = \Omega / W \) is a solution of (4).

**Proof.** Assume \( h \) is a nonvanishing solution of (4), then \( \psi = \sum_i Z^i \omega_i / h \), is a closed form. Hence, on a simply connected domain there exists a differentiable function \( \Omega \) such that \( d(\log \Omega) = \psi \). We define \( \Omega_i = d\Omega(e_i) \) and
$W = \Omega/h$. Then $dh(e_i) = \Omega_i(1 + \Omega \lambda^i/W)/W$ and (17) holds. Moreover, it follows from (4) that (16) and (18) are satisfied. Conversely if (16)-(18) hold, considering $Z^i = \Omega_i/W$ one concludes that (4) is satisfied. We define $h = \Omega/W$, then it follows that $dh(e_i) = Z^i(1 + h \lambda^i)$.

We observe that it follows from the proof of Proposition 1.2 that

\[
W^2 = \frac{\Omega_i}{h}(1 + \Omega \lambda^i/W) \quad Z^i = \frac{\Omega_i}{W} \quad \Delta = \frac{\sum j(\Omega_j)^2}{(W)^2}.
\]

Hence $dh(e_i) \neq 0$ if and only if $\Omega_i \neq 0$. For each solution $\Omega_i$, $1 \leq i \leq 2$, of (16), there exists a 2-parameter family of solutions of the system (17), (18). In fact, Equation (16) is the integrability condition of the system of equations (17), (18) for $\Omega$ and $W$.

The Ribaucour transformation of a surface is given in terms of the solutions of the above system.

**Theorem 1.3.** Let $M$ be an orientable surface of $\mathbb{R}^3$, without umbilic points, parametrized by $X : U \subset \mathbb{R}^2 \rightarrow M$. Assume $e_i$, $1 \leq i \leq 2$ are orthogonal principal directions, $\lambda^i$ the corresponding principal curvatures and $N$ is a unit vector field normal to $M$. A surface $\tilde{M}$ is locally associated to $M$, by a Ribaucour transformation, if and only if, there exist differentiable functions $W, \Omega, \Omega_i : V \subset U \rightarrow \mathbb{R}$, which satisfy (16)-(18), such that $W(W + \Omega \lambda^i)(\Omega_i S - \Omega dS(e_i)) \neq 0$, $\forall i$ and $\tilde{X} : V \subset \mathbb{R}^2 \rightarrow \tilde{M}$, is a parametrization of $\tilde{M}$ given by

\[
\tilde{X} = X - \frac{2\Omega}{S} \left( \sum_i \Omega_i e_i - WN \right),
\]

where

\[
S = \sum_i (\Omega_i)^2 + W^2.
\]

Moreover, the normal map of $\tilde{X}$ is given by

\[
\tilde{N} = N + \frac{2W}{S} \left( \sum_i \Omega_i e_i - WN \right)
\]

and the principal curvatures of $\tilde{X}$ for each $1 \leq i \leq 2$ are given by

\[
\tilde{\lambda}^i = \frac{dS(e_i)W + \Omega_i \lambda^i S}{\Omega_i S - \Omega dS(e_i)} \quad \text{if} \ \Omega_i \neq 0.
\]

**Proof.** Let $X$ and $\tilde{X}$ be parametrizations of $M$ and $\tilde{M}$. We have seen in Theorem 1.1 that the normal vector field $\tilde{N}$ is given by (2). Hence it follows from (19) and (9) that (22) holds. The expression (20) follows directly from (1) and (22). The condition $W(W + \Omega \lambda^i) \neq 0$, $\forall i$ follows from the fact that $h = \Omega/W$ and $1 + h \lambda^i \neq 0$. 

If \( \Omega_i \neq 0 \), then we consider (15). It follows from (7)-(9) and (19) that

\[
L_i^3 = 2 \frac{W}{S^2} (WdS(e_i) + \Omega_i S \lambda^i).
\]

Therefore, using (19) and (9) we have that

\[
dh(e_i)(1 - b^3) - hL_i^3 = 2 \frac{W}{S^2} (\Omega_i S - \Omega dS(e_i)).
\]

From the last two relations we conclude that (23) holds.

We observe that, eventually, the parametrization of \( \tilde{M} \) given by (20) may extend regularly to wherever \( W(W + \Omega \lambda^i) \) vanishes (see for example Section 2). From now on, whenever we say that a surface \( \tilde{M} \) is locally associated by a Ribaucour to a surface \( M \), we are assuming that there are functions \( \Omega, \Omega_i \) and \( W \) locally defined, satisfying (16)-(18).

We now provide a sufficient condition for a Ribaucour transformation to transform a linear Weingarten surface into another such surface.

**Theorem 1.4.** Let \( M \) be a surfaces of \( \mathbb{R}^3 \), without umbilic points and let \( \tilde{M} \) be associated to \( M \) by a Ribaucour transformation, such that the normal lines intersect at a distance function \( h \). Assume that \( h = \Omega/W \) is not constant along the lines of curvature and the functions \( \Omega_i, \Omega \) and \( W \) satisfy the additional relation

\[
S = 2c(\alpha \Omega^2 + \beta \Omega W + \gamma W^2),
\]

where \( S \) is defined by (21), \( c \neq 0 \) and \( \alpha, \beta, \gamma \) are real constants. Then \( \tilde{M} \) is a linear Weingarten satisfying \( \alpha + \beta \tilde{H} + \gamma \tilde{K} = 0 \), if and only if \( \alpha + \beta H + \gamma K = 0 \) holds for the surface \( M \), where \( K, H \) and \( \tilde{K}, \tilde{H} \) are the Gaussian and mean curvatures of \( M \) and \( \tilde{M} \) respectively. Moreover, \( \tilde{M} \) has no umbilic points.

**Proof.** We will introduce the following notation for the right-hand side of the algebraic equation (24):

\[
P = \alpha \Omega^2 + \beta \Omega W + \gamma W^2.
\]

Since \( S \) satisfies \( S = 2cP \), it follows from (17) and (18) that

\[
dS = 2c dP = 2c \sum_i [(2\alpha \Omega + \beta W) \omega_i + (2\gamma W + \beta \Omega) \omega^3_i] \Omega_i.
\]

Therefore,

\[
WdS(e_i) + S \Omega_i \lambda^i = 2c \Omega_i \{ [(2\alpha \Omega + \beta W) - (2\gamma W + \beta \Omega) \lambda^i] W + \lambda^i P \}
\]

\[
S \Omega_i - \Omega dS(e_i) = 2c \Omega_i (P - \Omega [(2\alpha \Omega + \beta W) - (2\gamma W + \beta \Omega) \lambda^i]) .
\]
By assumption $dh(e_i) \neq 0$, i.e., $\Omega_i \neq 0$, for all $i$, therefore from (23), (25) and (27) we get

\[
(28) \quad \tilde{\lambda}^i = \frac{2\alpha\Omega W + \beta W^2 + \lambda^i(\alpha\Omega^2 - \gamma W^2)}{(2\gamma\Omega W + \beta\Omega^2)\lambda^i - (\alpha\Omega^2 - \gamma W^2)}.
\]

In order to conclude the proof we introduce the following notation:

\[
L = 2\alpha\Omega W + \beta W^2 \quad T = \alpha\Omega^2 - \gamma W^2 \quad Q = 2\gamma\Omega W + \beta\Omega^2.
\]

Then, $\tilde{\lambda}^i = (L + \lambda^i T)/(Q\lambda^i - T)$ and hence, the numerator of $\alpha - \frac{\beta}{T}(\lambda^1 + \lambda^2) + \gamma\lambda^1\lambda^2$ is equal to

\[
\alpha T^2 + \beta LT + \gamma L^2 + H(2\alpha T Q - \beta T^2 + \beta L Q - 2\gamma LT) + K(\alpha Q^2 - \beta T Q + \gamma T^2).
\]

By substituting $\alpha$, $\beta H$ and $\gamma K$, on the right-hand side of this last equality, by the two other terms of the expression $\alpha + \beta H + \gamma K = 0$, we get

$\alpha + \beta \tilde{H} + \gamma \tilde{K} = 0$ if and only if $(\beta T + \gamma L - \alpha Q)(L - 2TH - QK) = 0$,

where the last equality follows from the fact that the expression $\beta T + \gamma L - \alpha Q$ is identically zero.

We conclude the proof of the theorem by observing that

\[
\tilde{\lambda}^2 - \tilde{\lambda}^1 = \frac{LQ + T^2}{(Q\lambda^1 - T)(Q\lambda^2 - T)}(\lambda^2 - \lambda^1).
\]

Since $LQ + T^2 = P^2 = S^2/(4c^2) \neq 0$ and $M$ has no umbilic points, it follows that $\tilde{M}$ has no umbilic points. □

The natural question one poses is if the system (16)-(18) with the additional condition (24) is integrable, whenever we start with a linear Weingarten surface. The following theorem answers this question affirmatively:

**Theorem 1.5.** Let $M$ be a surface of $\mathbb{R}^3$, which satisfies $\alpha + \beta H + \gamma K = 0$ and $H^2 - K > 0$. Then the system of equations (16)-(18) and (24) is integrable and the solution is uniquely determined on a simply connected domain $U$ by any given initial condition satisfying (24). Moreover, whenever $\alpha \neq 0$, any solution of the system defined on $U$ is either identically zero and hence annihilates $S$ or else the function $S$ does not vanish on $U$. 


Proof. We initially observe that as a consequence of (16)-(18) and (24) we only need to prove that the system

\[ d\Omega = \sum_i \Omega_i \omega_i \]
\[ dW = \sum_i \Omega_i \omega_{i3} \]
\[ d\Omega_i = \Omega_{ij} \omega_{ij} + c(2\alpha \Omega + \beta W) \omega_i - [(1 - 2c\gamma)W - c\beta \Omega] \omega_{i3} \quad i \neq j \]

is integrable.

We start by showing that if (29) holds then \( S - 2cP \) is a constant function, where \( S \) and \( P \) are defined by (21) and (25) respectively. In fact,

\[ dS - 2cdP = 2 \sum_i \Omega_i d\Omega_i + 2(W - c\beta \Omega - 2c\gamma W) dW - 2c(2\alpha \Omega + \beta W) d\Omega \]

\[ = 2 \sum_{i,j} \Omega_i \Omega_j \omega_{ij} = 0. \]

Therefore, by choosing the initial condition at a point such that \( S = 2cP \), we will have (24) identically satisfied on a connected domain.

Now we consider the ideal \( \mathcal{I} \) generated by the 1-forms

\[ \theta = d\Omega - \sum_i \Omega_i \omega_i \]
\[ \varphi = dW - \sum_i \Omega_i \omega_{i3} \]
\[ \theta_i = d\Omega_i - \Omega_{ij} \omega_{ij} - c(2\alpha \Omega + \beta W) \omega_i + [(1 - 2c\gamma)W - c\beta \Omega] \omega_{i3} \quad i \neq j. \]

A straightforward computation shows that \( d\theta = -\sum_i \theta_i \wedge \omega_i \) and \( d\varphi = -\sum_i \theta_i \wedge \omega_{i3} \). Similarly, using (30) we obtain that

\[ d\theta_i = -\theta_j \wedge \omega_{ij} + \varphi \wedge [(1 - 2c\gamma) \omega_{i3} - c\beta \omega_i] - c\theta \wedge (\beta \omega_{i3} + 2\alpha \omega_i) + 2c\Omega_j (\alpha + \beta H + \gamma K) \omega_i \wedge \omega_j \]

where \( i \neq j \). Since the surface is linear Weingarten, it follows that \( \mathcal{I} \) is closed under exterior differentiation, hence the system (29) is integrable.

Assume that \( S(p_0) = 0 \) for \( p_0 \in U \). Then it follows from (21) that \( \Omega_1, \Omega_2 \) and \( W \) vanish at \( p_0 \). Since (24) holds we conclude that if \( \alpha \neq 0 \), then \( \Omega(p_0) = 0 \). Since \( U \) is simply connected, the uniqueness of solutions for the system implies that \( \Omega \equiv \Omega_1 \equiv \Omega_2 \equiv W \equiv 0 \) and hence \( S \equiv 0 \). This concludes the proof of the theorem.

As a consequence of Theorems 1.3 and 1.4 we obtain:

**Theorem 1.6.** Let \( M \) be a linear Weingarten surface, without umbilic points, satisfying \( \alpha + \beta H + \gamma K = 0 \) and locally parametrized by \( X : U \subset \)
$R^2 \rightarrow M \subset R^3$. Any linear Weingarten parametrized surface, locally associated to $X$ by a Ribaucour transformation as in Theorem 1.5 is given by

$$\tilde{X} = X - \frac{2\Omega}{S} \left( \sum \Omega_i e_i - WN \right),$$

where $e_i$ are orthogonal principal directions, $\Omega$, $\Omega_i$, $W$ are solutions of (16)-(18) and (24), and $\tilde{X}$ is an immersed surface defined on

$$\tilde{U} = \{(u_1, u_2) \in U; \ T^2 + 2TQH + Q^2K \neq 0\}$$

where $T = \alpha\Omega^2 - \gamma W^2$ and $Q = 2\gamma \Omega W + \beta \Omega^2$.

**Proof.** We only need to show that $\tilde{X}$, defined by (31), is a parametrized surface of $R^3$. With the same notation introduced in the proof of Theorem 1.4, we observe that $S = 2cP$, where $P$ is defined by (25) and the differential of $S$ is given by (26). Therefore, we get that

$$d \left( \frac{\Omega}{S} \right) = \frac{1}{2cP^2} \sum_k \Omega_k \eta_k$$

where $\eta_k$ for $k = 1, 2$ is a 1-form defined by

$$\eta_k = (\gamma W^2 - \alpha \Omega^2) \omega_k - (\beta \Omega^2 + 2\gamma \Omega W) \omega_{k3}.$$

It follows from this expression and the last two equations of system (29) that

$$d\tilde{X} = \frac{1}{P} \sum_{i=1}^2 \eta_i \tilde{e}_i$$

where

$$\tilde{e}_1 = \frac{1}{cP} [(cP - \Omega_1^2)e_1 - \Omega_1 \Omega_2 e_2 + W \Omega_1 e_3]$$

$$\tilde{e}_2 = \frac{1}{cP} [-\Omega_1 \Omega_2 e_1 + (cP - \Omega_2^2)e_2 + W \Omega_2 e_3].$$

A simple computation shows that these vectors are orthonormal. Therefore, $\tilde{X}$ is an immersion wherever $\eta_1 \wedge \eta_2 \neq 0$, i.e., on the subset $\tilde{U}$ described by (32).

**Remark 1.7.** We observe that, as a consequence of the proof of Theorem 1.6, the principal directions $\tilde{e}_1$, $\tilde{e}_2$, of $\tilde{X}$ are given by (35) and (36). Moreover, its dual forms, which are determined by (34), are given by $\tilde{\omega}_i = \frac{1}{P} (\gamma W^2 - \alpha \Omega^2 + (\beta \Omega + 2\gamma W) \Omega \lambda) \omega_i$. We also observe that the surfaces described in Theorem 1.6 depend on 4 parameters. However, in some cases the number of parameters may reduce to one (the parameter $c$), if we exclude surfaces which are congruent by rigid motions of $R^3$. 

□
Our next results give the $H$-cmc case, which is obtained by considering $H$ a nonzero constant, $\alpha = -H$, $\beta = 1$ and $\gamma = 0$ on Theorems 1.4-1.6. For later use, we will explicitly give the corresponding results.

**Corollary 1.8.** Let $M$ be a regular surface of $\mathbb{R}^3$, with no umbilic point. Let $\tilde{M}$ be associated by a Ribaucour transformation to $M$, such that the normal lines intersect at a distance function $h$. Assume that $h = \Omega/W$ is not constant along the lines of curvature and the functions $\Omega_i$, $\Omega$ and $W$ satisfy the additional relation

$$S = 2c\Omega(-H\Omega + W),$$

where $S$ is defined by (21), $c \neq 0$ and $H \neq 0$ are real constants. Then $\tilde{M}$ is an $H$-cmc surface, if and only if, $M$ is an $H$-cmc surface. Moreover, $\tilde{M}$ has no umbilic points.

**Corollary 1.9.** Let $M$ be an $H$-cmc surface of $\mathbb{R}^3$. Then the system of Equations (16)-(18) and (37) is integrable and the solution is uniquely determined on a simply connected domain $U$ by any given initial condition satisfying (37). Moreover, for any solution of the system defined on $U$ the function $S$ does not vanish on $U$.

As a consequence of Theorem 1.6, we get our next result. We observe that in the case of $H$-cmc surfaces, by considering $\alpha = -H$, $\beta = 1$ and $\gamma = 0$, we conclude that (32) reduces to $\Omega^4(K - H^2) \neq 0$.

**Corollary 1.10.** Let $M$ be an $H$-cmc surface with no umbilic points, locally parametrized by $X : U \subset \mathbb{R}^2 \to M \subset \mathbb{R}^3$. Any $H$-cmc parametrized surface, locally associated to $X$ by a Ribaucour transformation as in Corollary 1.9 is given by

$$\tilde{X} = X - \frac{1}{c(W - H\Omega)} \left( \sum_i \Omega_i e_i - W N \right),$$

where $e_i$ are orthogonal principal directions, $\Omega$, $\Omega_i$, $W$ are solutions of (16)-(18) and (37) and the constant $c$ satisfies $c(c - 2H) > 0$.

**Proof.** We only need to prove that $c$ satisfies $c(c - 2H) > 0$. This follows from the algebraic condition (37), which can be written as

$$\sum_i (\Omega_i)^2 + (W - c\Omega)^2 - c(c - 2H)\Omega^2 = 0.$$
where \( P = \Omega(W - H\Omega) \). Its dual forms are,

\[
\bar{\omega}_i = \frac{(H + \lambda^i)\Omega}{W - H\Omega} \omega_i.
\]

Moreover, it follows from (28) that

\[
\bar{\lambda}_i = \frac{W^2 - H\Omega(2W + \lambda^i\Omega)}{\Omega^2(\lambda^i + H)}
\]

and the Gaussian curvature is given by

\[
\tilde{K} = H^2 + \frac{(W - H\Omega)^4}{\Omega^4(K - H^2)}.
\]

2. Families of linear Weingarten surfaces associated to the cylinder.

In this section, by applying Theorem 1.6 to the cylinder, we obtain a two parameter \((c, \gamma)\) family of complete linear Weingarten surfaces. The parameters belong to a region composed by two connected components of \( R^2 \). One of these components contains curves which provide \( n \)-bubble surfaces (Weingarten and cmc) which are 1-periodic, have genus zero and two ends of finite geometric index. We also show that their total curvature vanishes, while the total absolute curvature is \( 8\pi n \).

**Proposition 2.1.** Consider the cylinder parametrized by

\[
X(u_1, u_2) = (\cos(u_2), \sin(u_2), u_1) \quad (u_1, u_2) \in R^2
\]

as a linear Weingarten surface satisfying \(-1/2 + H + \gamma K = 0\). A parametrized surface is a linear Weingarten surface locally associated to \( X \) by a Ribaucour transformation as in Theorem 1.5, if and only if, it is given by

\[
\tilde{X}_{c\gamma} = X - \frac{2(f + g)}{c[(2\gamma + 1)g^2 - f^2]}(f'X_{u_1} + g'X_{u_2} - gN)
\]

where \( N \) is the inner unit normal vector field of the cylinder, \( c \neq 0 \) and \( \gamma \) are real constants such that

\[
\xi(c, \gamma) = 1 - c(2\gamma + 1)
\]

and \( c \) are not simultaneously positive, and \( f(u_1), g(u_2) \) are solutions of the equations

\[
f'' + cf = 0,
\]

\[
g'' + \xi g = 0
\]

with initial conditions satisfying

\[
((f')^2 + (g')^2 + \xi g^2 + cf^2) (u_1^0, u_2^0) = 0.
\]
Moreover, $\tilde{X}_{c\gamma}$ is a regular surface defined on the subset of $U$ where
\begin{equation}
(f + g)^2 + 2\gamma g^2 \left( f^2 + 2(2\gamma + 1)fg + (2\gamma + 1)g^2 \right) \neq 0.
\end{equation}

**Proof.** The first fundamental form of the cylinder is given by $ds^2 = du_1^2 + du_2^2$ and $\lambda^1 = 0$, $\lambda^2 = -1$. In order to obtain the Ribaucour transformations, we need to solve the following system of equations, which is obtained from (16)-(18):
\begin{align*}
\frac{\partial \Omega_i}{\partial u_j} &= 0, \\
\frac{\partial \Omega}{\partial u_i} &= \Omega_i, \\
\frac{\partial W}{\partial u_i} &= -\Omega_i \lambda^i, \quad 1 \leq i \neq j \leq 2.
\end{align*}

The associated surface will be linear Weingarten when $\Omega_1$ and $\Omega_2$ satisfy $\partial \Omega_1 / \partial u_1 = c(W - \Omega)$ and $\partial \Omega_2 / \partial u_2 = (c - 1 + 2c\gamma)W$.

Since $\Omega_{u_1u_2} = 0$, it follows that $\Omega = f(u_1) + g(u_2)$, where $f$ and $g$ are functions of $u_1$ and $u_2$ respectively. Therefore $\Omega_1 = f'$ and $\Omega_2 = g'$. Moreover, $W = g + a$, where $a$ is a real constant and the functions $f$ and $g$ satisfy the following equations:
\begin{align*}
f'' + cf - ca &= 0, \\
g'' + \xi(g + a) &= 0.
\end{align*}

It follows from these equations and the expressions of $\Omega$ and $W$, that without loss of generality we can consider $a = 0$. Therefore, $f$ and $g$ must satisfy Equations (45) and (46) and the algebraic condition (24), which reduces to (47). Moreover, since this last condition should be identically satisfied by the nontrivial solution functions $f$ and $g$, we conclude that the constants $c \neq 0$ and $\gamma$ are such that $c$ and $\xi$ cannot be simultaneously positive.

Moreover, from (20) we conclude that the associated linear Weingarten surface is given by (43). From (32) we obtain the domain where $\tilde{X}$ is regular, which is described by (48). \qed

The family of linear Weingarten surfaces given by (43) includes the cylinder. In fact, if we choose the initial conditions such that $f \equiv 0$, $g \neq 0$ for $\xi \leq 0$ or $f \neq 0$, $g \equiv 0$ for $c < 0$, we get a reparametrization of the cylinder.

Each linear Weingarten surface associated to the cylinder as in Proposition 2.1, is parametrized by lines of curvature and the metric is given (see Remark 1.7) by $ds^2 = \psi_1^2 du_1^2 + \psi_2^2 du_2^2$, where
\begin{align*}
\psi_1 &= \frac{(f + g)^2 + 2\gamma g^2}{(1 + 2\gamma)g^2 - f^2}, \\
\psi_2 &= \frac{(1 + 2\gamma)g^2 + 2(1 + 2\gamma)fg + f^2}{(1 + 2\gamma)g^2 - f^2}.
\end{align*}

These expressions show that the Ribaucour transformation, applied to the cylinder for $\gamma \neq 0$, is not a Darboux transformation.
We now introduce a notation, which will be useful in the following results.

A rotation of angle \( \theta \) in the \( xy \) plane of \( \mathbb{R}^3 \) will be denoted by

\[
R_\theta = \begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

We denote by \( T_\delta \) the translation defined by

\[
T_\delta(x, y, z) = (x, y, z + \delta).
\]

**Proposition 2.2.** Consider the linear Weingarten surfaces associated to the cylinder and parametrized by (43). Excluding the cylinder:

i) If \( c\xi \geq 0 \), then any surface \( \tilde{X}_{c,\gamma} \) has curves of singularities.

ii) If \( c\xi < 0 \), then, up to rigid motions of \( \mathbb{R}^3 \), the surface \( \tilde{X}_{c,\gamma} \) is determined by the functions

\[
f = \varepsilon_1 \sqrt{|\xi|} \sin(\sqrt{|c|} u_1) \quad g = \varepsilon_2 \sqrt{|c|} \cos(\sqrt{|\xi|} u_2)
\]

if \( c > 0, \xi < 0 \)

\[
f = \varepsilon_1 \sqrt{\xi} \cosh(\sqrt{|c|} u_1) \quad g = \varepsilon_2 \sqrt{|c|} \sin(\sqrt{\xi} u_2)
\]

if \( c < 0, \xi > 0 \)

where \( \varepsilon_i = \pm 1, c \neq 0 \) and \( \gamma \) are real numbers and \( \xi(c, \gamma) \) is defined by (44).

**Proof.** We observe that the functions \( f \) and \( g \) of the family of surfaces described by (43) are given by

\[
f = \begin{cases}
    a_1 \cos(\sqrt{|c|} u_1) + b_1 \sin(\sqrt{|c|} u_1) & \text{if } c > 0, \\
    a_1 \cosh(\sqrt{|c|} u_1) + b_1 \sinh(\sqrt{|c|} u_1) & \text{if } c < 0,
\end{cases}
\]

\[
g = \begin{cases}
    a_2 u_2 + b_2 & \text{if } \xi = 0, \\
    a_2 \cosh(\sqrt{|c|} u_2) + b_2 \sinh(\sqrt{|\xi|} u_2) & \text{if } \xi < 0, \\
    a_2 \cos(\sqrt{\xi} u_2) + b_2 \sin(\sqrt{\xi} u_2) & \text{if } \xi > 0,
\end{cases}
\]

and the constants satisfy the algebraic relation given by (47),

\[
a_1 = b_1 = a_2 = 0 \quad \text{if } c > 0 \text{ and } \xi = 0,
\]

\[
c(b_1^2 - a_2^2) - a_2^2 = 0 \quad \text{if } c < 0 \text{ and } \xi = 0,
\]

\[
c(b_1^2 + a_2^2) + \xi(a_2^2 - b_2^2) = 0 \quad \text{if } c > 0 \text{ and } \xi < 0,
\]

\[
c(b_1^2 - a_2^2) - \xi(a_2^2 - b_2^2) = 0 \quad \text{if } c < 0 \text{ and } \xi < 0,
\]

\[
c(b_1^2 + a_2^2) - \xi(a_2^2 + b_2^2) = 0 \quad \text{if } c < 0 \text{ and } \xi > 0.
\]

If \( c > 0 \) and \( \xi = 0 \), then the surface \( \tilde{X} \) reduces to the cylinder. If \( c < 0 \) and \( \xi = 0 \), there are curves in \( \mathbb{R}^2 \) where (48) vanishes.

If \( c\xi > 0 \), since by Proposition 2.1 \( c \) and \( \xi(c, \gamma) \) cannot be simultaneously positive, then we may only have \( c < 0 \) and \( \xi < 0 \). In this case, the functions \( f \) and \( g \) are defined as above, \( \gamma < -1/2 \) and there are four curves on \( \mathbb{R}^2 \) determined by (48)

\[
f + g \pm \sqrt{2|\gamma|} g = 0 \quad f + (2\gamma + 1) g \pm \sqrt{2\gamma(2\gamma + 1)} g = 0
\]
where $\tilde{X}$ is not regular.

If $c > 0$ and $\xi < 0$, then by choosing $a_1 = b_1 = 0$ we have $b_2 = \pm a_2$, $f = 0$, $g = a_2 \exp(\pm \sqrt{|\xi|}u_2)$ and the surface $X$ reduces to a reparametrization of the cylinder. Therefore, excluding the cylinder, we may assume $a_2^2 + b_2^2 \neq 0$ and

$$f = \varepsilon_1 \sqrt{|\xi|} \sin(A + \sqrt{c} u_1) \quad g = \varepsilon_2 \sqrt{c} \cosh(B + \sqrt{|\xi|} u_2).$$

Similarly, if $c < 0$ and $\xi > 0$, then excluding the cylinder, we may assume $a_2^2 + b_2^2 \neq 0$ and hence

$$f = \varepsilon_1 \sqrt{\xi} \cosh(A + \sqrt{|c|} u_1) \quad g = \varepsilon_2 \sqrt{|c|} \sin(B + \sqrt{\xi} u_2).$$

We conclude the proof by observing that the constants $A$ and $B$, without loss of generality, may be considered to be zero. One can verify that the surfaces with different values of $A$, $B$ are congruent by rigid motions of $\mathbb{R}^3$.

In fact, using the notation $\tilde{X}_{c\gamma AB}$ for the surface $\tilde{X}_{c\gamma}$ with fixed constants $A$ and $B$, we have

$$\tilde{X}_{c\gamma AB} = R_{- \frac{b}{\sqrt{|\xi|}}} \tilde{X}_{c\gamma 00} \circ h + T_{- \frac{a}{\sqrt{|\xi|}}}$$

where $h(u_1, u_2) = (u_1 + A/\sqrt{|c|}, u_2 + B/\sqrt{|\xi|})$. \hfill $\square$

We observe that it follows from the above expressions that the function $P$ defined by (25), reduces to

$$(54) \quad P = [(2\gamma + 1)g^2 - f^2]/2$$

and it does not vanish on $\mathbb{R}^2$. This will be useful for considering global properties of the surfaces obtained in Proposition 2.2.

In order to study the regularity of the surfaces obtained in Proposition 2.2, we introduce the following notation:

$$(55) \quad h_1(c, \gamma) = 2c(2\gamma + 1) \left( \sqrt{2\gamma(2\gamma + 1)} - 2\gamma \right) - 1,$$

$$(56) \quad h_2(c, \gamma) = 2c \left( \sqrt{2\gamma} + 2\gamma \right) - 1,$$

$$(57) \quad h_3(c, \gamma) = -2c(2\gamma + 1) \left( \sqrt{2\gamma(2\gamma + 1)} + 2\gamma \right) - 1,$$

$$(58) \quad h_4(c, \gamma) = 2c \left( \sqrt{2\gamma} - 2\gamma \right) + 1,$$

$$(59) \quad h_5(c, \gamma) = -2c(2\gamma + 1) \left( \sqrt{2\gamma(2\gamma + 1)} - 2\gamma \right) + 1.$$

Our next result shows that a surface $\tilde{X}_{c\gamma}$ given by (43) is an immersion of $\mathbb{R}^2$ if and only if the pair $(c, \gamma)$ belongs to a region with two connected components of $\mathbb{R}^2$ determined by the functions $\xi(c, \gamma)$ and $h_1(c, \gamma) \ldots h_5(c, \gamma)$ (see Figure 1).
Figure 1. Any pair \((c, \gamma)\), in each of the two connected components, generates a complete linear Weingarten surface, which satisfies the relation \(-1/2 + H - \gamma K = 0\) and it is cmc when \(\gamma = 0\). The dashed curves in the left region, given by \(1 - c(2\gamma + 1) = n^2/m^2\), generate 1-periodic \(n\)-bubble surfaces with two ends of geometric index \(m\) (see Figure 2).

Proposition 2.3. A Weingarten surface \(\tilde{X}_{c,\gamma}\) given by Proposition 2.2 is an immersion of \(R^2\), if and only if, \(c\xi(c, \gamma) < 0\) and the pair \((c, \gamma)\) belongs to one of the following subsets of \(R^2\):

i) \(c > 0\) and one of the following holds:
   a) \(\gamma \geq 0\) and \(\xi(c, \gamma) < 0\), \(h_1(c, \gamma) < 0\);
   b) \(-1/2 < \gamma < 0\) and \(\xi(c, \gamma) < 0\), \(h_2(c, \gamma) < 0\).

ii) \(c < 0\) and one of the following holds:
   a) \(\gamma \geq 0\) and \(h_3(c, \gamma) < 0\);
   b) \(-1 \leq \gamma < 0\) and \(h_4(c, \gamma) > 0\);
   c) \(\gamma \leq -1\) and \(h_5(c, \gamma) > 0\).

Proof. From Proposition 2.2, we have seen that if the surface \(\tilde{X}\) is a regular immersion of \(R^2\), then excluding the cylinder, we only need to consider \(c\xi < 0\).
i) If $c > 0$ and $ξ < 0$ then the functions $f$ and $g$ are given by (52). If $γ ≥ 0$, $X_{cγ}$ is an immersion of $R^2$ if and only if the second factor of (48) does not vanish. It follows from (44) that this is equivalent to having $h_1(c, γ) < 0$, where $h_1$ is defined by (55).

If $γ < 0$, then $1/c - 1 < 2γ < 0$. Hence, $c > 1$ and $-1/2 < γ < 0$. Then $X_{cγ}$ is a regular immersion of $R^2$, if and only if the first factor of (48) does not vanish i.e., $1/(2γ + 1) < c < 1/(\sqrt{2|γ|} + 2γ)$, or equivalently $h_2(c, γ) < 0$, where $h_2$ is defined by (56). This concludes the proof of i).

ii) If $c < 0$ and $ξ(c, γ) < 0$, then the surface $X_{cγ}$ is defined by (43) and the functions $f$ and $g$ are given by (53). We need to consider four cases for $γ$.

If $γ ≥ 0$, then the nonvanishing of the second factor of (48) is equivalent to $√|c| (2γ + 1 + √2γ(2γ + 1)) < √ξ$, i.e., $h_3(c, γ) < 0$, where $h_3$ is defined by (57). If $-1/2 < γ < 0$, from the first factor of (48) we conclude that the pair $(c, γ)$ must satisfy $h_4(c, γ) > 0$, where $h_4$ is defined by (58).

If $γ ≤ -1/2$, then both factors of (48) should not vanish on $R^2$. This occurs if and only if the following inequalities hold:

\[ |1 ± \sqrt{2|γ|}|\sqrt{|c|} < \sqrt{ξ} \]
\[ |2γ + 1 ± \sqrt{2γ(2γ + 1)}|\sqrt{|c|} < \sqrt{ξ}. \]

If $-1 ≤ γ ≤ -1/2$ then $\sqrt{2γ(2γ + 1)} - 2γ - 1 ≤ 1 + \sqrt{2|γ|}$. Hence, the system (60) holds if and only if $h_4(c, γ) > 0$. If $γ ≤ -1$ then $1 + \sqrt{2|γ|} ≤ \sqrt{2γ(2γ + 1)} - 2γ - 1$. Therefore, (60) holds if and only if

\[ \left( \sqrt{2γ(2γ + 1)} - 2γ - 1 \right) \sqrt{|c|} < \sqrt{ξ} \]

i.e., $h_5(c, γ) > 0$, where $h_5$ is defined by (59). This concludes the proof. □

Our next result shows that the linear Weingarten surfaces, locally associated to the cylinder by a Ribaucour transformation, are asymptotically close to cylinders.

**Proposition 2.4.** Let $X(u_1, u_2)$ be the parametrized cylinder given by (42). Any linear Weingarten surface $\tilde{X}_{cγ}$ given by Proposition 2.3, satisfies the following:

i) If $c < 0$ and $ξ(c, γ) > 0$ then $∀ε > 0$ there exists $L > 0$ such that

\[ |\tilde{X}_{cγ}(u_1, u_2) - X(u_1 ± 2/\sqrt{|c|}, u_2)| < ε \quad ∀(u_1, u_2) ∈ R^2 \text{ with } ± u_1 ≥ L \]

and

\[ \left| \frac{∂^{i+j} \tilde{X}_{cγ}}{∂u_1^i ∂u_2^j} (u_1, u_2) - \frac{∂^{i+j} X}{∂u_1^i ∂u_2^j} (u_1, u_2) \right| < ε, \]

where $1 ≤ i + j ≤ 2$, and $i, j$ are nonnegative integer numbers.
ii) If $c > 0$ and $\xi(c, \gamma) < 0$ then $\forall \varepsilon > 0$ there exists $L > 0$ such that

$$|\tilde{X}_{c\gamma}(u_1, u_2) - X(u_1, u_2 \pm \theta)| < \varepsilon \quad \forall (u_1, u_2) \in \mathbb{R}^2$$

with $u_2 \geq L$.

where $\theta$ is such that $\cos \theta = 1 - 2/(c(1+2\gamma))$ and $\sin \theta = -2\sqrt{-\xi}/(c(1+2\gamma))$.

Proof. If $c < 0$ and $\xi(c, \gamma) > 0$ then the functions $f$ and $g$ are given by (53).

It follows from a straightforward computation that

$$\frac{c^2}{4} \left| \tilde{X}_{c\gamma}(u_1, u_2) - X(u_1 \pm 2/\sqrt{|c|}, u_2) \right|^2 = S_1^2 + S_2^2 + S_3^2,$$

where

$$S_1 = \frac{(f + g)E'}{2P} \pm \sqrt{|c|}, \quad S_2 = \frac{(f + g)E'}{2P}, \quad S_3 = \frac{(f + g)E'}{2P},$$

where $P$ is given by (54). Since $g$ and $g'$ are bounded functions, we have that $\lim_{u_1 \to \pm \infty} S_i(u_1, u_2) = 0$ uniformly with respect to $u_2$. Similarly, considering the difference of the first and second derivatives of $\tilde{X}_{c\gamma}$ and $X$ as a linear combination of the vectors $X_{u_1}, X_{u_2}$ and $N$, one can show that each coefficient tends to 0 uniformly in $u_2$, when $u_1 \to \pm \infty$. This concludes the proof of i). Similar arguments prove ii). \qed

Our next result shows that the regular surfaces $\tilde{X}_{c\gamma}$ given in Proposition 2.3 are complete. Moreover, the connected region described by ii) contains an infinite number of curves (determined by considering $\sqrt{1 - c(2\gamma + 1)}$ to be a rational number) such that the corresponding surfaces are 1-periodic $n$-bubbles whose total absolute curvature is $8\pi n$.

**Proposition 2.5.** Any linear Weingarten surface $\tilde{X}_{c\gamma}$, given by Proposition 2.3 is complete.

a) If $c < 0$ and $\sqrt{\xi(c, \gamma)} = n/m$ is an irreducible rational number, then $\tilde{X}_{c\gamma}$ is an immersion of a cylinder into $\mathbb{R}^3$, with two ends of geometric index $m$ and $n$ isolated points of maximum (respectively minimum) for the Gaussian curvature. Moreover, the total curvature of $\tilde{X}_{c\gamma}$ is zero, while its total absolute curvature is $8\pi n$. The ends are embedded if and only if $m = 1$. In this case they are cylindrical ends.

b) If $c > 0$ or $c < 0$ and $\sqrt{\xi}$ is not a rational number then $\tilde{X}_{c\gamma}$ is an immersion of $\mathbb{R}^2$ into $\mathbb{R}^3$ (not periodic in any variable) with an infinite number of isolated critical points of its Gaussian curvature.

Proof. Assume $\xi > 0$ and $c < 0$ then the functions $f$ and $g$ are given by (53) and the coefficients of the first fundamental form $\psi_i$ of $\tilde{X}_{c\gamma}$ are given by (49). Therefore, $\lim_{|u_1| \to \infty} |\psi_i| = 1$ for $i = 1, 2$ uniformly in $u_2$. Hence, there
exits $k > 0$ such that $|\psi(u_1, u_2)| > 1/2$ for all $(u_1, u_2) \in \mathbb{R}^2$ with $|u_1| > k$. Let

$$m_i = \min \left\{ |\psi_i(u_1, u_2)|, (u_1, u_2) \in [-k, k] \times \left[ 0, \frac{2\pi}{\sqrt{|\xi|}} \right] \right\}.$$  

Since $X_{c\gamma}$ is regular, $m_i > 0$. Moreover, $g(u_2) = g(u_2 + 2\pi/\sqrt{|\xi|})$, therefore $|\psi_i(u_1, u_2)| \geq m_i$ in $[-k, k] \times R$. Now consider $m_0 = \min\{m_1, m_2, 1/2\}$, then $|\psi_i| \geq m_0$ in $R^2$. We conclude that $X_{c\gamma}$ is a complete surface. The case $\xi < 0$ and $c > 0$ is analogous.

For $c < 0$ if $\sqrt{\xi(c, \gamma)} = n/m$ is an irreducible rational number, then $X_{c\gamma}$ is periodic in the variable $u_2$ with period $2m\pi$. Hence it is an immersion of a cylinder into $R^3$. Moreover, the surface has two ends $\mathcal{F}^\pm$ corresponding to $u_1 \to \pm \infty$.

It follows from (41) that the Gaussian curvature $\mathcal{K}$ of $X_{c\gamma}$, is given by

$$\mathcal{K} = \frac{2fg(f^2 + (1 + 2\gamma)^g^2)}{((f + g)^2 + 2\gamma g^2)(f^2 + (1 + 2\gamma)g(2f + g))}.$$  

Therefore, the domain $R \times [0, 2m\pi]$ of $X$ is composed of $2n$ horizontal strips where $\mathcal{K}$ changes sign from positive to negative at each open consecutive strip and vanishes on the bordering straight lines. Moreover, $\lim_{u_1 \to \pm \infty} \mathcal{K} = 0$ uniformly in $u_2$ in each strip. The critical points of $\mathcal{K}$ are determined by the points $(u_1, u_2)$ which anihilate $f'$ and $g'$, which occur at $X(0, u_2^0)$, where $u_2^0 = m(2k + 1)\pi/2n$, $0 \leq k \leq 2n - 1$. We conclude that the image of each of the $n$ regions of positive (resp. negative) curvature has an isolated point of maximum (resp. minimum) Gaussian curvature.

As an immediate consequence of $\mathcal{K}$, it follows that the total curvature vanishes. Moreover, a straightforward computation shows that on a horizontal strip with positive curvature, the total curvature is $4\pi$. Hence, the total absolute curvature is $8\pi n$.

In order to show that the ends $\mathcal{F}^\pm$ have geometric index $m$, we consider for each $\lambda \in R$, the intersection curve $\Gamma_\lambda$ of the surface with the horizontal plane $z = \lambda$. Let

$$\gamma_\lambda = \left\{ (u_1, u_2) \in \mathbb{R}^2; \tilde{z}(u_1, u_2) = u_1 + \frac{2f'(f + g)}{c[f^2 - (1 + 2\gamma)g^2]} = \lambda \right\}.$$  

It is not difficult to prove that $\gamma_\lambda$ is a regular connected curve, which is the graph of a function $u_1 = \beta_\lambda(u_2)$, for $\lambda > 0$ sufficiently large. Moreover, it follows from Proposition 2.4 that the curvature $k_\lambda$ of the curve $\Gamma_\lambda(u_2) = \tilde{X}(\beta_\lambda(u_2), u_2)$, has the following property: $\lim_{\lambda \to \infty} k_\lambda(u_2) = 1$ uniformly in $u_2$. 

If $\sqrt{\xi(c,\gamma)} = n/m$ then $\tilde{z}$ and hence $\beta_\lambda$ are periodic functions in $u_2$ with period $2m\pi$. Therefore, for $\lambda$ sufficiently large, $\Gamma_\lambda$ is a closed curve and
\[ \lim_{\lambda \to \infty} \frac{1}{2\pi} \int_0^{2m\pi} k_\lambda(u_2)\,du_2 = m. \]
The curve $\Gamma_\lambda$ will be a simple closed (convex) curve if and only if $m = 1$. We conclude that the end $\mathcal{F}^+$ has geometric index $m$ and it is embedded if and only if $m = 1$. Using the symmetry of the surface $(\tilde{X}_{c,\gamma}(-u_1,u_2)$ is obtained by $\tilde{X}_{c,\gamma}(u_1,u_2)$ by reflecting with respect to the $x0y$ plane) we get the result for $\mathcal{F}^-$. If $c > 0$ or $c < 0$ and $\sqrt{\xi}$ is not a rational number then $\tilde{X}_{c,\gamma}$ is a nonperiodic immersion of the plane into $R^3$, with an infinite number of isolated critical points of its Gaussian curvature. \[\square\]

Our next two results provide the symmetries of the complete linear Weingarten surfaces given by Proposition 2.3. In Figure 2, one can visualize some of the surfaces given by $\tilde{X}_{c,\gamma}$. In order to describe the symmetries of the surfaces we introduce the following notation:

A reflection with respect to a plane $z = z_0$ will be denoted by
\[ Z_{z_0}(x,y,z) = (x,y,-z + 2z_0). \]

Let $V_\beta = (-\sin\beta,\cos\beta,0)$ be a unit vector determined by a constant $\beta$. The reflection with respect to the plane orthogonal to $V_\beta$ which passes through the origin is denoted by
\[ S_\beta(p) = p - 2(p,V_\beta)V_\beta, \quad p \in R^3, \]
where $\langle , \rangle$ denotes the euclidean inner product of $R^3$.

**Proposition 2.6.** Any complete linear Weingarten surface $\tilde{X}_{c,\gamma}$, given by Proposition 2.3 with $c < 0$ satisfies the following symmetries:
\[ \tilde{X}_{c,\gamma}(u_1, u_2 + \theta) = R_\theta \tilde{X}_{c,\gamma}(u_1, u_2), \]
where $\theta = 2\pi/\sqrt{\xi(c,\gamma)},$
\[ \tilde{X}_{c,\gamma}(u_1, u_2 + \beta_k) = S_{\beta_k} \tilde{X}_{c,\gamma}(u_1, -u_2 + \beta_k), \]
where $\beta_k = (2k+1)\pi/2\sqrt{\xi(c,\gamma)},$
\[ \tilde{X}_{c,\gamma}(-u_1, u_2) = Z_0 \tilde{X}_{c,\gamma}(u_1, u_2), \]
where $\xi$ is given by (44) and $R_\theta$, $Z_0$ and $S_{\beta_k}$ are defined by (50)-(62).

**Proof.** The surface $\tilde{X}_{c,\gamma}$ with $c < 0$ is described by (43) where $f$ and $g$ are given by (53). Moreover, $\tilde{X}_{c,\gamma}(u_1, u_2) = R_{u_2}Y(u_1, u_2)$ where $Y = (1 - g\Lambda, -g\Lambda, u_1 - f\Lambda)$, $\Lambda = (f + g)/cP$ and $P$ is given by (54). Therefore, we have
\[ \tilde{X}_{c,\gamma}(u_1, u_2 + \theta) = R_{u_2+\theta}Y(u_1, u_2 + \theta) = R_\theta R_{u_2}Y(u_1, u_2) = R_\theta \tilde{X}_{c,\gamma}(u_1, u_2), \]
Figure 2. Complete Weingarten surfaces $\tilde{X}_{c\gamma}$ which satisfy the relation $-1/2 + H + \gamma K = 0$ and they are associated to the cylinder by Ribaucour transformations. a), b) and c) are 1-periodic cmc surfaces obtained by considering $\gamma = 0$ and $\sqrt{1-c} = n/m$ a rational number equal to $2/1$, $3/2$ and $7/6$ respectively. d) and e) are 1-periodic Weingarten surfaces for which $\gamma = 0.2$, $\sqrt{1-c(2\gamma+1)} = 14/13$ and $\gamma = -1/2$, $c = -0.1$ respectively. f) is a cmc surface obtained by considering $c = 2.8$ and $\gamma = 0$. 
which proves (63).

From the definition of $S_{\beta_k}$ we have $S_{\beta_k}(x, y, z) = R_{2\beta_k}S_0(x, y, z)$. Moreover, $g(-u_2 + \beta_k) = g(u_2 + \beta_k)$ and $g'(-u_2 + \beta_k) = -g'(u_2 + \beta_k)$. Hence,
\[
S_{\beta_k}\tilde{X}_{c\gamma}(u_1, -u_2 + \beta_k) = R_{2\beta_k}S_0R_{-u_2 + \beta_k}Y(u_1, -u_2 + \beta_k)
\]
\[
= R_{2\beta_k}S_0R_{-u_2 + \beta_k}S_0Y(u_1, u_2 + \beta_k)
\]
\[
= R_{u_2 + \beta_k}Y(u_1, u_2 + \beta_k) = \tilde{X}_{c\gamma}(u_1, u_2 + \beta_k),
\]
where we have used the identity $S_0R_{-u_2 + \beta_k}S_0 = R_{u_2 - \beta_k}$. This proves (64).

The proof of (65) follows from the equalities
\[
\tilde{X}_{c\gamma}(-u_1, u_2) = R_{u_2}Y(-u_1, u_2) = Z_0\tilde{X}_{c\gamma}(u_1, u_2).
\]

\[\square\]

**Proposition 2.7.** Any complete linear Weingarten surface $\tilde{X}_{c\gamma}$, given by Proposition 2.3 with $c > 0$ satisfies the following symmetries:

\[(66)\quad \tilde{X}_{c\gamma}(u_1 + \delta, u_2) = T_{\delta}\tilde{X}_{c\gamma}(u_1, u_2), \quad \text{where} \quad \delta = \frac{2\pi}{\sqrt{c}};
\]

\[(67)\quad \tilde{X}_{c\gamma}(u_1 + z_0, u_2) = Z_{z_0}\tilde{X}_{c\gamma}(-u_1 + z_0, u_2), \quad \text{where} \quad z_0 = \frac{(2k + 1)\pi}{2\sqrt{c}},
\]

\[(68)\quad \tilde{X}_{c\gamma}(u_1, u_2) = S_0\tilde{X}_{c\gamma}(u_1, -u_2),
\]

where $k$ is an integer, $T_{\delta}$, $Z_{z_0}$ and $S_0$ are defined by (51), (61) and (62) respectively.

**Proof.** The surface $\tilde{X}_{c\gamma}$ is described by (43) where $f$ and $g$ are given by (52). Therefore, (66) follows immediately. In order to prove (67), we observe that $f(-u_1 + z_0) = f(u_1 + z_0)$, and $f'(u_1 + z_0) = -f'(-u_1 + z_0)$. Therefore, we conclude that
\[
\tilde{X}_{c\gamma}(u_1 + z_0, u_2) = R_{u_2}Y(u_1 + z_0, u_2) = Z_{z_0}\tilde{X}_{c\gamma}(-u_1 + z_0, u_2),
\]
where we have used the function $Y$ introduced in the proof of Proposition 2.6.

Finally, we prove (68) by observing that
\[
S_0\tilde{X}_{c\gamma}(u_1, -u_2) = S_0R_{-u_2}Y(u_1, -u_2) = S_0R_{-u_2}S_0Y(u_1, u_2)
\]
\[
= R_{u_2}Y(u_1, u_2) = R_{u_2}Y(u_1, u_2) = \tilde{X}_{c\gamma}(u_1, u_2).
\]

\[\square\]

We conclude by observing that the linear Weingarten surfaces given by $\tilde{X}_{c\gamma}$ are tubular surfaces when $\gamma = -1/2$, since they satisfy $\Delta = \beta^2 - 4\alpha\gamma = 0$, and they provide examples of complete surfaces with $\Delta < 0$ and $\Delta > 0$. The lines of curvature $\tilde{X}_{c\gamma}(u_1, u_2^0)$ are planar, while the curves $\tilde{X}_{c\gamma}(u_1^0, u_2)$, when $f'(u_1^0) \neq 0$, are contained on a sphere centered at $(0, 0, f/f'(u_1^0))$ with radius $\sqrt{1 + (f/f')^2}$. Whenever $f'(u_1^0) = 0$ (it can only occur for $c > 0$)
then \( \tilde{X}_{\gamma}(u_1^0, u_2) \) are planar lines of curvature. Surfaces with one family of planar curvature lines while the other family of curvature lines is spherical is said to be of Joachimsthal type by Wente [W2].

3. Families of cmc surfaces associated to the cylinder.

In this section we describe a one parameter family of 1/2-cmc surfaces obtained from the cylinder by Ribaucour transformations. The surfaces are contained in the class of linear Weingarten surfaces described in Proposition 2.1 where we restrict \( \gamma = 0 \). These surfaces could also be obtained directly from the cylinder by applying Ribaucour transformations as in Corollaries 1.9 and 1.10.

In [PS], Pinkal and Sterling introduced the notion of a solution of finite type of the equation \( w_{zz} + \frac{\sinh(2w)}{2} = 0 \), where \( z \) is the complex variable \( z = u_1 + iu_2 \). The cmc surfaces associated to such solutions, i.e., those parametrized by isothermal coordinates such that the metric is given by \( ds^2 = 4e^{2w}(du_1^2 + du_2^2) \), are also called cmc surfaces of finite type. In particular a solution \( w \) is of finite type 1, if considering \( \varphi_1 = w_z \) and \( \varphi_2 = w_{zzz} - 2w_z^3 \), there exist complex numbers \( a \) and \( b \) such that \( \varphi_2 = a\varphi_1 + b\varphi_1 \). We will show that the cmc surfaces we obtain associated to the cylinder are of finite type 1.

**Proposition 3.1.** Excluding the cylinder, any 1/2-cmc parametrized surface locally associated to the cylinder as in Theorem 1.5 is given, up to rigid motions of \( R^3 \), by

\[
X_c = X + \frac{2}{c(f - g)}(f'X_{u_1} + g'X_{u_2} - gN)
\]

where \( N \) is the inner unit normal field of the cylinder parametrized by (42), \( c \) is a real constant such that \( c < 0 \) or \( c > 1 \), and the functions \( f(u_1) \) and \( g(u_2) \) are given by

\[
f = \varepsilon_1 \sqrt{1 - c} \sin(\sqrt{|c|} u_1) \quad g = \varepsilon_2 \sqrt{1 - c} \cosh(\sqrt{|c|} u_1) \quad \text{if } c > 1,
\]

\[
f = \varepsilon_1 \sqrt{1 - c} \cosh(\sqrt{|c|} u_1) \quad g = \varepsilon_2 \sqrt{|c|} \sin(\sqrt{1 - c} u_2) \quad \text{if } c < 0,
\]

where \( \varepsilon_i = \pm 1 \). These surfaces are of finite type 1.

**Proof.** By considering \( \gamma = 0 \) in (43)-(47), we get the right-hand side of (69), where \( f \) and \( g \) are solutions of

\[
f'' + cf = 0 \quad \quad g'' + (1 - c)g = 0
\]

and the initial conditions for \( f \) and \( g \) must satisfy

\[
((f')^2 + (g')^2 + (1 - c)g^2 + cf^2)(u_1^0, u_2^0) = 0.
\]

Moreover, since this last condition should be identically satisfied by the nontrivial solution functions \( f \) and \( g \), we conclude that the constant \( c \neq 0 \)
is such that $c(1 - c) \leq 0$, hence $c < 0$ or $c \geq 1$. However, when $c = 1$, it follows from (73) that $g = b_2$ where $b_2 \neq 0$ is a real constant and $f \equiv 0$. In this case $\tilde{X}$ reduces to the cylinder and therefore $c \neq 1$.

From (44) we have that $\xi = 1 - c$. Therefore, using Corollary 2.2, we conclude that if $c > 1$ then (52) reduces to (70) and if $c < 0$ then (53) reduces to (71). Each $1/2$-cmc surface described by (69) is parametrized by isothermal coordinates where the metric is given (see (49)) by $ds^2 = \psi^2(du_1^2 + du_2^2)$, with $\psi = (f + g)/(g - f)$. Any $1/2$-cmc surface described above is of finite type 1. In fact, considering $w = \log(\psi/2)$, $\varphi_1 = w_z$, $\varphi_2 = w_{zzz} - 2w_z^2$, it is a straightforward computation to verify that $\varphi_2 = (1/2 - c)\varphi_1 + \varphi_1$.

Proposition 3.2. Any $1/2$-cmc surface $\tilde{X}_c$ given by (69) is a complete surface asymptotically close to the cylinder. Moreover,

a) If $c < 0$ and $\sqrt{1 - c} = n/m$ is an irreducible rational number, then $\tilde{X}_c$ is an immersion of a cylinder into $\mathbb{R}^3$, with two ends of geometric index $m$ and $n$ isolated points of maximum (respectively minimum) for the Gaussian curvature. The total curvature of $\tilde{X}_c$ is zero, while its total absolute curvature is $8\pi m$. The ends are embedded if and only if $m = 1$. In this case they are cylindrical ends.

b) If $c > 1$ or $c < 0$ and $\sqrt{1 - c}$ is not a rational number then $\tilde{X}_c$ is a nonperiodic immersion of $\mathbb{R}^2$ into $\mathbb{R}^3$, with an infinite number of isolated critical points of its Gaussian curvature.

Proof. The properties of any surface $\tilde{X}_c$ and its asymptotic behaviour are consequences of Propositions 2.4 and 2.5, where we consider $\gamma = 0$. We observe that in this case the Gaussian curvature $\tilde{K}_c$ of $\tilde{X}_c$, is given by $\tilde{K}_c = 2fg(f^2 + g^2)/(f + g)^4$.

Proposition 3.3. Any $1/2$-cmc surface $\tilde{X}_c$ with $c < 0$ satisfies the symmetries defined by (63)-(65) where $\theta = 2\pi/\sqrt{1 - c}$ and $\beta_k = (2k+1)\pi/(2\sqrt{1 - c})$.

Proposition 3.4. Any $1/2$-cmc surface $\tilde{X}_c$ with $c > 1$ satisfies the symmetries defined by (66)-(68).

We observe that the family of cmc surfaces considered in this section, is a special case ($\gamma = 0$) of the linear Weingarten surfaces of the previous section. Therefore, they are cmc surfaces of Joachimsthal type (see [A, Wa] and [W2]).

4. Families of cmc surfaces associated to Delaunay surfaces.

In this section, by using Ribaucour transformations, we will obtain families of cmc surfaces associated to the Delaunay surfaces. We will consider the
Delaunay surfaces parametrized by
\[ X(u_1, u_2) = (\rho(u_1) \cos(u_2), \rho(u_1) \sin(u_2), \phi(u_1)), \]
where \( a \neq 0 \) is a real constant such that \( 1 - 4aH > 0 \),
\[ (74) \quad \rho(u_1) = \frac{1}{\sqrt{2H}} \left( 1 + \sqrt{1 - 4aH} \right)^{1/2}, \]
\[ (75) \quad \phi(u_1) = \int_{0}^{u_1} \left( \frac{H\rho}{\rho} + \frac{a}{\rho} \right) dt \quad u_1 \in \mathbb{R}. \]
The generating curves are unduloids when \( a > 0 \) and they are nodoids when \( a < 0 \).

**Proposition 4.1.** A parametrized surface is a \( H \)-cmc surface locally associated to a Delaunay surface \( X \), given by (74), by a Ribaucour transformation as in Corollary 1.10, if and only if, it is given by
\[ (77) \quad \tilde{X} = X - \frac{\rho}{c[ag + (\ell - Hf)\rho]} \cdot \left( (\rho'g + f')X_{u_1} + \frac{g'}{\rho}X_{u_2} - \left( \frac{ag}{\rho} + \ell + H\rho g \right) N \right) \]
where \( N \) is the inner unit normal vector field of the Delaunay surface, \( c \neq \frac{1}{2a} \) is such that \( c < 0 \) or \( c > 2H \) and \( \ell(u_1), f(u_1), g(u_2) \) are solutions of the equations
\[ (78) \quad f'' - (c + \lambda^1)\ell - c\lambda^2 f = 0, \]
\[ (79) \quad g'' + (1 - 2ac)g = 0, \]
\[ (80) \quad -\rho' f' + c\rho \lambda^1 f + \rho(c + \lambda^2)\ell = 0, \]
where
\[ (81) \quad \lambda^1 = -H + a/\rho^2 \quad \lambda^2 = -H - a/\rho^2 \]
and the initial conditions must satisfy
\[ (82) \quad ((g')^2 + (1 - 2ac)g^2 + (f')^2 + \ell^2 + 2cHf^2 - 2cf\ell) (u_1^0, u_2^0) = 0. \]

**Proof.** The first fundamental form of the Delaunay surface is given by \( ds^2 = du_1^2 + \rho^2(u_1)du_2^2 \) and the eigenvalues \( \lambda^1 \) and \( \lambda^2 \) of \( dN \) are given by (81). There are no umbilic points and the 1-forms dual to the principal directions are \( \omega_1 = du_1, \omega_2 = \rho du_2 \). Moreover, the connection forms are given by \( \omega_12 = \rho' du_2, \omega_{13} = -\lambda^1 du_1 \) and \( \omega_{23} = -\lambda^2 \rho du_2 \).
In order to obtain the Ribaucour transformations of the Delaunay surface, we need to solve the integrable system (16)-(18), and (37) (see Corollary 1.10), which reduces to the following system of differential equations:

(83) \[ \Omega_{1,u_1} = \Omega_1 \quad \Omega_{1,u_2} = \Omega_2 \rho \]

(84) \[ W_{1,u_1} = -\Omega_1 \lambda^1 \quad W_{1,u_2} = -\Omega_2 \lambda^2 \rho \]

(85) \[ \Omega_{1,u_1} = c\lambda^2 \Omega + (c + \lambda^1)W \quad \Omega_{1,u_2} = \rho'\Omega_2 \]

(86) \[ \Omega_{2,u_1} = 0 \quad \Omega_{2,u_2} = -\rho'\Omega_1 + c\rho\lambda^1\Omega + \rho(c + \lambda^2)W \]

where the initial condition for the solution must satisfy (37) at a given point.

From (86) and (85) we get

(87) \[ \Omega_2 = g'(u_2) \quad \Omega_1 = \rho'g + f'(u_1) \]

where \( g \) and \( f \) are functions of \( u_2 \) and \( u_1 \) respectively, and it follows from (83) and (84) that

(88) \[ \Omega = \rho g + f \quad W = -\rho g\lambda^2 + \ell(u_1), \]

where

(89) \[ \ell'(u_1) = -\lambda^1 f'. \]

Substituting (87) and (88) into Equations (85) and (86), we get

\[ (\rho'' + \rho\lambda^1\lambda^2)g = -f'' + c\lambda^2f + (c + \lambda^1)\ell \]

\[ g'' = (-(\rho')^2 + c\rho^2\lambda^1 - \rho^2(c + \lambda^2)\lambda^2)g - \rho'f' + c\rho\lambda^1f + \rho(c + \lambda^2)\ell. \]

From the expression of \( \rho \), we have \( \rho'' = -\frac{1}{\rho^3}(H^2\rho^4 - a^2) \) and \( -(\rho')^2 + c\rho^2\lambda^1 - \rho^2(c + \lambda^2)\lambda^2 = -1 + 2ac \neq 0. \)

On the other hand, using the expressions of \( \lambda^1 \) and \( \lambda^2 \), we have \( \rho\lambda^1\lambda^2 = (H^2\rho^4 - a^2)/\rho^3. \) Therefore, we conclude that \( f \) satisfies Equation (78) and

\[ g'' + (1 - 2ac)g = B \]

\[ -\rho'f' + c\rho\lambda^1f + \rho(c + \lambda^2)\ell = B \]

where \( B \) is a real constant, which we may consider, without loss of generality, to be zero. Hence \( g \) satisfies (79) and the functions \( f(u_1), \ell(u_1) \) must satisfy (80). Moreover, from Corollary 1.10 \( c(c - 2H) > 0 \) and since \( H > 0 \), it follows that \( c < 0 \) or \( c > 2H \).

Finally, from (87)-(88) and using the equalities \( (\rho')^2 + \rho^2(\lambda^2)^2 = 1 \) and

\[ \rho'f' + c\left(\rho\frac{H}{\rho} - \frac{a}{\rho}\right)f - \rho(\lambda^2 + c)\ell = 0, \]

we conclude that (37) reduces to (82).

We observe that in the open subset of \( R^2 \) where \( c + \lambda^2 \) does not vanish \( \ell \) is defined by (80). Moreover, \( \ell \) can be extended continuously to \( R^2 \). It is a straightforward computation to verify that \( \ell \) satisfies (89).
The cmc-surfaces given by (77) are called of Enneper type by Wente [W2], since one family of curvature lines is spherical. In fact, the curves \( \tilde{X}(u_1^n, u_2) \) are contained on a sphere centered at \((0, 0, \varphi(u_1^n) - \phi(u_1^n))\) with radius \(\sqrt{\rho^2 + \phi^2}\), where \(\phi = \left[ c \rho^2 (\lambda_1 f + \ell) - f \right] / (\rho' \ell + \lambda_2 \rho f')\).

The cmc surfaces described in Proposition 4.1 by \( \tilde{X} \) depend on \( \ell \) which is well-defined on \( \mathbb{R}^2 \). However, the points where \( c + \lambda^2 = 0 \) may introduce singularities for the function \( f \) which must satisfy (78). In the following result, we will restrict the range of the parameter \( c \) providing a sufficient condition for the surfaces \( \tilde{X} \) to be defined on \( \mathbb{R}^2 \).

**Proposition 4.2.** Let \( \tilde{X}(u_1^n, u_2) \) be an \( H \)-cmc surface locally associated to a Delaunay surface by a Ribaucour transformation as in Proposition 4.1. If \( c \) satisfies

\[
(90) \quad c < \frac{1}{2a} - \frac{\sqrt{1 - 4aH}}{2|a|}, \quad \text{or} \quad c > \frac{1}{2a} + \frac{\sqrt{1 - 4aH}}{2|a|}
\]

then \( \tilde{X} \) is defined on \( \mathbb{R}^2 \) and it is given by (77), where \( f(u_1) \) is a solution of the equation

\[
(91) \quad \rho^2 (c + \lambda^2) f'' - \rho \ell' (c + \lambda^1) f' + 2ac(c - 2H) f = 0
\]

g\((u_2)\) satisfies (79), \( \ell(u_1) \) is given by (80), \( \lambda^1 \) and \( \lambda^2 \) are given by (81) and the initial conditions for \( f \) and \( g \) satisfy (82).

**Proof.** Since \( c(c - 2H) > 0 \) we have \( c \neq H \), and the hypothesis of \( c \) satisfying (90), is equivalent to

\[
\left| \frac{c - H - 2acH}{(c - H) \sqrt{1 - 4aH}} \right| > 1.
\]

This inequality occurs, if and only if, the function \( c + \lambda^2 \) does not vanish for any real value of \( u_1 \). Moreover, we observe that whenever \( c \) satisfies the first inequality of (90) then \( c + \lambda^2 < 0 \) and if \( c \) satisfies the second one then \( c + \lambda^2 > 0 \). It follows from (80) that the differential equation (78) for the function \( f \) reduces to (91). \( \square \)

In order to show that the \( H \)-cmc surfaces obtained in Proposition 4.2 are complete, we will need the following result, where we consider an assumption on the functions \( f \) and \( g \) which is equivalent to the non-vanishing of \( S \) given by (21) (see Corollary 1.9).

**Lemma 4.3.** Let \( f(u_1) \) and \( g(u_2) \) be solutions of (91) and (79) respectively. Assume that \( \rho g + f \) does not vanish, then:
i) If $1 - 2ac < 0$ then

$$f = c_1 r(u_1)e^{i\alpha u_1} + c_2 s(u_1)e^{-i\alpha u_1},$$  \hspace{1cm} (92)  

where $A \neq 0$ and $\alpha$ are real numbers, $\alpha \neq kH$, $k \in \mathbb{Z} \setminus \{0\}$, $c_1$, $c_2 \in \mathbb{C}$ and $r(u_1)$, $s(u_1)$ are complex valued periodic functions of period $\pi/H$ or $2\pi/H$.

ii) If $1 - 2ac > 0$ then

$$f = c_1 r(u_1)e^{\delta u_1} + c_2 s(u_1)e^{-\delta u_1},$$  \hspace{1cm} (94) 

$$g = A \sin(\sqrt{1-2ac} u_2),$$  \hspace{1cm} (95)  

where $A \neq 0$, $c_1$, $c_2$, $\delta \neq 0$ are real numbers, $r(u_1)$, $s(u_1)$ are real valued periodic functions of period $\pi/H$ such that for all $u_1$, $c_1 r(u_1)c_2 s(u_1) > 0$.

Proof. We start observing that $f$ is a solution of (91), which is of type $f'' - \Lambda_1f' - \Lambda_2f = 0$, where $\Lambda_1$ and $\Lambda_2$ are real periodic functions of $u_1$, with period $\pi/H$ and $\Lambda_1$ is an odd function. Therefore, it follows from Floquet’s theory [Le] that, given any real initial conditions, the only solution of this equation is real and it is of the form (92), (94) or of type

$$f = c_1 a(u_1) + c_2 (u_1a(u_1) + b(u_1)),$$  \hspace{1cm} (96)  

where $c_1$, $c_2$ are complex numbers and $r(u_1)$, $s(u_1)$ are complex valued periodic functions of period $\pi/H$ or $2\pi/H$. If $f$ is of type (96), then there exits $(u_1^0, u_2^0)$ which annihilates the function $\rho g + f$, which is a contradiction. It follows that $c_2 = 0$ and hence $f$ is of type (92) or (94).

Since $f$ and $g$ satisfy (82) in $R^2$, it follows from the expression of $\ell$ given by (80) that $f$ satisfies the equation

$$\begin{align*}
(1 + h_1^2)(f')^2 + 2h_1(h_2 - c)f f' + [(h_2 - c)^2 - c(c - 2H)]f^2 + A^2(1 - 2ac) &= 0, \\
\end{align*}$$  \hspace{1cm} (97)  

where

$$h_1 = \frac{\rho'}{\rho(c + \lambda^2)} \quad \text{and} \quad h_2 - c = \frac{-c(c - 2H)}{c + \lambda^2}.$$  

By considering (97) as a quadratic equation for $f'$, its discriminant is given by

$$\Delta = 4(1 - 2ac) \left[ \frac{c(c - 2H)f^2}{\rho^2(c + \lambda^2)^2} - (1 + h_1^2)A^2 \right] \geq 0,$$  \hspace{1cm} (98)  

Now if $1 - 2ac < 0$, it follows from (98) that $f^2 \leq A^2 \left( \rho^2 + \frac{1-2ac}{c(c-2H)} \right)$. Since the right-hand side of this inequality is a bounded function, we conclude that
$f$ is bounded, hence it is of the form (92). Moreover, we can show that \( g \) is given by (93).

If \( 1 - 2ac > 0 \), then \( f \) is of type (94). In fact, otherwise, \( f \) is of the form (92) and it follows from Floquet’s theory that there exists \( u_1^0 \) such that \( f(u_1^0) = 0 \). Since \( g(0) = 0 \), it follows that the pair \((u_1^0, 0)\) annihilates the function \( \rho g + f \), which contradicts the hypothesis. Therefore, \( f \) is of type (94) and for any integer \( n \),

\[
(99) \quad f \left( u_1^0 + n \frac{\pi}{H} \right) = c_1r(u_1^0)e^{\lambda n^1 e^{\lambda n^2}} + c_2s(u_1^0)e^{-\lambda n^1 e^{-\lambda n^2}}.
\]

We claim that \( \forall u_1 \in R \), we have \( c_1r(u_1)c_2s(u_1) > 0 \). Otherwise, suppose there exists \( u_1^0 \) such that \( r(u_1^0)s(u_1^0) = 0 \), then it follows from (99) that \( f(u_1^0 + n\pi/H) \) tends to zero when \( n \) tends to \( \pm\infty \). This is a contradiction because (98) implies that \( f^2 \geq A^2 \left( \rho^2 + \frac{1-2ac}{e^{c-2H}} \right) \). Similarly, we get a contradiction if \( c_1r(u_1^0)c_2s(u_1^0) < 0 \), since in this case, it follows from (94) that \( f(R) = R \).

**Proposition 4.4.** Any surface of the family \( X \) given in Proposition 4.2 is a complete \( H \)-cmc surface. Moreover, if \( 1 - 2ac > 0 \) and \( \sqrt{1-2ac} \) is an irreducible rational number, then \( X \) is an immersion of a cylinder into \( R^3 \), otherwise it is a immersion of \( R^2 \) into \( R^3 \).

**Proof.** The first fundamental form of any surface of the family \( X \) is given by (see Remark 1.11) \( ds^2 = \psi_1^2 du_1^2 + \psi_2^2 du_2^2 \) where

\[
\psi_i = \frac{a(\rho g + f)a_i}{a\rho g + \rho^2(\ell - Hf)} \quad a_1 = 1, \text{ and } a_2 = \rho.
\]

Moreover, the functions \( f \) and \( \ell \) determined by (91) and (89) satisfy the following relation identically:

\[
(1 - 2ac)A^2 + (f')^2 + 2cf(Hf - \ell) + \ell^2 = 0.
\]

i) If \( 1 - 2ac > 0 \), then \( f \) and \( g \) are given by (94) (95). It follows from the above equation that \( f \) and \( Hf - \ell \) do not vanish for any \( u_1 \in R \) and there exists \( \varepsilon > 0 \) such that \( |\ell/f - H| \geq \varepsilon \) for all \( u_1 \in R \). Using the arguments of the proof of Lemma 4.3, by considering Equation (97) as a quadratic equation for \( f'/f \), we get that \( f'/f \) is a bounded function and hence \( \ell/f \) is also a bounded function. Hence,

\[
\lim_{|u_1| \to \infty} \left( \psi_i - \frac{a a_i f}{\rho^2(\ell - Hf)} \right) = 0
\]

uniformly in the variable \( u_2 \). Now, with arguments similar to those used in the proof of Proposition 2.5, we get \( m > 0 \) such that \( |\psi_i| \geq m \) for all \( (u_1, u_2) \in R^2 \) and hence we conclude that \( X \) is complete.

ii) If \( 1 - 2ac < 0 \), then \( f \) and \( g \) are given by (92) and (93). Hence

\[
\lim_{|u_2| \to \infty} |\psi_i - a_i| = 0
\]
uniformly in $u_1$. Therefore, there exists $m_0$ and $L$ positive real numbers such that $|\psi_i| \geq m_0$ for all $(u_1, u_2)$ such that $|u_2| > L$. One can show that for any $\alpha$, there exist $m_i > 0$ such that $|\psi_i(u_1, u_2)| \geq m_i > 0$ for all $(u_1, u_2) \in R \times [-L, L]$. We conclude that $\tilde{X}$ is complete.

We conclude this section by observing that with the same arguments of the proof of Propositions 2.6 and 2.7, one can show that any $H$-cmc surface given in Proposition 4.2 satisfies the following symmetries:

\[ \tilde{X}(u_1, u_2 + \theta) = R_\theta \tilde{X}(u_1, u_2) \quad \text{where } \theta = \frac{2\pi}{\sqrt{1 - 2ac}}, \]
\[ \tilde{X}(u_1, u_2 + \beta_k) = S_{\beta_k} \tilde{X}(u_1, -u_2 + \beta_k) \quad \text{where } \beta_k = \frac{(2k + 1)\pi}{2\sqrt{1 - 2ac}}, \quad k \in \mathbb{Z}, \]
whenever $1 - 2ac > 0$ and

\[ \tilde{X}(u_1, u_2) = S_0 \tilde{X}(u_1, -u_2) \]
whenever $1 - 2ac < 0$.

One can also show that the complete $H$-cmc surfaces of the family $\tilde{X}$ given by Proposition 4.1, whenever $1 - 2ac < 0$, are asymptotically close to the Delaunay surface. In fact, $\forall \varepsilon > 0$ there exists $L > 0$ such that

\[ |\tilde{X}(u_1, u_2) - X(u_1, u_2 \pm \theta)| < \varepsilon \quad \forall (u_1, u_2) \in \mathbb{R}^2 \text{ with } \pm u_2 \geq L, \]
where $\theta$ is such that $\cos \theta = 1 - 1/(ac)$ and $\sin \theta = -\sqrt{2ac - 1}/(ac)$.

References


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GALOIS GROUPS OF ORDER $2n$ THAT CONTAIN A CYCLIC SUBGROUP OF ORDER $n$

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Let $n$ be any integer with $n > 1$, and let $F \subseteq L$ be fields such that $[L : F] = 2$, $L$ is Galois over $F$, and $L$ contains a primitive $n^{th}$ root of unity $\zeta$. For a cyclic Galois extension $M = L(\alpha^{1/n})$ of $L$ of degree $n$ such that $M$ is Galois over $F$, we determine, in terms of the action of $\text{Gal}(L/F)$ on $\alpha$ and $\zeta$, what group occurs as $\text{Gal}(M/F)$. The general case reduces to that where $n = p^e$, with $p$ prime. For $n = p^e$, we give an explicit parametrization of those $\alpha$ that lead to each possible group $\text{Gal}(M/F)$.

1. Introduction.

Let $F \subseteq L$ be fields with $[L : F] = 2$ and $L$ Galois over $F$, and let $n > 1$ be a positive integer. Assume $L$ contains a primitive $n^{th}$ root of unity. Let $M$ be a cyclic Galois field extension of $L$ of degree $n$. So $M = L(\alpha^{1/n})$ for some $\alpha \in L^*$, by Kummer theory. Let $\text{Gal}(L/F) = \{\sigma, 1\}$. It is easy to verify that $M$ is Galois over $F$ just when $\sigma(\alpha) = \alpha^t \beta^n$ for some $\beta \in L^*$ with $t^2 \equiv 1 \mod n$ (that is, the cyclic group $\langle \alpha L^n \rangle \subseteq L^*/L^n$ is stable under the action of $\text{Gal}(L/F)$). The goal of this paper is to describe explicitly in terms of $\alpha$, $\beta$, and $t$ what group arises for $\text{Gal}(M/F)$.

To do this, we first classify in §2 the possible groups that can arise as $\text{Gal}(M/F)$. These are the groups of order $2n$ containing a cyclic subgroup of order $n$. There are too many of them for arbitrary $n$ (the number is given in Proposition 2.7). We show in §3 that the general question of determining $\text{Gal}(M/F)$ can be reduced to the same question when $n$ is a prime power. When $n = p^e$ with $p$ an odd prime, there are just two groups: Cyclic and dihedral. When $n = 2^e$ with $e \geq 3$ there are six groups: One cyclic, four semidirect products, and a generalized quaternion group. We give in Theorem 3.4 a general description of the group $\text{Gal}(M/F)$ in terms of $\alpha$, $\beta$, and $t$. Since we assume that the group $\mu_n$ of $n^{th}$ roots of unity lies in $L$, but not necessarily in $F$, we must take into account the action of $\text{Gal}(L/F)$ on $\mu_n$. In order to make the determination of $\text{Gal}(M/F)$ more explicit, we obtain in §4 precise descriptions of the $\alpha$ satisfying $\sigma(\alpha) = \alpha^t \beta^n$. This allows us
in §§5 and 6 to pin down in detail the circumstances under which a given group arises.

There has been much work over the years on the realization of groups as Galois groups. This is still a very active topic of research (see, e.g., [V] and [MM]). For larger groups the question has often been whether the group can be realized at all over a given field. For small groups, there are criteria for exactly when the group appears as a Galois extension, see, e.g., [GSS]. For nonsimple groups one approach has been to examine the embedding problem: Given a Galois field extension \( L/F \), when can we find a field \( M \supseteq L \) Galois over \( F \) with \( \text{Gal}(M/F) \) a given group that has \( \text{Gal}(L/F) \) as a homomorphic image. Most often in this approach \( M/L \) is of prime degree (as in [K] and [GSS]). The work here can be thought of as analyzing an extension problem, but now with \( [L : F] \) as small as possible, and \( [M : L] \) arbitrarily large, but \( M \) cyclic Galois over \( L \).

In the papers by Damey et. al. [D1], [D2], [DP] and [DM], there is an examination of when dihedral and quaternion groups of 2-power order appear as Galois groups; the 2-power case of Proposition 5.2 below appears as Prop. 1 and Cor. 1 in [D1]. The focus in those papers is primarily on when a quaternion group can occur as a Galois group, particularly over an algebraic number field. Also, the paper by Jensen, [J], especially pp. 447-449, considers all four nonabelian groups of order \( 2^e+1 \) containing a cyclic subgroup of order \( 2^e \); but, while Jensen is primarily interested in when the groups of order \( 2^e \) are realizable over a given base field, we give a full classification of the fields \( M \supseteq L \) that yield these groups as \( \text{Gal}(M/F) \), assuming \( L \) contains all \( 2^e \)th roots of unity.

2. Groups of order \( 2n \) that contain a cyclic subgroup of order \( n \).

In this section we classify groups of order \( 2n \) that contain a cyclic subgroup of order \( n \). When \( n \) is a power of 2, this classification is well-known. A good reference for this case is [G], pp. 191-193. The general case of describing finite metacyclic groups has been considered in [B].

**Proposition 2.1.** Let \( G \) be a group of order \( 2n \) that contains a cyclic subgroup of order \( n \). Then there exist \( \tau, \sigma \in G \) and nonnegative integers \( j, l \) such that \( G = \langle \tau, \sigma \rangle \) and:

1. \(|\tau| = n, \sigma \notin \langle \tau \rangle \),
2. \( \sigma \tau \sigma^{-1} = \tau^j, \sigma^2 = \tau^l \),
3. \( j^2 \equiv 1 \mod n \) and \( l(j - 1) \equiv 0 \mod n \).

**Proof.** Let \( \tau \) be an element of order \( n \) and let \( \sigma \in G \), but \( \sigma \notin \langle \tau \rangle \). Then \( G = \langle \tau, \sigma \rangle \) and \( \langle \tau \rangle \triangleleft G \). Thus \( \sigma \tau \sigma^{-1} = \tau^j \) for some \( j \geq 0 \), and \( \sigma^2 \in \langle \tau \rangle \) since \( G/\langle \tau \rangle \) has order 2. Let \( \sigma^2 = \tau^l \), where \( 0 \leq l \leq n - 1 \). Since

\[
\tau = \sigma^2 \tau \sigma^{-2} = \sigma (\sigma \tau \sigma^{-1}) \sigma^{-1} = \sigma \tau^j \sigma^{-1} = (\sigma \tau \sigma^{-1})^j = \tau^{j^2},
\]

\[
\tau^j = \tau^{j^2} \equiv 1 \mod n,
\]

and

\[
\sigma \tau \sigma^{-1} = \tau^j,
\]

it follows that \( j^2 \equiv 1 \mod n \). Therefore, \( j^2 \equiv 1 \mod n \) and \( l(j - 1) \equiv 0 \mod n \).
it follows \( j^2 \equiv 1 \mod n \). Since
\[
\tau^l = \sigma^2 = \sigma \tau \sigma^{-1} = (\sigma \tau \sigma^{-1})^l = (\tau^j)^l = \tau^{jl},
\]
it follows \( jl \equiv l \mod n \) and thus \( l(j - 1) \equiv 0 \mod n \).
\[\square\]

**Definition 2.2.** Let \( (G, j, l) \) denote a group of order \( 2n \) as described in Proposition 2.1. We always assume that \( j \) and \( l \) satisfy the conditions in Proposition 2.1(3).

For each ordered pair \((j, l)\) mod \( n\) satisfying Condition (3) of Proposition 2.1, there does in fact exist a group \( G \) as in Proposition 2.1 with such an ordered pair \((j, l)\). A quick construction of such a group is to take any field \( k \) containing a primitive \( n^{th} \) root of unity \( \zeta_n \), and let \( G \) be the subgroup of \( \text{GL}_2(k) \) generated by \( \tau = \left( \begin{array}{cc} \zeta_n & 0 \\ 0 & \zeta_n \end{array} \right) \) and \( \sigma = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \).

The groups \((G, j, l)\) are clearly determined up to isomorphism by \( j \) and \( l \mod n \), but different values of \( l \) can yield isomorphic groups. In the rest of this section, we will determine the isomorphism classes of the \((G, j, l)\). Let us note immediately the obvious isomorphisms arising from different choices of generators of \((G, j, l)\).

**Remark 2.3.** If for the group \((G, j, l)\) described in Proposition 2.1 we replace the generator \( \sigma \) by \( \sigma' = \sigma^s \), for any integer \( k \), then \( \sigma' \tau (\sigma')^{-1} = \tau^j \) and \((\sigma')^2 = \tau^l\), where \( l' = k(j + 1) + l \). Of course also, \( \tau^l = \tau^{sn + l} \) for any integer \( s \). Hence, \((G, j, l) \cong (G, j, l')\) whenever \( l' = k(j + 1) + sn + l \), i.e., whenever \( l' \equiv l \mod \gcd(j + 1, n) \). On the other hand, if we take another generator \( \tilde{\tau} \) of \( \langle \tau \rangle \), say \( \tau = (\tilde{\tau})^u \), where \( \gcd(u, n) = 1 \), then \( \sigma \tilde{\tau} \sigma^{-1} = (\tilde{\tau})^j \) and \( \sigma^2 = (\tilde{\tau})^l \), where \( \tilde{l} = ul \). So, \((G, j, l) \cong (G, j, \tilde{l})\). But this is an isomorphism we already have, since in fact \( \tilde{l} \equiv l \mod \gcd(j + 1, n) \). To see this congruence, let \( d = \gcd(j + 1, n) \). Then, \( d \mid n \mid (j - 1)l \) and \( d \mid (j + 1) \mid (j + 1)l \), so \( d \mid 2l \). If \( u \) is odd, then \( d \mid (u - 1)l = \tilde{l} - l \). If \( u \) is even, then \( n \) must be odd, so \( d \) is odd. Then \( d \mid 2l \) implies \( d \mid l \); likewise, \( d \mid \tilde{l} \), so again \( d \mid (\tilde{l} - l) \).

Let \( n = p_0^{e_0} p_1^{e_1} \cdots p_m^{e_m} \) be the prime decomposition of \( n \) where \( 2 = p_0 < p_1 < \cdots < p_m, m \geq 0, e_0 \geq 0, \) and \( e_i \geq 1 \) for all \( i \geq 1 \). Then, the Chinese Remainder Theorem shows
\[
j^2 \equiv 1 \mod n \text{ if and only if } \begin{cases} j^2 \equiv 1 \mod 2^{e_0} \\ j^2 \equiv 1 \mod p_i^{e_i}, & 1 \leq i \leq m. \end{cases}
\]
If \( p_i \) is an odd prime, then \( j - 1 \) or \( j + 1 \) must be a unit of the ring \( \mathbb{Z}/p_i^{e_i} \mathbb{Z} \), so
\[
j^2 \equiv 1 \mod p_i^{e_i} \text{ if and only if } j \equiv \pm 1 \mod p_i^{e_i}.
\]
For $p_0 = 2$, since $j - 1$ or $j + 1$ is not a multiple of 4,

$$j^2 \equiv 1 \mod 2^{e_0} \text{ if and only if } \begin{cases} j \equiv 1 \mod 2, & \text{if } e_0 = 1, \\ j \equiv 1, 3 \mod 4, & \text{if } e_0 = 2, \\ j \equiv \pm 1, 2^{e_0-1} \pm 1 \mod 2^{e_0} & \text{if } e_0 \geq 3. \end{cases}$$

Now, fix $j$ with $j^2 \equiv 1 \mod n$. To see how many different groups $(G, j, l)$ might exist for different choices of $l$, let $A = \{ l \in \mathbb{Z} \mid lj \equiv l \mod n \}$ and $B = \{ l \in \mathbb{Z} \mid \gcd(j + 1, n) \mid l \}$.

**Lemma 2.4.** With the notation above:

1. $B \subset A$ and $|A/B| = \begin{cases} 2, & \text{if } n \text{ is even and } j \equiv \pm 1 \mod 2^{e_0}, \\ 1, & \text{otherwise}. \end{cases}$

2. The number of isomorphism classes of groups $(G, j, l)$ with given $j$ (and $n$) is at most $|A/B|$.

**Proof.** (1) If $l \in B$, then $l \equiv k(j + 1) \mod n$, for some $k \in \mathbb{Z}$. Then, $l(j - 1) \equiv k(j + 1)(j - 1) \equiv 0 \mod n$, so $l \in A$. Thus, $B \subset A$.

Let $d_1 = \gcd(j - 1, n)$ and $d_2 = \gcd(j + 1, n)$. Then $l \in A \iff n | l(j - 1) \iff n/d_1 | l(j - 1)/d_1 \iff n/d_1 | l$. But, $l \in B$ just when $d_2 | l$. So, $A/B = (n/d_1)\mathbb{Z}/d_2\mathbb{Z}$, and $|A/B| = d_1d_2/n$. For $p_i$ an odd prime, we have $p_i^{e_i} \mid n \mid (j^2 - 1)$, but $p_i$ cannot divide both $j - 1$ and $j + 1$. Hence, the power of $p_i$ in one of $d_1, d_2$ is $p_i^{e_i}$ and the power of $p_i$ in the other is $p_i^{0}$. So, $p_i \nmid (d_1d_2/n)$. Thus, if $n$ is odd, we have $d_1d_2/n = 1$. If $n$ is even and $j \equiv \pm 1 \mod 2^{e_0}$, then the power of 2 in one of $d_1, d_2$ is $2^{e_0}$, and the power of 2 in the other is 2; thus $d_1d_2/n = 2$. The only remaining case is $e_0 \geq 3$ and $j \equiv 2^{e_0-1} \pm 1$. In this case, the power of 2 in one of $d_1, d_2$ is $2^{e_0-1}$, and in the other is $2^{1}$; then $d_1d_2/n = 1$.

(2) is clear from Proposition 2.1 and Remark 2.3. \(\square\)

**Proposition 2.5.** Let $G = (G, j, l)$.

1. If $n$ is odd, then $G \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2n\mathbb{Z}$.

2. Suppose $j \equiv -1 \mod n$.

   (a) If $n$ is odd, then $l \equiv 0 \mod n$ and $G \cong D_n$, the dihedral group of order $2n$.

   (b) If $n$ is even, then $n/2 \mid l$ and

   $$G \cong \begin{cases} (G, -1, 0) \cong D_n, & \text{if } l \equiv 0 \mod n, \\ (G, -1, n/2) = Q_n, & \text{if } l \equiv n/2 \mod n, \end{cases}$$

   where $Q_n$ is the generalized quaternion group of order $2n$. 

Proof. (1) $G$ is abelian just when $\tau$ and $\sigma$ commute, which occurs if and only if $j \equiv 1 \mod n$. Assume this holds. If $n$ is odd, there is only one abelian group of order $2n$ containing a cyclic group of order $n$. Now, suppose $n$ is even. If $l$ is even, then Remark 2.3 shows that $G \cong (G, j, 0) \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$; if $l$ is odd, then $G \cong (G, j, 1)$, which is cyclic, as $\sigma$ then has order $2n$.

(2) Assume $j \equiv -1 \mod n$. The condition $lj \equiv l \mod n$ of Proposition 2.1 forces $n \mid 2l$. If $n \mid l$, then $G \cong (G, -1, 0) \cong D_n$. This always holds if $n$ is odd. But, if $n$ is even, we have $n/2 \mid l$. So, when $n \nmid l$, we have $l \equiv n/2 \mod n$, and Remark 2.3 shows that $G \cong (G, -1, n/2) \cong Q_n$. (Our terminology in calling this a generalized quaternion group follows [CR], p. 23. Unlike some authors, we do not require a generalized quaternion group to be a 2-group.)

We are going to show how the study of the groups described in Proposition 2.1 can be reduced to the case where $n$ is a prime power. But let us first observe the (well-known) classification of these groups in the prime power situation. If $n = p^e$, where $p$ is an odd prime, then $j \equiv \pm 1 \mod n$, so the two possible groups $(G, j, l)$ are described in Proposition 2.5; one is abelian, the other is dihedral. The classification for $n$ a power of 2 is given in [G], Th. 4.3, p. 191 and Th. 4.4, p. 193: If $n = 2^e_0$ with $e_0 \leq 2$, then again $j \equiv \pm 1 \mod n$, and the possibilities for $(G, j, l)$ are given in Proposition 2.5. If $n = 2^e_0$ with $e_0 \geq 3$, there are two further groups besides the four given in Proposition 2.5. There is one group (and only one, by Lemma 2.4) with $j \equiv 2^{e_0-1} + 1 \mod 2^{e_0}$, which we write $(G, 2^{e_0-1}, 0)$ and is denoted $M_{e_0+1}(2)$ in [G]. There is also exactly one group with $j \equiv 2^{e_0-1} - 1 \mod 2^{e_0}$, which we write $(G, 2^{e_0-1}, 0)$ and Gorenstein calls the semidihedral group $S_{e_0+1}$. He proves in [G], Th. 4.3(iii), p. 191 that no two of the four non-abelian groups with $n = 2^e_0$ are isomorphic. This clearly applies to the two abelian groups, as well.

For any group $G = (G, j, l) = \langle \tau, \sigma \rangle$ as in Proposition 2.1, let $H_i$ be the unique subgroup of $\langle \tau \rangle$ of order $n/p_i^{e_i}$, $0 \leq i \leq m$. Then, each $H_i < G$ and $|G/H_i| = 2p_i^{e_i}$. Furthermore, if we let $\tau = \tau H_i$ and $\sigma = \sigma H_i$, then $G/H_i = \langle \tau, \sigma \rangle$, where $\langle \tau \rangle$ is a cyclic subgroup of order $p_i^{e_i}$, $\tau \tau^{-1} = \tau^j$, $\sigma^2 = \tau$, and $\tau \not\in \langle \tau \rangle$. Thus, $G/H_i$ is a group of the type described in Proposition 2.1, with $n$ replaced by $n' = p_i^{e_i}$. Note that every element of $G$ of odd order has trivial image in $G/\langle \tau \rangle$, so must lie in $\langle \tau \rangle$. Thus, $H_0$ consists of all the elements of $G$ of odd order.

**Theorem 2.6.** Suppose $(G, \langle \tau \rangle, \sigma, j, l)$ and $(G', \langle \tau' \rangle, \sigma', j', l')$ are each groups of order $2n$ as in Proposition 2.1 and with all the previous notation. Assume $(j, l)$ and $(j', l')$ satisfy Condition (3) in Proposition 2.1. Let $H_i$ and $H_i'$, $0 \leq i \leq m$, be the subgroups of $\langle \tau \rangle$ and $\langle \tau' \rangle$ defined before. Then the following statements are equivalent:
(1) \( G \cong G' \).
(2) \( j \equiv j' \mod n \) and \( l \equiv l' \mod \gcd(j + 1, n) \).
(3) \( G/H_i \cong G'/H'_i, 0 \leq i \leq m \).
(4) \( j \equiv j' \mod n/2^{e_0} \) and \( G/H_0 \cong G'/H'_0 \).

**Proof.** (2) \( \Rightarrow \) (1): This was done in Remark 2.3.

(1) \( \Rightarrow \) (4): Let \( \alpha: G \to G' \) be an isomorphism. Since \( H_0 \) (resp. \( H'_0 \)) consists of all the elements of \( G \) (resp. \( G' \)) of odd order, \( \alpha(H_0) = H'_0 \). Therefore, \( \alpha \) induces an isomorphism \( G/H_0 \cong G'/H'_0 \). Let \( h \) be any generator of \( H_0 \), and let \( h' = \alpha(h) \), which generates \( H'_0 \). The conjugacy class of \( h \) in \( G \) is \( \{h, h^3\} \), which must be mapped bijectively to the conjugacy class \( \{h', (h')^j\} \) of \( h' \) in \( G' \). If these classes contain only one element each, then \( j \equiv 1 \equiv j' \mod n/2^{e_0} \). If the classes contain two elements each, then \( (h')^j' = \alpha(h^j) = (h^j)^j = (h')^j \), so again \( j \equiv j' \mod n/2^{e_0} \).

(3) \( \Leftrightarrow \) (4): For \( i \geq 1 \), since \( \left( i/\langle \tau \rangle \right) \) is a power of an odd prime, we have \( G/H_i \) is either abelian or dihedral. The first case occurs just when \( j \equiv 1 \mod p_i^{e_i} \), and the second just when \( j \equiv -1 \mod p_i^{e_i} \). Thus, \( G/H_i \cong G/H'_i \) if and only if \( j \equiv j' \mod p_i^{e_i} \). By the Chinese Remainder Theorem, this occurs for all \( i \geq 1 \) if and only if \( j \equiv j' \mod n/2^{e_0} \).

(3) \( \Rightarrow \) (2): As observed above, \( G/H_i \) is a group of type \((j, l)\) with \( n \) replaced by \( p_i^{e_i} \). For \( p_i \) odd, we noted in the previous paragraph that \( G/H_i \cong G'/H'_i \) implies \( j \equiv j' \mod p_i^{e_i} \); then Lemma 2.4 shows that the conditions \( lj \equiv l \) and \( lj' \equiv l' \mod p_i^{e_i} \) from Proposition 2.1 imply that \( l \equiv l' \mod \gcd(j + 1, p_i^{e_i}) \). For \( i = 0 \), [G], Th. 4.3(iii), p. 191, together with Proposition 2.5, shows that \( G/H_0 \cong G'/H'_0 \) implies \( j \equiv j' \mod 2^{e_0} \) and if \( j \equiv \pm 1 \), then \( l \) and \( l' \) must lie in the same congruence class \( \mod \gcd(j + 1, 2^{e_0}) \). (There are just two possible congruence classes, by Lemma 2.4.) When \( e_0 \geq 3 \) and \( j \equiv 2^{e_0-1} \pm 1 \mod 2^{e_0} \), Lemma 2.4 shows that the conditions \( j \equiv j', lj \equiv l \), and \( lj' \equiv l' \mod 2^{e_0} \) already imply \( l \equiv l' \mod \gcd(j + 1, 2^{e_0}) \). Thus, the Chinese Remainder Theorem yields \( j \equiv j' \mod n \) and \( l \equiv l' \mod \gcd(j + 1, n) \), as desired. \( \square \)

We can now count the number of isomorphism classes of groups of order \( 2n \) containing a cyclic subgroup of order \( n \). Let \( n = 2^{e_0}p_1^{e_1} \cdots p_m^{e_m} \), as usual. Let \( G \) be any such group, let \( H_0 \) be its unique (cyclic) subgroup of order \( n/2^{e_0} \), and let \( S \) be any 2-Sylow subgroup of \( G \). Since \( H_0 \vartriangleleft G \), \( |H_0 \cap S| = 1 \) (as \( \gcd(|H_0|, |S|) = 1 \)), and \( G = H_0S \) (as \( |G| = |H_0||S|/|H_0 \cap S| \)), \( G \) is the semidirect product of \( H_0 \) by \( S \). (We thank R. Guralnick for pointing out this semidirect product decomposition to us.) So, \( G \) is determined by \( H_0, S, \) and the map \( \gamma: S \to \text{Aut}(H_0), s \mapsto \text{conjugation by } s \). The image of \( \gamma \) consists of the identity map and the \( j^{th} \) power map. Theorem 2.6(4) shows that \( G \) is determined up to isomorphism by the isomorphism class of \( S \) (\( \cong G/H_0 \)) and by \( j \mod n/2^{e_0} \). By the results in [G] quoted above, the number of possible choices of \( S \) is \( 2^{e_0} \) if \( 0 \leq e_0 \leq 2 \) and is \( 6 \) if \( e_0 \geq 3 \). The number of
possible choices of \( j \mod n/2^{e_0} \) is \( 2^m \) since we must have \( j \equiv \pm 1 \mod p_i^{e_i} \) for \( 1 \leq i \leq m \). Every such choice of \( S \) and \( j \) yields a semidirect product that is a group of the desired type. (For, we obtain a cyclic group of order \( n \) in the semidirect product as the direct product of \( H_0 \) with a cyclic subgroup of \( S \) of order \( 2^{e_0} \) lying in \( \ker(\gamma) \).) Theorem 2.6 shows that different isomorphism classes of \( S \) or different choices of \( j \mod n/2^{e_0} \) yield nonisomorphic groups. Thus, we have proved:

**Proposition 2.7.** Let \( n \) have prime factorization \( n = 2^{e_0}p_1^{e_1} \cdots p_m^{e_m} \). The number of isomorphism classes of groups \( G \) of order \( 2n \) containing a cyclic subgroup of order \( n \) is \( \begin{cases} 2^{e_0+m}, & \text{if } 0 \leq e_0 \leq 2, \\ 6 \cdot 2^m, & \text{if } e_0 \geq 3. \end{cases} \)

### 3. Galois extensions with group \( G \).

Let \( F \) be a field with char \( F \nmid n \) and let \( L/F \) be a Galois quadratic extension. That is, if \( 2 \nmid n \) and char \( F = 2 \), assume that the quadratic extension \( L/F \) is also a separable extension.

Let \( G \) be a group of order \( 2n \) that contains a cyclic subgroup of order \( n \). We shall continue to use the notation from Section 2.

In this section, we shall determine when there exists a cyclic extension \( M/L \) of degree \( n \) such that \( M/F \) is a Galois extension with \( \text{Gal}(M/F) \cong G \). For most of this section, we shall assume that \( L \) contains a primitive \( n^{th} \) root of unity.

**Proposition 3.1.** Let \( G \) be a group of order \( 2n \) as in Proposition 2.1. Let \( \langle \tau \rangle \) be a cyclic subgroup of \( G \) of order \( n \) and let \( H_i, 0 \leq i \leq m \), be the subgroups of \( \langle \tau \rangle \) defined in Section 2. Let \( L/F \) be a Galois quadratic extension. Then the following statements are equivalent:

1. \( L/F \) extends to a Galois extension \( M/F \) with \( \text{Gal}(M/F) \cong G \).
2. For each \( i, 0 \leq i \leq m \), \( L/F \) extends to a Galois extension \( M_i/F \) with \( \text{Gal}(M_i/F) \cong G/H_i \) and \( \text{Gal}(M_i/L) \cong \langle \tau \rangle/H_i \).

**Proof.** It is clear that (1) implies (2) by letting \( M_i \) be the fixed field of \( H_i \) and recalling that \( H_i \triangleleft G \).

Now assume (2) holds. Then \( M_i/L \) is a cyclic Galois extension with \( [M_i : L] = p_i^{e_i} \), since \( [\langle \tau \rangle : H_i] = p_i^{e_i} \). Let \( M = M_0 \cdots M_m \). Then \( M/L \) is a cyclic Galois extension with \( [M : L] = p_0^{e_0} \cdots p_m^{e_m} = n \) and \( M/F \) is a Galois extension since \( M_i/F \) is a Galois extension, \( 0 \leq i \leq m \). Let \( G' = \text{Gal}(M/F) \), \( \langle \tau' \rangle = \text{Gal}(M/L) \), and \( H_i' = \text{Gal}(M_i) \). Then \( G/H_i \cong \text{Gal}(M_i/F) \cong G'/H_i' \), \( 0 \leq i \leq m \). Theorem 2.6 implies \( G \cong G' \).

Let \( \zeta \) denote a primitive \( n^{th} \) root of unity. From here on, assume that \( \zeta \in L \). Let \( \alpha \in L^* \) and let \( k \mid n \). Let \( L(\alpha^{1/k}) \) denote a field obtained by adjoining to \( L \) a root of the equation \( x^k - \alpha = 0 \). Since char \( L \nmid k \) and \( L \)}
contains a primitive \( k \)th root of unity, it follows \( L(\alpha^{1/k}) \) is a splitting field of \( x^k - \alpha \) over \( L \) and hence \( L(\alpha^{1/k})/L \) is a Galois extension. In particular, the field \( L(\alpha^{1/k}) \) does not depend on which \( k \)th root of \( \alpha \) is chosen. If \( [L(\alpha^{1/k}) : L] = k \), then \( \text{Gal}(L(\alpha^{1/k})/L) \cong \mathbb{Z}/k\mathbb{Z} \). However, when we write \( \alpha^{1/k} \), we will assume some specified \( k \)th root of \( \alpha \) has been selected and fixed throughout the discussion. Then \( \alpha^{s/k} \) will mean \( (\alpha^{1/k})^s \) for the given choice of \( \alpha^{1/k} \).

**Lemma 3.2.** Let \( \alpha, \beta \in L \). Let \( r, s \) be positive integers with \( \gcd(r, s) = 1 \) and assume \( rs \mid n \). Then \( L(\alpha^{1/r}, \beta^{1/s}) = L(\gamma^{1/(rs)}) \) where \( \gamma = \alpha^s \beta^r \).

**Proof.** We have \( L(\gamma^{1/(rs)}) \subseteq L(\alpha^{1/r}, \beta^{1/s}) \) since

\[
\gamma^{1/(rs)} = \alpha^{1/r} \beta^{1/s} \in L(\alpha^{1/r}, \beta^{1/s}).
\]

Choose \( a, b \in \mathbb{Z} \) such that \( ar + bs = 1 \). Then

\[
\alpha^{1/r} = \alpha^{(ar+bs)/r} = \alpha^a \alpha^{bs/r} = \alpha^a \beta^{-b} \alpha^{bs/r} \beta^b = \alpha^a \beta^{-b}(\alpha^s \beta^r)^{bs/r} = \alpha^a \beta^{-b}(\gamma^{1/(rs)})^{bs} \in L(\gamma^{1/(rs)}).
\]

Similarly, \( \beta^{1/s} \in L(\gamma^{1/(rs)}) \). \( \square \)

Let \( \text{Gal}(L/F) = \{1, \sigma\} \). Since \( \zeta \in L \) is a primitive \( n \)th root of unity, we have

\[
\sigma(\zeta) = \zeta^r,
\]

where \( \gcd(r, n) = 1 \). This equation defines \( r \) (mod \( n \)), which will be a significant invariant from here on. Note that \( r^2 \equiv 1 \) mod \( n \) since \( \zeta = \sigma^2(\zeta) = \sigma(\zeta^r) = \zeta^{r^2} \).

**Definition 3.3.** If \( L \subseteq M \), we will say that \( M/F \) realizes \( (G, j, l) \) if \( M/F \) is a Galois extension and \( \text{Gal}(M/F) = \langle \tau, \sigma \rangle \), where \( \text{Gal}(M/L) = \langle \tau \rangle \), \( \sigma \) denotes an extension of \( \sigma \in \text{Gal}(L/F) \) to an automorphism in \( \text{Gal}(M/F) \), \( \sigma \tau \sigma^{-1} = \tau^j \), and \( \sigma^2 = \tau^l \).

**Theorem 3.4.** Assume \( \zeta \in L \). Let \( M = L(\alpha^{1/n}) \), where \( \alpha \in L \), and assume \( [M : L] = n \). Then the following statements hold:

1. \( M/F \) is a Galois extension if and only if \( \sigma(\alpha) = \alpha^t \beta^n \), where \( \beta \in L \), \( \gcd(t, n) = 1 \). When this occurs, for any \( t' \equiv t \) mod \( n \) there is \( \beta' \in L \) with \( \sigma(\alpha) = \alpha^t (\beta')^n \).
2. If \( M/F \) is a Galois extension, then there exist integers \( j, l \) such that \( M/F \) realizes \( (G, j, l) \).
3. The following statements are equivalent:
   a. \( M/F \) realizes \( (G, j, l) \).
   b. \( \sigma(\alpha) = \alpha^t \beta^n \), with \( t \equiv jr \) mod \( n \) and \( \alpha^{(t^2-1)/n} \beta' \sigma(\beta) = \zeta^{l_1} \) where \( l_1 \equiv l \) mod \( \gcd(j+1, n) \).
(4) If $M/F$ realizes $(G, j, l)$ and we choose $\zeta$ so that $\tau(\alpha^{1/n}) = \zeta \alpha^{1/n}$ and $\beta$ so that $\sigma(\alpha^{1/n}) = \alpha^{t/n} \beta$, then $\alpha^{(t^2-1)/n} \beta' \sigma(\beta) = \zeta^t$. If we let $\beta' = \zeta^i \beta$, then $\alpha^{(t^2-1)/n}(\beta')^i \sigma(\beta') = \zeta^{t+i(r(j+1))}$.

Proof. (1) $M/F$ is a Galois extension $\Leftrightarrow (x^n - \alpha)(x^n - \sigma(\alpha))$ splits completely in $M$. Since $M = L/\alpha^{1/n}, \sigma(\alpha)^1/n$ $\Leftrightarrow L(\sigma(\alpha)^1/n)$, since $M = L(\alpha^{1/n})$ and $[L(\alpha^{1/n}) : L] = [L(\sigma(\alpha)^1/n) : L]$, $\Leftrightarrow \sigma(\alpha) = \alpha^{t/n} \beta^n$ with $\gcd(t, n) = 1$ and $\beta \in L$, by Kummer Theory. Finally, if $t' = t + dn$ and $\sigma(\alpha) = \alpha^{t/n} \beta^n$, then $\sigma(\alpha) = \alpha^{t'}(\beta')^n$ where $\beta' = \alpha^{-d} \beta$.

(2) Assume $M/F$ is a Galois extension and let $G = \text{Gal}(M/F)$. Since $|G| = [M : F] = 2n$ and $M/L$ is a cyclic extension of degree $n$, it follows that $\text{Gal}(M/L)$ is a cyclic subgroup of $G$ of order $n$ and thus $G$ is a group as in Proposition 2.1. Let $\text{Gal}(M/L) = \langle \tau \rangle$ and let $\sigma$ denote an extension of $\sigma \in \text{Gal}(L/F)$ to an automorphism $\sigma$ in $\text{Gal}(M/F)$. Then $G = \langle \tau, \sigma \rangle$ since $\sigma |L \neq 1$. Since $[G : \langle \tau \rangle] = 2$, we have $\sigma \sigma^{-1} \in \langle \tau \rangle$ and $\sigma^2 \in \langle \tau \rangle$. Thus $\sigma \tau \sigma^{-1} = \tau^j$ and $\sigma^2 = \tau^l$ and $M/F$ realizes $(G, j, l)$.

(3) and (4) Assume $M/F$ realizes $(G, j, l)$. Then $\sigma(\alpha) = \alpha^{t/n} \beta^n$ with $\beta \in L$, from the proof of (1). This equation implies $\sigma(\alpha^{1/n}) = \alpha^{t/n} \beta \omega$ where $\omega$ is an $n^{th}$ root of unity. We may replace $\beta \omega$ by $\beta$ so that we may assume that $\sigma(\alpha^{1/n}) = \alpha^{t/n} \beta$. We have $\tau(\alpha^{1/n}) = \zeta^{t} \alpha^{1/n}$, where $\zeta$ is a primitive $n^{th}$ root of unity, since $\tau$ has order $n$ and $M = L(\alpha^{1/n})$ is a cyclic extension of degree $n$. We can assume that $\zeta = \zeta^t$. We now apply the equation $\sigma \tau = \tau^j \sigma$ to $\alpha^{1/n}$.

$$\sigma \tau(\alpha^{1/n}) = \sigma(\zeta \alpha^{1/n}) = \zeta^t \sigma(\alpha^{1/n}) = \zeta^t \alpha^{t/n} \beta.$$  

Thus $\alpha^{t/n} \beta$ and $jt \equiv r \mod n$. Since $j^2 \equiv 1 \mod n$, it follows $t \equiv jr \mod n$ and $t^2 \equiv j^2 r^2 \equiv 1 \mod n$.

Next we apply the equation $\sigma^2 = \tau^l$ to $\alpha^{1/n}$. Since $$\sigma^2(\alpha^{1/n}) = \tau^l(\alpha^{1/n}) = \zeta^l \alpha^{1/n}$$ and $$\sigma^2(\alpha^{1/n}) = \sigma(\alpha^{t/n} \beta) = \sigma(\alpha^{1/n})^l \sigma(\beta) = \alpha^{t/n} \beta^l \sigma(\beta),$$ it follows $\alpha^{(t^2-1)/n} \beta^l \sigma(\beta) = \zeta^t$. We have now proved the first sentence of (4). For the rest of (4), observe that if $\beta' = \zeta^i \beta$, then $$\alpha^{(t^2-1)/n}(\beta')^i \sigma(\beta') = (\alpha^{(t^2-1)/n} \beta^l \sigma(\beta))^i = \zeta^{t+i(r(j+1))},$$ since $t + r \equiv jr + r \equiv r(j + 1) \mod n$.

To show (3)(a) $\Rightarrow$ (3)(b) we must see what happens if we make different choices of $\beta$ and $\zeta$. But, if $\sigma(\alpha^{1/n}) = \alpha^{t/n} \beta \omega$ and $\tau(\alpha^{1/n}) = \zeta^{t} \alpha^{1/n}$, then there is another generator $\tau_1$ of $\langle \tau \rangle$ and a $\sigma_1 = \sigma \tau^i$ such that $\sigma_1(\alpha^{1/n}) = \alpha^{t/n} \beta$ and $\tau_1(\alpha^{1/n}) = \zeta \alpha^{1/n}$. Then, the calculation made above (using $\sigma_1$
and $\tau_1$ and noting that $\sigma_1(\beta) = \sigma(\beta)$) shows that $\alpha^{(t^2-1)/n}\beta'\sigma(\beta) = \zeta^{l_1}$, where $\sigma^2 = \tau_1^{l_1}$. But, we saw in Remark 2.3 that $l_1 \equiv l \mod \gcd(j+1, n)$, so we have (3)(b).

Now assume the equations in (3)(b) hold. Then $M/F$ is a Galois extension by (1). Choose a generator $\tau$ of $\text{Gal}(M/L)$ such that $\tau(\alpha^{1/n}) = \zeta\alpha^{1/n}$ and choose $\sigma \in \text{Gal}(M/F)$ extending $\sigma \in \text{Gal}(L/F)$ such that $\sigma(\alpha^{1/n}) = \alpha^{t/n}\beta$. Then, (2) implies that $M/F$ realizes $(G, j', l')$, where $\sigma\tau\sigma^{-1} = \tau^j$, so $(j')^2 \equiv 1 \mod n$, and $\sigma^2 = \tau^l$. The equation $\sigma\tau(\alpha^{1/n}) = \tau^j\sigma(\alpha^{1/n})$ shows that $\zeta^{j't} = \zeta^r$, so $j't \equiv r \equiv jt \mod n$. Hence, $j' \equiv j \mod n$. Also, the calculation above for $\sigma^2(\alpha^{1/n})$ shows that $\alpha^{(t^2-1)/n}\beta'\sigma(\beta) = \zeta^{l'}$. Hence, $l' \equiv l_1 \equiv l \mod \gcd(j+1, n)$. But then, since $M/F$ realizes $(G, j', l')$, it also realizes $(G, j, l)$ with a different choice of $\sigma$, by Remark 2.3.


Let $L/F$ be a Galois quadratic extension and assume $\zeta \in L$ is a primitive $n^{th}$ root of unity. Thus $\text{char } F \nmid n$. Let $\sigma \in \text{Gal}(L/F)$ with $\sigma \neq 1$. Then $\sigma(\zeta) = \zeta^r$ where $r^2 \equiv 1 \mod n$. If $\text{char } F \neq 2$, let $L = F(\sqrt{a})$, $a \in F$.

In this section we study the problem of describing elements $\alpha \in L^*$ with the property $\sigma(\alpha) = \alpha^{t/n}\beta$, $\beta \in L$, for a given integer $t$ satisfying $t^2 \equiv 1 \mod n$. By Theorem 3.4(1), this is equivalent to describing elements $\alpha \in L^*$ with the property that $L(\alpha^{1/n})$ is a Galois extension of $F$. These results will be applied in Sections 5 and 6 to the problem of constructing the Galois extensions discussed in Section 3 with a given group as described in Proposition 2.1. Keeping in mind the intended applications in Sections 5 and 6, we shall consider only the cases $t \equiv \pm 1 \mod n$ and $t \equiv \pm 1, 2^{e-1} \pm 1 \mod 2^e$, $e \geq 3$, when $n = 2^e$.

We begin with a lemma to be used in the case $t \equiv 1 \mod n$.

Lemma 4.1.

1. If $\delta, \delta' \in L^*$ and $\sigma(\delta)/\delta = \sigma(\delta')/\delta'$, then $\delta' = b\delta$ with $b \in F$.
2. Suppose $\gamma = \sigma(\delta)/\delta$ with $\gamma, \delta \in L$. Then there exists $b \in F$ such that

$$\delta = \begin{cases} 
  b(1 + \sigma(\gamma)), & \text{if } \gamma \neq -1, \\
  b\sqrt{a}, & \text{if } \gamma = -1, \text{ char } F \neq 2, \\
  b & \text{if } \gamma = -1, \text{ char } F = 2.
\end{cases}$$

Proof. The equation in (1) implies $\sigma(\delta)/\delta = \delta'/\delta$ and thus $\delta'/\delta \in F$. This implies (1).

For (2), first assume $\gamma \neq -1$. Then $1 + \sigma(\gamma) \neq 0$. Since $\gamma\sigma(\gamma) = N_{L/F}(\gamma) = 1$, it follows $\sigma(\delta)/\delta = \gamma = \frac{1 + \gamma}{1 + \sigma(\gamma)} = \frac{\sigma(1 + \sigma(\gamma))}{1 + \sigma(\gamma)}$. Now (1) implies that $\delta = b(1 + \sigma(\gamma))$ with $b \in F$. Now assume $\gamma = -1$. If char $F \neq 2$,
then \( \sigma(\sqrt{a})/\sqrt{a} = -1 \), and so (1) implies that \( \delta = b\sqrt{a} \) with \( b \in F \). If \( \text{char} F = 2 \), then \( \sigma(\delta)/\delta = -1 = 1 \) and hence \( \delta \in F \).

The following proposition covers the case \( t \equiv 1 \mod n \):

**Proposition 4.2.** Let \( n \) be a positive integer and let \( \alpha \in L \). Then \( \sigma(\alpha) = \alpha \beta^n \), \( \beta \in L \), if and only if there exists \( b \in F \) such that

\[
\alpha = \begin{cases} 
  b(1 + \gamma^n), & \text{if } \sigma(\alpha)/\alpha \neq -1, \\
  b\sqrt{a}, & \text{if } \sigma(\alpha)/\alpha = -1, \text{ char } F \neq 2, \text{ and } -1 \in L^n, \\
  b, & \text{if } \sigma(\alpha)/\alpha = -1, \text{ char } F = 2,
\end{cases}
\]

where in the first case above, \( \gamma \in L \) and \( N_{L/F}(\gamma)^n = 1 \).

**Proof.** First suppose \( \sigma(\alpha) = \alpha \beta^n \), \( \beta \in L \). Then \( \beta^n = \sigma(\alpha)/\alpha \) and Lemma 4.1(2) implies there exists \( b \in F \) such that

\[
\alpha = \begin{cases} 
  b(1 + \sigma(\beta^n)), & \text{if } \sigma(\alpha)/\alpha \neq -1, \\
  b\sqrt{a}, & \text{if } \sigma(\alpha)/\alpha = -1, \text{ char } F \neq 2, \\
  b, & \text{if } \sigma(\alpha)/\alpha = -1, \text{ char } F = 2.
\end{cases}
\]

If \( \sigma(\alpha)/\alpha \neq -1 \), let \( \gamma = \sigma(\beta) \). Then

\[
N_{L/F}(\gamma)^n = N_{L/F}(\sigma(\beta)^n) = N_{L/F}(\beta^n) = N_{L/F}(\sigma(\alpha)/\alpha) = 1.
\]

If \( \sigma(\alpha)/\alpha = -1 \) and \( \text{char } F \neq 2 \), then \( -1 = \beta^n \in L \). Therefore the stated formula for \( \alpha \) holds.

Now assume that \( \alpha \) is given by the formula in the statement of this Proposition. If \( \alpha = b(1 + \gamma^n) \) and \( N_{L/F}(\gamma)^n = 1 \), then

\[
\frac{\sigma(\alpha)}{\alpha} = \frac{b(1 + \sigma(\gamma)^n)}{b(1 + \gamma^n)} = \sigma(\gamma)^n.
\]

Thus \( \sigma(\alpha) = \alpha \beta^n \), where \( \beta = \sigma(\gamma) \). If \( \alpha = b\sqrt{a} \) and \( -1 = \beta^n \in L^n \), then \( \sigma(\alpha)/\alpha = -1 = \beta^n \). If \( \alpha = b \), then \( \sigma(\alpha) = \alpha \cdot 1^n \).

If \( t \equiv -1 \mod n \) and \( \sigma(\alpha) = \alpha^{-1} \beta^n \), then \( N_{L/F}(\alpha) = \alpha \sigma(\alpha) = \beta^n \in F \cap L^n \). Thus to treat the case \( t \equiv -1 \mod n \), we shall first study \( F \cap L^n \) in Propositions 4.3-4.5. There does not seem to be a good description of \( F \cap L^n \) when \( L = F(\sqrt{-1}) \) and \( n = 2^e, e \geq 3 \), but the result in Proposition 4.5 is sufficient for our purposes.

**Proposition 4.3.** If \( n \) is odd, then \( F \cap L^n = F^n \).

**Proof.** It is clear that \( F^n \subseteq F \cap L^n \). Now let \( \lambda \in L \) and suppose \( \lambda^n = b \in F \). Then \( b^2 = N_{L/F}(b) = N_{L/F}(\lambda)^n \in F^n \). Since \( b^2 \) and \( b^n \) lie in \( F^n \), it follows that \( b \in F^n \). Thus \( F \cap L^n \subseteq F^n \).

**Proposition 4.4.** Assume \( n \) is even and let \( n = 2^e m, m \) odd, \( e \geq 1 \). If \( a \notin -F^2 \) (i.e., \( L \neq F(\sqrt{-1}) \)), then \( F \cap L^n = F^n \cup a^n/2 F^n \).
Recall that if $s$ and $t$ are any two relatively prime integers and $A$ is any abelian group (written additively) then $sA \cap tA = stA$. Consequently, if $E$ is any field, then $E^s \cap E^t = E^{st}$ (by taking $A = E^*$).

It is clear that $F^n \cup a^{n/2}F^n \subseteq F \cap L^n$ since $a^{n/2} = (\sqrt[n]{a})^n$. To prove the other inclusion take any nonzero $b \in F \cap L^n = F \cap L^{2e} \cap L^m = (F \cap L^{2e}) \cap F^m$ (by Proposition 4.3). Then, $b = \beta^{2e} = \sigma(\beta)^{2e}$ for some $\beta \in L^\ast$. Let $\omega = \sigma(\beta)/\beta$. So, $\omega^2 = 1$ and $1 = N_{L/F}(\omega) = \omega \sigma(\omega)$. So, $\sigma(\omega) = \omega^{-1}$. If $\omega = 1$, then $\beta \in F$, so $b \in F^{2e} \cap F^m = F^m$. If $\omega = -1$ then $\sigma(\beta) = -\beta$, so $\beta = c\sqrt{a}$ for some $c \in F$. Then, $b = \beta^{2e} \in a^{2e-1}F^{2e} = a^{2e-1}F^2$, so $b \in F^m \cap a^{2e-1}mF^{2e} = a^{2e-1}m(F^m \cap F^{2e}) = a^{n/2}F^n$. If $\omega \neq \pm 1$, then $\omega^k = \sqrt{-1}$ for some integer $k$, so $\sigma(\sqrt{-1}) = (\sqrt{-1})^{-1} = -\sqrt{-1}$. But then, $\sqrt{-1} = d\sqrt{a}$ for some $d \in F^\ast$, yielding $-a = d^{-2} \in F^2$, contrary to our hypothesis. Thus, in every case that can occur, $b \in F^n \cup a^{n/2}F^n$, as desired.

**Proposition 4.5.** Let $L = F(\sqrt{-1})$ and assume $\zeta \in L$ is a primitive $(2e)^{th}$ root of unity, $e \geq 2$. Then $F \cap L^{2e-1} = F^{2e-1} \cup -F^{2e-1}$.

**Proof.** The proof is by induction on $e$. The case $e = 2$ is well-known to be true. Now assume $e \geq 3$. We have $F^{2e-1} \cup -F^{2e-1} \subseteq F \cap L^{2e-1}$, since $-1 = \zeta^{2e-1}$. Suppose $\lambda^{2e-1} \in F$, with $\lambda \in L$. Then $(\lambda^{2e-2})^2 \in F$ and this implies $\lambda^{2e-2} \in F \cup \sqrt{-1}F = F \cup \zeta^{2e-2}F$.

First suppose $\lambda^{2e-2} \in F$. Then $\lambda^{2e-2} \in F \cap L^{2e-2} = F^{2e-2} \cup -F^{2e-2}$, by induction. Thus $\lambda^{2e-2} = \pm b^{2e-2}$, $b \in F$, and this implies $\lambda^{2e-1} = b^{2e-1} \in F^{2e-1}$.

On the other hand, if $\lambda^{2e-2} \in \zeta^{2e-2}F$, then $(\lambda/\zeta)^{2e-2} \in F$. The argument in the first part implies $(\lambda/\zeta)^{2e-2} \in F^{2e-1}$. Thus $\lambda^{2e-1} \in -F^{2e-1}$, since $-1 = \zeta^{2e-1}$. □

**Remark 4.6.** Under the hypotheses of Proposition 4.5, there does not seem to be a simple description of $F \cap L^2$. As already noted, for $e = 1$ we have $F \cap L^2 = F^2 \cup -F^2$. For $e = 2$ it is easy to show $F \cap L^4 = F^4 \cup -4F^4$. For $e \geq 3$, the descriptions become more awkward.

The next proposition characterizes the condition $N_{L/F}(\alpha) \in F^n$ and $N_{L/F}(\alpha) \in a^{n/2}F^n$ when $n$ is even. In light of Propositions 4.3 and 4.4, this covers the case $t \equiv -1 \mod n$, except when $L = F(\sqrt{-1})$ and $n$ is even.

**Proposition 4.7.** Let $\alpha \in L$, $\alpha \neq 0$.

1. $N_{L/F}(\alpha) \in F^n$ if and only if there exist $b \in F$ and $\beta, \gamma \in L$ such that

   \[
   \alpha = \begin{cases} 
   b^{n/2}N_{L/F}(\gamma)/\gamma^2, & \text{if } n \text{ is even } (e_0 \geq 1), \\
   N_{L/F}(\beta)^{(n-1)/2}/\beta, & \text{if } n \text{ is odd } (e_0 = 0). 
   \end{cases}
   \]
(2) $N_{L/F}(\alpha) \in a^{n/2}F^n$ if and only if there exist $b \in F$ and $\gamma, \delta \in L$ such that
\[
\alpha = \begin{cases} 
(b\sqrt{a})^{n/2}N_{L/F}(\gamma)/\gamma^2, & \text{if } n \equiv 0 \mod 4 \ (e_0 \geq 2), \\
(b\sqrt{a})^{n/2}\delta, & \text{if } n \equiv 2 \mod 4 \ (e_0 = 1).
\end{cases}
\]

Proof. The proofs of each of the cases are very similar and straight-forward. We will give one of the proofs. Suppose $N_{L/F}(\alpha) = a^{n/2}b^n$, with $n \equiv 2 \mod 4$. Let $\delta = \alpha/(b\sqrt{a})^{n/2}$. So, $\alpha = (b\sqrt{a})^{n/2}\delta$ and
\[
N_{L/F}(\delta) = N_{L/F}(\alpha/(b\sqrt{a})^{n/2}) = a^{n/2}b^n/(b^n(-a)^{n/2}) = (-1)^{n/2} = -1.
\]
The converse is easy as are the other cases. Note that for (1), if $N_{L/F}(\alpha) = b^n \in F^n$, then when $n$ is even we can (by Hilbert 90) choose $\gamma$ so that $ab^{-n/2} = \sigma(\gamma)/\gamma$; when $n$ is odd, choose $\beta = ab^{-(n-1)/2}$. For (2), if $N_{L/F}(\alpha) = a^{n/2}b^n$ with $n \equiv 0 \mod 4$, then choose $\gamma$ so that $aa^{-n/4}b^{-n/2} = \sigma(\gamma)/\gamma$. □

Now we assume $n = 2^e$, with $e \geq 3$. If $t^2 \equiv 1 \mod 2^e$, then
\[
t \in \{ \pm 1, 2^{e-1} \pm 1 \} \mod 2^e.
\]
The case $t \equiv 1 \mod 2^e$ is covered in Proposition 4.2 and the case $t \equiv -1 \mod 2^e$ is covered in Propositions 4.3-4.7, with a small gap in the case $L = F(\sqrt{-1})$. These cases do not depend on $r$, where $\sigma(\zeta) = \zeta^r$. Since $r^2 \equiv 1 \mod n$, in general, we have $r \in \{ \pm 1, 2^{e-1} \pm 1 \} \mod 2^e$ when $n = 2^e, e \geq 3$. The next two results characterize the value of $r$ when $t \equiv 2^{e-1} \pm 1 \mod 2^e, e \geq 3$.

**Proposition 4.8.** $L \neq F(\sqrt{-1})$ (i.e., $\sqrt{-1} \in F$) if and only if $r \equiv 1 \mod 2^{e-1}$. When this occurs, $\zeta^2 \in F$; furthermore, $\zeta \in F$ if and only if $r \equiv 1 \mod 2^e$.

Proof. Recall that $r \equiv \pm 1 \mod 2^{e-1}$. If $r \equiv -1 \mod 2^{e-1}$, then $\sigma(\zeta^2) = (\zeta^2)^{-1}$, so $\sigma(\sqrt{-1}) = (\sqrt{-1})^{-1} = -\sqrt{-1}$, as $\sqrt{-1} = \pm (\zeta^2)^{2^{e-3}}$. Hence, $\sqrt{-1} \notin F$, so $L = F(\sqrt{-1})$. On the other hand, if $r \equiv 1 \mod 2^{e-1}$ then $\sigma(\zeta^2) = \zeta^2$, so $\zeta^2 \in F$. Then, $\sqrt{-1} = \pm (\zeta^2)^{2^{e-3}} \in F$, so $L \neq F(\sqrt{-1})$. Clearly, $\zeta \in F$ if and only if $\zeta = \sigma(\zeta) = \zeta^r$, if and only if $r \equiv 1 \mod 2^e$. □

**Proposition 4.9.** Assume $L = F(\sqrt{-1})$. Then $r \equiv -1 \mod 2^{e-1}$ and the following statements hold:

1. The following statements are equivalent:
   a) $r \equiv -1 \mod 2^e$.
   b) $N_{L/F}(\zeta) = 1$.
   c) $\zeta \in F \cdot L^2$.

2. The following statements are equivalent:
   a) $r \equiv 2^{e-1} - 1 \mod 2^e$. 

Assume Proposition 4.10.

\[ \alpha N \to \gamma, \eta \] there exist \( 310 \) Y.-S. HWANG, D.B. LEEP, AND A.R. WADSWORTH

\[ \text{Proof.} \] Since \( r \equiv 1 \mod 2^e \), then \( \sigma(\zeta) = \zeta^{-1} \). This is equivalent to \( N_{L/F}(\zeta) = 1 \) and hence \( \zeta \in F \cdot L^2 \).

\[ \text{If } r \equiv 2^e - 1 \mod 2^e, \text{ then } \sigma(\zeta) = \zeta^{2^e - 1} = \zeta^{-1} \text{ and this is equivalent } \]
\[ \text{to } N_{L/F}(\zeta) = -1. \text{ Since } -1 \notin F^2, \text{ it follows that } \zeta \notin F \cdot L^2. \]

**Proposition 4.10.** Assume \( t \equiv 2^{e-1} + 1 \mod 2^e, e \geq 3, \text{ and let } \alpha \in L, \alpha \neq 0. \) Suppose \( \sigma(\zeta) = \zeta^t \). Then \( \sigma(\alpha) = \alpha^{2^e - 1 + \beta^{2^e}}, \beta \in L, \text{ if and only if } \) there exist \( \gamma, \eta \in L^* \text{ with } \eta^2 \in F \text{ such that } \)
\[ \alpha = \varphi N_{L/F}(\gamma)\eta^2 / \gamma^{2^e - 1}, \]
where
\[ \varphi = \begin{cases} 
1, & \text{if } r \equiv 1, -1, \text{ or } 2^e - 1 \mod 2^e, \\
1 \text{ or } \zeta, & \text{if } r \equiv 2^e - 1 + 1 \mod 2^e.
\end{cases} \]

**Proof.** Let \( k = 2^{e-1} \). Let
\[ A = \{ \alpha \in L^* \mid \sigma(\alpha) = \alpha^{k+1} \beta^{2k} \text{ for some } \beta \in L \} \]
and let
\[ B = \{ \alpha \in L^* \mid \alpha = N_{L/F}(\gamma)\eta^2 / \gamma^{2^e - 1} \text{ where } \gamma, \eta \in L^* \text{ and } \eta^2 \in F \} \].

Clearly \( A \) and \( B \) are groups.

Let \( \alpha = \varphi N_{L/F}(\gamma)\eta^2 / \gamma^{2^e - 1} \) and let \( \beta = \gamma^{2^e - 1} / \sigma(\gamma)\eta \). Then \( \alpha \beta^2 = \varphi \gamma / \sigma(\gamma) \). We have \( \sigma(\varphi) = \varphi^{2^e - 1 + 1} \) in all cases since the case \( r \equiv 2^e - 1 + 1 \mod 2^e \) and \( \varphi = \zeta \) implies \( \sigma(\zeta) = \zeta^r = \zeta^{2^e - 1 + 1} \). It now follows that
\[ (1) \quad \sigma(\alpha)/\alpha = (\gamma/\sigma(\gamma))^{k}(\varphi/\varphi) = ((\alpha \beta^2)^k / \varphi^k) \sigma(\varphi/\varphi) = (\alpha \beta^2)^k. \]

This implies that \( \alpha \in A \). The case \( \varphi = 1 \) shows that \( B \subseteq A \). If \( r \equiv k + 1 \mod 2k \), then \( \zeta \in A \), so \( B \cup \zeta B \subseteq A \). We must show that \( A = B \cup \zeta B \) if \( r \equiv k + 1 \mod 2k \) and \( A = B \) otherwise.

Take any \( \alpha \in A \); so \( \sigma(\alpha) = \alpha^{k+1} \beta^{2k}, \text{ i.e., } \sigma(\alpha)/\alpha = (\alpha \beta^2)^k \). Let \( \omega = N_{L/F}(\alpha \beta^2) \). Then, \( \omega^k = N_{L/F}(\sigma(\alpha)/\alpha) = 1 \), so, since \( \omega \) is a power of \( \zeta^2 \), we have \( \omega \in L^2 \cap F = F^2 \cup aF^2 \).

Let \( \epsilon = \omega^{k/2} \). Then \( \epsilon^2 = \omega^k = 1 \) and thus \( \epsilon = \pm 1 \). In either case, \( \epsilon = \omega^{k/2} \in F^2 \), since \( \omega \in F \) and \( k/2 \) is even. Let \( \delta = \alpha \beta^2 \). Then,
\[ \sigma(\alpha \delta^{k/2}) = \alpha \delta^{k} \sigma(\delta^{k/2}) = \alpha \delta^{k/2} (N_{L/F}(\delta))^{k/2} = \alpha \delta^{k/2} \omega^{k/2} = \alpha \delta^{k/2} \epsilon. \]

If \( \epsilon = 1 \), then \( \alpha \delta^{k/2} \in F \). From this we conclude \( N_{L/F}(\alpha \delta^{k/2}) \in F^2, \)
\[ N_{L/F}(\alpha) \in F^2, \quad N_{L/F}(\alpha \beta^2) \in F^2, \text{ and finally } \omega \in F^2. \]
If \( \epsilon = -1 \), then \( \alpha \delta^{k/2} \in \sqrt{aF} \). From this we conclude \( N_{L/F}(\alpha \delta^{k/2}) \in -aF^2 \), \( N_{L/F}(\alpha) \in -aF^2 \), \( N_{L/F}(\alpha \beta^2) \in -aF^2 \), and finally \( \omega \in -aF^2 = aF^2 \), since \( -1 = \epsilon \in F^2 \).

**Case 1.** Assume \( \epsilon = 1 \). Then \( N_{L/F}(\alpha \beta^2) = \omega \in F^2 \). Therefore \( \alpha \beta^2 = b\gamma^2 \) for some \( b \in F^* \), \( \gamma \in L^* \). Since, \( b^2 N_{L/F}(\gamma)^2 = N_{L/F}(b\gamma^2) = N_{L/F}(\alpha \beta^2) = \omega \), we have \( b^k N_{L/F}(\gamma)^k = \omega^{k/2} = \epsilon = 1 \). This gives

\[
\sigma(\alpha)/\alpha = (\alpha \beta^2)^k = (b\gamma^2)^k = \gamma^{2k}/N_{L/F}(\gamma)^k = \gamma^k/\sigma(\gamma)^k.
\]

Thus, \( \sigma(\alpha^{\gamma^k}) = \alpha^{\gamma^k} \) and we have \( \alpha^{\gamma^k} = d \in F \).

Since \( (\alpha \beta^2)^k = (\gamma/\sigma(\gamma))^k \), we have \( \alpha \beta^2 = \omega^j \gamma/\sigma(\gamma) = \omega^j c/\sigma(\gamma)^2 \), where \( (\omega^j)^k = 1 \) and \( c = N_{L/F}(\gamma) \). Note that \( \alpha/d = \gamma^{-k} \in L^2 \) and \( \alpha/c = \omega^j/(\sigma(\gamma)^2 \beta^2) \in L^2 \); so \( d/c \in L^2 \cap F \). Let \( \eta^2 = d/c \). Then, \( \alpha = c\eta^2/\gamma^k \in B \).

**Case 2.** Now assume \( \epsilon = -1 \). Then \( \omega \in aF^2 \) and \(-1 \in F^2 \). This implies \( r \equiv 1 \mod k \) (see Proposition 4.8). Since \( \omega \notin F^2 \), it follows \( \zeta \notin F \) and \( r \equiv 1 \mod 2k \). Hence, \( r \equiv k + 1 \mod 2k \). Because \( L = F(\zeta) \) and \( \zeta^2 \in F \) (see Proposition 4.8), we can take \( a = \zeta^2 \). The congruence condition on \( r \) says that \( \sigma(\zeta) = \zeta^{1+k} \), showing that \( \zeta \in A \). Since \( \sigma(\alpha)/\alpha = (\alpha \beta^2)^k \) and \( \sigma(\zeta)/\zeta = \zeta^k \), we have \( \sigma(\alpha/\zeta)/(\alpha/\zeta) = ((\alpha/\zeta) \beta^2)^k \). Also, \( N_{L/F}(\alpha \zeta \beta^2) = \omega(-\zeta^{-2}) = a^{-1} \omega \in F^2 \). This shows that \( \alpha/\zeta \in A \), and that Case 1 above applies to \( \alpha/\zeta \). Hence, \( \alpha/\zeta \in B \), so \( \alpha \in \zeta B \). Since Case 2 occurs for \( \alpha \) only when \( r \equiv k + 1 \mod 2k \), the proof is complete.

**Proposition 4.11.** Assume \( t \equiv 2e^{-1} - 1 \mod 2^e \), \( e \geq 3 \), and let \( \alpha \in L \), \( \alpha \neq 0 \). Let \( \sigma(\zeta) = \zeta^r \). Then \( \sigma(\alpha) = \alpha^{2e^{-1} - 1} \beta^{2^e} \), \( \beta \in L \), if and only if there exist \( c \in F \), \( \gamma \in L \), with \( N_{L/F}(\gamma) = \pm c \), such that \( \alpha = \theta c^{2e^{-2}+1}/\gamma^2 \) where

\[
\theta = \begin{cases} 1, & \text{if } L \neq F(\sqrt{-1}), \\ 1, & \text{if } L = F(\sqrt{-1}), r \equiv -1 \mod 2^e, \\ 1 \text{ or } \zeta, & \text{if } L = F(\sqrt{-1}), r \equiv 2e^{-1} - 1 \mod 2^e. \end{cases}
\]

**Proof.** First assume \( \alpha = \theta c^{2e^{-2}+1}/\gamma^2 \) where \( N_{L/F}(\gamma) = \pm c \) and \( \theta = 1 \) or \( \zeta \), as above. We see that \( \theta^{2e^{-1}} = N_{L/F}(\theta) \) in all three cases since \( \zeta^{2e^{-1}} = \zeta^{2e^{-1}-1} = N_{L/F}(\zeta) \) in the third case. Let \( \beta = \gamma/c^{2e^{-3}} \). Then \( \alpha \beta^2 = \theta^2 c \) and

\[
N_{L/F}(\alpha) = N_{L/F}(\theta)c^{2e^{-1}+2}/c^2 = N_{L/F}(\theta)c^{2e^{-1}} = \theta^{2e^{-1}}c^{2e^{-1}} = (\alpha \beta^2)^{2e^{-1}}.
\]

Thus \( \sigma(\alpha) = \alpha^{2e^{-1}-1} \beta^{2^e} \).

Now assume \( \sigma(\alpha) = \alpha^{2e^{-1}-1} \beta^{2^e} \), \( \beta \in L \). Then \( N_{L/F}(\alpha) = (\alpha \beta^2)^{2e^{-1}} \). Since

\( (\alpha \beta^2)^{2e^{-1}} \in F \cap L^{2e^{-1}} = \begin{cases} F^{2e^{-1}} \cup a^{2e^{-2}} F^{2e^{-1}} , & \text{if } L \neq F(\sqrt{-1}), \\ F^{2e^{-1}} \cup -F^{2e^{-1}} , & \text{if } L = F(\sqrt{-1}), \end{cases} \)
by Propositions 4.4 and 4.5, there exists \( c \in F \) such that \( \alpha \beta^2 \in \{ \omega, \sqrt{\alpha} \omega, \zeta \omega \} \) where \( \omega^{2e-1} = 1 \). Since \( \omega \in L^2 \), by replacing \( \beta \) by \( \beta \omega^{-1/2} \) we can assume
\[
\alpha \beta^2 = \begin{cases} 
  c \text{ or } \sqrt{\alpha} c, & \text{if } L \neq F(\sqrt{-1}), \\
  c \text{ or } \zeta c, & \text{if } L = F(\sqrt{-1}), 
\end{cases}
\]
without affecting the equation \( \sigma(\alpha) = \alpha^{2e-1} \beta^{2e} \).

If \( L \neq F(\sqrt{-1}) \), then \(-1 \in F^2 \) (since \(-1 \in L^2 \)) and
\[
N_{L/F}(\alpha) \in F^{2e-1} \cup a^{2e-2} F^{2e-1} \subseteq F^2,
\]
since \( e \geq 3 \). If \( \alpha \beta^2 = \sqrt{ac} \), then \( N_{L/F}(\alpha) = -aF^2 = aF^2 \neq F^2 \), a contradiction. Thus \( \alpha \beta^2 = c \).

If \( L = F(\sqrt{-1}) \) and \( \alpha \beta^2 = \zeta c \), then
\[
N_{L/F}(\alpha) = (\alpha \beta^2)^{2e-1} = (\zeta c)^{2e-1} = -c^{2e-1} \in -F^2 \neq F^2.
\]
Then the equation \( \alpha \beta^2 = \zeta c \) implies \( N_{L/F}(\zeta) \notin F^2 \), and thus \( N_{L/F}(\zeta) = -1 \) and \( r \equiv 2e-1 \mod 2e \) by Proposition 4.9.

We conclude \( \alpha \beta^2 = \theta c \), where
\[
\theta = \begin{cases} 
  1, & \text{if } L \neq F(\sqrt{-1}), \\
  1, & \text{if } L = F(\sqrt{-1}), r \equiv -1 \mod 2e, \\
  1 \text{ or } \zeta, & \text{if } L = F(\sqrt{-1}), r \equiv 2e-1 \mod 2e. 
\end{cases}
\]

Let \( \gamma = c^{2e-3} \beta \). Then
\[
\alpha = \theta c/\beta^2 = \theta c^{2e-2+1}/(c^{2e-2} \beta^2) = \theta c^{2e-2+1}/\gamma^2.
\]
Since \( N_{L/F}(\alpha) = (\alpha \beta^2)^{2e-1} = \theta^{2e-1} c^{2e-1} \) and \( \theta^{2e-1} = N_{L/F}(\theta) \) in all cases, we have
\[
N_{L/F}(\gamma^2) = c^{2e-1} N_{L/F}(\beta^2) = c^{2e-1} N_{L/F}(\theta c/\alpha) = \frac{c^{2e-1} \theta^{2e-1} N_{L/F}(c)}{N_{L/F}(\alpha)} = c^2.
\]
Thus \( N_{L/F}(\gamma) = \pm c \). \( \square \)

5. Explicit constructions of Galois extensions \( M/F \).

Proposition 3.1 and Lemma 3.2 let us reduce the problem of describing explicit constructions of Galois extensions \( M/F \) as in Section 3 to the case \( n = p^e \), where \( p \) is a prime number. In this section, we treat the case when \( p \) is an odd prime. The case \( p = 2 \) will be handled in Section 6. Recall that \( r \mod n \) is defined by \( \sigma(\zeta) = \zeta^r \), where \( \zeta \) is a primitive \( n^{th} \) root of unity. Since \( j^2 \equiv r^2 \equiv 1 \mod p^e \), it follows that if \( p \) is odd, then \( j \equiv \pm 1 \mod p^e \) and \( r \equiv \pm 1 \mod p^e \). Since it is no extra trouble, instead of considering only the case \( n = p^e \) with \( p \) odd, we will consider the more general case where
n is arbitrary and \( j \equiv \pm 1 \mod n \) and \( r \equiv \pm 1 \mod n \). Of course the case \( r \equiv 1 \mod n \) occurs if and only if \( \zeta \in F \). Recall from Proposition 2.5 that when \( j \equiv 1 \mod n \), either \( G \cong \mathbb{Z}/2n\mathbb{Z} \) or \( G \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), and when \( j \equiv -1 \mod n \), either \( G \cong D_n \) or \( G \cong Q_{2n} \).

We saw in Theorem 3.4(3) that when \( M = L(\alpha^{1/n}) \) realizes \((G, j, l)\), then \( \sigma(\alpha) = \alpha^t\beta^n \), where \( t = jr \mod n \). So, \( t \equiv \pm 1 \mod n \). By Theorem 3.4(1), we can assume that \( t = \pm 1 \). To be able to handle the two possible values of \( t \) at the same time, and to bring out the similarities in the two cases, we consider a modified group action. It will be convenient to use the language of group cohomology, though everything in this section can be derived easily without mentioning cohomology.

Let \( C = \text{Gal}(L/F) = \{1, \sigma\} \). Let \( t = \pm 1 \). For any multiplicative group \( Q \) on which \( C \) acts, we have a “twisted” \( t \)-action of \( C \) on \( Q \) defined by

\[
\sigma \ast q = (\sigma \cdot q)^t.
\]

(Here \( \cdot \) denotes the original action and \( \ast \) denotes the \( t \)-action.) Of course, when \( t = 1 \) the \( t \)-action coincides with the original action. Let \( \mu_n = \langle \zeta \rangle \) denote the group of \( n \)th roots of unity in \( L \). The short exact sequences

\[
1 \rightarrow L'^n \rightarrow L^s \rightarrow L'^s/L'^n \rightarrow 1 \quad \text{and} \quad 1 \rightarrow \mu_n \rightarrow L^s \rightarrow L'^n \rightarrow 1
\]

are compatible with the usual Galois action of \( \text{Gal}(L/F) \), but also with the \( t \)-action. They lead to connecting homomorphisms in cohomology (using the \( t \)-action):

\[
f: H^0(C, L^s/L'^n) \rightarrow H^1(C, L'^n) \quad \text{and} \quad g: H^1(C, L'^n) \rightarrow H^2(C, \mu_n).
\]

We describe the maps \( f \) and \( g \): First, \( H^0(C, L^s/L'^n) \) consists of the elements \( [\alpha] = \alpha L'^n \subseteq L^s/L'^n \) stable under the \( t \)-action of \( C \), i.e., those \([\alpha]\) such that \( \sigma \ast [\alpha] = [\alpha] \), i.e.,

\[
\sigma \ast \alpha = \alpha^t\gamma^n \quad \text{for} \quad \gamma \in L^s, \quad \text{i.e.,} \quad \sigma(\alpha) = \alpha^t\beta^n, \quad \text{where} \quad \beta = \gamma^t.
\]

The connecting map \( f \) takes the class of the 0-cocycle \([\alpha]\) to the class of the 1-cocycle \( c_{\gamma^n}: C \rightarrow L'^n \) mapping \( 1 \mapsto 1 \) and \( \sigma \mapsto \gamma^n \). Let \( N_t \) denote the “\( t \)-norm,” given by

\[
N_t(x) = x^t \sigma \ast x = x^t \sigma(x). \tag{1}
\]

Note that by applying \( N_t \) to the equation \( \sigma \ast \alpha = \alpha^t\gamma^n \) we find that \( N_t(\gamma^n) = 1 \). Let

\[
\omega = N_t(\gamma) = \gamma \sigma(\gamma)^t = \beta^t \sigma(\beta) \in \mu_n.
\]

The map \( g \) takes the class of \( c_{\gamma^n} \) to the class of the 2-cocycle \( h_\omega: C \times C \rightarrow \mu_n \) given by \( h_\omega(\sigma, \tau) = N_t(\gamma) = \omega \) and \( h_\omega(1, 1) = h_\omega(\sigma, 1) = h_\omega(1, \sigma) = 1 \). Thus, \( g \circ f[\alpha] = [h_\omega] \in H^2(C, \mu_n) \).

Now, the \( t \)-action of \( C \) on \( \mu_n \) is determined by \( \sigma \ast \zeta = \sigma(\zeta)^t = \zeta^t = \zeta^j \), where \( j = rt \). The group extension of \( C \) by \( \mu_n \) determined by the 2-cocycle \( h_\omega \) is the group \( \mathcal{G} = \mu_n x_1 \cup \mu_n x_\sigma \), with the multiplication given by (cf. [R], p. 154) \( (\xi^i x_\rho)(\xi^k x_\psi) = \xi^i(\rho \ast \xi^k)h_\omega(\rho, \psi)x_{\rho\psi} \). If \( \omega = \xi^i \), then
\[ \mathfrak{G} \] is the group of order \( 2n \) generated by \( \zeta, \, x_\sigma \) with the relations \( \zeta^n = 1, \quad x_\sigma x_{\sigma^{-1}} = \sigma * \zeta = \zeta^j, \) and \( x_\sigma^2 = \zeta^j. \) That is, \( \mathfrak{G} \cong (G, j, l). \) Observe also that for this \( j \) and \( l, \) we have \( (G, j, l) \cong \text{Gal}(L(a^{1/n})/F), \) by Theorem 3.4(3) (assuming \( [L(a^{1/n}) : L] = n. \)) Now, \( \mathfrak{G} \) is the trivial group extension (i.e., a semidirect product, i.e., \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \)) when \( j \equiv 1 \mod n \) and \( D_n \) when \( j \equiv -1 \mod n) \) just when \( [h, \omega] = 0 \in H^2(C, \mu_n). \) This occurs just when \( \omega \) is the \( t \)-norm of an element of \( \mu_n \) (cf. \( [R] \), Th. 10.35, p. 297), i.e., just when \( \omega = \zeta^j \in \langle N_l(\xi) \rangle = \langle \zeta^{j+1} \rangle. \) Note in any case that since \( \omega = N_l(\gamma), \omega = \sigma * \omega = \omega^j. \) When \( j \equiv -1 \mod n \) this says that \( \omega = \pm 1, \) and \( \mathfrak{G} \) is the trivial extension just when \( \omega = 1. \) When \( j \equiv 1 \mod n, \) \( \mathfrak{G} \) is the trivial extension just when \( \omega \in \langle \zeta^j \rangle. \) When \( n \) is odd, we have \( H^2(C, \mu_n) = 0 \) as \( \gcd(|C|, |\mu_n|) = 1, \) so then \( \mathfrak{G} \) is always the trivial extension.

When \( t = 1 \) we can say a little more. Then, the \( t \)-action is the usual \( C \)-action. Since \( H^1(C, L^*) = 0 \) (Hilbert 90), the exact sequence \( H^1(C, L^*) \rightarrow H^1(C, L^{*n}) \Rightarrow H^2(C, \mu_n) \) shows that the map \( g \) is injective. But, we also have the exact sequence \( H^0(C, L^*) \rightarrow H^1(C, L^{*}/L^{*n}) \Rightarrow H^1(C, L^{*n}). \) Thus, \( [\alpha] \in H^0(C, L^{*}/L^{*n}) \) determines the trivial group extension \( \Leftrightarrow g \circ f[\alpha] = 0 \) in \( H^2(C, \mu_n) \Leftrightarrow f[\alpha] = 0 \Leftrightarrow [\alpha] \in \text{im}(H^0(C, L^*) \rightarrow H^0(C, L^{*}/L^{*n})) = F^*L^{*n}/L^n \Leftrightarrow \alpha \in F^*L^{*n}. \) When \( n \) is odd, this always holds because then \( H^2(C, \mu_n) = 0. \)

The following propositions summarize what the preceding discussion has shown.

**Proposition 5.1.** Assume that \( M/F \) is a Galois extension that realizes \( (G, j, l). \) Thus \( \sigma(\alpha) = \alpha^j \beta^n, \) with \( \alpha, \beta \in L. \) Assume \( j \equiv 1 \mod n \) and \( r \equiv \pm 1 \mod n. \) Then, \( t \equiv r \mod n. \) Assume \( t = \pm 1 \) (and adjust \( \beta \) accordingly). Then, \( \beta^r \sigma(\beta) \) is an \( n^{th} \) root of unity. Furthermore:

1. The following statements are equivalent:
   (a) \( \text{Gal}(M/F) \cong \mathbb{Z}/2n\mathbb{Z}. \)
   (b) The order of \( \beta^r \sigma(\beta) \) is divisible by \( 2^e_0. \)
   (c) \( \beta^r \sigma(\beta) \in \zeta \langle \zeta^2 \rangle. \)
   (d) \( n \) is odd or \( l \) is even.
   If \( n \) is odd, then (a)-(d) always hold. If \( n \) is even, then (a)-(d) are equivalent to the following statement:
   (e) \( (\beta^r \sigma(\beta))^{n/2} = -1. \)

2. The following statements are equivalent:
   (a) \( \text{Gal}(M/F) \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \)
   (b) If \( n \) is even, then the order of \( \beta^r \sigma(\beta) \) is not divisible by \( 2^e_0. \)
   (c) \( \beta^r \sigma(\beta) \in \langle \zeta^2 \rangle. \)
   (d) \( n \) is odd or \( l \) is even.
   If \( r \equiv 1 \mod n \) (i.e., \( \zeta \in F), \) so \( t = 1, \) then (a)-(d) are equivalent to the following statement:
   (e) \( \alpha \in F \cdot L^n. \)
   If \( n \) is odd, then (a)-(e) always hold.
Proposition 5.2. Assume $M/F$ is a Galois extension that realizes $(G, j, l)$ with $j \equiv -1 \mod n$ and $r \equiv 1 \mod n$. Then, $t \equiv -1 \mod n$, and we assume $t = -1$. Suppose $\sigma(\alpha) = \alpha^t \beta^n$ with $\beta \in L$.

(1) The following statements are equivalent:
(a) $\text{Gal}(M/F) \cong D_n$.
(b) $l \equiv 0 \mod n$.
(c) $N_{L/F}(\alpha) \in F^n$.
(d) $\beta \in F$.

If $n$ is odd, then (a)-(d) always hold.

(2) Assume $n$ is even (and hence $\text{char } F \neq 2$). Let $L = F(\sqrt{a})$. Then $\beta \in F \cup \sqrt{a}F$ and $N_{L/F}(\alpha) \in F^n \cup a^{n/2}F^n$. In addition, the following statements are equivalent:
(a) $\text{Gal}(M/F) \cong Q_{2n}$.
(b) $l \equiv n/2 \mod n$.
(c) $N_{L/F}(\alpha) \in a^{n/2}F^n$.
(d) $\beta \in \sqrt{a}F$.

Proof. In addition to the observations preceding Proposition 5.1, note the following: Because $t = -1$, we have $\sigma(\alpha) = \alpha^{-1} \beta^n$, so $N_{L/F}(\alpha) = \beta^n$. Since $j \equiv -1 \mod n$, $\beta/\sigma(\beta) = N_L(\beta) \in \{\pm 1\} \cap \mu_n$. So $\sigma(\beta) = \pm \beta$. The Galois group is $D_n$ just when $\sigma(\beta) = \beta$, i.e., $\beta \in F$; then $N_{L/F}(\alpha) = \beta^n \in F^n$. We have $\text{Gal}(M/F) \cong Q_{2n}$ just when $\sigma(\beta) = -\beta$, i.e., $\beta \in \sqrt{a}F$; then $n$ is necessarily even since $-1 \in \mu_n$, and $N_{L/F}(\alpha) \in a^{n/2}F^n \neq F^n$. \[\square\]

Proposition 5.3. Assume $M/F$ is a Galois extension that realizes $(G, j, l)$ with $j \equiv -1 \mod n$ and $r \equiv -1 \mod n$. Then, we may assume $t = 1$. Suppose $\sigma(\alpha) = \alpha^t \beta^n$ with $\beta \in L$.

(1) The following statements are equivalent:
(a) $\text{Gal}(M/F) \cong D_n$.
(b) $l \equiv 0 \mod n$.
(c) $N_{L/F}(\beta) = 1$.
(d) $\alpha \in F \cdot L^n$.

If $n$ is odd, then (a)-(d) always hold.

(2) The following statements are equivalent:
(a) $\text{Gal}(M/F) \cong Q_{2n}$.
(b) $l \equiv n/2 \mod n$.
(c) $N_{L/F}(\beta) = -1$.

6. The case when $n = 2^e$ with $e \geq 3$.

We now study the problem of constructing Galois extensions $M/F$, which were considered in Section 3, when $n = 2^e$ with $e \geq 1$. We have $L = F(\sqrt{a})$, $a \in F$, since $\text{char } F \neq 2$. We continue to assume that $\zeta \in L$ is a primitive $(2^e)^{\text{th}}$ root of unity and that $\sigma(\zeta) = \zeta^r$. We shall assume $e \geq 3$ since the
cases when \( e \leq 2 \) are covered in Propositions 5.1-5.3 when \( j \equiv \pm 1 \mod n \) and \( r \equiv \pm 1 \mod n \). If \( M/F \) is a Galois extension that realizes \((G,j,l)\) with \( n = 2^e \) and \( e \geq 3 \), then by Theorem 3.4(3), \( \sigma(\alpha) = \alpha^l\beta^n \) with \( \beta \in L \), \( t \equiv jr \mod 2^e \) and \( t,j,r \in \{0,1,2^{e-1} + 1,2^{e-1} - 1\} \mod 2^e \). By Theorem 3.4(1), we may assume that \( t \in \{1,-1,2^{e-1} + 1,2^{e-1} - 1\} \). If \( j \equiv 2^{e-1} + 1 \) or \( 2^{e-1} - 1 \mod 2^e \), then the group \( \text{Gal}(M/F) \) is uniquely determined up to isomorphism, by Lemma 2.4. Therefore, we shall focus only on values of \( t \) and \( r \) that give \( j \equiv 1 \) or \(-1 \mod 2^e \). So, if \( t \in \{1,-1\} \), then \( r \equiv 1 \) or \(-1 \mod 2^e \) since \( t \equiv jr \mod 2^e \). These cases have already been discussed in \( \S 5 \). Thus, we can assume in this section that \( t \in \{2^{e-1} + 1,2^{e-1} - 1\} \). The interesting cases are when \( r \equiv 2^{e-1} + 1 \) or \( 2^{e-1} - 1 \mod 2^e \).

**Proposition 6.1.** Suppose \( M = L(\alpha^{1/2^e}) \) is a Galois extension of \( F \) of degree \( 2^{e+1} \) (\( e \geq 3 \)) that realizes \((G,j,l)\) with \( t = 2^{e-1} + 1 \), i.e., \( \sigma(\alpha) = \alpha^{2^{e-1} + 1}\beta^e \) for some \( \beta \in L \). So, \( \alpha = \varphi N_{L/F}(\gamma)\eta^2/\gamma^{2e-1} \), where \( \gamma \in L^* \), \( \eta \in F \cup \sqrt{\alpha}F \) and

\[
\varphi = \begin{cases} 
1, & \text{if } r \equiv 1, -1, \text{ or } 2^{e-1} - 1 \mod 2^e, \\
1 \text{ or } \zeta, & \text{if } r \equiv 2^{e-1} + 1 \mod 2^e.
\end{cases}
\]

(1) Suppose \( r \equiv 2^{e-1} + 1 \mod 2^e \) (so \( j \equiv 1 \mod 2^e \)). Then,

(a) \( \text{Gal}(M/F) \cong \mathbb{Z}/2^{e+1}\mathbb{Z} \) if and only if \( \varphi = \zeta \), if and only if \( N_{L/F}(\alpha) \in aF^2 \).

(b) \( \text{Gal}(M/F) \cong \mathbb{Z}/2^e\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) if and only if \( \varphi = 1 \), if and only if \( N_{L/F}(\alpha) \in F^2 \).

(2) Suppose \( r \equiv 2^{e-1} - 1 \mod 2^e \) (so \( j \equiv -1 \mod 2^e \)). Then,

(a) \( \text{Gal}(M/F) \cong D_{2^e} \) if and only if \( \eta \in F \).

(b) \( \text{Gal}(M/F) \cong Q_{2e+1} \) if and only if \( \eta \in \sqrt{\alpha}F \).

**Proof.** The description of \( \alpha \) is given in Proposition 4.10. We have \( t^2 - 1 = 2^{e-2} + 1 \) and \( \alpha = \varphi N_{L/F}(\gamma)\eta^2/\gamma^{2e-1} \). Equation (1) in the proof of Proposition 4.10 shows that \( \sigma(\alpha)/\alpha = (\alpha(\beta^2))^{2e-1} \), where \( \beta = \gamma^{2e-2}/(\sigma(\gamma)\eta) \). Thus, we may let \( \beta = \beta' \) here. Let \( \rho = \alpha^{(l^2 - 1)/2^e}\beta^l \sigma(\beta) \). By Theorem 3.4(3), \( \rho = \zeta^l \), where \( l \equiv l \mod \gcd(j + 1,2^e) \). Now, \( \alpha\beta^2 = \varphi\gamma/\sigma(\gamma) \), which yields

\[
\rho = (\alpha\beta^2)^{2e-2} \alpha N_{L/F}(\beta) = \varphi^{2e-2+1}\eta/\sigma(\eta).
\]

Note that since \( \eta^2 \in F \), we have \( \eta/\sigma(\eta) = \pm 1 \in (\zeta^2) \). Also, the formula for \( \alpha \) shows that \( N_{L/F}(\alpha) \in N_{L/F}(\varphi)F^2 \).

For (1), suppose \( r \equiv 2^{e-1} + 1 \mod 2^e \). Then, as \( t \equiv jr \mod 2^e \), we have \( j \equiv 1 \mod 2^e \). So, \( \rho = \varphi^{2e-2}+1\eta/\sigma(\eta) \in \varphi(\zeta^2) \). We have \( \text{Gal}(M/F) \cong \mathbb{Z}/2^e\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) just when \( l \) is even, which (since \( \varphi = 1 \) or \( \zeta \)) occurs just when \( \varphi = 1 \). In this case, \( N_{L/F}(\alpha) \in F^2 \). The only other possibility
is that $\text{Gal}(M/F) \cong \mathbb{Z}/2^{e+1}\mathbb{Z}$, which occurs just when $l$ is odd, so just when $\varphi = \zeta$. Since $\sigma(\zeta) = -\zeta$ and $-1 \in F^2$ by Proposition 4.8, we have $\zeta^2 F^2 = a F^2 = -a F^2$. Thus, when $\varphi = \zeta$ we have $N_{L/F}(\alpha) = N_{L/F}(\varphi) F^2 = -\zeta^2 F^2 = a F^2 \neq F^2$.

For (2), suppose $r \equiv 2^{e-1} - 1 \mod 2^e$. Then, $j \equiv -1 \mod 2^e$ and $\varphi = 1$, so $\rho = \eta/\sigma(\eta) = \pm 1$. We have $\text{Gal}(M/F) \cong D_{2^e}$ just when $l \equiv 0 \mod 2^e$, which occurs just when $\rho = 1$; this occurs just when $\sigma(\eta) = \eta$, i.e., $\eta \in F$. The only other possibility is that $\text{Gal}(M/F) \cong Q_{2e+1}$, which occurs just when $l \equiv 2^{e-1} \mod 2^e$. This holds just when $\rho = -1$, i.e., $\sigma(\eta) = -\eta$, i.e., $\eta \in \sqrt{a}F$.

\begin{proposition}
Suppose $M = L(\alpha^{1/2^e})$ is a Galois extension of $F$ of degree $2^{e+1}$ ($e \geq 3$) that realizes $(G,j,l)$ with $t = 2^{e-1} - 1$, i.e., $\sigma(\alpha) = \alpha^{2^{e-1}-1} \beta^{2^e}$. So, $\alpha = \theta c^{2e-2+1}/\gamma^2$ where $\gamma \in L^*$, $N_{L/F}(\gamma) = \pm c$, and
\[
\theta = \begin{cases}
1, & \text{if } r \equiv 1, \ -1, \ or \ 2^{e-1} + 1 \mod 2^e, \\
1 \text{ or } \zeta, & \text{if } r \equiv 2^{e-1} - 1 \mod 2^e.
\end{cases}
\]

(1) Suppose $r \equiv 2^{e-1} - 1 \mod 2^e$ (so $j \equiv 1 \mod 2^e$). Then,
\begin{enumerate}
\item[(a)] $\text{Gal}(M/F) \cong \mathbb{Z}/2^{e+1}\mathbb{Z}$ if and only if $\theta = \zeta$, if and only if $N_{L/F}(\alpha) \in -F^2$.
\item[(b)] $\text{Gal}(M/F) \cong \mathbb{Z}/2^e \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if and only if $\theta = 1$, if and only if $N_{L/F}(\alpha) \in F^2$.
\end{enumerate}

(2) Suppose $r \equiv 2^{e-1} + 1 \mod 2^e$ (so $j \equiv -1 \mod 2^e$). Then,
\begin{enumerate}
\item[(a)] $\text{Gal}(M/F) \cong D_{2^e}$ if and only if $N_{L/F}(\gamma) = c$.
\item[(b)] $\text{Gal}(M/F) \cong Q_{2e+1}$ if and only if $N_{L/F}(\gamma) = -c$.
\end{enumerate}
\end{proposition}

\begin{proof}
The proof is very similar to the proof of Proposition 6.1. The description of $\alpha$ is given in Proposition 4.11. Since $\alpha = \theta c^{2e-2+1}/\gamma^2$, the first paragraph of the proof of Proposition 4.11 shows that we can take $\beta = \gamma/c^{2e-3}$. Let $\rho = \alpha^{(t^2-l_1)/2^e} \beta^l \sigma(\beta) = \zeta^{l_1}$, where $l_1 \equiv l \mod \gcd(j + 1, 2^e)$. Since $(t^2 - 1)/2^e = 2^{e-2} - 1$ and $\alpha \beta^2 = \theta c$, we have
\[
\rho = (\alpha \beta^2)^{2^{e-2}} \sigma(\beta)/(\alpha \beta) = \theta^{2e-2} N_{L/F}(\gamma)/c.
\]

The rest of the proof is left to the reader.
\end{proof}

In Propositions 6.1 and 6.2 it was assumed that $[L(\alpha^{1/2^e}) : L] = 2^e$. The next three results will allow us to identify when this occurs.

\begin{lemma}
If $r \equiv 2^{e-1} \pm 1 \mod 2^e$, $e \geq 3$, and $c \in F^*$, then $\zeta c \notin L^2$.
\end{lemma}

\begin{proof}
First assume that $r \equiv 2^{e-1} + 1 \mod 2^e$. If $\zeta c \in L^2$, then $N_{L/F}(\zeta) \in F^2$, but we saw in the proof of Proposition 6.1 that $N_{L/F}(\zeta) \in a F^2 \neq F^2$. Hence, $\zeta c \notin L^2$.
\end{proof}
Now assume that \( r \equiv 2^{e-1} - 1 \mod 2^e \). Proposition 4.8 implies that \( L = F(\sqrt{-1}) \). Now Proposition 4.9(2) implies that \( \zeta \not\in F \cdot L^2 \) and thus \( \zeta c \not\in L^2 \).

**Corollary 6.4.** Let \( \alpha = \varphi N_{L/F}(\gamma)\gamma^{2e-1} / \gamma^{2e-1} \) as in Propositions 4.10 and 6.1. Then \( [L(\alpha^{1/2^e}) : L] = 2^e \) if and only if \( \alpha \not\in L^2 \), which holds if and only if \( \varphi = \zeta \) or \( N_{L/F}(\gamma) \not\in F \cap L^2 = F^2 \cup aF^2 \).

**Proof.** Since \(-1 \in L^2\), it is standard that \([L(\alpha^{1/2^e}) : L] = 2^e\) if and only if \( \alpha \not\in L^2 \), see, e.g., [L], Theorem 9.1, p. 297. The formula for \( \alpha \) shows that this is equivalent to: \( \varphi N_{L/F}(\gamma) \not\in L^2 \). This holds if \( \varphi = \zeta \) by Lemma 6.3, since then \( r \equiv 2^{e-1} + 1 \mod 2^e \); if \( \varphi = 1 \), this holds just when \( N_{L/F}(\gamma) \not\in L^2 \cap F \).

**Corollary 6.5.** Let \( \alpha = \theta c^{2^e-2} \gamma^2 / \gamma^2 \) as in Propositions 4.11 and 6.2. Then, \([L(\alpha^{1/2^e}) : L] = 2^e \) if and only if \( \alpha \not\in L^2 \), which holds if and only if \( \theta = \zeta \) or \( c \not\in F \cap L^2 = F^2 \cup aF^2 \).

**Proof.** The formula for \( \alpha \) shows that \( \alpha \not\in L^2 \) just when \( \theta c \not\in L^2 \). The rest of the proof is analogous to the proof of Corollary 6.4. \( \square \)

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COTILTING VERSUS PURE-INJECTIVE MODULES

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Let $R$ and $S$ be arbitrary associative rings. A left $R$-module $RW$ is said to be cotilting if the class of modules cogenerated by $RW$ coincides with the class of modules for which the functor $\text{Ext}^1_R(-, W)$ vanishes. In this paper we characterize the cotilting modules which are pure-injective. The two notions seem to be strictly connected: Indeed all the examples of cotilting modules known in the literature are pure-injective. We observe that if $RW_S$ is a pure-injective cotilting bimodule, both $R$ and $S$ are semiregular rings and we give a characterization of the reflexive modules in terms of a suitable “linear compactness” notion.

Introduction.

Cotilting modules first appeared as vector space duals of tilting modules over finite dimensional algebras [12, IV, 7.8]. Recently they have been introduced [5] in the framework of modules over arbitrary associative rings, acquiring a proper independent role. The cotilting modules generalize the notion of injective cogenerator: They are injectives with respect to short exact sequences of modules cogenerated by them.

For arbitrary rings $R$ and $S$, a Morita duality between left $R$-modules and right $S$-modules is given by the contravariant Hom functors associated to a Morita bimodule, i.e., a faithfully balanced bimodule $RW_S$ with $RW$ and $W_S$ both injective cogenerators. One of the major component in the theory of Morita dualities is Müller’s theorem [13] which states that the reflexive modules are precisely the linearly compact modules. If $RW_S$ is a Morita bimodule, both $R$ and $S$ are semiperfect rings [16, Theorem 2.7].

For arbitrary rings $R$ and $S$, a cotilting duality between left $R$-modules and right $S$-modules is given by the contravariant Hom functors and the contravariant Ext functors associated to a cotilting bimodule, i.e., a faithfully balanced bimodule $RW_S$ with both $RW$ and $W_S$ cotilting modules (see [4]).

All known examples of cotilting modules are pure-injective. In this paper we characterize the pure-injective cotilting modules. We observe that if $RW_S$ is a pure-injective cotilting bimodule, both $R$ and $S$ are semiregular
rings and we give a characterization of the reflexive modules in terms of a suitable “linear compactness” notion.

1. About the pure-injectivity of a cotilting module.

Let $R$ be an associative ring with $1 \neq 0$. We denote by $R$-Mod the category of left unitary $R$-modules and their homomorphisms. Given a left $R$-module $W$, we consider the following classes:

- $\text{Cogen}_W$ denotes the class of all left $R$-modules cogenerated by $R^W$, that is all $M$ in $R$-Mod such that there exist a cardinal $\lambda$ and a monomorphism $M \hookrightarrow W^\lambda$;
- $\perp W$ denotes the class of all left $R$-modules $M$ such that $\text{Ext}^1_R(M, W) = 0$.

A left $R$-module $R^W$ is said to be cotilting [5] if $\text{Cogen}_R^W = \perp W$. The cotilting modules generalize injective cogenerators: Clearly $R^W$ is an injective cogenerator if and only if both the classes $\text{Cogen}_R^W$ and $\perp W$ coincide with the whole category of left $R$-modules. A short exact sequence

$$0 \to K \to L \to M \to 0$$

is said to be pure if any morphism $P \to M$, with $P$ finitely presented, lifts to a morphism $P \to L$.

**Definition 1.1.** A module $R^W$ is pure-injective if it is injective with respect to any pure exact sequence.

All known examples of cotilting modules are pure-injective. It naturally arises the question how the two notions are related.

**Proposition 1.2.** Let $R^W$ be a cotilting module. If the class $\text{Cogen}_W$ is closed under direct limits, then $W$ is pure-injective.

**Proof.** Let us show that for any pure exact sequence $0 \to A \to B \to C \to 0$ and for any map $f : A \to W$ there exists a map $g$ making the following diagram commute:

$$
\begin{array}{cccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
& & \downarrow f & & \downarrow g & & \\
& & W & & & \\
\end{array}
$$

Replacing $A \hookrightarrow B$ by $A/\text{Rej}_W A \hookrightarrow B/\text{Rej}_W A$, we can assume that $\text{Rej}_W A = 0$. The pure exact sequence $0 \to A \to B \to C \to 0$ is a direct limit of split exact sequences $0 \to A_i \to B_i \to C_i \to 0$ with $C_i$ finitely presented (cf. 34.2 [14]). For each index $i$ we have the commutative diagram with exact rows
and columns:

\[
\begin{array}{c}
\text{Rej}_W B_i \cong \leftarrow \text{Rej}_W C_i \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
0 \rightarrow A \rightarrow B_i \rightarrow C_i \rightarrow 0 \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
0 \rightarrow A \rightarrow B_i/\text{Rej}_W B_i \rightarrow C_i/\text{Rej}_W C_i \rightarrow 0.
\end{array}
\]

Applying the direct limit functor we get

\[
\begin{array}{c}
\lim \text{Rej}_W B_i \cong \lim \text{Rej}_W C_i \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
0 \rightarrow A \rightarrow \lim(B_i/\text{Rej}_W B_i) \rightarrow \lim(C_i/\text{Rej}_W C_i) \rightarrow 0.
\end{array}
\]

Since \(\lim \text{Rej}_W B_i\) is in the kernel of \(\text{Hom}(-, W)\) and \(\lim(B_i/\text{Rej}_W B_i)\) belongs to \(\text{Cogen}_W\) by the assumption, we infer that \(\lim(B_i/\text{Rej}_W B_i) \cong B/\text{Rej}_W B\). Also \(\lim(C_i/\text{Rej}_W C_i)\) belongs to \(\text{Cogen}_W = \perp W\). So \(f\) can be extended to a morphism \(g': B/\text{Rej}_W B \rightarrow W\); the composition of \(g'\) with the canonical projection \(B \rightarrow B/\text{Rej}_W B\) yields the desired map \(g\). \(\square\)

In the cotilting case, since \(\text{Cogen}_W = \perp W\), the hypothesized closure under direct limits of the class of modules cogenerated by \(W\) is suggested by the following proposition:

**Proposition 1.3.** If \(R_W\) is a pure-injective module, then \(\perp R_W\) is closed under direct limits.

*Proof.* Consider a direct system \(\{M_i : i \in I\}\) in \(\perp W\). The canonical exact sequence

\[0 \rightarrow K \rightarrow \oplus_{i \in I} M_i \rightarrow \lim_{i \in I} M_i \rightarrow 0\]

is pure (cf. [14, 33.9, (2)]). Applying \(\text{Hom}_R(-, W)\) we get the long exact sequence

\[
\cdots \rightarrow \text{Hom}(\oplus_{i \in I} M_i, W) \xrightarrow{f} \text{Hom}(K, W) \rightarrow \text{Ext}^1_R(\lim_{i \in I} M_i, W) \rightarrow \text{Ext}^1_R(\oplus_{i \in I} M_i, W) = 0.
\]

Since \(W\) is pure-injective, \(f\) is surjective; so \(\lim_{i \in I} M_i\) belongs to \(\perp W\). \(\square\)

This result has been used in [9, Lemma 9] to prove that, if \(\mathcal{C}\) is a class of pure-injective modules, every module \(M\) which has a \(\perp \mathcal{C}\)-precover has a
\(C\)-cover (see [15] for an extensive introduction to theory of (pre)covers and (pre)envelopes of modules, including various recent results).

**Corollary 1.4.** If \(RW\) is a cotilting module, then \(W\) is pure-injective if and only if \(\text{Cogen} W\) is closed under direct limits. In such a case any module has a \(\text{Cogen} W\)-cover.

**Proof.** The first claim follows by Propositions 1.2 and 1.3. The second one follows by [1, Corollary 2.6].

**Open Problem 1.5.** Are all cotilting (bi)modules pure-injective?

It is well-known that the endomorphism rings of a Morita bimodule are semiperfect; indeed a Morita bimodule is injective and finitely cogenerated on both sides (see [16, Theorem 2.7, Proposition 1.19]). We are able to give an analogous result for cotilting bimodules, assuming the closure under direct limits of the classes cogenerated by them. We recall that a ring \(R\) is said to be semiregular if \(R/J(R)\) is regular and the idempotents lift over the Jacobson radical \(J(R)\).

**Proposition 1.6.** Let \(RW_S\) be a pure-injective cotilting bimodule, i.e., faithfully balanced and cotilting and pure-injective on both sides. Then both \(R\) and \(S\) are semiregular rings.

**Proof.** The notions of pure-injective and algebraically compact module coincide (cf. [14, 34.4]). Then, by [17, Theorem 9], \(R\) and \(S\) are both semiregular.

**Remark 1.7.** Observe that if the ring \(R\) is regular, the classes of pure-injective and of injective \(R\)-modules coincide [14, 37.6]. Therefore pure-injective cotilting bimodules which are not Morita bimodules “live in the space between semiregular and regular rings”.

2. Characterizing the reflexive modules.

Let \(RW_S\) be an arbitrary bimodule. In the sequel we denote by \(\Delta\) the functors \(\text{Hom}_R(-, W)\), and by \(\Gamma\) the functors \(\text{Ext}_1^R(-, W)\), where ? stands for \(R\) or \(S\). We denote by \(\Delta^2\) both the compositions \(\text{Hom}_R(\text{Hom}_S(-, W), W)\) and \(\text{Hom}_S(\text{Hom}_R(-, W), W)\). Given a (left \(R\), or right \(S\))-module \(M\), we denote by \(\delta_M\) the canonical homomorphism \(M \to \Delta^2(M)\) defined by \(m \mapsto (f \mapsto f(m))\). A module \(M\) is called reflexive (resp. torsionless) if \(\delta_M\) is an isomorphism (resp. monomorphism).

Clearly a torsionless module \(M\) is reflexive if and only if the evaluation map \(\delta_M\) is surjective. Endowed \(M\) and \(\Delta^2M\) with any topology, the surjectivity of \(\delta_M\) can be tested in a topological way asking for \(\text{Im} \delta_M\) to be both dense and closed in \(\Delta^2M\). As the approach of Müller [13] to the classical

\(^1\)Recently Silvana Bazzoni proved that any cotilting module is pure-injective [2].
case of Morita dualities suggests, we introduce topological tools in order to characterize the reflexive modules.

Let us endow $\Delta^2 M$ with the finite topology $\varphi$: The linear topology for which the family of submodules $V(F) = \{\alpha \in \Delta^2 M : \alpha(F) = 0\}$, where $F$ is a finite subset of $\Delta M$, is a base for the filter of neighbourhoods of zero.

Let us endow $W$ with the discrete topology. Given any torsionless module $M$, we associate with each subset $A$ of $\Delta M$ the weak topology with respect to morphisms in $A$, denoted by $\tau_A$. By definition $\tau_A$ is the coarsest topology on $M$ such that all morphisms in $A$ are continuous: It is a linear topology with a base for its filter of neighbourhoods of zero formed by finite intersections of kernels of morphisms in $A$. In the sequel the topology $\tau_{\Delta M}$ will be shortly denoted by $\tau$. Note that $\tau$ is the maximum element of the set of linear topologies $\{\tau_A : A \subseteq \Delta M\}$ partially ordered by inclusion. Let $L_\sigma$ be a linearly topologized module. Denote by $H$ the $\sigma$-closure of zero in $L$. Note that $H$ is equal to the intersection of all neighbourhoods of zero and, since $\sigma$ is a linear topology, $\sigma$ is Hausdorff if and only if $H = 0$. A $\sigma$-Cauchy net in $L$ is a family $X_\lambda$, $\lambda \in \Lambda$, indexed by the upwards directed partially ordered set $\Lambda$, such that for every neighbourhood $U$ of zero there exists an upper subset $\Lambda'$ of $\Lambda$ with $x_\lambda - x_{\lambda'} \in U$ for every $\lambda, \lambda' \in \Lambda'$. The topology $\sigma$ is complete, i.e., any $\sigma$-Cauchy net in $L$ converges in $L$, if and only if the topological quotient $L/H$ is complete (see [3, Chap. 3, §2]). Note that a closed submodule of a complete module is complete. The completion of $L/H$ is called the Hausdorff completion of $L$: Denoted by $J = \{J_\lambda : \lambda \in \Lambda\}$ a base for the filter of neighbourhoods of zero in $L_\sigma$ consisting of open submodules, it coincides with the inverse limit $\varprojlim L/J_\lambda$ (see [10, Proposition 13.7]).

**Proposition 2.1.** Let $M$ be a torsionless module.

(i) The topologies $\tau$ on $M$ and $\varphi$ on $\Delta^2 M$ are Hausdorff.

(ii) $\delta_M : M_{\tau} \to \Delta^2 M_\varphi$ is a topological embedding.

(iii) The topology $\varphi$ on $\Delta^2 M$ is complete.

**Proof.** (i) Since $M$ is cogenerated by $W$, there exists a set $X$ and the following maps:

$$M \overset{i}{\hookrightarrow} W^X \overset{\pi_x}{\twoheadrightarrow} W.$$  

Clearly $\{0\}$ is the intersection of $\text{Ker}(\pi_x \circ i)$, $x \in X$. Since any $\text{Ker}(\pi_x \circ i)$ is $\tau$-open and hence $\tau$-closed, $\tau$ is Hausdorff. Let us consider the topology $\varphi$. The open submodules $V(F) = \{\alpha \in \Delta^2 M : \alpha(F) = 0\}$, with $F$ finite subset of $\Delta M$, have intersection zero: Hence $\varphi$ is Hausdorff.

(ii) It follows from the fact that for any finite subset $F$ of $\Delta M$ we have

$$V(F) \cap \delta_M(M) = \{\delta_M(m) : f(m) = 0 \forall f \in F\} = \delta_M(\cap_{f \in F} \text{Ker } f).$$

(iii) The topological module $\Delta^2 M_\varphi$ is a closed submodule of $W^{\Delta M}$ endowed with the product of the discrete topologies. Since the product of complete topologies is also complete, we can conclude. $\square$
Corollary 2.2. Each reflexive module $M$ endowed with the topology $\tau$ is complete.

Proof. If $M$ is reflexive, $\delta_M$ is an isomorphism. Therefore, by Proposition 2.1, (ii), it is a topological isomorphism. We conclude by Proposition 2.1, (iii).

□

Proposition 2.3. For a torsionless module $M$ the following statements are equivalent:

(i) $M$ is reflexive.

(ii) $\tau$ is a complete topology and $\delta_M(M)$ is dense in $\Delta^2M_\varphi$.

Proof. (i) $\Rightarrow$ (ii): It follows by Corollary 2.2.

(ii) $\Rightarrow$ (i): Since $\tau$ is complete, by Proposition 2.1 $\delta_M(M)$ is a complete, and hence closed, topological submodule of $\Delta^2M_\varphi$. Being $\delta_M(M)$ dense in $\Delta^2M_\varphi$ and $\varphi$ an Hausdorff topology, $\delta_M$ is an isomorphism.

□

As suggested by Müller [13] in the case of Morita dualities, we look for a suitable notion of compactness for a module $M$ in order to guarantee the completeness of the topology $\tau$ on $M$.

Note that $\delta_M(M)$ is dense in $\Delta^2M_\varphi$ if and only if for each $\alpha$ in $\Delta^2M$ and $f_1, \ldots, f_n$ in $\Delta M$ there exists $m$ in $M$ such that $\alpha(f_i) = f_i(m)$ for each $i = 1, \ldots, n$. Following [11], a module $M$ satisfying the above property will be called $W$-dense.

Definition 2.4. Let $M$ be a torsionless left $R$-module. A submodule $K$ of $M$ is called $W$-closed if $M/K$ is torsionless (see [11, §2]). A linear topology on $M$ is said to be a $W$-topology if it has a basis of neighbourhoods of zero consisting of $W$-closed submodules.

Each $W$-closed submodule $K$ of $M$ is closed in $M_\tau$. Indeed, for a suitable set $X$, there exist the following maps:

$$M \xrightarrow{\pi} M/K \xrightarrow{i} W^X \xrightarrow{\pi_X} W.$$  

Then, $K = \bigcap_{x \in X} \text{Ker}(\pi_x \circ i \circ \pi)$ is closed, since it is an intersection of open and hence closed submodules. The converse is not true in general.

Example 2.5. Let $R$ denote the $k$-algebra given by the quiver $1 \rightarrow 2 \rightarrow 3$. It is easy to verify that $_RR_R$ is a cotilting bimodule. Consider the projective $R$-module $P(2)$. The topology $\tau_{\Delta P(2)}$ is discrete since $P(2)$ embeds in $R_R$. Therefore each submodule of $P(2)$, in particular the simple module $S(3)$, is closed. Nevertheless $S(3)$ is not a $R$-closed submodule of $P(2)$, since $P(2)/S(3) \cong S(2)$ is not cogenerated by $R_R$.

Definition 2.6. A left $R$-module $M$ is said to be $W$-linearly compact (see [11, §3]), briefly $W$-lc, (resp. $HW$-linearly compact, briefly $HW$-lc) if it is complete in any $W$-topology (resp. in any Hausdorff $W$-topology).

...
Analogously to the usual linear compactness, a left $R$-module $M$ is $W$-linear compact if and only if any finitely solvable system of congruences $x \equiv x_\lambda \mod M_\lambda$, where $\{M_\lambda : \lambda \in \Lambda\}$ is a downwards directed collection of $W$-closed submodules of $M$, is solvable. Similarly a module $M$ is $HW$-linearly compact if and only if it satisfies the previous condition restricted to downwards directed collections of $W$-closed submodules of $M$ with intersection equal to zero.

**Proposition 2.7.** Let $R^M$ be a torsionless left $R$-module. If $M$ is $HW$-linearly compact, then $M$ is complete in the topology $\tau$.

**Proof.** Since the intersection of the kernels of a finite number of elements of $\Delta M$ is a $W$-closed submodule of $M$, $\tau$ is a $W$-topology. Since $M$ is torsionless, by Proposition 2.1 the topology $\tau$ is Hausdorff. Since $M$ is $HW$-lc, $\tau$ is complete. $\square$

**Corollary 2.8.** Let $R^M$ be a torsionless left $R$-module. If $M$ is $HW$-linearly compact and $W$-dense, then $M$ is reflexive.

**Proof.** It follows by Propositions 2.3 and 2.7. $\square$

We can obtain a more precise result for cotilting bimodules.

**Theorem 2.9.** Let $R^W_S$ be a cotilting bimodule. For a torsionless left $R$-module $M$ the following statements are equivalent:

(a) $M$ is $HW$-linearly compact and $W$-dense.

(b) $M$ is reflexive and $\tau$ is the unique Hausdorff topology among those induced by subsets of $\Delta M$.

**Proof.** (a $\Rightarrow$ b): By Corollary 2.8 we only have to prove that $\tau$ is the unique Hausdorff topology induced by subsets of $\Delta M$. Since $\tau$ is the maximum element in $\{\tau_A : A \subseteq \Delta M\}$, it is sufficient to prove that if $\tau_A$ is Hausdorff, then $\tau$ is coarser than $\tau_A$ and hence $\tau_A = \tau$.

Let $F$ be a finite subset of $A$. We denote by $f_F : M \to W^F$ the diagonal morphism. Let $M_F := \bigcap_{f \in F} \ker f = \ker f_F$ and $N_F := M/M_F$. By [4, Proposition 5] both the left $R$-modules $M_F$ and $N_F$ are reflexive. We call $\pi_F$ the induced map $M \to N_F$. Since $\{M_F : F \subseteq A, F \text{ finite}\}$ is a base for the filter of $\tau_A$-neighbourhoods of zero consisting of open submodules, $\lim N_F$ is the Hausdorff completion of $M$ endowed with the topology $\tau_A$.

But, since $\tau_A$ is an Hausdorff $W$-topology and, by hypothesis, $M$ is $HW$-lc, $\tau_A$ is complete. Thus $M \cong \lim N_F$.

Applying the functors $\Delta$ and $\lim$ to the exact sequences

$$0 \to M_F \to M \xrightarrow{\pi_F} N_F \to 0$$

we get the exact sequence of right $S$-modules

$$\xymatrix{0 \ar[r] & \lim \Delta N_F \ar[r]^{\lim (\Delta(\pi_F))} & \Delta M \ar[r] & \lim \Delta M F \ar[r] & 0}$$

\[\star\]
Now $\Delta(\lim \Delta(\pi_F)) \cong \lim \Delta^2(\pi_F) \cong \lim \pi_F$ is an isomorphism. Then, from the exact sequence

$$0 \to \Delta \lim \Delta M_F \longrightarrow \Delta^2 M \longrightarrow \Delta \lim \Delta N_F \longrightarrow \Gamma \lim \Delta M_F \to 0,$$

we get $\lim \Delta M_F$ belongs to $\Ker \Delta \cap \Ker \Gamma = 0$. Hence $\Delta M \cong \lim \Delta N_F$.

Let now $g$ be in $\Delta M$. Since $g$ belongs to $\Delta(\pi_F)(\Delta(N_F))$ for some finite subset $F$ of $A$, there exists a morphism $h : N_F \to W$ such that $g = h \circ \pi_F$. Then since $\Ker g \supseteq \Ker \pi_F = \bigcap_{f \in F} \Ker f$, $\Ker g$ is $\tau_A$-open. Therefore $\tau$ is coarser than $\tau_A$.

(b $\Rightarrow$ a): We only have to prove that $M$ is $HW$-lc. Let $\sigma$ be a Hausdorff $W$-topology on $M$. By definition $\sigma$ has a basis $\mathcal{B}$ for the filter of neighbourhoods of zero consisting of $W$-closed submodules; since $\sigma$ is Hausdorff, the intersection of elements in $\mathcal{B}$ is equal to zero. Observe that any element $V$ of $\mathcal{B}$ is the intersection of the kernels of a (not necessarily finite) subset $A_V$ of $\Delta M$. Let $A$ the union $\cup_{V \in \mathcal{B}} A_V$. If $f$ belongs to $A_V$, $\Ker f$ contains $V$ and hence it is $\sigma$-open. Therefore the topology $\tau_A$ is coarser than $\sigma$. Since

$$\bigcap_{f \in A} \Ker f = \bigcap_{V \in \mathcal{B}} V = \{0\},$$

$0$ is a closed subset of $M_{\tau_A}$, i.e., $\tau_A$ is Hausdorff. By hypothesis $\tau_A = \tau$ and, since $M$ is reflexive, by Proposition 2.3 $\tau_A$ is complete. By [3, Proposition III.3.10], also the topology $\sigma$ is complete. \hfill $\Box$

**Lemma 2.10.** Let $R WS$ be a cotilting bimodule with $\Cogen W_S$ closed under direct limits. Let $M$ be a reflexive left $R$-module. Then $\tau$ is the unique Hausdorff topology among those induced by subsets of $\Delta M$.

**Proof.** We can follow the first two paragraphs of the proof of Theorem 2.9, (a $\Rightarrow$ b). Applying the functors $\Delta$ and $\lim$ to the exact sequence

$$0 \to M_F \to M \xrightarrow{\pi_F} N_F \to 0$$

we get the exact sequence

$$0 \to \lim \Delta N_F \longrightarrow \lim \Delta(\pi_F) \longrightarrow \Delta M \longrightarrow \lim \Delta M_F \to 0$$

of right $S$-modules. Observe that $\lim \Delta M_F$ again belongs to $\Ker \Gamma = \Cogen W$ since, by hypothesis, $\Cogen W$ is closed under direct limits. Moreover

$$\Delta \lim \Delta M_F \cong \lim \Delta^2 M_F \cong \lim M_F = \bigcap_{f \in A} \Ker f = 0.$$

Therefore, since $\Ker \Delta \cap \Ker \Gamma = 0$, we get $\lim \Delta M_F = 0$. Hence $\Delta M \cong \lim \Delta N_F$. We can thus conclude following the last paragraph of the proof of Theorem 2.9, (a $\Rightarrow$ b). \hfill $\Box$

Thus we obtain the characterization of reflexive modules for pure-injective cotilting bimodules.
Theorem 2.11. Let \( R \otimes W \otimes S \) be a pure-injective cotilting bimodule. For a torsionless module \( M \) the following are equivalent:

(a) \( M \) is reflexive.
(b) \( M \) is \( HW \)-linearly compact and \( W \)-dense.
(c) \( M \) is \( W \)-linearly compact and \( W \)-dense.

Proof. (c \( \Rightarrow \) b \( \Rightarrow \) a): They follow by Definition 2.6 and by Theorem 2.9.

(a \( \Rightarrow \) c): Trivially \( M \) is \( W \)-dense. Let \( \sigma \) be a \( W \)-topology on \( M \) and let \( H \leq M \) be the \( \tau \)-closure of zero. Since \( H \) is a \( W \)-closed submodule of \( M \), \( M/H \) is reflexive (see [4, Proposition 5]). By Theorem 2.9 and Lemma 2.10 \( M/H \) is \( HW \)-lc and hence complete endowed with the quotient topology of \( \tau \). Therefore \( M \) is complete.

In [13] M"uller proved that if \( R \otimes W \otimes S \) is a Morita bimodule, a module \( M \) is reflexive if and only if it is linearly compact in the discrete topology if and only if it is complete in any Hausdorff linear topology. In such a case \( W \) cogenerates the whole category of modules, hence any submodule is \( W \)-closed. Therefore the notions of \( W \)-linear compactness, of \( HW \)-linear compactness and of linear compactness in the discrete topology coincide. In our setting a density condition comes out. Let us better investigate its role.

Proposition 2.12. Let \( R \otimes W \otimes S \) be a bimodule such that \( \text{Cogen} W \subseteq \text{Ker} \Gamma \). A left \( R \)-module \( M \) is \( W \)-dense if and only if \( \text{Im}(f) \) is a reflexive left \( R \)-module for every \( f \in \text{Hom}_R(M, W^n) \), \( n \in \mathbb{N} \).

Proof. Let us consider for each \( f \) in \( \text{Hom}(M, W^n) \), \( n \in \mathbb{N} \), the following commutative diagram of linearly topologized modules and continuous morphisms:

\[
\begin{array}{ccc}
M \tau & \xrightarrow{f} & W^n \\
\downarrow \delta_M & & \downarrow \delta_{\text{Im} f} \\
\Delta^2 M \varphi & \xrightarrow{\Delta^2 \epsilon_f} & \Delta^2 \text{Im} f \omega'
\end{array}
\]

where \( \varphi' \) is the finite topology on \( \Delta^2 \text{Im} f \). Since \( \text{Im} f \leq W^n \), \( \tau_{\Delta \text{Im} f} \) is the discrete topology; in particular it is complete.

Suppose that \( M \) is \( W \)-dense. Applying \( \Delta \) to the exact sequence \( 0 \to \text{Ker} f \to M \xrightarrow{\epsilon_f} \text{Im} f \to 0 \), we get the exact sequence \( 0 \to \Delta \text{Im} f \xrightarrow{\Delta(\epsilon_f)} \Delta M \to C \to 0 \) with \( C \) in \( \text{Cogen} W \). Since \( \text{Cogen} W \subseteq \text{Ker} \Gamma \), \( \Delta^2(\epsilon_f) \) is an epimorphism. Therefore, the \( W \)-density of \( M \) implies the \( W \)-density of \( \text{Im} f \). By Proposition 2.3, \( \text{Im} f \) is reflexive.
Conversely, let \( f_1, \ldots, f_n \) be in \( \Delta M \). We denote by \( f : M \rightarrow W^n \) their diagonal morphism, \( m \mapsto (f_1(m), \ldots, f_n(m)) \), and by \( \mu_f \circ \varepsilon_f \) the usual factorization of \( f \) through \( \text{Im} \, f \). Since \( \text{Im} \, f \) is reflexive, for each \( \alpha \) in \( \Delta^2 M \) there exists \( m_\alpha \) in \( M \) such that

\[
\Delta^2(\varepsilon_f)(\alpha) = \delta_{\text{Im} \, f}(f(m_\alpha)).
\]

In particular, denoted by \( p_i : \text{Im} \, f \rightarrow W \) the \( i \)-th projection, we have

\[
(\Delta^2(\varepsilon_f)(\alpha))(p_i) = \alpha(\Delta(\varepsilon_f))(p_i) = \alpha(p_i \circ \varepsilon_f) = \alpha(f_i)
\]

\[
(\delta_{\text{Im} \, f}(f(m_\alpha))(p_i) = p_i(f(m_\alpha)) = f_i(m_\alpha);
\]

therefore \( \alpha \) and \( \delta_M(m_\alpha) \) coincide on \( f_1, \ldots, f_n \). Therefore \( M \) is \( W \)-dense. \( \square \)

If \( RW \) is a Morita bimodule, then the class of reflexive modules contains \( W \) and it is closed under submodules and finite direct sums. Therefore the \( W \)-density condition is always satisfied: Any module \( M \) is \( W \)-dense. This is not the case for cotilting bimodules.

**Example 2.13.** Let \( k \) be an algebraically closed field. Denote by \( A \) the *generalized Kronecker algebra* of dimension \( \aleph_0 \) over \( k \) given by the quiver

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\uparrow & & \uparrow \\
& & \\
\end{array}
\]

with a countable set of arrows from 1 to 2, i.e., the ring of lower triangular matrices

\[
\begin{pmatrix}
k & 0 \\
V & k
\end{pmatrix} = \left\{ \begin{pmatrix}
a & 0 \\
v & b
\end{pmatrix} : a, b \in k, v \in V \right\}
\]

where \( V \) is a \( k \)-vector space of dimension \( \aleph_0 \) (see [7, 8]). Then, by [8, Lemma 2.2], \( A \) is a hereditary, coherent and perfect ring. It is easily verified that \( AA_A \) is a cotilting bimodule and \( \text{Cogen}(A) \) consists of projective modules, while \( \text{Ker}(\Delta) \) contains exactly the modules without projective direct summands. The reflexive modules coincide with the finitely generated projective modules [8, Lemma 2.3]. Denote by \( e_1, e_2 \) the primitive idempotents, i.e.,

\[
e_1 = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}, \quad e_2 = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}.
\]

Let \( P_i = Ae_i \) and \( Q_i = e_iA, i = 1, 2 \). The socle \( S \) of \( P_1 \) is isomorphic to \( P_2(\aleph_0) \); therefore it is a not reflexive submodule of \( A \). By Proposition 2.12, \( S \) is not \( A \)-dense.

In [6, Definition 1.2] Colpi and Fuller introduced the \( W \)-*torsionless linear compactness*, a generalization of the notion of linear compactness with respect to the torsion theories associated to a cotilting bimodule \( RW \). They prove that if a module is \( W \)-torsionless linearly compact, then it is reflexive, i.e., (see Proposition 2.3) \( \tau \) is a complete topology and \( M \) is \( W \)-dense.
Our notion of $W$-linear compactness is strong enough to assure the completeness, but to obtain the $\Delta$-reflexivity we need to assume explicitly the $W$-density. Assuming $RW$ and $WS$ pure-injective (as in all examples known in the literature), these two notions together completely characterize the classes of reflexive modules. The notion of $W$-torsionless linear compactness is too strong to characterize the classes of reflexive modules in the general case; this happens if and only if the classes of reflexive left $R$- and right $S$-modules are closed under submodules [6, Corollary 1.9]. Observe that in this case, by Proposition 2.12, any module is $W$-dense. Adding the hypotheses of both the contexts we get:

**Corollary 2.14.** Let $RW_S$ be a pure-injective cotilting bimodule. Assume the classes of reflexive left $R$- and right $S$-modules being closed under submodules. For a module $M$, the following statements are equivalent:

(a) $M$ is reflexive;
(b) $M$ is $W$-torsionless linearly compact;
(c) $M$ is $W$-linearly compact.

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GROWTH PROPERTIES FOR MODIFIED POISSON INTEGRALS IN A HALF SPACE

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Dedicated to Professor Hidenobu Yoshida on the occasion of his sixtieth birthday.

Our aim in this paper is to deal with growth properties at infinity for modified Poisson integrals (of fractional power) in the half space of $\mathbb{R}^n$. We also discuss weighted boundary limits for the modified Poisson integrals.

1. Introduction and statement of results.

Let $\mathbb{R}^n$ ($n \geq 2$) denote the $n$-dimensional Euclidean space with points $x = (x_1, \ldots, x_{n-1}, x_n)$. Let $D = \{x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n; x_n > 0\}$, whose boundary is usually identified with $\mathbb{R}^{n-1}$.

For $\lambda > 0$ and $x \in \mathbb{R}^n$, consider the kernel function

$$K_\lambda(x) = |x|^{-\lambda}.$$ 

The Poisson integral is defined by

$$P[f](x) = \alpha_n x_n \int_{\mathbb{R}^{n-1}} K_n(x-y)f(y)dy,$$ 

where $f$ is a locally integrable function on $\mathbb{R}^{n-1}$ and $\alpha_n = 2/(n\sigma_n)$ with $\sigma_n = \pi^{n/2}/\Gamma(1 + n/2)$ being the volume of the unit $n$-ball. The Poisson integrals are used to solve the Dirichlet problem in the half space $D$. Further, Sjögren ([15] and [16]), Rönning [11] and Brundin [3] treated fractional Poisson integrals with respect to the fractional power of the Poisson kernel

$$P_\theta[f](x) = \int_{\mathbb{R}^{n-1}} \{\alpha_n x_n K_n(x-y)\}^\theta f(y)dy;$$ 

if $n = 2$, then it defines a solution of the hyperbolic Laplacian

$$x_2^2 \Delta u = \theta(\theta - 1)u.$$ 

The Poisson integral $P[f]$ will be harmonic in $D$ if

$$\int_{\mathbb{R}^{n-1}} |f(y)|(1 + |y|)^{-n} dy < \infty$$

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In this paper, we consider functions $f$ satisfying
\[
\int_{\mathbb{R}^{n-1}} |f(y)|^p (1 + |y|)^{-\gamma} dy < \infty
\]
for $1 \leq p < \infty$ and a real number $\gamma$. To obtain the Dirichlet solution for the boundary data $f$, as in [13, 14] and [19], we use the following modified kernel function defined by
\[
K_{\lambda, m}(x, y) = \begin{cases} 
K_{\lambda}(x-y) & \text{when } |y| < 1, \\
K_{\lambda}(x-y) - \sum_{|j| \leq m-1} \frac{x^j}{j!} \left[ (\partial/\partial x)^j K_{\lambda}(x-y) \right]_{x=0} & \text{when } |y| \geq 1
\end{cases}
\]
for a nonnegative integer $m$ and a point $x = (x_1, \ldots, x_n)$, where $j = (j_1, \ldots, j_n)$ is a multiindex with length $|j| = j_1 + \cdots + j_n$, $j! = j_1! \cdots j_n!$, $x^j = x_1^{j_1} \cdots x_n^{j_n}$ and $(\partial/\partial x)^j = (\partial/\partial x_1)^{j_1} \cdots (\partial/\partial x_n)^{j_n}$. In the papers mentioned above, it is expressed by use of Gegenbauer polynomials ([18]). Write
\[
K_{\lambda, m}f(x) = \int_{\mathbb{R}^{n-1}} K_{\lambda, m}(x, y) f(y) dy
\]
and
\[
U_{\lambda, m}f(x) = \alpha_n x_n K_{\lambda, m}f(x).
\]
Here note that $U_{0,0}f$ is nothing but the Poisson integral $P[f]$.

Recently Siegel-Talvila ([14, Theorem 2.1 and Corollary 2.1]) proved the following:

**Theorem A.** Let $f$ be a continuous function on $\mathbb{R}^{n-1}$ satisfying (1.4) with $p = 1$ and $\gamma = n + m$. Then the function $U_{n, m}f(x)$ satisfies
\[
U_{n, m}f \in C^2(D) \cap C^0(\overline{D}),
\]
\[
\Delta U_{n, m}f = 0 \quad \text{in } D,
\]
\[
U_{n, m}f = f \quad \text{on } \partial D,
\]
\[
U_{n, m}f(x) = o(x_1^{1-n}|x|^{n+m}) \quad \text{as } |x| \to \infty, \ x \in D.
\]

Our first aim in this paper is to establish the following theorem (cf. [14, Theorem 2.1], [13, Theorem 5.1]):

**Theorem 1.** Let $1 \leq p < \infty$, $\lambda > 0$, $\gamma > - (n-1)(p-1)$ and
\[
-\lambda + \gamma \leq m < -\lambda + \gamma + 1 \quad \text{in case } p > 1,
\]
\[
-n - \lambda - 1 - (n-\gamma-1)/p < m \leq n - \lambda - (n-\gamma-1)/p \quad \text{in case } p = 1.
\]
If $f$ is a measurable function on $\mathbb{R}^{n-1}$ satisfying (1.4), then
\[
\lim_{|x| \to \infty, x \in D} x_1^\lambda |x|^{1-n+(n-\gamma-1)/p} K_{\lambda, m}f(x) = 0
\]
when \( m < n - \lambda - (n - \gamma - 1)/p \), and
\[
\lim_{|x| \to \infty, x \in D} x_n^\lambda |x|^{1-n+(n-\gamma-1)/p} (\log |x|)^{-1/p'} \mathcal{K}_{\lambda,m} f(x) = 0
\]
when \( m = n - \lambda - (n - \gamma - 1)/p > 1 \) and \( p' = p/(p-1) \).

**Remark 1.** Siegel-Talvila [14, Theorem 2.1] treated the case \( p = 1 \) and \( m = -\lambda + \gamma \) (see also [13, Theorem 5.1]).

**Corollary 1.** Let \( p > 1 \), \( \gamma > -(n-1)(p-1) \) and
\[-1 - (n - \gamma - 1)/p \leq -(n - \gamma - 1)/p.

If \( f \) is a measurable function on \( \mathbb{R}^{n-1} \) satisfying (1.4), then
\[
\lim_{|x| \to \infty, x \in D} x_n^{n-1} |x|^{1-n+(n-\gamma-1)/p} U_{n,m} f(x) = 0
\]
when \( m < -(n - \gamma - 1)/p \),
\[
\lim_{|x| \to \infty, x \in D} x_n^{n-1} |x|^{1-n+(n-\gamma-1)/p} (\log |x|)^{-1/p'} U_{n,m} f(x) = 0
\]
when \( m = -(n - \gamma - 1)/p \).

Next we are concerned with minimally fine limits at infinity for \( U_{\lambda,m} f \), as an extension of Lelong-Ferrand [7]. For related results, we refer the reader to the papers by Aikawa [1], Essén-Jackson [4], Miyamoto-Yoshida [8] and the first author [9]. For this purpose, consider the kernel function
\[
k_{\beta,\lambda}(x,y) = x_n^{1-\beta} |x-y|^{-\lambda}.
\]

To evaluate the size of exceptional sets, for a set \( E \subset D \) and an open set \( G \subset \mathbb{R}^{n-1} \), we consider the capacity
\[
C_{k_{\beta,\lambda},p}(E;G) = \inf \int_{\mathbb{R}^{n-1}} g(y)^p \, dy,
\]
where the infimum is taken over all nonnegative measurable functions \( g \) such that \( g = 0 \) outside \( G \) and
\[
\int_{\mathbb{R}^{n-1}} k_{\beta,\lambda}(x,y) g(y) \, dy \geq 1 \quad \text{for all} \ x \in E.
\]

We say that \( E \subset D \) is (minimally) \( (k_{\beta,\lambda},p) \)-thin at infinity if
\[
\sum_{i=1}^{\infty} 2^{-i((\beta+\lambda-n)p+n-1)} C_{k_{\beta,\lambda},p}(E_i;D_i) < \infty,
\]
where \( E_i = \{ x \in E : 2^i \leq |x| < 2^{i+1} \} \) and \( D_i = \{ x \in \mathbb{R}^{n-1} : 2^{i-1} < |x| < 2^{i+2} \} \).
Theorem 2. Let $p$, $\lambda$ and $\gamma$ be as in Theorem 1. If $f$ is a measurable function on $\mathbb{R}^{n-1}$ satisfying (1.4) and $\beta \leq 1$, then there exists a set $E \subset D$ such that $E$ is $(k_\beta, \lambda, p)$-thin at infinity and
\[
\lim_{|x| \to \infty, x \in D - E} x_n^{-\beta}|x|^{\beta+\lambda-n(n-\gamma-1)/p}U_{\lambda,m}f(x) = 0.
\]

It is well-known that the Poisson integral $U_{n,0}f = P[f]$ has nontangential boundary limits $f$ at almost all boundary points. Our final goal is to show that $U_{\lambda,m}f$ has weighted boundary limits. For this purpose, we discuss the existence of boundary limits for
\[
P_{\lambda}f(x) = \frac{K_{\lambda}f(x)}{K_{\lambda}\chi_G(x)},
\]
where $\lambda \geq n-1$, $G$ is a bounded open set in $\mathbb{R}^{n-1}$, $1 \leq p < \infty$, $f \in L^p(G)$, $\chi_G$ denotes the characteristic function of $G$ and
\[
K_{\lambda}f(x) = \int_G K_{\lambda}(x-y)f(y)dy.
\]

For a nonnegative function $h$ on the interval $\mathbb{R}^+ = [0, \infty)$, let
\[
A_h(\xi) = \{x \in D : |x - \xi| < h(x_n)\}.
\]

Theorem 3. Let $1 \leq p < \infty$ and $f \in L^p(G)$. For a.e. $\xi \in G$, $P_{\lambda}f(x) \rightarrow f(\xi)$ as $x \rightarrow \xi$ along $A_h(\xi)$, where
\[
h(t) = C \begin{cases} 
t & (\lambda > n-1), \\
t \left(\log \frac{1}{t}\right)^{p/(n-1)} & (\lambda = n-1)
\end{cases}
\]
for fixed $C > 0$.

In the unit disc, this result was proved for $\lambda = 1$ by Sjögren [15] and [16], Rönning [11] and Brundin [3].

2. Proof of Theorem 1.

Throughout this paper, let $M$ denote various constants independent of the variables in question.

First we note the following properties for the kernel functions $K_{\lambda,m}(x,y)$:

Lemma 1. For $t > 0$, set
\[
f(t) = f(t,x,y) = tx_n|tx - y|^{-\lambda}
\]
and
\[
g(t) = g(t,x,y) = |tx - y|^{-\lambda}.
\]
Then \( f^{(\ell)}(0) = \ell x_n g^{(\ell-1)}(0) \) for \( \ell = 1, 2, \ldots, m \), and

\[
\begin{align*}
f(1) - \left( f(0) + f'(0) + \frac{1}{2!} f''(0) + \cdots + \frac{1}{m!} f^{(m)}(0) \right) \\
= x_n \left\{ g(1) - \left( g(0) + g'(0) + \frac{1}{2!} g''(0) + \cdots + \frac{1}{(m-1)!} g^{(m-1)}(0) \right) \right\} \\
= x_n K_{\lambda,m}(x, y)
\end{align*}
\]

when \( |y| \geq 1 \).

**Corollary 2.** \( U_{n,m}(x, y) = \alpha_n x_n K_{n,m}(x, y) \) is harmonic in \( D \) for each fixed \( y \in \mathbb{R}^{n-1} \).

In our discussions, the following estimates for the kernel functions \( K_{\lambda,m} \) are fundamental (see [6, Lemma 4.2] and [12, Section 3]):

**Lemma 2.** Let \( m \) be a nonnegative integer and \( \lambda > 0 \).

1. If \( 1 \leq |y| \leq |x|/2 \), then \( |K_{\lambda,m}(x, y)| \leq M|x|m^{-1}|y|^{-\lambda-m+1} \).
2. If \( |x|/2 \leq |y| \leq 2|x| \), then \( |K_{\lambda,m}(x, y)| \leq M|x-y|^{-\lambda} \leq Mx^{-\lambda} \).
3. If \( |y| \geq 2|x| \) and \( |y| \geq 1 \), then \( |K_{\lambda,m}(x, y)| \leq M|x|^m|y|^{-\lambda-m} \).

**Proof of Theorem 1.** We prove only the case \( p > 1 \); the proof of the case \( p = 1 \) is similar. For fixed \( x \in D \), \( |x| > 2 \), we write

\[
K_{\lambda,m} f(x) = \int_{G_1} K_{\lambda,m}(x, y) f(y) \, dy + \int_{G_2} K_{\lambda,m}(x, y) f(y) \, dy \\
+ \int_{G_3} K_{\lambda,m}(x, y) f(y) \, dy + \int_{B(0,1)} K_{\lambda,m}(x, y) f(y) \, dy
\]

\[
= U_1(x) + U_2(x) + U_3(x) + U_4(x),
\]

where \( B(x, r) \) denotes the open ball centered at \( x \) with radius \( r > 0 \), and

\[
G_1 = \{ y \in \mathbb{R}^{n-1} : |y| \geq 2|x| \},
\]

\[
G_2 = \{ y \in \mathbb{R}^{n-1} : 1 \leq |y| < |x|/2 \},
\]

\[
G_3 = \{ y \in \mathbb{R}^{n-1} : |x|/2 \leq |y| < 2|x| \}.
\]

First note that

\[
|U_4(x)| \leq (|x|/2)^{-\lambda} \int_{B(0,1)} |f(y)| \, dy,
\]

so that

\[
(2.1) \quad \lim_{|x| \to \infty, x \in D} |x|^\lambda\gamma - n + 1 + (n-\gamma-1)/p U_4(x) = 0
\]

since \( \gamma > -(n-1)(p-1) \).
Lemma 3. If \( m > n - \lambda - 1 - (n - \gamma - 1)/p \), then

\[
|U_1(x)| \leq M |x|^{-\lambda n - 1 - (n - \gamma - 1)/p} \left( \int_{G_1} |f(y)|^p |y|^{-\gamma} \, dy \right)^{1/p}.
\]

Proof. If \( m > n - \lambda - 1 - (n - \gamma - 1)/p \), then \((-\lambda - m + \gamma/p)p' + n - 1 < 0\), so that we obtain by Lemma 2 (3) and Hölder’s inequality

\[
|U_1(x)| \leq M |x|^m \int_{G_1} |y|^{-\lambda - m} |f(y)| \, dy
\]

\[
\leq M |x|^m \left( \int_{G_1} |y|^{(-\lambda - m + \gamma/p)p'} \, dy \right)^{1/p'} \left( \int_{G_1} |f(y)|^p |y|^{-\gamma} \, dy \right)^{1/p}
\]

\[
\leq M |x|^{-\lambda n - 1 - (n - \gamma - 1)/p} \left( \int_{G_1} |f(y)|^p |y|^{-\gamma} \, dy \right)^{1/p},
\]

where \( 1/p + 1/p' = 1 \). This proves the lemma.

By Lemma 3, we have

\[
\lim_{|x| \to \infty, x \in D} |x|^{\lambda - n + 1 + (n - \gamma - 1)/p} U_1(x) = 0.
\]

Lemma 4. If \( m < n - \lambda - (n - \gamma - 1)/p \), then

\[
|U_2(x)| \leq M |x|^{-\lambda + n - 1 - (n - \gamma - 1)/p} \left( \int_{G_2} |f(y)|^p |y|^{-\gamma} \, dy \right)^{1/p};
\]

if \( m = n - \lambda - (n - \gamma - 1)/p \), then

\[
|U_2(x)| \leq M |x|^{-\lambda + n - 1 - (n - \gamma - 1)/p} \left( \int_{G_2} |f(y)|^p |y|^{-\gamma} \, dy \right)^{1/p}.
\]

Proof. If \( m < n - \lambda - (n - \gamma - 1)/p \), then \((-\lambda - m + 1 + \gamma/p)p' + n - 1 > 0\), so that we obtain by Lemma 2 (1) and Hölder’s inequality

\[
|U_2(x)| \leq M |x|^m \int_{G_2} |y|^{-\lambda - m + 1} |f(y)| \, dy
\]

\[
\leq M |x|^m \left( \int_{G_2} |y|^{(-\lambda - m + 1 + \gamma/p)p'} \, dy \right)^{1/p'} \left( \int_{G_2} |f(y)|^p |y|^{-\gamma} \, dy \right)^{1/p}
\]

\[
\leq M |x|^{-\lambda + n - 1 - (n - \gamma - 1)/p} \left( \int_{G_2} |f(y)|^p |y|^{-\gamma} \, dy \right)^{1/p},
\]

as required.

The remaining case can be proved similarly.
For $r > 1$, we have
\[ U_2(x) = \int_{G_2} K_{\lambda,m}(x,y) f(y) \, dy \]
\[ = \int_{G_2 \cap B(0,r)} K_{\lambda,m}(x,y) f(y) \, dy + \int_{G_2 - B(0,r)} K_{\lambda,m}(x,y) f(y) \, dy \]
\[ = U_{21}(x) + U_{22}(x). \]

If $|x| > 2r$ and $m < n - \lambda - (n - \gamma - 1)/p$, then
\[ |U_{21}(x)| \leq M|x|^{m-1} \int_{B(0,r) - B(0,1)} |y|^{-\lambda - m + 1} |f(y)| \, dy, \]
so that
\[ \lim_{|x| \to \infty, x \in D} |x|^{\lambda-n+1+(n-\gamma-1)/p} U_{21}(x) = 0. \]

Moreover, we have by Lemma 4
\[ |U_{22}(x)| \leq M|x|^{-\lambda+n-1-(n-\gamma-1)/p} \left( \int_{R^{n-1-B(0,r)}} |f(y)|^p |y|^{-\gamma} \, dy \right)^{1/p}. \]

Hence, in case $m < n - \lambda - (n - \gamma - 1)/p$, we find
\[ \limsup_{|x| \to \infty, x \in D} |x|^{\lambda-n+1+(n-\gamma-1)/p} |U_2(x)| \]
\[ \leq M \left( \int_{R^{n-1-B(0,r)}} |f(y)|^p |y|^{-\gamma} \, dy \right)^{1/p}, \]
which implies by arbitrariness of $r$ that
\[ (2.3) \lim_{|x| \to \infty, x \in D} |x|^{\lambda-n+1+(n-\gamma-1)/p} U_2(x) = 0. \]

Similarly, in case $m = n - \lambda - (n - \gamma - 1)/p$, we find
\[ (2.4) \lim_{|x| \to \infty, x \in D} |x|^{\lambda-n+1+(n-\gamma-1)/p} (\log |x|)^{-1/p'} U_2(x) = 0. \]

Finally, by Lemma 2 (2) and Hölder’s inequality, we obtain
\[ |U_3(x)| \leq M x_n^{-\lambda} \int_{G_3} |f(y)| \, dy \]
\[ \leq M x_n^{-\lambda} |x|^{n-1-(n-\gamma-1)/p} \left( \int_{G_3} |f(y)|^p |y|^{-\gamma} \, dy \right)^{1/p}. \]

Hence we have
\[ (2.5) \lim_{|x| \to \infty, x \in D} x_n^{\lambda} |x|^{-n+1+(n-\gamma-1)/p} U_3(x) = 0. \]

Thus, collecting (2.1)-(2.5), we complete the Proof of Theorem 1. \qed
**Corollary 3** (cf. [14, Corollary 2.1]). Let \( f \) be a continuous function on \( \mathbb{R}^{n-1} \) satisfying (1.4) with \( \gamma > -(n-1)(p-1) \). Let
\[-1 - (n - \gamma - 1)/p < m < -(n - \gamma - 1)/p.\]
Then the function \( U_{n,m} f(x) \) satisfies
(i) \( \Delta U_{n,m} f = 0 \) in \( D \),
(ii) \( U_{n,m} f = f \) on \( \partial D \),
(iii) \( U_{n,m} f(x) = o\left(x_1^{-\gamma}|x|^{n-1-(n-\gamma-1)/p}\right) \) as \( |x| \to \infty, x \in D \).

**Proof.** We show only (iii). For \( r > 2 \) and \( x \in B(0, r) \cap D \), we write
\[ U_{n,m} f(x) = \alpha_n x_n \int_{R^{n-1} \cap B(0, 2r)} K_{n,m}(x, y) f(y) dy \]
\[ + \alpha_n x_n \int_{R^{n-1} - B(0, 2r)} K_{n,m}(x, y) f(y) dy = u_1(x) + u_2(x). \]

In view of Lemma 2 (3), we find
\[ \lim_{x \to \xi, x \in D} u_2(x) = 0 \]
for every \( \xi \in \mathbb{R}^{n-1} \cap B(0, r) \). Further,
\[ \lim_{x \to \xi, x \in D} u_1(x) = \lim_{x \to \xi, x \in D} \alpha_n x_n \int_{R^{n-1} \cap B(0, 2r)} K_n(x - y) f(y) dy = f(\xi) \]
for every \( \xi \in \mathbb{R}^{n-1} \cap B(0, r) \) (see [17]), so that (iii) follows.

### 3. Proof of Theorem 2.

As in the Proof of Theorem 1 we write
\[ U_{\lambda,m} f(x) = \alpha_n x_n \{ U_1(x) + U_2(x) + U_3(x) + U_4(x) \}. \]

By (2.1) we see that
\[ \lim_{|x| \to \infty, x \in D} x^{1-\beta}|x|^\beta + \lambda - n + (n-\gamma-1)/p U_4(x) = 0 \]
since \( 1 - \beta \geq 0 \). Moreover, by (2.2) and (2.3) we have
\[ \lim_{|x| \to \infty, x \in D} x^{1-\beta}|x|^\beta + \lambda - n + (n-\gamma-1)/p \{ U_1(x) + U_2(x) \} = 0. \]

Note that by Lemma 2 (2)
\[ x_n |U_3(x)| \leq M x_n \int_{G_3} |x - y|^{-\lambda}|f(y)| dy \]
\[ = M x_n \int_{G_3} k_{\beta, \lambda}(x, y)|f(y)| dy. \]
In view of (1.4), we can find a sequence \( \{a_i\} \) of positive numbers such that 
\[
\sum_{i=1}^{\infty} a_i \int_{D_i} |f(y)|^p |y|^{-\gamma} dy < \infty;
\]
recall \( D_i = \{ y \in \mathbb{R}^{n-1} : 2^{i-1} < |y| < 2^{i+2} \} \). Consider the sets
\[
E_i = \left\{ x \in D : 2^i \leq |x| < 2^{i+1}, x_n^{1-\beta}|U_3(x)| \geq a_i^{-1/p} 2^{-i(\beta+\lambda-n+(n-\gamma-1)/p)} \right\}
\]
for \( i = 1, 2, \ldots \). If \( x \in E_i \), then
\[
a_i^{-1/p} \leq 2^{i(\beta+\lambda-n+(n-\gamma-1)/p)} x_n^{1-\beta}|U_3(x)| \leq M 2^{i(\beta+\lambda-n+(n-\gamma-1)/p)} \int_{D_i} k_{\beta,\lambda}(x,y)|f(y)|dy,
\]
so that it follows from the definition of \( C_{k_{\beta,\lambda,p}} \) that
\[
C_{k_{\beta,\lambda,p}}(E_i; D_i) \leq M a_i 2^{i(\beta+\lambda-n+(n-\gamma-1)/p)p} \int_{D_i} |f(y)|^p dy
\]
\[
\leq M a_i 2^{i(\beta+\lambda-n)p+n-1} \int_{D_i} |f(y)|^p |y|^{-\gamma} dy.
\]
Define \( E = \bigcup_{i=1}^{\infty} E_i \). Then \( E \cap B(0, 2^{i+1}) - B(0, 2^i) = E_i \) and
\[
\sum_{i=1}^{\infty} 2^{-i(\beta+\lambda-n)p+n-1} C_{k_{\beta,\lambda,p}}(E_i; D_i) < \infty.
\]
Clearly,
\[
\lim_{|x| \to \infty, x \in D-E} x_n^{1-\beta}|x|^{\beta+\lambda-n+(n-\gamma-1)/p} U_3(x) = 0.
\]
Thus the proof of Theorem 2 is completed. \( \square \)

**Remark 2.** Suppose \( \lambda > 1 - \beta + (n-1)/p' \). Then we can find a measurable function \( f \) on \( \mathbb{R}^{n-1} \) satisfying (1.4) such that
\[
(3.1) \quad \limsup_{|x| \to \infty, x \in D} x_n^{-\beta}|x|^{\beta+\lambda-n+(n-\gamma-1)/p} U_{\lambda,m} f(x) = \infty.
\]

To show this, take a positive number \( \delta \) such that \( n-\lambda-\beta < \delta < (n-1)/p \).
Letting \( e_j = (2^j, 0, \ldots, 0) \) and \( r_j = 2^{j-1} \), we consider
\[
f(y) = \sum_{j=1}^{\infty} 2^{-j(n-\gamma-1)/p} |e_j - y|^{-\delta} \chi_{B(e_j, r_j)} \cap \mathbb{R}^{n-1}(y),
\]
where \( \chi_{E} \) denotes the characteristic function of \( E \). Then
\[
(3.2) \quad \int_{\mathbb{R}^{n-1}} f(y)^p (1 + |y|)^{-\gamma} dy \leq M \sum_{j} 2^{-j(n-1)} r_j^{-\delta p+n-1} < \infty.
\]
Moreover, if \( x \in B(e_j, r_j) \cap D \), then
\[
x_n^{-\beta} |x|^{\beta + \lambda - n + (n-\gamma -1)/p} U_{\lambda, m} f(x) \geq M x_n^{n-\lambda - \beta - \delta 2^j (\beta + \lambda - n)}.
\]
so that
\[
\lim_{x \to e_j, x \in D} x_n^{-\beta} |x|^{\beta + \lambda - n + (n-\gamma -1)/p} U_{\lambda, m} f(x) = \infty.
\]
This proves (3.1). Thus \( f \) has all the required conditions.

4. Proof of Theorem 3.

Recall that \( \lambda \geq n - 1 \), \( G \) is a bounded open set in \( \mathbb{R}^{n-1} \), \( 1 \leq p < \infty \), \( f \in L^p(G) \) and \( \chi_G \) denotes the characteristic function of \( G \).

For a proof of Theorem 3, we need some lemmas.

**Lemma 5.** Consider the function
\[
H(t) = C \left\{ \begin{array}{ll} t^{n-1-\lambda} & (\lambda > n-1), \\
\log \frac{1}{t} & (\lambda = n-1), \end{array} \right.
\]
where \( C = K_{\lambda \chi_{\mathbb{R}^{n-1}}} (e) \) with \( e = (0, \ldots, 0, 1) \) when \( n-1 < \lambda \) and \( C = (n-1) \sigma_{n-1} \) when \( \lambda = n-1 \). Then
\[
K_{\lambda \chi_G}(x) = H(x_n) + O(1) \quad \text{as } x \in D \text{ tends to } \xi \in G.
\]

**Proof.** We give a proof only when \( \lambda > n-1 \), because the case \( \lambda = n-1 \) can be treated similarly. In this case, let \( x = (x', x_n) \in D, \xi \in G \) and note that
\[
K_{\lambda \chi_G}(x) = \int_{\mathbb{R}^{n-1}} (x_n^{2} + |x' - y|^{2})^{-\lambda/2} dy + O(1) \quad \text{(as } x \to \xi \text{)}
\]
\[
= x_n^{-\lambda+n-1} \int_{\mathbb{R}^{n-1}} (1+|z|^{2})^{-\lambda/2} dz + O(1),
\]
which proves the required case. \( \square \)

For fixed \( \xi \in G \) and \( g \in L^p(G) \), write
\[
K_{\lambda} g(x) = \int_{G} K_{\lambda}(x - y) g(y) \ dy
\]
\[
= \int_{\{y \in G : |x - y| \leq 2r\}} K_{\lambda}(x - y) g(y) \ dy
\]
\[
+ \int_{\{y \in G : |x - y| > 2r\}} K_{\lambda}(x - y) g(y) \ dy
\]
\[
= I_1(x) + I_2(x),
\]
where \( x \in D \) and \( r = |x - \xi| \).
Lemma 6. Let \( g \in L^p(G) \). For \( x = (x', x_n) \in D \) and \( r = |x - \xi| \), we have
\[
|I_1(x)| \leq M x_n^{-\lambda + (n-1)/p'} \left( \int_{\{|y| \leq 2r\}} |g(y)|^p dy \right)^{1/p}.
\]

Proof. Since \(-\lambda p' + n - 1 < 0\), we have by Hölder’s inequality
\[
|I_1(x)| \leq \left( \int_{G \cap B(\xi, 2r)} \left( x_n^2 + |x' - y|^2 \right)^{-\lambda p'/2} dy \right)^{1/p'} \left( \int_{G \cap B(\xi, 2r)} |g(y)|^p dy \right)^{1/p}.
\]
\[
\leq x_n^{-\lambda + (n-1)/p'} \left( \int_{\mathbb{R}^{n-1}} (1 + |z|^2)^{-\lambda p'/2} dz \right)^{1/p'} \left( \int_{G \cap B(\xi, 2r)} |g(y)|^p dy \right)^{1/p},
\]
which implies the required inequality. \( \square \)

Note that
\[
|I_2(x)| \leq M \int_{G - B(\xi, 2r)} |\xi - y|^{-\lambda} |g(y)| \ dy.
\]

Lemma 7. If \( \lim_{t \to 0} t^{1-n} \int_{G \cap B(\xi, t)} |g(y)| dy = 0 \), then
\[
\lim_{r \to 0} \frac{1}{[H(r)]^{-1}} \int_{G \cap B(\xi, 2r)} |\xi - y|^{-\lambda} |g(y)| \ dy = 0.
\]

Proof. For \( r > 0 \), set
\[
\varepsilon(r) = \sup_{0 < t < r} t^{1-n} \int_{G \cap B(\xi, t)} |g(y)| dy;
\]
then \( \lim_{r \to 0} \varepsilon(r) = 0 \) by our assumption. Hence we have
\[
\limsup_{r \to 0} \frac{1}{[H(r)]^{-1}} \int_{G \cap B(\xi, 2r)} |\xi - y|^{-\lambda} |g(y)| \ dy = \limsup_{r \to 0} \frac{1}{[H(r)]^{-1}} \int_{B(\xi, 2r) \cap B(\xi, 2r)} |\xi - y|^{-\lambda} |g(y)| \ dy
\]
\[
\leq \limsup_{r \to 0} \frac{1}{[H(r)]^{-1}} \left( \delta^{-\lambda} \int_{G \cap B(\xi, \delta)} |g(y)| dy + \lambda \int_{2r}^{\delta} \left( \int_{G \cap B(\xi, t)} |g(y)| dy \right) t^{-\lambda - 1} dt \right)
\]
\[
\leq M \limsup_{r \to 0} \frac{1}{[H(r)]^{-1}} \varepsilon(\delta) \int_{2r}^{\delta} t^{n-1-\lambda - 1} dt
\]
\[
\leq M \varepsilon(\delta)
\]
for \( \delta > 0 \), which gives the required equality. \( \square \)
Now we are ready to prove Theorem 3.

**Proof of Theorem 3.** Letting $\xi$ be a point such that

\[
\lim_{t \to 0} t^{1-n} \int_{G \cap B(\xi,t)} |f(y) - f(\xi)|^p \, dy = 0,
\]

almost every $\xi \in G$ has this property. Note that

\[
P_\lambda f(x) - f(\xi) = K_\lambda f(x) - f(\xi) \chi_{G}(x) = \int_{G \cap B(\xi,2r)} (f(y) - f(\xi)) \chi_{G - B(\xi,2r)}(y) \, dy
\]

where

\[
K_\lambda(x) = K_\lambda(x) \chi_{G}(x) + \int_{G \cap B(\xi,2r)} (f(y) - f(\xi)) \chi_{G - B(\xi,2r)}(y) \, dy
\]

By Lemmas 5 and 6, we have

\[
|J_1(x)| \leq M \left( r^{1-n} \int_{G \cap B(\xi,2r)} |f(y) - f(\xi)|^p \, dy \right)^{1/p}
\]

for $x \in A_h(\xi)$ and small $r > 0$. Hence it follows from (4.2) that

\[
\lim_{x \to \xi, x \in A_h(\xi)} J_1(x) = 0.
\]

On the other hand, we have by Lemma 5 and (4.1)

\[
|J_2(x)| \leq M[H(r)]^{-1} \int_{G - B(\xi,2r)} |\xi - y|^{-\lambda} |f(y) - f(\xi)| \, dy
\]

for $x \in A_h(\xi)$ and small $r > 0$, so that we see that by (4.2) and Lemma 7

\[
\lim_{x \to \xi, x \in A_h(\xi)} J_2(x) = 0.
\]

Thus the Proof of Theorem 3 is completed. \qed

**Remark 3.** Let $1 \leq p < \infty$ and $f$ be a measurable function on $\mathbb{R}^{n-1}$ satisfying (1.4) for some number $\gamma$. Then, taking $m$ as in Theorem 1, we may consider the function $K_{\lambda,m}f(x)$ instead of $K_\lambda f(x)$, and see that

\[
\lim_{x \to \xi, x \in A_h(\xi)} H(|x - \xi|)^{-1} K_{\lambda,m}f(x) = f(\xi)
\]

for a.e. $\xi \in \mathbb{R}^{n-1}$. 

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A FAMILY OF TRIPLY PERIODIC COSTA SURFACES

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We derive global Weierstrass representations for complete minimal surfaces obtained by substituting the ends of the Costa surface by symmetry curves.

1. Introduction.

Among all known complete embedded minimal surfaces in $\mathbb{R}^3$ the triply periodic ones form the richest class of examples regarding variety of genus and symmetry group. For instance, the gyroid can be pointed out as a complete minimal surface containing neither straight lines nor reflectional symmetry curves. Another curiosity is the associate family from the Schwarz P-Surface till its conjugate, the D-Surface, which has the gyroid as an intermediate embedded member (see [4], p. 25). Another associate family with such a behaviour is unknown outside the triply periodic class.

The variety of triply periodic minimal surfaces has been known much earlier than most of the non-triply periodic examples, even if the existence proof of the formers took longer to be concluded. The first five examples came out in 1890 due to a work of H.A. Schwarz and his students [18]. This work inspired A.H. Schoen [17] who presented 17 other such surfaces in 1970. Later in 1989 H. Karcher [9] proved the existence of these triply periodic examples and found many others with the conjugate surface method.

In this paper we enrich this class, not only by presenting a new family of triply periodic minimal surfaces in $\mathbb{R}^3$, but also with a full study of this family which includes uniqueness and limits. Such a thorough study is rare to be found among other triply periodic examples. Moreover, in this new family the boundary contour of the conjugate surfaces patch does not project onto any convex domain. Hence, the classical Plateau approach fails for these particular surfaces and they are perhaps the simplest such examples.

In order to find new surfaces one can make use of already known examples like the Costa surface, which gave rise to the “$M_k$-Costa-Hoffman-Meeks” families [7], later generalised by D. Hoffman and H. Karcher [5]. In this present paper, once more the Costa surface turns out to be a source of new results.
2. Background.

In this section we state some well-known theorems on minimal surfaces. For details we refer the reader to [9], [12] and [14]. In this paper all surfaces are required to be regular.

**Theorem 2.1 (Weierstrass representation).** Let $S$ be a minimal surface in $\mathbb{R}^3$ and $R$ the underlying Riemann surface of $S$. Let $dh$ be a meromorphic 1-form on $R$ and $g : R \to \hat{\mathbb{C}} := \mathbb{C} \cup \infty$ a meromorphic function. Then $X : R \to \mathbb{R}^3$ given by
\[
X(p) := \frac{1}{2} \text{Re} \int^p \left( g - g^{-1}, i(g + g^{-1}), 2 \right) dh
\]
is a conformal regular minimal immersion provided the poles and zeros of $g$ coincide with the zeros of $dh$. Conversely, every regular conformal minimal immersion $X : R \to \mathbb{R}^3$ can be expressed in this form for some meromorphic function $g$ and meromorphic 1-form $dh$.

**Definition 2.2.** The pair $(g, dh)$ is the **Weierstrass data** on $R$ of the minimal immersion $X : R \to X(R) = S \subset \mathbb{R}^3$.

**Definition 2.3.** A complete, orientable minimal surface $S$ is **algebraic** if it admits a Weierstrass representation such that $R = \hat{R} - \{p_1, \ldots, p_r\}$, where $\hat{R}$ is compact and both $g$ and $dh$ extend meromorphically to $\hat{R}$.

**Definition 2.4.** An **end** of $S$ is the image of a punctured neighbourhood $V_p$ of a point $p \in \{p_1, \ldots, p_r\}$ such that $(\{p_1, \ldots, p_r\} - p) \cap \overline{V}_p = \emptyset$. The end is **embedded** if this image is embedded for a sufficiently small neighbourhood of $p$.

**Theorem 2.5.** Let $S$ be an algebraic minimal surface whose genus of $\hat{R}$ is $k$ and the number of ends is $r$ (all of them embedded). Then
\[
\text{deg}(g) = k + r - 1.
\]

**Theorem 2.6.** Let $S$ be a complete minimal surface in $\mathbb{R}^3$. Then $S$ is algebraic if, and only if, it can be obtained from a piece $S_t$ of finite total curvature by applying a finitely generated translation group $G_t$ of $\mathbb{R}^3$.

From now on we consider only algebraic surfaces. The function $g$ is the stereographic projection of the Gauß map $N : R \to S^2$ of the minimal immersion $X$. This minimal immersion is well-defined in $\mathbb{R}^3/G_t$, but is allowed to be a multivalued function in $\mathbb{R}^3$. The function $g$ is a covering map of $\hat{C}$, and hence the total curvature of $S_t$ is $-4\pi \text{deg}(g)$.

3. The Costa surface.

We describe the Costa surface, which is the starting point of our constructions. Details are found in [3] and [6].
Theorem 3.1 (The Costa surface). Let $\mathbf{R}$ be the square torus whose algebraic equation is

$$\wp'^2 = \wp(1 - \wp)(1 + \wp).$$

For some positive $\mu$, define $g = \mu \wp'$ and $dh = \wp d\wp / \wp'$. Then there exists a unique positive value of $\mu$ such that $(g, dh)$ is the Weierstrass pair on $R = \mathbf{R} - \wp^{-1}(\{-1, 1, \infty\})$ of a complete minimal embedding of $R$ in $\mathbb{R}^3$.

The next picture represents the image of the minimal embedding referred to by the previous theorem.

![Figure 1](image)

**Figure 1.** (a) The Costa surface; (b) the fundamental piece $P$ of a triply periodic Costa surface.

After a suitable rigid motion in $\mathbb{R}^3$ we can position the Costa surface in such a way that it will have the following symmetries:

\[
\begin{align*}
\sigma_1 & := (x_1, x_2, x_3) \rightarrow (-x_1, x_2, x_3); \\
\sigma_2 & := (x_1, x_2, x_3) \rightarrow (x_1, -x_2, x_3); \\
\sigma_3 & := (x_1, x_2, x_3) \rightarrow (x_2, x_1, -x_3) \quad \text{and} \\
\sigma_4 & := (x_1, x_2, x_3) \rightarrow -(x_2, x_1, x_3). 
\end{align*}
\]

Notice that $\sigma_2 = \sigma_3 \circ \sigma_1 \circ \sigma_3$ and $\sigma_4 = \sigma_1 \circ \sigma_3 \circ \sigma_1$. We call $G$ the group of symmetries of the Costa surface. In our case,

$$G = \langle \sigma_1, \sigma_3 \rangle.$$

We remark that the Costa surface is invariant under a $180^\circ$-rotation around the $x_3$-axis. This rotation can be given by $\sigma_1 \circ \sigma_2$. 

\[\]

We begin by considering the possibility of existence of triply periodic minimal surfaces in \( \mathbb{R}^3 \) fully generated by a fundamental piece \( P \) which looks like a Costa surface with its planar end replaced by symmetry curves (see Figure 1(a)). We called \( G \) the symmetry group of the Costa surface, which is isomorphic to the symmetry group of \( P \).

The main goal of this paper is then to prove the following:

**Theorem 4.1.** There exists a one-parameter family of complete embedded triply periodic minimal surfaces in \( \mathbb{R}^3 \) such that, for any member of this family the following hold:

(a) The quotient by its translation group has genus 5.
(b) The whole surface is generated by a fundamental piece \( P \), which is a surface with border in \( \mathbb{R}^3 \). The border consists of four planar curves of vertical reflectional symmetry and two planar curves of horizontal reflectional symmetry. The fundamental piece has a symmetry group isomorphic to \( G \), where \( G \) is the symmetry group of the Costa surface.
(c) By successive reflections in the border of \( P \) one obtains the triply periodic surface.

![Figure 2. The triply periodic Costa surface for \( \lambda = 0.5 \) (see below).](image)

5. The symmetries of the surface and the elliptic \( Z \)-function.

From this point on we shall make use of heuristic arguments which will just help us in the formal demonstration of Theorem 4.1. Let us consider the surface represented in Figure 2. The quotient by its translation...
A FAMILY OF TRIPLY PERIODIC COSTA SURFACES

The group generates a compact Riemann surface of genus 5 that we call \( S \) (see Figure 3(a)). Let \( \rho \) be the quotient of \( S \) by its 180°-rotational symmetry around the \( x_3 \)-axis. Then, the Euler-Poincaré characteristic of \( \rho(S) \) is given by \( \chi(\rho(S)) = \frac{\chi(S)}{2} + 4 = 0 \). Because of this, \( \rho(S) \) is a torus that we call \( T \). This torus must be rectangular because of the following argument. The horizontal reflectional symmetries of \( S \) are induced by \( \rho \) in \( T \). These turn out to be reflectional symmetries of \( T \) as well, and there are two curves which remain invariant under any of these symmetries. Then, the fixed-point set has two components and this only happens for the rectangular torus.

The surface \( S \) has two other 180°-rotational symmetries, namely the ones around the \( x_1 \) - and \( x_2 \)-axes. The torus \( T \) has these two symmetries as well. Let \( \tilde{\rho} \) be the 180°-rotational symmetry around the \( x_1 \)-axis. The quotient of \( T \) by \( \tilde{\rho} \) is conformally \( S^2 \). After we fix an identification of \( S^2 \) with \( \hat{C} \), we finally obtain an elliptic function \( Z : T \to S^2 \). In the following we are going to use the same notation as in [1], pp. 170-171, to facilitate comparisons.

**Figure 3.** (a) Half of \( S \); (b) the torus \( T = \rho(S) \).

Consider Figure 3(b) and the points of the torus \( T \) represented there. These points correspond to special points of \( S \), represented on Figure 3(a) (they were given the same names). Let \( Z : T \to S^2 \) be the elliptic function with \( Z(A) = \infty \), \( Z(M_1) = Z(M_2) = -1 \) and \( Z(B) = 0 \) (this function is the same defined in [1], p. 171). The points \( e_1 \) and \( e_2 \) correspond to the ones in [1], p. 170, but in our case they will not lead to ends of the minimal surface.

On the surface \( S \), the unitary normal vector at \( e_1, e_2, v_1, v_2 \) is vertical. That is, \( g + g^{-1} = \infty \). The other points at which \( g + g^{-1} = \infty \) are labelled...
as \( e_{1l}, e_{1f}, e_{2l}, e_{2f} \) (“\( l \)” and “\( f \)” mean “lateral” and “front”). In [1], p. 170, one sees that the vertical points of the saddles, namely \( v_1 \) and \( v_2 \), do not coincide in general with the middle points \( M_1 \) and \( M_2 \) between \( A \) and \( B \), represented in Figure 4. But, in our case, the vertical reflectional symmetries of \( \mathcal{S} \) imply \( M_2 = v_2 \) and \( M_1 = v_1 \).

Next we are going to summarize the important properties of the function \( Z \) (see Figure 4).

![Figure 4](image-url)

**Figure 4.** The torus \( T \) with values of \( Z \) at special points on it.

The function \( Z \) is real on the bold lines (and nowhere else), and \(|Z| = 1\) on the dashed lines (and nowhere else). It has exactly four branch points, marked with \( \times \) on Figure 4. Two of them are \( A \) and \( B \), where \( Z \) assumes the values \( \infty \) and \( 0 \), respectively. At the centre \( Z \) takes a value \( \lambda \in (0,1) \) and at the other branch point it takes the value \( \lambda^{-1} \). Let \( x = Z(e_{1l}) = Z(e_{2l}) \), then \( \lambda < x < 1 \). Now we write the most important values of \( Z \) together:

\[
Z(e_1) = Z(e_2) = 1, \quad Z(v_1) = Z(v_2) = -1, \quad Z(e_{1l}) = Z(e_{2l}) = x, \quad \text{and} \quad Z(e_{1f}) = Z(e_{2f}) = x^{-1}.
\]

6. The \( z \)-function on \( \mathcal{S} \) and the Gauss map in terms of \( z \).

In this section we start by studying the necessary conditions for the existence of a minimal surface like in Figure 2. They will lead to an algebraic equation for the compact Riemann surface \( \mathcal{S} \), together with Weierstrass data on it. From this point on, our problem will be concrete. We shall have to prove that the algebraic equation really corresponds to \( \mathcal{S} \) in terms of its genus and symmetries. Afterwards, we shall have to prove that the Weierstrass data really lead to a minimal embedding of \( \mathcal{S} \) in \( \mathbb{R}^3 \) with the desired properties: Symmetry curves, periodicity, etc.

Let us call \( S \) the surface represented in Figure 2 and suppose that it is a minimal immersion of \( \mathcal{S} \) in \( \mathbb{R}^3 \). In this case, we make use of the previous section and consider the functions \( \rho : \mathcal{S} \to T \) and \( Z : T \to \mathbb{C} \). Let us define
$z := Z \circ \rho$. Both functions $Z$ and $\rho$ have degree 2, then $z$ is a function on $\mathcal{S}$ of degree 4 (see Figure 5(a)).

![Diagram](image)

**Figure 5.** (a) Values of $z$ at special points; (b) the corresponding values of $g$ at these points; (c) the corresponding values of $Z$ on $T$.

We are supposing that $S$ is a minimal immersion of $\mathcal{S}$ in $\mathbb{R}^3$. In this case, the Gauss map on $S$ must lead to a meromorphic function $g$ on $\mathcal{S}$, as Figure 5(b) suggests. We are going to define multiplicity as the branch order plus one. Then, the expected correspondence between the values of $z$ and $g$ (including their multiplicities) is indicated in Figures 5(a) and 5(b). Notice that Figure 5 shows the particular case in which $g$ takes the values $\pm i$ and $\pm 1$ with multiplicity 3 at the points $z = 0$. For this special case we expect the surface to have a four-fold symmetry saddle at these points.

In the general case, we need to introduce a new parameter. Consider $y \in (-1, \lambda)$ such that $z = y$ implies $g(z) = \pm i$ with multiplicity 1. For $y = 0$ the multiplicity is 3 (see Figure 5). The $180^\circ$-rotational symmetries correspond to the map $Z \to \frac{1}{Z}$. Because of this, one has that $z = 1/y$ implies $g = \pm 1$. Based on what Figure 5 suggests and on this last remark we obtain the following relation between $g$ and $z$:

$$
(g + \frac{1}{g})^2 = \frac{cz(z-y)^2(z-\lambda^{-1})}{(z^2-1)(z-x)(z-x^{-1})},
$$

where $c$ is a real positive constant. Notice that the functions $g$ and $z$ at both sides of (1) have the same poles and zeros, including their multiplicities, while $c$, $x$, $y$ and $\lambda$ are free parameters.

Now we have a concrete problem: We must show that (1) really represents the surface $\mathcal{S}$ in terms of its genus and symmetries. This will be true if the variables $c$, $x$, $y$ and $\lambda$ satisfy certain conditions which are going to be presented soon. During our demonstration of this fact we deduce that $c = 4$. This is consistent with the expected position of the unitary normal vector at $z = \infty$ (see Figure 5(a)), which must correspond to $g = \pm 1$. 


Now we show that the function \( g \) on \( S \), defined by (1) represents the Gauss map on the symmetry lines of the surface \( S \). First of all, for the 180°-rotational symmetries of \( S \), which correspond to \( |z| \equiv 1 \) on \( S \), we want \( g^2 \) to be pure imaginary. Define \( X := x + x^{-1} \). Then (1) leads to:

\[
g^2 + g^{-2} = \frac{c(z - y)^2(1 - \lambda^{-1}z^{-1}) - 2(z - z^{-1})(z - X + z^{-1})}{(z - z^{-1})(z - X + z^{-1})}.
\]

On the right-hand side of (2) the denominator is pure imaginary. For \( g^2 + g^{-2} \) (and consequently \( g^2 \)) to be pure imaginary as well we must have

\[
\text{Im}\{c(z - y)^2(1 - \lambda^{-1}z^{-1})\} = \text{Im}\{2(z - z^{-1})(z - X + z^{-1})\}.
\]

For \( z = e^{it}, t \in \mathbb{R} \), (3) leads to:

\[
c(2 \cos t - 2y - \lambda^{-1} + \lambda^{-1}y^2) = 4(2 \cos t - X).
\]

This is possible if, and only if \( c = 4 \) and

\[
X = \frac{1 + (2\lambda - y)y}{\lambda}, \quad \text{where } X := x + x^{-1}.
\]

Recall that the variables \( x, y \) and \( \lambda \) must satisfy the following inequalities (see Figure 5(a)):

\[
-1 < y < \lambda < x < 1 \quad \text{and} \quad 0 < \lambda.
\]

We can always choose \( y \) and \( \lambda \) with \(-1 < y < \lambda < 1\) and \( 0 < \lambda \) to get the value \( x \) from (4). Nevertheless, the condition \( x < 1 \) will not always be satisfied, unless \( y > 2\lambda - 1 \) (we prove this assertion later). This will reduce the domain of \( y \). In fact,

\[
y > 2\lambda - 1 \quad \text{implies } X > 2, \quad \text{which implies } 0 < x < 1.
\]

The remaining condition from (5), namely \( \lambda < x \), is always valid for \( 0 < x < 1 \) as a consequence of (4).

At this point we have shown that, under conditions (4) and \( y > 2\lambda - 1 \), the phase of the function \( g \) in (1) is constant and equals \( \pm \pi/4 \) or \( \pm 3\pi/4 \) on the symmetry lines of the Riemann surface \( \overline{S} \) which we want to be the straight lines on the minimal surface \( S \). Now we are going to prove that the 180°-rotations around the straight lines lead to the expected change of the unitary normal vector. From Figures 5(a) and 5(b), these rotations suggest that, for example, \( g = 1 \) is mapped to \( g = \pm i \). Let us now verify this fact.

An important consequence of (4) is that it implies:

\[
4\varepsilon(z - y)^2(z - \lambda^{-1}) - 4(z^2 - 1)(z - x)(z - x^{-1}) = 4(1 - yz)^2(1 - \lambda^{-1}z).
\]

With (2), (7) and the algebraic equality

\[
(g + g^{-1})^2 - 4 = (g - g^{-1})^2,
\]
one obtains:

\[(g^2 - 1) = 4(1 - \lambda)z^2(1 - \lambda^{-1}z).\]  

The 180°-rotational symmetries around the straight lines of the surface \(S\) are represented by means of the map \(z \rightarrow 1/z\) on \(S\) (notice that the points \(|z| = 1\) remain fixed). If we calculate \(g(1/z)\) from (8), we obtain:

\[\left( g - \frac{1}{g} \right)^2 \bigg|_{z=1} = \left( \frac{1}{g} + \frac{1}{g} \right)^2 \bigg|_{z},\]

or equivalently, \(g(1/z) = \pm i\bar{g}(z)\). This means: The 180°-rotations of \(S\) lead to the following maps for the function \(g\): either \(g \rightarrow i\bar{g}\) or \(g \rightarrow -i\bar{g}\). The fixed-point set of the former is given by \(\{(z, g) : |z| = 1\text{ and } g = e^{i\pi/4}|g|\}\). For the latter, the fixed-point set is \(\{(z, g) : |z| = 1\text{ and } g = e^{-i\pi/4}|g|\}\). Both maps correspond to an inversion of the surface orientation.

Now we prove that the compact Riemann surface (8) has genus 5. From (1) and (8) we have that each value \(z \in \{-1, 1, x, x^{-1}, \lambda, \lambda^{-1}, 0, \infty\}\) represents 2 different branch points of order 1 (multiplicity 2) on the compact Riemann surface. The function \(g\) is a four-sheet-covering and because of this, from the Riemann-Hurwitz formula the genus of \(S\) is:

\[\frac{8 \cdot 2 \cdot (2 - 1)}{2} - 4 + 1 = 5.\]

We still must show the following:

**Proposition 6.1.** Given \(\lambda\) and \(y\) in the interval \((-1, 1)\) such that \(y > 2\lambda - 1\), if \(x\) and \(c\) are determined by (4) then \(x < 1\).

**Proof.** This is a consequence of the equivalence of the following statements:

(a) \(y > 2\lambda - 1\);
(b) \((1 - \lambda)^2 > (\lambda - y)^2\);
(c) \(2\lambda < 1 + (2\lambda - y)\);
(d) \(X > 2\).

\(\square\)

At this point we have proved that the correspondence between \(z\) and \(g\), given by (1) or (8), is consistent for the straight lines. Now we focus our attention on the remaining symmetry lines. Let us call \(r^2\) the left-hand side of (1). Then, for every complex value \(r\) and some branch of the square root one has:

\[g^{-1} + g = r \quad \text{implies} \quad g = \frac{r \pm \sqrt{r^2 - 4}}{2}.\]
Therefore, from (1) and based on Figure 5 we can briefly verify the values of $g$ on the planar symmetry curves as follows:

<table>
<thead>
<tr>
<th></th>
<th>$-1 &lt; z &lt; 0$</th>
<th>$r \in i\mathbb{R}$</th>
<th>$g \in i\mathbb{R}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\infty &lt; z &lt; -1$</td>
<td>$r \in (2, \infty)$</td>
<td>$g \in \mathbb{R}$</td>
</tr>
<tr>
<td>2</td>
<td>$\lambda^{-1} &lt; z &lt; \infty$</td>
<td>$r \in (0, 2)$</td>
<td>$</td>
</tr>
<tr>
<td>3</td>
<td>$1 &lt; z &lt; \lambda^{-1}$</td>
<td>$r \in i\mathbb{R}$</td>
<td>$g \in i\mathbb{R}$</td>
</tr>
<tr>
<td>4</td>
<td>$x^{-1} &lt; z &lt; x^{-1}$</td>
<td>$r \in \mathbb{R}$</td>
<td>$g \in \mathbb{R}$</td>
</tr>
<tr>
<td>5</td>
<td>$x &lt; z &lt; 1$</td>
<td>$r \in i\mathbb{R}$</td>
<td>$g \in i\mathbb{R}$</td>
</tr>
<tr>
<td>6</td>
<td>$\lambda &lt; z &lt; x$</td>
<td>$r \in (2, \infty)$</td>
<td>$g \in \mathbb{R}$</td>
</tr>
<tr>
<td>7</td>
<td>$0 &lt; z &lt; \lambda$</td>
<td>$r \in (0, 2)$</td>
<td>$</td>
</tr>
<tr>
<td>8</td>
<td>$x &lt; z &lt; \lambda$</td>
<td>$r \in i\mathbb{R}$</td>
<td>$g \in i\mathbb{R}$</td>
</tr>
</tbody>
</table>

We have just proved that the values of $g$ on all symmetry curves of the Riemann surface $\mathcal{S}$ are consistent with the expected unitary normal vector on the minimal surface $S$ in $\mathbb{R}^3$.

**7. The height differential $dh$ in terms of $z$.**

Now we need an expression for the differential form $dh$. The surface has no ends and because of this $dh$ is holomorphic. Its zeros are exactly the ones where $g = 0$ or $g = \infty$ and all have multiplicity 1 (i.e., branch order 0). If we consider the differential form $dz$, then it would be sufficient to divide it by a function on the surface whose zeros were simple at $z \in \{0, \lambda, \lambda^{-1}\}$ and with a unique pole (of multiplicity 3) at $z = \infty$. This function will turn out to be the pull-back by $\rho$ of another function, that we call $V$, on the torus $T$.

Since $0, \lambda, \lambda^{-1}$ and $\infty$ are the only branch values of $Z$, all of them of order one, then the torus $T$ can be algebraically described by the equation $V^2 = Z(Z - \lambda)(Z - \lambda^{-1})$, and $V \circ \rho$ has exactly the zeros and poles on $\mathcal{S}$ with the desired multiplicities. We can define $v := V \circ \rho$. This means that $v$ is a well-defined square root of the function $z(z - \lambda^{-1})(1 - \lambda^{-1}z)$ on $\mathcal{S}$.

Another way to see this is to observe that (1) and (8) imply:

\[
z(z - \lambda^{-1})(1 - \lambda^{-1}z) = \frac{(z^2 - 1)^2(z^2 - Xz + 1)^2}{16(z - y)^2(yz - 1)^2} \left( g + \frac{1}{g} \right)^2 \left( g - \frac{1}{g} \right)^2.
\]

Then, we can define

\[
\sqrt{z(z - \lambda^{-1})(1 - \lambda^{-1}z)} := v = \frac{(z^2 - 1)(z^2 - Xz + 1)}{4(z - y)(yz - 1)} \left( g + \frac{1}{g} \right) \left( g - \frac{1}{g} \right).
\]

Finally, we need to find a proportional constant to determine $dh$ by means of $\frac{dz}{v}$. On the straight lines of the surface, where $|z| \equiv 1$, the coordinate $x_3 = \text{Re} \int dh$ must be constant. Then $\text{Re}\{dh\}$ is zero there. Because of this
we choose the proportional constant to be 1, namely
\[ dh = \frac{dz}{v} = \frac{dz}{\sqrt{z(z - \lambda^{-1})(1 - \lambda^{-1}z)}} = \frac{\lambda^{1/2}z^{-1}dz}{\sqrt{\lambda + \lambda^{-1} - z - z^{-1}}}. \]

At this point we have reached concrete Weierstrass data \((g, dh)\) on \(S\), defined by (8) and (10), with \(x, y\) and \(\lambda\) satisfying (4) for \(y\) in the interval \((2\lambda - 1, \lambda)\). Now our task will be the demonstration of the following: Let \(S\) be the minimal immersion of \(\tilde{S}\) by these Weierstrass data. Then \(S\) leads to the desired surface represented in Figure 2. In other words, we need to show that \(S\) really has all the symmetry curves and lines of our initial assumptions and \(S\) has no other periods except the ones indicated in Figure 2. This second task will be discussed in the next section. Now we analyse the symmetries of \(S\).

From (9) and (10) we see that all \(z\)-curves listed in (9) are geodesics because \(g(z)\) is contained either in a meridian or in the equator of \(S^2\) and \(dh(\dot{z})\) is contained in a meridian of \(S^2\). Moreover, these geodesics are planar because \(\frac{dg(\dot{z})}{g(z)} \cdot dh(\dot{z}) \in \mathbb{R}\). The expected straight lines of the surface, where \(|z| \equiv 1\), come from (4) together with \(y \in (2\lambda - 1, \lambda)\). We have already proved that (4) leads to \(g/|g| = e^{\pm i\pi/4}\) on the straight lines. This means, \(|z| \equiv 1\) implies \(\frac{dg(\dot{z})}{g(z)} \cdot dh(\dot{z}) \in i\mathbb{R}\). Therefore, \(S\) has all desired symmetries.

8. Solution of the period problems.

The quotient of the triply periodic minimal surface by its translation group leads to a compact surface \(\tilde{S}\). The left half of \(\tilde{S}\) is shown in Figure 6(a). The fundamental domain for the full symmetry group of the minimal surface is the shaded region represented on Figure 6(a).

\[\text{Figure 6. (a) The left half of } \tilde{S}; \text{ (b) its image under } \rho.\]

We just need to analyse the period vector given by \(\text{Re} \oint (\phi_1, \phi_2, \phi_3)\) on the curves of the homology of \(\tilde{S}\). Let us consider the curves (1) to (8) from
Table (9) as closed curves. The curve (1) crosses the planar geodesics (2) and (8), which are in orthogonal planes. Therefore, the period on (1) is zero. The curve (3) crosses the planar geodesics (2) and (4). Thus, the period on (3) is zero. The same conclusion is valid for (4), which crosses (3) and (5). The straight lines on $S$ bring (1), (3) and (4) respectively to (2), (8) and (7). Then, the periods on these three last curves is zero as well.

Nevertheless, (5) crosses orthogonally (4) and (6), which stay in planes parallel to $x_1 = 0$. Hence, (5) has a period exactly in the $x_1$-direction. Due to the straight lines on $S$, which interchange (5) and (6), the period on (6) is exactly in the $x_2$-direction. Now consider Figure 3(b) and the line segment which contains $A$ and $B$ represented there. Its inverse image by $\rho$ is a closed curve on $S$ which crosses (3) and (8) orthogonally. Since (3) and (8) are parallel to $x_3 = 0$, the period on the curve will be exactly in the $x_3$-direction. A simple calculation can show us that the periods we are mentioning in this paragraph are not zero. But, if we show that they are the only periods of $S$, then the half-space theorem automatically guarantees that none of them is zero (see [5], p. 29).

Due to the straight lines on $S$, there remains just one curve of $S$ on which we must analyse the period vector $\text{Re} \oint \phi_1, \phi_2, \phi_3$. This curve we call $\gamma$ and it is represented on Figure 6(a). The curve $\gamma$ can be explicitly given by $\gamma(s) = s, -1 < s < \lambda$. From Figure 6(a), we see that $\gamma$ crosses orthogonally the geodesics (1) and (7), which lie in planes parallel to $x_2 = 0$. Therefore, our task is reduced to the solution of the following equality:

$$\text{Re} \int_{\gamma} \phi_2 = \text{Re} \int_{\gamma} \frac{i}{2}(g + g^{-1})dh = 0. \quad (11)$$

To interpret (11) geometrically, consider the bold curves in Figure 6(a). They are supposed to belong to the same plane, and this condition is represented by (11). The integrand $\phi_2$ has two free parameters, namely $\lambda$ and $y$. If we fix the parameter $\lambda$, we can vary the $y$-parameter and try to make (11) valid. In other words, we then get the two bold curves in the same plane. Otherwise, they remain in distinct parallel planes.

Now observe Figure 6(b). We define $\Gamma := \rho \circ \gamma$. Hence, $Z \circ \Gamma(s) = z \circ \gamma(s) = s$. We need to calculate the integrand from (11) on $\gamma(s)$. To make this task easier, we split up the curves $\gamma$ and $\Gamma$ into two pieces, one for $-1 < s < 0$ and the other for $0 < s < \lambda$. The branches of the square root need to be chosen in accordance with Figures 6(a) and 6(b). Herewith we recall that the torus $T$ can be algebraically described by

$$V^2 = Z(Z - \lambda)(Z - \lambda^{-1}). \quad (12)$$

Let us define $\gamma_1(s) = \gamma(s), \Gamma_1(s) = \Gamma(s)$ for $-1 < s < 0$ and $\gamma_2(s) = \gamma(s), \Gamma_2(s) = \Gamma(s)$ for $0 < s < \lambda$. For the stretch $-1 < s < 0$ we take
\[ s(t) = -t, 0 < t < 1. \] Then, for \( \gamma_1 \) and \( \Gamma_1 \) we have:

\[
\left( \frac{1}{g} + g \right)^2 \bigg|_{\gamma_1(s(t))} = -4i(t + y)^2(t + \lambda^{-1}) \frac{1}{(1 - t^2)(t + x)(t + x^{-1})} < 0, 0 < t < 1
\]

\[ V^2(\Gamma_1(s(t))) = t^2(-(t + t^{-1}) - (\lambda + \lambda^{-1})) < 0. \]

We need to choose the square roots with signs compatible with the choice of the integration path represented on Figures 6(a) and 6(b). From now on consider positive the square root of positive reals. Now observe the following:

\[ -1 = \frac{d}{dt}(Z(s(t))) = Z'(\Gamma_1(t)) \cdot \Gamma'_1(t). \]

This means that \( Z'(\Gamma_1(t)) \cdot \Gamma'_1(t) \) is real. From Figure 6(b) and Equations (13), our choice leads to \( i\Gamma'_1(t) > 0 \). Hence, one has \( iZ'(\Gamma_1(t)) > 0 \), namely, \( Z'(t) = -i\sqrt{t(t + \lambda)(t + \lambda^{-1})} \).

Then \( dh = -\frac{idt}{Z(t)} > 0, 0 < t < 1 \), which is in accordance with our choice indicated in Figures 6(a) and 6(b). Of course, this means that the 3rd coordinate of our minimal surfaces \( x_3 = \text{Re} \int dh \) is increasing along this path, which goes from \( z = -1 \) until \( z = 0 \) (see Figure 6(a)). But there are two different positions at which \( z = 0 \) (at the back and at the front of the half piece). The one we want is at the front, as represented in Figure 6(a). There we see that on \( \gamma_1 \) the unitary normal vector on the surface leads to \( -ig < 0 \). Then we choose our square root for \( g^{-1} + g \) at (13) in such a way that \( -i(g^{-1} + g) \circ \gamma_1 > 0 \). Hence:

\[
\phi_2(\gamma_1(s(t))) = \frac{-(t + y)dt}{\sqrt{(1 - t^2)(t + x)(t + x^{-1})(t + \lambda)}}.
\]

For \( \gamma_2 \) and \( \Gamma_2 \) we take \( s(t) = t, 0 < t < \lambda \). Therefore:

\[
\left( \frac{1}{g} + g \right)^2 \bigg|_{\gamma_2(s(t))} = 4i(t - y)^2(\lambda^{-1} - t) \frac{1}{(1 - t^2)(t - x)(t^{-1} - x)} > 0.
\]

Based on (12) and Figure 6(b) we have that \( \Gamma'_2(t) > 0 \) and \( \frac{d}{dt}(Z(t)) = 1 \) imply \( Z'(\Gamma_2(t)) > 0 \). With an analogous argument as before we choose \( g^{-1} + g < 0 \). Hence:

\[
\phi_2(\gamma(t)) = \frac{(t - y)dt}{\sqrt{(1 - t^2)(x - t)(x^{-1} - t)(\lambda^{-1} - t)}}.
\]

At this point we are ready to write down (11) as an equality between two real integrals. But first we recall (see (4)):

\[ (-2\lambda + y) = 1 - (x + x^{-1})\lambda \ implies \ y = \lambda - \sqrt{(x - \lambda)(x^{-1} - \lambda)}. \]

We shall see that the period problem can be solved for every \( \lambda \) in the interval \( (0; 0.6 + \varepsilon) \), where \( \varepsilon > 0 \). Moreover, we shall see that the solution
is unique for every fixed $\lambda$ in this interval. The theoretical value of $\varepsilon$ is not known yet, but numerical computations reach the approximate value of $\varepsilon = 0.05$. Remember that $y \in (2\lambda - 1, \lambda)$ for (5) to be valid. The following proposition is proved in [16], p. 3:

**Proposition 8.1.** For a certain positive $\varepsilon < 0.4$ one has that every $\lambda \in (0; 0.6 + \varepsilon)$ admits a unique $y_\lambda \in (2\lambda - 1, \lambda)$ such that, if $X_\lambda$ is the corresponding value of $x = x(\lambda, y)$ from (4), then $\int_0^1 \phi_2(\gamma_1(t)) + \int_0^\lambda \phi_2(\gamma_2(t)) = 0$. Or equivalently, from (14) and (15):

$$\int_0^1 \frac{(t + y_\lambda)dt}{\sqrt{(1 - t^2)(t^2 + X_\lambda t + 1)}(t + \lambda)} = \int_0^\lambda \frac{(t - y_\lambda)dt}{\sqrt{(1 - t^2)(t^2 - X_\lambda t + 1)}(\lambda - t)}.$$

In fact, it is possible to show that the period problem is unsolvable for $\lambda > 0.8$, hence $\varepsilon < 0.2$. Furthermore, one can prove the following: If $\varepsilon_0$ is the biggest value of $\varepsilon$, then the period problem is unsolvable for every $\lambda > 0.6 + \varepsilon_0$. In other words, the family of these triply periodic Costa surfaces is unique in the sense that there are no other subintervals of $(0,1)$, except $(0; 0.6 + \varepsilon_0)$, in which one can find such surfaces. These results can be found in [16].


This section is strongly based on the ideas used in [9], pp. 60-62, where the author shows a demonstration for the embeddedness of the Costa surface in $\mathbb{R}^3$. We remark that now we have an explicit definition of $S$ (given by $g$ and $dh$ on $S$ in (1)), and $S$ is represented in Figure 2 with no other periods but the ones suggested by the picture. In this section, we would like to verify if $S$ is embedded in $\mathbb{R}^3$. As we have already defined before, $\mathcal{S}$ is the quotient of $S$ by its translation group. Half of $\mathcal{S}$ is again reproduced in Figure 7(a). The shaded region indicated in this picture represents the fundamental domain of the surface, namely $(x_1, x_2, x_3) = \text{Re} \int \phi_{1,2,3} : \{z \in \mathbb{C} : |z| < 1 \text{ and } \text{Im}(z) > 0\} \rightarrow \mathbb{R}^3$.

Now define $\mathcal{A} := \{z \in \mathbb{C} : |z| < 1 \text{ and } \text{Im}(z) > 0\}$. By using the fact that $g$ is an open map of degree 4, which corresponds to the unitary normal vector on the surface, it is then easy to verify the consistency of Figure 8. Since $g(\mathcal{A})$ is contained in a half-sphere, there is a direction in which the orthogonal projection of the fundamental domain is an immersion. In our case, we even have infinitely many such directions, but the most convenient is $\vec{x}_2$. Therefore, $(x_1, x_3) : \mathcal{A} \rightarrow \mathbb{R}^2$ is an immersion. This is the first important argument for the demonstration of the embeddedness.

The orthogonal projection of $g(\mathcal{A})$ in the plane $x_2 = 0$ is represented on Figure 7(b). We want to prove that Figure 7(b) is consistent, namely, that the image curve of $z(t) = t$, $\lambda < t < x$, under the minimal immersion is well represented in Figure 7(b). Firstly, it has no intersections with the
Figure 7. (a) The left half of $S$; (b) the orthogonal projection of the fundamental domain on $x_2 = 0$.

Figure 8. (a) The half-circle $A$; (b) the corresponding image $g(A)$.

$x_3$-axis because $(x_1, x_3) : A \to \mathbb{R}^2$ is an immersion, hence open, and $A$ is precompact. Secondly, the curve is convex because $g$ varies monotonely on it.

Simple calculations show that the other curves of the contour of $(x_1, x_3)(A)$ are monotone as well. Since $(x_1, x_3)$ is an immersion which fulfils the inside of this contour, and $(x_1, x_3)(A)$ is simply connected, it follows that $(x_1, x_2, x_3) : A \to \mathbb{R}^3$ is a graph. In particular, it is an embedding. One easily shows that the fundamental domain is inside a prism in $\mathbb{R}^3$. Together with a $180^\circ$-rotation around the straight line segment $(x_1, x_3)(\partial A \setminus \mathbb{R})$, one gets an embedded piece of surface which is again inside a prism in $\mathbb{R}^3$ and whose border, now consisting only of reflectional symmetry curves, is contained in the border of the prism. By successive reflections in the border one
obtains the whole triply periodic surface without self-intersections. Hence, it is an embedding because the immersion is proper.

Our next section will be devoted to the study of some limits of the triply periodic Costa surfaces.

10. Limits of the triply periodic Costa surfaces family.

In this section we are going to analyse what happens at the extremes of the triply periodic Costa surfaces family. In [16] we show the existence of a positive $\varepsilon_0 < 0.2$ such that the family exists only for $\lambda \in (0; 0.6 + \varepsilon_0)$. Let us define $\lambda_0 = 0.6 + \varepsilon_0$. We are going to show the following: If we fix the diameter of the closed curve (1) from Table (9), the convergence $\lambda \rightarrow \lambda_0$ implies that the triply periodic Costa surfaces converge to the $M_3$-Callahan-Hoffman-Meeks’ surface in compact subsets of $\mathbb{R}^3$. If we fix the length of the straight line segments on $S$ which correspond to $|z| \equiv 1$, the convergence $\lambda \rightarrow 0$ implies that the triply periodic Costa surfaces converge to a pair of doubly periodic Scherk’s surfaces in compact subsets of $\mathbb{R}^3$.

Let us first analyse the case $\lambda \rightarrow \lambda_0$. As demonstrated in [16], pp. 11-12, the functions

$$I_1(\lambda, y) = \int_0^1 \frac{(t + y)dt}{\sqrt{(1 - t^2)(t^2 + Xt + 1)(t + \lambda)}}$$

and

$$I_2(\lambda, y) = \int_0^\lambda \frac{(t - y)dt}{\sqrt{(1 - t^2)(t^2 - Xt + 1)(\lambda - t)}}$$

are increasing and decreasing with $y$, respectively. As a matter of fact, it is a little bit more, namely: $\frac{\partial I_1}{\partial y} > 0$ and $\frac{\partial I_2}{\partial y} < 0$. Although we had restricted the $y$-domain to $(2\lambda - 1, \lambda)$, the functions $I_1$ and $I_2$ are defined for every $(\lambda, y) \in (0, 1) \times (0, 1)$. Of course, in this case we do not consider the variable $x$ with $x + x^{-1} = X$, but just $X = \frac{1 + (2\lambda - y)y}{\lambda}$, which can assume any positive value.

Now consider the function $F := (I_1 - I_2) : (0, 1) \times (0, 1) \rightarrow \mathbb{R}$. By Proposition 8.1, for every $\lambda \in (0, \lambda_0)$ there exists a unique value $y_\lambda = y(\lambda)$ in the interval $(2\lambda - 1, \lambda)$ such that $F(\lambda, y(\lambda)) = 0$. From the fact that $\frac{\partial I_1}{\partial y} > 0$ and $\frac{\partial I_2}{\partial y} < 0$, we have $\frac{\partial F}{\partial y} > 0$. Then, the *implicit function theorem* guarantees that the function $y_\lambda = y(\lambda)$ is differentiable. We can apply this theorem for $F$ at $(\lambda_0, 2\lambda_0 - 1)$. Let us define $I_j := I_j(\lambda, 2\lambda - 1), j = 1, 2$. From [16], pp. 15-16, we have that $I_1(\lambda_0) = I_2(\lambda_0)$. Therefore, $F(\lambda_0, 2\lambda_0 - 1) = I_1(\lambda_0) - I_2(\lambda_0) = 0$. By the implicit function theorem, there is a neighbourhood of $\lambda_0$ in which one can define $y = y(\lambda)$, such that $F(\lambda, y(\lambda)) = 0$ for every $\lambda$ in this neighbourhood. Due to the uniqueness of
y_{\lambda}, we conclude that:

\begin{equation}
\lim_{\lambda \to \lambda_0} y(\lambda) = 2\lambda_0 - 1.
\end{equation}

Because of (4) and (16), we conclude that \( \lambda \to \lambda_0 \) implies \( x \to 1 \). Now consider the algebraic Equation (1) of the compact Riemann surface \( \overline{S} \), which associates the Gauss map \( g \) on \( S \) and the meromorphic function \( z \):

\begin{equation}
\left( g + \frac{1}{g} \right)^2 = \frac{4z(z - y)^2(z - \lambda^{-1})}{(z^2 - 1)(z - x)(z - x^{-1})}.
\end{equation}

Let us take a compact set \( K \subset \mathbb{A} = \{ z \in \mathbb{C} : |z| \leq 1 \text{ and } \text{Im}(z) \geq 0 \} \). This set is contained in the upper half of the unitary disk in \( \mathbb{C} \), where we can consider \( z \) as a variable on it, \( g \) and \( dh/dz \) as functions of \( z \). We are going to fix the diameter of the closed curve (1) from Table (9), whose length is given by

\[ d(\lambda) = 2 \int_{z(t)} dh(\dot{z}(t)), \quad z(t) = t, -1 < t < 0. \]

Therefore, we are going to work with the Weierstrass data \((g, d^{-1}dh)\) on the compact set \( K \subset \mathbb{C} \). Choose any point \( p_0 \in K \setminus \{1\} \) and for every \( p \in K \setminus \{1\} \) an integrable curve in \( K \setminus \{1\} \) connecting \( p_0 \) and \( p \). Of course, the integrands \( \phi_j, j \in \{1, 2, 3\} \) have singularities at \( z \in \{-1, 0, \lambda\} \), but the changes of variable \( z \to 1 + z^2, z \to z^2 \) and \( z \to z^2 - \lambda \) can be used to make them bounded in \( K \). Hence, the coordinate functions \((x_1, x_2, x_3)_{\lambda}(p) = \text{Re} \int_{p_0}^p (\phi_1, \phi_2, \phi_3)\) are uniformly bounded in \( \mathbb{R}^3 \), for every \( \lambda \) in a neighbourhood of \( \lambda_0 \). We can fix this neighbourhood to be \((\frac{1}{2}, \lambda_0)\). Then, the coordinates are inside a compact subset of \( \mathbb{R}^3 \), for every \( \lambda \in (\frac{1}{2}, \lambda_0) \).

Moreover, a simple calculation shows that the convergence of the coordinate functions \((x_1, x_2, x_3)_{\lambda} \to (x_1, x_2, x_3)_{\lambda_0}\) is uniform on \( K \). Let us then analyse the special case \( \lambda = \lambda_0, y = 2\lambda_0 - 1 \) and \( x = 1 \) for (1). In this case we have:

\begin{equation}
\left( g + \frac{1}{g} \right)^2 = \frac{4z(z - 2\lambda_0 + 1)^2(z - \lambda_0^{-1})}{(z + 1)(z - 1)^3}.
\end{equation}

By the Riemann-Hurwitz formula, Equation (18) represents a compact Riemann surface of genus 3, which we call \( \overline{S}_{\lambda_0} \). Table (9) remains valid for \( \overline{S}_{\lambda_0} \) (of course, the stretches \( x < z < 1 \) and \( 1 < z < x^{-1} \) must not be considered). This means, except for these stretches, all the other symmetries of the triply periodic Costa surfaces are symmetries of \( \overline{S}_{\lambda_0} \) as well. In particular, \( \overline{S}_{\lambda_0} \) has a \( 180^\circ \)-rotational symmetry around the \( x_3 \)-axis. The left half of \( \overline{S}_{\lambda_0} \) is represented in Figure 9(a) (compare Figure 9 with Figure 5).
Figure 9. (a) The left half of $S_{\lambda_0}$ with $z = x = 1$ at infinity; (b) the torus $T = \rho(S_{\lambda_0})$.

If $\rho$ is the projection of $S_{\lambda_0}$ by the 180°-rotational symmetry around the $x_3$-axis, then the Euler-Poincaré characteristic of $\rho(S_{\lambda_0})$ is given by

$$\chi(\rho(S_{\lambda_0})) = \frac{\chi(S_{\lambda_0})}{2} + 2 = 0.$$ 

Then $\rho(S_{\lambda_0})$ is a torus that we call $T$ as well. The surface $S_{\lambda_0}$ has reflectional symmetries whose fixed point set consists of two components. After the projection $\rho$, the torus $T$ will also have these symmetries, with the fixed point set consisting of two components. Therefore, $T$ must be a rectangular torus.

Some special values of $z = Z \circ \rho$ are represented on Figure 9(a), together with their multiplicity. Now, the Gauss map $g$ and the differential holomorphic form:

$$dh = \frac{dz}{\nu} = \frac{dz}{\sqrt{z(z - \lambda^{-1}_0)(1 - \lambda^{-1}_0 z)}},$$

constitute a Weierstrass pair $(g, dh)$ on $S_{\lambda_0}$. At the point $z = 1$, on the one hand we have $g = 0$ or $g = \infty$ (both of multiplicity 3). On the other hand, $dh = 0$ at this point. This characterizes a planar end. In Section 9 we saw that every member of the triply periodic Costa surfaces family is embedded in $\mathbb{R}^3$. In the same way as in Section 9, one can prove that the minimal immersion of $S_{\lambda_0} \setminus z^{-1}(\{1\})$ in $\mathbb{R}^3$, defined by $(g, dh)$, is embedded as well. The fact that this embeddedness leads to the $M_3$-Callahan-Hoffman-Meeks’
A FAMILY OF TRIPLY PERIODIC COSTA SURFACES

Proposition 10.1. Let \( M \) be a properly embedded minimal surface in \( \mathbb{R}^3 \) satisfying the following:

1. \( M \) has an infinite number of annular ends;
2. \( M \) is invariant under a cyclic group of translations \( \tau \);
3. \( M/\tau \) has genus \( k \geq 2 \) and 2 ends;
4. \( |\text{Iso}(M/T)| \geq 4(k + 1) \).

Then, \( k \) is odd and \( M \) is the \( M_k \)-Callahan-Hoffman-Meeks’ surface.

Now we are going analyze the consistence of our results with [2]. The numeric value of \( \lambda_0 \) for our surface \( S_{\lambda_0} \) is approximately 0.65 (see Figure 12). In [2], the period problem is solved with means of another parameter whose value is approximately 0.4. We want to relate these two parameters and verify if their relation is consistent with these values.

Consider Figure 10(a) which represents the left half of \( M_3 \) divided by its vertical translation \( \tau \). The surface \( M_3 \) is invariant under the \( 180^\circ \)-rotational symmetry around the \( x_3 \)-axis, and because of the same arguments used for \( \rho(\overline{S}_{\lambda_0}) \), this torus is rectangular. We are going to represent this torus by \( T \). Since \( M_3 \) has two other \( 180^\circ \)-rotational symmetries, namely the ones around the \( x_1 \)- and \( x_2 \)-axes, the torus \( T \) has these two symmetries as well. Let \( \overline{\rho} \) be its \( 180^\circ \)-rotational symmetry around the \( x_1 \)-axis. The quotient of \( T \) by \( \overline{\rho} \) is conformally \( S^2 \). After we fix an identification of \( S^2 \) with \( \hat{\mathbb{C}} \), this will define an elliptic function \( U : T \to S^2 \).

\[
\begin{align*}
M/\tau \quad \text{(half of)} & \quad \overline{\rho} \\
\alpha = 0 & \quad \alpha = 0 \\
\epsilon_1 = -1 & \quad \epsilon_2 = -1 \\
\epsilon_1 = -1 & \quad \epsilon_2 = -1 \\
\end{align*}
\]

Figure 10. (a) The left half of \( M_3/\tau \); (b) the torus \( T = \rho(\overline{S}_{\lambda_0}) \).
Consider Figure 10(b) and the points of the torus $T$ represented there. These points correspond to special points of $M_3/\tau$, represented on Figure 10(a) (they were given the same names). Let $U : T \rightarrow S^2$ be the elliptic function with $U(A) = U(B) = 1$ and $U(v_1) = 1/U(v_2) = \alpha$, where $\alpha$ is a real value in $(0,1)$ (these functions are the same functions $z$ defined in [2], p. 501).

Next we are going to summarize some important properties of the function $U$ (see Figure 11).}

$$
\begin{array}{c|c|c|c}
 A & U(A) = 1 & U(D) = 1 & U(e_2) = -1/\alpha \\
 U(v_1) & U(C) = 1 & U(e_1) = -1 & \\
 U(B) & \times & \times & \\
 U(v_2) = \alpha & \times & \times & \\
\end{array}
$$

**Figure 11.** The torus $T$ with values of $U$ at special points on it.

The function $U$ is real on the bold lines (and nowhere else), and $|U| = 1$ on the dashed lines (and nowhere else). It has exactly four branch points, marked with $\times$ on Figure 11. At the points $C$ (centre) and $D$, $U$ takes the value $-1$. At the other branch points $e_1$ and $e_2$, it takes the values $-\alpha^{-1}$ and $-\alpha$, respectively. Now we write the most important values of $U$ together:

$$
\begin{align*}
U(A) &= U(B) = 1, \\
U(C) &= U(D) = -1, \\
U(v_2) &= 1/U(v_1) = \alpha, \quad \text{and} \\
U(e_2) &= 1/U(e_1) = -\alpha.
\end{align*}
$$

On the surface $M_3/\tau$, the unitary normal vector at $e_1, e_2, v_1, v_2$ is vertical, that is, $g(v_2) = g(e_2) = 0$ and $g(v_1) = g(e_1) = \infty$. From Theorem 2.5 we have

$$
\deg(g) = 3 + 2 - 1 = 4.
$$

Let us define the function $u := U \circ \rho$ on $M_3/\tau$. The most important values of $u$ on $M_3/\tau$ are indicated on Figure 10(a). Based on Figure 10(a), the following is the relation which must hold between the Gauss map and $u$:

$$
g^4 \approx \frac{(u - \alpha)(u + 1/\alpha)^3}{(u - 1/\alpha)(u + \alpha)^3}.
$$

This algebraic equation is the same as in [2], p. 502 (our parameter $\alpha$ and our function $u$ are represented there by “$\lambda$” and “$z$”, respectively). Since
inversion through \( |u| = 1 \) is an isometry for \( M_3 \), the proportional constant at (20) must be \( \alpha^2 \) (this is the same argument from [2], p. 503).

By the same reasons explained for \( T \), the torus \( T \) can be algebraically described by

\[
U'^2 = (U^2 - \alpha^2)(U^2 - 1/\alpha^2).
\]

Since \( M_3/\tau \) has just two ends, both planar, the height-differential \( dh \) for the Weierstrass pair \((g, dh)\) on \( M_3/\tau \) must be:

\[
dh \sim \frac{du}{U' \circ \rho}.
\]

(21)

On the straight lines of the surface, \( \text{Re} \int dh \) must be constant. On these lines, \( U' \) is imaginary and \( u \) is real. Because of this, we choose the proportional constant in (21) to be 1.

At this point, we would like to comment that there is a unique \( \alpha \in (0,1) \) which solves the period problem for \( M_3 \). This fact is not proved in [2], but in [11].

**Remark.** The tori \( T \) and \( T' \) are biholomorphic if their cross ratios are equal. In other words, if there is a Möbius transformation \( w: S^2 \to S^2 \) such that \( Z = w \circ U \). On the one hand, an algebraic equation for \( T \) is

\[
V^2 = Z(Z - \lambda_0)(Z - \lambda^{-1}_0).
\]

Therefore, the cross ratio of \( T \) is

\[
\frac{\lambda^{-1}_0 - \infty}{\lambda^{-1}_0 - 0} \cdot \frac{\lambda_0 - 0}{\lambda_0 - \infty} = \lambda^2_0.
\]

On the other hand, the cross ratio of \( T' \) is

\[
\frac{1/\alpha + 1/\alpha}{1/\alpha + \alpha} \cdot \frac{\alpha + \alpha}{\alpha + 1/\alpha} = \frac{4\alpha^2}{(1 + \alpha^2)}.
\]

Because of this, the variables \( \lambda_0 \) and \( \alpha \) must fulfil the following condition:

(22)

\[
\lambda_0 = \frac{2\alpha}{1 + \alpha^2}.
\]

From Proposition 10.1, Equations (18) and (19) lead to the \( M_3 \)-Callahan-Hoffman-Meeks’ surface. Therefore, (22) holds. In [2], p. 503, there is a graph which indicates that \( \alpha \approxeq 0.4 \). This implies that \( \lambda_0 \approxeq 0.65 \) and vice versa, which is consistent with Figure 12.

Now we are going to analyse the case \( \lambda \to 0 \). This is the last task of this section. From [16], p. 3, we have

(23)

\[
\lim_{\lambda \to 0} \mathbb{I}_1(\lambda, \lambda) = \lim_{\lambda \to 0} \mathbb{I}_2(\lambda, \lambda) = 0.
\]
From [16], p. 13, one concludes that
\[ \lim_{\lambda \to 0} I_1(\lambda, 2\lambda - 1) = -\infty \quad \text{and} \quad \lim_{\lambda \to 0} I_2(\lambda, 2\lambda - 1) = 0. \]

We have proved in [16], pp. 11-12, that \( I_1(\lambda, y) \) is increasing while \( I_2(\lambda, y) \) is decreasing with \( y \). From (23) and (24) we conclude that
\[ \lim_{\lambda \to 0} y(\lambda) = 0. \]

From (4) and (25) it follows that \( \lambda \to 0 \) implies \( X \to \infty \) and consequently \( x \to 0 \). We want to write down (1) for the special case \( \lambda = x = y = 0 \). From (4) and (25) one easily sees that \( \lambda \to 0 \) implies \( x^{-1}\lambda \to 0 \). Because of this, the case \( \lambda \to 0 \) for (1) leads to
\[ \left( g + \frac{1}{g} \right)^2 = \frac{4z^2}{z^2 - 1}, \]
which we rewrite, for instance, as
\[ g^2 = \frac{z + 1}{z - 1}. \]

Of course, we are again considering \( z \) as a variable on the compact set \( \mathcal{K} \) with \( g \) and \( dh/dz \) as functions of \( z \). We are going to fix the length of the straight line segment on \( S \) which corresponds to the image in \( \mathbb{R}^3 \) of the curve \( z(t) = e^{it}, t \in [0, \pi] \). This length is given by
\[ l(\lambda) = \sqrt{2} \left| \int_0^1 \phi_1(z(t)) \right|, \]
and it is easy to see that \( L_0 := \lim_{\lambda \to 0} \frac{l(\lambda)}{\lambda} \) is positive and finite. Recalling (1) and (10), we are going to work with the Weierstrass data \( \left( g, \frac{L_0}{l(\lambda)} dh \right) \) on the compact set \( K \subset \mathbb{C} \). From (10) we have that the differential \( \lambda^{-1} dh \) for \( \lambda \to 0 \) is:

\[
(28) \quad dh = \frac{dz}{z}.
\]

Choose any point \( p_0 \in K \setminus \mathbb{R}_+ \) and for every \( p \in K \setminus \mathbb{R}_+ \) an integrable curve in \( K \setminus \mathbb{R}_+ \) connecting \( p_0 \) and \( p \). Then, the coordinate functions \( (x_1, x_2, x_3)_\lambda(p) = \text{Re} \int_{p_0}^p (\phi_1, \phi_2, \phi_3) \) are uniformly bounded in \( \mathbb{R}^3 \), for every \( \lambda \) in a neighbourhood of 0. We can fix this neighbourhood to be \((0, \frac{1}{2})\). Then, the coordinates are inside a compact subset of \( \mathbb{R}^3 \), for every \( \lambda \in (0, \frac{1}{2}) \). A simple calculation shows that the convergence \( (x_1, x_2, x_3)_\lambda \to (x_1, x_2, x_3)_0 \) is uniform on \( K \).

Applying the Möbius transformation \( z = \frac{w + 1}{w - 1} \) we get

\[
(29) \quad g^2 = w \quad \text{and} \quad dh = \frac{-2dw}{w^2 - 1} = \frac{-4dg}{g^2 - g^{-2}},
\]

which are the the Weierstrass data on the sphere \( g^2 = w \) for the doubly periodic Scherk’s surface. This concludes our last section.

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IMMERSIONS OF SURFACES WITH BOUNDARY INTO THE PLANE

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Let $f: M \to \mathbb{R}^2$ be a smooth immersion of a compact connected oriented surface with boundary $M$ into $\mathbb{R}^2$. Kauffman defined an equivalence relation called image homotopy and classified the set of all orientation preserving immersions of $M$ into $\mathbb{R}^2$ up to image homotopy. When $M$ is of genus one and the number of boundary components is strictly greater than one, Kauffman’s result requires a correction. In this paper we will study this particular case.

1. Introduction.

Let $M$ be a compact connected oriented surface of genus $g$ with $k + 1$ boundary components $c_0, \ldots, c_k$, which are oriented as the boundary of $M$ ($g \geq 0, k \geq 0$). We choose an orientation of $M$ and oriented simple closed curves $a_1, b_1, \ldots, a_g, b_g$ as in Figure 1. We take the orientations of $a_i$ and $b_i$ so that the intersection number $b_i \cdot a_i$ in $M$ is equal to 1.

Let $f: M \to \mathbb{R}^2$ be an orientation preserving immersion and $\alpha: S^1 \to M$ be an oriented simple closed curve in $M$. We define $D_f(\alpha) \in \mathbb{Z}$ by the rotation number of $f \circ \alpha$ (see [W]). Note that $\mathbb{R}^2$ and $S^1$ are oriented counterclockwise direction.

In [K], Kauffman studied the classification of all orientation preserving immersions of $M$ into $\mathbb{R}^2$ up to regular homotopy and up to image homotopy. The definition of the image homotopy is as follows: Let $f_1, f_2: M \to \mathbb{R}^2$ be two orientation preserving immersions. We say that $f_1$ and $f_2$ are image homotopic if there exists an orientation preserving diffeomorphism $h: M \cong M$.
such that \( f_1 \) is regularly homotopic to \( f_2 \circ h \). But he incorrectly stated that as in the case where \( g \geq 2 \), the Arf-invariant and a collection of the rotation numbers of the boundary curves became a complete invariant of the image homotopy where \( g = 1, k \geq 1 \). This is because [K, Lemma 2.2] which is the formula about the sum of the rotation numbers of the boundary curves of \( M \) is incorrect. The purpose of this paper is to have a correct complete invariant of image homotopy classes where \( g = 1, k \geq 1 \).

We will obtain the required complete invariant in the following theorem:

**Theorem 1.1.** Let \( M \) be a compact connected oriented surface of genus one with \( k + 1 \) boundary components \( c_0, c_1, \ldots, c_k \), which are oriented as the boundary of \( M \) \((k \geq 1)\). Let \( f_1, f_2 : M \to \mathbb{R}^2 \) be two orientation preserving immersions and \( \mathfrak{S}_{k+1} \) be the symmetric group of degree \( k + 1 \). Then \( f_1 \) is image homotopic to \( f_2 \) if and only if:

1. \((Df_1(c_0), \ldots, Df_1(c_k)) = (Df_2(c_{\varepsilon(0)}), \ldots, Df_2(c_{\varepsilon(k)})) \) for some \( \varepsilon \) in \( \mathfrak{S}_{k+1} \), and
2. \( \gcd(Df_1(a_1), Df_1(b_1), Df_1(c_0) + 1, \ldots, Df_1(c_k) + 1) = \gcd(Df_2(a_1), Df_2(b_1), Df_2(c_0) + 1, \ldots, Df_2(c_k) + 1) \).

Here \( a_1 \) and \( b_1 \) are the oriented curves as depicted in Figure 1, and any integers are the divisors of 0.

We have that the case where \( g = 1, k = 0 \) is contained in Theorem 1.1. This complete invariant where \( g = 1 \) is quite different from the complete invariants where \( g \geq 2 \) or \( g = 0 \).

This paper is organized as follows: In Section 2, we recall some basic facts about the regular homotopy and the mapping class group of \( M \) and study the changes of rotation numbers for an immersion under the action of the mapping class group. In Section 3, we prove Theorem 1.1 using the results in Section 2. This method is the same one used in [K]. In Section 4, we will define another invariant, Arf-invariant, and give an example that the Arf-invariant is not a complete invariant in general where \( g = 1 \). This becomes a counterexample of Theorem 4.2 (iii) in [K].

Throughout the paper, all surfaces and maps are differentiable of class \( C^\infty \).

### 2. Preliminaries.

Suppose \( M \) is a surface as in Theorem 1.1. The following proposition is well-known (see [C], [K] and [W]):

**Proposition 2.1.**

1. Let \( \mathcal{R}(M) \) denote the set of regular homotopy classes of all orientation preserving immersions of \( M \) into \( \mathbb{R}^2 \) and \([f]_{\mathcal{R}}\in\mathcal{R}(M)\) denote the regular homotopy class of an orientation preserving immersion \( f : M \to \mathbb{R}^2 \).
Then we have the following one-to-one mapping \( \Phi : \mathcal{R}(M) \rightarrow \mathbb{Z}^{2+k} \) where

\[
\mathcal{R}(M) \ni [f] \mapsto \Phi([f]) = [D_f(a_1), D_f(b_1), \ldots, D_f(c_k)] \in \mathbb{Z}^{2+k}.
\]

(2) Let \( \mathcal{M}(M) \) denote the mapping class group of \( M \) and \( \mathcal{I}(M) \) denote the set of image homotopy classes of all orientation preserving immersions of \( M \) into \( \mathbb{R}^2 \). Then \( \mathcal{M}(M) \) acts on \( \mathcal{R}(M) \) by composition, and \( \mathcal{I}(M) \) is in one-to-one correspondence with the orbit space \( \mathcal{R}(M)/\mathcal{M}(M) \) as a set.

(3) Let \( f : M \rightarrow \mathbb{R}^2 \) be an orientation preserving immersion and \( \chi(M) \) denote the Euler characteristic of \( M \). Then \( \sum_{i=0}^{k} D_f(c_i) = \chi(M) \).

In order to study the set of image homotopy classes \( \mathcal{I}(M) \), let us study the mapping class group \( \mathcal{M}(M) \). The following proposition has been proved in \([B2, \text{Theorem 3}]\):

**Proposition 2.2.** Let \( a_1, b_1 \) be the oriented simple closed curves as in Figure 1 (\( g = 1 \)). Let \( \rho_i, \tau_i \), and \( \sigma_m \) be the oriented loops illustrated in Figure 2 (a) and Figure 10 in \([B1]\) (\( 0 \leq i \leq k, 0 \leq m \leq k - 1 \)). Then \( \mathcal{M}(M) \) is generated by the isotopy classes of the following maps:

- The \( \eta \)-twists \( \eta_{a_1}, \eta_{b_1} \), the \( \xi \)-twists \( \xi_{\rho_i}, \xi_{\tau_i} \), and the \( \zeta \)-twists \( \zeta_{\sigma_m} \) (\( 0 \leq i \leq k, 0 \leq m \leq k - 1 \)).

For the precise definitions of the twists, see \([B2, \text{Section 2}]\). In \([B2]\), these twists are homeomorphisms. Therefore in this paper, we deform them to diffeomorphisms by smoothing.

The following lemma was implicitly stated in \([K]\):

**Lemma 2.3.** Let \( M \) be a compact connected oriented surface of genus one with \( k+1 \) boundary components \( c_0, \ldots, c_k \), which are oriented as the boundary of \( M \) (\( k \geq 0 \)). Let \( [f] \) be the regular homotopy class of an orientation-preserving immersion \( f : M \rightarrow \mathbb{R}^2 \) and \( \Phi : \mathcal{R}(M) \rightarrow \mathbb{Z}^{2+k} \) be the one-to-one mapping in Proposition 2.1 (f). For \( \Phi([f]) = [\alpha_1, \beta_1, \gamma_1, \ldots, \gamma_k] \in \mathbb{Z}^{2+k} \), we have the following equations:

\[
\Phi([[f \circ \eta_{a_1}^\pm]] \mathcal{R}) = [\alpha_1, \mp \alpha_1 + \beta_1 | \gamma_1, \ldots, \gamma_k],
\]
\[
\Phi([[f \circ \eta_{b_1}^\pm]] \mathcal{R}) = [\alpha_1 \pm \beta_1, \beta_1 | \gamma_1, \ldots, \gamma_k],
\]
\[
\Phi([[f \circ \xi_{\rho_i}^\pm]] \mathcal{R}) = [\alpha_1, \beta_1 \pm (\gamma_i + 1) | \gamma_1, \ldots, \gamma_k],
\]
\[
\Phi([[f \circ \xi_{\tau_i}^\pm]] \mathcal{R}) = [\alpha_1 \mp (\gamma_i + 1), \beta_1 | \gamma_1, \ldots, \gamma_k] \quad (0 \leq i \leq k),
\]
\[
\Phi([[f \circ \zeta_{\sigma_m}^{\pm}]] \mathcal{R}) = [\alpha_1, \beta_1 | \gamma_i, \gamma_{i+1}, \ldots, \gamma_k] \quad (1 \leq m \leq k - 1), \quad \text{and}
\]
\[
\Phi([[f \circ \zeta_{\gamma_0}^{\pm}]] \mathcal{R}) = [\alpha_1, \beta_1 | \gamma_0, \gamma_2, \ldots, \gamma_k],
\]

where \( \gamma_0 = -\sum_{i=1}^{k} \gamma_i - (k + 1) \) and double signs in the same order.
Proof. To prove these equations, we chase the change of each of the curves \(a_1, b_1, c_0, \ldots, c_k\) under the action of the generators of the mapping class group \(\mathcal{M}(M)\) and calculate the rotation numbers of the resulting curves. They can be easily checked. □

3. Proof of Theorem 1.1.

Let \([f]_\mathcal{R}\) be the regular homotopy class of an orientation preserving immersion \(f : M \to \mathbb{R}^2\), and \(\Phi : \mathcal{R}(M) \to \mathbb{Z}^{2+k}\) be the one-to-one mapping in Proposition 2.1 (1) (\(\mathcal{R}(M)\) is the set of regular homotopy classes). Set \(\Phi([f]_\mathcal{R}) = [\alpha_1, \beta_1|\gamma_1, \ldots, \gamma_k] \in \mathbb{Z}^{2+k}\). We define the bijection \(\Psi : \mathbb{Z}^{2+k} \to \mathbb{Z}^{2+k}\) by \(\Psi[\alpha_1, \beta_1|\gamma_1, \ldots, \gamma_k] = (\alpha_1, \beta_1|\gamma_1 + 1, \ldots, \gamma_k + 1)\), and we transfer the relations of Lemma 2.3 under this map as follows:

\[
(\alpha_1, \beta_1|\gamma_1 + 1, \ldots, \gamma_k + 1)
\]

(i) \(\approx(\alpha_1 \pm \beta_1, \beta_1|\gamma_1 + 1, \ldots, \gamma_k + 1)\)

(ii) \(\approx(\alpha_1, \mp \alpha_1 + \beta_1|\gamma_1 + 1, \ldots, \gamma_k + 1)\)

(iii) \(\approx(\alpha_1 \mp (\gamma_i + 1), \beta_1|\gamma_1 + 1, \ldots, \gamma_i + 1, \ldots, \gamma_k + 1)\)

(iv) \(\approx(\alpha_1, \beta_1 \pm (\gamma_i + 1)|\gamma_1 + 1, \ldots, \gamma_i + 1, \ldots, \gamma_k + 1)\)

(v) \(\approx(\alpha_1, \beta_1|\gamma_1, \ldots, \gamma_{m+1} + 1, \gamma_m + 1, \ldots, \gamma_k + 1)\)

(vi) \(\approx(\alpha_1, \beta_1|\gamma_0 + 1, \gamma_2 + 1, \ldots, \gamma_k + 1)\)

where \(\gamma_0 = -\sum_{i=1}^k (\gamma_i + 1)\) and for \(x, y\) in \(\mathcal{R}(M)\), \(\Psi \circ \Phi(x) \approx \Psi \circ \Phi(y)\) means that \(x\) and \(y\) represent image homotopic immersions. From Proposition 2.1 (2) and Proposition 2.2, we have that two elements in \(\mathcal{R}(M)\) represent the same element in the set of image homotopy classes if and only if they change each other by a finite iteration of (i)-(vi).

First, suppose that two orientation preserving immersions \(f_1\) and \(f_2\) are image homotopic. It is easy to see that Conditions (1) and (2) of Theorem 1.1 are the invariants of an image homotopy class.

Next, suppose that Conditions (1) and (2) of Theorem 1.1 are satisfied. Let \([f_1]_\mathcal{R}\) and \([f_2]_\mathcal{R}\) be the regular homotopy classes and set \(\Phi([f_j]_\mathcal{R}) = [\alpha_j^1, \beta_j^1|\gamma_j^1, \ldots, \gamma_j^k]\) where \(\gamma_j^0 = -\sum_{i=1}^k (\gamma_j^i + 1)\) and \(j = 1, 2\). By Theorem 1.1 (1) and Lemma 2.3, we may assume that \(\gamma_j^1 = \gamma_j^2 = \gamma_i\) (0 ≤ \(i\) ≤ \(k\)).

We use the above relations to \(\Psi \circ \Phi([f_j]_\mathcal{R})\). By (i), (ii), (iii), and (iv), we have
\[(\alpha^j_1, \beta^j_1 | \gamma_1 + 1, \ldots, \gamma_k + 1) \approx (\delta^j_1, 0 | \gamma_1 + 1, \ldots, \gamma_k + 1) \approx (\delta^j_2, 0 | \gamma_1 + 1, \ldots, \gamma_k + 1)\]

where \(\delta^j_1 = \gcd(\alpha^j_1, \beta^j_1)\), \(\delta^j_2 = \gcd(\delta^j_1, \gamma_0 + 1, \ldots, \gamma_k + 1) = \gcd(\alpha^j_1, \beta^j_1, \gamma_0 + 1, \ldots, \gamma_k + 1)\) and \(j = 1, 2\). Then by Condition (2) of Theorem 1.1, we have \(\Psi \circ \Phi(f_1|R) \approx \Psi \circ \Phi(f_2|R)\). This completes the proof. \(\square\)

**Remark 3.1.** If \(k = 0\), we have \(D_f(c_0) = -1\) for any orientation preserving immersion \(f : M \to \mathbb{R}^2\). Thus, when we apply the above proof, Theorem 1.1 also holds in the case where \(k = 0\).

### 4. Supplement.

We define another invariant of image homotopy classes. Let \(f : M \to \mathbb{R}^2\) be an orientation preserving immersion. If all \(D_f(c_i)\) are odd \((0 \leq i \leq k)\), the definition of the Arf-invariant is \(A(f) : \equiv (D_f(a_1) + 1)(D_f(b_1) + 1) \pmod{2}\). On the other hand, if there exists \(c_i\) with \(D_f(c_i) \equiv 0 \pmod{2}\), we cannot define the Arf-invariant (see [KB]).

The following example shows that the Arf-invariant is not a complete invariant of image homotopy classes in general even if there is no \(c_i\) with \(D_f(c_i) \equiv 0 \pmod{2}\).

**Example 4.1.** Assume that \(k = 1\) and \((D_f_j(c_0), D_f_j(c_1)) = (-5, 3)\) \((j = 1, 2)\). Set \(\Phi([f_1|R]) = [2, 0|3] \) and \(\Phi([f_2|R]) = [4, 0|3] \) (\(\Phi\) is the map in Proposition 2.1 and see Figure 2). From Theorem 1.1, \(f_1\) and \(f_2\) are not image homotopic. But their Arf-invariants are equal to 1, we have that the Arf-invariant is not a complete invariant in this case.

![Figure 2.](image-url)
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CONTRACTION CRITERIA FOR REDUCIBLE RATIONAL CURVES WITH COMPONENTS OF LENGTH ONE IN SMOOTH COMPLEX THREEFOLDS

Tom Zerger

Let $X$ be a smooth complex threefold and $C$ a linear chain of $n$ smooth rational curves in $X$, each intersecting the canonical sheaf $\mathcal{K}_X$ trivially, and each having length 1, where the length is Kollár’s invariant. Formal criteria will be given to determine when $C$ contracts, when $C$ deforms, and when $C$ neither contracts or deforms in $\hat{X}$, the formal completion of $X$. It is shown precisely, using the curve $C$, its components, and their defining ideals, how the behavior of $C$ coincides with the deformation theory of the compound $A_n$ singularity.

1. Introduction.

Let $C = \bigcup_{i=1}^{n} C_i$ be a linear chain of $n$ smooth rational curves $C_i$ of length 1 in a smooth complex threefold $X$. Furthermore, assume $\mathcal{K}_X \cdot C_i = 0$ for each $i$ and that each $C_i$ has a rational formal neighborhood in $X$.

The main purpose of this paper is to describe explicitly the deformations of $C$ in $\hat{X}$, the formal completion of $X$. Formal criteria will be given to determine when $C$ contracts formally and when $C$ deforms formally in $\hat{X}$. Before contraction criteria can be stated it is important to make clear what is meant by a formal contraction and a formal deformation. In particular, the definitions of formal cDV modification and formal cDV contraction will be made precise. These definitions are motivated by the deformations of DuVal singularities and by the formal constructions that were utilized by Reid [14] and Jiménez [6] in determining contraction criteria for a single smooth rational curve; Reid considered length 1 curves and Jiménez curves of length greater than 1.

Assume $f : X \to Y$ of $C = \bigcup_{i=1}^{n} C_i$ is an analytic contraction with $f(C) = q$ so that $f : X \setminus C \to Y \setminus q$ is an isomorphism. Let $m_{q,Y}$ be the maximal ideal at the point $q \in Y$. 

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Definition 1.1 ([3, Kollár, p. 95]). Let $f : X \to Y$ contract $C$ to the point $q \in Y$. The length of the component $C_i$ is the length of the scheme with structure sheaf $\mathcal{O}_X/f^{-1}(m_q,Y)$ at a generic point of $C_i$.

Now $C$ is a closed subscheme of $X$ and $q$ is a closed subscheme of $Y$, and, therefore, $\hat{X}$ is supported on $C$ and $\hat{Y}$ is supported on $q$. Reid [14] has shown that $q$ is a compound DuVal (cDV) singularity, that is, a general hyperplane section $Y_0$ of $q$ has a DuVal singularity at $q$. Therefore, $q$ is a singularity of type $cA_n$ ($n \geq 1$), $cD_n$ ($n \geq 4$), $cE_6$, $cE_7$, or $cE_8$. Furthermore, the induced map on surfaces $X_0 = f^*Y_0 \to Y_0$ is a factor of the minimal resolution of $q \in Y_0$.

Definition 1.2. A formal cDV modification consists of a map $\hat{f} : \hat{X} \to \hat{Y}$ of formal threefolds, with $\hat{X}$ supported on a curve $C$ and $\hat{Y}$ supported on a point $q$, such that a general section $s \in m_q$, the maximal ideal at $q$, defines a formal DuVal surface singularity, while $\hat{f}^{-1}(s)$ defines a formal partial resolution.

The formal length of $C_i$ is analogous to Definition 1.1 with $\hat{f}$ a formal cDV modification. Certainly what is preferred is that a formal cDV modification is equivalent to a formal modification as defined by Artin [2, Defn. 1.7], because Artin shows in [2, Thrm. 3.1] and [1, Thrm. 6.2] that the existence of a formal modification implies $C$ contracts or deforms in the analytic category. At the end of this section, after more evidence is acquired, it will be conjectured that these two definitions are equivalent, at least for the curve considered in this paper.

From the definition of a formal cDV modification and the semi-universal property of deformations of DuVal singularities and their simultaneous partial resolutions, a formal cDV contraction can be defined. This theory has been described by Tyurina [15], Kas [7], Pinkham [13], Reid [14], as well as Katz and Morrison [8]. All of the works cited are in the analytic category, so for ease in referring to these, the general brief discussion following and the defining equations given explicitly in Section 2.2 for the $A_n$ singularity, which mainly follows the notation in [8], will be in the analytic category. However, by taking formal completions of these spaces, analogous results are obtained in the formal category for formal DuVal singularities. When necessary for clarity, these formal completions will be described in more detail.

Let $(\pi, Y, \Sigma)$ and $(\sigma, X, T)$ be the semi-universal families of the deformations of $Y_0$ and the partial resolution $X_0$, respectively. It is known, see [15], that from the defining equation for the singular space $Y_0$ the threefold $Y$ can be viewed as the total space of a one-parameter family $\{Y_t\}$. Therefore, there is a map $h : \Delta_t \to \Sigma$ that classifies $Y$. The space $X$, then, can be viewed as the corresponding one-parameter family $\{X_t\}$ described by $g : \Delta_t \to T$ in the diagram below.
More precisely, $X = g^*(\mathcal{X})$ and $Y = h^*(\mathcal{Y})$. The curve $C$ is the exceptional set in $\mathcal{X}$ that lies over the discriminant in the base space $T$. Determining if $C$ contracts or components of $C$ deform is accomplished by looking at the discriminant locus of $T$ and observing when $g$ factors through.

If $\hat{f} : \hat{X} \to \hat{Y}$ is a formal cDV modification, then $\{s = t\}$ gives a one-parameter family of hypersurfaces defining a formal deformation of the singular space, $\hat{Y}_0$, given by $s = 0$. Therefore, the inverse image of this family under $\hat{f}$ is a formal partial resolution of singularities. The semi-universal property of deformations of DuVal singularities and their partial resolutions shows that $\hat{f} : \hat{X} \to \hat{Y}$ is isomorphic to the induced map $g^*(\hat{\mathcal{X}}) \to h^*(\hat{\mathcal{Y}})$, where $\hat{\mathcal{X}}$ and $\hat{\mathcal{Y}}$ are the semi-universal families of the formal deformation of $X_0$ and $Y_0$, respectively. Taking formal completions of the spaces $\Delta_t$ and $T$ we have the following situation:

Definition 1.3. $\hat{f} : \hat{X} \to \hat{Y}$ is a formal cDV contraction if $\hat{f} : \hat{X} \to \hat{Y}$ is a formal cDV modification such that the general section $s \in m_q$ defines a map $\hat{g} : \Delta_t \to \hat{T}$ which does not factor through the inclusion of the discriminant locus in $\hat{T}$.

With these definitions of formal cDV modification and formal cDV contraction, formal results have been established that are analogous to Artin’s results in the analytic case.

Proposition 1.4. If $\hat{f} : \hat{X} \to \hat{Y}$ is a formal cDV modification, then $\hat{f}$ is either a formal cDV contraction, or some component of $C$ has a formal deformation.

Proof. As $C$ lies over the discriminant locus, the components of $C$ that deform can be determined from its locally closed subsets. Over each subset is a flat family of deformations of some corresponding subset of $C$. So, if $\hat{g}$ factors through, then pulling back the flat family to $\hat{\mathcal{X}}$ is the formal deformation of this component of $C$. 

Figure 1.
If \( \hat{g} \) does not factor through the discriminant locus, then by definition \( \hat{f} \) is a formal cDV contraction. □

**Conjecture 1.5.** If \( \hat{f} : \hat{X} \to \hat{Y} \) is a formal cDV modification, then \( \hat{f} \) is a formal modification in the sense of Artin [2, Defn. 1.7].

With these definitions established, contraction criteria can be established. To know which cDV deformation space to utilize it must first be known which cDV singularity results if \( C \) contracts. This analytic result is immediate from the construction of the formal cA\(_n\) modification \( \hat{f} \) in Section 3, which utilizes the conormal sheaf on \( C \) and its restrictions to the components \( C_i \).

**Theorem 1.6.** If \( f : X \to Y \) is an analytic contraction map with \( f(C) = q \) and \( C = \bigcup_{i=1}^{n} C_i \) with all components having length 1, then a general hyperplane section of \( q \) has an A\(_n\) type singularity at \( q \).

Section 4, then, explicitly determines when \( C \) deforms formally in \( \hat{X} \) and when a formal cDV contraction exists. More precisely, a method similar to Reid’s “Pagoda” construction in [14] shows that the contractibility of \( C \) can be detected in its higher order neighborhoods. The construction results in a sequence of defining ideals

\[
K_m \subset K_{m-1} \subset \cdots \subset K_2 \subset K_1 = \mathcal{I}
\]

where \( \mathcal{I} \) is the reduced ideal sheaf of \( C \) in \( X \) and

\[
0 \longrightarrow \mathcal{O}_C \longrightarrow K_i/\mathcal{I}K_i \longrightarrow \omega_C^* \longrightarrow 0
\]

is exact for each \( m \leq i \leq 1 \) with \( \omega_C^* \) the dual of the dualizing sheaf on \( C \). This sequence of ideals can be extended if and only if the the exact sequence splits. Comparing the construction of this sequence of ideals defining \( C \) with the semi-universal deformation of the A\(_n\) singularity at \( q \), the following two theorems are proved:

**Theorem 1.7.** \( C \) deforms formally in \( X \) if and only if there exists an infinite chain of subsheaves \( \cdots \subset K_{m+1} \subset K_m \subset \cdots \subset K_2 \subset \mathcal{I} \) such that \( K_m/K_{m+1} \cong \mathcal{O}_C \) and \( K_{m+1}/\mathcal{I}K_m \cong \omega_C^* \), where \( \omega_C^* \) is the dual of the dualizing sheaf.

**Theorem 1.8.** A formal cA\(_n\) contraction of \( C \) exists if and only if there is no infinite chain of subsheaves \( \cdots \subset K_{m+1} \subset K_m \subset \cdots \subset K_2 \subset \mathcal{I}_D \) satisfying \( K_m/K_{m+1} \cong \mathcal{O}_D \) and \( K_{m+1}/\mathcal{I}K_m \cong \omega_D^* \) for any \( D = \bigcup_{j=1}^{k} C_j \) (\( 1 \leq i \leq k \leq n \)), where \( \mathcal{I}_D \) is the ideal sheaf of \( D \) in \( X \) and \( \omega_D^* \) is the dual of the dualizing sheaf.

Some immediate consequences of these theorems and the construction involved is that not all chains of (1,1) curves will contract, though each
component could be contracted separately (see Example 4.13). Also, unlike the single component case, it is not true that a chain of length 1 curves will either contract or deform.

2. Preliminaries: Length and equations for deformations of $A_n$ singularities.

In Section 2.1, we briefly discuss the significant role that the length of the components of $C$ plays in determining the higher order neighborhoods of $C$. Section 2.2 gives a more precise discussion of the semi-universal deformations of $A_n$ singularities and their simultaneous resolutions than in Section 1. See Figure 1 in Section 1 in reference to the maps and spaces discussed. Again, the notation utilized here is most similar to that in [8].

2.1. Length. Let $\mathcal{I}_i/\mathcal{I}_i^2$ be the conormal sheaf of $C_i$ in $X$, which is locally free of rank two, so $\mathcal{I}_i/\mathcal{I}_i^2 \cong \mathcal{O}_{C_i}(a) \oplus \mathcal{O}_{C_i}(b)$ for integers $a$ and $b$. It will be written that $C_i$ is a curve of type $(a,b)$. The assumption that $K_X \cdot C_i = 0$ implies, by adjunction, that $a + b = 2$, and the assumption that $C$ has a rational formal neighborhood in $X$, $H^1(X, \mathcal{O}_X) = 0$, implies each $C_i$ is a curve of type $(1,1), (0,2)$ or $(-1,3)$. It is known, see [3, page 95], that since each component has length 1, they are all curves of type $(1,1)$ or $(0,2)$. As in the work of Jiménez in [6], it is because of the existence of a projection to $\mathcal{O}(-1)$ that the $(-1,3)$ curve has length greater than 1.

As the methods in this paper are for a curve $C$ with multiple components, it is necessary to confirm that the existence of $\mathcal{O}_i(-1)$ factors has a similar effect on the length of each component curve $C_i$. Assume there is a finite sequence of defining ideals $\mathcal{I} = \mathcal{J}_1 \supset \cdots \supset \mathcal{J}_k$ with $\mathcal{J}_l/\mathcal{J}_{l+1} \cong \mathcal{O}_{m(l)}(-1)$ for all $1 \leq l \leq k-1$ and some $m(l) \in \{1, 2, \ldots, n\}$, and assume that the sequence cannot be extended to $\mathcal{J}_{k+1}$. If $\mathcal{J}_k/\mathcal{J}_k^2$ is generated by global sections and these can be lifted to global sections of $\hat{\mathcal{J}}_k$, then, as in [6], they define a formal cDV modification $\hat{f} : \hat{X} \to \hat{Y}$ with $\hat{f}^{-1}(m_q) = \hat{\mathcal{J}}_k$. The significance of all quotients being $\mathcal{O}_{m(l)}(-1)$ for some $m(l)$ is that $\mathcal{H}^0(\mathcal{J}_k) = \mathcal{H}^0(\mathcal{I})$. By definition, the formal length of a component $C_i$ is the length of $\mathcal{O}_X/\hat{\mathcal{J}}_k$ at a generic point of $C_i$. Let $p \in C_i$ with $p \notin C_j$ for $j \neq i$ be a generic point. We have $\mathcal{J}_t = \mathcal{J}_{t+1}$ unless $\mathcal{J}_i/\mathcal{J}_{i+1} \cong \mathcal{O}_i(-1)$. In this case, then, there is a longest subsequence $\mathcal{I} = \mathcal{J}_1 \supset \cdots \supset \mathcal{J}_l$, such that $\mathcal{J}_{t+1} \cong \mathcal{O}_i(-1)$ for $1 \leq t \leq j - 1$. The length of $C_i$ is the length of $\mathcal{O}_X/\mathcal{J}_j$, which is $l_j$. In conclusion, this means that if we have a sequence $\mathcal{I} = \mathcal{J}_1 \supset \cdots \supset \mathcal{J}_l$ and there is an ideal $\mathcal{J}_{l+1} \subset \mathcal{J}_l$ such that $\mathcal{J}_l/\mathcal{J}_{l+1} \cong \mathcal{O}_i(-1)$, then the formal length of the component $C_i$ increases by 1.

2.2. Equations for deformations of $A_n$ singularities. We are interested in the case where $Y_0$ is a singular surface having an $A_n$ singularity. Near the singularity with coordinates $\{x, y, z\}$, $Y_0$ is the hypersurface in
$\mathbb{C}^3(x, y, z)$ defined by the equation $-xy + z^{n+1}$. The analytic space $\mathcal{Y}$ is defined as the hypersurface in $\mathbb{C}^3(x, y, z) \times \mathbb{C}^n(\sigma_1, \ldots, \sigma_n)$ defined by

$$-xy + z^{n+1} + \sigma_1z^{n-1} + \cdots + \sigma_{n-1}z + \sigma_n = 0.$$ 

The base space is $\Sigma = \mathbb{C}^n(\sigma_1, \ldots, \sigma_n)$ and the map $\pi : \mathcal{Y} \to \Sigma$ is the map induced by projection.

The resolution corresponding to the semi-universal family can also be explicitly described. Let $T$ be the hyperplane in $\mathbb{C}^{n+1}(t_1, \ldots, t_{n+1})$ defined by $\sum_{i=1}^{n+1} t_i = 0$. The map on the base spaces, $\phi : T \to \Sigma$, is defined by $\sigma_i = $ the $(i+1)$st symmetric polynomial in the $t_i$. Notice that by definition $\sigma_0 = \sum_{i=1}^{n+1} t_i = 0$. The smooth deformation $\sigma : V \to T$ induced by $\phi$ is defined in $\mathbb{C}^3(x, y, z) \times \mathbb{C}^{n+1}(t_1, \ldots, t_{n+1})$ by the equations

$$\sum_{i=1}^{n+1} t_i = 0, \quad -xy + \prod_{i=1}^{n+1} (z + t_i) = 0.$$ 

Now, define a mapping $V \to (\mathbb{P}^1)^n$ by

$$(x, y, z, t_1, \ldots, t_{n+1}) \to \left\{ x, \prod_{j=1}^{i} (z + t_j) \right\}_{i}$$

for $i = 1, \ldots, n$. The analytic space $\mathcal{X}$, then, is defined to be the closure of the graph of this map, and the mapping $\sigma : \mathcal{X} \to T$ is defined by projection. If $(u_k, v_k)$ are the homogeneous coordinates on the $k$th $\mathbb{P}^1$ from the resolution, then the equations defining $\mathcal{X}$ are

$$-xy + \prod_{i=1}^{n+1} (z + t_i) = 0,$$ 

$$xv_j = u_j \prod_{i=1}^{j} (z + t_i) \quad (1 \leq j \leq n),$$ 

$$\prod_{i=k+1}^{j} (z + t_i)u_jv_k = u_kv_j \quad (1 \leq k < j \leq n).$$

From these equations it has been shown that $C$ is the exceptional set of the fiber of $\mathcal{X}$ over $(t_1, t_2, \ldots, t_{n+1}) = \vec{0}$ and the component $C_i$ is defined by $x = y = z = 0$, $u_j = 0$ for $j < i$ and $v_k = 0$ for $k > i$. Furthermore, the curve $C_i + C_{i+1} + \cdots + C_j$ deforms when $t_i = t_{j+1}$.

Since $X$ is being viewed as the space of a one-parameter family of deformations of a resolution of an $A_n$ singularity, $X$ is recovered, as described by Pinkham in [13], from $g : \Delta_t \to T$. The coordinates $t_i$ of $T$, then, can be expressed as functions of $t$ vanishing at $t = 0$. Let $g_i(t) = t_i$ under this parameterization, where the $g_i$ are holomorphic functions on a neighborhood
These functions have a power series expansion near $t = 0$. Let
\[
g_i(t) = \sum_{j=1}^{\infty} a_{ij} t^j \quad (1 \leq i \leq n + 1).
\]

Note that for our situation, we can only assume that the $g_i$ are formal functions. Pulling back $X$ via $g$, $X$ can now be described from the equations defining the resolution with coordinates $\{x, y, z, (u_i, v_i), t\}$. In particular, we will be interested in defining $X$ near an intersection point of two components, $C_i$ and $C_i+1$ with $1 \leq i \leq n-1$. Now $C_i \cong \mathbb{P}^1(u_i, v_i)$, $C_i+1 \cong \mathbb{P}^1(u_{i+1}, v_{i+1})$ and $X$ is defined near this point of intersection by the transition functions on the coordinate patches $(u_{i-1}, v_i, t)$, $(u_i, v_{i+1}, t)$ and $(u_{i+1}, v_{i+2}, t)$, with the intersection point being in the coordinate patch $(u_i, v_{i+1}, t)$. These transition functions are:
\[
\begin{align*}
  u_{i-1} &= u_i^2 v_{i+1} + u_i(g_i(t) - g_{i+1}(t)) \\
  v_i &= 1/u_i \\
  t &= t \\
  u_{i+1} &= 1/v_{i+1} \\
  v_{i+2} &= v_{i+1}^2 u_i + v_{i+1}(g_{i+2}(t) - g_{i+1}(t)) \\
  t &= t
\end{align*}
\]
with the convention that if $i = 1$ then $u_{i-1} = x$, and if $i = n - 1$ then $v_{i+2} = y$.

Since the deformation of $\bigcup_{i=j}^{k} C_i$ occurs when $t_j = t_{k+1}$, the deformation of this curve is determined by whether it coincides with the discriminant locus of $g$ in $T$, which is $g_j(t) = g_{k+1}(t)$. In particular, the whole curve $C$ deforms when $g_1(t) = g_{n+1}(t)$. By viewing these functions as formal functions, we have:

**Theorem 2.1.** $C = \bigcup_{i=1}^{n} C_i$ deforms formally in $\hat{X}$ if and only if $g_1(t) = g_{n+1}(t)$.

**Theorem 2.2.** $C = \bigcup_{i=1}^{n} C_i$ can be contracted via a formal cDV contraction if and only if $g_i(t) \neq g_j(t)$ for any $1 \leq i < j \leq n + 1$.

These results and the transition functions defining $X$, Equation (2), will be utilized in Section 4.

### 3. The singularity $q$.

As mentioned above, in this case it is necessarily assumed that $I_i/I_i^2 = (1, 1)$ or $(0, 2)$ for all $i$. We will identify the rational double point by investigating...
the general section of \( \tilde{I} \), which is obtained by lifting it from the globally generated conormal sheaf, \( \mathcal{I}/\mathcal{I}^2 \), on \( C \).

The exact sequence of sheaves obtained by restricting the conormal sheaf of \( C \) to the component \( C_i \),

\[
0 \longrightarrow \mathcal{I}_i\mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{I}/\mathcal{I}_i \longrightarrow 0,
\]

and its long exact cohomology sequence allow us to determine this global information.

The defining ideal sheaves for the component curves can be defined in local coordinates \( \{x, y, z\} \) at the point of intersection \( p = (0, 0, 0) \) of \( C_i \) and \( C_j \) by \( \mathcal{I}_i = (x, z) \), \( \mathcal{I}_j = (y, z) \) and \( \mathcal{I} = (xy, z) \). Then \( \mathcal{I}/\mathcal{I}^2 \big|_{C_i} = \mathcal{I}/\mathcal{I}_i \mathcal{I} \) is locally free of rank 2 on \( C_i \), generated by \( \{xy, z\} \) at a point of intersection. \( \mathcal{I}_i/\mathcal{I}_i^2 \) and \( \mathcal{I}_j/\mathcal{I}_j^2 \) are generated by \( \{x, z\} \) and \( \{y, z\} \), respectively, and \( \mathcal{I}_i/\mathcal{I}_i^2 \) is locally free of rank 2 on \( \bigcup_{j \neq i} C_j \). The map on generators of the inclusion map \( \mathcal{I}/\mathcal{I}_i \mathcal{I} \hookrightarrow \mathcal{I}_i/\mathcal{I}_i^2 \) at a point of intersection is given by \( xy \mapsto y \cdot x \), \( z \mapsto z \), with \( y \) a local coordinate on \( C_i \). The determinant of this map, then, vanishes to order one at each point of intersection. Since \( \mathcal{I}_i/\mathcal{I}_i^2 \) has degree two, \( \mathcal{I}/\mathcal{I}_i \mathcal{I} \) has degree 0 if there are two points of intersection and degree 1 if there is just one. We have,

\[
\mathcal{I}/\mathcal{I}_i \mathcal{I} = \begin{cases} 
(0,0), (-1,1), or (-2,2) & \text{if } 2 \leq i \leq n - 1 \\
(0,1) or (-1,2) & \text{if } i = 1 or n.
\end{cases}
\]

But, since each component has a rational formal neighborhood there can be no \( \mathcal{O}(-2) \) factors and since it is assumed that each component has length 1, there can be no \( \mathcal{O}(-1) \) factors, as discussed in Section 2.1. Therefore, it is necessary that \( \mathcal{I}/\mathcal{I}_i \mathcal{I} = (0, 0) \) for \( 2 \leq i \leq n - 1 \) and \( (0, 1) \) for \( i = 1, n \).

The exact sequence, from the normalization of \( C \), is

\[
0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \bigoplus_{i=1}^{n} \mathcal{I}/\mathcal{I}^2{}_{|C_i} \longrightarrow \bigoplus_{p \in C_i \cap C_j} \mathcal{I}/\mathcal{I}^2{}_{|p} \longrightarrow 0.
\]

That is, the middle term in the sequence is \( \pi_*\pi^*\mathcal{F} \), where \( \pi : \tilde{C} \to C \) is the normalization of \( C \) such that \( \tilde{C} \) is the disjoint union of the smooth component curves. Therefore, the first map is an isomorphism away from the intersection points and the second map is defined by \( (s_i) \mapsto \bigoplus_{p \in C_i \cap C_j} ((s_i)_p - (s_j)_p) \).
Since all of the rank 2 sheaves $\mathcal{I}/\mathcal{I}^2|_{C_i} \cong \mathcal{I}/\mathcal{I}\mathcal{I}$ are all $(0,0)$ or $(0,1)$, the group $H^1(\bigoplus_{i=1}^n \mathcal{I}/\mathcal{I}^2|_{C_i})$ is the trivial group and

$$0 \to H^0(\mathcal{I}/\mathcal{I}^2) \to H^0\left(\bigoplus_{i=1}^n \mathcal{I}/\mathcal{I}^2|_{C_i}\right) \to H^0 \left(\bigoplus_{p \in C_i \cap C_j} \mathcal{I}/\mathcal{I}^2|_p\right) \to H^1(\mathcal{I}/\mathcal{I}^2) \to 0,$$

is the resulting cohomology exact sequence.

Now, $\mathcal{I}/\mathcal{I}^2|_p$ is a two dimensional vector space, as it is a rank two vector bundle over a point $p$, and $\mathcal{I}/\mathcal{I}^2|_{C_i}$ is locally free of rank two on $C_i$, so $H^0\left(\bigoplus_{i=1}^n \mathcal{I}/\mathcal{I}^2|_{C_i}\right) \to H^0 \left(\bigoplus_{p \in C_i \cap C_j} \mathcal{I}/\mathcal{I}^2|_p\right)$ is a surjective map, or, equivalently, $H^1(\mathcal{I}/\mathcal{I}^2) = 0$. Furthermore, since $\mathcal{I}/\mathcal{I}\mathcal{I}$ decomposes as $(0,1)$ on both $C_1$ and $C_n$ and as $(0,0)$ on each of $C_2, \ldots, C_{n-1}$, $h^0(\bigoplus_{i=1}^n \mathcal{I}/\mathcal{I}\mathcal{I}) = 3 + 2(n - 2) + 3 = 2n + 2$. Also, we have $h^0(\mathcal{I}/\mathcal{I}^2|_p) = 2$ at each point of intersection, so $h^0 \left(\bigoplus_{p \in C_i \cap C_j} \mathcal{I}/\mathcal{I}^2|_p\right) = 2(n - 1)$ since there are $n - 1$ points of intersection. In particular, then, it has been shown that $h^0(\mathcal{I}/\mathcal{I}^2) = 2n + 2 - (2n - 2) = 4$ and $H^1(\mathcal{I}/\mathcal{I}^2) = 0$.

As the map on global sections, $H^0(\mathcal{I}/\mathcal{I}^2) \longrightarrow H^0(\bigoplus_{i=1}^n \mathcal{I}/\mathcal{I}^2|_{C_i})$ is an isomorphism away from the singular points, $H^0(\mathcal{I}/\mathcal{I}^2) \longrightarrow H^0(\mathcal{I}/\mathcal{I}^2|_{C_i})$ is surjective for each $i$. From the normalization of $C$ and the higher order neighborhoods of $C$,

$$0 \longrightarrow \mathcal{I}^m/\mathcal{I}^{m+1} \longrightarrow \bigoplus_{i=1}^n \mathcal{I}^m/\mathcal{I}^{m+1}|_{C_i} \longrightarrow \bigoplus_{p \in C_i \cap C_j} \mathcal{I}^m/\mathcal{I}^{m+1}|_p \longrightarrow 0$$

is exact, where $\mathcal{I}^m/\mathcal{I}^{m+1}|_{C_i} = S^m(\mathcal{I}/\mathcal{I}\mathcal{I})$ has $h^1 = 0$. Therefore, $H^0(\mathcal{I}/\mathcal{I}^2) \longrightarrow H^0(\mathcal{I}/\mathcal{I}^2)$ is surjective as well.

Recall that $\mathcal{I}_i/\mathcal{I}_i^2 = (1, 1)$ or $(0, 2)$, and $\mathcal{I}/\mathcal{I}\mathcal{I} = (0, 1)$ or $(0, 0)$, are both generated by global sections. So, by lifting to global sections of $\mathcal{I}/\mathcal{I}^2$:

**Lemma 3.1.** $\mathcal{I}/\mathcal{I}^2$ is generated by global sections.

These four global sections of $\mathcal{I}/\mathcal{I}^2$, lifted to global sections of $\mathcal{H}$, define a formal dDV modification $\hat{f} : \hat{X} \to \hat{C}^4$ for which $\hat{f}^{-1}(0) = \hat{D}$, so to determine the singularity from contracting $C$, the general section of the ideal sheaf $\hat{I}$ must be determined.

**Proposition 3.2.** The zero scheme of a general section of $\hat{I}$ is a smooth surface along $C$. 
Proof. A general section of $\hat{\mathcal{I}}$ at any point of intersection $p$ is of the form $g \cdot xy + h \cdot z$ with $g, h \in \mathcal{O}_{p, X}$ and $g$ or $h$ is a unit. Considering this as a local section of $\mathcal{I}/\mathcal{I}^2$, there exists a global section $s \in \mathcal{I}/\mathcal{I}^2$ that does not vanish at $p$, and, therefore, $h(p) \neq 0$. So, $s$ is nonsingular at $p$. The condition $h(p) \neq 0$ defines an open dense subset of $X$ on which $h$ is non-vanishing. Since $h(p) \neq 0$ at each point of intersection, and the intersection of these sets is open and dense in $X$, a general section of $\mathcal{I}/\mathcal{I}^2$ is nonsingular at each point of intersection. As $H^0(\hat{\mathcal{I}}) \to H^0(\mathcal{I}/\mathcal{I}^2)$ is surjective, lift this to a global section of $\hat{\mathcal{I}}$.

At a smooth point of $C$, a general section of $\hat{\mathcal{I}}$ is of the form $g \cdot x + h \cdot z$ with $g$ or $h$ a unit. Therefore, a general section of $\hat{\mathcal{I}}$ is smooth away from the singular points of $C$ as well. □

Take a general nonzero section $s \in H^0(\mathcal{I}/\mathcal{I}^2)$. With the first map being multiplication by $s$, we have

$$0 \to \mathcal{O}_C \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{I}_{C,S}/\mathcal{I}^2_{C,S} \to 0,$$

where $S$ is the smooth surface defined by $s$. Restricting to $C_i$, then, $\mathcal{I}_{C,S}/\mathcal{I}^2_{C,S}|_{C_i} \cong \mathcal{O}_{C_i}(1)$ for $i = 1, n$ and $\mathcal{I}_{C,S}/\mathcal{I}^2_{C,S}|_{C_i} \cong \mathcal{O}_{C_i}$ for $2 \leq i \leq n - 1$. Invertible sheaves on $C$ are completely determined by their degree on each component, so it will be written $\mathcal{I}_{C,S}/\mathcal{I}^2_{C,S} \cong \mathcal{O}_C(1, 0, \ldots, 0, 1)$. Since $S$ is smooth, at a point of intersection $p \in C$, coordinates can be chosen so that $(s)$ is defined by $(z = 0)$ and $\mathcal{I}_{C,S} = (xy)$. From the injection $\mathcal{I}_{C,S}/\mathcal{I}^2_{C,S}|_{C_i} \hookrightarrow \mathcal{I}_{C_i,S}/\mathcal{I}^2_{C_i,S}$, which is an isomorphism away from the singular points of $C$, local coordinates show that the determinant map vanishes to order 1 at each point of intersection. Therefore, $\mathcal{I}_{C_i,S}/\mathcal{I}^2_{C_i,S} \cong \mathcal{O}_{C_i}(2)$ for all $1 \leq i \leq n$. We have:

**Theorem 3.3.** If $f : X \to Y$ is an analytic contraction map with $f(C) = q$ and $C = \bigcup_{i=1}^n C_i$ with all components having length 1, then a general hyperplane section of $q$ has an $A_n$ type singularity at $q$.

**4. Contraction criteria for $C$.**

In this section it will be shown how the splitting of Sequence (3) is equivalent to the existence of an infinitesimal deformation of $C$. Then, using the theory of deformations of the $A_n$ singularity, it will be shown that the continued splitting of related sequences determine higher order deformations of $C$.

The discussion following Sequence (3) in Section 3 shows $\mathcal{I}_{C,S}/\mathcal{I}^2_{C,S} \cong \mathcal{O}_C(1, 0, \ldots, 0, 1)$, which is the dual of the dualizing sheaf on $C$, $\omega_C^*$. Sequence (3), then, will be written

$$0 \to \mathcal{O}_C \to \mathcal{I}/\mathcal{I}^2 \to \omega_C^* \to 0.$$
Lemma 4.1. Sequence (4) splits if and only if there exists an ideal \( \mathcal{K}_2 \) satisfying \( \mathcal{I}^2 \subset \mathcal{K}_2 \subset \mathcal{I}, \) \( \mathcal{I}/\mathcal{K}_2 \cong \mathcal{O}_C \) and \( \mathcal{K}_2/\mathcal{I}^2 \cong \omega_C^* \).

Proof. If this sequence splits define
\[
\mathcal{K}_2 = \text{Ker}(\mathcal{I} \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_C).
\]

By definition, then, \( \mathcal{I}^2 \subset \mathcal{K}_2 \subset \mathcal{I} \) and \( \mathcal{I}/\mathcal{K}_2 \cong \mathcal{O}_C \). The sheaf \( \mathcal{K}_2/\mathcal{I}^2 \) is invertible on \( C \) and restricting to each component \( C_i \) we have
\[
0 \to \mathcal{K}_2/\mathcal{I}^2|_{C_i} \to \mathcal{I}/\mathcal{I}^2|_{C_i} \to \mathcal{I}/\mathcal{K}_2|_{C_i} \to 0.
\]

But, \( \mathcal{I}/\mathcal{K}_2|_{C_i} \cong \mathcal{O}_i \) for all \( 1 \leq i \leq n \), \( \mathcal{I}/\mathcal{I}^2|_{C_i} \cong \mathcal{O}_i(1) \oplus \mathcal{O}_i \) for \( i = 1, n \), and \( \mathcal{I}/\mathcal{I}^2|_{C_i} \cong \mathcal{O}_i \oplus \mathcal{O}_i \) for \( 2 \leq i \leq n - 1 \). Therefore, this sequence must split for all \( 1 \leq i \leq n \) and \( \mathcal{K}_2/\mathcal{I}^2 \cong \mathcal{O}_C(1, 0, \ldots, 0, 1) \cong \omega_C^* \).

The converse is immediate from the properties of \( \mathcal{K}_2 \) since
\[
0 \to \mathcal{K}_2/\mathcal{I}^2 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{I}/\mathcal{K}_2 \to 0
\]
is exact. \( \square \)

The following proposition relates the splitting of Sequence (4), or, equivalently, the existence of the defining ideal sheaf \( \mathcal{K}_2 \), to the infinitesimal deformation of \( C \).

Proposition 4.2. \( C \) has an infinitesimal deformation if and only if there exists an ideal sheaf \( \mathcal{K}_2 \) satisfying \( \mathcal{I}^2 \subset \mathcal{K}_2 \subset \mathcal{I}, \) \( \mathcal{I}/\mathcal{K}_2 \cong \mathcal{O}_C \) and \( \mathcal{K}_2/\mathcal{I}^2 \cong \omega_C^* \).

Proof. This second condition is equivalent to the splitting of the dual of Sequence (4), which is
\[
0 \to \omega_C \to \mathcal{N}_C \to \mathcal{O}_C \to 0,
\]
and the infinitesimal deformations are classified by \( H^0(C, \mathcal{N}_C) \). Now
\[
\text{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_C, \omega_C) \cong H^1(C, \omega_C) \cong C,
\]
and the geometric genus of \( C \) is 0, so the long exact cohomology sequence can be written
\[
0 \to H^0(\mathcal{N}_C) \to C \xrightarrow{\delta} C \to H^1(\mathcal{N}_C) \to 0,
\]
where \( \delta \) is the coboundary map given by \( 1 \mapsto \text{extension class of } \mathcal{O}_C \) by \( \omega_C \). Therefore, \( \delta \) is either an isomorphism or the zero map. Sequence (5) splits if and only if \( \delta \) is the zero map, which is equivalent to \( H^0(\mathcal{N}_C) = C \). \( \square \)

It will now be shown that the sheaf \( \mathcal{K}_2/I\mathcal{K}_2 \) determines the existence of higher order deformations of \( C \). Calculating in local coordinates \( \{x, y, z\} \) at a singular point of \( C \), there are two possible forms for \( \mathcal{K}_2 \).

The invertible sheaf \( \mathcal{K}_2/\mathcal{I}^2 \) is a subsheaf of \( \mathcal{I}/\mathcal{I}^2 \), which is generated by \( \{xy, z\} \). Therefore, \( \mathcal{K}_2/\mathcal{I}^2 \) is generated locally by an element of the form \( f_0xy + f_1z \) with \( f_0, f_1 \in \mathcal{O}_{p, X} \) with either \( f_0 \) or \( f_1 \) a unit. If \( f_0 \) is a unit,
then, dividing by \( f_0 \), \( \mathcal{K}_2/\mathcal{I}^2 \) is generated by an element of the form \( xy + \lambda_1 z \). So,
\[
\mathcal{K}_2 = (xy + \lambda_1 z) + \mathcal{I}^2 = (xy + \lambda_1 z, z^2).
\]
On the other hand, if \( f_1 \) is a unit, then, dividing by \( f_1 \), \( \mathcal{K}_2/\mathcal{I}^2 \) is generated by an element of the form \( g_0 xy + z \). In this case, the analytic change of coordinates inverse to \( (x, y, z) \mapsto (x, y, g_0 xy + z) \) gives \( \mathcal{K}_2/\mathcal{I}^2 \) being generated by \( z \), and it does not affect the description of \( \mathcal{I} \) as \( (xy, z) \). It can now be seen that
\[
\mathcal{K}_2 = (z) + \mathcal{I}^2 = (x^2 y^2, z).
\]

In either case, though, the kernel of the map \( \mathcal{O}_C \oplus \mathcal{O}_C \to \mathcal{K}_2/\mathcal{I}\mathcal{K}_2 \) defined by \( (f, g) \mapsto f(xy + \lambda_1 z) + g^2 \) or \( (f, g) \mapsto f^2 y^2 + gz \) is \( I_C \oplus \mathcal{O}_C \). Therefore, \( \mathcal{K}_2/\mathcal{I}\mathcal{K}_2 \) is locally free of rank 2 on \( C \).

Furthermore, the invertible sheaf \( \mathcal{I}^2/\mathcal{I}\mathcal{K}_2 \cong \mathcal{I}/\mathcal{K}_2 \otimes \mathcal{I}/\mathcal{K}_2 \cong \mathcal{O}_C \), so the exact sequence
\[
0 \to \mathcal{I}^2/\mathcal{I}\mathcal{K}_2 \to \mathcal{K}_2/\mathcal{I}\mathcal{K}_2 \to \mathcal{K}_2/\mathcal{I}^2 \to 0
\]
can be written
\[
(6) \quad 0 \to \mathcal{O}_C \to \mathcal{K}_2/\mathcal{I}\mathcal{K}_2 \to \omega_C^* \to 0,
\]
which is the same as Sequence (4) with \( \mathcal{K}_2/\mathcal{I}\mathcal{K}_2 \) replacing \( \mathcal{I}/\mathcal{I}^2 \). So, the process continues to determine higher order deformations of \( C \).

**Lemma 4.3.** Let \( \mathcal{K}_m \subset \mathcal{K}_{m-1} \subset \cdots \subset \mathcal{K}_3 \subset \mathcal{K}_2 \subset \mathcal{K}_1 = \mathcal{I} \) be a sequence of ideals satisfying \( \mathcal{I}\mathcal{K}_{i-1} \subset \mathcal{K}_i \subset \mathcal{K}_{i-1}, \mathcal{K}_{i-1}/\mathcal{K}_i \cong \mathcal{O}_C \) and \( \mathcal{K}_i/\mathcal{I}\mathcal{K}_{i-1} \cong \omega_C^* \). Then:

1. In local coordinates at \( p \) on \( C \), \( \mathcal{K}_i = (xy + \lambda_1 z + \cdots + \lambda_{i-1} z^{i-1}, z^i) \) for all \( 1 \leq i \leq m \), or \( \mathcal{K}_i = (x^i z^i, z) \) for all \( 1 \leq i \leq m \).
2. \( \mathcal{I}\mathcal{K}_{i-1}/\mathcal{I}\mathcal{K}_i \cong \mathcal{I}/\mathcal{K}_2 \otimes \mathcal{K}_{i-1}/\mathcal{K}_i \cong \mathcal{O}_C \) for all \( 1 \leq i \leq m \).
3. \( \mathcal{K}_i/\mathcal{I}\mathcal{K}_i \) is locally free of rank 2 for all \( 1 \leq i \leq m \).
4. The sequence \( 0 \to \mathcal{I}\mathcal{K}_{i-1}/\mathcal{I}\mathcal{K}_i \to \mathcal{K}_i/\mathcal{I}\mathcal{K}_i \to \mathcal{K}_i/\mathcal{I}\mathcal{K}_{i-1} \to 0 \) is split exact for all \( 2 \leq i \leq m-1 \).

We will now establish the relationship between the existence of the defining ideals \( \mathcal{K}_i \) and the deformation of the curve \( C \) as determined from the semi-universal deformations of \( Y_0 \) and \( X_0 \). In particular, it will be shown explicitly, from the defining equations for \( X \) (see Equation (2) in Section 2.2) in the deformation theory of the \( A_n \) singularity, that the existence of \( \mathcal{K}_i \) corresponds to an \( i \)th order deformation of \( C \). From Theorems 2.1 and 2.2 in Section 2.2, and the paragraph preceding them, it will be shown how the existence of the higher order neighborhoods of \( C \), defined by the \( \mathcal{K}_i \), coincide with certain vanishing properties of the \( t^i \) terms of the functions \( g_i \) (see Equation (1), Section 2.2) defining the discriminant locus of \( \tilde{g} : \text{Spec} \mathcal{O}[[t]] \to \text{Spec} \mathcal{O}[[t_1, \ldots, t_{n+1}]] \). An inductive argument on the number of components of \( C \) and on the order of the terms will be utilized.
Theorem 4.4. \( C \) deforms formally in \( \hat{X} \) if and only if there exists an infinite chain of subsheaves \( \cdots \subset K_{m+1} \subset K_m \subset \cdots \subset K_2 \subset I \) such that \( K_m/K_{m+1} \cong \mathcal{O}_C \) and \( K_{m+1}/IK_m \cong \omega_C^* \), where \( \omega_C^* \) is the dual of the dualizing sheaf.

Proof. The case where \( C \) is a smooth rational curve has been proved by Reid in [14], the “Pagoda” construction. A brief discussion of Reid’s results will be given to compare to the multiple component case.

Case 1. \( C = C_1 \).

Using the transition functions defining \( X \) in the deformation of an \( A_1 \) singularity from Section 2.2, Equation (2), and expanding the \( g_i \) in power series form as in Equation (1), Section 2, we have

\[
x = u_1^2 y + u_1 \sum_{j=1}^{\infty} (a_{1j} - a_{2j}) t^j
\]

\[
t = t
\]

\[
v_1 = 1/u_1.
\]

The curve \( C \) is given by \( y = t = 0 \) in the \( \{u_1, y, t\} \) coordinate patch, and by \( x = t = 0 \) in the \( \{v_1, x, t\} \) patch. In other words, the ideal sheaf of \( C \) in \( X \) is \( I = (y, t) = (x, t) \).

Since \( I/I^2 \) decomposes as \((1, 1)\) or \((0, 2)\), there is a surjection \( I/I^2 \to \mathcal{O}_C \) if and only if \( I/I^2 = (0, 2) \). It has been shown this is also equivalent to the existence of \( K_2 \) (Lemma 4.1) and the existence of an infinitesimal deformation of \( C \) (Proposition 4.2).

Lemma 4.5. There is a surjection \( I/I^2 \to \mathcal{O}_C \) if and only if \( a_{11} = a_{21} \).

Proof. It suffices to show \( I/I^2 = (0, 2) \) if and only if \( a_{11} = a_{21} \). In calculating the decomposition of the conormal sheaf from the transition function, since \( t^j \in I^2 \) for \( j \geq 2 \), it is only necessary to consider \( x = u_1^2 y + u_1(a_{11} - b_{21}) t \). \( I/I^2 \) is generated locally by \( \{y, t\} \) and \( \{x, t\} \) in these two coordinate patches and

\[
(x, t) = \begin{pmatrix} u_1^2 & (a_{11} - a_{21}) u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ t \end{pmatrix}.
\]

This matrix, in particular the entry \( (a_{11} - a_{21}) \), determines \( I/I^2 \). That is, \( I/I^2 = (0, 2) \) if and only if \( a_{11} = a_{21} \) (see [12, pp. 519-520]). \( \square \)

The map \( I/I^2 \to \mathcal{O}_C \), if it exists, can be calculated explicitly in coordinates to determine the ideal \( K_2 \). Reid [14] shows that \( K_2/I^2 \cong \mathcal{O}_C(2) \cong \omega_C^* \) and \( K_2 = (y, t^2) = (x, t^2) \) in local coordinates. From Sequence (5), then, \( K_2/IK_2 = (1, 1) \) or \((0, 2)\), and the process can be continued.
Assume that we have $K_m \subset K_{m-1} \subset \cdots \subset K_2 \subset \mathcal{I}$ with $K_{i-1}/K_i \cong \mathcal{O}_C$, $K_i/\mathcal{I}_K_{i-1} \cong \mathcal{O}_C(2)$ and $K_i/\mathcal{I}_K_i = (0,2)$ if and only if $a_{1,i} = a_{2,i}$. Induction, with details eliminated, proves:

**Lemma 4.6.** There is a surjection $K_{m+1}/\mathcal{I}K_{m+1} \rightarrow \mathcal{O}_C$ if and only if $a_{1(m+1)} = a_{2(m+1)}$.

From this inductive argument and the fact that $C$ deforms in $X$ if and only if $g_1(t) = g_2(t)$ (Theorem 2.1, Section 2.2), the following results of Reid [14] have been established:

**Theorem 4.7** ([14, Reid]). $C \cong \mathbb{P}^1$ deforms in $X$ if and only if there exists an infinite chain $\cdots \subset K_{m+1} \subset K_m \subset \cdots \subset K_2 \subset \mathcal{I}$ satisfying $K_m/K_{m+1} \cong \mathcal{O}_C$ and $K_m/\mathcal{I}K_m \cong \mathcal{O}_C(2)$.

Furthermore, if for some $m$, $K_m/\mathcal{I}K_m = (1,1)$, then $g_1(t) \neq g_2(t)$. Therefore:

**Theorem 4.8** ([14, Reid]). $C \cong \mathbb{P}^1$ contracts if and only if the chain $\cdots \subset K_m \subset \cdots \subset K_2 \subset \mathcal{I}$ terminates.

**Remark 4.9.** Reid, in [14], showed not only that $C$ contracts or deforms in this formal structure, but also that there is actually an analytic deformation or contraction of $C$.

To extend to multiple components, it will first be shown for two components and it is without loss of generality (and to avoid sub-subscripts) that the first two components are used.

**Case 2.** $C = C_1 \cup C_2$.

From the description of $X$ by transition functions in Equation (2), if $i = 1$ and $n = 2$, then $X$ is defined by the transition functions:

$$
\begin{align*}
x &= u_1^2v_2 + u_1(g_1(t) - g_2(t)) & u_2 &= 1/v_2 \\
v_1 &= 1/u_1 & y &= v_1^2u_1 + v_2(g_3(t) - g_2(t)) \\
t &= t & t &= t
\end{align*}
$$

where $\mathcal{I} = (u_1,v_2,t) = (x,t), \mathcal{I}_1 = (v_2,t) = (x,t)$ and $\mathcal{I}_2 = (u_1,t) = (y,t)$ in the coordinate patches $(u_1,v_2,t)$, $(x,v_1,t)$ and $(u_2,y,t)$.

**Lemma 4.10.** There is a surjection $\mathcal{I}/\mathcal{I}^2 \rightarrow \mathcal{O}_C$ if and only if $a_{11} = a_{31}$.

**Proof.** Assume that there is a surjection $\mathcal{I}/\mathcal{I}^2 \rightarrow \mathcal{O}_C$. Defining this in local coordinates on the patch $(u_1,v_2,t)$ containing the point of intersection, let $u_1v_2 \mapsto h_1(u_1,v_2)$ and $t \mapsto h_2(u_1,v_2)$ where the $h_i$ are holomorphic functions in $u_1$ and $v_2$. Since $t^2 \in \mathcal{I}^2$ in each patch, it suffices to assume $g_i(t) = a_{11}t$ for $1 \leq i \leq 3$. Then, in the remaining coordinate patches, $\mathcal{I}/\mathcal{I}^2 \rightarrow \mathcal{O}_C$ is given by

$$
\begin{align*}
x &\mapsto u_1h_1 + u_1(a_{11} - a_{21})h_2 \\
y &\mapsto v_2h_1 + v_2(a_{31} - a_{21})h_2 \\
t &\mapsto h_2.
\end{align*}
$$
The images of the generators \{x, t\} and \{y, t\} of \(\mathcal{I}/\mathcal{I}^2\) must be holomorphic in the coordinate patches \((x, v_1, t)\) and \((u_2, y, t)\) respectively. In particular, \(h_2\) must be holomorphic in the coordinate \(v_1 = 1/u_1\) and in \(u_2 = 1/v_2\). This can only be possible if \(h_2\) is a constant function. Let \(h_2 = c\) where \(c \in \mathbb{C}\) and \(c \neq 0\) for a nontrivial map.

Viewing \(h_1\) as a power series in \(u_1\) and \(v_2\), the image of \(x, u_1h_1 + u_1(a_{11} - a_{21})c\), can only be holomorphic in \(v_1\) if it is the zero function. Therefore, \(h_1 = (a_{21} - a_{11})c\). The surjection can exist, then, only if in the coordinates \{y, t\},

\[
\begin{align*}
y & \mapsto v_2(a_{21} - a_{11})c + v_2(a_{31} - a_{21})c \\
t & \mapsto c
\end{align*}
\]

with the image of \(y\) holomorphic in \(u_2 = 1/v_2\). Again, this is only possible if it is the zero function, which is equivalent to \((a_{21} - a_{11}) = (a_{21} - a_{31})\) or \(a_{11} = a_{31}\).

Furthermore, assuming \(c = 1\) (since \(c \neq 0\)) and letting \(\lambda_1 = a_{11} - a_{21} = a_{31} - a_{21}\), the surjection is defined on the generators by the equations

\[
\begin{align*}
u_1v_2 & \mapsto -\lambda_1 \\
x & \mapsto 0 \\
y & \mapsto 0 \\
t & \mapsto 1
\end{align*}
\]

Conversely, if \(a_{11} = a_{31}\) define \(\mathcal{I}/\mathcal{I}^2 \rightarrow \mathcal{O}_C\) by the above equations. \(\square\)

The subsheaf \(\mathcal{K}_2\) of \(\mathcal{I}\) satisfying the conditions of Lemma 4.1 can also be calculated explicitly using this local description. By definition, \(\mathcal{K}_2 = \text{Ker}(\mathcal{I} \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow \mathcal{O}_C)\), so \(t^2 \in \mathcal{I}^2\) and \(\{u_1v_2 + \lambda_1t, t^2\}\) generate \(\mathcal{K}_2\) in the \((u_1, v_2, t)\) patch. Similarly, from the equations of the map \(\mathcal{I}/\mathcal{I}^2 \rightarrow \mathcal{O}_C\) above \\{\(x, t^2\)\} and \\{\(y, t^2\)\} generate \(\mathcal{K}_2\) in the \((x, v_1, t)\) and \((u_2, y, t)\) patches, respectively. Notice that this local description of \(\mathcal{K}_2\) is equivalent to one of the forms that was determined without the use of the equations from the deformations of the An singularity (see Lemma 4.3, Part 1). To complete the induction:

**Lemma 4.11.** There is a surjection \(\mathcal{K}_{m+1}/\mathcal{IK}_{m+1} \rightarrow \mathcal{O}_C\) if and only if \(a_{1(m+1)} = a_{3(m+1)}\).

**Proof.** Assume there is a surjection \(\mathcal{K}_k/\mathcal{IK}_k \rightarrow \mathcal{O}_C\) if and only if \(a_{1k} = a_{3k}\) for all \(k \leq m\).

To extend this to \(m+1\), notice that \(\mathcal{K}_{m+1}/\mathcal{IK}_{m+1}\) is generated by \(\{u_1v_2 + \lambda_1t + \cdots + \lambda_mt^m, t^{m+1}\}\) at the point of intersection, where \(\lambda_i = a_{1i} - a_{2i} = a_{3i} - a_{2i}\), and by \(\{x, t^m\}\) and \(\{y, t^m\}\) on the other coordinate patches. Since \(t^{m+2} \in \mathcal{IK}_{m+1}\) in each patch, to calculate the surjection \(\mathcal{K}_{m+1}/\mathcal{IK}_{m+1} \rightarrow \mathcal{O}_C\), it suffices to consider \(g_i(t) = \sum_{j=1}^{m+1} a_{ij}t^j\) for \(i = 1, 2, 3\).
Defining this map on generators in the coordinates \((u_1, v_2, t)\), let

\[
\begin{align*}
    u_1 v_2 + \lambda_1 t \cdots + \lambda_m t^m & \mapsto h_1(u_1, v_2) \\
    t^{m+1} & \mapsto h_2(u_1, v_2)
\end{align*}
\]

where \(h_1\) and \(h_2\) are holomorphic in \(u_1\) and \(v_2\). In the other patches we also have the generator \(t^{m+1}\) mapping to the function \(h_2\). The exact reasoning from Lemma 4.10 shows that \(h_2\) must be the constant function, and it can be assumed to be the constant 1. So, the surjection on the remaining generators is given by

\[
\begin{align*}
    x & \mapsto u_1(h_1 + a_{1(m+1)} - a_{2(m+1)}) \\
    t^m & \mapsto 1 \\
    y & \mapsto v_1(h_1 + a_{3(m+1)} - a_{2(m+1)}) \\
    t^{m+1} & \mapsto 1.
\end{align*}
\]

As for the surjection from \(\mathcal{I}/\mathcal{I}^2\) done previously, we can conclude that \(h_1 = a_{2(m+1)} - a_{1(m+1)} = a_{2(m+1)} - a_{3(m+1)}\), and this surjection can occur if and only if \(a_{1(m+1)} = a_{3(m+1)}\). \(\square\)

Therefore, for the case where \(C = C_1 \cup C_2\), \(C\) deforms if and only if there is an infinite chain \(\cdots \subset \mathcal{K}_m \subset \cdots \mathcal{K}_2 \subset \mathcal{I}\) with \(\mathcal{K}_m/\mathcal{K}_{m+1} \cong \mathcal{O}_C\) and \(\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_m \cong \omega_C^*\).

Induction on more components of \(C\) is immediate as it is necessary to show that the surjection is well-defined where \(C_{m+1}\) intersects \(\bigcup_{i=1}^m C_i\), which is just the point \(p = C_m \cap C_{m+1}\). This completes the proof of Theorem 4.4. \(\square\)

**Theorem 4.12.** A formal c\(A_n\) contraction of \(C\) exists if and only if there is no infinite chain of subsheaves \(\cdots \subset \mathcal{K}_{m+1} \subset \cdots \mathcal{K}_2 \subset \mathcal{I}_D\) satisfying \(\mathcal{K}_m/\mathcal{K}_{m+1} \cong \mathcal{O}_D\) and \(\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_m \cong \omega_D^*\) for any \(D = \bigcup_{i=1}^k C_i\) \((1 \leq i \leq k \leq n)\), where \(\mathcal{I}_D\) is the ideal sheaf of \(D\) in \(X\).

**Proof.** For every \(i\) and \(k\) we can conclude from the proof of Theorem 4.4 that \(g_i \neq g_{k+1}\). Therefore, the curve \(D\) is not contained in the discriminant locus, which is equivalent to the induced formal map, \(\text{Spec} \mathbb{C}[t] \to \text{Spec} \mathbb{C}[[t_1, \ldots, t_n]]\) not factoring through the discriminant locus in \(\text{Spec} \mathbb{C}[[t_1, \ldots, t_n]]\). \(\square\)

Notice from this theorem, it can be concluded that even if every component of \(C\) can be contracted, this is not enough to ensure that \(C\) contracts.

**Example 4.13.** Using the description of \(X\) by transition functions in Equation (2) of Section 2.2, with \(C = C_1 \cup C_2\), let \(g_1(t) = 2t\), \(g_2(t) = t\) and \(g_3(t) = 2t\). Since \(g_1(t) = g_3(t)\), the curve \(C\) deforms in \(X\) and so is not contractible. However, since \(g_1(t) \neq g_2(t)\), \(C_1\) can be contracted, and since \(g_2(t) \neq g_3(t)\), \(C_2\) can also be contracted. In fact, \(\mathcal{I}_1/\mathcal{I}_1^2 = (1, 1)\) and \(\mathcal{I}_2/\mathcal{I}_2^2 = (1, 1)\) (see [10]). The conormal sheaves of each component being ample implies that \(C_1\) and \(C_2\) can each be contracted separately.
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References


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