REAL PALEY-WIENER THEOREMS FOR THE INVERSE FOURIER TRANSFORM ON A RIEMANNIAN SYMMETRIC SPACE

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We prove real Paley-Wiener theorems for the inverse Fourier transform on a semisimple Riemannian symmetric space $G/K$ of the noncompact type. The functions on $G/K$ whose Fourier transform has compact support are characterised by a $L^2$ growth condition. We also obtain real Paley-Wiener theorems for the inverse spherical transform.

1. Introduction.

The classical Fourier transform $\mathcal{F}_{cl}$ is an isomorphism of the Schwartz space $S(\mathbb{R}^k)$ onto itself. The space $C_c^\infty(\mathbb{R}^k)$ of smooth functions with compact support is dense in $S(\mathbb{R}^k)$, and the classical Paley-Wiener theorem characterises the image of $C_c^\infty(\mathbb{R}^k)$ under $\mathcal{F}_{cl}$ as rapidly decreasing functions having an holomorphic extension to $\mathbb{C}^k$ of exponential type. Since $\mathbb{R}^k$ is self-dual, the same theorem also applies to the inverse Fourier transform.

Let $G$ be a noncompact semisimple Lie group and $K$ a maximal compact subgroup of $G$. The Fourier transform $\mathcal{F}$ on the Riemannian symmetric space $X = G/K$ is an analogue of the classical Fourier transform on $\mathbb{R}^k$. A Paley-Wiener theorem for the Fourier transform $\mathcal{F}$, which characterises the image of $C_c^\infty(X)$ under $\mathcal{F}$ in terms of holomorphic extensions and growth behaviour, as in the classical case, was proved by Helgason, see [7]. Furthermore, the $L^2$-Schwartz space $S^2(X)$ contains $C_c^\infty(X)$ as a dense subspace and $\mathcal{F}$ is an isomorphism of $S^2(X)$ onto some generalised Schwartz space, see [4].

Unlike the classical case, however, we cannot use a duality argument to deduce a Paley-Wiener theorem for the inverse Fourier transform. So how can we characterise the functions whose Fourier transform $\mathcal{F}$ has compact support?

The Fourier transform on $X$ reduces to the spherical transform $\mathcal{H}$ on $G$ when restricted to $K$-invariant functions. The paper [8] provides an answer to the above question for the spherical transform on Schwartz functions in the rank one and complex cases. The characterisation is in analogy with the classical Paley-Wiener theorem given in terms of meromorphic extensions and growth conditions.
In this paper we prove (real) Paley-Wiener theorems for the inverse Fourier transform for general Riemannian symmetric spaces, i.e., we characterise, as a subset of $L^2(X)$, the set of functions $f$ on $X$ whose Fourier transform $\mathcal{F}f$ has compact support. More precisely, $f \in C^\infty(X)$ has to satisfy

$$\lim_{n \to \infty} \|\Delta^n f\|_{1/2n}^2 < \infty,$$

where $\Delta$ is the Laplace-Beltrami operator (and $(1 + |\cdot|)^n f \in L^2(X)$ for all $n \in \mathbb{N} \cup \{0\}$ if we also want the Fourier image to be smooth). Specialising to bi-$K$-invariant functions yields (real) Paley-Wiener theorems for the inverse spherical transform for general noncompact semisimple Lie groups.

Our approach is based on real analysis techniques developed by H. H. Bang, see [2] and [3], and V.K. Tuan, see [10]. Also see [11] for a history and overview of (real) Paley-Wiener theorems for certain transforms (Fourier, Mellin, Hankel...) on $\mathbb{R}$. In particular we use Parseval’s formula, intertwining properties of $\mathcal{F}$, and the following characterisation of the radius of the support of a function $g$ on $\mathbb{R}^n$:

$$\sup_{\lambda \in \text{supp } g} \|\lambda\| = \lim_{n \to \infty} \left\{ \int_{\mathbb{R}^n} \|\lambda\|^{4n} |g(\lambda)|^2 d\lambda \right\}^{1/4n}.$$

For completeness and comparison, we first consider the Fourier transform on $\mathbb{R}^k$. The results here are originally due to H.H. Bang, see [2] and [3], and V.K. Tuan, see [10]. Notice the beautiful symmetry between the (statements of the) results for the various transforms.

### 2. The Fourier transform on $\mathbb{R}^k$.

For background and details, please see [9, Chapter 7]. Let $\mathcal{F}_{cl}$ denote the classical Fourier transform on $\mathbb{R}^k$:

$$\mathcal{F}_{cl} f(\lambda) := \int_{\mathbb{R}^k} f(x) e^{-i\lambda \cdot x} dx,$$

defined for nice functions $f$, for all $\lambda \in \mathbb{C}^k$ for which the above integral makes sense. Let $\Delta = \frac{d^2}{dx_1^2} + \cdots + \frac{d^2}{dx_k^2}$ denote the Laplacian on $\mathbb{R}^k$ and let $\mathcal{S}(\mathbb{R}^k)$ denote the Schwartz space of rapidly decreasing differentiable functions. Then $\mathcal{F}_{cl}(\Delta f)(\lambda) = -\|\lambda\|^2 \mathcal{F}_{cl} f(\lambda)$, $(\lambda \in \mathbb{R}^k)$, for all $f \in \mathcal{S}(\mathbb{R}^k)$, and the Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^k)$ onto itself, with inverse given by:

$$\mathcal{F}_{cl}^{-1} g(x) = (2\pi)^{-k} \int_{\mathbb{R}^k} g(\lambda) e^{i\lambda \cdot x} d\lambda, \quad (x \in \mathbb{R}^k)$$
for \(g \in \mathcal{S}(\mathbb{R}^k)\). Parseval’s formula states that

\[
\langle f_1, f_2 \rangle := \int_{\mathbb{R}^k} f_1(x) \overline{f_2(x)} \, dx = (2\pi)^{-k} \int_{\mathbb{R}^k} \mathcal{F}_{cl} f_1(\lambda) \overline{\mathcal{F}_{cl} f_2(\lambda)} \, d\lambda
\]

for \(f_1, f_2 \in \mathcal{S}(\mathbb{R}^k)\), which implies that \(\|f\|_2 = \|\mathcal{F}_{cl} f\|_2\), for all \(f \in \mathcal{S}(\mathbb{R}^k)\), and hence that the Fourier transform extends to an isometry from \(L^2(\mathbb{R}^k)\) onto itself.

Let \(f \in C^\infty(\mathbb{R}^k)\) such that \(\Delta^n f \in L^2(\mathbb{R}^k)\) for all \(n \in \mathbb{N} \cup \{0\}\) and let \(f_2 \in C_c^\infty(\mathbb{R}^k)\). Then:

\[
\langle \mathcal{F}_{cl}(\Delta f), \mathcal{F}_{cl} f_2 \rangle = \langle \Delta f, f_2 \rangle = \langle f, \Delta f_2 \rangle = \langle \mathcal{F}_{cl} f, \mathcal{F}_{cl}(\Delta f_2) \rangle = \langle \mathcal{F}_{cl} f, -\|\lambda\|^2 \mathcal{F}_{cl} f_2 \rangle = \langle -\|\lambda\|^2 \mathcal{F}_{cl} f, \mathcal{F}_{cl} f_2 \rangle,
\]

and we conclude that \(\mathcal{F}_{cl}(\Delta f)(\lambda) = -\|\lambda\|^2 \mathcal{F}_{cl} f(\lambda)\) a.e., by a density argument, whence \(\mathcal{F}_{cl}(\Delta^n f)(\lambda) = (-1)^n \|\lambda\|^{2n} \mathcal{F}_{cl} f(\lambda)\) a.e., and

\[
\int_{\mathbb{R}^k} |\Delta^n f(x)|^2 \, dx = (2\pi)^{-k} \int_{\mathbb{R}^k} \|\lambda\|^{4n} |\mathcal{F}_{cl} f(\lambda)|^2 \, d\lambda,
\]

for all \(n \in \mathbb{N} \cup \{0\}\).

We define the support, \(\text{supp} \, g\), of \(g \in L^2(\mathbb{R}^k)\) to be the smallest closed set, outside which the function \(g\) vanishes almost everywhere, and \(R_g := \sup_{\lambda \in \text{supp} \, g} \|\lambda\|\) to be the radius of the support of \(g\); \(R_g\) is finite if, and only if, \(g\) has compact support.

**Lemma 2.1.** Let \(g \in L^2(\mathbb{R}^k)\) such that \(\|\lambda\|^{2n} g(\lambda) \in L^2(\mathbb{R}^k)\) for all \(n \in \mathbb{N} \cup \{0\}\). Then

\[
R_g = \lim_{n \to \infty} \left\{ \int_{\mathbb{R}^k} \|\lambda\|^{4n} |g(\lambda)|^2 \, d\lambda \right\}^{1/4n}.
\]

**Proof.** Assume \(g\) has compact support with \(R_g > 0\). Then:

\[
\limsup_{n \to \infty} \left\{ \int_{\mathbb{R}^k} \|\lambda\|^{4n} |g(\lambda)|^2 \, d\lambda \right\}^{1/4n} \leq R_g \limsup_{n \to \infty} \left\{ \int_{\|\lambda\| \leq R_g} |g(\lambda)|^2 \, d\lambda \right\}^{1/4n} = R_g.
\]

On the other hand,

\[
\int_{R_g - \varepsilon \leq \|\lambda\| \leq R_g} |g(\lambda)|^2 \, d\lambda > 0,
\]
for any $\varepsilon > 0$, hence
\[
\liminf_{n \to \infty} \left\{ \int_{\mathbb{R}^k} \|\lambda\|^{4n} |g(\lambda)|^2 d\lambda \right\}^{1/4n} \\
\geq \liminf_{n \to \infty} \left\{ \int_{R_g - \varepsilon \leq \|\lambda\| \leq R_g} \|\lambda\|^{4n} |g(\lambda)|^2 d\lambda \right\}^{1/4n} \\
\geq (R_g - \varepsilon) \liminf_{n \to \infty} \left\{ \int_{R_g - \varepsilon \leq \|\lambda\| \leq R_g} |g(\lambda)|^2 d\lambda \right\}^{1/4n} = R_g - \varepsilon,
\]
and thus
\[
\lim_{n \to \infty} \left\{ \int_{\mathbb{R}^k} \|\lambda\|^{4n} |g(\lambda)|^2 d\lambda \right\}^{1/4n} = R_g.
\]

Now assume that $g$ has unbounded support. Then
\[
\int_{\|\lambda\| \geq N} |g(\lambda)|^2 d\lambda > 0,
\]
for any $N > 0$, so:
\[
\liminf_{n \to \infty} \left\{ \int_{\mathbb{R}^k} \|\lambda\|^{4n} |g(\lambda)|^2 d\lambda \right\}^{1/4n} \\
\geq \liminf_{n \to \infty} \left\{ \int_{\|\lambda\| \geq N} \|\lambda\|^{4n} |g(\lambda)|^2 d\lambda \right\}^{1/4n} \\
\geq N \liminf_{n \to \infty} \left\{ \int_{\|\lambda\| \geq N} |g(\lambda)|^2 d\lambda \right\}^{1/4n} = N,
\]
for arbitrary $N > 0$, and we conclude that
\[
\liminf_{n \to \infty} \left\{ \int_{\mathbb{R}^k} \|\lambda\|^{4n} |g(\lambda)|^2 d\lambda \right\}^{1/4n} = \infty.
\]

Let $L^2_c(\mathbb{R}^k)$ denote the subspace of $L^2(\mathbb{R}^k)$ of functions with compact support and let $L^2_R(\mathbb{R}^k) := \{ g \in L^2(\mathbb{R}^k) \mid R_g = R \}$. Let also $C^\infty_R(\mathbb{R}^k) := \{ g \in C^\infty(\mathbb{R}^k) \mid R_g = R \}$.

**Definition 2.2.** We define the $L^2$-Paley-Wiener space $PW^2(\mathbb{R}^k)$ to be the space of all functions $f \in C^\infty(\mathbb{R}^k)$ satisfying:

(a) $\Delta^n f \in L^2(\mathbb{R}^k)$ for all $n \in \mathbb{N} \cup \{0\}$.

(b) $R^n_f := \lim_{n \to \infty} \|\Delta^n f\|_2^{1/2n} < \infty$. 

\[\square\]
Let also \( PW^2_R(\mathbb{R}^k) := \{ f \in PW^2(\mathbb{R}^k) | R_\lambda^f = R \} \), for \( R \geq 0 \).

The proof of Theorem 2.3 below shows that the limit in (b) above is well-defined. The real version of the \( L^2 \)-Paley-Wiener theorem for the inverse Fourier transform can now be formulated as follows:

**Theorem 2.3.** The inverse Fourier transform \( \mathcal{F}^{-1}_{\text{cl}} \) is a bijection of \( L^2(\mathbb{R}^k) \) onto \( PW^2(\mathbb{R}^k) \), mapping \( L^2_{\text{cl}}(\mathbb{R}^k) \) onto \( PW^2(\mathbb{R}^k) \).

**Proof.** Let \( g \in L^2_{\text{cl}}(\mathbb{R}^k) \). Then \( \| \lambda \|^n g(\lambda) \in L^1(\mathbb{R}^k) \) for all \( n \in \mathbb{N} \cup \{0\} \), and \( \mathcal{F}^{-1}_{\text{cl}} g \in C^\infty(\mathbb{R}^k) \). We also have \( \Delta^n(\mathcal{F}^{-1}_{\text{cl}} g) = \mathcal{F}^{-1}((-1)^n \| \lambda \|^{2n} g) \in L^2(\mathbb{R}^k) \) for all \( n \in \mathbb{N} \cup \{0\} \), by the formula for \( \mathcal{F}^{-1}_{\text{cl}} \), and (1) thus yields:

\[
\lim_{n \to \infty} \left( \frac{1}{(2\pi)^{-k}} \int_{\mathbb{R}^k} |\Delta^n(\mathcal{F}^{-1}_{\text{cl}} g)(x)|^2 dx \right)^{1/4n} = R,
\]

whence \( \mathcal{F}^{-1}_{\text{cl}} g \in PW^2(\mathbb{R}^k) \).

Let now \( f \in PW^2_{\text{cl}}(\mathbb{R}^k) \). Then \( \mathcal{F}(\Delta^n f)(\lambda) = (-1)^n \| \lambda \|^{2n} \mathcal{F}_{\text{cl}} f(\lambda) \in L^2(\mathbb{R}^k) \) for all \( n \in \mathbb{N} \), and another application of (1) shows that

\[
\lim_{n \to \infty} \left( \frac{1}{(2\pi)^{-k}} \int_{\mathbb{R}^k} |\Delta^n(\mathcal{F}_{\text{cl}} f)(\lambda)|^2 d\lambda \right)^{1/4n} = R,
\]

and we conclude that \( \mathcal{F}_{\text{cl}} f \) has compact support with \( R_{\mathcal{F}_{\text{cl}} f} = R \). \( \square \)

**Remark 2.4.** The classical (complex) \( L^2 \)-Paley-Wiener theorem implies that \( PW^2_{\text{cl}}(\mathbb{R}^k) \) exactly consists of those \( L^2(\mathbb{R}^k) \) functions that can be extended to holomorphic functions of exponential type \( R \) on \( \mathbb{C}^k \).

**Remark 2.5.** Let \( f \in PW^2(\mathbb{R}) \). Then \( \frac{d^n}{dx^n} f \in L^p(\mathbb{R}) \) for all \( n \in \mathbb{N} \cup \{0\} \), and:

\[
\lim_{n \to \infty} \left\| \frac{d^n}{dx^n} f \right\|_{L^p}^{1/n} = R_{\mathcal{F}_{\text{cl}} f} = R_\lambda^f,
\]

for all \( 1 \leq p \leq \infty \). This follows from [2, Theorem 1]. Similar results hold for \( \mathbb{R}^k \), \( k > 1 \), see [3, Theorem 3] and [10, Theorem 4].

**Definition 2.6.** We define the Paley-Wiener space \( PW(\mathbb{R}^k) \) as the space of all functions \( f \in C^\infty(\mathbb{R}^k) \) satisfying:

(a) \( (1 + |x|)^m \Delta^n f \in L^2(\mathbb{R}^k) \) for all \( m, n \in \mathbb{N} \cup \{0\} \).

(b) \( R_\lambda^f := \lim_{n \to \infty} \| \Delta^n f \|_2^{1/2n} < \infty \).

Let again \( PW_R(\mathbb{R}^k) := \{ f \in PW(\mathbb{R}^k) | R_\lambda^f = R \} \), for \( R \geq 0 \).
We notice that the only difference between \( \text{PW}^2(\mathbb{R}^k) \) and \( \text{PW}(\mathbb{R}^k) \) is an extra requirement of polynomial decay, to help ensure that \( \mathcal{F}_{cl} f \in C^\infty(\mathbb{R}^k) \).

The real version of the Paley-Wiener theorem for the inverse Fourier transform is the following:

**Theorem 2.7.** The inverse Fourier transform \( \mathcal{F}_{cl}^{-1} \) is a bijection of \( C^\infty_c(\mathbb{R}^k) \) onto \( \text{PW}(\mathbb{R}^k) \), mapping \( C^\infty_k(\mathbb{R}^k) \) onto \( \text{PW}_R(\mathbb{R}^k) \).

**Proof.** Let \( g \in C^\infty_k(\mathbb{R}^k) \), then \( \mathcal{F}_{cl}^{-1} g \in \mathcal{S}(\mathbb{R}^k) \), and \( \mathcal{F}_{cl}^{-1} g \in \text{PW}_R^2(\mathbb{R}^k) \) by Theorem 2.3.

Let \( f \in \text{PW}_R(\mathbb{R}^k) \subset \text{PW}^2_R(\mathbb{R}^k) \). Then \( \mathcal{F}_{cl} f \in C^\infty(\mathbb{R}^k) \) since \( f \) has polynomial decay, and \( \mathcal{F}_{cl} f \) has compact support with \( R_{\mathcal{F}_{cl}} f = R \) by Theorem 2.3. \( \square \)

### 3. Lie group notation.

In this section we introduce the Lie group notation we need in the next sections. We refer to [5], [6] and [7] for further details.

Let \( G \) be a real connected noncompact semisimple Lie group with finite center and let \( \theta \) be a Cartan involution of \( G \). Then the fixed point group \( K := G^\theta \) is a maximal compact subgroup. Let \( g \) and \( \mathfrak{k} \) denote their Lie algebras, and let \( g = \mathfrak{k} \oplus \mathfrak{p} \) be the Cartan decomposition of \( g \) into the \( \pm 1 \) eigenspaces of \( \theta \). The Killing form on \( g \) induces an \( AdK \)-invariant scalar product on \( \mathfrak{p} \) and hence a \( G \)-invariant Riemannian metric on \( X := G/K \). With this structure, \( X \) becomes a Riemannian globally symmetric space of the noncompact type.

Fix a maximal abelian subspace \( \mathfrak{a} \) of \( \mathfrak{p} \). Denote its real dual by \( \mathfrak{a}^* \) and its complex dual by \( \mathfrak{a}_C^* \). The Killing form of \( g \) induces a scalar product \( \langle \cdot, \cdot \rangle \) and hence a norm \( \| \cdot \| \) on \( \mathfrak{a}_C \) and \( \mathfrak{a}_C^* \). Let \( \Sigma \subset \mathfrak{a}^* \) denote the root system of \( (g, \mathfrak{a}) \) and let \( W \) be the associated Weyl group. Choose a set \( \Sigma_+ \subset \Sigma \) of positive roots, let \( n := \bigoplus_{\alpha \in \Sigma_+} \mathfrak{g}_\alpha \) be the corresponding nilpotent subalgebra of \( g \) and let \( a_+ := \{ H \in \mathfrak{a} | \alpha(H) > 0 \forall \alpha \in \Sigma_+ \} \) be the positive Weyl chamber with \( \overline{a_+} \) it's closure. Denote by \( a_+^\ast \) and \( \overline{a_+} \) the similar cones in \( \mathfrak{a}^* \), and define the element \( \rho \in \mathfrak{a}^* \) by: \( \rho(H) := \frac{1}{2} \sum_{\alpha \in \Sigma_+} m_\alpha \alpha(H), H \in \mathfrak{a} \), where \( m_\alpha = \dim \mathfrak{g}_\alpha \).

Let \( g = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \) be the Iwasawa decomposition of \( g \) and \( G = KAN = NAK \) the corresponding Iwasawa decompositions of \( G \), where \( A \) and \( N \) are the Lie groups generated by \( \mathfrak{a} \) and \( \mathfrak{n} \) respectively. Every \( g \in G \) can be represented as: \( g = K \exp H(g)N = N \exp A(g)K \), where the projections onto the \( A \)-parts \( A(g) \in \mathfrak{a} \) and \( H(g) \in \mathfrak{a} \) are uniquely determined. We note that \( A(g) = -H(g^{-1}) \). Let \( M := Z_K(\mathfrak{a}) \), then \( B := K/M \) is a compact homogeneous space. We define the vector \( A(x, b) \in \mathfrak{a} \) as \( A(x, b) := A(k^{-1}g) \), for \( x = gK \in X \) and \( b = kM \in B \).

Put \( A_+ = \exp(a_+) \), then \( A_+ = \exp(\overline{a_+}) \). The Cartan decomposition implies that the natural mapping from \( K/M \times A_+ \times K \) into \( G = K\overline{A_+}K \) is a
diffeomorphism onto its dense open image. We define the norm of an element $g \in G$ as: $|g| = |k_1 \exp(H)k_2| = \|H\|$, with $H \in \mathfrak{a}_+^\perp$; this is the $K$-invariant geodesic distance to the origin $eK$. We denote by $B_R := \{g \in G \mid |g| \leq R\}$ the $K$-invariant ball of radius $R$ around $e$.

We identify functions on $X$ with right-$K$-invariant functions on $G$. We normalise the invariant measure on $X$ as:

$$\int_X f(x)dx = \int_K \int_{\mathfrak{a}_+} \int_K f(k_1 \exp(H)k_2)J(H)dk_1dHdk_2,$$

for $f \in C^\infty_c(X)$, where the Jacobian $J$ is given by: $J(H) = \prod_{\alpha \in \Sigma_+} (e^{\alpha(H)} - e^{-\alpha(H)})^{m_\alpha}$, $dH$ is the Lebesgue measure on $\mathfrak{a}$ and $dk$ is the measure on $K$ such that $\int_K dk = 1$. We notice that $0 \leq J(H) \leq Ce^{2\rho(H)}$, for $H \in \mathfrak{a}_+^\perp$, where $C$ is a positive constant.

The spherical functions $\varphi_\lambda$, $\lambda \in \mathfrak{a}_C^*$, on $G$ are defined as:

$$\varphi_\lambda(g) := \int_K e^{i(\lambda + \rho)A(k^{-1}g)}dk = \int_K e^{i(\lambda + \rho)H(g^{-1}k)}dk.$$

We note that $\varphi_\lambda$ is Weyl group invariant, $\varphi_{w\lambda} = \varphi_\lambda$, $w \in W$. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. We write $Df(g)$ for the action of $D \in U(\mathfrak{g})$ on $f \in C^\infty(G)$ from the left at $g \in G$. The $L^p$-Schwartz space $S^p(X)$, $0 < p \leq 2$, is defined as the space of all functions $f \in C^\infty(X)$ such that:

$$\sup_{g \in G} (1 + |g|)^m \varphi_\lambda(g)^{-\frac{2}{p}}|Df(g)| < \infty,$$

for all $D \in U(\mathfrak{g})$ and $m \in \mathbb{N} \cup \{0\}$. We can also characterise $S^p(X)$ as the space of all functions $f \in C^\infty(X)$ satisfying:

$$(1 + |g|)^m Df(g) \in L^p(X),$$

for all $D \in U(\mathfrak{g})$ and $m \in \mathbb{N} \cup \{0\}$. We note that $S^p(X) \not\subset L^p(X)$ for $0 < q < p \leq 2$.

### 4. The Fourier transform.

In this section, we recall some facts and theorems for the Fourier transform on a noncompact semisimple Riemannian symmetric space, see [7, Chapter 3] for details and references.

The Fourier transform of a function $f$ on $X$ is defined as:

$$\mathcal{F}f(\lambda, b) := \int_X f(x)e^{-(i\lambda + \rho)(A(x,b))}dx,$$

for all $\lambda \in \mathfrak{a}_C^*$, $b \in B$ for which the integral exists. In particular, $\mathcal{F}f$ extends to a smooth function on $\mathfrak{a}_C^* \times B$, holomorphic in the first variable, for $f \in C^\infty_c(X)$, see also below.
The plane wave eigenfunction

\[ e_{\lambda,b}(x) := e^{(i\lambda + \rho)(A(x,b))}, \]

is a joint eigenfunction of \( \mathbb{D}(X) \), the commutative algebra of \( G \)-invariant differential operators on \( X \), for all \( \lambda \in \mathfrak{a}_C^* \), \( b \in B \), or, more precisely

\[ De_{\lambda,b} = \Gamma(D)(i\lambda)e_{\lambda,b}, \quad \forall D \in \mathbb{D}(X), \quad (\lambda \in \mathfrak{a}_C^*, b \in B) \]

where \( \Gamma : \mathbb{D}(X) \rightarrow S(\mathfrak{a}^*)^W \) is the Harish-Chandra isomorphism. In particular,

\[ \Delta e_{\lambda,b} = -((\lambda, \lambda) + \|\rho\|^2)e_{\lambda,b}, \quad (\lambda \in \mathfrak{a}_C^*, b \in B) \]

for the Laplace-Beltrami operator \( \Delta \) on \( X \), and hence

\[ \mathcal{F}(\Delta f)(\lambda, b) = -((\lambda, \lambda) + \|\rho\|^2)\mathcal{F}f(\lambda, b), \quad (\lambda \in \mathfrak{a}_C^*, b \in B) \]

for all \( f \in C^\infty_c(X) \), by self-adjointness of \( \Delta \), see also (6).

A \( C^\infty \)-function \( \psi(\lambda, b) \) on \( \mathfrak{a}_C^* \times B \), holomorphic in \( \lambda \), is called a holomorphic function of uniform exponential type \( R \), if there exists a constant \( R \geq 0 \), such that, for each \( N \in \mathbb{N} \), we have:

\[ \sup_{\lambda \in \mathfrak{a}_C^*, b \in B} e^{-R|\Im(\lambda)|} |\psi(\lambda, b)| < \infty. \]

The space of holomorphic functions of uniform exponential type \( R \) will be denoted \( \mathcal{H}_R(\mathfrak{a}_C^* \times B) \) and we denote by \( \mathcal{H}(\mathfrak{a}_C^* \times B) \) their union over all \( R > 0 \).

Let furthermore \( \mathcal{H}(\mathfrak{a}_C^* \times B)^W \) denote the space of all functions \( \psi \in \mathcal{H}(\mathfrak{a}_C^* \times B) \) satisfying the symmetry condition:

\[ \int_B e^{i(w\lambda + \rho)(A(x,b))} \psi(w, \lambda, b) db = \int_B e^{i(\lambda + \rho)(A(x,b))} \psi(\lambda, b) db, \]

for \( w \in W \) and all \( \lambda \in \mathfrak{a}_C^*, x \in X \).

The Paley-Wiener theorem states that the Fourier transform is a bijection of the space \( C^\infty_c(X) \) onto the space \( \mathcal{H}(\mathfrak{a}_C^* \times B)^W \), with the following inversion formula:

\[ f(x) = \int_{\mathfrak{a}_C^*} \int_B e^{i(\lambda + \rho)(A(x,b))} \mathcal{F}f(\lambda, b)|c(\lambda)|^{-2} d\lambda db, \quad (x \in X) \]

where \( c(\lambda) \) is the Harish-Chandra \( c \)-function, for \( f \in C^\infty_c(X) \). Moreover, \( \mathcal{F}f \in \mathcal{H}(\mathfrak{a}_C^* \times B)^W \) if, and only if, \( \text{supp} f \subset B_R \). We note that \( |c(\lambda)|^{-2} \) is bounded by some polynomial for \( \lambda \in \mathfrak{a}^* \).

Let \( f_1, f_2 \in C^\infty_c(X) \), then Parseval’s formula for \( \mathcal{F} \) is given by:

\[ \int_X f_1(x)f_2(x) dx = \int_{\mathfrak{a}_C^*} \int_B \mathcal{F}f_1(\lambda, b)\overline{\mathcal{F}f_2(\lambda, b)}|c(\lambda)|^{-2} d\lambda db. \]

We conclude that the Fourier transform extends to an isometry of \( L^2(X) \) onto \( L^2(\mathfrak{a}_C^* \times B, |c(\lambda)|^{-2} d\lambda db) \). In the following we adopt the convention \( L^2(\mathfrak{a}_C^* \times B) := L^2(\mathfrak{a}_C^* \times B, |c(\lambda)|^{-2} d\lambda db) \).
Let \( f \in C^\infty(X) \) such that \( \Delta^n f \in L^2(X) \) for all \( n \in \mathbb{N} \cup \{0\} \) and let \( f_2 \in C_c^\infty(X) \). Then self-adjointness of the Laplace-Beltrami operator \( \Delta \):

\[
\int_X \Delta^n f(x)f_2(x)dx = \int_X f(x)\Delta^n f_2(x)dx,
\]

Parseval’s formula (5) and density of \( C_c^\infty(X) \) imply, as in the classical case, that

\[
\mathcal{F}(\Delta^n f)(\lambda, b) = (-1)^n(\|\lambda\|^2 + \|\rho\|^2)^n \mathcal{F}f(\lambda, b),
\]
a.e., for all \( n \in \mathbb{N} \cup \{0\} \).

5. The inverse Fourier transform.

We define the inverse Fourier transform \( \mathcal{F}^{-1} \) of a function \( g \) on \( a_+^* \times B \) via (4):

\[
\mathcal{F}^{-1}g(x) := \int_{a_+^*} \int_B e^{i(\lambda+\rho)(A(x,b))} g(\lambda, b)|e(\lambda)|^{-2}d\lambda db,
\]

for all \( x \in X \) for which the integral exists.

We define the support, \( \text{supp} \, g \), of \( g \in L^2(a_+^* \times B) \) to be the smallest closed set in \( a_+^* \times B \), outside which the function \( g \) vanishes almost everywhere, and \( R_g := \sup(\lambda, b) \in \text{supp} \, g \|\lambda\| \) to be the ‘radius’ of the support of \( g \).

**Lemma 5.1.** Let \( g \in L^2(a_+^* \times B) \) such that \( \|\lambda\|^{2n}g(\lambda, b) \in L^2(a_+^* \times B) \) for all \( n \in \mathbb{N} \cup \{0\} \). Then

\[
R_g = \lim_{n \to \infty} \left\{ \int_{a_+^*} \int_B \|\lambda\|^{4n}|g(\lambda, b)|^2|e(\lambda)|^{-2}d\lambda db \right\}^{1/4n}.
\]

**Proof.** As for Lemma 2.1. \( \square \)

Let \( L^2_c(a_+^* \times B) \) denote the subspace of \( L^2(a_+^* \times B) \) of functions with bounded support and let \( L^2_R(a_+^* \times B) := \{ g \in L^2_c(a_+^* \times B) \mid R_g = R \} \).

**Definition 5.2.** We define the \( L^2 \)-Paley-Wiener space \( \text{PW}^2(X) \) as the space of all functions \( f \in C^\infty(X) \) satisfying:

(a) \( \Delta^n f \in L^2(X) \) for all \( n \in \mathbb{N} \cup \{0\} \).

(b) \( R_f^2 := \lim_{n \to \infty} \|\Delta + \|\rho\|^2\|^n f\|_2^{1/2n} < \infty \).

Let also \( \text{PW}^2_R(X) := \{ f \in \text{PW}^2(X) \mid R_f^2 = R \} \), for \( R \geq 0 \).

The real \( L^2 \)-Paley-Wiener theorem for the inverse Fourier transform can now be formulated as follows:

**Theorem 5.3.** The inverse Fourier transform \( \mathcal{F}^{-1} \) is a bijection of \( L^2_c(a_+^* \times B) \) onto \( \text{PW}^2(X) \), mapping \( L^2_R(a_+^* \times B) \) onto \( \text{PW}^2_R(X) \).
Proof. Let $g \in L^2_R(a^*_+ \times B)$. Then $\mathcal{F}^{-1}g \in C^\infty(X)$ by Lebesgue’s dominated convergence theorem. Equation (2) gives, for $D \in \mathbb{D}(X)$,

$$D(\mathcal{F}^{-1}g)(x) = \int_{a^*_+} \int_B \Gamma(D)(i\lambda)e^{(\lambda+\rho)(A(x,b))}g(\lambda,b)|c(\lambda)|^{-2}d\lambda db,$$

which in particular shows that $(\Delta + \|\rho\|)^n \mathcal{F}^{-1}g = \mathcal{F}^{-1}((-1)^n\|\lambda\|^{2n}g) \in L^2(X)$ for all $n \in \mathbb{N} \cup \{0\}$. Parseval’s formula (5) with

$$f_1 = f_2 = \mathcal{F}^{-1}((-1)^n\|\lambda\|^{2n}g)$$

yields:

$$\lim_{n \to \infty} \left\{ \int_X |(\Delta + \|\rho\|)^n(\mathcal{F}^{-1}g)(x)|^2 dx \right\}^{1/4n} = \lim_{n \to \infty} \left\{ \int_{a^*_+} \int_B \|\lambda\|^{4n}|g(\lambda,b)|^2|c(\lambda)|^{-2}d\lambda db \right\}^{1/4n} = R,$$

whence $\mathcal{F}^{-1}g \in PW^2_R(X)$.

Let now $f \in PW^2_R(X)$. Then $\mathcal{F}((\Delta + \|\rho\|)^nf)(\lambda,b) = (-1)^n\|\lambda\|^{2n}F_f(\lambda,b) \in L^2(a^*_+ \times B)$ for all $n \in \mathbb{N}$ by (7). Another application of Parseval’s formula as above with $f_1 = f_2 = (\Delta + \|\rho\|)^nf$ shows that $R_{\mathcal{F}f} = R_{\mathcal{F}^2} = R$, and we conclude that $\mathcal{F}f$ has bounded support. \qed

**Corollary 5.4.** Let $f \in C^\infty(X)$ be such that $\Delta^n f \in L^2(X)$ for all $n \in \mathbb{N} \cup \{0\}$. It then follows that $\lim_{n \to \infty} \|\Delta^n f\|_2^{1/2n} < \infty$ if, and only if, $\lim_{n \to \infty} \|\Delta + \|\rho\||^{2n}f\|_2^{1/2n} < \infty$. Furthermore, $\lim_{n \to \infty} \|\Delta^n f\|_2^{1/2n} = (R^2 + \|\rho\|^2)^{1/2}$, for $f \in PW^2_R(X)$ with $R > 0$.

**Proof.** Let $f \in PW^2_R(X)$, with $R > 0$, then $\mathcal{F}f \in L^2_R(a^*_+ \times B)$. Parseval’s formula and an easy adaption of the proof of Lemma 2.1 shows that

$$\lim_{n \to \infty} \|\Delta^n f\|_2^{1/2n} = \lim_{n \to \infty} \left\{ \int_{a^*_+} \int_B (\|\lambda\|^2 + \|\rho\|^2)^{2n}|\mathcal{F}f(\lambda,b)|^2|c(\lambda)|^{-2}d\lambda db \right\}^{1/4n} = (R^2 + \|\rho\|^2)^{1/2}.$$
Assume that \( \lim_{n \to \infty} \|\Delta^n f\|_2^{1/2n} < \infty \). Then \( \mathcal{F}(\Delta^n f)(\lambda, b) = (-1)^n(\|\lambda\|^2 + \|\rho\|^2)^n \mathcal{F}f(\lambda, b) \in L^2(a_+^* \times B) \), for all \( n \in \mathbb{N} \), and

\[
\lim_{n \to \infty} \left\{ \int_{a_+^-} \int_B \|\lambda\|^{4n} |\mathcal{F}f(\lambda, b)|^2 |c(\lambda)|^{-2} d\lambda db \right\}^{1/4n} \\
\leq \lim_{n \to \infty} \left\{ \int_{a_+^-} \int_B (\|\lambda\|^2 + \|\rho\|^2)^{2n} |\mathcal{F}f(\lambda, b)|^2 |c(\lambda)|^{-2} d\lambda db \right\}^{1/4n} \\
= \lim_{n \to \infty} \|\Delta^n f\|_2^{1/2n} < \infty,
\]
that is, \( \mathcal{F}f \) has bounded support.

\[\textbf{Remark 5.5.} \text{ Assume that } f \in \mathcal{S}^p(X), \text{ with } 0 < p < 2, \text{ then } \mathcal{F}f \text{ extends to an analytic function on a small tube domain around } a^* \times B \text{ in } a_+^* \times B. \text{ Hence } \mathcal{F}f \text{ cannot have compact support on } a^* \times B \text{ and we conclude that } \mathcal{S}^p(X) \cap \mathcal{P}^2(X) = \{0\} \text{ for any } 0 < p < 2.\]

\[\textbf{Definition 5.6.} \text{ We define the Paley-Wiener space } \mathcal{P}^2(X) \text{ as the space of all functions } f \in C^\infty(X) \text{ satisfying:}
\]

(a) \((1 + |x|)^m \Delta^n f \in L^2(X) \) for all \( m, n \in \mathbb{N} \cup \{0\}\).

(b) \( R^2 f = \lim_{n \to \infty} \|\Delta + \|\rho\|^2\| f\|_2^{1/2n} < \infty \).

Let also \( \mathcal{P}^2(X) := \{ f \in \mathcal{P}(X) \mid R^2 f = R \} \), for \( R \geq 0 \).

Here \( |x| := |g| \), for \( x = gK \in X \). Again, the only difference between the Paley-Wiener spaces \( \mathcal{P}^2(X) \) and \( \mathcal{P}^2(X) \) is the polynomial decay condition (a), ensuring that \( \mathcal{F}f \in C^\infty(a^* \times B)^W \) (see below).

The space \( C^\infty_c(a^* \times B)^W \) is defined as the subspace of functions \( \psi \in C^\infty_c(a^* \times B) \) satisfying the symmetry condition (3) for all \( w \in W \) and all \( \lambda \in a^*, x \in X \). Let finally \( C^\infty_c(a^* \times B)^W := \{ f \in C^\infty_c(a^* \times B) \mid R_g = R \} \).

The real Paley-Wiener theorem for the inverse Fourier transform then is:

\[\textbf{Theorem 5.7.} \text{ The inverse Fourier transform } \mathcal{F}^{-1} \text{ is a bijection of } C^\infty_c(a^* \times B)^W \text{ onto } \mathcal{P}^2(X), \text{ mapping } C^\infty_c(a^* \times B)^W \text{ onto } \mathcal{P}^2(X).\]

\[\textbf{Proof.} \text{ Let } g \in C^\infty_c(a^* \times B)^W, \text{ then } g \in L^2_R(a_+^* \times B) \text{ and thus } \mathcal{F}^{-1}g \in \mathcal{P}^2(X) \text{ by Theorem 5.3. We furthermore see that } \mathcal{F}^{-1}g \in \mathcal{S}^2(X) \text{ by [4, Theorem 4.1.1]}, whence } \mathcal{F}^{-1}g \text{ satisfies the polynomial decay condition (a).}

\text{Let now } f \in \mathcal{P}^2(X). \text{ The basic estimate } \|A(g)\| \leq C|g|, \text{ for all } g \in G, \text{ gives us a polynomial estimate (in } x \text{) of the derivatives (with respect to } \lambda \text{) of the plane wave eigenfunctions } e_{\lambda,b}(x). \text{ It is also well-known that } (1 + |x|)^{-r}\varphi_0 \in L^2(X) \text{ for some large } r \in \mathbb{N}. \text{ All this, the polynomial decay condition (a), the Cauchy-Schwartz theorem and Lebesgue's dominated convergence theorem imply that } \mathcal{F}f \in C^\infty_c(a^* \times B)^W. \text{ Furthermore } \mathcal{F}f \text{ has the desired compact support by Theorem 5.3.} \]
6. The inverse spherical transform.

In this section, we specialise our results to bi-$K$-invariant functions, that is, we consider the (inverse) spherical transform. We refer to [1], [5] and [6] for background concerning Paley-Wiener theorems for the spherical transform. Let $C^\infty(K\backslash G/K) \subset C^\infty(G)$ denote the subspace of bi-$K$-invariant differentiable functions on $G$. We will use similar notation for the $L^2$, Paley-Wiener and Schwartz spaces of $K$-invariant differentiable functions.

Let $f \in C^\infty_c(K\backslash G/K)$. The spherical transform $\mathcal{H}f$ of $f$ is defined as:

$$\mathcal{H}f(\lambda) := \int_G f(x)\varphi_{-\lambda}(x)dx,$$

for $\lambda \in a^*_C$. We note that $\mathcal{F}f(\lambda, b) = \mathcal{H}f(\lambda)$ for all $\lambda \in a^*_C$ and all $b \in B$. This follows from left-$K$-invariance of $f$, the identity $A(k \cdot x, b) = A(x, k^{-1} \cdot b)$ and integrating over $K$.

The spherical transform is an isomorphism of $\mathcal{S}^2(K\backslash G/K)$ onto $S(a^*_W)$, the Weyl group invariant Schwartz functions on $a^*$. The inversion formula is given by:

$$f(x) = \frac{1}{|W|} \int_{a^*} \mathcal{H}f(\lambda)\varphi_{\lambda}(x)|c(\lambda)|^{-2}d\lambda, \quad (x \in G) \tag{8}$$

for $f \in \mathcal{S}^2(K\backslash G/K)$. We use (8) to define the inverse spherical transform $\mathcal{H}^{-1}g$ for a general function $g$ on $a^*$:

$$\mathcal{H}^{-1}g(x) := \int_{a^*} g(\lambda)\varphi_{\lambda}(x)|c(\lambda)|^{-2}d\lambda.$$

Let $f \in C^\infty(K\backslash G/K)$ be such that $\Delta^nf \in L^2(K\backslash G/K)$ for all $n \in \mathbb{N} \cup \{0\}$. Then $\mathcal{H}((\Delta + \|\rho\|^2)^nf)(\lambda) = (-1)^n\|\lambda\|^{2n}\mathcal{H}f(\lambda)$ a.e., and Parseval’s formula for $\mathcal{H}$ gives:

$$\int_G |(\Delta + \|\rho\|^2)^nf(x)|^2dx = \frac{1}{|W|} \int_{a^*} \|\lambda\|^{4n}|\mathcal{H}f(\lambda)|^2|c(\lambda)|^{-2}d\lambda,$$

for all $n \in \mathbb{N} \cup \{0\}$. It also follows that the spherical transform extends to an isometry from $L^2(K\backslash G/K)$ onto $L^2(a^*, \frac{1}{|W|}|c(\lambda)|^{-2}d\lambda)^W$, where superscript $W$ denotes Weyl group invariance.

Let $L^2_c(a^*_W)$ denote the Weyl group invariant $L^2$-functions on $a^*$ with compact support and let subscript $R$ denote the radius of the support. The real versions of the Paley-Wiener theorems for the inverse spherical transform then becomes:

**Theorem 6.1.** The inverse spherical transform $\mathcal{H}^{-1}$ is a bijection of $L^2_c(a^*_W)$ onto $PW^2(K\backslash G/K)$, mapping $L^2_{2R}(a^*_W)$ onto $PW^2_{2R}(K\backslash G/K)$.

**Theorem 6.2.** The inverse spherical transform $\mathcal{H}^{-1}$ is a bijection of $C^\infty_{cR}(a^*_W)$ onto $PW(K\backslash G/K)$, mapping $C^\infty_{2R}(a^*_W)$ onto $PW_{2R}(K\backslash G/K)$. 
Proof. The above theorems are special cases of Theorem 5.3 and Theorem 5.7. We note, however, that we can prove them independently using Parseval’s formula and intertwining properties of $\mathcal{H}$. □

Remark 6.3. Let $f \in \mathrm{PW}(K\backslash G/K)$ and consider $f$ as a function on $\mathfrak{a}$ by the application $H \mapsto f(\exp(H))$. Then $f$ does not extend to an entire function on $\mathfrak{a}_C$, due to the poles of the spherical function $\varphi_\lambda(\exp(H))$. There is, however, a description of the Paley-Wiener space $\mathrm{PW}(K\backslash G/K)$ as functions having an explicit meromorphic extension and satisfying some exponential growth conditions for the rank 1 and the complex cases, see [8] for details.

References


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ON SURFACES OF PRESCRIBED $F$-MEAN CURVATURE

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Hypersurfaces of prescribed weighted mean curvature, or $F$-mean curvature, are introduced as critical immersions of anisotropic surface energies, thus generalizing minimal surfaces and surfaces of prescribed mean curvature. We first prove enclosure theorems in $\mathbb{R}^{n+1}$ for such surfaces in cylindrical boundary configurations. Then we derive a general second variation formula for the anisotropic surface energies generalizing corresponding formulas of do Carmo for minimal surfaces, and Sauvigny for prescribed mean curvature surfaces. Finally we prove that stable surfaces of prescribed $F$-mean curvature in $\mathbb{R}^3$ can be represented as graphs over a planar strictly convex domain $\Omega$, if the given boundary contour in $\mathbb{R}^3$ is a graph over $\partial \Omega$.

1. Introduction and main results.

Let $X : M \to \mathbb{R}^{n+1}$, $n \geq 2$, be an immersion of class $C^3(M, \mathbb{R}^{n+1})$ of an $n$-dimensional smooth manifold $M = M^n$ with boundary $\partial M$ into $\mathbb{R}^{n+1}$. We denote the corresponding unit normal by $N$ and the induced area element by $dA$, and consider general parametric variational functionals $F$ of the form

$$\mathcal{F}(X) := \int_M F(X, N) \, dA. \tag{1.1}$$

The integrand $F$ of class $C^0(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\}))$ is a parametric Lagrangian characterized by the homogeneity condition

$$(H) \quad F(y, tz) = tF(y, z) \quad \text{for all } t > 0, (y, z) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}. \tag{1.2}$$

Note that $(H)$ implies

$$F_{zz}(y, z) = 0 \quad \text{for all } (y, z) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\}); \tag{1.3}$$

hence we will identify the symmetric endomorphism $F_{zz}(y, z) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with its restriction to the space

$$(1.3) \quad z^\perp := \{\zeta \in \mathbb{R}^{n+1} : \langle \zeta, z \rangle = 0\}.$$ 

Important examples of parametric Lagrangians are given by the area integrand

$$A(z) := |z|, \quad A(z) := \sqrt{\sum_{i=1}^{n+1} z_i^2}, \tag{1.4}$$
and the integrand
\[ E(y, z) := |z| + \langle Q(y), z \rangle \]
appearing in the theory of capillary surfaces. Here, \( Q \) can be chosen as a differentiable vector field in \( \mathbb{R}^{n+1} \) with \( \text{div}_{\mathbb{R}^{n+1}} Q(y) = H(y) \), where \( H(y) \) is a given function representing the prescribed mean curvature. Critical immersions of the corresponding functionals
\[ A(X) := \int_M A(N) \, dA = \int_M dA \]
and
\[ E(X) := \int_M E(X, N) \, dA \]
are minimal surfaces and surfaces of prescribed mean curvature \( H(X) \), respectively.

Another interesting example is
\[ F(z) = \sum_{j=1}^{3} \sqrt{\delta^2 |z|^2 + z_j^2}, \quad \delta > 0, \]
which serves as a regularized version of the discrete \( l^1 \)-norm used for numerical computations involving the anisotropic mean curvature flow [7]. Furthermore, in surface processing [3] such parametric functionals have become an increasingly important tool to enhance edge structures within a suitable surface evolution based on (1.6) and (1.8). For more examples of integrands and applications in numerical analysis we refer to [8] and [6].

For general parametric integrals we recall the notion of the \( F \)-mean curvature
\[ H_F(X, N) = H_F := -\text{tr}(A_F S), \]
as introduced in [2] and [4]. Here, \( S \in \text{End}(TM) \) is the shape operator defined by \( DX \circ S := DN \) on the tangent bundle \( TM \), and \( A_F \in \text{End}(TM) \) is the symmetric endomorphism field given by
\[ A_F := (DX)^{-1}(F_{zz}(X, N)DX) \quad \text{on} \quad TM. \]
For the special parametric Lagrangians in (1.6) and (1.7) the \( F \)-mean curvature \( H_F \) reduces to the classical mean curvature \( H \), since \( A_F|_{T_w M} = \text{Id}_{T_w M} \) for each \( w \in M \) and \( F(y, z) = A(z) \), or \( F(y, z) = E(y, z) \), respectively. Here \( T_w M \) denotes the tangent space of \( M \) at \( w \in M \).

The first author proved in [2] that the Euler equation for \( F \) can be written as
\[ H_F = \sum_{i=1}^{n+1} F_{y^i z^i}(X, N). \]
Consequently, given a general parametric Lagrangian \( F = F(y, z) \), critical immersions of the corresponding parametric functional \( \mathcal{F} \) may be viewed as \textit{surfaces of prescribed \( F \)-mean curvature}. In particular, we will regard critical immersions of the specific parametric functional

\[
\mathcal{F}_0(X) := \int_M F(N) \, dA + \int_M \langle Q(X), N \rangle \, dA,
\]

where \( \text{div}_{\mathbb{R}^{n+1}} Q(y) = \mathcal{H}_F(y) \in C^0(\mathbb{R}^{n+1}) \) is a given function, as \textit{surfaces of prescribed \( F \)-mean curvature}. This class of surfaces yields a natural generalization of minimal surfaces if \( \mathcal{H}_F(y) \equiv 0 \), or of surfaces of constant mean curvature if \( \mathcal{H}_F(y) \equiv \mathcal{H}^0_F \in \mathbb{R} \). Let us point out that the parametric Lagrangian \( F(z) \) in (1.12) depends on \( z \) only, and that in case \( \mathcal{H}_F(y) \equiv \mathcal{H}^0_F \in \mathbb{R} \) the second integrand in (1.12) is linear in \( y \) and \( z \) and can be interpreted as a volume term.

As a starting point for our investigations we will derive in Section 2 a differential equation for the surface normal of an arbitrary immersion in terms of the \( F \)-Laplace-Beltrami operator

\[
\Delta_F := \text{div} (A_F \text{grad}(\cdot)),
\]

where the differential operators are taken with respect to the induced metric

\[
g_w(V, W) = g(V, W) := \langle DX(V), DX(W) \rangle \quad \text{for} \quad V, W \in T_wM, w \in M,
\]

i.e., \( \text{div} = \text{div}_M \) and \( \text{grad} = \text{grad}_M \).

\textbf{Theorem 1.1.} Let \( N \) be the normal of an arbitrary immersion \( X \) of class \( C^3(M, \mathbb{R}^{n+1}) \) and let \( F \in C^0(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})) \) be a parametric Lagrangian. Then

\[
\Delta_F N + \text{tr} (A_F S^2) N = DX(\text{div} (S A_F)).
\]

Here, \( \text{div} (S A_F) \) denotes the divergence of the endomorphism field \( S A_F \); see Section 2 for details.

In Section 3 we consider hypersurfaces with bounded \( F \)-mean curvature spanning\(^1\) a given Jordan curve \( \Gamma \subset \mathbb{R}^{n+1} \), i.e., we take an immersion \( X : M \to \mathbb{R}^{n+1} \) mapping the boundary \( \partial M \) topologically onto \( \Gamma \).

A parametric Lagrangian \( F(y, z) \) is said to be (uniformly) \textit{elliptic}, if there exists a constant \( M_1 > 0 \) such that

\[
|z| \langle \zeta, F_{zz}(y, z) \zeta \rangle \geq M_1 |\zeta|^{\tan}|^2
\]

\(^1\)The existence of conformally parametrized \( \mathcal{F} \)-minimizing surfaces under Plateau type boundary conditions was proven in [14] and [15] for \( n = 2 \) and arbitrary co-dimension, but these solutions might have branch points. For the restricted class of boundary contours considered in Theorem 1.2, White [24] has constructed an embedded \( \mathcal{F} \)-minimizing disk in \( \mathbb{R}^3 \).
for all \((y,z) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\}), \zeta \in \mathbb{R}^{n+1},\) where \(\zeta^{\tan} := \zeta - \langle \zeta, z \rangle z/|z|^2.\) Notice that \(A(z)\) and \(E(y, z)\) as defined in (1.4) and (1.5) are elliptic satisfying (E) with \(M_1 = 1.\)

Surfaces of vanishing \(F\)-mean curvature, where \(F\) is elliptic, have the convex hull property as proven in [2, Thm. 2.3]:

**Theorem 1.2.** Let \(F = F(z) \in C^0(\mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1} \setminus \{0\})\) be a parametric Lagrangian satisfying (E). Suppose \(X \in C^0(M, \mathbb{R}^{n+1}) \cap C^2(M, \mathbb{R}^{n+1})\) is an immersion of vanishing \(F\)-mean curvature, i.e., with \(H_F(X, N) = \mathcal{H}_F(X) \equiv 0.\) If \(X\) spans a Jordan curve \(\Gamma \subset \mathbb{R}^{n+1}\) contained in the boundary of a closed convex set \(K \subset \mathbb{R}^{n+1},\) then \(X(M) \subset K.\)

For surfaces of bounded (but not necessarily vanishing) \(F\)-mean curvature spanning a given Jordan curve \(\Gamma\) within the infinite cylinder
\[(1.16)\]
\[\mathcal{Z}_h := \left\{ (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : h\sqrt{(x^1)^2 + \cdots + (x^n)^2} \leq 1 \right\}, \quad h \geq 0,\]
we restrict our attention to Jordan curves \(\Gamma \subset \mathbb{R}^{n+1}\) with an orthogonal projection onto an \(h\)-convex domain \(\overline{\Omega} \subset B_{h^{-1}}(0) \subset \mathbb{R}^n.\) Following Sauvigny [21] we call a bounded convex domain \(\Omega \subset \mathbb{R}^n\) \(\kappa\)-convex for some \(\kappa > 0,\) if for every \(w_0 \in \partial \Omega\) there is a point \(\xi_0 = \xi_0(w_0) \in \mathbb{R}^n\) such that the ball \(B_{1/\kappa}(\xi_0) \subset \mathbb{R}^n\) contains \(\Omega\) and such that \(w_0 \in \partial B_{1/\kappa}(\xi_0).\)

**Theorem 1.3.** Let \(F = F(y, z) \in C^0(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\}))\) be a parametric Lagrangian satisfying (E). Suppose \(X : \overline{M} \to \mathcal{Z}_h\) of class \(C^0(M, \mathbb{R}^{n+1}) \cap C^2(M, \mathbb{R}^{n+1})\) is an immersion of prescribed \(F\)-mean curvature \(\mathcal{H}_F \in C^0(\mathbb{R}^{n+1}),\) where \(\mathcal{H}_F(y)\) satisfies\n\[(1.17)\]
\[\|\mathcal{H}_F\|_{C^0(\mathbb{R}^{n+1})} \leq M_1 h(n-1),\]
and \(X\) spans a curve \(\Gamma \subset \mathcal{Z}_h,\) whose orthogonal projection onto \(\mathbb{R}^n\) lies in an \(h\)-convex domain \(\Omega \subset B_{h^{-1}}(0) \subset \mathbb{R}^n.\) Then\n\[(1.18)\]
\[X(B) \subset \mathcal{Z}_\Omega := \left\{ (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : (x^1, \ldots, x^n) \in \Omega \right\}.\]

In general one cannot expect that surfaces of bounded \(F\)-mean curvature satisfying the conditions of Theorem 1.3 can be represented as a graph over the \(h\)-convex domain \(\Omega \subset \mathbb{R}^n.\) For \(n = 2\) and stable surfaces of bounded mean curvature \(\mathcal{H}(y) \in C^{1,\alpha}(\mathbb{R}^3),\) however, Sauvigny was able to prove such a result [21] under a sign condition on \(\frac{\partial}{\partial y^3} \mathcal{H},\) and it turns out that the same is true for stable surfaces of prescribed \(F\)-mean curvature in \(\mathbb{R}^3;\) see Theorem 1.4 below.

Before defining stability in Section 4 we generalize do Carmo’s [1] second variation formula for the area functional (1.6) to the parametric functional (1.12). That is, we derive a general formula for the second variation \(\delta^2 F^0(X, \Xi)\) of the functional (1.12) at critical immersions \(X : M \to\)
In the direction of an arbitrary compactly supported vector field $\Xi \in C_0^2(M,\mathbb{R}^{n+1})$ containing normal and tangential terms\(^2\); see Theorem 4.1. For immersions $X : M \to \mathbb{R}^{n+1}$ of prescribed $F$-mean curvature $\mathcal{H}_F$, however, the tangential term drops out (see Corollary 4.2), which additionally implies a simplified differential equation for the normal $N$ of such surfaces derived in Corollary 4.3:

\[
(1.19) \quad \Delta_F N + \left[ \text{tr} (A_F S^2) - \langle \nabla_{\mathbb{R}^{n+1}} \mathcal{H}_F(X), N \rangle \right] N = -\nabla_{\mathbb{R}^{n+1}} \mathcal{H}_F(X).
\]

By means of this equation we are able to generalize Sauvigny’s result \cite{21} for surfaces of bounded mean curvature mentioned above to stable surfaces of prescribed $F$-mean curvature in $\mathbb{R}^3$:

**Theorem 1.4.** Let $F = F(z) \in C^0(\mathbb{R}^3) \cap C^3(\mathbb{R}^3 \setminus \{0\})$ be an elliptic parametric Lagrangian satisfying (E). Suppose $X : \overline{B} \to \mathcal{Z}_h$ is of class $C^3(\overline{B},\mathbb{R}^3) \cap C^{1,\alpha}(\overline{B},\mathbb{R}^3)$ for some $\alpha \in (0,1)$ and a stable immersion of prescribed $F$-mean curvature $\mathcal{H}_F \in C^{1,\alpha}(\mathbb{R}^3)$, where $\mathcal{H}_F$ satisfies

\[
(1.20) \quad \|\mathcal{H}_F\|_{C^0(\mathbb{R}^3)} \leq M_1 h.
\]

We assume that $X$ spans $\Gamma \subset \mathcal{Z}_h$, where $\Gamma$ is a Jordan curve given as a graph over the boundary $\partial \Omega$ of an $h$-convex domain $\Omega \subset \mathbb{R}^2$. Then $X(B) \subset \mathcal{Z}_\Omega$, and $X(\overline{B})$ can be represented as a graph over $\Omega$, if $\frac{\partial}{\partial y^3} \mathcal{H}_F(y) \geq 0$ for all $y = (y^1, y^2, y^3) \in \mathbb{R}^3$.

The proof of this result can be found in Section 5. For minimal surfaces this result is due to Radó \cite{19}. Gulliver and Spruck \cite{13} generalized Radó’s theorem to surfaces of constant mean curvature.

**Remark.** For simplicity of presentation we have assumed throughout this paper that the surfaces are immersed up to the boundary. The strong smoothness hypotheses of Theorem 1.4, however, allow us to exclude boundary branch points for the specific boundary configurations considered in Theorems 1.2 and 1.4 with $n = 2$; see the corresponding remarks in Sections 3 and 5. That is, a conformally parametrized surface of class $C^{1,\alpha}(\overline{B},\mathbb{R}^3)$ without interior branch points does not have boundary branch points if it either has vanishing $F$-mean curvature with boundary contour $\Gamma \subset \partial K$ for some convex set $K \subset \mathbb{R}^3$, or if it has prescribed $F$-mean curvature $\mathcal{H}_F$ satisfying (1.20), with boundary contour $\Gamma \subset \mathcal{Z}_h$ as in Theorem 1.4.

A general boundary regularity result, however, guaranteeing $C^{1,\alpha}$-smoothness up to the boundary is currently only available for $F$-minimizers; see \cite{16}, but not for $F$-critical points.

\(^2\)Theorem 4.1 contains as special cases the corresponding second variation formulas of Sauvigny \cite{21} and Räwer \cite{20} who consider only normal, or $F$-normal variations, respectively.
2. Preliminaries and a differential equation for the normal.

In terms of the induced metric $g : T_wM \times T_wM \to \mathbb{R}$ defined in (1.14) we can express an arbitrary tangent vector $V \in T_wM$ as
\begin{equation}
V = g^{kj}g(V, \partial_j)\partial_k,
\end{equation}
and its image under the isomorphism $DX : T_wM \to T_{X(w)}M$, where $M := X(M) \subset \mathbb{R}^{n+1}$, as
\begin{equation}
DX(V) = g^{kj}g(V, \partial_j)\partial_k X.
\end{equation}
Here $g^{kj}$ are the coefficients of the inverse of the metric tensor $g_{kj}$ and \{$\partial_1, \ldots, \partial_n$\} := \{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\} is the coordinate basis spanning $T_wM$. Let $\chi(M)$ be the space of vector fields of class $C^2$ on $M$ and denote by $\nabla_V$ the covariant derivative in the direction of $V \in \chi(M)$. We set $\nabla_i := \nabla_{\partial_i}$, $i = 1, \ldots, n$. We will frequently use the following versions of the product rule:
\begin{align}
(2.3) & \quad U(g(V,W)) = g(\nabla_U V, W) + g(V, \nabla_U W), \\
(2.4) & \quad \nabla_U (AV) = (\nabla_U A)V + A(\nabla_U V),
\end{align}
for all $U, V, W \in \chi(M)$ and all differentiable endomorphism fields $A \in \text{End}(TM)$. As a consequence of (2.3) we obtain for symmetric $A \in \text{End}(TM)$ and $\phi \in C^2(M)$
\begin{equation}
(2.5) \quad U(d\phi(AV)) = g(\nabla_U (A \text{ grad } \phi), V) + d\phi(A \nabla_U V),
\end{equation}
where $g(\text{grad } \phi, V) := d\phi(V)$, $V \in T_wM$, defines the gradient of the function $\phi$ on $M$ as usual. Using the fact that $\langle DX(V), N \rangle = 0$ one can show that
\begin{equation}
(2.6) \quad U(DX(V)) = DX(\nabla_U V) - \langle DX(V), DX \circ S(U) \rangle N
\end{equation}
for all $U, V \in \chi(M)$. The trace of an endomorphism $A \in \text{End}(TM)$ in local coordinates is given by
\begin{equation}
(2.7) \quad \text{tr } A = g^{ik}g(A\partial_i, \partial_k).
\end{equation}
In particular, we will denote
\begin{equation}
(2.8) \quad \text{tr } (A\nabla_V V) = g^{ik}g(A\nabla_i V, \partial_k) \quad \text{for } A \in \text{End}(TM), V \in \chi(M).
\end{equation}
For $A := \text{Id}$ we obtain the usual \textit{divergence of a vector field} $W \in \chi(M)$
\begin{equation}
(2.9) \quad \text{div } W = \text{div } M W := \text{tr } (\nabla_W W) = g^{ik}g(\nabla_i W, \partial_k).
\end{equation}
The \textit{divergence} $\text{Div } Z$ of a (not necessarily tangential) vector field $Z : M \to \mathbb{R}^{n+1}$ is given in local coordinates by
\begin{equation}
(2.10) \quad \text{Div } Z = g^{ik}\langle DZ(\partial_i), DX(\partial_k) \rangle.
\end{equation}
ON SURFACES OF PRESCRIBED $F$-MEAN CURVATURE

For $Z := DX(W)$, $W \in \chi(M)$, we get $\text{Div} Z = \text{div} W$ by (2.9) and (2.10). We will also use the notion of the divergence of an endomorphism field $\text{div} A$, $A \in \text{End} (TM)$, with adjoint $A^*$, given by

\begin{equation}
g(\text{div} A, V) := \text{tr} (\nabla_i A^* V) = g^{ik} g((\nabla_i A^*) V, \partial_k).
\end{equation}

In local coordinates we can write

\begin{equation}
\text{div} A = g^{ik}(\nabla_i A) \partial_k,
\end{equation}

where $\nabla_i A$ denotes the covariant derivative of the tensor $A$; see [9, Def. 2.60].

If we denote the coefficients of the second fundamental form of $(M, g)$ with $h_{ij} := -g(\partial_i, S(\partial_j))$, and, correspondingly, the coefficients of the $F$-second fundamental form by $h_{Fij} := -g(\partial_i, A_F S(\partial_j)) = -(F_{zz} \partial_i X, \partial_j N)$,

\begin{equation}
H_F = -\text{tr} (A_F S) = -g^{ij} g(\partial_i, A_F S(\partial_j)) = g^{ij} h_{Fij}.
\end{equation}

Introducing the second order differential operator

\begin{equation}
\Theta_F := \Delta_F - \text{div} A_F,
\end{equation}

where $\Delta_F$ is given by (1.13), the first author could prove in [2] that

\begin{equation}
\Theta_F X = H_F N
\end{equation}

holds for any immersion $X \in C^2(M, \mathbb{R}^{n+1})$. This equation reduces to the classical identity $\Delta X = \text{div} \text{grad} X = HN$, if $F(y, z) = A(z)$, or $F(y, z) = E(y, z)$, respectively; see (1.4), (1.5). Moreover, $\Theta_F$ is uniformly elliptic if $F$ satisfies the ellipticity condition (E), which leads to the enclosure theorems proven in [2], and which will be used in the proofs of Sections 3 and 5.

Now we will conclude this section with:

**Proof of Theorem 1.1.** Apply (2.5) to $\phi := N^i$, $i = 1, \ldots, n + 1$, and $A := A_F \in \text{End} (TM)$ to obtain by (2.6) and (2.4)

\begin{align*}
g(\nabla_U (A_F \text{grad} N), V) &= U(DN(A_F V)) - DN(A_F \nabla_U V) \\
&= U(DX(SA_F V)) - DX(SA_F \nabla_U V) \\
&= DX(\nabla_U (SA_F V)) \\
&= \langle DX(SA_F V), DX \circ S(U) \rangle N \\
&= DX(SA_F \nabla_U V) \\
&= DX(\nabla_U (SA_F V)) - g(SA_F(V), S(U)) N.
\end{align*}
Choosing \( U = \partial_i, \ V = \partial_k \) we obtain by (1.13), (2.9), (2.12) and (2.7)

\[
\Delta_F N = \text{div} (A_F \text{grad} N) \\
= g^{ik} g(\nabla_i (A_F \text{grad} N), \partial_k) \\
= g^{ik} DX(\nabla_i (SA_F) \partial_k) - g^{ik} g(SA_F(\partial_k), S(\partial_i)) N \\
= DX(\text{div} (SA_F)) - \text{tr} (A_F S^2) N,
\]

where we have used the symmetry of \( A_F \) and \( S \) to obtain the last term. \( \square \)

Using the Codazzi equation (cf. [18, p. 30])

\[
(\nabla_V S)W = (\nabla_W S)V 
\]

one can show that

\[
\text{div} S = -\text{grad} H. 
\]

Thus in case of the functionals (1.6) or (1.7), where \( A_F \) is the identity, we obtain

\[
\Delta N + \text{tr} (S^2)N = -DX(\text{grad} H).
\]

Therefore (1.15) is a generalization of [21, Hilfssatz 1].

### 3. Proofs of the enclosure theorems.

For the convenience of the reader we recall the Proof of Theorem 1.2 from [2].

**Proof of Theorem 1.2.** Since \( H_F(X, N) = H_F(X) = 0 \), we infer from (2.16) that

\[
\Theta_F(t(X)) = 0
\]

for all affine linear functions

\[
t(y) := \langle a, y \rangle + b, \ a \in \mathbb{R}^{n+1}, b \in \mathbb{R}. 
\]

Taking an arbitrary supporting half plane of the convex body \( K \) characterized by an affine linear function \( t_K \), we have \( t_K(X) \leq 0 \) on \( \partial M \), and hence by (3.1) and the maximum principle [11, p. 32], \( t_K(X) \leq 0 \) on \( \overline{M} \), i.e., \( X(M) \subset K \). \( \square \)

**Remark.** For \( n = 2, M := B = B_1(0) \subset \mathbb{R}^2 \), and \( X \) immersed only in the interior of \( B \) but given in conformal parameters, i.e., with

\[
|X_u|^2 = |X_v|^2 \quad \text{and} \quad \langle X_u, X_v \rangle = 0 \quad \text{on} \ \overline{B},
\]
we can exclude boundary branch points. In fact, introducing polar coordinates \((r, \theta)\) in \(B\) and fixing \(w_0 \in \partial B\) we can apply Hopf’s boundary point lemma [11, p. 34] together with (3.1) to obtain for \(t : \mathbb{R}^3 \to \mathbb{R}\) as in (3.2)

\[
\frac{\partial}{\partial \theta} \left[ t(X(w)) \right] \big|_{w=w_0} = \langle a, X_r(w_0) \rangle > 0.
\]

Therefore we have \(|X_r(w_0)| > 0\). Rewriting (3.3) in polar coordinates we conclude \(|X_\theta(w_0)| > 0\) which shows that \(w_0\) is not a branch point.

**Proof of Theorem 1.3.** For the function \(R(x) := (x^1)^2 + \cdots + (x^n)^2\) we compute similarly as in [5, p. 7] using (2.1), (2.15), (2.16), (2.12) and (E)

\[
\frac{1}{2} \Theta_F(R(X)) = \sum_{i=1}^{n} X^i \text{div} (A_F \text{grad}(X^i)) \\
+ \sum_{i=1}^{n} g(\text{grad}(X^i), A_F \text{grad}(X^i)) \\
- \sum_{i=1}^{n} X^i (\text{div} A_F)(X^i)
\]

\[
= \sum_{i=1}^{n} X^i \Theta_F(X^i) + \sum_{i=1}^{n} g(\text{grad}(X^i), A_F \text{grad}(X^i)) \\
\geq \sum_{i=1}^{n} H_F(X, N)X^iN^i + M_1 \sum_{i=1}^{n} g(\text{grad}(X^i), \text{grad}(X^i)) \\
\geq -|H_F(X)|\sqrt{R(X)} + M_1(n-1) \\
\geq M_1(n-1) - \|H_F\|_{C^0(\mathbb{R}^3)} h^{-1}
\]

on \(M\), since \(X(M) \subset \mathcal{Z}_h\); see (1.16). Notice that we have used the relation \(p^{ii} = g(\text{grad}(X^i), \text{grad}(X^i))\), where \(P = P(w) = (p^{ij})(w) : \mathbb{R}^{n+1} \to T_{X(w)}M\) is the orthogonal projection onto the \(n\)-dimensional tangent plane of \(M = X(M)\), with \(\sum_{i=1}^{n+1} p^{ii} = n\), so that

\[
\sum_{i=1}^{n} g(\text{grad}(X^i), \text{grad}(X^i)) = n - g(\text{grad}(X^{n+1}), \text{grad}(X^{n+1})) \geq n - 1.
\]

Thus \(\Theta_F(R(X)) \geq 0\) on \(M\) due to (1.17), and the maximum principle implies \(R(X(w)) < h^{-2}\) for all \(w \in M\), since \(R(X) \not\equiv h^{-2}\) in \(M\). Following Sauvigny [21] we now argue as follows: Assuming that there is some point \(w^* \in M\) with \(X(w^*) \not\in \mathcal{Z}_\Omega\) we infer that \(x^* := (X^1(w^*), X^2(w^*)) \not\in \Omega\). Let \(y^* \in \partial \Omega\) be a point with \(|y^* - x^*| = \text{dist}(x^*, \Omega) \geq 0\). (If \(x^* \in \partial \Omega\) take \(y^* := x^*\).) Since \(\Omega\) is \(h\)-convex there is a point \(\eta_0 \in \mathbb{R}^n\) such that \(\Omega \subset B_{1/h}(\eta_0)\) and
\[ y^* \in \partial B_{1/\epsilon}(\eta_*) \text{.} \] Thus \( X(w^*) \notin Z_{B_{1/\epsilon}(\eta_*)} \) and we can look at the 1-parameter family of cylinders \( \{ Z(\lambda) := Z_{B_{1/\epsilon}(\lambda \eta_*)} \}_{0 \leq \lambda \leq 1} \), for which

\[ X(B) \subset Z(0) = Z_h, \]

and

\[ X(B) \cap \partial Z(1) = X(B) \cap \partial Z_{B_{1/\epsilon} \eta_*} \neq \emptyset. \]

By continuity we find \( \lambda_0 \in [0, 1] \) with

\[ X(B) \subset Z(\lambda_0) \text{ and } X(B) \cap \partial Z(\lambda_0) \neq \emptyset. \]

(3.4)

With the same computation as before we deduce for \( R_0(x) := (x^1 - \lambda_0 \eta_1)^2 + \cdots + (x^n - \lambda_0 \eta_n)^2 \) the inequality \( \Theta_F(R_0(X)) \geq 0 \) on \( B \); hence by (3.4) and the maximum principle \( R_0(X(w)) \equiv h^{-2} \), which is absurd. Thus we have shown (1.18).

\[ \square \]

4. A general second variation formula and stability.

In this section we consider \( C^3 \)-perturbations \( X(., \epsilon) : M \times (-\epsilon_0, \epsilon_0) \to \mathbb{R}^{n+1} \) of an immersed hypersurface \( X \in C^3(M, \mathbb{R}^{n+1}) \) with

\[ X(., 0) = X \text{ and } \]

\[ \partial \frac{\partial X(., \epsilon)}{\partial \epsilon} \bigg|_{\epsilon=0} = \varphi N + DX(V) =: \Xi, \]

where \( \varphi \in C^2_0(M), V \in \chi(M) \) with compact support. Notice that we admit a non-vanishing tangential component in the variational field \( \Xi \in C^2_0(M, \mathbb{R}^{n+1}) \) as in [1] but in contrast to [21, p. 64]. The second variation \( \delta^2 \mathcal{F}^0(X, \Xi) \) of the functional \( \mathcal{F}^0 \) defined in (1.12) at \( X \) in the direction of \( \Xi \) is defined as

\[ \delta^2 \mathcal{F}^0(X, \Xi) := \left. \frac{d^2}{d\epsilon^2} \mathcal{F}^0(X(., \epsilon)) \right|_{\epsilon=0}. \]

Theorem 4.1. Let \( F = F(z) \in C^0(\mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1}\{0\}) \) be a parametric Lagrangian. Suppose \( X \in C^3(M, \mathbb{R}^{n+1}) \) is a critical immersion for the functional (1.12) and \( \Xi \in C^2_0(M, \mathbb{R}^{n+1}) \) is a variational field of the form (4.2). Then

\[ \delta^2 \mathcal{F}^0(X, \Xi) \]

\[ = \int_M \left\{ g(A_F \text{ grad } \varphi, \text{ grad } \varphi) - \varphi^2 (\text{tr } (A_F S^2) \right. \]

\[ - \langle \nabla_{\mathbb{R}^{n+1}} \mathcal{H}_F(X), N \rangle + \varphi g(\text{div } (SA_F) + \text{grad } \mathcal{H}_F(X), V) \} dA. \]

Note that only first order derivatives of \( X(., \epsilon) \) with respect to \( \epsilon \), i.e., merely \( \Xi \) defined in (4.2) enters the formula for the second variation which justifies the notation on the left-hand side of (4.3).
Proof of Theorem 4.1. Using the identity
\[
\frac{\partial}{\partial \eta} X(., \epsilon + \eta)|_{\eta=0} = \frac{\partial}{\partial \epsilon} X(., \epsilon)|_{\epsilon=\epsilon}
\]
we obtain
\[
\delta^2 F^0(X, \Xi) = \frac{d^2}{d\epsilon^2} F^0(X(., \epsilon))|_{\epsilon=0} = \frac{d}{d\epsilon} \left( \frac{d}{d\eta} F^0(X(., \epsilon + \eta))|_{\eta=0} \right)|_{\epsilon=0}
\]
\[
= \frac{d}{d\epsilon} \left( \delta F^0 \left( X(., \epsilon), \frac{\partial}{\partial \epsilon} X(., \epsilon)|_{\epsilon=\epsilon} \right) \right)|_{\epsilon=0}.
\]
Hence, by the first variation formula proved in [2, pp. 5,6] applied to (1.12) and evaluated at \( X(., \epsilon) \) in the direction \( \frac{\partial}{\partial \epsilon} X(., \epsilon)|_{\epsilon=\epsilon} \),
\[
\delta^2 F^0(X, \Xi) = \frac{d}{d\epsilon} \left( \int_M \left\{ \frac{\partial}{\partial \epsilon} X(., \epsilon)|_{\epsilon=\epsilon}, N(., \epsilon) \right\} \cdot [H_F(X(., \epsilon), \frac{\partial}{\partial \epsilon} X(., \epsilon)|_{\epsilon=\epsilon}) - H_F(X(., \epsilon), N(., \epsilon))] \right\} dA \right|_{\epsilon=0}.
\]
where \( N(., \epsilon) \) is the unit normal and \( H_F(X(., \epsilon), N(., \epsilon)) \) the \( F \)-mean curvature of the perturbed immersion \( X(., \epsilon) \in C^3(M, \mathbb{R}^{n+1}) \).

According to [2, Lemma 1.1] one has
\[
\frac{\partial}{\partial \epsilon} N(., \epsilon)|_{\epsilon=0} = -DX(\text{grad } \varphi) + DN(V),
\]
where \( \varphi \in C^2_0(M) \) and \( V \in \chi(M) \) with compact support determine the normal and tangential component of \( \Xi \) defined in (4.2). From (2.14), on the other hand, we infer
\[
\frac{\partial}{\partial \epsilon} H_F(X(., \epsilon), N(., \epsilon))|_{\epsilon=0} = (2.14) \left[ \left( \frac{\partial}{\partial \epsilon} g^{ij}(\epsilon) \right) h_{Fi} + g^{ij} \left( \frac{\partial}{\partial \epsilon} h_{Fi}(\epsilon) \right) \right] |_{\epsilon=0} = : I + II,
\]
where the argument \( \epsilon \) indicates that the corresponding quantity belongs to the perturbed immersion \( X(., \epsilon) \). In particular, we write, e.g., \( g^{ij}(0) = g^{ij} \), \( \partial_t X(., 0) = \partial_t X \), etc. On account of \( g^{ij}(\epsilon) g_{js}(\epsilon) = \delta^i_s \) for all \( \epsilon \in (-\epsilon_0, \epsilon_0) \) one has
\[
\frac{\partial}{\partial \epsilon} g^{ij}(\epsilon) = -g^{ik}(\epsilon) \left( \frac{\partial}{\partial \epsilon} g_{kl}(\epsilon) \right) g^{lj}(\epsilon),
\]
and therefore by (4.2) and (2.6) for \( U := \partial_k, \partial_l \), respectively,

\[
\frac{\partial}{\partial \epsilon} g^{ij}(\epsilon) \big|_{\epsilon=0} = \left. g^{ij}(\epsilon) \right|_{\epsilon=0} = 0
\]

\[
= -g^{ik} \left\{ \left. \left( \frac{\partial}{\partial \epsilon} (\partial_k X(\epsilon)) \right) \right|_{\epsilon=0} + \left. \left( \frac{\partial}{\partial \epsilon} (\partial_l X(\epsilon)) \right) \right|_{\epsilon=0} \right\} = 0
\]

\[
= -g^{ik} \{ \langle \phi \partial_k N + \partial_k (DX(V)), \partial_l X \rangle + \langle \partial_k X, \phi \partial_l N + \partial_l (DX(V)) \rangle \} g^{lj}
\]

\[
= 2\varphi g^{ik} h_{kl} g^{lj} - g^{ik} \langle DX(\nabla_k V), \partial_l X \rangle g^{lj} - g^{ik} \langle DX(\nabla_l V), \partial_k X \rangle g^{lj}.
\]

Thus we obtain for the expression \( I \) in (4.7) by the symmetry of the mappings \( A_F \) and \( S \)

\[
(4.8) \quad I = \left. \frac{\partial}{\partial \epsilon} g^{ij}(\epsilon) \right|_{\epsilon=0} = h_{Fij}
\]

\[
= 2\varphi g^{ik} h_{kl} g^{lj} h_{Fij} - g^{ik} \langle DX(\nabla_k V), \partial_l X \rangle g^{lj} h_{Fij}
\]

\[
= 2\varphi g^{ik} g(\partial_k, S(\partial_l)) g^{lj} g(\partial_l, A_F S(\partial_l))
\]

\[
+ g^{ik} g(\nabla_k V, \partial_l) g^{lj} g(\partial_l, A_F S(\partial_l))
\]

\[
+ g^{ik} g(\nabla_l V, \partial_k) g^{lj} g(\partial_k, A_F S(\partial_l))
\]

\[
= 2\varphi g^{ik} g(\partial_k, S(\partial_l)) g^{lj} g(\partial_l, A_F S(\partial_l))
\]

\[
+ g^{ik} g(\nabla_k V, \partial_l) g^{lj} g(\partial_l, A_F S(\partial_l))
\]

\[
= 2\varphi g^{ik} g(A_F S^2(\partial_k), \partial_l) + g^{ik} g(SA_F(\partial_l), \nabla_k V)
\]

\[
+ g^{lj} g(\nabla_l V, A_F S(\partial_l))
\]

\[
= 2\varphi \text{tr} (A_F S^2) + \text{tr} ((SA_F + A_F S) \nabla V).
\]

Furthermore we need to compute

\[
(4.9) \quad \frac{\partial}{\partial \epsilon} h_{Fij}(\epsilon) \big|_{\epsilon=0}
\]

\[
= -\left. \left( \frac{\partial}{\partial \epsilon} (F_{zz}(N(\epsilon))) \right) \right|_{\epsilon=0} \partial_i X, \partial_j N
\]

\[
- \left. \left( F_{zz}(N) \partial_i \left[ \frac{\partial}{\partial \epsilon} X(\epsilon) \big|_{\epsilon=0} \right] \right) \right|_{\epsilon=0} \partial_j N
\]

\[
- \left. \left( F_{zz}(N) \partial_i X, \partial_j \left[ \frac{\partial}{\partial \epsilon} N(\epsilon) \big|_{\epsilon=0} \right] \right) \right|_{\epsilon=0}
\]
\[
(4.13) \quad \frac{\partial}{\partial \epsilon} H_{F}(X(\cdot, \epsilon), N(\cdot, \epsilon))|_{\epsilon=0} = \varphi \text{ tr } (A_{F}S^{2}) + g^{ij} \langle \partial_{i}X, F_{zz}(N)\partial_{j}DX(\text{grad } \varphi) \rangle - g^{ij} \left\langle \frac{\partial}{\partial \epsilon} (F_{zz}(N(\cdot, \epsilon)))|_{\epsilon=0}, \partial_{i}X, \partial_{j}N \right\rangle - \varphi \text{ tr } (A_{F} \circ [\nabla_{\cdot}S]).
\]
By virtue of (1.10), (2.6), (2.9) and (1.13) we may rewrite the second term on the right-hand side as

\[
\begin{align*}
(4.14) \quad g^{ij} & \langle \partial_i X, F_{zz}(N) \partial_j DX(\text{grad } \varphi) \rangle \\
& = g^{ij} \langle \partial_i X, \partial_j \{F_{zz}(N) DX(\text{grad } \varphi)\} \rangle \\
& \quad - g^{ij} \langle \partial_i X, \partial_j (F_{zz}(N)) DX(\text{grad } \varphi) \rangle \\
& = g^{ij} (\text{grad } \varphi) - g^{ij} \langle \partial_i X, \partial_j (F_{zz}(N)) DX(\text{grad } \varphi) \rangle \\
& \quad - \Delta_F \varphi - g^{ij} \langle \partial_i X, \partial_j (F_{zz}(N)) DX(\text{grad } \varphi) \rangle.
\end{align*}
\]

Moreover, by the symmetry of \( F_{zz} \) we have

\[
(4.15) \quad g^{ij} \langle \partial_i X, \partial_j (F_{zz}(N)) DX(\text{grad } \varphi) \rangle = g^{ij} \langle \partial_j (F_{zz}(N)) \partial_i X, DX(\text{grad } \varphi) \rangle,
\]

and by (2.6), (1.10), (2.4) and (2.11) for general \( W \in T_w M \)

\[
(4.16) \quad g^{ij} \langle \partial_i X, \partial_j (F_{zz}(N)) DX(W) \rangle \\
& = g^{ij} \langle \partial_j (F_{zz}(N)) DX(\partial_i), DX(W) \rangle \\
& = g^{ij} \langle DX((\nabla_j A_F) \partial_i), DX(W) \rangle \\
& = g^{ij} g((\nabla_j A_F) \partial_i, W) = g(\text{div } A_F, W).
\]

Summarizing (4.13), (4.14) and (4.16) for \( W := \text{grad } \varphi \) we arrive at

\[
(4.17) \quad \frac{\partial}{\partial \epsilon} H_F(X(\cdot, \epsilon), N(\cdot, \epsilon))|_{\epsilon=0} \\
& = \Delta_F \varphi + \varphi \text{tr } (A_F S^2) - g(\text{div } A_F, \text{grad } \varphi) \\
& \quad - g^{ij} \left\langle \frac{\partial}{\partial \epsilon} (F_{zz}(N(\cdot, \epsilon)))|_{\epsilon=0} \partial_i X, \partial_j N \right\rangle - \text{tr } (A_F \circ [(\nabla \cdot S)V]).
\]

Now writing out components one calculates

\[
\begin{align*}
\frac{\partial}{\partial \epsilon} (F_{zz}^{(k)}(N(\cdot, \epsilon)))|_{\epsilon=0} & = \partial_i X^k \partial_j N^l \\
& = F_{zz}^{(k)}(N) \left( \frac{\partial}{\partial \epsilon} N^s(\cdot, \epsilon) |_{\epsilon=0} \right) \partial_i X^k \partial_j N^l \\
& = \partial_j (F_{zz}^{(k)}(N)) \left( \frac{\partial}{\partial \epsilon} N^s(\cdot, \epsilon) |_{\epsilon=0} \right) \partial_i X^k,
\end{align*}
\]
whence by (4.6), (4.15) and (4.16) for \( W := S(V) - \text{grad} \varphi \),

\[
g^{ij} \left\langle \frac{\partial}{\partial \epsilon} (F_{zz}(N(\cdot, \epsilon))) \right\rangle_{\epsilon=0} \partial_i X, \partial_j N \rightangle
= g^{ij} \left( \partial_j (F_{zz}(N)) \partial_i X, DX \circ S(V) - DX(\text{grad} \varphi) \right)
= g(\text{div} A_F, S(V) - \text{grad} \varphi).
\]

Next we claim that for any \( V \in T_w M \)

\[
g(\text{div} \left( S A_F \right), V) = \text{tr} \left( A_F \circ (\nabla \bullet S) V \right) + g(\text{div} A_F, S(V)).
\]

This together with (4.18) and (4.17) leads to

\[
\frac{\partial}{\partial \epsilon} H_F(X(\cdot, \epsilon), N(\cdot, \epsilon)) \big|_{\epsilon=0} = \Delta_F \varphi + \varphi \text{tr} (A_F S^2) - g(\text{div} (S A_F), V).
\]

By (4.5) we then conclude using (4.2)

\[
\delta^2 F^0(X, \Xi) = - \int_M \varphi \{ \Delta_F \varphi + \varphi \text{tr} (A_F S^2) - \varphi \langle \nabla_{\mathbb{R}^{n+1}} H_F(X), N \rangle - g(\text{div} (S A_F) + \text{grad} H_F(X), V) \} \, dA,
\]

which proves Theorem 4.1. Notice that the other terms obtained by carrying out the differentiation with respect to \( \epsilon \) in (4.5) and evaluating at \( \epsilon = 0 \) vanish, since

\[
H_F(X(\cdot, 0), N(\cdot, 0)) = H_F(X, N) \equiv H_F(X)
\]
because \( X \) is a critical immersion for (1.12).

It remains to show (4.19). By (2.11) and the symmetry of \( A_F \) and \( S \)

\[
g(\text{div} (S A_F), V) = \begin{aligned}
g^{ik} g((\nabla_i (S A_F)^* V, \partial_k)
= g^{ik} g((\nabla_i A_F^*) S^* V, \partial_k) + g^{ik} g(A_F^* (\nabla_i S^*) V, \partial_k)
= g(\text{div} A_F, S^* V) + g^{ik} g((\nabla_i S^*) V, A_F(\partial_k))
= g(\text{div} A_F, V) + g^{ik} g(V, (\nabla_i S) A_F(\partial_k)).
\end{aligned}
\]

The Codazzi Equation (2.17) and the symmetry of \( S \) and \( A_F \) imply now

\[
g^{ik} g(V, (\nabla_i S) A_F(\partial_k)) = g^{ik} g(A_F \circ (\nabla_i S) V, \partial_k)
= \text{tr} (A_F \circ [(\nabla \bullet S) V]),
\]

which proves the claim. \( \square \)

As a consequence of Theorem 4.1 we can state:
**Corollary 4.2.** Let $X \in C^3(M, \mathbb{R}^{n+1})$ be a immersion of prescribed $F$-mean curvature $H_F \in C^1(\mathbb{R}^{n+1})$, where

$$F = F(z) \in C^0(\mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1}\{0\})$$

is a parametric Lagrangian. Then

\begin{equation}
\text{div} \left( S \frac{A_F}{F} \right) = -\nabla H_F(X), \quad \text{and}
\end{equation}

\begin{equation}
\delta^2 \mathcal{F}_0(X, \Xi) = \int_M \left\{ g(A_F \text{grad } \varphi, \text{grad } \varphi) - \left[ \text{tr} (A_F S^2) - \langle \nabla_{\mathbb{R}^{n+1}} H_F(X), N \rangle \right] \varphi^2 \right\} dA,
\end{equation}

where $\Xi = \varphi N + DX(V), \varphi \in C^2_0(M)$, and $V \in \chi(M)$ with compact support. In particular, the second variation of a parametric integrand depends on normal variations only.

**Proof.** The symmetry argument we use here is due to White [23]. Consider the surfaces

$$X(\cdot, \epsilon, \eta) = X + \epsilon(\varphi N + DX(V)) + \eta(\psi N + DX(W)),$$

where $\varphi, \psi \in C^\infty_0(M)$ and $V, W \in \chi(M)$ with compact support. Similarly as in (4.5) we have

\begin{equation}
\frac{d}{d\epsilon} \bigg|_{\epsilon=0} \left[ \frac{d}{d\eta} \bigg|_{\eta=0} \mathcal{F}_0(X(\cdot, \epsilon, \eta)) \right]
= \frac{d}{d\epsilon} \left( \int_M \left\{ \left( \frac{\partial}{\partial \eta} X(\cdot, \epsilon, \eta) \right)_{\eta=0}, N(\cdot, \epsilon, 0) \right) \right)
\left[ H_F(X(\cdot, \epsilon, 0)) - H_F(X(\cdot, \epsilon, 0), N(\cdot, \epsilon, 0)) \right] dA \bigg|_{\epsilon=0}.
\end{equation}

Hence by (4.20) we obtain

$$\frac{d}{d\epsilon} \bigg|_{\epsilon=0} \left[ \frac{d}{d\eta} \bigg|_{\eta=0} \mathcal{F}_0(X(\cdot, \epsilon, \eta)) \right]$$
$$= \int_M \psi \left\{ -\Delta_F \varphi - \varphi \text{tr} (A_F S^2) + g(\text{div} (S A_F), V) \right\}
+ \psi \langle \nabla_{\mathbb{R}^{n+1}} H_F(X), \varphi N + DX(V) \rangle dA.$$

Since

$$\frac{d}{d\epsilon} \bigg|_{\epsilon=0} \left[ \frac{d}{d\eta} \bigg|_{\eta=0} \mathcal{F}_0(X(\cdot, \epsilon, \eta)) \right] = \frac{d}{d\eta} \bigg|_{\eta=0} \left[ \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \mathcal{F}_0(X(\cdot, \epsilon, \eta)) \right],$$
we arrive at
\begin{equation}
\int_M \psi g(\text{div } (SA_F), V) + \psi g(V, \text{grad } \mathcal{H}_F(X)) \, dA \\
= \int_M \varphi g(\text{div } (SA_F), W) + \varphi g(W, \text{grad } \mathcal{H}_F(X)) \, dA
\end{equation}
for all \( \varphi, \psi \in C^\infty_0(M) \) and \( V, W \in \chi(M) \) with compact support, where we used that
\begin{equation}
\int_M \psi (-\Delta_F \varphi - \varphi \text{tr } (A_F S^2)) \, dA = \int_M \varphi (-\Delta_F \psi - \psi \text{tr } (A_F S^2)) \, dA.
\end{equation}
Equation (4.25) is only possible if \( \text{div } (SA_F) = -\text{grad } H_F(X) \), for if not, we could choose \( W \equiv 0 \) to have a vanishing right-hand side in (4.25), and \( \psi \) and \( V \) appropriately to obtain a positive left-hand side and thus a contradiction.

Inserting (4.22) into formula (1.15) of Theorem 1.1 for the normal of an \( F^0 \)-critical immersion we obtain:

**Corollary 4.3.** Let \( N \) be the normal of an immersion \( X \in C^3(M, \mathbb{R}^{n+1}) \) of prescribed \( F \)-mean curvature \( \mathcal{H}_F(y) \in C^1(\mathbb{R}^{n+1}) \), where \( F = F(z) \) is a parametric Lagrangian of class \( C^0(\mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1}\{0\}) \). Then
\begin{equation}
\Delta_F N + \left[ \text{tr } (A_F S^2) - \langle \nabla_{\mathbb{R}^{n+1}} \mathcal{H}_F(X), N \rangle \right] N = -\nabla_{\mathbb{R}^{n+1}} \mathcal{H}_F(X).
\end{equation}

The above corollary generalizes \([21, \text{ Satz 1}]\).

The notion of stability is defined as follows:

**Definition 4.4.** Let \( X \in C^3(M, \mathbb{R}^{n+1}) \) be an \( F^0 \)-critical immersion, where \( F^0 \) is defined in (1.12) with a parametric Lagrangian \( F = F(z) \in C^0(\mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1}\{0\}) \). Then \( X \) is called stable if \( \delta^2 F^0(X, \Xi) \geq 0 \) for all \( \Xi \in C^2_0(M, \mathbb{R}^{n+1}) \). If \( \delta^2 F^0(X, \Xi) > 0 \) we say \( X \) is strictly stable.

**5. Graph representation of prescribed \( F \)-mean curvature surfaces.**

The Proof of Theorem 1.4 is based on a maximum principle for elliptic equations of the form \( Lu = (\alpha^{ij} u_{x^i})_{x^j} + \beta^i u_{x^i} + cu \). Usually it is required that the coefficient \( c \) be nonpositive. As was carried out in \([21]\) for the Laplace operator this condition may be replaced by assuming that the first eigenvalue of \( L \) is nonnegative. Our proof of the corresponding lemma for general elliptic equations is related to \([12, \text{ Lemma 1}]\), but we assume less regularity of the coefficients:

**Lemma 5.1.** Let \( Lu = (\alpha^{ij} u_{x^i})_{x^j} + \beta^i u_{x^i} + cu \leq 0 \) be a linear elliptic equation in a domain \( \Omega \subset \mathbb{R}^n \) with smooth boundary, where \( \alpha^{ij}, \beta^i, c \in C^{0,\mu}(\Omega) \).
and with $\alpha^{ij} = \alpha^{ji}$ for $i, j = 1, \ldots, n$. Assume that the first eigenvalue of $L$ is nonnegative on $\Omega$. If for $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ we have $Lu \leq 0$ and $u|_{\partial \Omega} > 0$, then $\inf_{\Omega} u > 0$.

**Proof.** On account of the continuity of $u$ we can assume that there is a smoothly bounded domain $\Omega_2 \subset \subset \Omega$ with $u|_{\partial \Omega_2} > 0$. The first eigenvalue $\lambda$ of $L$ in a domain $\Omega_1$ with $\Omega_2 \subset \subset \Omega_1 \subset \subset \Omega$ is simple and therefore strictly positive, since otherwise we could apply [11, Thm. 8.38] to find a positive eigenfunction $\xi \in \dot{H}^{1,2}(\Omega_1)$ on $\Omega_1$ with $\xi = 0$ on $\partial \Omega_1$. Extending $\xi$ by zero outside of $\Omega_1$ we would obtain $\tilde{\xi} \in \dot{H}^{1,2}(\Omega)$ with $L\tilde{\xi} + \lambda \tilde{\xi} = 0$, $\tilde{\xi} \neq 0$ on $\Omega$ with $\tilde{\xi} \equiv 0$ on $\Omega \setminus \Omega_1$, contradicting [11, Thm. 8.38].

The regularity of the coefficients $\alpha^{ij}$, $\beta^i$, $c$, leads to $\xi \in C^{1, \mu}(\Omega_1)$; see e.g., [10, Theorem 3.2].

Thus in $\Omega_2$ we can write $u = \xi v$, and due to the regularity of $u$ and $\xi$ we obtain a.e. on $\Omega_2$:

$$
L(\xi v) = v(\alpha^{ij}\xi_{x^i})_{x^j} + \xi(\alpha^{ij}v_{x^j})_{x^i} + 2\alpha^{ij}v_{x^j}\xi_{x^i} + v\beta^i\xi_{x^i} + \xi\beta^iv_{x^i} + cv\xi \\
= vL\xi + \xi[(\alpha^{ij}v_{x^j})_{x^i} + \beta^i v_{x^i} + 2\alpha^{ij}v_{x^j}\xi_{x^i}] \\
= \xi[(\alpha^{ij}v_{x^j})_{x^i} + \beta^i v_{x^i} + (2/\xi)\alpha^{ij}\xi_{x^j}v_{x^i} - \lambda v].
$$

Thus we obtain an elliptic differential inequality for $v$:

$$
0 \leq \int_{\Omega_2} [\alpha^{ij}v_{x^j}\varphi_{x^i} - (\tilde{\beta}^i v_{x^i} - \lambda v)\varphi] \, dx
$$

for all nonnegative $\varphi \in C^1_0(\Omega_2)$, where $\tilde{\beta}^i := \beta^i + (2/\xi)\alpha^{ij}\xi_{x^j} \in L^\infty(\Omega_2)$ for $i = 1, \ldots, n$. Thus the weak maximum principle [11, Thm 8.1] holds for $v$ and we have $\inf_{\Omega_2} v \geq 0$. By the strong minimum principle [11, Thm. 8.19] we obtain the strict relation $\inf_{\Omega_2} v > 0$. \hfill \square

**Proof of Theorem 1.4.** According to our assumptions on $\Gamma$ and $\Omega$ in Theorem 1.3 there is a function $f \in C^2(\partial \Omega)$, such that $\Gamma = \{(x, f(x)) : x \in \partial \Omega\}$ is (positively) oriented by setting

$$
P_k := (x_k, f(x_k)), \quad k = 1, 2, 3,
$$

where $x_k \in \partial \Omega$, $k = 1, 2, 3$, are chosen in positive orientation with respect to $\mathbb{R}^2$.

Since $X$ is immersed on $\overline{B}$ we may assume without loss of generality that $X$ is conformally parametrized, i.e., satisfies the conformality relations (3.3). (Otherwise we can perform a diffeomorphism $w : \overline{B} \to \overline{B}$ of class $C^{2, \alpha}(B, \mathbb{R}^2) \cap C^{1, \alpha}(\overline{B}, \mathbb{R}^2)$ such that $\tilde{X} := X \circ w^{-1}$ is conformally parametrized; see e.g., [17, Corollary 3.1.2].) Performing a suitable Möbius
transformation on $\partial B$ we may assume that $X$ satisfies the three-point condition
\begin{equation}
X(w_k) = P_k \quad k = 1, 2, 3,
\end{equation}
where $w_1, w_2, w_3$ are fixed distinct points on $\partial B$.

Fix some point $w_0 = e^{i\vartheta_0} \in \partial B$. Since $\Omega$ is $h$-convex there is a point $\eta_0 \in \mathbb{R}^2$ such that $\Omega \subset B_{1/h}(\eta_0)$ and such that $y_0 := (X^1(w_0), X^2(w_0)) \in \partial \Omega$ is contained in $\partial B_{1/h}(\eta_0)$. Without loss of generality we may assume that $\eta_0 = 0$. By Hopf’s boundary point lemma we then obtain for the function $R(x) := (x^1)^2 + (x^2)^2$
\begin{equation}
\frac{\partial}{\partial \nu} R(X(w))|_{w = w_0} > 0,
\end{equation}
i.e., in polar coordinates $(r, \vartheta)$
\begin{equation}
2 \sum_{i=1}^2 X_i(w) \frac{\partial}{\partial r} X_i(w)|_{w = w_0} > 0,
\end{equation}
which implies $|X_r(w_0)| > 0$, and by conformality also
\begin{equation}
|X_\vartheta(w_0)| > 0.
\end{equation}
Since $R(X(w_0)) = h^{-2} \geq R(X(w))$ for all $w = e^{i\vartheta} \in \partial B$ we obtain
\begin{equation}
0 = \frac{\partial}{\partial \vartheta}|_{\vartheta = \vartheta_0} (R(X(e^{i\vartheta}))) = 2 \sum_{i=1}^2 X_i(e^{i\vartheta}) \frac{\partial}{\partial \vartheta} X_i(e^{i\vartheta})|_{\vartheta = \vartheta_0}.
\end{equation}
Since $\Gamma = \{(x, f(x)) : x \in \partial \Omega\}$, $f \in C^2(\partial \Omega)$, we have
\begin{equation}
|X_\vartheta^3| = |f_x^1 X_\vartheta^1 + f_x^2 X_\vartheta^2| \leq \|\nabla \tilde{f}\|_{C^0(\mathbb{R}^2)} \sqrt{(X_\vartheta^1)^2 + (X_\vartheta^2)^2},
\end{equation}
where $\tilde{f} \in C^2(\mathbb{R}^2)$ is an extension of $f$ onto $\mathbb{R}^2$ with controlled $C^2$-norm; see [11, p. 137]. Hence, by (5.4),
\begin{equation}
0 < |X_\vartheta(w)|^2|_{w = w_0} \leq (1 + \|\nabla \tilde{f}\|_{C^0(\mathbb{R}^2)}) \left[ (X_\vartheta^1(w))^2 + (X_\vartheta^2(w))^2 \right]|_{w = w_0}.
\end{equation}
By (5.2) the mapping $(X^1, X^2) : \partial B \to \partial \Omega$ respects the positive orientation, thus we infer from (5.5) and (5.6) that there is a constant $\sigma > 0$ such that
\begin{equation}
X_\vartheta^1(w_0) = -\sigma X^2(w_0), \quad X_\vartheta^2(w_0) = \sigma X^1(w_0).
\end{equation}
Therefore, by (5.3)
\begin{equation}
X_\vartheta^1(w_0) X_\vartheta^2(w_0) - X_x^2(w_0) X_\vartheta^1(w_0)
= \sigma (X^1(w_0) X_\vartheta^1(w_0) + X^2(w_0) X_\vartheta^2(w_0)) > 0,
\end{equation}
which means that \( X_u^1(w_0) X_v^2(w_0) - X_u^1(w_0) X_v^2(w_0) > 0 \), i.e., the third component of \( N^3 \) of the normal \( N \) is positive on \( \partial B \). Moreover, by Corollary 4.3, \( N^3 \) satisfies the elliptic differential equation
\[
\Delta_F N^3 + \langle \nabla_{\mathbb{R}^3} \mathcal{H}_F(X), N \rangle N^3 = -\frac{\partial \mathcal{H}_F}{\partial y^3}(X).
\]

Using the assumption on \( \mathcal{H}_F(X) \) this relation is given in coordinates by
\[
\mathcal{L} N^3 := \partial_i(\sqrt{g} g^{ij} a_{jk} g^{kl} \partial_l N^3) + \sqrt{g} (\text{tr} (A_F S^2) - \langle \nabla_{\mathbb{R}^3} \mathcal{H}_F(X), N \rangle) N^3 \leq 0,
\]
which we regard as a linear elliptic equation for \( N^3 \) with the differential operator \( \mathcal{L} \) associated to the second variation formula (4.23). Here, \( g = \det(g_{ij}) \), \( a_{jk} = \langle F_{zz}(N) \partial_j X, \partial_k X \rangle \) and the remaining coefficients are of class \( C^{0,\alpha}(\overline{B}) \), and the leading coefficients of \( \mathcal{L} \) are symmetric. Since \( X \) is stable we have \( \delta^2 \mathcal{F}^0(X, \Xi) \geq 0 \); hence the first eigenvalue of \( \mathcal{L} \) is nonnegative. Thus Lemma 5.1 is applicable and we have \( N^3 > 0 \) on \( B \).

Since \( X : \partial B \to \Gamma \) is a topological mapping, we can apply Sauvigny’s reasoning involving degree theory as in [21, pp. 53,54] to conclude the proof.

\[ \square \]

**Remark.** We have seen in (5.3) and (5.4) that there are no branch points on the boundary by the simple Hopf maximum principle argument, which is applicable because of our regularity assumptions up to the boundary. Consequently, it would suffice to assume that \( X \) is conformal and has no interior branch points and maps the boundary \( \partial B \) only weakly monotonically onto \( \Gamma \), but at this point it is an open question if one can relax the smoothness assumptions to \( X \in C^0(\overline{B}, \mathbb{R}^3) \cap C^3(B, \mathbb{R}^3) \).

**References**


ON SURFACES OF PRESCRIBED F-MEAN CURVATURE


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We calculate the Artin–Greenberg function of an irreducible plane curve germ. We deduce from this calculation that the Artin–Greenberg function, together with the multiplicity, determines the topological type of the curve and vice versa.

1. Introduction.

Soient \((X, x)\) un germe d’espace analytique complexe et \(\mathcal{O}_{X,x}\) son algèbre locale. Nous désignerons par \(\mathcal{N}_{X,x}\) l’espace des arcs analytiques tracés sur \((X, x)\), c’est à dire, \(\mathcal{N}_{X,x}\) est l’ensemble des germes de morphismes analytiques \(\varphi : (\mathbb{C}, 0) \rightarrow (X, x)\), ou bien de façon équivalente, l’ensemble des morphismes de \(\mathbb{C}\)-algèbres locales \(\varphi^* : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_1\). Rappelons maintenant la définition de la fonction d’Artin-Greenberg \(\beta\) de \((X, x)\) (cf. \[8\] Def. 2). Pour cela, \(\mathcal{O}_1\) étant munie de sa valuation naturelle et considérant deux éléments \(\varphi, \psi\) d’un \(\mathcal{N}_{Y,y}\), nous noterons \(\text{val } (\varphi - \psi)\) la valuation de l’idéal de \(\mathcal{O}_1\), \((\varphi^* - \psi^*)(m_{Y,y})\), où \(m_{Y,y}\) est l’idéal maximal de \(\mathcal{O}_{Y,y}\). Soit maintenant \(I\) un idéal d’un certain \(\mathcal{O}_n\) tel que \(\mathcal{O}_{X,x} \simeq \mathcal{O}_n/I\). La fonction suivante \(\beta : \mathbb{N} \rightarrow \mathbb{N}\), est dite la fonction d’Artin-Greenberg de \((X, x)\):

- Si \(I = (0)\), on pose \(\beta(i) = i\), pour tout \(i \in \mathbb{N}\).
- Si \(I \neq (0)\), \(\beta(i)\) est le plus petit entier tel que,

\[
\forall \varphi \in \mathcal{N}_{\mathbb{C}^n,0}, \text{val } \varphi^*(I) \geq \beta(i) + 1 \Rightarrow \exists \psi \in \mathcal{N}_{\mathbb{C}^n,0}/\psi^*(I) = 0
\]

et \(\text{val } (\varphi - \psi) \geq i + 1\).

La définition ne dépend en fait que de l’algèbre locale \(\mathcal{O}_{X,x}\) et la fonction d’Artin-Greenberg est une mesure de la singularité de \((X, x)\), puisqu’elle est égale à l’identité si et seulement si \((X, x)\) est lisse. Dans \[8\], pour un germe d’hypersurface \(f = 0\) à l’origine de \(\mathbb{C}^n\), nous relions la fonction d’Artin-Greenberg de \(\mathcal{O}_n/f\) à celle de son lieu singulier \(\mathcal{O}_n/\langle f, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \rangle\), et nous en déduisons, pour une singularité isolée, une majoration optimale de \(\beta\) en fonction des invariants polaires de l’hypersurface. Pour d’autres résultats nous renvoyons le lecteur à \[5\], \[8\] et \[9\].

Nous proposons ici le calcul complet et exact de la fonction d’Artin-Greenberg d’une branche plane (cf. Th. 2.1). Ce calcul fait apparaître que
la fonction d’Artin-Greenberg, jointe à la multiplicité, détermine le type topologique de la branche et réciproquement.

2. Fonction d’Artin-Greenberg d’une branche plane.

Soit $(C,0)$ un germe, à l’origine de $\mathbb{C}^2$, de courbe plane irréductible de multiplicité $n$. Désignons par $(r_1,n_1),\ldots,(r_g,n_g)$ ses paires de Puiseux et par $(\beta_0 = n,\beta_1,\ldots,\beta_g)$ la caractéristique de $(C,0)$ (cf. [15]). On a

$$\frac{\delta_i}{n} = \frac{r_i}{n_1\ldots n_i}, \quad 1 \leq i \leq g.$$ 

Considérons les entiers $(\beta_0 = n,\beta_1,\ldots,\beta_g)$ définis par:

$$\beta_{q+1} - \beta_q = n_q/\beta_q - \beta_q, \quad 0 \leq q \leq g - 1 \quad n_0 = 1.$$ 

Rappelons que $(\beta_0,\beta_1,\ldots,\beta_g)$ est un système minimal de générateurs du semi-groupe $\Gamma$ des valuations de $(C,0)$ (cf. [15]). Par ailleurs, considérons la courbe polaire $P(C)$ associée à $C$ et à une direction $L$ assez générale. Ecrivons $P(C) = \bigcup_{1 \leq i \leq l} \Gamma_i$, où les $\Gamma_i$ sont les composantes irréductibles de $P(C)$. Soient alors:

$m_i = m(\Gamma_i)$ la multiplicité de $\Gamma_i$ à l’origine,

$e_i + m_i = (C,\Gamma_i)$ la multiplicité d’intersection de $C$ et $\Gamma_i$ à l’origine.

D’après M. Merle [10], on peut regrouper les $\Gamma_i$ en $g$ paquets, $\bigcup_{i \in I_q} \Gamma_i$, $1 \leq q \leq g$, de telle sorte que pour $i \in I_q$, on ait:

$$\frac{e_i + m_i}{m_i} = \frac{\beta_q}{n_1\ldots n_{q-1}}.$$ 

Nous supposerons dans la suite la numérotation des $\Gamma_i$ choisie de telle sorte que $q \in I_q$. Par ailleurs, le symbole $[x]$ désignera la partie entière du nombre réel $x$.

Le résultat suivant donne le calcul complet de la fonction d’Artin-Greenberg d’une branche plane $(C,0)$.

**Théorème 2.1.**

1) Soit $i$, $1 \leq i \leq n - 1$. Considérons l’unique entier $q$, $1 \leq q \leq g$, tel que $n_1\ldots n_{q-1} \leq i < n_1\ldots n_q$. Alors:

$$\beta(i) = \text{Max}\{\beta_1i,\beta_2[i/n_1],\ldots,\beta_q[i/n_1\ldots n_{q-1}]\}.$$ 

En particulier pour $i \equiv 0 \pmod{(n_1\ldots n_{q-1})}$ et $i < n_1\ldots n_q$, on a:

$$\frac{\beta(i)}{i} = \frac{\beta_q}{n_1\ldots n_{q-1}} = \frac{e_q}{m_q} + 1$$

où les $(e_q,m_q)$ sont les invariants polaires ordonnés comme précédemment.
2) Soit $i \geq n$. Alors:
$$\beta(i) = \operatorname{Max}\{ni, \bar{\beta}_1 k_1(i), \bar{\beta}_2 k_2(i), \ldots, \bar{\beta}_g k_g(i), \theta(i, q), 0 \leq q \leq g\}$$

où
$$k_q(i) = \begin{cases} 
\frac{i}{n_1 \ldots n_{q-1}} & \text{si } \frac{i}{n_1 \ldots n_{q-1}} \neq 0 \text{ et } n_q \ldots n_g \\
\frac{i}{n_1 \ldots n_{q-1}} - 1 & \text{dans le cas contraire}
\end{cases}$$
$$\theta(i, q) = \begin{cases} 
\frac{n}{\beta_q} \left(\frac{\beta_q}{n_1 \ldots n_q} - \frac{\beta_q}{n_1 \ldots n_{q-1}}\right) + \frac{ni}{n_1 \ldots n_q} & \text{si } \frac{i}{\beta_q} < \frac{i}{\beta_q+1} \\
0 & \text{si } \frac{i}{\beta_q+1} = \frac{i}{\beta_q}
\end{cases}$$

(par convention $\beta_{g+1} = +\infty$).

La première assertion de 2.1.1 a été obtenue indépendamment dans [4]. Je remercie le referee de m’avoir communiquer cette référence. C. Cuesta-Sainz donne aussi un exemple où $\beta(i)$ prend toutes les valeurs dans le Max de 2.1.1.

**Remarque 2.2.** Soit $\nu = \operatorname{Max}(e_q/m_q) = e_q/m_q = \frac{\beta_q}{n_1 \ldots n_{q-1}} - 1$. On peut vérifier à l’aide du calcul ci-dessus et de la définition des $\beta_q$ que,
$$\forall i \in \mathbb{N}, \beta(i) \leq \nu i + i.$$ 

De plus pour $i \equiv 0 \ (n_1 \ldots n_{q-1})$ et $i \not\equiv 0 \ (n), \beta(i) = \nu i + i$.

**Exemple 2.3.** Si $(C, 0)$ a une seule paire de Puiseux $(m, n)$, on a:
- Si $m \not= n + 1$, $\beta(i) = mi$, si $i \not\equiv 0 \ (n)$ et $\beta(i) = m(i - 1)$ si $i \equiv 0 \ (n)$
- Si $m = n + 1$, $\beta(i) = mi$ si $i \not\equiv 0 \ (n)$, $\beta(n) = n^2$, $\beta(kn) = m(kn - 1)$, $k > 1$.

**Corollaire 2.4.** Deux branches planes $(C, 0)$ et $(C', 0)$ ont même type topologique si et seulement si elles ont même multiplicité et même fonction d’Artin-Greenberg.

**Preuve.** Si $(C, 0)$ et $(C', 0)$ ont même type topologique, elles ont même multiplicité et même fonction d’Artin-Greenberg, car celle-ci se calcule entièrement par la donnée de la caractéristique $(n, \beta_1, \ldots, \beta_g)$. Réciproquement si $(C, 0)$ et $(C', 0)$ ont même multiplicité et fonction d’Artin-Greenberg. On a $\beta(1) = \beta_1 = \beta'(1) = \beta'_1$. Par suite $\frac{\beta_2}{n} = \frac{\beta'_2}{n}$, et $C, C'$ ont donc même première paire. Ensuite par $\beta(n_1) = \beta_2 = \beta'(n_1) = \beta'_2$, on obtient $\frac{\beta_3}{n} = \frac{\beta'_3}{n}$, d’où la coïncidence de la seconde paire. La coïncidence de la $q$-ième paire de $(C, 0)$ et $(C', 0)$ est obtenue en écrivant $\beta(n_1 \ldots n_{q-1}) = \beta_q = \beta'(n_1 \ldots n_{q-1})$.

**Remarque 2.5.** En fait $(C, 0)$ et $(C', 0)$ ont même type topologique si et seulement si $(n, \beta(i), i < n) = (m, \beta(i), i < m)$. 
Remarque 2.6. D’après un résultat de J.Pas ou de M.Presburger (cf. [5] Remarque p. 231), il existe deux entiers positifs $d, N$ tels que, pour $i > N$, $\beta(i) = a_{\rho(i)} + b_{\rho(i)}$, où $\rho(i)$ désigne la classe de $i$ modulo $d$. Dans le cas présent on a: $\beta(i) = \overline{\beta}_g k_g(i)$ pour $i \gg 0$.

En effet pour $q < g$, $\overline{\beta}_q k_q(i) \leq \overline{\beta}_q k_q(i) \leq \overline{\beta}_q k_q(i) = \overline{\beta}_q n_q \ldots n_{g-1} \leq \overline{\beta}_g - 1$. D’où $\overline{\beta}_q k_q(i) \leq \overline{\beta}_q k_q(i) \leq \overline{\beta}_g k_g(i) \leq \overline{\beta}_g k_g(i)$. D’autre part $\theta(i, q) \leq n \overline{\beta}_q n_{1 \ldots n_{g-1}}$. Donc pour $q < g$ et $i \gg 0$, $\theta(i, q) \leq \overline{\beta}_g k_g(i)$. De plus $\theta(i, g) \leq \overline{\beta}_g k_g(i)$. Par suite pour $i \gg 0$ on a: $\theta(i, g) \leq \overline{\beta}_g k_g(i)$.

Maintenant $\overline{\beta}_g k_g(i)$ est du type Pas-Presburger précédemment évoqué, avec $d = n_1 \ldots n_{g-1}$. Je remercie le referee de m’avoir fait part de cette observation.

\textit{Preuve du théorème 2.1.} 1) Soient $i \in \mathbb{N}, 0 \leq i \leq n - 1$, et $f \in \mathcal{O}_2$ un germe d’équation réduite et irréductible de $(C, 0)$. On remarque d’abord que:

$$\beta(i) = \text{Max} \{ \text{val} (f \circ \varphi, \varphi \in \mathcal{N}_{C, 0} \text{ et } \text{val} (\varphi) \leq i) \}.$$ 

Soit donc $\varphi \in \mathcal{N}_{C, 0}$ avec $\text{val} (\varphi) \leq i$. Notons $(D, 0)$ le germe de courbe image de $\varphi$. Après un choix convenable d’une uniformisante $t$, on a $\varphi(t) = \varphi'(t^k)$ où $k \geq 1$ et $\varphi'$ est une paramétrisation irréductible de $(D, 0)$. (Nous disons qu’une paramétrisation $t \to \varphi(t)$ d’une branche $(D, 0)$ est irréductible, s’il n’existe pas de de $l > 1$ et de $\theta \in \mathcal{N}_{C, 0}$ tel que $\varphi(t) = \theta(t^j)$.) On a alors:

$$\text{val} (f \circ \varphi) = k \text{val} (f \circ \varphi') = k(C, D)$$

où $(C, D)$ est la multiplicité d’intersection à l’origine de $C$ et $D$. On est donc amené à calculer les valeurs possibles de $(C, D)$ pour $0 < m(D) < n$, où $m(D)$ désigne la multiplicité de $D$ à l’origine. Pour cela, choisissons un système de coordonnées $(x, y)$ à l’origine de $\mathbb{C}^2$, tel que, désignant par $f$ et $g$ des équations respectives de $C$ et $D$, on ait $f$ et $g$ régulières d’ordre leur multiplicité dans la direction $y$. Par le théorème de préparation de Weierstrass, on peut supposer que:

$$f(x, y) = y^n + \sum_{1 \leq i \leq n} a_i(x)y^{n-i}, \quad a_i(0) = 0,$$

$$g(x, y) = y^m + \sum_{1 \leq i \leq m} b_i(x)y^{m-i}, \quad b_i(0) = 0.$$

Ensuite, quitte à faire un changement de variables, $y \leftarrow y - \sum_{1 \leq j \leq d} c_j x^j, \quad x \leftarrow x$, on peut toujours supposer que $y = 0$ a le contact maximal (au sens
usuel) avec \((C, 0)\). Considérons alors:
\[
y_1(x^{1/n}) = \sum_{j \geq \beta_1} a_j x^{j/n},
\]
\[
y_1'(x^{1/m}) = \sum_{j \geq 1} b_j x^{j/m}
\]
des développements de Puiseux racines respectivement de \(f(x, y) = 0\) et \(g(x, y) = 0\). Si \(\xi\) (resp. \(\theta\)) est une racine primitive \(n\)-ième (resp. \(m\)-ième) de l’unité, on pose:
\[
(2.1.1)\quad y_l(x^{1/n}) = y_l(\xi^{l-1} x^{1/n}), \quad y_l'(x^{1/m}) = y_l'(\theta^{l'-1} x^{1/m}), \quad 1 \leq l \leq n, 1 \leq l' \leq m.
\]
Par définition, on appellera ordre de coïncidence des développements de Puiseux de \(C\) et \(D\) le nombre:
\[
c(D) = \max \{\text{ord}_x(y_l(x^{1/n}) - y_l'(x^{1/m})) \mid 1 \leq l \leq n, 1 \leq l' \leq m\}.
\]
De même le nombre \(\alpha(D) = nc(D)\) sera dit l’ordre de contact de \(C\) et \(D\) à l’origine (cf. [10] Def. 2.1 et Remarque 2.2 p. 106).

Soit alors \(s = \max\{k \in \{0, \ldots, g - 1\}/m = m(D) \equiv 0 (n_1 \ldots n_k)\}.\)

L’ordre de coïncidence entre développements de Puiseux \(c(D)\) est inférieur ou égal à \(\beta_{s+1}^{1/n}\). En effet \(\beta_{s+1}^{1/n} = \frac{r_{s+1}}{n_1 \ldots n_{s+1}} \notin \frac{1}{m} \mathbb{N}\) car, \(m \equiv 0 (n_1 \ldots n_s)\), \(m \neq 0 (n_1 \ldots n_{s+1})\), et \(\text{pgcd}(n_{s+1}, r_{s+1}) = 1\). Par suite \(\alpha(D) \leq \beta_{s+1}\). D’autre part, on a \(\alpha(D) = \beta_{s+1}\) pour les courbes \(D_{d,m}\), images de \(\rho_{d,m} : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)\) où:
\[
\rho_{d,m}(t) = (t^m, \theta_{d,m}(t)) \quad \text{et} \quad \theta_{d,m}(t) = \sum_{j < \beta_{s+1}} \xi^j a_j \frac{t^n}{n^{j+1}}, \quad 0 \leq d < n.
\]

On a bien \(\theta_{d,m}(t) \in \mathbb{C}\{t\}\), car \(\frac{im}{n} \in \mathbb{N}\) si \(a_j \neq 0\). En effet \(m \equiv 0 (n_1 \ldots n_s)\) et, \(\frac{j}{n} \notin \frac{1}{n_1 \ldots n_s} \mathbb{N}\), lorsque \(a_j \neq 0\) et \(j < \beta_{s+1}\), par définition de la caractéristique.

D’après la Proposition 2.4 p. 106 de [10] on a:
\[
(C, D_{d,m}) = m \left(\frac{\beta_{s+1}}{n_1 \ldots n_s}\right).
\]

D’autre part, pour une courbe \(D\) de multiplicité \(m = m(D)\), soit \(q\) l’unique entier \(0 \leq q \leq g\) tel que \(\beta_q \leq \alpha(D) = \alpha < \beta_{q+1}\) (rappelons que \(\beta_{g+1} = \infty\)).

On a d’après la proposition de [10] précitée:
\[
\frac{(C, D)}{m(D)} = \frac{\beta_q}{n_1 \ldots n_{q-1}} + \frac{\alpha - \beta_q}{n_1 \ldots n_q}.
\]

Puisque \(\alpha \leq \beta_{s+1}\), on a: \(q \leq s + 1 \leq g\). Si \(q \leq s\), on a:
\[
\frac{\beta_q}{n_1 \ldots n_{q-1}} + \frac{\alpha - \beta_q}{n_1 \ldots n_q} < \frac{\beta_q}{n_1 \ldots n_{q-1}} + \frac{\beta_{q+1} - \beta_q}{n_1 \ldots n_q} = \frac{\beta_{q+1}}{n_1 \ldots n_q} \leq \frac{\beta_{s+1}}{n_1 \ldots n_s}.
\]
Si \( q = s + 1 \), on a \( \alpha = \beta_{s+1} \), donc:

\[
\frac{(C, D)}{m(D)} = \frac{\beta_{s+1}}{n_1 \ldots n_s}.
\]

Au vue de ce qui précède on a donc établi:

**Lemme 2.7.** Soient \( m \) un entier premier à \( n \) et \( (D, 0) \) un germe de branche plane de multiplicité \( m(D) = m \). Soit \( s = \text{Max}\{k \in \{0, \ldots, g - 1\}/m \equiv 0 \ (n_1 \ldots n_k)\} \). Alors:

\[
\frac{(C, D)}{m(D)} \leq \frac{\beta_{s+1}}{n_1 \ldots n_s}.
\]

De plus, il existe des courbes \( D \) de multiplicité \( m \) telles que:

\[
\frac{(C, D)}{m(D)} = \frac{\beta_{s+1}}{n_1 \ldots n_s}.
\]

Revenons au calcul de \( \beta(i) \). Soient donc \( i \leq n - 1 \), et \( q \) l’unique entier tel que: \( 1 \leq q \leq g \) et \( n_1 \ldots n_{q-1} \leq i < n_1 \ldots n_q \). Il s’agit d’après ce qui précède de calculer \( \text{Max}\{k(C, D), D \text{ telle que } km(D) \leq i\} \).

Pour une telle courbe \( s = s(D) \leq q - 1 \). Donc en vertu du lemme:

\[
(C, D) \leq m(D) \frac{\beta_{s(D)+1}}{n_1 \ldots n_{s(D)}}.
\]

Or

\[
\frac{km(D)}{n_1 \ldots n_{s(D)}} \leq \frac{i}{n_1 \ldots n_{s(D)}}.
\]

Mais \( \frac{km(D)}{n_1 \ldots n_{s(D)}} \) étant un entier, on a:

\[
\frac{km(D)}{n_1 \ldots n_{s(D)}} \leq \left\lfloor \frac{i}{n_1 \ldots n_{s(D)}} \right\rfloor.
\]

Par conséquent:

\[
k(C, D) \leq \left\lfloor \frac{i}{n_1 \ldots n_{s(D)}} \right\rfloor \beta_{s(D)+1}.
\]

D’où: \( \beta(i) \leq \text{Max}\{\beta_1 i, \beta_2 [i/n_1], \ldots, \beta_q [i/n_1 \ldots n_{q-1}]\} \).

D’autre part soit \( p, 1 \leq p \leq q \). Posons \( k_p = [i/n_1 \ldots n_{p-1}] \). Il existe un arc \( \varphi_p \) tel que \( \text{val} \varphi_p = k_p n_1 \ldots n_{p-1} \leq i \) et \( \text{val} (f \circ \varphi_p) = \beta_p [i/n_1 \ldots n_{p-1}] \). Il suffit en effet de considérer l’arc \( \varphi_p \) défini par:

\[
\varphi_p(t) = (t^{n_1 \ldots n_{p-1} k_p}, \theta_{0,n_1 \ldots n_{p-1}}(t^{k_p})).
\]

Alors \( \text{val} (f \circ \varphi_p) = k_p (C, D_{0,n_1 \ldots n_{p-1}}) = \beta_p [i/n_1 \ldots n_{p-1}] \). D’où l’égalité annoncée dans 2.1.

Maintenant si \( i \equiv 0 \ (n_1 \ldots n_{q-1}) \) et \( i < n_1 \ldots n_q \), les nombres \( i/n_1 \ldots n_{p-1}, 1 \leq p \leq q \), sont entiers, et donc \( \beta(i) = \text{Max}\{\beta_1 i, \beta_2 i/n_1, \ldots, \beta_q i/n_1 \ldots n_{q-1}\} \).
Par conséquent \( \beta(i) = \overline{\beta}_q \frac{i}{n_1 \ldots n_{q-1}} \), car pour \( p \leq p' \) on a:
\[
\overline{\beta}_p \frac{n_1 \ldots n_{p-1}}{n_{p+1} \ldots n_{p+1-1}} \leq \overline{\beta}_{p'} \frac{n_1 \ldots n_{p'-1}}{n_{p'+1} \ldots n_{p'-1}}.
\]
Ainsi \( \beta(i) / i = \overline{\beta}_q \frac{i}{n_1 \ldots n_{q-1}} = e_q / m_q + 1 \) d’après [10].

Passons maintenant au second point de 2.1. Pour cela, étant donnés \( \varphi \in \mathcal{N}_{C,0} \) et \( i \in \mathbb{N} \), on écrira \( \varphi \equiv^i \mathcal{N}_{C,0} \) si et seulement si il existe \( \psi \in \mathcal{N}_{C,0} \) tel que \( \text{val} (\varphi - \psi) > i \). Si \( (x, y) \) est un système de coordonnées satisfaisant les conditions précédentes, on notera \( \varphi_1 \) (resp. \( \varphi_2 \)) la composante en \( x \) (resp. en \( y \)) de \( \varphi \). On a alors:

**Lemme 2.8.** Soient \( i \in \mathbb{N}^* \), \( \varphi \in \mathcal{N}_{C,0} \) tel que \( \text{val} \varphi \leq i \), et \((D, 0)\) l’image de \( \varphi \). \( \varphi \equiv^i \mathcal{N}_{C,0} \) si et seulement si \( \text{val} \varphi = \text{val} \varphi_1 \), \( \varphi \) est un multiple de \( n \), et l’ordre de coincidence des développements de Puiseux \( c(D) \) est strictement supérieur à \( i / \text{val} (\varphi) \).

**Preuve.** Supposons qu’il existe \( \psi \in \mathcal{N}_{C,0} \) tel que \( \text{val} (\varphi - \psi) > i \). Puisque \( i \geq \text{val} (\varphi) \), \( \varphi \) et \( \psi \) ont même valuation et même tangente. La tangente de \( \psi \) étant \( y = 0 \), on a donc:

\[
\text{val} (\varphi) = \text{val} (\varphi_1) < \text{val} (\varphi_2) \quad \text{et} \quad \text{val} (\varphi) = \text{val} (\psi) = np.
\]

Pour un choix convenable d’une uniformisante \( t \), on a:
\[
\varphi(t) = (t^{np}, \varphi_2(t)) \quad \text{et} \quad \psi(t) = (u(t)^n, y_1(u(t))) \quad \text{où} \quad u \in O_1.
\]

Ecrivons \( u(t) = v(t)^p \), où \( v(t) = at + r(t) \) et \( \text{val} (r) \geq 2 \). Comme \( np \leq i \), on a: \( a^p = 1 \). Par suite \( a^p = \xi^{k-1} \), pour un certain \( k \), \( 1 \leq k \leq n \). D’autre part puisque \( t^{np} - v(t)^{np} \in (t)^{i+1} \), un calcul élémentaire montre que:

\[
\text{val} (r) \geq 2 + i - np.
\]

Maintenant:
\[
y_1(v(t)^p) - y_1(\xi^{k-1}, t^p) = \sum_{j \geq h} a_j \sum_{1 \leq l \leq np} \binom{np}{l} r(t)^l (at)^{jp - l}.
\]

Mais: \( l \text{val} (r) + np - l \geq l(\text{val} (r) - 1) + np \geq i + 1 - np + np \geq i + 1 \). Par conséquent: \( y_1(v(t)^p) - y_1(\xi^{k-1}, t^p) \in (t)^{i+1} \). Ainsi les notations étant celles de (2.1.1) on a : \( \varphi_2(t) - y_k(t)^p \in (t)^{i+1} \). Ecrivons maintenant \( (t^{np}, \varphi_2(t)) = (t^{ms}, \theta(t)) \) où l’arc \( (t^m, \theta(t)) \) est irréductible. Puis posons:
\[
\theta(t) = \sum_{j > m} b_j t^j.
\]

Ainsi:
\[
y'_1(x^{\frac{1}{m}}) = \sum_{j > m} b_j x^{\frac{j}{m}}
\]
est un développement de Puiseux correspondant à l’image $D$ de $\varphi$. On a:

$$c(D) \geq \text{ord}_x(y_1'(x^{\frac{1}{m}}) - y_k(x^{\frac{1}{n}})) = \frac{1}{ms} \text{ord}_x(y_1'(x^{s}) - y_k(x^p))$$

$$= \frac{1}{ms} \text{val}(\varphi_2(t) - y_k(t^p)) \geq \frac{i+1}{ms}.$$  

Et donc, $c(D) \geq \frac{i+1}{\text{val}(\varphi)} > \frac{i}{\text{val}(\varphi)}$. Réciproquement si $c(D) > \frac{i}{\text{val}(\varphi)}$, on prouve de manière similaire que $\varphi \equiv^i N_{C,0}$ en tenant compte du fait que:

$$c(D) = \text{Max}_{1 \leq l \leq n} \text{ord}_x(y_1'(x^{\frac{1}{m}}) - y_l(x^{\frac{1}{n}})).$$

\[\square\]

Revenons au calcul de $\beta(i)$. On a par définition de la fonction d’Artin-Greenberg:

$$\beta(i) = \text{Max}\{\text{val } f \circ \varphi, \varphi \not\equiv^i N_{C,0}\}$$

$$= \text{Max}\{\text{val } f \circ \varphi, \varphi \not\equiv^i N_{C,0} \text{ et } \text{val } (\varphi) \leq i\}.$$  

Pour un $\varphi$ tel que $\text{val } (\varphi) \leq i$ et $\varphi \not\equiv^i N_{C,0}$, trois cas peuvent se produire en vertu du lemme.

1\textsuperscript{er cas}: $\text{val } (\varphi) < \text{val } (\varphi_1)$.  
On a alors: $\text{val } (f \circ \varphi) \leq n\text{val } (\varphi) \leq ni$. De plus l’égalité est obtenue pour l’arc $t \to (0,t^i).$

2\textsuperscript{iem cas}: $\text{val } (\varphi) = \text{val } (\varphi_1)$ et $\text{val } (\varphi_1)$ n’est pas un multiple de $n$.  
Écrivons $\varphi(t) = (t^{mp}, \theta(t^p))$ où l’arc $(t^m, \theta(t))$ est irréductible. Soit $s = \text{Max}\{k \in \{0, \ldots, g - 1\}/ m \equiv 0 (n_1 \ldots n_k)\}$. En vertu du Lemme 2.7, $\text{val } (f \circ \varphi) \leq mp\frac{1}{n_1 \ldots n_s}$. Mais par définition $k_{s+1}(i)$ est le plus grand entier $l$ tel que: $n_1 \ldots n_is \leq i$ et $n_1 \ldots n_is \not\equiv 0 (n)$. Par suite $mp = n_1 \ldots n_is$, avec $l \leq k_{s+1}(i)$, et donc $\text{val } (f \circ \varphi) \leq l\frac{1}{n_1 \ldots n_s} \leq k_{s+1}(i)\frac{1}{n_1 \ldots n_s}$. De plus l’égalité est obtenue pour l’arc $t \to \rho_{0,n_1 \ldots n_s}(t^{k_{s+1}(i)})$ qui a été défini au cours de la preuve du Lemme 2.7.

3\textsuperscript{iem cas}: $\text{val } (\varphi) = \text{val } (\varphi_1)$ et $\text{val } (\varphi_1)$ est un multiple de $n$.  
Posons $\varphi(t) = (t^{mp}, \varphi_2(t)) = (t^{mk}, \theta(t^p))$, où l’arc $(t^m, \theta(t))$ est irréductible. Puis considérons l’unique entier $q$ tel que: $q \in \{0, \ldots, g\}$ et $\beta_q \leq \frac{i}{p} < \beta_{q+1}$.  
On a donc:

$$\frac{i}{\beta_{q+1}} < p \leq \frac{i}{\beta_{q}}$$

i.e.,

$$\left\lfloor \frac{i}{\beta_{q+1}} \right\rfloor + 1 \leq p \leq \left\lfloor \frac{i}{\beta_{q}} \right\rfloor.$$
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Soit maintenant $\alpha(D) = nc(D)$, l’ordre de contact entre $C$ et l’image $D$ de $\varphi$. On a d’après [10] Proposition 2.4:

\[
\text{val } (f \circ \varphi) = km \left( \frac{\overline{\beta}_q}{n_1 \ldots n_{q-1}} + \frac{\alpha - \beta q'}{n_1 \ldots n_{q'}} \right)
\]

où $q'$ est l’unique entier tel que: $0 \leq q' \leq g$ et $\beta q' \leq \alpha < \beta q'+1$. En vertu du Lemme 2.8, $\alpha = nc(D) \leq \frac{ni}{\text{val } (\varphi)} = \frac{i}{p}$. Par suite $q' \leq q$. Si $q' < q$, on a:

\[
\text{val } (f \circ \varphi) = np \left( \frac{\overline{\beta}_q}{n_1 \ldots n_{q-1}} + \frac{\alpha - \beta q'}{n_1 \ldots n_{q'}} \right)
\]

\[
\leq np \left( \frac{\overline{\beta}_q}{n_1 \ldots n_{q-1}} + \frac{\beta q'+1 - \beta q'}{n_1 \ldots n_{q'}} \right).
\]

Mais le dernier terme de l’inégalité ci-dessus vaut:

\[
np \frac{\beta q'+1}{n_1 \ldots n_{q'}}.
\]

Donc pour $q' < q$, on a:

\[
\text{val } (f \circ \varphi) \leq np \frac{\overline{\beta}_q}{n_1 \ldots n_{q-1}} \leq n \left[ \frac{i}{\beta q} \right] \frac{\overline{\beta}_q}{n_1 \ldots n_{q-1}}
\]

\[
\leq n \left[ \frac{i}{\beta q} \right] \left( \frac{\overline{\beta}_q}{n_1 \ldots n_{q-1}} + \frac{i - \beta q}{n_1 \ldots n_{q}} \right) = \theta(i, q).
\]

Si $q' = q$. Puisque $\alpha \leq \frac{i}{p}$, on a:

\[
\text{val } (f \circ \varphi) = np \left( \frac{\overline{\beta}_q}{n_1 \ldots n_{q-1}} + \frac{\alpha - \beta q}{n_1 \ldots n_{q}} \right)
\]

\[
\leq np \left( \frac{\overline{\beta}_q}{n_1 \ldots n_{q-1}} + \frac{i/p - \beta q}{n_1 \ldots n_{q}} \right).
\]

Comme $\overline{\beta}_q - \frac{\beta q}{n_q} \geq 0$, le dernier terme de l’inégalité ci-dessus est une fonction croissante de $p$. Mais $p \leq \left[ \frac{i}{\beta_q} \right]$, donc:

\[
\text{val } (f \circ \varphi) \leq n \left[ \frac{i}{\beta q} \right] \left( \frac{\overline{\beta}_q}{n_1 \ldots n_{q-1}} - \frac{\beta q}{n_1 \ldots n_{q}} \right) + \frac{ni}{n_1 \ldots n_q} = \theta(i, q).
\]

Pour conclure, il nous reste à voir qu’il existe $\varphi \in N_{C^2,0}$, $\varphi \neq i N_{C,0}$ tel que $\text{val } (\varphi) \leq i$ et $\text{val } (f \circ \varphi) = \theta(i, q)$. Pour cela posons $p = \left[ \frac{i}{\beta_q} \right]$, il revient au même de voir qu’il existe $\varphi$ tel que, $\text{val } (\varphi) = np$, et l’ordre de coïncidence
entre les développements de Puiseux de $C$ et de l’image $D$ de $\varphi$ soit égal exactement à $\frac{1}{np}$. Considérons $\varphi(t) = (t^{np}, \theta(t))$, où:

$$\theta(t) = \sum_{j \geq \beta_1, pj < i} a_j t^{pj} + rt^i.$$ 

Avec:

- si $i$ non multiple de $p$, $r \neq 0$,
- si $i = lp$, $r \neq \alpha \xi^k \eta^d$, $1 \leq k \leq n$, $1 \leq d \leq np$, où $\xi$ (resp. $\eta$) est une racine primitive $n$-ième (resp. $np$-ième) de l’unité.

Ecrivons: $\varphi(t) = (t^{ms}, \rho(t^s))$, où l’arc $(t^m, \rho(t))$ est irréductible. Alors:

$$y_1'(x_1^{\frac{1}{m}}) = \sum_{j \geq \beta_1, pj < i} a_j x_1^{\frac{j}{n}} + rx_1^{\frac{i}{np}}$$

est un développement de Puiseux racine correspondant à $D$. Or par construction:

$$ord_x(y_1'(x_1^{\frac{1}{m}}) - y_1(x_1^{\frac{1}{n}})) = \frac{i}{np}$$

et

$$ord_x(y_k'(x_1^{\frac{1}{m}}) - y_k(x_1^{\frac{1}{n}})) \leq \frac{i}{np}, 1 \leq k \leq n, 1 \leq k' \leq m.$$ 

D’où $c(D) = \frac{1}{np}$, comme annoncé.

References


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We give sufficient conditions in terms of the Melnikov functions in order that an analytic or a polynomial differential system in the real plane has a period annulus.

We study the first nonzero Melnikov function of the analytic differential systems in the real plane obtained by perturbing a Hamiltonian system having either a nondegenerate center, a heteroclinic cycle, a homoclinic cycle, or three cycles obtained connecting the four separatrices of a saddle. All the singular points of these cycles are hyperbolic saddles.

Finally, using the first nonzero Melnikov function we give a new proof of a result of Roussarie on the finite cyclicity of the homoclinic orbit of the integrable system when we perturb it inside the class of analytic differential systems.

1. Introduction and statement of the main results.

We consider the planar vector fields $X_\epsilon$ associated to the system:

\begin{align*}
\dot{x} & = X(x, y, \lambda, \epsilon) = p(x, y) + \epsilon P(x, y, \lambda, \epsilon), \\
\dot{y} & = Y(x, y, \lambda, \epsilon) = q(x, y) + \epsilon Q(x, y, \lambda, \epsilon),
\end{align*}

where $X, Y$ depend analytically on their variables and parameters $\lambda \in \Lambda$, and $\epsilon \in \mathbb{R}$, $\Lambda \subset \mathbb{R}^r$ is an open region. Assume that for $\epsilon = 0$, system (1) has a period annulus; i.e., a continuous family of periodic orbits. As usual, the dot denotes derivative with respect to the time variable $t$. We say that system (1) with $\epsilon = 0$ is the unperturbed system, while system (1) with $\epsilon \neq 0$ is the perturbed one.

Given any compact subset $D$ of $\Lambda$ and $\epsilon_0 > 0$ small, we assume that there is a transversal section $J$ to the vector fields $X_\epsilon$ in the region covered by the period annulus for $|\epsilon| < \epsilon_0$ and $\lambda \in D$. Let $u$ be an analytical parameterization of $J$. Then there is a subsection $\Sigma \subset J$ such that the Poincaré return map $(u, \lambda, \epsilon) \mapsto \Pi(u, \lambda, \epsilon)$ is defined from $\Sigma \times D \times (-\epsilon_0, \epsilon_0)$ to $J$. The displacement function $d(u, \lambda, \epsilon)$ is defined as $d(u, \lambda, \epsilon) = \Pi(u, \lambda, \epsilon) - u$. Since system (1) has a period annulus for $\epsilon = 0$, we have $d(u, \lambda, 0) \equiv 0$, 

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and thus for \( u \in \Sigma, \lambda \in D \) and \( \epsilon_0 > 0 \) sufficiently small, we have

\[
d(u, \lambda, \epsilon) = \sum_{i=1}^{\infty} M_i(u, \lambda) \epsilon^i.
\]

The function \( M_i \) is called the \( i \)-th Melnikov function. In what follows the notation \( |\epsilon| \ll 1 \) means for all \( \epsilon \) such that \( |\epsilon| < \epsilon_0 \) with \( \epsilon_0 > 0 \) sufficiently small. The first part of this paper is dedicated to period annulus.

**Theorem 1.** For any compact set \( D \subset \Lambda, \epsilon_0 > 0 \) and a transversal section \( \Sigma \) for which the displacement function (2) is defined, there exists a natural number \( N \) depending on \( D \) such that for any \( \lambda_0 \in D \), if \( M_i(u, \lambda_0) \equiv 0 \), for \( u \in \Sigma, 1 \leq i \leq N \), then system (1) has a period annulus for \( \lambda = \lambda_0 \) and \( |\epsilon| \ll 1 \).

**Theorem 2.** Assume that

\[
P(x, y, \lambda, \epsilon) = P(x, y, \epsilon) = \sum_{i=0}^{l} P_i(x, y) \epsilon^i,
\]

\[
Q(x, y, \lambda, \epsilon) = Q(x, y, \epsilon) = \sum_{i=0}^{l} Q_i(x, y) \epsilon^i,
\]

and \( P_i(x, y), Q_i(x, y) \) are polynomials in the variables \( x \) and \( y \) of degree at most \( n \), then there exists a natural number \( N \) depending on the unperturbed system \( X_0 \) and on the natural numbers \( l, n \) such that if \( M_i(u) \equiv 0 \) for \( 1 \leq i \leq N \), then system (1) has a period annulus for \( |\epsilon| \ll 1 \).

The second part of this paper is concerned with the properties of Melnikov functions near a nondegenerate center and a hyperbolic heteroclinic or homoclinic cycle for the perturbed Hamiltonian systems.

We first recall some definitions. Let \( X \) be a vector field in the plane. A center is a singular point of \( X \) for which there is a neighbourhood filled of periodic orbits with the exception of the singular point. A center of \( X \) is called nondegenerate if it has a pair of pure imaginary eigenvalues. A heteroclinic cycle \( \Gamma \) for \( X \) is a finite collection of separatrices of hyperbolic sectors \( \gamma_1, \gamma_2, \ldots, \gamma_n \) and a finite collection of singular points \( p_1, p_2, \ldots, p_n \) such that the \( \alpha \)-limit set of \( \gamma_i \) is \( p_i \) for \( i = 1, \ldots, n \), the \( \omega \)-limit set of \( \gamma_i \) is \( p_{i+1} \) for \( i = 1, 2, \ldots, n-1 \) and the \( \omega \)-limit set of \( \gamma_n \) is \( p_1 \). Moreover, some of the \( p_i \) can be repeated. A heteroclinic cycle \( \Gamma \) is called hyperbolic, if all its singular points are hyperbolic saddles. A heteroclinic cycle becomes a homoclinic one, if it consists of one singular point and one separatrix. Now
we consider the following perturbed Hamiltonian system:

\[
\begin{align*}
\dot{x} &= \frac{\partial H(x, y)}{\partial y} + \epsilon P(x, y, \epsilon), \\
\dot{y} &= -\frac{\partial H(x, y)}{\partial x} + \epsilon Q(x, y, \epsilon),
\end{align*}
\]

where \( H, P, Q \) are analytical functions in the variables \((x, y) \in \mathbb{R}^2\) and in the parameter \( \epsilon \in (\mathbb{R}, 0) \). Here \((\mathbb{R}, 0)\) denotes a small neighbourhood of zero, and \( C^\omega(\mathbb{R}, 0) \) denotes the set of analytic functions in a small neighbourhood of zero. In this case, as usual, we parameterize the transversal section \( J \) by the Hamiltonian constant \( h = H \).

**Theorem 3.** For system (3) assume that when \( h \downarrow 0, \gamma_h \to (0, 0) \), a non-degenerate center of the unperturbed system. Then the following hold:

1. \( M_1(h) \) can be analytically continued to \( h = 0 \), and

\[
M_1(0) = 0, \quad M'_1(0) = \frac{2\pi}{\beta} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \bigg|_{(x, y, \epsilon) = (0, 0, 0)},
\]

where \( \pm i\beta \) with \( \beta > 0 \) are the eigenvalues of the center.

2. If \( M_i(h) \equiv 0 \) for \( 1 \leq i \leq k-1 \), then \( M_k(h) \) can be analytically continued to \( h = 0 \) and \( M_k(0) = 0 \).

**Theorem 4.** For system (3) assume that when \( h \downarrow 0, \gamma_h \to \gamma_0 \), a heteroclinic cycle of the unperturbed system consisting of \( n \) hyperbolic saddles \( p_1, p_2, \ldots, p_n \) (eventually they can be repeated) and the corresponding \( n \) separatrices. Then the following hold:

1. There exist analytical functions \( a_1(h), b_1(h) \in C^\omega(\mathbb{R}, 0) \) such that

\[
M_1(h) = a_1(h) + b_1(h) \ln h, \quad 0 < h \ll 1,
\]

with

\[
a_1(0) = \int_{\gamma_0} P(x, y, 0) \, dy - Q(x, y, 0) \, dx, \\
b_1(0) = 0,
\]

\[
b'_1(0) = -\sum_{i=1}^n \frac{1}{\lambda_i} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \bigg|_{(x, y) = p_i, \epsilon = 0},
\]

where \(-\lambda_i < 0 < \lambda_i\) are the eigenvalues of the saddle \( p_i \). Moreover, if

\[
\left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \bigg|_{(x, y) = p_i, \epsilon = 0} = 0, \quad \text{for} \quad i = 1, \ldots, n,
\]
then
\[ a'_1(0) = \int_{\gamma_0} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \bigg|_{\epsilon=0} dt. \]

(2) If \( n = 1 \), and \( \gamma_0 \) is a homoclinic cycle of a hyperbolic saddle, and \( M_i(h) \equiv 0 \) for \( 1 \leq i \leq k - 1 \), then there exist analytical functions \( a_k(h), b_k(h) \in C^\omega(\mathbb{R}, 0) \) with \( b_k(0) = 0 \), such that
\[ M_k(h) = a_k(h) + b_k(h) \ln h, \quad 0 < h \ll 1. \]

In general, Statement (2) of Theorem 4 cannot be generalized to heteroclinic cycles with two saddles or more. Now we consider the so-called 8-figure heteroclinic cycles, i.e., the cycles consisting of one saddle and its two homoclinic orbits. Assume that for \( \epsilon = 0 \), system (3) has two homoclinic orbits \( \gamma_0^\pm \) of a hyperbolic saddle, called the 8-figure cycle, and three families of periodic orbits:
\[ \gamma_h \subset \{ H^{-1}(h), h > 0 \}, \gamma^+_h \cup \gamma^-_h \subset \{ H^{-1}(h), h < 0 \}, \]
such that \( \gamma_h \to \gamma^+_0 \cup \gamma^-_0 \) when \( h \searrow 0 \), and \( \gamma^+_h \to \gamma^+_0 \) when \( h \not\searrow 0 \), see Figure 1.

Three classes of Melnikov functions are defined corresponding to the three period annuli: \( M_k(h), M^+_k(h), M^-_k(h) \) for \( k \geq 1 \). Then we have:

**Theorem 5.** If \( M_i = M^+_i = M^-_i \equiv 0 \) for \( 0 \leq i \leq k - 1 \) with \( k \geq 1 \), then the following hold:

1. If one of three functions \( M_k, M^+_k, M^-_k \) can be analytically continued to \( h = 0 \), then the other two can also be continued.
2. If two of three functions \( M_k, M^+_k, M^-_k \) are identically zero, then the third one is identically zero.
3. There exist analytical functions \( a_k(h), b_k(h) \in C^\omega(\mathbb{R}, 0) \) with \( b_k(0) = 0 \) such that \( M_k(h) = a_k(h) + b_k(h) \ln h \) for \( 0 < h \ll 1 \).

**Figure 1.** The two 8-figure heteroclinic cycles.

We remark that by using Melnikov functions we can determinate the cyclicity of a center or of a homoclinic cycle. Assume that the origin \((0,0)\)
is a nondegenerate center of system (3) for $\epsilon = 0$. Without loss of generality, we can assume that

$$P(0, 0, \epsilon) = Q(0, 0, \epsilon) \equiv 0,$$

which means that our perturbation preserves the singular point $(0, 0)$ fixed.

Let $f(h) \in C^\omega(\mathbb{R}, 0)$. If $f(h) = ah^n + O(h^{n+1})$ with $a \neq 0$, we define $m(f) = n$. If $f \equiv 0$, we define $m(f) = \infty$.

**Remark 6.** Let the origin $(0, 0)$ be a nondegenerate center of system (3) for $\epsilon = 0$. Assume that (4) holds, and there exist integers $k \geq 1$, $m \geq 0$ such that $M_i(h) \equiv 0$ for $0 \leq i \leq k - 1$, and $m(M_k(h)) = n + 1$ with $0 \leq n < \infty$.

Then system (3) has at most $n$ (taking into account their multiplicity) limit cycles in some neighbourhood of the origin for $|\epsilon| \ll 1$.

**Remark 7.** Assume that when $h \searrow 0$, $\gamma_h \to \gamma_0$, where $\gamma_0$ is a homoclinic cycle of a hyperbolic saddle. Let $k \geq 1$ be such an integer that $M_i(h) \equiv 0$ for $0 \leq i \leq k - 1$ and $M_k(h) = a_k(h) + b_k(h) \ln h$ is not identically zero, then there exists a neighbourhood $U$ of $\gamma_0$ such that for $|\epsilon| \ll 1$, system (3) in $U$ has at most $2m(b_k) - 1$ limit cycles if $m(a_k) \geq m(b_k)$; and $2m(a_k)$ limit cycles if $m(a_k) < m(b_k)$, these estimates hold taken into account the multiplicity of the limit cycles.

We point out that Roussarie in [10] obtained the result of Remark 7 for $k = 1$ and the method works for $k \geq 2$ also. As an application of Remark 7, we will give a simple proof of the finite cyclicity of the homoclinic cycle of infinite codimension under a one-parameter perturbation, which is a particular case of a result due to Roussarie [11]. First we recall that a homoclinic cycle $\gamma$ is said to be of infinite codimension if there exists a continuous family of periodic orbits tending to the cycle $\gamma$.

**Theorem 8.** Let $X_\epsilon$ be an one parameter analytic family of planar vector fields. Assume that $X_0$ has a homoclinic cycle $\gamma$ of a hyperbolic saddle of infinite codimension, then there exists a neighbourhood $U$ of $\gamma$ and a natural number $N$ such that $X_\epsilon$ has at most $N$ limit cycles (taking into account their multiplicity) in $U$ for $|\epsilon| \ll 1$.

This paper is organized as follows: In Section 2, we prove Theorems 1 and 2. In Section 3 we first recall three important results, one is about the formula for computing Melnikov functions of arbitrary order, the other two are about the normalization of planar Hamiltonian vector fields near a nondegenerate center or a hyperbolic saddle, which are the main tools in this paper, and then prove Theorem 3. In Section 4 we prove Theorems 4 and 5. In Section 5 (after Proposition 25) we prove Theorem 8.
2. Proofs of Theorems 1 and 2.

Lemma 9. Let \( d(u, \lambda, \epsilon) \) be the displacement function as defined in Section 1. Assume that, for \( \lambda = \lambda_0 \in D \), \( d(u, \lambda_0, \epsilon) \equiv 0 \), then there exists a neighbourhood \( U \) of \( \lambda_0 \) and a natural number \( N \) such that for any \( \lambda \in U \), if \( M_i(u, \lambda) \equiv 0, 1 \leq i \leq N \), then \( M_i(u, \lambda) \equiv 0 \) for all natural number \( i \), i.e., system (1) has a period annulus for \( |\epsilon| \ll 1 \).

Proof. By the assumption, \( M_i(u, \lambda_0) \equiv 0 \) for all \( i \). For \( u_0 \in \Sigma \), let

\[
M_i(u, \lambda) = \sum_{j=0}^{\infty} a_i^j(\lambda, u_0)(u - u_0)^j.
\]

Then \( a_i^j(\lambda_0, u_0) = 0, i \geq 1, j \geq 0 \). Denote by \( \mathcal{A} \) the ring of germs of analytic functions at \( \lambda_0 \) and \( \mathcal{I} = \mathcal{I}\{a_i^j(\cdot, u_0)\}_{i \geq 1, j \geq 0} \) the ideal generated by the germs of the analytical functions \( a_i^j \) at \( \lambda = \lambda_0 \). Since the ring \( \mathcal{A} \) is Noetherian (see for instance [3], p. 161, Theorem 6.3.3), and so \( \mathcal{I} \) is generated by a finite number of germs \( \hat{a}_i^j \):

\[
\mathcal{I} = \mathcal{I}\{\hat{a}_1^{i_1}, \hat{a}_2^{i_2}, \ldots, \hat{a}_n^{i_n}\}.
\]

Let \( N = \max\{i_1, i_2, \ldots, i_n\} \). Then there exist analytical functions \( h_{i,k}(\lambda, u) \), \( 1 \leq k \leq N, i \geq 1 \) defined in some neighborhood \( U \times V \) of \( (\lambda_0, u_0) \) such that

\[
M_i(u, \lambda) = \sum_{k=1}^{n} a_{j_k}^{i_k}(\lambda, u_0)h_{i,k}(\lambda, u).
\]

(See [12], p. 79, Proposition 4.) For \( \lambda \in U \), if \( M_i(u, \lambda) \equiv 0, 1 \leq i \leq N \), then

\[
a_i^j(\lambda, u_0) = 0, \text{ for } 1 \leq i \leq N, j \geq 0,
\]

which, by (5), implies \( M_i(u, \lambda) \equiv 0 \) for \( i \geq 1 \). The proof of Lemma 9 is complete. \( \square \)

Proof of Theorem 1. Let

\[
D_i = \{ \lambda \in D | \exists k, k \leq i \text{ with } M_k(u, \lambda) \text{ not identically vanishing} \}
\]

\[
D = \{ \lambda \in D | M_i(u, \lambda) \equiv 0, \forall i > 0 \}.
\]

Then

\[
D = \left( \bigcup_{i=1}^{\infty} D_i \right) \bigcup D.
\]

If the conclusion is not true, then there exists a sequence of parameter values \( \lambda_n \in D \) such that

\[
M_i(u, \lambda_n) \equiv 0 \text{ for } 1 \leq i \leq n \text{ and } d(u, \lambda_n, \epsilon) \text{ is not identically zero}.
\]

By the compactness of \( D \), we can assume that \( \lambda_n \to \bar{\lambda} \in D \). Since \( D_i \) are open subsets of \( D \), \( \bar{\lambda} \in D \), which is a contradiction with Lemma 9. \( \square \)
Proof of Theorem 2. Let
\[ P_i(x, y) = \sum_{0 \leq j + k \leq n} p_{i,j,k} x^j y^k, \quad Q_i(x, y) = \sum_{0 \leq j + k \leq n} q_{i,j,k} x^j y^k. \]
We consider the coefficients of the polynomials \( p_{i,j,k}, q_{i,j,k} \) and \( \epsilon \) as the parameters. Note that system (4) preserves unchanged under the parameter change \( \epsilon \to \delta^{-1}\epsilon, p_{i,j,k} \to \delta^i p_{i,j,k}, q_{i,j,k} \to \delta^i q_{i,j,k} \). Therefore, we can assume that \( |p_{i,j,k}| \leq 1, |q_{i,j,k}| \leq 1 \). Hence Theorem 2 becomes a corollary of Theorem 1.

Example 1. For the quadratic perturbations of Bagdanov-Takens system (see [4]):
\[
\begin{align*}
\dot{x} &= y + \epsilon P(x, y) \\
\dot{y} &= -x - x^2 + \epsilon Q(x, y)
\end{align*}
\]
where \( P, Q \) are polynomials of degree at most 2, \( N = 4 \).

Example 2. For the quadratic perturbation of quadratic Hamiltonian system which preserves the center fixed (see [7]):
\[
\begin{align*}
\dot{x} &= \frac{\partial H(x, y)}{\partial y} + \epsilon P(x, y) \\
\dot{y} &= -\frac{\partial H(x, y)}{\partial x} + \epsilon Q(x, y)
\end{align*}
\]
where \( H \) is a polynomial of degree 3, the origin \((0, 0)\) is a center of the unperturbed system and \( P, Q \) are polynomials of degree \( \leq 2 \) with \( P(0, 0) = Q(0, 0) = 0, N = 6 \).

3. Analyticity of Melnikov functions at a center.

We first recall three results which are necessary in the proof of our theorems. The first one is about the computation of the Melnikov functions of system (3). We consider now the equivalent form of system (3):
\[ \omega_\epsilon = \left( \frac{\partial H}{\partial x} + \epsilon Q \right) dx + \left( \frac{\partial H}{\partial y} - \epsilon P \right) dy = 0. \]
Let
\[ \omega_\epsilon = \sum_{i=0}^{\infty} \omega_i \epsilon^i. \]
Then \( \omega_0 = dH \), and \( \omega_i \)'s are analytical 1-form. The following result is due to Poggiale [9], its proof can be found in [12].

Proposition 10.
(1) \( M_1(h) = -\int_{\gamma_0} \omega_1; \)
(2) If $M_i(h) \equiv 0$ for $1 \leq i \leq k$, then
\[
M_{k+1}(h) = \int_{\gamma_h} \left( \sum_{i=1}^{k} g_i \omega_k - \omega_{k+1} \right),
\]
where the analytic functions $g_i, i = 1, 2, \ldots, k,$ are defined inductively by
\[
\omega_i - g_i dH = \sum_{j=1}^{i-1} g_j \omega_{i-j} + dR_i.
\]

The next two classical results are about the normalization of planar Hamiltonian system near a nondegenerate center or a hyperbolic saddle respectively (for the proofs, see, for instance, [6] and [8]).

Consider now the following planar Hamiltonian system:
\[
\dot{x} = \frac{\partial H(x,y)}{\partial y},
\]
\[
\dot{y} = - \frac{\partial H(x,y)}{\partial x},
\]
where $H$ is an analytical function defined in some neighbourhood of the origin $(0,0)$.

**Proposition 11.** Assume that the origin $(0,0)$ is a nondegenerate center of system (7) with eigenvalues $\pm \beta, \beta > 0$, then there exist an analytical area-preserving transformation of variables: $(x,y) = G(u,v)$ in some neighbourhood of the origin and a function $f \in C^\omega(\mathbb{R},0)$ with $f(0) = 0, f'(0) = \frac{\beta}{2}$ such that $f(u^2 + v^2) = H \circ G(u,v)$ and system (7) is changed to the form:
\[
\dot{u} = 2vf'(u^2 + v^2),
\]
\[
\dot{v} = -2uf'(u^2 + v^2).
\]

**Proposition 12.** Assume that the origin $(0,0)$ is a hyperbolic saddle of system (7) with eigenvalues $\pm \lambda, \lambda > 0$, then there exist an analytical area-preserving transformation of variables: $(x,y) = G(u,v)$ in some neighbourhood of the origin and a function $f \in C^\omega(\mathbb{R},0)$ with $f(0) = 0, f'(0) = \lambda$ such that $f(uv) = H \circ G(u,v)$ and system (7) is changed to the form:
\[
\dot{u} = uf'(uv),
\]
\[
\dot{v} = -vf'(uv).
\]

**Lemma 13.** Assume that $f \in C^\omega(\mathbb{R},0), f(0) = 0, f'(0) > 0$, and $F \in C^\omega(\mathbb{R}^2,0)$. Define function
\[
M(h) := \iint_{f(x^2+y^2) \leq h} F(x,y) \, dx \, dy, \quad 0 < h \ll 1.
\]
Then the following statements hold:
(1) $M(h)$ can analytically be continued to $h = 0$, and

$$M(0) = 0, \quad M'(0) = \frac{\pi}{f'(0)} F(0, 0).$$

(2) If

$$F(x, y) = \sum_{n=0}^{\infty} F_n(x, y), \quad F_n(x, y) = \sum_{i=0}^{n} b_{i, n} x^{n-i} y^{i},$$

then

$$M(h) \equiv 0 \iff C_m := \sum_{k=0}^{m} (2m - 2k - 1)!! (2k - 1)!! b_{2k, 2m} = 0, \forall m \geq 0,$$

where $(-1)!! := 1$.

Proof. Assume that series (8) is convergent in the square $D = \{(x, y) \in \mathbb{C}^2 \mid |x| \leq R, |y| \leq R\}$. Let $K = \sup_D |F|$. By Cauchy inequality, $|b_{2k, 2m}| \leq K R^{-2m}$. Let $d_m = \frac{C_m}{(2m+2)!!}$, then

$$|d_m| \leq (m + 1)KR^{-2m},$$

which implies the function $g(r) := \sum_{m=0}^{\infty} d_m r^{m+1}$ is analytic in the region $|r| \leq R^2$. Now we calculate the function $M(h)$. By introducing the polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, $s = \sqrt{f^{-1}(h)}$, we have

$$M(h) = \int_{0}^{s} dr \int_{0}^{2\pi} r F(r \cos \theta, r \sin \theta) d\theta$$

$$= \sum_{n=0}^{\infty} \int_{0}^{s} r^{n+1} dr \int_{0}^{2\pi} \sum_{i=0}^{n} b_{i, n} \cos^{n-i} \theta \sin^{i} \theta d\theta$$

$$= \sum_{m=0}^{\infty} \int_{0}^{s} r^{2m+1} dr \int_{0}^{2\pi} \sum_{i=0}^{2m} b_{i, 2m} \cos^{2m-i} \theta \sin^{i} \theta d\theta$$

$$= \sum_{m=0}^{\infty} \int_{0}^{s} r^{2m+1} dr \int_{0}^{2\pi} \sum_{k=0}^{m} b_{2k, 2m} \cos^{2m-2k} \theta \sin^{2k} \theta d\theta$$

$$= 2\pi \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{(2m - 2k - 1)!! (2k - 1)!! b_{2k, 2m} s^{2m+2}}{(2m + 2)!!}$$

$$= 2\pi \sum_{m=0}^{\infty} d_m (f^{-1}(h))^{m+1} = 2\pi g(f^{-1}(h)),$$

which is analytical at $h = 0$, and satisfies the following:

$$M(0) = 2\pi g(0) = 0, \quad M'(0) = 2\pi g'(0)(f'(0))^{-1} = \frac{\pi}{f'(0)} F(0, 0).$$
In the computation above, we have used that
\[\int_0^{2\pi} \cos^{2m-2k} \theta \sin^{2k} \theta d\theta = \frac{(2m-2k-1)!! (2k-1)!!}{(2m)!!}.\]
For a proof, see [2]. Statement (2) is obvious by noting that
\[M(h) \equiv 0 \iff g(r) \equiv 0 \iff d_m = 0, \forall m.\]
\[\square\]

**Remark 14.** Condition (9) is equivalent to
\[\int_0^{2\pi} F(r \cos \theta, r \sin \theta) d\theta \equiv 0, \ 0 \leq r \ll 1.\] (11)

Now we consider an analytical system
\[\dot{u} = \frac{\partial H}{\partial v}, \quad \dot{v} = -\frac{\partial H}{\partial u}.\] (12)

**Lemma 15.** Assume that system (12) has a family of periodic orbits \(\gamma_h : H(u, v) = h, \ 0 < h < \bar{h}\). The origin \((0, 0) = H^{-1}(0)\) is a nondegenerate center with eigenvalues \(\pm i \beta\) with \(\beta > 0\). Let \(\omega = -P(u, v) dv + Q(u, v) du\) be an analytical 1-form defined in some neighbourhood of the origin, then the function \(M(h) := \int_{\gamma_h} \omega\) can be analytically continued to \(h = 0\), and
\[M(0) = 0, \quad M'(0) = \frac{2\pi}{\beta} \left( \frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v} \right) \bigg|_{(u,v)=(0,0)}.\]

**Proof.** By Proposition 11, there exist an area-preserving transformation
\[u = u(x, y), \ v = v(x, y), \ u(0, 0) = 0, \ v(0, 0) = 0,\] (13)
and a function \(f \in C^\infty(\mathbb{R}, 0)\) with \(f(0) = 0, \ f'(0) = \frac{\beta}{2}\), such that
\[H(u(x, y), v(x, y)) = f(x^2 + y^2).\]
Thus, by Green’s formula, we obtain
\[M(h) = \iint_{H(u,v) \leq h} \left( \frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v} \right) du dv = \iint_{f(x^2+y^2) \leq h} F(x,y) dx dy, \quad F(x,y) = \left( \frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v} \right) \bigg|_{(u,v)=(x,y)}.\]
By Lemma 13, \(M(h)\) can be analytically continued to \(h = 0\), with
\[M(0) = 0, \quad M'(0) = \frac{2\pi}{\beta} \left( \frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v} \right) \bigg|_{(u,v)=(0,0)}.\]
\[\square\]
Statement (1) of Theorem 3 follows from Proposition 10 and Lemma 15. Next, we prove statement (2) of this theorem.

**Lemma 16.** Let \( H, \gamma_h, \omega \) be defined as in Lemma 15, then \( M(h) = \int_{\gamma_h} \omega \equiv 0 \) if and only if there exists a real analytical function \( z = z(u, v) \) defined in some neighbourhood of the origin \((0, 0)\) satisfying the following linear partial differential equation:

\[
\frac{\partial H}{\partial u} \frac{\partial z}{\partial u} - \frac{\partial H}{\partial v} \frac{\partial z}{\partial v} = \frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v}.
\]

(14)

**Proof.** Sufficiency. Assume \( z(u, v) \) satisfies Equation (14), then \( \omega - zdH \) is a total differential of some function, i.e., there exists an analytical function \( R \) defined in some neighbourhood of the origin such that \( \omega - zdH = dR \). Therefore, \( \int_{\gamma_h} \omega = \int_{\gamma_h} (zdH + dR) \equiv 0 \).

Necessity. Let \( u = u(x, y), v = v(x, y) \) be the area-preserving normalization transformation (13). We denote by \( z(x, y) = z(u(x, y), v(x, y)) \), \( H(x, y) = f(x^2 + y^2) = H(u(x, y), v(x, y)) \), \( F(x, y) = \left( \frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v} \right)_{u=u(x, y), v=v(x, y)} \).

Then
\[
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}.
\]

Consequently
\[
\begin{pmatrix}
\frac{\partial z}{\partial u} \\
\frac{\partial z}{\partial v}
\end{pmatrix} =
\begin{pmatrix}
\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \\
-\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial z}{\partial x} \\
\frac{\partial z}{\partial y}
\end{pmatrix}.
\]

Substituting it into (14), we get
\[
\left( \frac{\partial u}{\partial x} \frac{\partial z}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial z}{\partial x} \right) \frac{\partial H}{\partial u} + \left( \frac{\partial v}{\partial x} \frac{\partial z}{\partial y} - \frac{\partial v}{\partial y} \frac{\partial z}{\partial x} \right) \frac{\partial H}{\partial v} = F(x, y),
\]
or equivalently
\[
\frac{\partial H}{\partial x} \frac{\partial z}{\partial y} - \frac{\partial H}{\partial y} \frac{\partial z}{\partial x} = F(x, y).
\]

(15)

By using \( H(x, y) = f(x^2 + y^2) \), (15) can be written in the form
\[
\frac{\partial z}{\partial y} - \frac{\partial z}{\partial x} = R(x, y), \quad R(x, y) := \frac{F(x, y)}{2f'(x^2 + y^2)}.
\]

(16)
If
\[ R(x, y) = \sum_{n=0}^{\infty} R_n(x, y), \quad R_n(x, y) = \sum_{i=0}^{n} b_{i, n} x^{n-i} y^i, \]
then, by Lemma 13 and Remark 14,
\[ \int_{0}^{2\pi} R(r \cos \theta, r \sin \theta) \, d\theta = \frac{1}{2f'(r^2)} \int_{0}^{2\pi} F(r \cos \theta, r \sin \theta) \, d\theta \equiv 0. \]
This implies that the coefficients \( b_{i, n} \) must satisfy (9). Let
\[ \bar{z} = \sum_{n=0}^{\infty} z_n, \quad z_n = \sum_{k=0}^{n} a_{k, n} x^{n-k} y^k. \]
Substituting (17) into (16), we get
\[ \frac{\partial z_n}{\partial y} x - \frac{\partial z_n}{\partial x} y = R_n, \quad n = 0, 1, 2, \ldots. \]
Setting \( a_{-1, n} = a_{n+1, n} = 0 \), from (18), we obtain
\[ \sum_{k=0}^{n} [(k+1)a_{k+1, n} - (n-k+1)a_{k-1, n}] x^{n-k} y^k = \sum_{k=0}^{n} b_{k, n} x^{n-k} y^k, \]
or
\[ (k+1)a_{k+1, n} - (n-k+1)a_{k-1, n} = b_{k, n}, \quad k = 0, 1, \ldots, n. \]
The determinant of system (19) is
\[ \Delta = \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -n & 0 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -n & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-1 & 0 \\ 0 & 0 & 0 & \cdots & -2 & 0 & n \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \end{vmatrix} = \begin{cases} 0 & \text{if } n = 2m; \\ [(2m+1)!!]^2 & \text{if } n = 2m + 1. \end{cases} \]
Therefore, system (19) has a unique solution for \( n \) odd. For \( n = 2m \) even, system (19) can be divided into two independent systems:
\[ 2k a_{2k, m} - 2(m-k+1)a_{2k-2, 2m} = b_{2k-1, 2m}, \quad k = 1, 2, \ldots, m, \]
\[ (2k+1)a_{2k+1, 2m} - (2m-2k+1)a_{2k-1, 2m} = b_{2k, 2m}, \quad k = 0, 1, \ldots, m. \]
System (20) contains \( m \) equations and \( m+1 \) unknown numbers and its matrix of coefficients has rank \( m \). This implies that it has a solution of one
System (21) contains \(m + 1\) equations and \(m\) unknown numbers, and its matrix of coefficients has rank \(m\). Note that the determinant of the augmented matrix of system (21) is

\[
\begin{vmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & b_{0,2m} \\
1 - 2m & 3 & 0 & \cdots & 0 & 0 & b_{2,2m} \\
0 & 3 - 2m & 5 & \cdots & 0 & 0 & b_{4,2m} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2m - 3 & 0 & b_{2m-4,2m} \\
0 & 0 & 0 & \cdots & -3 & 2m - 1 & b_{2m-2,2m} \\
0 & 0 & 0 & \cdots & 0 & -1 & b_{2m,2m}
\end{vmatrix}
\]

\[
= \sum_{k=0}^{m} (2m - 2k - 1)!! (2k - 1)!! b_{2k,2m}
\]

the last equality follows from (9). Therefore, system (21) has a unique solution. The argument above shows that system (19) always has solutions, and if we set \(a_{0,2m} = 0\), the solution is unique. Next we prove that the series (17) defined by the unique solution is convergent in some neighbourhood of the origin \((0,0)\). Assume that \(R(x,y)\) is convergent in the square \(D = \{(x,y) \in \mathbb{C}^2 \mid |x| \leq \bar{r}, |y| \leq \bar{r}\}\). Let \(C = \sup_D |R|\), then by Cauchy inequality,

\[
|b_{i,n}| \leq \bar{r}^{-n} C.
\]

We claim that

\[
|a_{i,n}| \leq 2^n \bar{r}^{-n} C.
\]

We will prove (23) only for \(n = 2m + 1, i = 2k + 1\). All other cases can be proved in a similar way. Indeed, by (19),

\[
a_{-1,n} = 0, \quad a_{1,n} = b_{0,n}, \quad a_{3,n} = \frac{1}{3} b_{2,n} + \frac{n - 1}{3} b_{0,n},
\]

\[
a_{5,n} = \frac{1}{5} b_{4,n} + \frac{n - 3}{3 \cdot 5} b_{2,n} + \frac{(n - 1)(n - 3)}{3 \cdot 5} b_{0,n},
\]

and in general,

\[
a_{2k+1,n} = \frac{1}{2k + 1} b_{2k,n} + \frac{n - 2k + 1}{2k + 1} a_{2k-1,n}.
\]

Let

\[
e_n = \max_k \left\{ \frac{(n - 1)(n - 3) \cdots (n - 2k + 1)}{(2k + 1)!!} \right\}.
\]

By induction,

\begin{equation}
\alpha_{2k+1,n} = \sum_{j=0}^{k} l_{j,n} b_{2j,n},
\end{equation}

where \( l_{j,n} \) are some constants with \( |l_{j,n}| \leq e_n \). Now we calculate the value of \( e_n \). For \( m = 2p \) even,

\[
e_n = \max_k \left\{ \frac{4p(4p-2) \cdots (4p-2k+2)}{(2k+1)!!} \right\} = \frac{4p(4p-2) \cdots (2p+2)}{(2p+1)!!} = \frac{2^m}{m+1},
\]

Similarly, for \( m = 2p + 1 \) odd, we also have \( e_n = \frac{2^m}{m+1} \). Now from (24),

\[
\left| \alpha_{2k+1,n} \right| \leq \sum_{j=0}^{k} e_n |b_{2j,n}| \leq 2^m \max_j |b_{2j,n}|
\]

\[
\leq 2^m \frac{r}{p} \leq \left( \frac{2}{p} \right)^n C,
\]

which implies that (17) is convergent in the square \( \{|x| < \frac{r}{2}, |y| < \frac{r}{2}\} \). □

Next we prove statement (2) of Theorem 3 by induction with respect to \( k \).

Suppose \( k = 1 \). The 1-form \( \omega_1 - g_1 dH \) with \( g_1 = z(x,y) \) is a total differential of some function if and only if \( z(x,y) \) is a solution of (14). By Lemma 16, if \( \int_{\gamma_h} \omega_1 \equiv 0 \), then there exists an analytical function \( g_1 = z(x,y) \) defined in some neighbourhood of the origin satisfying (14). This implies that \( \omega_1 - g_1 dH = dR_1 \) for some analytical function \( R_1 \) defined in some neighbourhood of the origin. Therefore, \( g_1 \omega_1 - \omega_2 \) is an analytical 1-form defined in some neighbourhood of the origin. By Lemma 15 and Proposition 10, \( M_2(h) = \int_{\gamma_h} g_1 \omega_1 - \omega_2 \) can be analytically continued to \( h = 0 \) and \( M_2(0) = 0 \). Now we assume that

\[
M_j(h) = \int_{\gamma_h} \left( \sum_{i=1}^{j-1} g_{i} \omega_{j-i} - \omega_{j} \right) \equiv 0 \text{ for } 1 \leq j \leq k - 1.
\]

Applying Lemma 16 to the function \( M_{k-1}(h) \), we get an analytical function \( g_{k-1} \) defined in some neighbourhood of the origin \( (0,0) \) such that (6) holds for \( i = k - 1 \). By Lemma 15 and Proposition 10, \( M_k(h) = \int_{\gamma_h} \left( \sum_{i=1}^{k-1} g_{i} \omega_{k-i} - \omega_{k} \right) \) can be analytically continued to \( h = 0 \) and \( M_k(0) = 0 \). Therefore, the proof of Theorem 3 is now completed.
4. Melnikov functions near homoclinic and heteroclinic cycles.

In this section we shall prove Theorem 4.

**Lemma 17.** Assume that $f \in C^\omega(\mathbb{R}, 0)$ with $f(0) = 0$, $f'(0) > 0$; $P(u, v)$ and $Q(u, v)$ are analytical functions in the square $\{(u, v) \in C^2 \mid |u| \leq \delta_1, |v| \leq \delta_1\}$:

\[
P(u, v) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} p_{i, n} u^{n-i} v^i, \quad Q(u, v) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} q_{i, n} u^{n-i} v^i.
\]

Let

\[
D = [-\delta, \delta] \times [-\delta, \delta] \subset \mathbb{R}^2, \quad 0 < \delta < \delta_1,
\]

\[
\gamma_h = \{(u, v) \in D \mid f(uv) = h, u \geq 0, v \geq 0\}.
\]

Define the function

\[
M(h) = \int_{\gamma_h} \omega, \quad \omega = -P \, dv + Q \, du.
\]

Let $s = f^{-1}(h)$. Then there exist functions $a(s), b(s) \in C^\omega(\mathbb{R}, 0)$ such that

\[
M(h) = a(s) + b(s) \ln s, \quad 0 < h \ll 1,
\]

where

\[a(0) = \int_{\gamma_0} \omega, \quad b(s) = -\sum_{m=0}^{\infty} (p_{m, 2m+1} + q_{m+1, 2m+1}) s^{m+1}.\]

**Proof.** Calculating straightforwardly, we have

\[
\int_{\gamma_h} Q \, du = \sum_{n=0}^{\infty} \sum_{i=0}^{n} q_{i, n} \int_{s^{\delta-1}}^{\delta} u^{n-2i} s^i \, du
\]

\[
= \sum_{m=0}^{\infty} \sum_{i=0}^{2m} q_{i, 2m} \int_{s^{\delta-1}}^{\delta} u^{2m-2i} s^i \, du
\]

\[
+ \sum_{m=0}^{\infty} \sum_{i=0}^{2m+1} q_{i, 2m+1} \int_{s^{\delta-1}}^{\delta} u^{2m+1-2i} s^i \, du
\]

\[
= \sum_{m=0}^{\infty} \sum_{i=0}^{2m} q_{i, 2m} \frac{2m}{2m-2i+1} (s^{2m-2i+1} s^i - s^{-2m+2i-1} s^{2m-i+1})
\]

\[
+ \sum_{m=0}^{\infty} \sum_{i=0}^{2m+1} \frac{q_{i, 2m+1}}{2m-2i+2} (s^{2m-2i+2} s^i - s^{-2m+2i-2} s^{2m-i+2})
\]

\[
+ 2 \ln \delta \sum_{m=0}^{\infty} q_{m+1, 2m+1} s^{m+1} - \sum_{m=0}^{\infty} q_{m+1, 2m+1} s^{m+1} \ln s
\]
\[
I(s) - \left( \sum_{m=0}^{\infty} q_{m+1, 2m+1} s^{m+1} \right) \ln s.
\]

Similarly, we can get
\[
\int_{\gamma_h} P \, dv = I_2(s) + \left( \sum_{m=0}^{\infty} p_{m, 2m+1} s^{m+1} \right) \ln s,
\]
which, together with (25), implies
\[
M(h) = \int_{\gamma_h} -P \, dv + Q \, du = a(s) + b(s) \ln s,
\]
where
\[
a(s) = I_1(s) - I_2(s), \quad b(s) = -\sum_{m=0}^{\infty} (p_{m, 2m+1} + q_{m+1, 2m+1}) s^{m+1}.
\]

By using the Cauchy inequality, it is easy to prove that the functions \(a(s)\) and \(b(s)\) are analytical in some neighbourhood of \(s = 0\).

From Lemma 17 it follows:

**Corollary 18.** Under the assumption of Lemma 17, the function \(M(h)\) can be analytically continued to \(h = 0\) if and only if
\[
p_{m, 2m+1} + q_{m+1, 2m+1} = 0, \quad \forall \, m \geq 0.
\]

We remark that Condition (27) is equivalent to say that \(b(s) \equiv 0\) for \(0 < s \ll 1\).

**Lemma 19.** Let \(\gamma_0\) be as in Theorem 4 a heteroclinic cycle of system (3) consisting of \(n\) hyperbolic saddles \(p_1, p_2, \ldots, p_n\) (eventually they can be repeated) and \(n\) separatrices, \(\omega = -P(x, y) \, dy + Q(x, y) \, dx\) an analytical 1-form defined in some neighbourhood of \(\gamma_0\), and let \(M(h) := \int_{\gamma_h} \omega\), then there exist analytical functions \(a(h), b(h) \in C^\omega(R, 0)\) with
\[
a(0) = \int_{\gamma_0} \omega, \quad b(0) = 0, \quad b'(0) = -\sum_{i=1}^{n} \frac{1}{\lambda_i} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \bigg|_{(x, y)=p_i}
\]
where \(-\lambda_i < 0 < \lambda_i\) are the eigenvalues of the saddle \(p_i\), such that
\[
M(h) = a(h) + b(h) \ln h, \quad 0 < h \ll 1.
\]

Moreover, if
\[
\left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \bigg|_{(x, y)=p_i} = 0 \quad \text{for} \, i = 1, 2, \ldots, n,
\]
then
\[
a'(0) = \int_{\gamma_0} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \, dt.
\]
Proof. According to Proposition 12, for every $1 \leq i \leq n$, there exist an analytical function $H_i \in C^\omega(\mathbb{R}, 0)$ with $H_i(0) = 0$, $H'_i(0) = \lambda_i$, and an area-preserving normalization coordinate transformation 

$$F_i : x = x(u, v, i), \ y = y(u, v, i),$$

from some neighbourhood of the origin $(0, 0)$ to some neighbourhood $U_i$ of $p_i$ such that in the new coordinate $(u, v)$ system (3) for $\epsilon = 0$ takes the form 

$$\dot{u} = \frac{\partial G_i}{\partial v}, \ \dot{v} = -\frac{\partial G_i}{\partial u},$$

where 

$$G_i(u, v) = H_i(uv) = H \circ F_i(u, v).$$

Denote by $D = \{|u| \leq \delta, |v| \leq \delta\}$, and fix $\delta > 0$ small enough such that $F_i(D) \subset U_i$, $i = 1, 2, \ldots, n$. We note that $u = 0$ and $v = 0$ are the separatrices of the saddle $(0, 0)$ for the system $(\dot{u}, \dot{v})$. Let 

$$\Gamma^+_i = F_i(\{u = \delta\}), \ \Gamma^-_i = F_i(\{v = \delta\}).$$

Any closed orbits near $\gamma_0$ is separated by $\Gamma^+_i$, $i = 1, 2, \ldots, n$ into $2n$ segments: $\gamma^+_h$, $i = 1, 2, \ldots, 2n$, in which $\gamma^2_{hi}$ are close to $p_i$ and $\gamma^1_{hi}$ connects $\gamma_h \cap \Gamma^+_i$ and $\gamma_h \cap \Gamma^-_i$, see Figure 2. Then 

$$(30) \quad M(h) = \int_{\gamma_h} \omega = \sum_{i=1}^{2n} \int_{\gamma^+_h} \omega = \sum_{i=1}^{n} \int_{\gamma^2_{hi}} \omega + \sum_{i=1}^{n} \int_{\gamma^1_{hi}} \omega.$$ 

Since $\gamma^1_{hi}$ depend analytically on $h$, $\int_{\gamma^1_{hi}} \omega$ are analytical at $h = 0$. Next we consider the integrals $\int_{\gamma^2_{hi}} \omega$. Let $\Sigma_h = \{(u, v) \in D \ | \ H_i(uv) = h\}$. Substituting $x = x(u, v, i)$, $y = y(u, v, i)$ into the integral $\int_{\gamma^2_{hi}} \omega$, we obtain 

$$\int_{\gamma^2_{hi}} \omega = \int_{\Sigma_h} -P_i(u, v) \ dv + Q_i(u, v) \ du,$$

where 

$$P_i(u, v) = P \circ F_i \frac{\partial y}{\partial v} - Q \circ F_i \frac{\partial x}{\partial v}, \ \ Q_i(u, v) = Q \circ F_i \frac{\partial x}{\partial u} - P \circ F_i \frac{\partial y}{\partial u}. $$
Figure 2.

Computing straightforward, we get

\[
\frac{\partial P_i}{\partial u} = \left( \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y} \frac{\partial y}{\partial u} \right) \frac{\partial y}{\partial v} + P \circ F_i \frac{\partial^2 y}{\partial u \partial v} - \left( \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial y} \frac{\partial y}{\partial u} \right) \frac{\partial x}{\partial v} - Q \circ F_i \frac{\partial^2 x}{\partial u \partial v},
\]

\[
\frac{\partial Q_i}{\partial u} = \left( \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial y} \frac{\partial y}{\partial u} \right) \frac{\partial x}{\partial v} + Q \circ F_i \frac{\partial^2 x}{\partial u \partial v} - \left( \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y} \frac{\partial y}{\partial u} \right) \frac{\partial y}{\partial v} - P \circ F_i \frac{\partial^2 y}{\partial u \partial v},
\]

which implies

\[
(31) \quad \frac{\partial P_i}{\partial u} + \frac{\partial Q_i}{\partial v} = \frac{\partial P}{\partial x} \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) + \frac{\partial Q}{\partial y} \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) = \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \circ F_i.
\]

The last equality above follows from the fact that \( F_i \) is area-preserving. Let \( s = H^{-1}_i(h) \). From Lemma 17, there exist analytical functions \( a_i(s) \), \( b_i(s) \in C^\omega(\mathbb{R}, 0) \) with \( a_i(0) = \int_{\gamma_0} -P_i \, dv + Q_i \, du \), \( b_i(0) = 0 \) and \( b_i'(0) = \)
\[
\left( \frac{\partial P_i}{\partial u} + \frac{\partial Q_i}{\partial y} \right) \bigg|_{(u,v)=(0,0)} \text{ such that }
\int_{\gamma^2_i h} \omega = \int_{\gamma^1_i h} -\overline{P_i} \, dv + \overline{Q}_i \, du = a_i(s) + b_i(s) \ln s, \quad 0 < h \ll 1.
\]

Consequently,
\[
\int_{\gamma^2_i h} \omega = a_i(h) + b_i(h) \ln h,
\]

where
\[
\tilde{a}_i(h) = a_i \circ H_i^{-1}(h) + b_i \circ H_i^{-1}(h) \ln \left( \frac{H_i^{-1}(h)}{h} \right) \quad \text{and} \quad \tilde{b}_i(h) = b_i \circ H_i^{-1}(h),
\]

are analytical at \( h = 0 \) and satisfy
\[
\tilde{b}_i(0) = b_i(0) = 0,
\]
\[
\tilde{b}'_i(0) = \frac{b'_i(0)}{\lambda_i} = -\frac{1}{\lambda_i} \left( \frac{\partial P}{\partial u} + \frac{\partial Q}{\partial y} \right) \bigg|_{(u,v)=(0,0)}
\]
\[
= -\frac{1}{\lambda_i} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \bigg|_{(x,y)=p_i}.
\]

Substituting (32) into (30), we obtain
\[
M(h) = a(h) + b(h) \ln h,
\]

where
\[
a(h) = \sum_{i=1}^{n} \int_{\gamma^2_i h} \omega + \sum_{i=1}^{n} \tilde{a}_i(h), \quad b(h) = \sum_{i=1}^{n} \tilde{b}_i(h)
\]

are analytical at \( h = 0 \) and satisfy
\[
a(0) = \sum_{i=1}^{n} \int_{\gamma^2_i h} \omega + \sum_{i=1}^{n} \tilde{a}_i(0) = \sum_{i=1}^{n} \int_{\gamma^2_i h} \omega + \sum_{i=1}^{n} \int_{\gamma^0 h} \omega = \int_{\gamma^0} \omega,
\]
\[
b(0) = \sum_{i=1}^{n} \tilde{b}_i(0) = 0,
\]
\[
b'(0) = \sum_{i=1}^{n} \frac{1}{\lambda_i} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \bigg|_{(x,y)=p_i}.
\]

Now we prove (29). First we point out that if (28) holds, it follows from \( b(0) = b'(0) = 0 \) that \( M(h) \in C^1 \). We claim that the integral in (29) is convergent. Indeed, let \( p(t) \subset \gamma_0 \) be a solution of system (3) for \( \epsilon = 0 \) and assume that \( \lim_{t \to +\infty} p(t) = p_i, \lim_{t \to -\infty} p(t) = p_{i-1} \). Note that \( p_i \) is a
hyperbolic saddle, we have, as \( t \to +\infty \), \( \| p(t) - p_i \| = O(\exp(-ct)) \) for some \( c > 0 \). Hence,

\[
\left\| \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \circ p(t) \right\| = \left\| \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \circ p(t) - \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \right\|_{p_i} = O(\| p(t) - p_i \|) = O(\exp(-ct)), \quad \text{as} \quad t \to +\infty.
\]

So, the integral \( \int_0^{+\infty} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dt \) is convergent. Similarly, the integral \( \int_{-\infty}^0 \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dt \) is convergent too. Our claim is proved. From \( H(x, y) = h \), we get

\[
\frac{\partial H}{\partial x} \frac{\partial x}{\partial h} = 1, \quad \frac{\partial H}{\partial y} \frac{\partial y}{\partial h} = 1,
\]

which implies

\[
\frac{\partial x}{\partial h} dy = -dt, \quad \frac{\partial y}{\partial h} dx = dt.
\]

Thus,

\[
M'(h) = \frac{\partial}{\partial h} \int_{\gamma_h} (-P(x, y) dy + Q(x, y) dx)
= \int_{\gamma_h} \left( - \frac{\partial P}{\partial x} \frac{\partial x}{\partial h} dy + \frac{\partial Q}{\partial y} \frac{\partial y}{\partial h} dx \right) = \int_{\gamma_h} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dt.
\]

Let \( h \to 0 \), we get

\[
a'(0) = M'(0) = \lim_{h \to 0} M'(h) = \lim_{h \to 0} M'(h) = \int_{\gamma_0} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dt.
\]

From Lemma 19, we get immediately Statement (1) of Theorem 4. Next we prove Statement (2).

**Lemma 20.** Let \( \omega = -P(x, y) dy + Q(x, y) dx \) be an analytical 1-form defined in some neighbourhood of \( \gamma_0 \), where \( \gamma_0 \) is a homoclinic orbit of a hyperbolic saddle \( p = (0, 0) \) of system (7), then \( M(h) := \int_{\gamma_h} \omega \) can be analytically continued to \( h = 0 \) if and only if for any area-preserving normalization coordinate transformation \( F(u, v) \) near \( p \) given by Proposition 12, the Taylor series of the function \( \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \circ F(u, v) \) at \( (0, 0) \) does not contain the terms \( u^m v^n \) for any integers \( m \geq 0 \).

**Proof.** According to Proposition 12, there exists a function \( f \in C^\omega(\mathbb{R}, 0) \) with \( f(0) = 0, f'(0) > 0 \) such that \( f(uv) = H \circ F(u, v) \). Let

\[
\Gamma^+ = F(\{ u = \delta \}), \quad \Gamma^- = F(\{ v = \delta \}).
\]
Any closed orbits near $\gamma_0$ is separated by $\Gamma$ into two segments: $\gamma_1^h$ and $\gamma_2^h$, in which $\gamma_2^h$ is close to the saddle $p$ and $\gamma_1^h$ connects $\gamma_h \cap \Gamma^+$ and $\gamma_h \cap \Gamma^-$ in the complement of some neighbourhood of $p$. Since the integral $\int_{\gamma_h} \omega$ is analytical at $h = 0$,

$$M(h) = \int_{\gamma_h} \omega = \int_{\gamma_1^h} \omega + \int_{\gamma_2^h} \omega$$

can be analytically continued to $h = 0$ if and only if the integral $\int_{\gamma_2^h} \omega$ can be analytically continued. Let $\tau_2^h = \{(u,v)|f(uv) = h, 0 < u, v \leq \delta\}$, then as in the proof of Lemma 19, we have

$$\int_{\gamma_2^h} \omega = \int_{\tau_2^h} -P(u,v) \, dv + \overline{Q}(u,v) \, du,$$

where

$$P(u,v) = P \circ F \frac{\partial y}{\partial v} - Q \circ F \frac{\partial x}{\partial v}, \quad \overline{Q}(u,v) = Q \circ F \frac{\partial x}{\partial u} - P \circ F \frac{\partial y}{\partial u}.$$  

Let

$$\overline{P}(u,v) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} p_{i,n} u^{n-i} v^i, \quad \overline{Q}(u,v) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} q_{i,n} u^{n-i} v^i.$$

From Corollary 18, the integral $\int_{\gamma_2^h} \omega$ can be analytically continued to $h = 0$ if and only if (27) holds which is equivalent to say that the coefficients of the terms $u^m v^m$ for any $m \geq 0$ in the Taylor series of the function $\frac{\partial P}{\partial u} + \frac{\partial \overline{Q}}{\partial v}$ at $(0,0)$ are zero. Now the statement of Lemma 20 follows from (31).

**Lemma 21.** Let $\omega$ and $M(h)$ be defined as in Lemma 20, then the function $M(h)$ can be analytically continued to $h = 0$ if and only if there exists an analytical function $z = z(x,y)$ defined in some neighbourhood of the saddle $p$, such that $z(x,y)$ satisfies the following linear partial differential equation:

$$\frac{\partial z}{\partial x} \frac{\partial H}{\partial y} - \frac{\partial z}{\partial y} \frac{\partial H}{\partial x} = \frac{\partial P}{\partial u} + \frac{\partial \overline{Q}}{\partial v}. \quad (34)$$

**Proof.** Let $(x,y) = F(u,v)$ be the area-preserving normalization coordinate transformation near $p$ given by Proposition 12 and let $\overline{z} = z \circ F, f(uv) = H \circ F(u,v)$, then (34) can be changed to the form

$$\frac{\partial \overline{z}}{\partial u} - \frac{\partial \overline{z}}{\partial v} = R(u,v), \quad R(u,v) = \frac{1}{f'(uv)} \left( \frac{\partial P}{\partial x} + \frac{\partial \overline{Q}}{\partial y} \right) \circ F(u,v). \quad (35)$$

Obviously, the Taylor series of $f'(uv)R(u,v)$ at $(0,0)$ does not contain the terms $u^m v^m$ for any integers $m \geq 0$ if and only if $R(u,v)$ has the same
property. Let
\[ R(u, v) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} b_{i,n} u^{n-i} v^i, \]
(36)
\[ z(u, v) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} a_{i,n} u^{n-i} v^i, \]
(37)
Substituting them into (35), we get
\[ \sum_{n=0}^{\infty} \sum_{i=0}^{n} [(n-i)a_{i,n} - a_{i,n}] u^{n-i} v^i = \sum_{n=0}^{\infty} \sum_{i=0}^{n} b_{i,n} u^{n-i} v^i, \]
or equivalently
\[ (n-2i)a_{i,n} = b_{i,n}, \quad \text{for } n = 0, 1, 2, \ldots, \text{ and } i = 0, 1, \ldots, n. \]
(38)
System (38) has solutions if and only if
\[ b_{m,2m} = 0, \quad \forall m \geq 0. \]
(39)
Moreover, if (39) holds, we can choose
\[ a_{i,n} = \begin{cases} \frac{b_{i,n}}{n-2i}, & \text{if } n \neq 2i; \\ 0, & \text{if } n = 2i. \end{cases} \]
Since \(|a_{i,n}| \leq |b_{i,n}|\), the convergence radius of (37) is at least equal to the convergence radius of (36). So the function \( z \) defined in (37) is analytical in some neighbourhood of the origin. Now, the lemma follows using Lemma 20.

Lemma 22. Let \( \omega \) and \( M(h) \) be defined as in Lemma 20, then the function \( M(h) \) is constant for \( 0 < h \ll 1 \) if and only if there exists an analytical function \( z = z(x, y) \) defined in some neighbourhood of \( \gamma_0 \) such that (34) holds.

Proof. Sufficiency. We consider the characteristic equation of (34):
\[ \dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}, \quad \dot{z} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}. \]
(40)
Assume that there exists an analytical function \( z = z(x, y) \) defined in some neighbourhood of \( \gamma_0 \) such that \( z(x, y) \) satisfies Equation (34). This implies that the surface \( S = \{(x, y, z) \in \mathbb{R}^3 \mid z = z(x, y)\} \) is invariant under the flow of (40). Therefore, \( S \cap \{(x, y, z) \in \mathbb{R}^3 \mid H(x, y) = h\} \) for \( 0 < h \ll 1 \) is a periodic orbit of (40). So from (33), we have
\[ M'(h) = \int_{\gamma_h} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dt = \int_{\gamma_h} \dot{z} \, dt = 0, \]
(41)
which implies that \( M(h) \) is constant for \( 0 < h \ll 1 \).
Necessity. Assume now $M(h)$ is constant. By Lemma 21, Equation (34) has an analytical solution $z(x, y)$ in some neighbourhood of the saddle $p = (0, 0)$. We claim that this solution $z(x, y)$ can be extended continuously to a single valued analytic function in some neighbourhood of $\gamma_0$. Indeed, from (34),\[
abla \cdot \begin{pmatrix} \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \end{pmatrix}_{(x, y) = (0, 0)} = 0,
\]so the straight line $\{(0, 0, z) | z \in \mathbb{R}\}$ consists of singular points of (40). Then the invariant surface $S$ of (40) contains local stable and unstable manifolds of the singular point $p_0 = (0, 0, z(0, 0))$. Let $\Gamma^-$ and $\Gamma^+$ be the planes transversal to the flow of (40) at some point of local stable manifold and local unstable one respectively. Then $l^\pm := S \cap \Gamma^\pm$ are analytical curves in $\Gamma^\pm$, see Figure 3.

![Figure 3.](image)

Let $A$ be the projection from $\Gamma^+$ to $\Gamma^-$ along the orbits of (40). Then $l^\prime_- := Al^\prime_+$ is an analytical curve in $\Gamma^-$. Introducing the set $U = \{(x, y, z) \in \mathbb{R}^3 | H(x, y) > 0\}$, by (41) we have $U$ is filled with periodic orbits. This implies $l^\prime_- \cap U = l^- \cap U$. Therefore, by the analyticity, $l^\prime_- = l^-$. Thus, the union of the orbits passing through $l^+$ and $S$ constructs an analytical invariant surface of system (40), which is the graph of an analytical function $z(x, y)$ defined in some neighbourhood of $\gamma_0$. From the invariance, $z(x, y)$ is a solution of Equation (34). \hfill \Box

**Lemma 23.** Let $\omega$ and $M(h)$ be defined as in Lemma 20. Then $M(h) \equiv 0$ if and only if there exist analytical functions $z(x, y)$ and $R(x, y)$ defined in some neighbourhood of $\gamma_0$ such that

\[(42) \quad \omega = zdH + dR.\]
Proof. Sufficiency. Assume that formula (42) holds. Since the function $H$ is constant along the closed curve $\gamma$, so $dH = 0$. This implies that 
$$\int_{\gamma}^{} \omega = \int_{\gamma}^{} zdH + dR = \int_{\gamma}^{} dR = 0.$$ 

Necessity. By Lemma 22, there exists an analytical function $z(x, y)$ defined in some neighbourhood of $\gamma_0$ such that Equation (34) holds. Let 
$$Mdx + Ndy = \omega - zdH.$$ 

From (34), 
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$ 

Now we define the function $R(x, y)$ in the following way: For any point $(x, y)$ near $\gamma_0$, let 
$$R(x, y) = \int_{(0, 0)}^{(x, y)} Mdx + Ndy.$$ 

By (43) and the fact 
$$\int_{\gamma}^{} Mdx + Ndy = \int_{\gamma}^{} (\omega - zdH) = \int_{\gamma}^{} \omega = M(h) \equiv 0,$$
the integral in (44) defines a single valued analytical function in some neighbourhood of $\gamma_0$ satisfying (42). \hfill \square

The proof of statement (2) of Theorem 4. Suppose $k = 2$. Then $M_1(h) = -\int_{\gamma}^{} \omega_1 \equiv 0$. By Lemma 23, there exist analytical functions $z_1, R_1$ defined in some neighbourhood of $\gamma_0$ such that $\omega_1 = z_1 dH + dR_1$. By Proposition 10, $M_2(h) = \int_{\gamma}^{} (z_1 \omega_1 - \omega_2)$. Now by using Lemma 19, we get that statement (2) holds for $k = 2$. Similarly, assume that $M_i(h) \equiv 0$, $1 \leq i \leq k - 1$. Again from Lemma 23, there exist analytical functions $a_{k-1}, R_{k-1}$ defined in some neighbourhood of $\gamma_0$ satisfying (6). Now by Proposition 10 and Lemma 19, Statement (2) holds for $k$. This completes the proof of Theorem 4. \hfill \square

Proof of Theorem 5. Statement (1) is just a corollary of Lemma 20. Statement (2) is a corollary of Lemma 24 below. Finally Statement (3) follows easily from Lemma 24 and Lemma 19. \hfill \square

All notations used in Lemma 24 below are the same as in the statement of Theorem 5.

**Lemma 24.** Let $\omega$ be an analytical 1-form defined in some neighbourhood of the eight figure cycle, then two of the three integrals $\int_{\gamma}^{} \omega, \int_{+}^{} \omega, \int_{-}^{} \omega$ are identically zero if and only if there exist analytical functions $z(x, y), R(x, y)$ defined in some neighbourhood of $\gamma_0^+ \cup \gamma_0^-$ such that $\omega = zdH + dR$. 


Proof. The sufficiency can be proved by using the same argument as the proof of Lemma 23. We now prove the necessity. First by Lemma 21, Equation (34) has an analytical solution \( z(x, y) \) in some neighbourhood of the saddle \( p \). By using the same argument of the proof of Lemma 22, the function \( z(x, y) \) can be extended continuously to a single valued analytic function in the some neighbourhood of the eight figure. Let \( Mdx + Ndy = \omega - zdH \). Then from (34), formula (43) holds. Now we define the function \( R \) by the integral in (44). Then by (43) and the assumption that two of the three integrals \( \int_{\gamma_h} \omega, \int_{\gamma^+_h} \omega, \int_{\gamma^-_h} \omega \) are identically zero, the function \( R \) is a single-valued analytical one in some neighbourhood of eight figure \( \gamma^+_0 \cup \gamma^-_0 \) and satisfies (42). □

5. Proof of Theorem 8.

We shall need the following result:

**Proposition 25.** Let \( X \) be an analytical vector field defined in some open region of \( \mathbb{R}^2 \). Assume that \( X \) has a continuous family of periodic orbits (the period annulus) \( \gamma_s, 0 < s < \bar{s}, \) and \( \gamma_0 \) is a nondegenerate center \( p \) or a homoclinic orbit of a hyperbolic saddle \( p \) such that \( \lim_{s \downarrow 0} \gamma_s = \gamma_0 \). Then for any \( s \in [0, \bar{s}) \), there exists an analytical function \( \rho > 0 \) defined in some neighbourhood of \( \gamma_s \) such that \( \text{div}(\rho X) \equiv 0 \), i.e., \( \rho X \) is a Hamiltonian vector field.

Proposition 25 will be proved through two lemmas.

**Lemma 26.** Under the assumptions of Proposition 25 the following hold:
For any \( s \in [0, \bar{s}) \), the vector field \( X \) has an analytical first integral \( H \) defined in some neighbourhood of \( \gamma_s \) such that
\[
\det D^2H(p) \neq 0 \text{ and } DH = 0 \iff X = 0.
\]

**Proof.** If \( \gamma_s \) is a periodic orbit, the lemma is trivial. For \( \gamma_0 \) being a nondegenerate center, by the Poincaré Normal Form Theorem (for a proof, see [1]), there exists an analytic change of coordinates that brings the initial system to the normal form
\[
\begin{align*}
\dot{x} &= -yf(x^2 + y^2), \\
\dot{y} &= xf(x^2 + y^2).
\end{align*}
\]
Obviously, the system above has a first integral \( H = x^2 + y^2 \) satisfying (45). Now we assume that \( \gamma_0 \) is a homoclinic orbit of a hyperbolic saddle. Since there exists a family of periodic orbits tending to \( \gamma_0 \), the saddle values of any order must be zero. Therefore under the normalized coordinate the vector field \( X \) near the saddle takes the following form (see [1]):
\[
\begin{align*}
\dot{x} &= -\lambda x(1 + R(xy)) \\
\dot{y} &= \lambda y(1 + R(xy)), \quad R \in C^\omega(\mathbb{R}, 0), \ R(0) = 0.
\end{align*}
\]


\( H = xy \) is a first integral of system (46). This implies that the vector field \( X \) has an analytical first integral in some neighbourhood \( U \) of the saddle satisfying (45). We claim that the first integral \( H \) can be extended to some neighbourhood of \( \gamma_0 \). Indeed, let \( I, J \subset U \) be two transversal sections to \( X \) at some point of the local stable manifold and the local unstable one, respectively. Sections \( I \) and \( J \) can be parameterized by using \( h = H \).

Without loss of generality, we assume that the intersections of the periodic orbits \( \gamma_s \) with \( I \) and \( J \) correspond to points with \( h > 0 \). Let \( f : J \to I \) denote the Poincaré map along the orbits of \( X \), then \( f \in C^\infty(\mathbb{R}, 0) \). Let \( G(h) = H(f(h)) - h \). Since all orbits starting from the points of \( J \) with \( h > 0 \) are periodic, we have \( G(h) \equiv 0 \) for \( 0 < h \ll 1 \). By the analyticity, \( G(h) \equiv 0 \) for \( |h| \ll 1 \).

Now we define the function \( H \) in some neighbourhood of \( \gamma_0 \) as follows: For any \( x \) close to \( \gamma_0 \), denote by \( \gamma(x) \) the orbit of \( X \) passing through \( x \), then \( H(x) := H(\gamma(x) \cap I) = H(\gamma(x) \cap J) \). Obviously, \( H(x) \) takes a constant on each orbit, i.e., \( H \) is a first integral. \( \square \)

Let \( H \) be the first integral in some neighbourhood of \( \gamma_s \) as in Lemma 26. Consider the Hamiltonian vector field
\[
Y := JDH, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
Then \( X \) and \( Y \) define the same direction field. Let \( \rho = \|Y\|/\|X\| \), then \( \rho \) is positive and analytical at any regular point. Obviously, \( Y = \rho X \) or \( Y = -\rho X \). The next lemma shows that the function \( \rho \) can be analytically continued to the saddle \( p \).

**Lemma 27.** Assume that the analytical vector fields \( X_1 = (P_1, Q_1) \) and \( X_2 = (P_2, Q_2) \) define the same direction field and the origin \((0,0)\) is an isolated singular point of \( X_1 \) and \( X_2 \). If

\[
\prod_{i=1}^{2} \left[ \left( \frac{\partial P_i}{\partial x} \right)^2 + \left( \frac{\partial P_i}{\partial y} \right)^2 \right]_{(0,0)} \neq 0
\]

or

\[
\prod_{i=1}^{2} \left[ \left( \frac{\partial Q_i}{\partial x} \right)^2 + \left( \frac{\partial Q_i}{\partial y} \right)^2 \right]_{(0,0)} \neq 0,
\]

then there exists a positive analytical function \( \rho \) defined in some neighbourhood of the origin such that \( \|X_1\| = \rho \|X_2\| \).

**Proof.** We will prove the lemma under the assumption \( \frac{\partial P_1}{\partial x}(0,0) \neq 0 \). The proofs for other cases are similar. Since \( X_1 \) and \( X_2 \) define the same direction field, they have the same vertical isocline:
\[
V : \{(x,y) \mid P_1(x,y) = 0\} = \{(x,y) \mid P_2(x,y) = 0\}.
\]
By using Implicit Function Theorem for $P_1$ at $(0,0)$, there exists a function $g(y) \in C^\omega(\mathbb{R}, 0)$ with $g(0) = 0$, such that

\begin{equation}
V = \{ (x, y) \mid x = g(y) \}.
\end{equation}

Thus,

$$P_1(x, y) = (x - g(y))^2 + \epsilon f(x, y),$$

We claim that $\frac{\partial P_2}{\partial x}(0, 0) \neq 0$. Indeed, if $\frac{\partial P_2}{\partial x}(0, 0) = 0$, then by (47), $\frac{\partial P_2}{\partial y}(0, 0) \neq 0$. By the Implicit Function Theorem, the curve $V$ is tangent to the $x$-axis at the origin, which is a contradiction with (48). Thus, we get

$$P_2(x, y) = (x - g(y))^2 + \epsilon f(x, y),$$

Therefore, the function $p(x, y) := \frac{P_1(x, y)}{P_2(x, y)}$ is analytical and has definite sign in some neighbourhood of the origin. Let $\rho = |p|$, then the proof of the lemma is completed.

**Proof of Proposition 25.** It is a corollary of Lemmas 26 and 27.

**Proof of Theorem 8.** Consider the following one parameter family of analytical systems:

\begin{equation}
\dot{x} = f(x) + \epsilon g(x, \epsilon), \quad x \in \mathbb{R}^2.
\end{equation}

We assume that for $\epsilon = 0$ system (49) has a homoclinic orbit $\gamma_0$ of a hyperbolic saddle of infinite codimension, i.e., there is a continuous family of periodic orbits tending to $\gamma_0$. By Proposition 25, there exists a positive analytical function $\rho(x)$ defined in some neighbourhood of $\gamma_0$ such that $\text{div}(\rho f) \equiv 0$. Now we consider the following perturbed Hamiltonian system which is orbitally equivalent to system (49):

\begin{equation}
\dot{x} = \rho(x)f(x) + \epsilon \rho(x)g(x, \epsilon).
\end{equation}

We assume, without loss of generality, that there exists an analytical function $H$ defined in some neighbourhood of $\gamma_0$ such that $JDK = \rho f$, $\gamma_0 \subset H^{-1}(0)$, and for $0 < h \ll 1$, $\gamma_h \subset H^{-1}(h)$ is a family of periodic orbits with $\gamma_h \to \gamma_0$ as $h \to 0$. Let $M_k(h)$ denote the $k$-th Melnikov function of system (50).

**Case 1.** $M_i(h) \equiv 0$, $\forall i \geq 1$. Then for $|\epsilon| \ll 1$, system (50) has a period annulus and hence has no limit cycles near $\gamma_0$.

**Case 2.** There exists an integer $k \geq 1$ such that $M_i(h) \equiv 0$, $0 \leq i \leq k-1$ and $M_k(h) = a_k(h) + b_k(h) \ln h$ is not identically vanishing. Then by Remark 7, system (50) in some neighbourhood of $\gamma_0$ has at most $2 \min\{m(a_k), m(b_k)\}$ limit cycles for $|\epsilon| \ll 1$.

This completes the Proof of Theorem 8.
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References

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SOME REMARKS ON VARIETIES WITH DEGENERATE
GAUSS IMAGE

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We consider projective varieties with degenerate Gauss image whose focal hypersurfaces are non-reduced schemes. Examples of this situation are provided by the secant varieties of Severi and Scorza varieties. The Severi varieties are moreover characterized by a uniqueness property.

1. Introduction.

A classical theorem on surfaces states that, if the tangent plane to a ruled projective surface $S$ remains fixed along a general line of the ruling, then $S$ is a developable surface, i.e., a cone or the tangent developable of a curve.

In the case of higher dimensional varieties, this generalizes to the problem of giving a structure theorem for projective varieties with degenerate Gauss image. More precisely, let $X \subset \mathbb{P}^N$ be a projective variety of dimension $n$. The Gauss map of $X$ is the rational map $\gamma$ from $X$ to the Grassmannian $G(n,N)$ of $n$-dimensional subspaces of $\mathbb{P}^N$, associating to a smooth point $x$ of $X$ the embedded tangent space $T_xX$ to $X$ at $x$. The Gauss image $\gamma(X)$ of $X$ is, by definition, the closure of $\gamma(X_{\text{sm}})$, where $X_{\text{sm}}$ is the smooth locus of $X$.

Clearly $\dim \gamma(X) \leq n$, and for “general” varieties equality holds. Several terms have been used recently to denote the varieties such that $\dim \gamma(X) < \dim X$: They are called varieties with degenerate Gauss image by Landsberg ([L2]), developable by Piontkowski ([P]), tangentially degenerate by Akivis-Goldberg ([AG]). We will follow Landsberg’s convention.

Several general facts are well-known for these varieties (see for instance the classical article [S]). First of all, general fibres of $\gamma$ are linear. This means that, if the dimension of the Gauss image is $r < n$, then the general fibre is a linear space of dimension $n - r$ along which the tangent space is constant. Secondly, if $X$ is not linear, then it is singular and its singular locus cuts a general fibre along a codimension 1 subscheme.

The theory of foci for families of linear varieties well applies to the family of fibres of $\gamma$. This allows us to consider and study the focal locus on $X$ and on the fibre $\Lambda$ of $\gamma$, which we will denote by $F$ and $F_\Lambda$ respectively. Both loci have a natural scheme structure.
Several classification results have recently been established for these varieties (e.g., [GH], [AGL], [MT], [P] and [AG]) but a general structure theorem is still missing. In [AG] a structure theorem is proved, but under rather strong assumptions on a general $F_\Lambda$, for example that it is reduced or set-theoretically linear. Moreover, Akivis and Goldberg state the problem of constructing examples of projective varieties with degenerate Gauss image such that $F_\Lambda$ is not reduced and $(F_\Lambda)_\text{red}$ is not linear (or proving that such varieties do not exist).

Püntkowski (\cite{P}) proved a uniqueness theorem for developable varieties $X$ with Gauss image of dimension 2, such that the focal locus has codimension 2 in $X$. Those for which the focal conic is integral are precisely the varieties of secant lines of the Veronese surface $v_2(\mathbb{P}^2)$ or of cones over it. Inspired by this example, we construct a series of examples where $F_\Lambda$ is not reduced and $(F_\Lambda)_\text{red}$ has arbitrarily large degree. The examples are the secant varieties of Severi varieties of dimension $> 2$ and some natural generalizations of them.

Zak’s theorem on linear normality (\cite{Z1}) states that a smooth non-degenerate $m$-dimensional subvariety of $\mathbb{P}^N$, with $N < 3m/2 + 2$, cannot be isomorphically projected to $\mathbb{P}^{N-1}$. Zak also classified in \cite{Z2} the varieties which are borderline cases in his theorem and called them Severi varieties. He proved that there are only 4 smooth examples: The Veronese surface $v_2(\mathbb{P}^2)$ in $\mathbb{P}^5$, $\mathbb{P}^2 \times \mathbb{P}^2$, $G(1,5)$ and the non-classical variety $E$ of dimension 16. For any of these varieties $F$, its secant variety $X = SF$ is a cubic hypersurface in $\mathbb{P}^N$ with Gauss image isomorphic to $X$.

We prove in Example 2 that on a general fibre of the Gauss map for the secant variety of a Severi variety, the focal scheme is a quadric of maximal rank with multiplicity 1, 2, 4, 8 respectively.

In Section 4 we show that these examples can be generalized in two ways, taking instead of the Severi varieties the following three series of varieties $Y$: $v_2(\mathbb{P}^m)$, $\mathbb{P}^m \times \mathbb{P}^m$ and $G(1,2m+1)$, of dimensions $m$, $2m$, $4m$ respectively (they are Scorza varieties according to Zak \cite{Z3}). If we consider their higher secant variety $S^{m-1}Y$ of $m$-secant $(m-1)$-planes, it turns out that they are hypersurfaces of degree $m + 1$, whose dual is $Y$, and whose singular locus is $S^{m-2}Y$ and appears as focal locus in each fibre with multiplicity resp. 1, 2, 4. If we consider $SY$ instead, we get on each fibre a focal quadric with an arbitrarily high multiplicity, more precisely, $m - 1$, $2(m - 1)$, $4(m - 1)$ respectively.

It is interesting to observe that the fourth Severi variety does not generalize to give a complete class of examples \cite{L1}).

We observe also that there is an upper bound on the codimension of the focal scheme and that there are restrictions on the set of possible codimensions if we require that $F$ is not reduced (see Theorem 3.1). For “high” codimension we prove in Theorem 3.3 some uniqueness results, in the case
in which the general focal hypersurface is a quadric of maximal rank with multiplicity 2. We extend in this way the result found by Piontkowski in the case of the secant variety of the Veronese surface. Our results rely on the fact, stated by Zak in [Z4], that if $F$ is a variety whose general linear section is a Severi variety, then $F$ is a cone over a Severi variety.

The plan of the article is as follows: In §2 we recall the language of focal schemes for families of linear spaces, as introduced in [MT]. We also prove Proposition 2.2, giving a lower bound, based on the codimension of the focal locus, for the multiplicity of the focal scheme. In §3 we prove Theorems 3.1 and 3.3. Finally in §4, we give two series of examples, showing that the focal hypersurfaces can have arbitrarily high degree and multiplicity.

1.1. Notations and conventions. In this paper a variety will be an integral closed subscheme of a projective space over an algebraically closed field $K$, char $K = 0$.

If $\Lambda \subset \mathbb{P}^N$ is a projective linear subspace, $\hat{\Lambda} \subset \mathbb{K}$ will denote the linear subspace associated to $\Lambda$ such that $\Lambda = \mathbb{P}(\hat{\Lambda})$. $T_xX$ will denote the Zariski tangent space to the variety $X$ at its point $x$, while we will denote by $T_xX \subset \mathbb{P}^N$ the embedded tangent space to $X$ at $x$.

We will always use the same symbol to denote the points of a Grassmannian and the corresponding linear subspaces.

2. The focal scheme and its multiplicity.

Let $X \subset \mathbb{P}^N$ be a projective variety of dimension $n < N$. Let us assume that $X$ is covered by an $r$-dimensional family of linear spaces of dimension $k := n - r$. Let $B$ be the subvariety of the Grassmannian $\mathbb{G}(k, N)$ parametrizing that family.

Let us denote by $\mathcal{I}$ the incidence correspondence of $B$, with the natural projections:

\begin{align*}
B & \xleftarrow{p_1} B \times \mathbb{P}^N \xrightarrow{p_2} \mathbb{P}^N \\
B & \xleftarrow{g} \mathcal{I} \xrightarrow{f} X.
\end{align*}

We will associate to the family $B$ its focal subscheme $\Phi \subset \mathcal{I}$ and its focal locus $F \subset X$. For their definitions and for the proof of results cited in this section, we refer to [MT].

We recall moreover that the characteristic map of the family $B$ is the map $\chi := \beta \circ \alpha$,

$$(p_1^*(TB)) \xrightarrow{|\mathcal{I}} T_\mathcal{I} \xrightarrow{\beta} \mathcal{N}_{\mathcal{I}|B \times \mathbb{P}^N},$$

where $\alpha$ comes from the exact sequence expressing the tangent sheaf to the product variety $B \times \mathbb{P}^N$ as a product of tangent sheaves, and $\beta$ from that defining the normal sheaf to $\mathcal{I}$ inside $B \times \mathbb{P}^N$. 

For every smooth point $\Lambda$ of $B$, the restriction of $\chi$ to $g^{-1}(\Lambda)$ is called the \textit{characteristic map of $B$ relative to $\Lambda$}. Finally the \textit{focal scheme} on $\Lambda$, denoted $F_\Lambda$, is the scheme-theoretic intersection of the Cartier divisor $\Phi \subset I$ with $\{\Lambda\} \times \Lambda \subset I$.

Let $\gamma : X \dashrightarrow \mathbb{G}(n,N)$ be the (rational) Gauss map of $X$, regular on $X_{\text{sm}}$. It is a consequence of biduality ([K]) that its fibres are linear spaces, i.e., that a general embedded tangent space is tangent to $X$ along a linear space. In the following, we will apply the above construction to the family of the fibres of $\gamma$, assuming that they are of positive dimension.

\textbf{Definition 2.1.} The \textit{Gauss rank} of a variety $X$ is the number $r = \dim \gamma(X)$. If $r < n$, then $X$ is called a \textit{variety with degenerate Gauss image}.

Varieties with degenerate Gauss image are characterized by the fact that the focal scheme on a general $\Lambda \in B$ is a hypersurface of degree $r$. Another important property is that the focal locus of $B$ is always contained in the singular locus of $X$.

Let us denote by $g$ (respectively, $f$) the restriction of $g$ (resp., $f$) to $\Phi \subset I$. Note that the fibre $g^{-1}(\Lambda)$ coincides with $F_\Lambda$, which has in general dimension $n-r-1$, hence every irreducible component of $\Phi$ has dimension $n-1$.

\textbf{Proposition 2.2.} Let $X$ be a variety of dimension $n$ with Gauss image of dimension $r < n$. Let $\Phi$ be an irreducible component of the focal scheme of the family $B$ considered as a reduced variety. Let $F \subset X$ be its scheme-theoretic image and let $c$ be the codimension of $F$ in $X$. For a general point $P = (\Lambda, x) \in \Phi$, the Cartier divisor $\Phi \subset I$ has Samuel multiplicity $\mu(P, \Phi) \geq c - 1$.

\textbf{Corollary 2.3.} For a general element $(\Lambda, x) \in \Phi$, either $F_\Lambda = \{\Lambda\} \times \Lambda$ or the Cartier divisor $F_\Lambda \subset \{\Lambda\} \times \Lambda$ has Samuel multiplicity $\mu(x, F_\Lambda) \geq c - 1$.

\textbf{Proof.} Let $P = (\Lambda, x) \in \Phi$ be a point that projects to $x$. Considering the differentials of the maps $f$ and $\tilde{f}$ at $P$, we get the diagram:

\[
\begin{array}{ccc}
T_P I & \xrightarrow{df} & T_x X \\
\cup & & \cup \\
T_P \Phi & \xrightarrow{d\tilde{f}} & T_x F.
\end{array}
\]

(2)

We consider also the characteristic map $\chi(\Lambda)$ of the family $B$ relative to $\Lambda$. At the point $P$, it gives rise to a linear map

$\chi(\Lambda, x) : T_\Lambda B \to \hat{T}_x X/\hat{\Lambda}$.

Let $K_P$ denote the common kernel of both $df$ and $\chi(\Lambda, x)$. By generic smoothness $d_P \tilde{f}$ is surjective, hence

$\dim \ker d_P \tilde{f} = (n-1) - (n-c) = c - 1$. 

But \( \ker d_P f = K_P \cap T_P \Phi \), where \( T_P \Phi \) has dimension \( n - 1 \), so either \( K_P \subset T_P \Phi \), the two kernels coincide and \( \dim K_P = c - 1 \), or they are different and \( \dim K_P = c \). Since the structure of scheme on \( \Phi \) is given by the minors of the characteristic map, this proves that \( \mu(P, \Phi) \) is at least \( c - 1 \) in the former case, and at least \( c \) in the latter. \( \square \)

**Example 1.** The multiplicity \( \mu(x, F_\Lambda) \) can be strictly greater than \( c \). We see now an example of such a situation, in which the fibres of the Gauss map have special properties of tangency to the focal locus.

Let us consider a birational map \( \varphi : F \dashrightarrow S \) between two surfaces \( F, S \subset \mathbb{P}^6 \). Then \( \psi : F \dashrightarrow \mathbb{G}(3, 6) \) \( x \mapsto \langle T_x F, \varphi(x) \rangle \) defines a rational map on \( F \). We define \( X \) as the closure of the variety swept by the 3-planes in \( \mathbb{P}^6 \) belonging to the image of \( \psi \). If the choice of \( F, S \) and \( \varphi \) is general, then \( X \) is a variety of dimension 5 whose Gauss image has dimension 4, and its focal locus is \( F \), with codimension 3. If \( x \) is a general point of \( F \), then any line passing through \( x \) and contained in \( \langle T_x F, \varphi(x) \rangle \) is a fibre of the Gauss map.

Let \( P = (\Lambda, x) \) project to a general point of \( F \). By direct computation, \( x \) results to be the only focus on \( \Lambda \), with multiplicity \( 4 = c + 1 \).

It is nonetheless true that \( \dim K_P = 2 = c - 1 \). The kernels of \( d_P f \) and \( d_P \bar{f} \) indeed coincide and are generated by the directions of the curves in \( T \) of the form \( \langle \Lambda(t), x \rangle \), where \( \Lambda(t) \) varies in the star of lines of centre \( x \) contained in \( \psi(x) \).

The just constructed example belongs to the class of hyperbands of \( [AGL] \). It can be generalized to a whole series of analogous examples with larger dimensions and increasing difference between the multiplicity of the focal locus and the codimension \( c \).

**Example 2.** *The Severi varieties.*

Let \( F \) denote one of the four Severi varieties (see the Introduction). Let \( X = SF \) be its secant variety, i.e., the closure of the union of lines joining two distinct points of \( F \). There are several known facts about \( F \).

**Proposition 2.4 ([Z2]).** Let \( F \subset \mathbb{P}^N \) be a Severi variety. Then the following hold:

- \( F \) has dimension \( m \in \{2, 4, 8, 16\} \);
- \( F \) is embedded in a projective space of dimension \( N = 3m/2 + 2 \);
- the secant variety \( X \) of \( F \) is a normal cubic hypersurface in \( \mathbb{P}^N \);
- \( X \) is isomorphic to the dual variety of \( F \);
- the singular locus of \( X \) coincides with \( F \).

As \( X \) is a hypersurface, its Gauss image is the same as its dual variety, which is \( F \). Then \( X \) has Gauss rank \( m \) and the fibres of the Gauss map
γ of X are linear subspaces of dimension \( m/2 + 1 \). As shown in [Z2], the intersection with F of the secant lines of F passing through a general point \( x \in X \) is a quadric generating a space of dimension \( m/2 + 1 \). This space is precisely the fibre of the Gauss map passing through \( x \), and \( B \) is just the family of such spaces. As the quadric depends on the fibre \( \Lambda \) to which \( x \) belongs and not on \( x \), we will denote it by \( Q_\Lambda \).

The degree of the focal scheme on \( \Lambda \) must equal the Gauss rank, so it is \( m \), i.e., 2, 4, 8 or 16 respectively. But the focal locus of \( B \) on \( \Lambda \) has to be \( Q_\Lambda \), of degree 2, because it has to be contained in the singular locus of \( X \). This shows that \( F_\Lambda \) coincides with \( Q_\Lambda \) set-theoretically, but, as a scheme, appears with multiplicity 1, 2, 4, 8 respectively in the four cases. This fact is confirmed by Proposition 2.2, because in this case \( c = m/2 + 1 \) and the multiplicity \( \mu(x, F_\Lambda) \) is equal to \( m/2 = c - 1 \).

3. Gaps on the codimension and uniqueness of Severi examples.

The results of the previous section allow us to state some bounds on the possible codimension of the focal locus for the family of the fibres of the Gauss map and some uniqueness results.

**Theorem 3.1.** Let \( X \) be a variety of dimension \( n \) with degenerate Gauss image \( \gamma(X) \) of dimension \( r \) with \( r \geq 2 \). Let \( c \) denote the codimension of the focal locus \( F \) in \( X \). Then:

(i) \( c \leq r + 1 \);
(ii) if \( c = r + 1 \), then \( X \) is a cone of vertex a space of dimension \( (n-r-1) \) over a variety of dimension \( r \);
(iii) if \( c \leq r \) and the reduced focal hypersurfaces are not linear, then \( c \leq r/2 + 1 \);
(iv) if \( c = r/2 + 1 \), then either the multiplicity of the focal hypersurfaces in the fibres of \( \gamma \) is at least \( c \) and they are set-theoretically linear, or the multiplicity is \( r/2 \) and the reduced focal hypersurfaces are quadrics.

**Proof.** (i) Let \( F_\Lambda \) be the focal scheme on a general fibre \( \Lambda \) of the Gauss map. Then \( F_\Lambda \) is a hypersurface of degree \( r \), and from Corollary 2.3, it has multiplicity \( \mu(x, F_\Lambda) \geq c-1 \) at a general point \( x \). Hence \( c-1 \leq \mu(x, F_\Lambda) \leq r \), which implies (i).

(ii) If \( c = r + 1 \), then a general \( F_\Lambda \) is a hyperplane in \( \Lambda \) with multiplicity \( r \).

On the other hand \( \dim F = n - r + 1 = \dim F_\Lambda \), so \( F_\Lambda \) is a fixed \( (n-r+1) \)-plane and \( X \) is a cone over it.

(iii) and (iv) Assume \( r/2 + 1 \leq c \leq r \): Then \( \mu(x, F_\Lambda) \geq c - 1 \geq r/2 \). So the degree of the reduced focal locus on \( \Lambda \) is \( r/\mu(x, F_\Lambda) \leq 2 \). If \( \mu(x, F_\Lambda) \geq c \), then the hypothesis implies that \( \mu(x, F_\Lambda) = r \), so \( (F_\Lambda)_{\text{red}} \) is linear. If \( \mu(x, F_\Lambda) = c - 1 \), then our hypothesis implies \( c = r/2 + 1 \) and \( \mu(x, F_\Lambda) = r/2 \). \( \square \)
Remark 3.2. The description of varieties with Gauss rank \( r = 1 \) is classical (see \([FP]\) for a modern account), the description of varieties with \( r = 2 \) has been accomplished recently by Piontkowski (\([P]\)). His uniqueness theorem for the case in which the focal locus has codimension \( 2 \) can be extended as follows:

Theorem 3.3. Let \( X \) be a variety of dimension \( n \) with degenerate Gauss image \( \gamma(X) \) of dimension \( r \geq 2 \) and let \( F \) be the focal locus of the family of fibres of \( \gamma \). Assume that the codimension of \( F \) in \( X \) is \( c = r/2 + 1 \) and that the focal hypersurfaces are quadrics counted with multiplicity \( r/2 \).

(i) If \( F \) is irreducible, then \( X \) coincides with \( SF \), the secant variety of \( F \), and \( n \geq 3r/2 + 1 \);
(ii) if moreover \( n = 3r/2 + 1 \) and a general quadric \( F_\Lambda \) is smooth, then \( F \) is a Severi variety;
(iii) if \( n \geq 3r/2 + 1 \), then the rank of the quadrics \( F_\Lambda \) is at most \( r/2 + 1 \), and, if equality generically holds, then \( F \) is (a cone over) a Severi variety of dimension \( r \).

Proof. (i) Since the focal hypersurfaces have set-theoretically degree 2, then the lines which are contained in the fibres of \( \gamma \) are all secant lines of \( F \). Hence \( X \subset SF \). On the other hand, the family of lines obtained in this way has dimension \( r + \dim G(1,n-r) \), which is equal to \( 2 \dim F \). This proves that \( X = SF \). A standard count of parameters shows that the family of quadrics passing through a general point of \( F \) has dimension \( r/2 \) and that the intersection of two focal quadrics is always nonempty.

Let us denote by \( B \) as usual the family of fibres of \( \gamma \). Let \( \Sigma \subset B \times B \times F \) be the set of triples \( (\Lambda,M,P) \) such that \( P \in F_\Lambda \cap F_M \). Considering the projections from \( \Sigma \) to \( B \times B \) and to \( F \), one gets easily that a general intersection \( F_\Lambda \cap F_M \) has dimension \( n - 3r/2 - 1 \). This proves (i).

(ii) We can assume without loss of generality that \( X \) is nondegenerate (we can always restrict to the projective subspace of minimal dimension in which \( X \) is contained). Suppose first that \( X \) is a hypersurface. We have that \( F \) is a variety of dimension \( r \) covered by a \( r/2 \)-dimensional family of nonsingular quadrics of dimension \( r/2 \), such that a general pair of such quadrics meet at a point. This property of the family of quadrics allows us to argue as in \([Z2, Lemma 5]\) and deduce, as is done there, that \( F \) is smooth. Therefore \( F \) is a Severi variety by definition.

Suppose now that \( X \subset P^N \), \( N = 3r/2 + 2 + k \), \( k > 0 \). The image of \( X \) in a projection with centre a general \((k-1)\)-plane of \( P^N \) is a hypersurface, which is the secant variety of a Severi variety \( F' \) by the previous argument. Since the focal locus of \( X \) projects to \( F' \), we have that the general section of \( F \) by a linear subspace of dimension \( 3r/2 + 2 \) is a Severi variety. By Corollary 1 of \([Z4]\) Severi varieties are unextendable, hence \( F \) must be a cone over \( F' \), which is impossible because in that case the quadrics \( F_\Lambda \) would be singular.
(iii) Assume that the rank of $F_{\Lambda}$ is $\geq r/2 + 1$. Let $h := (n-r/2 - 1) - (r+1) = n - 3r/2 - 2$, $h \geq -1$. Cutting $F$ with $L$, the intersection of $h + 1$ general hyperplanes, we get $F' := F \cap L$, of dimension $r$, containing a $r$-dimensional family of quadrics of dimension $r/2$ and generically of maximal rank $r/2 + 1$. Moreover the variety of secant lines of $F'$ coincides with $SF \cap L$, so it has dimension $3r/2 + 1$. By (ii) we obtain that $F'$ is a Severi variety. Again, since Severi varieties are unextendable, we can conclude that $F$ is a cone over $F'$ with vertex a linear space of dimension $h$. In particular, the rank of $F_{\Lambda}$ is $r/2 + 1$. □

**Remark 3.4.** Smooth varieties $F$ such that $X = SF$ satisfies the assumptions of Theorem 3.3 are precisely the varieties studied by Ohno in [O].

4. Examples of focal hypersurfaces of any degree or with arbitrarily high multiplicity.

In this section we give two series of examples: The former show that the focal hypersurfaces in the fibres of the Gauss map can appear with arbitrarily high multiplicity, the latter have focal hypersurfaces of arbitrary degree. Note that in Example 3 the variety $X$ is not a hypersurface if $m \geq 3$.

**Example 3. Secant varieties of Scorza varieties.**

Let $F$ be one of the following varieties: $v_2(\mathbb{P}^m)$, $\mathbb{P}^m \times \mathbb{P}^m$, $G(1, 2m + 1)$. They are contained in the projective spaces of dimension $(m+2)/2 - 1$, $(m + 1)^2 - 1$, $(2m+2)/2 - 1$ respectively, and have dimensions $m$, $2m$, $4m$. In all cases $F$ is defined by suitable minors of a matrix of variables, and precisely by $2 \times 2$ minors of a symmetric $(m + 1) \times (m + 1)$ matrix in the first case, by $2 \times 2$ minors of a generic $(m + 1) \times (m + 1)$ matrix in the second case, by Pfaffians of $4 \times 4$ minors of a skew-symmetric matrix of order $2m + 2$ in the third one. These varieties are considered by Zak in [Z3], and named by him Scorza varieties.

Let $X$ be the secant variety of $F$: It is defined by minors of the same matrix as $F$ of order one more. Then the fibres of the Gauss map of $X$ have respectively dimension $2$, $3$, $5$, the focal hypersurfaces are quadrics of dimensions $1$, $2$, $4$, which appear with multiplicity $m - 1$, $2m - 2$, $4m - 4$ respectively.

**Example 4. Higher secant varieties of Scorza varieties.**

Let $X = S^{m-1}F$ be the variety of $m$-secant $(m - 1)$-planes of $F$, where $F$ is one of the varieties appearing in the previous example. $X$ is the maximal proper secant variety of $F$ and it is the hypersurface defined by the determinant (or the Pfaffian) of the matrix considered in previous example. The varieties $X$ and $F$ result to be mutually dual, so $r = m, 2m, 4m$ respectively. The focal locus of the Gauss map is $S^{m-2}F$. The focal hypersurfaces are the maximal secant varieties of $v_2(\mathbb{P}^{m-1})$, $\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$, $G(1, 2m - 1)$.
respectively. They have degree $m$ and appear with multiplicity respectively $1, 2, 4$.

**Remark 4.1.** The class of Scorza varieties includes also $\mathbb{P}^m \times \mathbb{P}^{m-1}$ and $G(1, 2m)$, for all $m \geq 3$. Also their secant varieties have properties similar to above. They are defined by minors of matrices of type $(m+1) \times (m+2)$ and of skew-symmetric square matrices of type $2m+1$ respectively.

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TRANSLATING SOLUTIONS FOR GAUSS CURVATURE FLOWS WITH NEUMANN BOUNDARY CONDITIONS

We consider strictly convex hypersurfaces which are evolving by the non-parametric logarithmic Gauss curvature flow subject to a Neumann boundary condition. Solutions are shown to converge smoothly to hypersurfaces moving by translation. In particular, for bounded domains we prove that convex functions with prescribed normal derivative satisfy a uniform oscillation estimate.

1. Introduction.

In this paper, we evolve hypersurfaces represented as graphs of strictly convex functions over strictly convex bounded domains by the non-parametric logarithmic Gauss curvature flow subject to a Neumann boundary condition. We show that solutions exist for all time and converge smoothly to translating solutions.

To be more precise, we address the following slightly more general problem: Let \( u_0 \) be a strictly convex function over a smooth strictly convex bounded domain \( \Omega \subset \mathbb{R}^n \). We use the phrase strictly convex for functions whose Hessian is positive definite, and for domains for which all principal curvatures of the boundary are positive. Assume that \( u_0 \) is smooth up to the boundary, \( u_0 \in C^\infty(\Omega) \), and satisfies

\[
D_\nu u_0 = \varphi \quad \text{on } \partial \Omega,
\]

where \( \nu \) is the inner unit normal to \( \partial \Omega \) and \( \varphi \in C^\infty(\partial \Omega) \). Let \( f \in C^\infty(\overline{\Omega} \times \mathbb{R}^n) \). We prove the following:

**Theorem 1.1.** For \( \Omega, \varphi, \nu, f \) and \( u_0 \) as introduced above, there exists a family \( u(\cdot, t), t \in [0, \infty) \), of strictly convex functions solving

\[
\begin{cases}
\frac{\partial}{\partial t} u = \log \det D^2 u - \log f(x, Du) & \text{in } \Omega \times [0, \infty), \\
D_\nu u(\cdot, t) = \varphi & \text{on } \partial \Omega, \ t > 0, \\
u(\cdot, 0) = u_0 & \text{in } \Omega,
\end{cases}
\]

where \( u \in C^\infty(\overline{\Omega} \times (0, \infty)) \), and \( u(\cdot, t) \) approaches \( u_0 \) in \( C^2(\Omega) \) as \( t \to 0 \). Moreover, \( u(\cdot, t) \) converges smoothly to a translating solution, i.e., to a solution with constant time derivative.

We remark that the parabolic maximum principle implies that the asymptotic solutions for different initial data \( u_0 \) are unique up to a constant.
For the Gauß curvature flow, mentioned in the beginning, the flow equation takes the form
\[
\frac{\partial}{\partial t} u = \log \det D^2 u \left( 1 + |Du|^2 \right)^{-\frac{n+2}{2}} = \log \det D^2 u - \frac{n+2}{2} \log \left( 1 + |Du|^2 \right).
\]

In the proof of Theorem 1.1, we generalize the result to the so-called oblique boundary condition
\[
D_\beta u = \varphi \quad \text{on } \partial \Omega,
\]
where \( \beta \) is a unit vector field which is \( C^1 \)-close to \( \nu \), i.e., such that there exists a small positive constant \( c_\beta > 0 \) for which \( \| \nu - \beta \|_{C^1} \leq c_\beta \). Such a generalization of a Neumann boundary condition is studied for the elliptic case in [7].

We base the barrier construction in the proof of Theorem 1.1 on solutions to a related elliptic problem given by the following:

**Theorem 1.2.** Consider \( \Omega, \nu, \varphi, \) and \( f \) as introduced before. There exist a unique \( v \in \mathbb{R} \) and a strictly convex function \( u \in C^\infty(\overline{\Omega}) \) solving the boundary value problem
\[
\begin{cases}
\det D^2 u = e^v \cdot f(x, Du) & \text{in } \Omega, \\
D_\nu u = \varphi & \text{on } \partial \Omega,
\end{cases}
\]
provided that there exists a smooth strictly convex function \( u_0 \) satisfying the boundary condition \( D_\nu u_0 = \varphi \). The function \( u \) is unique up to a constant.

We remark that translating solutions to (1.1) can be viewed as solutions to (1.2), where \( v \) denotes the speed.

A situation similar to Theorem 1.1 is considered for the mean curvature flow in [2]. Hypersurfaces of prescribed Gauß curvature subject to Neumann boundary conditions are found in the pioneering paper [4]. The extension to the oblique boundary value problem is made in [7]. Flows of Monge-Ampère type for the Neumann and the second boundary value problem are studied in [6]. In our setting, the situation is more degenerate as neither \( f \) nor \( \varphi \) do depend on \( u \). Thus, both \( f \) and \( \varphi \) fail to satisfy the crucial monotonicity requirement with respect to \( u \). For the second boundary value problem, translating solutions to flows of Gauß curvature type are considered in [5].

As mentioned in [5], methods of [6] can be adapted to the non-parametric logarithmic Gauß curvature flow subject to the second boundary condition. For the Neumann boundary value problem, however, the lack of monotonicity requires a new proof to uniformly bound the oscillation of a solution. Therefore, we establish a generalization of the spatial \( C^1 \)-estimates of [4]. Then, we use the translating solutions provided by Theorem 1.2, in particular its uniquely determined speed, to construct an auxiliary barrier function and to obtain uniform spatial \( C^2 \)-estimates. Hence, the results of Krylov, Safonov, Evans and Schauder imply uniform bounds on higher derivatives.
for all times, uniformly bounded away from 0. This allows to show smooth convergence to a translating solution.

The paper is organized as follows: As explained in [6], standard linear parabolic theory and the implicit function theorem imply short-time existence. We show uniform first-order estimates in Sections 2 and 3. Section 4 contains the proof of Theorem 1.2. Having the unique velocity of a translating solution, we can prove uniform a priori $C^2$-estimates in Section 5. Finally, in Section 6, we prove smooth convergence to a translating solution. In Appendix A we apply Theorem 1.2 to construct entire graphs of prescribed Gauß curvature. To illustrate the convergence of the flow, we carry out a numerical integration on a planar domain in Appendix B.

2. $\frac{\partial}{\partial t} u$-Estimate.

Notation 2.1. We write a dot to denote the time derivative, whereas we use indices for the spatial partial derivatives. Let $f_{pi}$ denote a derivative of $f$ with respect to the gradient. For a vector $\xi$ we define $u_\xi \equiv \xi^i u_i$. For the logarithm of $f$ we use $\hat{f} \equiv \log f$. We use the Einstein summation convention and sum over repeated upper and lower indices. The inverse of the Hessian of $u$ is denoted by $(u_{ij}) = (u_{ij})^{-1}$. We remark that – besides in the case $u_{ij}$ – indices are lifted with respect to the Euclidean metric. The letter $c$ denotes a generic positive constant. Furthermore, we may assume that $0 \in \Omega$.

Lemma 2.2. Under the assumptions of Theorem 1.1, there holds

$$|\dot{u}| \leq \max_{t=0} |\dot{u}|,$$

as long as a smooth convex solution of (1.1) exists.

Proof. Similar to [6], we consider

$$r := (\dot{u})^2.$$

We get the evolution equation

$$\dot{r} = u^{ij} r_{ij} - 2u^{ij} \dot{u}_i \dot{u}_j - f_{pi} r_i.$$

Hence, $\dot{r} \leq 0$ at a maximum of $r$ in $\Omega \times [0,t]$. Now, assume that a maximum of $r$ on $\overline{\Omega} \times [0,t]$ occurs at $(x_0,t_0)$ with $x_0 \in \partial \Omega$. If $t_0 = 0$ the lemma holds. Thus, in the following, we may assume $t_0 > 0$. If $r$ is constant, then $u$ is a translating solution, and our lemma holds. Otherwise, we get $r_\beta(x_0) < 0$ from the Hopf boundary point lemma. At $x_0$ we compute

$$0 > r_\beta = \left((\dot{u})^2\right)_\beta = 2\dot{u}\dot{u}_\beta = 2\dot{u} \frac{\partial}{\partial \varphi} \varphi(x) = 0.$$

Contradiction. Note, that the assumption $t_0 > 0$ guarantees that $u$ is smooth near $(x_0,t_0)$ allowing to interchange differentiation with respect to time and space. $\square$
Integrating the last estimate yields:

**Corollary 2.3.** As long as a smooth convex solution of (1.1) exists, we obtain the estimate

\[ |u(x, t)| \leq \sup_{\Omega} |u_0| + \sup_{\Omega} |\dot{u}(\cdot, 0)| \cdot t. \]

### 3. Ice-cream cone estimate.

The following theorem generalizes the \( C^1 \)-estimates of [4]. It is essential for our situation, because the oscillation but not the \( C^0 \)-norm of the solution is expected to be bounded uniformly in time.

We wish to mention oscillation estimates of Urbas [9, 8]. There, the convexity of \( u \) and appropriate growth of \( f(x, p) \) in \( p \) is used, whereas Theorem 3.1 combines convexity and the boundary condition.

**Theorem 3.1** (Ice-cream cone estimate). Let \( \Omega \subset \mathbb{R}^n \) be a smooth bounded domain, \( u : \overline{\Omega} \to \mathbb{R} \) a smooth strictly convex function with \( |u_\beta| \) uniformly bounded on \( \partial \Omega \), where \( \beta \) is a unit vector field on \( \partial \Omega \) such that \( \langle \beta, \nu \rangle \geq \tilde{c}_\beta \) for a positive constant \( \tilde{c}_\beta > 0 \) (recall that \( \nu \) is the inner unit normal to \( \partial \Omega \)). Then there is a uniform bound for \( \sup |Du| \), independent of \( \sup |u| \).

In view of Lemma 2.2, the result above yields an estimate for the full \( C^1 \)-norm of solutions \( u \) to (1.1). Note, that only the estimate for the derivatives of \( u \) is uniform in time.

**Proof.** In the name of the theorem we want to emphasize that our proof uses balls and cones, similar to ice-cream placed in a cone of waffle. We argue by contradiction. Assume that there exists a point \( x_0 \), where \( |Du| \) is maximal and equal to \( M \). If \( M \) is larger than a suitably chosen constant \( M_0 \) we will find a contradiction. As \( u \) is strictly convex, we see that \( x_0 \in \partial \Omega \). At \( x_0 \) we find a tangential direction \( \xi_0 \) such that \( \langle Du(x_0), \xi_0 \rangle \) is maximal compared to all other tangential directions. Here and later, unit vectors are called directions. We wish to prove a lower estimate for \( \langle Du(x_0), \xi_0 \rangle \) in terms of \( M \).

Let \( \xi_1 \) be a direction such that \( \langle Du(x_0), \xi_1 \rangle = M \). Similar to [7], we decompose a direction \( \xi \) using \( \beta \) and a tangential vector \( \tau(\xi) \) as

\[ \xi = \tau(\xi) + \frac{\langle \nu, \xi \rangle}{\langle \beta, \nu \rangle} \beta, \]

where

\[ \tau(\xi) = \xi - \langle \nu, \xi \rangle \nu - \frac{\langle \nu, \xi \rangle}{\langle \beta, \nu \rangle} \beta^T, \quad \beta^T = \beta - \langle \beta, \nu \rangle \nu. \]
Note, that $|\tau(\xi)|$ is bounded by assumption. Decomposing $\xi_1$, we get

$$M = \langle Du, \xi_1 \rangle = \langle Du, \tau(\xi_1) \rangle + \langle \nu, \xi_1 \rangle \frac{\langle \beta, \nu \rangle}{|\beta|} \langle Du, \beta \rangle$$

$$\leq |\tau(\xi_1)| \cdot \max_{r \in \partial \Omega \cap \partial \Omega} \langle Du, \tau \rangle + c$$

$$= |\tau(\xi_1)| \cdot \langle Du(x_0), \xi_0 \rangle + c.$$  

Hence, we deduce that $\langle Du(x_0), \xi_0 \rangle \geq \frac{M}{c}$, as long as $M \geq M_0$ and $M_0$ is chosen sufficiently large. For a direction $\xi$ near $\xi_0$, say $|\xi - \xi_0| < \varepsilon = \frac{1}{2c} < 1$, we obtain

$$\langle Du(x_0), \xi \rangle \geq \varepsilon M. \tag{3.2}$$

From the convexity of $u$, we deduce that $\langle Du(y), \xi \rangle \geq \varepsilon M$ for all points $y \in \overline{\Omega}$ of the form $y = x_0 + \lambda \cdot \xi$. Here, $\lambda > 0$ and $\xi$ are chosen such that $|\xi - \xi_0| \leq \varepsilon$ and $x_0 + t \cdot \lambda \cdot \xi \in \overline{\Omega}$ for all $t \in [0, 1]$.

The uniform boundedness of the principal curvatures of $\partial \Omega \subset \mathbb{R}^n$ implies that there exist $R > 0$ and $x_1 \in \partial \Omega$ such that $|x_0 - x_1| > 2R$, and, especially, any $x \in B_R(x_1) \cap \partial \Omega$ can be written in the form $x_0 + t \cdot \lambda \cdot \xi$, as described above. Thus, according to (3.2), $|Du| \geq \varepsilon M$ in $\partial \Omega \cap B_R(x_1)$. Due to our construction, we have

$$\inf_{x \in B_R(x_1) \cap \partial \Omega} u(x) > u(x_0).$$

![Figure 1. Ice-cream cone estimate.](image)

Figure 1 shows a part of $\partial \Omega$ and two cones corresponding to the directions $\xi$ as well as two pairs of concentric balls. The larger ones are the balls $B_R$ mentioned above, the smaller ones are introduced in the following.
Now, we proceed iteratively. Note, that $R$ and $\varepsilon$ can be chosen as fixed constants, independent of the point $x_0$. As long as $|Du(x_i)| \geq M \varepsilon^i \geq M_0$, we can find a further point $x_{i+1}$ as we went from $x_0$ to $x_1$. Thus, for $M = \sup|Du|$ sufficiently large, we can construct a sequence of points $\{x_i\}_{i=0,\ldots,N}$ of arbitrarily large length $N$, satisfying for all $i \geq 1$

$$|Du| \geq M \varepsilon^i \quad \text{on } \partial \Omega \cap B_R(x_i)$$

and

$$\inf_{x \in B_R(x_i) \cap \partial \Omega} u(x) > u(x_{i-1}).$$

Since $\partial \Omega$ has finite measure and bounded principal curvatures, there is an upper bound $N_0(\rho)$ on the number of pairwise disjoint restricted balls $B_\rho(y_j) \cap \partial \Omega$ for fixed $\rho > 0$ and $y_j \in \partial \Omega$.

Hence, if $M = \sup|Du| > M_0 \varepsilon^{-N_0} \left(\frac{\rho}{2}\right)$, there will be two points $x_{i_0}, x_{j_0}$ with $i_0 > j_0 > 0$ such that

$$B_{\frac{\rho}{2}}(x_{i_0}) \cap B_{\frac{\rho}{2}}(x_{j_0}) \cap \partial \Omega \neq \emptyset.$$

But $x_{j_0} \in B_R(x_{i_0})$ implies

$$u(x_{j_0}) < u(x_{j_0+1}) < \cdots < u(x_{i_0-1}) < \inf_{x \in B_R(x_{i_0}) \cap \partial \Omega} u(x) \leq u(x_{j_0}).$$

Contradiction. $\square$

4. Existence of translating solutions.

This section is devoted to the proof of Theorem 1.2. That is, we construct solutions to the elliptic problem

\begin{equation}
\begin{aligned}
\begin{cases}
  v = \log \det D^2 u - \log f(x, Du) & \text{in } \Omega, \\
  u_{\beta} = \varphi & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\end{equation}

The absence of any monotonicity property in $u$, both in $f$ as well as in the boundary condition $\varphi$, limits seriously the existence of solutions.

Step 1. Here, we show that for given $\varepsilon > 0$ and $v \in \mathbb{R}$ there is a unique solution $u_{\varepsilon,v}$ of

\begin{equation}
\begin{aligned}
\begin{cases}
  \det D^2 u = e^v f(x, Du) e^{\varepsilon u} & \text{in } \Omega, \\
  u_{\beta} = \varphi & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\end{equation}

Note, that the dependence on $v$ is continuous and strictly decreasing. In fact, we have the explicit relation

$$u_{\varepsilon,v} = u_{\varepsilon,0} - \frac{v}{\varepsilon}.$$

To show the unique existence of $u_{\varepsilon,v}$, we will derive an a priori $C^0$-bound, then the ice-cream cone estimate, Theorem 3.1, yields the $C^1$-bound. Having controlled the full $C^1$-norm, we can estimate the $C^2$-norm exactly as in Urbas [7], since the strict monotonicity assumption on $\varphi$ is not used for this
part. A detailed argument, applied to the parabolic case, is given in the next section. Bounds for higher $C^k$-norms follow via the estimates due to Krylov, Safonov, Evans, and from Schauder theory.

To get the $C^0$-bound, we define suitable barriers for $(\ast_\varepsilon,0)$. Recall, that $u_0$ is a convex function satisfying $(u_0)_\beta = \varphi$ on $\partial \Omega$. Define $u_\varepsilon^\pm = u_0 \pm M/\varepsilon$, where $M > 0$ will be chosen later. Then

$$\frac{\det D^2 u_\varepsilon^\pm}{f(x,Du_\varepsilon^\pm) e^{u_\varepsilon^\pm}} = \frac{\det D^2 u_0}{f(x,Du_0) e^{u_0}} e^{-\varepsilon u_0} e^{\mp M},$$

with $e^{-1} < g(x) < c$. Hence, restricting ourselves to $\varepsilon < 1$, there exists a large constant $M > 0$, not depending on $\varepsilon$, such that $u_\varepsilon^+$ is a strict supersolution and $u_\varepsilon^-$ is a strict subsolution of $(\ast_\varepsilon,0)$. This implies, that

$$u_\varepsilon^- < u_\varepsilon,0 < u_\varepsilon^+,$$

or equivalently,

$$\left| u_{\varepsilon,v} - (u_0 - \frac{v}{\varepsilon}) \right| < \frac{M}{\varepsilon}.$$

**Step 2.** Now, we consider the limit $\varepsilon \to 0$. In general, we cannot expect that the sup bounds for $u_{\varepsilon,v}$ can be obtained uniformly in $\varepsilon$. In fact, it follows from the maximum principle that only for a unique $v$, there is a solution to $(\ast_\varepsilon,v)$ with $\varepsilon = 0$. Observe, that (4.2) implies that $u_{\varepsilon,v} < u_\varepsilon,0 < u_\varepsilon^+$.

Therefore, for every $\varepsilon > 0$ we can find a unique $v_\varepsilon \in (-M,M)$ such that $u_{\varepsilon,v}(0) = u_0(0)$. Note, that (4.2) does not suffice to control the oscillation of $u_{\varepsilon,v}$, uniformly in $\varepsilon$. We employ the ice-cream cone estimate to bound $u_{\varepsilon,v}$ uniformly in $C^1$. Again, uniform $C^1$-bounds imply uniform higher $C^k$-bounds.

Now, we choose a sequence $\varepsilon_i \to 0$ as $i \to \infty$. Since $v_\varepsilon$ is bounded, there exists a subsequence, relabeled, such that $v_{\varepsilon_i} \to v^\infty$ and $u_{\varepsilon_i,v_{\varepsilon_i}} \to u_{\infty}$ in any $C^k$-norm. This completes the proof of Theorem 1.2.

The extension $u^\infty(x,t) := u_{\infty}(x) + v^\infty t$ is a translating solution, as, by construction, $u^\infty$ satisfies

$$\left\{ \begin{array}{ll} v^\infty = u^\infty = \log \det D^2 u^\infty - \log f(x,Du^\infty) & \text{in } \Omega, \\
 u^\infty_\beta = \varphi & \text{on } \partial \Omega. \end{array} \right.$$

5. **Parabolic $C^2$-estimates.**

The following argument is a modification of the proofs in [4, 6] and [7]. We use the translating solution $u^\infty$, especially, its speed $v^\infty$, to construct an auxiliary barrier function.

Assume that $u_{\infty} > u_0$. We define

$$\tilde{\varphi}(x,z) = \varphi(x) + (z - u_{\infty}^\infty).$$
Due to uniform estimates on the gradient of \( u \), we can find a positive constant \( \mu_0 \) such that

\[
\min \{ f(x, Du), f(x, Du^{\infty}_{\text{ell}}) \} \geq \mu_0.
\]

For \( 0 < \rho < 1 \), consider the elliptic boundary value problem

\[
\begin{aligned}
\psi^{\infty} &= \log \det D^2 \psi - \log \frac{\mu_0}{2} \quad &\text{in } \Omega, \\
\psi_{\beta} &= \tilde{\varphi}(x, \psi + \rho \cdot |x|^2) - 2\rho \langle x, \beta \rangle \quad &\text{on } \partial \Omega.
\end{aligned}
\]  

(5.1)

We wish to show a uniform a priori \( C^2 \)-estimate for \( \psi \). Theorem 3.1 gives an estimate for the gradient. Similarly to [7], bounds on the second derivatives follow. It is here that the smallness of \( \| \nu - \beta \|_{C^1} \) is used. Thus, it remains to prove uniform \( C^0 \)-estimates. Note, that a convex solution \( \psi \) cannot satisfy \( \psi_{\beta}(x) > 0 \) for all \( x \in \partial \Omega \). Hence, the upper bound on \( \psi \) follows since \( \tilde{\varphi}(x, z) \to \infty \) uniformly as \( z \to \infty \). For the lower bound, we consider \( \psi - u^{\infty}_{\text{ell}} \). Applying the maximum principle to the differential inequality

\[
\begin{aligned}
0 > \log \det D^2 \psi - \log \det D^2 u^{\infty}_{\text{ell}} \quad &\text{in } \Omega, \\
(\psi - u^{\infty}_{\text{ell}})_{\beta} &= \psi - u^{\infty}_{\text{ell}} + \rho \cdot |x|^2 - 2\rho \langle x, \beta \rangle \quad &\text{on } \partial \Omega,
\end{aligned}
\]

we see that \( \psi - u^{\infty}_{\text{ell}} \) cannot attain an interior minimum. If a minimum occurs on \( \partial \Omega \), we get

\[
0 \leq (\psi - u^{\infty}_{\text{ell}})_{\beta} = \psi - u^{\infty}_{\text{ell}} + \rho \cdot |x|^2 - 2\rho \langle x, \beta \rangle.
\]

Thus, \( \psi \) is uniformly bounded below, a solution to (5.1) exists. Due to the uniform \( C^2 \)-estimates, we can fix \( \lambda > 0 \) such that

\[
(\psi_{ij}) \geq \lambda d.
\]  

(5.2)

Furthermore, these estimates allow to fix \( \rho > 0 \) such that \( \overline{\psi} := \psi + \rho \cdot |x|^2 \) satisfies

\[
\begin{aligned}
\psi^{\infty} &> \log \det D^2 \overline{\psi} - \log \mu_0 \quad &\text{in } \Omega, \\
\overline{\psi}_{\beta} &= \overline{\varphi}(x, \overline{\psi}) \quad &\text{on } \partial \Omega.
\end{aligned}
\]

Applying the maximum principle, we get \( u^{\infty}_{\text{ell}} \leq \overline{\psi} \). We extend \( \psi \) and \( \overline{\psi} \) by setting \( \psi(x, t) := \psi(x) + t \cdot \psi^{\infty} \), \( \overline{\psi}(x, t) := \overline{\psi}(x) + t \cdot \psi^{\infty} \), respectively. Thus, \( u \leq u^{\infty} \leq \overline{\psi} \), where \( u^{\infty} \) is the translating solution defined in Section 4. We get for \( x \in \partial \Omega \)

\[
(\overline{\psi}_{\beta} - u_{\beta})(x, t) = \overline{\psi}_{\beta}(x, 0) - \varphi(x) = (\overline{\psi} - u^{\infty})(x, 0) = (\overline{\psi} - u^{\infty})(x) \geq 0.
\]

(5.3)

Furthermore, for a sufficiently small \( \delta_0 > 0 \),

\[
(\psi - u)_{\beta} = (\overline{\psi} - \rho \cdot |x|^2 - u)_{\beta} \geq -2\rho \langle x, \beta \rangle \geq \delta_0 > 0 \quad \text{on } \partial \Omega,
\]

provided that \( \beta \) is \( C^0 \)-close to \( \nu \). Here, we used that \( 0 \in \Omega \) implying \( \langle x, \nu \rangle < 0 \). Using these preparations, we prove a priori \( C^2 \)-estimates similarly to [6] and [7]. For the reader’s convenience, we repeat the arguments incorporating the necessary modifications to the parabolic case.
5.1. Preliminary results. Assume, that a smooth solution \( u \) of our flow equation (1.1) exists on the time interval \([0, T]\). We will use the letter \( \tau \) to indicate a direction tangential to \( \partial \Omega \).

**Lemma 5.1** (Mixed \( C^2 \)-estimates at the boundary). Let \( u \) be a solution of (1.1). Then \( |u_{\tau \beta}| \) remains uniformly bounded on \( \partial \Omega \).

**Proof.** We represent \( \partial \Omega \) locally as graph \( \omega \) over its tangent plane at a fixed point \( x_0 \in \partial \Omega \) such that locally \( \Omega = \{ (\hat{x}, x^n) : x^n > \omega(\hat{x}) \} \). Let us extend \( \beta \) and \( \varphi \) smoothly. At \( x_0 \), differentiating the oblique boundary condition

\[
\beta^i(\hat{x})u_i(\hat{x}, \omega(\hat{x})) = \varphi(\hat{x}, \omega(\hat{x})), \quad \hat{x} \in \mathbb{R}^{n-1},
\]

with respect to tangential directions \( \hat{x}^j \), \( 1 \leq j \leq n-1 \),

\[
\beta^j u_i + \beta^i u_{ij} + \beta^i u_{in} \omega_j = \varphi_j + \varphi_n \omega_j,
\]

we obtain at \( x_0 \equiv (\hat{x}_0, \omega(\hat{x}_0)) \) a bound for \( \beta^i u_{ij} \). Thereby, we use the gradient estimate for \( u \) and \( D\omega(\hat{x}_0) = 0 \). Multiplying with \( \tau^j \) gives the result. \( \square \)

**Lemma 5.2** (Double oblique \( C^2 \)-estimates at the boundary). For any solution of (1.1), \( |u_{\beta \beta}| \) is uniformly bounded on \( \partial \Omega \).

**Proof.** Note that \( u_{\beta \beta} > 0 \) as \( u(\cdot, t) \) is strictly convex for each \( t \). We keep the geometric setting of the proof of Lemma 5.1 with \( x_0 \in \partial \Omega \). From (1.1) we obtain

\[
\dot{u}_k = u^{ij} u_{ijk} - (\dot{f}_k + \dot{f}_p u_{ik}).
\]

We define

\[
Lw := \dot{w} - u^{ij} w_{ij} + \dot{f}_p w_i.
\]

We can find appropriate extensions of \( \beta \) and \( \varphi \) such that

\[
\left| L \left( \beta^k u_k - \varphi(x) \right) \right| \leq c \cdot (1 + \text{tr} \left( u^{ij} \right)).
\]

We choose \( \delta > 0 \) sufficiently small and define \( \Omega_{\delta} := \Omega \cap B_{\delta}(x_0) \). Set

\[
\vartheta := d - \mu d^2,
\]

where \( \mu \gg 1 \) is chosen sufficiently large, and \( d \) denotes the distance to \( \partial \Omega \). We will show that in \( \Omega_{\delta} \) there holds \( L\vartheta \geq \frac{\varepsilon}{2} \text{tr} \left( u^{ij} \right) \) for a small constant \( \varepsilon > 0 \), depending only on a positive lower bound for the principal curvatures of \( \partial \Omega \). Next,

\[
L\vartheta = -u^{ij} d_{ij} + 2\mu u^{ij} d_i d_j + 2\mu u^{ij} d d_{ij} + \dot{f}_p (d_i - 2\mu d d_i)
\geq -u^{ij} d_{ij} + 2\mu u^{ij} d_i d_j - c\mu d \left( 1 + \text{tr} \left( u^{ij} \right) \right) - c.
\]
We use the strict convexity of $\partial \Omega$, $|Dd - e_n| \leq c \delta$, $|u^k| \leq \text{tr} (u^{ij})$, $1 \leq k, l \leq n$, and the inequality for arithmetic and geometric means

\[ L \vartheta \geq \varepsilon \text{tr} (u^{ij}) + \mu u^{nn} - c \mu \delta \left(1 + \text{tr} (u^{ij})\right) - c \]

\[ \geq \frac{n}{3} (\det (u^{ij}))^{\frac{1}{n}} \cdot \varepsilon^{\frac{n-1}{n}} \cdot \mu^{\frac{1}{n}} + \frac{2}{3} \varepsilon \text{tr} (u^{ij}) \]

\[ - c \mu \delta \left(1 + \text{tr} (u^{ij})\right) - c. \]

More precisely, we used

\[ \frac{\varepsilon \text{tr} (u^{ij}) + \mu u^{nn}}{3} \geq \frac{n}{3} \varepsilon^{\frac{n-1}{n}} \mu^{\frac{1}{n}} \left(\prod_{i=1}^{n} u^{ii}\right)^{\frac{1}{n}}, \]

and, assuming $(u^{ij})_{i,j<n}$ is diagonal,

\[ \det (u^{ij}) = \det \begin{pmatrix} u^{11} & 0 & \cdots & 0 & u^{1n} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & u^{n-1,n-1} & u^{n-1,n} \\ u^{1n} & \cdots & \cdots & u^{n-1,n} & u^{nn} \end{pmatrix} \]

\[ = \prod_{i=1}^{n} u^{ii} - \sum_{i<n} |u^{ii}|^2 \prod_{j\neq i, j<n} u^{jj} \leq \prod_{i=1}^{n} u^{ii}. \]

(5.4)

Since

\[ \det (u^{ij}) = \frac{1}{\det(u_{ij})} = \exp \left( -\hat{f} - \hat{u} \right), \]

det $(u^{ij})$ is uniformly bounded from below by a positive constant. We may choose $\mu$ so large, that

\[ \frac{n}{3} (\det (u^{ij}))^{\frac{1}{n}} \cdot \varepsilon^{\frac{n-1}{n}} \cdot \mu^{\frac{1}{n}} \geq c + 1. \]

For $\delta \leq \frac{1}{c \mu} \min \left\{ 1, \frac{1}{3} \varepsilon \right\}$, we get

\[ L \vartheta \geq \varepsilon \text{tr} (u^{ij}). \]

Furthermore, $\vartheta \geq 0$ on $\partial \Omega_\delta$, if $\delta$ is chosen smaller if necessary.

Let $l$ be an affine linear function such that $l(x_0) = 0$ and

\[ l \geq \beta^i (u_{0i}) - \varphi \quad \text{in} \ \Omega_\delta. \]

For constants $A, B > 0$, consider the function

\[ \Theta := A \vartheta + B |x - x_0|^2 - (\beta^i u_i - \varphi(x)) + l. \]

We fix $B \gg 1$, get $\Theta \geq 0$ on $\partial \Omega_\delta$, and deduce for $A \gg B$ that $L \Theta \geq 0$, since $\text{tr} (u^{ij})$ is bounded from below by a positive constant. The maximum principle yields $\Theta \geq 0$ in $\Omega_\delta$. As $\Theta(x_0) = 0$, we conclude that $\Theta_\beta(x_0) \geq 0$ implying $u_{\beta \beta} \leq c$. □
Lemma 5.3. For a solution of (1.1), there holds
\[ \min_{t \in [0, T]} \max_{x \in \partial \Omega} u_{\xi\xi}(x, t) > 0. \]

Proof. We have already seen that there is a uniform positive lower bound for \( \det D^2 u \). Again, at a fixed boundary point, we may choose a coordinate system, such that \( e_n \) is equal to the inner unit normal of \( \partial \Omega \) and \((u_{ij})_{i,j<n}\) is diagonal. Similarly to (5.4), we estimate
\[ \det (u_{ij}) = \prod_{i=1}^{n} u_{ii} - \sum_{i<n} |u_{ni}|^2 \prod_{j<n, j \neq i} u_{jj} \leq \prod_{i=1}^{n} u_{ii}. \]

We decompose \( \nu \) as
\[ \nu = \frac{1}{\langle \beta, \nu \rangle} (\beta - \beta^T), \]
\( \beta^T \) as in the proof of Theorem 3.1, and get in view of Lemmata 5.1 and 5.2
\[ u_{\nu\nu} \leq \frac{1}{\langle \beta, \nu \rangle^2} \left( \beta^T \right)^2 \cdot \max_{i<n} u_{ii} + c. \]

Finally, the claimed bound follows from (5.5).

5.2. Remaining \( C^2 \)-estimates. Similarly to [7], we define
\[ w(x, \xi, t) := e^{\alpha(\psi - u) + \gamma |Du|^2} \cdot u_{\xi\xi} \]
for \((x, \xi, t) \in \overline{\Omega} \times S^{n-1} \times [0, T]\) and positive constants \( \alpha, \gamma \) to be fixed later. We assume that \( w \), restricted to boundary points and tangential directions, attains its maximum at \( x_w \in \partial \Omega \) in a tangential direction which we may take to be \( e_1 \), and \( t_w \in [0, T] \). We may assume that \( t_w > 0 \). Furthermore, we fix Euclidean coordinates such that \( e_n \) is the inner normal direction and \((u_{ij})_{i,j<n}(x_w)\) is diagonal. Decompose \( e_1 \) as
\[ e_1 = \tau(e_1) + \frac{\langle \nu, e_1 \rangle}{\langle \beta, e_1 \rangle} \beta, \]
where
\[ \tau(e_1) = \tau = e_1 - \frac{\langle \nu, e_1 \rangle}{\langle \beta, e_1 \rangle} \nu - \frac{\langle \nu, e_1 \rangle}{\langle \beta, e_1 \rangle} \beta^T, \quad \beta^T = \beta - \frac{\langle \beta, e_1 \rangle}{\langle \beta, \nu \rangle} \nu. \]

Note that \( \tau \) is tangential, but not necessarily of unit length. For smoothly extended \( \beta \) and \( \varphi \), we differentiate the boundary condition and obtain on \( \partial \Omega \)
\[ 2\frac{\langle \nu, e_1 \rangle}{\langle \nu, \beta \rangle} u_{\beta \tau} = \frac{2\langle \nu, e_1 \rangle}{\langle \nu, \beta \rangle} (\varphi_j \tau^j - \tau^j \beta^j u_i) =: \chi(x, Du). \]

On the boundary, we get
\[ u_{11} = u_{\tau\tau} + \chi + \frac{\langle \nu, e_1 \rangle^2}{\langle \beta, \nu \rangle^2} u_{\beta \beta}. \]
Since $\chi(x_w, \cdot) = 0$, the function
\[ \tilde{w}(x, t) := e^{\alpha(\psi - u) + \gamma |Du|^2} (u_{11} - \chi) \]
satisfies $\tilde{w}(x_w, t_w) = w(x_w, \tau, t_w)$, moreover, for all $x \in \partial \Omega$ and $t \in [0, T]$}
(5.7) $\tilde{w}(x, t)$

\[= e^{\alpha(\psi - u) + \gamma |Du|^2} (x, t) \left\{ u_{\tau\tau}(x, t) + \frac{\langle \nu, e_1 \rangle^2}{\langle \beta, \nu \rangle^2} u_{\beta\beta}(x, t) \right\} \]
\[\leq \left\{ 1 - \langle \nu, e_1 \rangle^2 \left( 1 - \frac{|\beta^T|^2}{\langle \beta, \nu \rangle^2} \right) - 2 \langle \nu, e_1 \rangle \frac{\langle \beta^T, e_1 \rangle}{\langle \beta, \nu \rangle} \right\} \tilde{w}(x_w, t_w) \]
\[+ c \langle \nu, e_1 \rangle^2 e^{\alpha(\psi - u) + \gamma |Du|^2} (x, t) \]
\[\leq \left\{ 1 + c \langle \nu, e_1 \rangle^2 - \frac{2 \langle \nu, e_1 \rangle \langle \beta^T, e_1 \rangle}{\langle \beta, \nu \rangle} + c \frac{\langle \nu, e_1 \rangle^2}{\max_{\xi \in T_\nu \partial \Omega} u_{\xi\xi}(x)} \right\} \tilde{w}(x_w, t_w) \]
\[\leq \left\{ 1 + c_1 \langle \nu, e_1 \rangle^2 - \frac{2 \langle \nu, e_1 \rangle \langle \beta^T, e_1 \rangle}{\langle \beta, \nu \rangle} \right\} \tilde{w}(x_w, t_w). \]

Now, Lemma 5.3 ensures that we can choose $c_1$ in the last inequality independent of $\alpha$ and $\gamma$.

We may assume that $c_1$ in (5.7) is chosen sufficiently large and $\beta$ is sufficiently close to $\nu$ such that the expression in the last curly brackets in (5.7) is bounded below by $\frac{1}{2}$.

We define on $\Omega \times [0, T]$
\[ W := \frac{e^{\alpha(\psi - u) + \gamma |Du|^2} (u_{11} - \chi)}{1 + c_1 \langle \nu, e_1 \rangle^2 - \frac{2 \langle \nu, e_1 \rangle \langle \beta^T, e_1 \rangle}{\langle \beta, \nu \rangle}}. \]

Assume that $W$ attains its maximum at $(x_W, t_W)$ and $t_W > 0$.

First, we address the case $x_W \in \partial \Omega$. Observe that $W(x_W, t_W) \leq \tilde{w}(x_w, t_w) = W(x_w, t_w)$. At $(x_w, t_w)$, we get $W_{\beta} \leq 0$, which implies that
(5.8) $u_{11\beta} + \alpha \delta_0 u_{11} \leq c(1 + (1 + \gamma) u_{11})$,
using $\delta_0$ from (5.3). At $x_w$, keeping the notation of Lemma 5.1, we differentiate the boundary condition $u_{\beta} = \phi$ twice in direction $e_1$. The a priori estimates obtained so far, and the fact that $D^2 u$, restricted to tangential directions, is diagonal, yield
(5.9) $u_{\beta 11} \geq -c - 2\beta^T_1 u_{11} - 2\beta^T_1 u_{n1}$.

Then, combining (5.8) and (5.9) implies
\[ c(1 + (1 + \gamma) u_{11}) \geq \alpha \delta_0 u_{11} - c. \]
For $\alpha = \alpha(\gamma)$ sufficiently large, we get an upper bound on $u_{11}(x_w, t_w)$. This completes the $C^2$-estimates, if $W$ attains its maximum on $\partial \Omega$.

Now, we consider the case that $W$ attains its maximum at $(x_W, t_W)$, $x_W \in \Omega$. We use

$$\Gamma = - \log \left( 1 + c_1 \langle \nu, e_1 \rangle^2 - \frac{2 \langle \nu, e_1 \rangle \langle \beta^T, e_1 \rangle}{\langle \beta, \nu \rangle} \right)$$

in the following calculations. $\Gamma$ is well-defined as the argument of the logarithm is bounded below by a positive constant. Moreover, the $C^2(\overline{\Omega})$-norm of $\Gamma$ is uniformly bounded independent of $\alpha$ and $\gamma$. We use that

$$\log W = \alpha \cdot (\psi - u) + \gamma \cdot |D u|^2 + \log(u_{11} - \chi) + \Gamma$$

attains its maximum at $x_W$. Of course, we may assume that $1 \leq (u_{11} - \chi)(x_W, t_W)$. At $(x_W, t_W)$, we get

\[
0 \leq \dot{W} = \alpha \left( \dot{\psi} - \dot{u} \right) + 2\gamma u^k \dot{u}_k + \frac{\dot{u}_{11} - \frac{d}{dt} \chi}{u_{11} - \chi},
\]

\[
0 = \frac{W_i}{W} = \alpha (\psi - u)_i + 2\gamma u^k u_{ki} + \frac{u_{11i} - D_i \chi}{u_{11} - \chi} + \Gamma_i,
\]

and, in the matrix sense,

\[
0 \geq \frac{W_{ij}}{W} - \frac{W_i W_j}{W^2}
= \alpha (\psi - u)_{ij} + 2\gamma u^k u_{ki} + 2\gamma u^k u_{kij}
+ \frac{u_{11ij} - D_{ij} \chi}{u_{11} - \chi} - \frac{(u_{11i} - D_i \chi)(u_{11j} - D_j \chi)}{(u_{11} - \chi)^2} + \Gamma_{ij},
\]

where we have used that $\Gamma$ is time-independent. We use $D.$ and $\frac{d}{dt}$ to indicate that the chain rule has not yet been applied. In the rest of the section, we drop the argument, if we evaluate at $(x_W, t_W)$. We get

\[
0 \geq u^{ij}(\log W)_{ij} - \dot{W}.
\]

Estimates for the time derivatives of $\psi$ and $u$, the strict convexity of $\psi$ (5.2), the fact that $\Gamma \in C^2$ with uniform bounds, and the differentiated flow equation (1.1) yield

\[
(5.10) \quad 0 \geq 2\gamma \Delta u + \frac{1}{u_{11} - \chi} \left( u^{ir} u^{js} u_{ij1} u_{rs1} \right)
- u^{ij} \frac{(u_{11i} - D_i \chi)(u_{11j} - D_j \chi)}{(u_{11} - \chi)^2}
+ \frac{1}{u_{11} - \chi} \left( \dot{f}_p u_{i11} - c - c \cdot |D^2 u|^2 \right) + 2\gamma u^k \dot{f}_p u_{ik}
+ \frac{1}{u_{11} - \chi} \left( \frac{d}{dt} \chi - u^{ij} D_{ij} \chi \right) - c(\alpha + \gamma) + (\alpha \lambda - c) \text{tr} \left( u^{ij} \right).
\]
Direct calculations and (1.1) imply
\[
\frac{d}{dt} \chi - u^{ij} D_{ij} \chi \geq -c \left(1 + |D^2 u| + \text{tr} \left( u^{ij} \right) \right).
\]
We use \( \frac{W^2}{U^2} = 0 \) to get
\[
2 \gamma u^k \hat{f}_p u_{ik} + \frac{\hat{f}_p u_{11i}}{u_{11} - \chi} \geq -c \cdot \alpha - c \cdot \frac{(1 + |D^2 u|)}{u_{11} - \chi} - c.
\]
Now, these estimates are applied to (5.10). Let \( \vartheta \in (0, \frac{1}{2}) \) be a small constant, to be fixed later. First, we assume that, still at \((x_W, t_W)\),
\[
(1 - \vartheta) u_{\eta\eta} \equiv (1 - \vartheta) \max_{|\xi| = 1} |\eta| \leq u_{11} - \chi.
\]
Here, a direction \( \eta \), \( |\eta| = 1 \), is chosen which corresponds to a maximal eigenvalue. Schwarz’s inequality gives
\[
u_{ij}(u_{11i} - D_i \chi)(u_{11j} - D_j \chi) \leq (1 + \vartheta) u_{ij} u_{11i} u_{11j} + \frac{c}{\vartheta} u_{ij} D_i \chi D_j \chi.
\]
From the definition of \( \eta \), we get
\[
u^r u^s u_{ij1} u_{rs1} \geq \frac{\max_{|\xi| = 1} u^j u_{\xi j1} u_{\xi s1}}{u_{\eta \eta}} \geq \frac{1 - \vartheta}{u_{11} - \chi} u_{ij} u_{11i} u_{11j}.
\]
Using \( \vartheta \leq \frac{1}{2} \) and \( \frac{W^2}{U^2} = 0 \), we get
\[
u^r u^s u_{ij1} u_{rs1} - u_{ij} \frac{1}{u_{11} - \chi} (u_{11i} - D_i \chi)(u_{11j} - D_j \chi) \geq u^r u^s u_{ij1} u_{rs1} - (1 + \vartheta) \frac{1}{u_{11} - \chi} u_{ij} u_{11i} u_{11j} - \frac{c}{\vartheta(1 - \vartheta) u_{\eta \eta}} u_{ij} D_i \chi D_j \chi \geq -\vartheta \frac{2}{u_{11} - \chi} u_{ij} u_{11i} u_{11j} - \frac{2 c}{\vartheta u_{\eta \eta}} \left(1 + \text{tr} \left( u^{ij} \right) + |D^2 u| \right) \geq -c \vartheta (u_{11} - \chi) \left(\text{tr} \left( u^{ij} \right) + \alpha^2 \text{tr} \left( u^{ij} \right) + \gamma^2 |D^2 u| \right)
\]
\[-\frac{1}{\vartheta} \frac{c}{u_{\eta \eta}} \left(1 + \text{tr} \left( u^{ij} \right) + |D^2 u| \right).
\]
Combining this inequality with (5.10) gives
\[
0 \geq \left(2 \gamma - c \vartheta \gamma^2 - \frac{c}{\vartheta(\Delta u)^2} - c \right) \Delta u - c(1 + \alpha + \gamma) + \left(\alpha \lambda - c - c \vartheta \alpha^2 - \frac{c}{\vartheta(\Delta u)^2} - \frac{c}{\Delta u} \right) \text{tr} \left( u^{ij} \right).
\]
We fix \( \gamma, \alpha = \alpha(\gamma) \) sufficiently large, and finally \( \vartheta = \vartheta(\gamma, \alpha) \) sufficiently small. This implies an upper bound on \( u_{11} \). Note that as before, first we have fixed \( \gamma \) and then \( \alpha \).
Now, it remains to consider the case
\[(1 - \vartheta) u_{\eta\eta} \geq (u_{11} - \chi).
\]
We assume that we are in the nontrivial situation where
\[(1 - \vartheta^2) u_{\eta\eta} \geq u_{11}.
\]
Set
\[
\tilde{\Omega} \times S^{n-1} \times [0, T] \ni (x, \xi, t) \mapsto \tilde{W}(x, \xi, t) = \frac{e^{\alpha(\psi - u) + |Du|^2} (u_{\xi\xi} - \chi)}{1 + c_1 (\nu, e_1)^2 - 2(\nu, e_1)(\beta T, e_1)},
\]
where $\chi$ is introduced in (5.6). Assume that $\tilde{W}$ attains its maximum at a positive time $t_{\tilde{W}}$ at $x_{\tilde{W}} \in \tilde{\Omega}$ for a direction $\xi \in S^{n-1}$. Assume further, that $x_{\tilde{W}} \in \partial \Omega$. If $x_{\tilde{W}} \in \Omega$, a modification of the proof for the case, when $W$ attains its maximum in $\Omega \times (0, T]$, implies a bound for the second spatial derivatives of $u$. Using a decomposition of $\xi$ as in (3.1), we obtain for $\beta$ sufficiently $C^0$-close to $\nu$
\[
|\tau(\xi)|^2 \leq 1 + c \|\beta T\|_{C^0}.
\]
As a direct consequence of this decomposition, we see that
\[
u^\infty \cdot t \leq u \leq +\nu^\infty \cdot t.
\]
(6.1)
We apply Hölder estimates for the second derivatives due to Evans, Krylov, and Safonov, as well as Schauder estimates, see [3]. Since (1.1) has no explicit $u$-dependence, we get longtime existence with uniform bounds on all higher derivatives of $u$. 

So far, we have obtained uniform estimates on $\dot{u}$, $Du$, and $D^2u$ as long as a smooth solution exists. For $t = 0$, we enclose our initial value $u_0$ by translating solutions. The maximum principle implies, that $u$ will stay between the translating solutions. We obtain that
\[
(1 - \frac{\vartheta}{2}) u_{\eta\eta} \geq u_{11}.
\]
We apply Hölder estimates for the second derivatives due to Evans, Krylov, and Safonov, as well as Schauder estimates, see [3]. Since (1.1) has no explicit $u$-dependence, we get longtime existence with uniform bounds on all higher derivatives of $u$. 

To show convergence to a translating solution, we consider \( w := u - u^\infty \). The following argument is similar to \([1, 5]\). Using the mean value theorem, we see that \( w \) satisfies a parabolic flow equation of the form
\[
\begin{aligned}
\dot{w} &= a^{ij}w_{ij} + b^iw_i \quad \text{in } \Omega, \\
w_\beta &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]
Thus, the strong maximum principle implies, that \( w \) is constant or its oscillation is strictly decreasing in time. In the first case, \( u \) is already a translating solution. In the second case, we wish to exclude that the oscillation is strictly decreasing but does not tend to zero. If the oscillation of \( w(\cdot, t) \) tends to \( \varepsilon > 0 \) as \( t \to \infty \), we consider for \( t_i \to \infty \)
\[
u^i(x, t) := u(x, t + t_i) - v^\infty \cdot t_i.
\]
We have uniform estimates in any \( C^k \)-norm for the derivatives of \( \nu^i \), and locally (in time) uniform bounds for the \( C^0 \)-norm, see (6.1). Hence, a subsequence of the functions \( \nu^i \) converges locally (in time) uniformly in any \( C^k \)-norm to a solution \( u^* \) of our flow equation (1.1) that exists for all time. As the oscillation of \( w = u - u^\infty \) is monotone in \( t \), we see that the oscillation of \( u^* - u^\infty \) is equal to \( \varepsilon \), independent of \( t \). This is a contradiction to the strong maximum principle. If the oscillation of \( w \) tends to zero, we see that \( u \) converges to a translating solution in \( C^0 \) as \( t \to \infty \). Adding a constant to the translating solution \( u^\infty \), we may assume that \( u \to u^\infty \) in the \( C^0 \)-norm as \( t \to \infty \). Interpolation inequalities of the form
\[
\|Dw\|_{C^0(\overline{\Omega})}^2 \leq c(\Omega) \cdot \|w\|_{C^0(\overline{\Omega})} \cdot \left( \|D^2w\|_{C^0(\overline{\Omega})} + \|Dw\|_{C^0(\overline{\Omega})} \right)
\]
for \( w = u - u^\infty \) and its derivatives imply smooth convergence. The proof of Theorem 1.1 is complete.

**Appendix A. Prescribing Gauß curvature for entire graphs.**

Here, we present an application of our previous existence result on bounded domains to construct unbounded hypersurfaces with prescribed Gauß curvature. Assuming that the hypersurface is given as an entire graph, the problem is to find a solution of
\[
\frac{\det D^2u}{(1 + |Du|^2)^{n/4}} = g(x).
\]
(A.1)

Observe that this equation fits in the context of the present paper, cf. (4.1)
\[
\nu^\infty = \log \det D^2u - \log f(x, Du),
\]
by defining
\[
f(x, p) = g(x)/h(p) \quad \text{with} \quad h(p) = h(|p|) = (1 + |p|^2)^{-n/4}
\]
and looking for translating solutions with speed \( \nu^\infty = 0 \).
Following an argument of Altschuler and Wu [1], we will construct entire rotationally symmetric translating solutions from solutions on growing disk-type domains. Using the graph of the lower half sphere with suitably chosen radius, it is a direct consequence of the strong maximum principle that there cannot exist an entire strictly convex translating solution for any constant value of the Gauß curvature. In the rotationally symmetric case, we have the following result:

**Theorem A.1.** Let \( g(x) = g(|x|) \) be positive, smooth and integrable. There exists an entire strictly convex solution \( u \) to (A.1) if and only if

\[
v := \log \int_{\mathbb{R}^n} h(|p|) dp \int_{\mathbb{R}^n} g(|x|) dx \geq 0.
\]

The solution constructed here has a uniformly bounded gradient if and only if \( v > 0 \).

We remark that the rotationally symmetric setting does also allow for a proof by reducing the problem to an ordinary differential equation.

**Proof.** We will use that for given constants \( R > 0, \rho > 0 \), there is a unique, strictly convex solution \((v^\infty, u^\infty)\) to the following problem:

\[
\begin{cases}
  e^{v^\infty} = \det D^2 u^\infty \frac{h(|Du^\infty|)}{g(|x|)} & \text{in } B_R(0), \\
  u^\infty_\nu = -\rho & \text{on } \partial B_R(0), \\
  u^\infty(0) = 0.
\end{cases}
\]

This is a direct application of Theorem 1.2. Furthermore, the uniqueness of the solution implies its rotational symmetry. The solution can also be found by imposing a second boundary value condition.

From the strict convexity, we deduce that \( Du^\infty \) is a diffeomorphism from \( B_R(0) \) onto \( B_\rho(0) \). Integrating (\( \ast_{R, \rho} \)), we obtain

\[
e^{v^\infty} \int_{B_R} g(|x|) dx = \int_{B_R} \det D^2 u^\infty h(|Du^\infty|) dx = \int_{B_\rho} h(|p|) dp,
\]

which uniquely determines the speed

\[
v^\infty = v^\infty(R, \rho) := \log \frac{\int_{B_\rho} h(|p|) dp}{\int_{B_R} g(|x|) dx},
\]

as a function of the parameters \( R, \rho \). Note that \( v^\infty(R, \rho) \) is strictly decreasing in \( R \) and strictly increasing in \( \rho \) with \( v^\infty(R, \rho) \rightarrow -\infty \) for \( \rho \rightarrow 0 \).

1. Nonexistence for \( v < 0 \): We argue similarly as in the aforementioned case of constant Gauß curvature. Now, we replace lower half spheres by suitably constructed solutions on finite domains with arbitrarily large gradients at the boundary. Since \( v < 0 \), there exists a unique \( \hat{R} \) such that \( \int_{B_{\hat{R}}} g(|x|) dx = \int_{\mathbb{R}^n} h(|p|) dp \). Assuming that we have an entire solution \( u \) of (A.1), there is a
\( \rho > 0 \) such that \( |Du(x)| < \rho/2 \) for all \( |x| < \hat{R} \). Now, take the unique \( R < \hat{R} \) satisfying
\[
\int_{B_R} g(|x|)dx = \int_{B_\rho} h(|p|)dp
\]
and consider the solution \((v^\infty, u^\infty)\) of \((\ast_{R, \rho})\). By definition, we know that \( v^\infty = v^\infty(R, \rho) = 0 \), hence, \( u^\infty \) solves Equation (A.1) in \( B_R(0) \). Furthermore, the Neumann boundary condition and our choice of \( \rho \) yield \( |Du| < \rho/2 \) for all \( |x| < \hat{R} \). Thus, there is a translate \( u^\infty + m, m \in \mathbb{R} \), which is strictly greater than \( u \). Now we shift back until the graphs touch first at a point \( x \in B_R(0) \). By the strong maximum principle, this is impossible as \( u \) and \( u^\infty \) solve the same elliptic equation in \( B_R(0) \).

2. Existence for \( v \geq 0 \): We construct our solution by choosing a sequence of increasing radii \( R_k \) tending to \( \infty \). By the monotonicity properties of the function \( v^\infty(R, \rho) \), we can find for each \( R > 0 \) a unique \( \rho_R \) such that \( v^\infty(R, \rho_R) = 0 \). We remark that \( \rho_R \) is an increasing sequence. Again, Theorem 1.2 gives a unique smooth rotationally symmetric solution \( u_R \) to \((\ast_{R, \rho_R})\), which is defined on \( B_R(0) \) and satisfies \( v^\infty = 0 \). Note that for fixed \( R \), the speed \( v^\infty \) and the normal derivative at the boundary, \( -\rho_R \), are uniquely related. Hence, the solution \( u_R \) must coincide on smaller balls \( B_{R'}(0), R' < R \), with the previous solutions \( u_{R'} \) to \((\ast_{R', \rho_R})\). Therefore, as \( R \) tends to \( \infty \), the sequence \( \{u_R\} \) will converge uniformly on compact sets to a limit \( u \), defined on all of \( \mathbb{R}^n \). Clearly, \( u \) is a rotationally symmetric solution to (A.1). Observe that the sequence \( \rho_R \) will diverge in the case \( v = 0 \), whereas it stays bounded for \( v > 0 \). This proves the boundedness of \( |Du| \) in the latter case. \( \square \)

Proceeding as in the existence part of the proof, we get easily that non-integrable \( g(x) \) also allow for solutions provided that \( h(p) \) is non-integrable too.

This observation can be extended to the function \( h(p) \) arising in the equation of prescribed Gauß curvature in Minkowski space
\[
(A.2) \quad \frac{\det D^2 u}{(1 - |Du|^2)^{\frac{n+2}{2}}} = g(x).
\]
Hence, \( h(p) = h(|p|) = (1 - |p|^2)^{-\frac{n+2}{2}} \), which is not integrable on \( B_1(0) \).

**Theorem A.2.** For all positive and smooth functions \( g(x) = g(|x|) \), there exists an entire strictly convex solution \( u \) to (A.2) satisfying \( |Du| < 1 \). Moreover, for a solution constructed here, \( |Du| \leq 1 - \varepsilon, \varepsilon > 0 \), if and only if \( g \) is integrable.
Proof. We proceed similarly as in the proof of Theorem A.1 and use the notation introduced there. Here, the speed function \( v^\infty(R, \rho) \) is only defined for \( \rho < 1 \). But still, \( v^\infty(R, \rho) \) is strictly decreasing in \( R \) and strictly increasing in \( \rho \). In addition, \( v^\infty(R, \rho) \to -\infty \) for \( \rho \to 0 \) and \( v^\infty(R, \rho) \to \infty \) for \( \rho \to 1 \). As in part 2 of the proof of Theorem A.1, we can find for any \( R > 0 \) a unique \( \rho_R \in (0, 1) \) satisfying \( v^\infty(R, \rho_R) = 0 \). Since \( \rho_R < 1 \), we can choose a smooth function \( \tilde{h}(p) \), defined on \( \mathbb{R}^n \), such that \( h(p) = \tilde{h}(p) \) for all \( |p| \leq \rho_R \). Again, Theorem 1.2 gives a unique smooth convex rotationally symmetric solution \( u_R \) to \((R, \rho_R)\) with \( h \) replaced by \( \tilde{h} \), which is defined on \( B_R(0) \) and satisfies \( v^\infty = 0 \). We remark that the convexity of \( u \) implies that \( |Du| \leq \rho_R \) on \( B_R(0) \). Thus, \( u \) also solves the elliptic equation, if we replace \( \tilde{h} \) by the original \( h \). Again, for \( R > R' \), \( u_R \) will coincide with solutions \( u_{R'} \) obtained on smaller balls \( B_{R'}(0) \). Hence, for \( R \) tending to infinity, \( u_R \) converges to an entire solution \( u \) of (A.2). In the present case, the sequence \( \rho_R \) stays uniformly bounded away from 1 if \( \int_{\mathbb{R}^n} g(|x|) dx < \infty \), whereas \( \rho_R \) converges to 1 if \( g \) is non-integrable. \( \square \)

In the general case without rotational symmetry, a theorem corresponding to Theorem 1.1 in Minkowski space can be obtained easily from the techniques of this paper, provided that there holds a uniform a priori bound of the form \( |Du| < 1 - \varepsilon, \varepsilon > 0 \).

Appendix B. Illustrations.

To illustrate the convergence of solutions, we investigate numerically the flow equation

\[
\begin{align*}
\dot{u} &= \log \det D^2 u \quad \text{in } \Omega \times (0, \infty), \\
u(\cdot, t) &= (u_0)_{\nu} \quad \text{on } \partial \Omega, \ t > 0, \\
u(\cdot, 0) &= u_0 \quad \text{in } \Omega
\end{align*}
\]

on the ellipsoidal domain \( \Omega = \{(x, y) \in \mathbb{R}^2 : 1.1 \cdot (x^2 + (2y)^2) < 1\} \), where \( u_0(x, y) = 1.5x^2 + y^2 - 0.1y^4 \).

The numerical integration has been carried out on a \( 200 \times 100 \) grid corresponding to \([-1, 1] \times [-0.5, 0.5] \subset \mathbb{R}^2 \). Let \( \Omega_0 \) consist of all grid points contained in \( \Omega \), and \( \partial \Omega_0 \) denotes those grid points not contained in \( \Omega_0 \) such that one of the nearest neighbors belongs to \( \Omega_0 \). For simplicity, we keep the same notation for the discretized quantities.

We use an explicit scheme for time integration. The boundary condition is implemented as follows: For all \( x_0 \in \partial \Omega \) let \( y_0 := x_0 + \nu(x_0) \cdot \tau_0 \), where \( \nu(x_0) \) is the normalized negative gradient of \( x^2 + (2y)^2 \) and \( \tau_0 = \inf \{ \tau : x_0 + \nu(x_0) \cdot \tau \in \text{convex hull}(\Omega_0) \} \). We set \( u(x_0) := u_0(x_0) - u_0(y_0) - u(y_0) \). Here, \( u(y_0) \) is obtained by linear interpolation.
Figure 2. Time evolution on an ellipsoidal domain.

Figure 2 shows a gray-scale plot of the velocity \( \dot{u} \) at different times. It can be seen that the velocity tends to a constant, reflecting the convergence of \( u \) to a translating solution.

![Figure 2](image)

Figure 3. Convergence to constant velocity.

In Figure 3a, we show the decay of \( \delta(t) = ||\dot{u}(t) - \overline{v}(t)||_{L^2(\Omega)}^2 \), where \( \overline{v}(t) = \frac{1}{|\Omega|} \int \dot{u}(x,t) \, dx \) is the mean velocity. The expected exponential convergence can be seen from Figure 3b. Here, we plot the exponential rate \( \frac{1}{t} \log \delta(t) \), which saturates nicely for larger times.

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References


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ON A THEOREM OF LANCASTER AND SIEGEL

DANZHU SHI AND ROBERT FINN

In establishing conditions for continuity of the height of a capillary surface \( f(x, y) \) at a re-entrant corner point of the domain of definition, Lancaster and Siegel introduced a hypothesis of symmetry, which does not appear in corresponding conditions for a protruding corner. We show here that the hypothesis cannot be discarded. Starting with a symmetric configuration for which the surface height is continuous at the corner point in accordance with the hypotheses of those authors, we show that the height can be made discontinuous by an asymmetric domain perturbation that is in an asymptotic sense arbitrarily small, and for which all hypotheses other than that of symmetry remain in force.

In a downward directed gravity field, the height \( f(x, y) \) of a capillary surface interface is determined, up to an additive constant depending on an eventual volume constraint, as a solution of

\[
\nabla \cdot Tf = \kappa f, \quad Tf = \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}
\]

(1)

corresponding to a (physical) constant \( \kappa > 0 \). On the boundary \( \partial \Omega \) one finds the condition

\[
\nu \cdot Tu = \cos \gamma
\]

(2)

where \( \nu \) is unit exterior normal, and \( \gamma \) the (prescribed) contact angle between the solution surface and the bounding cylinder walls over \( \partial \Omega \). We may assume \( 0 \leq \gamma \leq \pi \). The unique existence of solutions of (1) and (2) in a piecewise smooth bounded domain \( \Omega \) follows from the material in [2], [4], [7] and [8]. It should be noted that Condition (2) is imposed only on the smooth portions of the boundary; no condition is required at corner points, nor is there any a-priori growth restriction at those points.

Lancaster and Siegel [6] have characterized remarkable properties of radial limits \( Rf \) of solutions \( f(x, y) \) at eventual corner points \( O \in \partial \Omega \). The limit exists for every direction of approach at the corner points, but its dependence on direction can vary considerably, depending on circumstances. By Corollary 1 of [6], if \( \partial \Omega \) is of class \( C^{(1)} \) except at the corner points, then the solutions are continuous up to all boundary points with the possible exception of corner points.
For a protruding corner of opening $2\alpha < \pi$, Corollary 4 of [6] yields that for all $\gamma$ in the range $\pi/2 - \alpha < \gamma < \pi/2 + \alpha$, $Rf$ is finite and is independent of direction, and thus $f$ is continuous at the corner. If $\gamma$ lies exterior to this range, it was shown by Concus and Finn [1] that no bounded solution exists.

For a re-entrant corner of opening $2\alpha > \pi$, the situation is more complicated. It follows from Theorem 5.2 of [3] that all solutions are bounded at such a point; however Korevaar showed [5] that solutions can be discontinuous at re-entrant corner points. Lancaster and Siegel proved in [6]:

**Corollary 2.** Suppose $\pi/2 \leq \alpha < \pi$ and that $\Omega$ is symmetric under reflection in the $x$-axis, with the $x$-axis bisecting the corner point. Suppose further that either

$$\alpha - \pi/2 \leq \gamma \leq \pi/2 \quad \text{or} \quad \pi/2 \leq \gamma \leq 3\pi/2 - \alpha.$$  

Then the (uniquely determined) solution $f(x,y)$ of (1) and (2) in $\Omega$ is symmetric under reflection in the $x$-axis, and continuous at $O$.

Condition (3) is analogous to the condition of Corollary 4 just mentioned for a protruding corner. The authors show that also in this case their result is sharp with respect to that condition, by providing a counterexample under the given symmetry when $\gamma$ is outside the specified range. However symmetry was not needed for Corollary 4, and the question arises, as to the extent it is essential for Corollary 2. This question is not addressed in [6].

In the present note, we show that in general (3) does not suffice for continuity, and the symmetry hypothesis cannot be significantly relaxed. In fact, in an asymptotic sense to be defined, there exist configurations that deviate arbitrarily little from symmetry, for which (3) holds and for which the solution $f(x,y)$ fails to be continuous at $O$.

We construct an example of such behavior as a modification of the counterexample just indicated. We assume $\pi/2 < \alpha < \pi$, and consider the case $\gamma = \alpha - \pi/2$, which satisfies (3). As starting point we repeat some steps in the reasoning of [6].

Figure 1 illustrates a particular configuration. This is basically Figure 10 in [6], the only change significant for us being that two vertical boundary lines in that figure are now introduced with finite slopes that are negatives of each other.

As in [6], we introduce the disk $D \subset \Omega$ of radius $R_0$, tangent to the $y$-axis at $O$. By Theorem 5.2 of [5], any solution $f(x,y)$ of (1) in $D$ satisfies

$$\sup_D f < \frac{2}{\kappa R_0} + R_0$$  

throughout $D$ (one needs a slightly strengthened version of the theorem, which is routine to obtain). The choice $R_0 = \sqrt{2/\kappa}$ yields the best possible
bound from (4); we restrict attention to domains $\Omega$ that contain this disk, and we are led to

$$\sup_{D} f < 2\sqrt{\frac{2}{\kappa}}$$

in $D$. Thus, (5) holds also for the radial limit $Rf$ along any line segment within $D$ leading to $O$.

We denote by $\varepsilon$ the distance between the lines $L$ and $L^*$ of Figure 1. We choose $\gamma^*$ in the range $\gamma \leq \gamma^* < \pi - \gamma$ and introduce the two symmetrically placed circular arcs $C_1, C_2$ of equal radii $r$, determined by

$$\varepsilon = r(\cos \gamma + \cos \gamma^*)$$

and meeting the line $L$ in angle $\gamma$. These arcs meet $L^*$ in the angle $\gamma^*$. We construct a torus $T$ of sectional radius $r$ which contains $C_1, C_2$, and we denote by $2R$ the distance between the centers of $C_1, C_2$. Consider the portion of $T$ that lies above the plane of the arcs, and choose the underside $g(x, y)$ of that surface as a “comparison surface”. This surface has the general appearance of a footbridge in a Japanese garden. Note that it is vertical on the circles $C_1, C_2$, and that it meets vertical walls through $L, L^*$ in the constant angles $\gamma, \gamma^*$. The surface $g(x, y)$ has at each of its points a mean curvature

$$H_g \geq \frac{1}{2} \left( \frac{1}{r} - \frac{1}{R-r} \right),$$

and $\cos \gamma^* \leq \cos \gamma$. Theorem 5.1 of [5] now yields that in the domain $D_T \subset \Omega$ onto which $T$ projects, there holds

$$\inf_{D_T} f > \frac{1}{\kappa} \left( \frac{1}{r} - \frac{1}{R-r} \right) - R.$$

We now change the perspective somewhat. Instead of starting with the channel width $\varepsilon$ and angle $\gamma^*$ as above, we choose $r = r_0$ so that

$$\frac{1}{\kappa} \left( \frac{1}{r_0} - \frac{1}{R-r_0} \right) - R > 2\sqrt{\frac{2}{\kappa}}.$$

We retain this choice for $r$ throughout the ensuing discussion. The choice imposes a relation between $\varepsilon$ and $\gamma^*$, according to (6), which is formally satisfied, for any $\varepsilon$ in the range $0 < \varepsilon \leq r_0(1 + \cos \gamma)$, by a unique value $\gamma^*$ for which $0 \leq \gamma^* < \pi - \gamma$. We observe that any $r_0$ chosen as above satisfies $r_0 < R_0$.

We would like to position $C_2$ so that the intersection with $L$ will be at $O$. From the relations (5), (8) and (9) we could then conclude that $f(x, y)$ is discontinuous at $O$. In the configuration of Figure 1 that cannot be done, as the other endpoint of $C_2$ would then no longer meet $L^*$, and Theorem 5.1 of [5] could no longer be applied. That is of course consistent with the
original result of [2], that in the symmetric configuration the solution must be continuous at $O$.

By abandoning symmetry, we can however modify the configuration to permit the construction. We tilt the $y$-axis clockwise through an angle $\delta$ to be determined, so that it becomes an inclined line $L^\delta$, and we shift the disk $D$ so that it is tangent to $L^\delta$ at $O$. We then extend $L^\gamma$ until it meets $\partial D$ at a point $p(\delta)$, and we shorten $L^{**}$ so that it does not extend beyond $L^\delta$. Finally, we choose $\delta$ so that the two circles $C_2$ and $\partial D$ meet at that same point $p(\delta)$, when $C_2$ meets $L$ at $O$. If $\delta$ is small enough, $D$ will continue to lie within $\Omega$. We will show that such a choice is always available, corresponding to the value of $r = r_0$ chosen above, and with $\gamma^*$ within the required range, if $\varepsilon$ is sufficiently small. For the moment we postpone verification of this requirement.

Local details of the construction are indicated in Figures 2 and 3. In this configuration, $T$ can be positioned to contain the arc $C_2$, and a segment extending to $O$ can be found within $D_T$. On such a segment, the lower bound (8) holds. However, on any segment extending to $O$ within $D$ we have the upper bound (5). In view of (9), the asserted discontinuity of $f$ at $O$ follows.

But all conditions for Corollary 2 are fulfilled except for symmetry. Subject to verification that $\delta$ is determined by the requirements and tends to zero with $\varepsilon$, we find that the symmetry hypothesis of the corollary is necessary for continuity at $O$.

The formal analytical criterion for determining $\delta(\varepsilon)$ is contained in the equations

\begin{align}
\cos \gamma + \cos \gamma^* &= \frac{\varepsilon}{r_0} \\
\cos(\gamma + \delta) + \cos(\gamma^* + \delta) &= \frac{\varepsilon}{R_0}.
\end{align}

Here (10a) is a repetition of (6); it follows immediately from the construction of $C_1, C_2$. We obtain (10b) similarly using the placement of $D$ and additionally that $C_2$ and $\partial D$ meet at $O$ and at $p$ in equal angles; see Figure 3 for the configuration.

Keeping $r_0$ fixed, we let $\varepsilon \to 0$, and observe that $\gamma^*$, as determined from (10a), then increases toward $\pi - \gamma$. Specifically, from (10a) a smooth function branch

$$
\gamma^* = \varphi(\varepsilon) \equiv \arccos \left( \frac{\varepsilon}{r_0} - \cos \gamma \right)
$$

is determined, for which $\varphi(0) = \pi - \gamma$. For this branch, we find

$$
\varphi'(\varepsilon) = -1/r_0 \sin \varphi(\varepsilon)
$$

and thus $\varphi'(0) = -1/r_0 \sin \gamma$. We conclude $\gamma^* < \pi - \gamma$, and $\gamma^* \geq \gamma$ if $\varepsilon$ is small enough. We place the function (11) into (10b), and solve the resulting
implicit relation for a branch $\delta(\varepsilon)$, with
\[
\delta(\varepsilon) = \arccos \left( \frac{\varepsilon}{2R_0 \cos \frac{\gamma - \gamma^*}{2}} \right) - \frac{\gamma + \gamma^*}{2},
\]
such that $\delta(0) = 0$. Since $\varphi(0) = \gamma = \pi - 2\gamma$ and $0 < \pi - 2\gamma < \pi$, this procedure determines the required $\delta(\varepsilon)$ explicitly.

For the distance $d(\varepsilon)$ to which $L^*$ must be extended to meet $L^\delta$, we obtain after some manipulation, using Figure 3,
\[
d(\varepsilon) = \varepsilon \frac{\cos \left( \frac{\gamma + \gamma^*}{2} \right)}{\sin \gamma \cos \left( \frac{\gamma - \gamma^*}{2} \right)}.
\]

The denominator in (14) is harmless, as just observed. For the numerator, we have
\[
\cos \frac{\gamma + \gamma^*}{2} = \left[ 1 + \cos \left( \gamma + \gamma^* \right) \right]^{\frac{1}{2}}
\]
\[= \frac{\sqrt{2}}{2} \left[ 1 + \cos \gamma \cos \gamma^* - \sin \gamma \sin \gamma^* \right]^{\frac{1}{2}}.
\]

In view of (10a) we obtain
\[
\cos \frac{\gamma + \gamma^*}{2} = \frac{\sqrt{2} \sin \gamma}{2} \left[ 1 + \frac{\varepsilon \cos \gamma}{r_0 \sin^2 \gamma} - \left( 1 + \frac{2\varepsilon \cos \gamma}{r_0 \sin^2 \gamma} - \frac{\varepsilon^2}{r_0^2 \sin^2 \gamma} \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}.
\]

From the inequality $\sqrt{1 + x} > 1 + (x/2) - (x^2/8)$, $x > 0$, applied to the inner root, we are led to
\[
\cos \frac{\gamma + \gamma^*}{2} < C \varepsilon
\]
for a fixed constant $C$, as $\varepsilon \to 0$, and hence $d(\varepsilon) = O(\varepsilon^2)$, from (14). A similar estimate applies to the amount $d^*$ by which $L^*$ must be shortened. Thus $d$ and $d^*$ decrease faster than $\varepsilon$. Under a coordinate normalization holding the width of the strips constant as $\varepsilon \to 0$, the new distance $\tilde{d}(\varepsilon)$ retains the order $\tilde{d}(\varepsilon) = O(\varepsilon)$. It is in that asymptotic sense that there exist configurations that deviate arbitrarily little from symmetry, for which the solution $f(x, y)$ fails to be continuous at $O$, as asserted above. This behavior is also evident geometrically, directly from the nature of the construction.

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Figure 1. Domain $\Omega$ leading to continuity at $O$ of capillary surface with contact angle $\gamma = \alpha - \pi/2$. 

**Figure 1.** Domain $\Omega$ leading to continuity at $O$ of capillary surface with contact angle $\gamma = \alpha - \pi/2$. 
Figure 2. Details of construction for example of discontinuity at $O$. The two circles are tangent respectively to the vertical and to $L^\delta$ at $O$, and both pass through $p$. 

References

Figure 3. Further details of the construction.

Figure 3. Further details of the construction.


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THE MACROSCOPIC SOUND OF TORI

Constantin Vernicos

Take a torus with a Riemannian metric. Lift the metric on its universal cover. You get a distance which in turn yields balls. On these balls you can look at the Laplacian. Focus on the spectrum for the Dirichlet or Neumann problem. We describe the asymptotic behaviour of the eigenvalues as the radius of the balls goes to infinity, and characterise the flat tori using the tools of homogenisation our conclusion being that “Macroscopically, one can hear the shape of a flat torus”. We also show how in the two dimensional case we can recover earlier results by D. Burago, S. Ivanov and I. Babenko on the asymptotic volume.

1. Introduction and claims.

Let \((T^n, g)\) be a Riemannian torus, lift its metric on its universal cover and use it to define first a distance, then the metric’s balls. The first thing one can observe is the volume of these balls as a function of their radius, indeed as the distance obtained arises from a compact quotient it is equivalent to an Euclidean distance hence the volume of these balls is equivalent to the Euclidean volume of an Euclidean ball i.e., proportional to the radius of the ball to the power of \(n\) (the dimension of our torus).

We are thus naturally led to wonder what happens if one looks at the following Riemannian function on the balls \((\mathcal{B}_g(\rho))\) is the ball of radius \(\rho\):

\[
\frac{\text{Vol}_g(\mathcal{B}_g(\rho))}{\rho^n} \quad \text{as} \quad \rho \to +\infty.
\]

If it is not very surprising that it converges to some constant for this limit can be seen as a mean value due to the periodicity of the metric (see for example Pansu [Pan82] and a slightly different and more analytical proof in this paper Section 2.3), it is quite remarkable that this constant, called asymptotic volume, is bounded from below by the constant arising from the flat tori and furthermore that the case of equality characterises the flat tori as D. Burago and S. Ivanov showed in [BI95].

The study of the balls of large radii on the universal cover of tori (and more generally of a nilmanifold) is what we call here the macroscopical geometry. Indeed in our case the universal cover is a real vector space, where
some lattices acts by translation (in the more general case of nilmanifolds one should consider a left action). Should one focus on the point of this lattice endowed with the distance arising from the torus, one gets an invariant metric on the lattice. Now if one looks at this lattice from a galaxy far, far away, one won’t be able to distinguish the lattice from the whole universal cover. Thus it is understandable that for this observer the distance obtained on the universal cover seems invariant by all translations (for general nilmanifolds one gets a left invariant distance).

In the case of tori this “seen from a far away galaxy” distance is a norm, called the stable norm and was first defined by Federer in homology. It is some kind of mean value of the metric. This asymptotic behaviour was generalized and proved by P. Pansu for all nilmanifold [Pan82] and precised by D. Burago [Bur92] for tori. Since then the stable normed appeared in many other works: For surfaces and the links with Aubry-Mather theory in D. Massart’s works, one can also find it in the weak KAM theory of A. Fathi. It is also worth mentioning the crucial role it plays in the proof by D. Burago and S. Ivanov [BI94] of the Hopf conjecture concerning tori without conjugate points. Here in Part 2 we show, for the case of tori, how one recovers the stable norm using homogenisation tools.

There is another interesting geometric invariant attached to the balls and linked with the volume, the spectrum of the Laplacian. Indeed if one knows the spectrum one knows the volume thanks to Weyl’s asymptotic formula. Here again one easily sees, comparing with the Euclidean case, that the eigenvalues converge to zero with a $1/\rho^2$ speed ($\rho$ being the radius). If one can expect a convergence when rescaled, it is quite surprising that as a limit we obtain the spectrum of an Euclidean and not a finsler metric, indeed the behaviour is described by the following theorem which is one of the aims of this paper:

**Theorem 1.** Let $(\mathbb{T}^n, g)$ be a Riemannian torus, $B_g(\rho)$ the induced metric ball on its universal cover and $\lambda_i(B_g(\rho))$ the $i^{th}$ eigenvalue of the Laplacian for the Dirichlet (resp. Neumann) problem.

There exists an elliptic operator $\Delta_{\infty}$, which is the Laplacian of some Euclidean metric on $\mathbb{R}^n$, such that if $\lambda_i^{\infty}$ is its $i^{th}$ eigenvalue for the Dirichlet (resp. Neumann) problem on the stable’s norm unit ball then

$$\lim_{\rho \to +\infty} \rho^2 \lambda_i(B_g(\rho)) = \lambda_i^{\infty}.$$
**Theorem 2.** Let \((\mathbb{T}^n, g)\) be a Riemannian torus, \(B_g(\rho)\) the induced metric ball on its universal cover and \(\lambda_1(B_g(\rho))\) the first eigenvalue of the Laplacian for the Dirichlet problem. Then:

1. \(\lim_{\rho \to +\infty} \rho^2 \lambda_1(B_g(\rho)) = \lambda_1^\infty \leq \lambda_{e,n}\).
2. equality holds if, and only if, the torus is flat,

where \(\lambda_{e,n}\) is the first eigenvalue of the Euclidean Laplacian on the Euclidean unit ball.

The proof, which is done in Section 6, involves some kind of transplan-
tation for the inequality mixed with \(\Gamma\)-convergence for the equality. For a better understanding of what happens we briefly give some informations related to the \(\Gamma\)-convergence and adapt it to our purpose in Section 5, following the general ideas of K. Kuwae and T. Shioya in [KS] (who in turn generalized U. Mosco’s paper [Mos94]), this section being completed by the proof of Section 8.

As the macroscopical spectrum involved rises from an Euclidean metric, we can use the Faber-Krahn inequality to obtain a new inequality regarding the asymptotic volume, this is done in Section 7.1:

**Proposition 3.** Let \((\mathbb{T}^n, g)\) be a Riemannian torus, \(B_g(\rho)\) the geodesic balls of radius \(\rho\) centred on a fixed point and \(\text{Vol}_g(B_g(\rho))\) their Riemannian volume induced on the universal cover, writing

\[
\text{Asvol}(g) = \lim_{\rho \to \infty} \frac{\text{Vol}_g(B_g(\rho))}{\rho^n}
\]

then:

1. \(\text{Asvol}(g) \geq \frac{\text{Vol}_g(\mathbb{T}^n)}{\text{Vol}_{\text{Al}}(\mathbb{T}^n)} \omega_n\).
2. In case of equality, the torus is flat.

Here \(\omega_n\) is the unit Euclidean ball’s Euclidean volume, and \(\text{Vol}_{\text{Al}}(\mathbb{T}^n)\) is the volume of the Albanese torus.

A surprising fact arises because this new inequality involves a constant which happened to be at the heart of the isosystolic inequality of two dimensional tori (see J. Lafontaine [Laf74]), hence we obtain an alternate proof of the asymptotic volume’s lower boundedness in dimension two:

**Corollary 4.** Let \((\mathbb{T}^2, g)\) be a 2-dimensional torus then:

1. \(\text{Asvol}(g) \geq \pi\).
2. In case of equality, the torus is flat.

It is worth mentioning that the case of equality in the previous claims relies on Theorem 33, which states that the stable norm coincides with the Albanese metric if and only if the torus is flat, and whose proof does not rely on the work of D. Burago and S. Ivanov [BI95] or I. Babenko [Bab91].
Thus we actually get an alternate proof of this theorem in the 2-dimensional case.

We also give a kind of generalised Faber-Krahn inequality for normed finite dimensional vector spaces, which implies that we cannot distinguish the Euclidean’s ones among them using the first generalised eigenvalue of the Dirichlet Laplacian (see Lemma 36 and its corollary):

**Theorem 5** (Faber-Krahn inequality for norms). Let $D$ be a domain of $\mathbb{R}^n$, with the norm $\| \cdot \|$ and a measure $\mu$ invariant by translation. Let $D^*$ be the norm’s ball with same measure as $D$, then

$$\lambda_1(D^*,\| \cdot \|) \leq \lambda_1(D,\| \cdot \|).$$

We finally explain in Section 7.2 how is our work related to works focused on the long time asymptotics of the heat kernel (see [KS00], [DZ00], [ZKON79]) and finally in Section 7.3 we state how Theorem 1 transposes to all graded nilmanifolds (subject which should be widely extended in a forthcoming article).

### 2. Stable norm and homogenisation.

In this section we show how the stable norm, the Gromov-Hausdorff convergence and the $\Gamma$-convergence of the homogenisation theory are linked and finish by re-spelling our goal. In what follows, $B_\rho(\rho)$ will be the metric ball of radius $\rho$ on the universal cover of a torus with the lifted metric. We first begin by two definitions.

#### 2.1. Convergences.

We recall the definition of $\Gamma$-convergence in a metric space:

**Definition 6.** Let $(X,d)$ be a metrics space. We say that a sequence of function $(F_j)$ from $X$ to $\mathbb{R}$, $\Gamma$-converges to a function $F : X \to \mathbb{R}$ if and only if for all $x \in X$ we have:

1. For all converging sequences $(x_j)$ to $x$
   
   $$F(x) \leq \liminf_{j \to \infty} F_j(x_j);$$

2. there exists a sequence $(x_j)$ converging to $x$ such that
   
   $$F(x) = \lim_{j \to \infty} F_j(x_j).$$

We now introduce the Gromov-Hausdorff measured convergence in the space $\mathcal{M}$ of compact metric and measured spaces $(X,d,m)$ modulo isometries. First if $X$ and $Y$ are in $\mathcal{M}$ then an application $\phi : X \to Y$ is called an $\epsilon$-Hausdorff approximation if and only if we have:

1. The $\epsilon$-neighbourhood of $\phi(X)$ in $Y$ is $Y$;
(2) for all \( x, y \in X \) we have
\[
|d(x, y) - d(\phi(x), \phi(y))| \leq \epsilon.
\]

We write \( C^0(X) \) for the space of continues functions from \( X \) to \( \mathbb{R} \) and \( A \) will be a partially directed space.

**Definition 7.** We say that a net \((X_\alpha, d_\alpha, m_\alpha)_{\alpha \in A}\) of spaces in \( \mathcal{M} \) converges to \((X, d, m)\) for the Gromov-Hausdorff measured topology if, and only if there exists a net of positive real numbers \((\epsilon_\alpha)_{\alpha \in A}\) decreasing to 0 and \(m_\alpha\) measurable \(\varepsilon_\alpha\)-Hausdorff approximations \(f_\alpha : X_\alpha \to X\) such that \((f_\alpha)_*(m_\alpha)\) converges vaguely to \(m\), i.e.,
\[
\int_{X_\alpha} u \circ f_\alpha \, dm_\alpha \to \int_X u \, dm \quad \forall u \in C^0(X).
\]

**2.2. The stable norm.** Let \((\mathbb{T}^n, g)\) a Riemannian torus. We will call rescaled metrics the metrics \(g_\rho = (1/\rho^2)(\delta_\rho)^* g\) and their lifts on the universal cover. We will also write \(\delta_\rho\) for the homothety of scale \(\rho\).

In the 80’s P. Pansu showed that the distance induced on \(\mathbb{R}^n\) as a universal cover of a torus, behaved asymptotically like the distance induced by a norm. In the 90’s D. Burago showed a similar result for periodic metrics on \(\mathbb{R}^n\). It is that norm which is called the stable norm. To be more precise let us write \(f_1(x) = d_g(0, x)\) the distance from the origin to \(x\) and \(f_\rho(x) = d_g(0, \delta_\rho(x))/\rho\), then P. Pansu’s result says that there exist a norm \(\|\cdot\|_\infty\) such that for all \(x \in \mathbb{R}^n\)
\[
\lim_{\rho \to \infty} f_\rho(x) = \|x\|_\infty
\]
and Burago’s says that there exists a constant \(C\) such that for all \(x \in \mathbb{R}^n\)
\[
|f_\rho(x) - \|x\|_\infty| \leq \frac{C}{\rho}
\]
in other words, Pansu’s results is a simple convergence and Burago’s is a uniform convergence result.

There is another proof of the simple convergence of the sequence \((f_\rho)\) as \(\rho\) goes to infinity, using homogenisation tools.

**Theorem 8.** Let \(\tilde{g}\) the induced metric on \(\mathbb{R}^n\) as a universal cover of a Riemannian torus \((\mathbb{T}^n, g)\). Then there exists a norm \(\|\cdot\|_\infty\) such that:

1. For every bounded open \(I \subset \mathbb{R}\) the sequence of functionals
\[
E_\rho(u) = \int_I \tilde{g}(\delta_\rho u(t))(u'(t), u'(t)) \, dt
\]
on \(W^{1,2}(I; \mathbb{R}^n), \Gamma\)-converge for the \(L^2\) norm toward the functional
\[
E_\infty(u) = \int_I \|u'(t)\|_\infty^2 \, dt;
\]
(2) the norm satisfies

\[ \| \xi \|_\infty = \lim_{t \to +\infty} \inf \left\{ \frac{1}{t} \int_0^t g(u+\xi \tau)(u' + \xi, u' + \xi) \; d\tau : u \in W^{1,2}_0([0, t]; \mathbb{R}^n) \right\}. \]

Furthermore if \( f_\rho(x) = d_\rho(0, \rho x)/\rho \) then for all \( x \in \mathbb{R}^n \)

\[ \lim_{\rho \to +\infty} f_\rho(x) = \| x \|_\infty. \]

Proof. We use Proposition 16.1, p. 142 of A. Braides and A. Desfranceschi [BD98]. It gives us the \( \Gamma \)-convergence of the sequence of functional \( (E_\rho) \) toward a functional \( E_\infty \) such that

\[ E_\infty(u) = \int_I \varphi(u'(t))dt \]

with \( \varphi \) convex and satisfying the asymptotic formula (1). It remains to show that \( \varphi \) is the square of a norm.

**Homogeneity:** Using the asymptotic formula (1) we easily get \( \varphi(0) = 0 \) and by a change of variables \( \varphi(\lambda x) = \lambda^2 \varphi(x) \).

**Separation:** Let us point out that:

1) The minimum of the energy of a path between 0 and \( t\xi \) in an Euclidean space is attained for the straight line. Thus if we put into the asymptotic formula (1) an Euclidean metric, we get the same metric.

2) Let \( g \) and \( h \) be two metrics such that for all \( s \) and \( \xi \)

\[ g_s(\xi, \xi) \leq h_s(\xi, \xi) \]

then for all \( u \in W^{1,2}_0([0, t]; \mathbb{R}^n) \) we get

\[ \frac{1}{t} \int_0^t g(u+\xi \tau)(u' + \xi, u' + \xi) \; d\tau \leq \frac{1}{t} \int_0^t h(u+\xi \tau)(u' + \xi, u' + \xi) \; d\tau \]

thus taking the infimum for \( u \) and taking the limit as \( t \) goes to infinity we get

\[ \lim_{t \to +\infty} \inf \left\{ \frac{1}{t} \int_0^t g(u+\xi \tau)(u' + \xi, u' + \xi) \; d\tau : u \in W^{1,2}_0([0, t]; \mathbb{R}^n) \right\} \]

\[ \leq \lim_{t \to +\infty} \inf \left\{ \frac{1}{t} \int_0^t h(u+\xi \tau)(u' + \xi, u' + \xi) \; d\tau : u \in W^{1,2}_0([0, t]; \mathbb{R}^n) \right\}. \]

Now let us also remark that \( g \) being periodic, there exists two strictly positive constants \( \alpha \) and \( \beta \) such that

\[ \alpha |\xi|^2 \leq g_s(\xi, \xi) \leq \beta |\xi|^2 \]

now applying the three remarks we get

\[ \alpha |\xi|^2 \leq \varphi(\xi) \leq \beta |\xi|^2 \]

thus \( \varphi(\xi) = 0 \) if and only if \( \xi = 0 \).
Triangle inequality: First note that
\[ \{ \xi \in \mathbb{R}^n \mid \varphi(\xi) \leq 1 \} = \{ \xi \in \mathbb{R}^n \mid \sqrt{\varphi(\xi)} \leq 1 \} = S_n. \]

It follows that if \( \sqrt{\varphi}(\xi) = 1 = \varphi(\xi) \) and \( \sqrt{\varphi}(\eta) = 1 = \varphi(\eta) \) then for all \( 0 \leq \lambda \leq 1 \) by the convexity of \( \varphi \)
\[ \varphi(\lambda \xi + (1 - \lambda)\eta) \leq \lambda \varphi(\xi) + (1 - \lambda)\varphi(\eta) = 1 \]
so
\[ \sqrt{\varphi}(\lambda \xi + (1 - \lambda)\eta) \leq 1. \]
Thus for all non-null \( x, y \)
\[ \sqrt{\varphi}(\lambda x + (1 - \lambda)y) \leq 1 \]
now taking \( \lambda = \sqrt{\varphi}(x)/(\sqrt{\varphi}(x) + \sqrt{\varphi}(y)) \) and using \( \sqrt{\varphi} \) homogeneity we finally get the triangle inequality and we are able to conclude that \( \| \cdot \|_\infty = \sqrt{\varphi}(\cdot) \) is a norm.

The final assertion comes from the fact that \( \| \xi \|^2_\infty \) is the limit of the energies’ infimum along the paths between 0 and \( \xi \) for the rescaled metrics \((1/t^2)(\delta_t^*)g\), which are attained along the geodesics. \( \square \)

This theorem easily induces the following assertion:

**Corollary 9.** For all \( x \) and \( y \in \mathbb{R}^n \) we have
\[ \lim_{\rho \to +\infty} \frac{d_g(\rho x, \rho y)}{\rho} = \| x - y \|_\infty. \]

From now on we will write \( d_\rho(x, y) = d_g(\rho x, \rho y)/\rho \), and we are now going to see what can be deduced for the balls \( B_\rho(\cdot) \) in terms of Gromov-Hausdorff convergence.

### 2.3. Gromov-Hausdorff convergence of metric balls.

We will write \( \mu_g \) (resp. \( \mu_\rho \)) the measure induced by \( \bar{g} \) (resp. \( g_\rho \)). \( \mu_\infty \) will be the measure of Lebesgue such that for a fundamental domain \( D_f \) we have \( \mu_\infty(D_f) = \mu_g(D_f) \). Finally let
\[ B_\rho(R) = \{ x \in \mathbb{R}^n \mid d_\rho(0, x) \leq R \} = \frac{1}{\rho} B_g(R \cdot \rho), \]
and
\[ B_\infty(R) = \{ x \in \mathbb{R}^n \mid \| x \|_\infty \leq R \}. \]

**Theorem 10.** The net of measured metric spaces \((B_\rho(1), d_\rho, \mu_\rho)\) converges in the Gromov-Hausdorff measured topology to \((B_\infty(1), \| \cdot \|_\infty, \mu_\infty)\) as \( \rho \) goes to infinity.
Proof. Let us choose an $\epsilon > 0$. We first show that the identity is an $\epsilon$-approximation if $\rho$ is large enough. It suffice to show that there is a finite family of points $(x_1, \ldots, x_N)$ such that its $\epsilon$-neighbourhood in $(B_\infty(1), d_\infty)$ and, for $\rho$ large enough, in $(B_\rho(1), d_\rho)$ is respectively $B_\infty(1)$ and $B_\rho(1)$ and such that for all $i, j = 1, \ldots, N$ we have
\[ ||x_i - x_j||_\infty - d_\rho(x_i, x_j) || \leq \epsilon. \]

Let $r > 0$ and let $(\gamma_1, \ldots, \gamma_N)$ be all the images of 0 by the action of $\mathbb{Z}^n$, such that for $i = 1, \ldots, N$, $\gamma_i \in B_\infty(r)$. Then we take for $i = 1, \ldots, N$, $x_i = \gamma_i/r$. Let us remark that for $\rho$ large enough these points will all be in $B_\rho(1)$.

Now let us point out that, because of the invariance by the $\mathbb{Z}^n$ action, there are two constants $\alpha$ and $\beta$ such that for all $x$ and $y \in \mathbb{R}^n$ we have
\[ \alpha ||x - y||_\infty \leq d_g(x, y) \leq \beta ||x - y||_\infty; \]
thus for every $x \in B_\infty(1)$ take the closest point $x_i$ (thus $\gamma_i$ is the closest point of $\mathbb{Z}^n \cdot 0$ from $rx$) then there is a constant $C$ (the diameter of the fundamental domain) such that
\[ ||x - x_i||_\infty \leq \frac{1}{\alpha r} d_g(rx, \gamma_i) \leq \frac{1}{\alpha r} C. \]
we also get
\[ d_\rho(x, x_i) \leq \frac{\beta}{\alpha r} C. \]
thus, for $r$ large enough $(x_1, \ldots, x_N)$ is an $\epsilon$-neighbourhood of $(B_\infty(1), \| \cdot \|_\infty)$. Furthermore if $\rho$ is large enough it is also an $\epsilon$-neighbourhood of $(B_\rho(1), d_\rho)$ and by Corollary 9
\[ ||x_i - x_j||_\infty - d_\rho(x_i, x_j) || \leq \epsilon. \]

Now let us take a continuous function from $B_\infty(1)$ to $\mathbb{R}$. Let $z_1, \ldots, z_k$ and $\zeta_1, \ldots, \zeta_l$ in the orbit of 0 by the $\mathbb{Z}^n$ action such that $\zeta_j + D \cap B_\infty(\rho) \neq \emptyset$ for $j = 1, \ldots, l$ and
\[ \bigcup_i z_i + D_f \subset B_\infty(\rho) \subset \bigcup_k \zeta_k + D_f \]
(where we took all $z_i$ such that $z_i + D_f \subset B_\infty(\rho)$) then we get
\[
\sum_i \inf_{\rho x \in z_i + D_f} f(x) \mu_\infty(D_f) \leq \int_{B_\infty(\rho)} f(x/\rho) \, d\mu_\rho(x) \\
\leq \sum_j \sup_{\rho x \in (\zeta_j + D_f) \cap B_\infty(\rho)} f(x) \mu_\infty(D_f)
\]
now dividing by \( \rho^n \) we find
\[
\sum_i \inf_{x \in \frac{1}{\rho}(z_i + D_f)} f(x) \mu_{\infty}\left((1/\rho)D_f\right) \\
\leq \int_{B_{\infty}(1)} f \ d\mu_{\rho}(x) \\
\leq \sum_j \sup_{x \in \frac{1}{\rho}(\zeta_j + D_f) \cap B_{\infty}(1)} f(x) \mu_{\infty}\left((1/\rho)D_f\right).
\]
The middle term is surrounded by two sums of Riemann, which converges to \( \int_{B_{\infty}(1)} f \ d\mu_{\infty} \), thus it also converges. To conclude, notice that the net of characteristic function \( \chi_{B_{\rho}(1)} \) converges simply to \( \chi_{B_{\infty}(1)} \) inside of \( B_{\infty}(1) \).

2.4. What shall we finally study? As we said we are now going to focus on the spectrum of the balls \( B_g(\rho) \). As we already mentioned we know that the eigenvalues are converging to zero with a \( 1/\rho^2 \) speed. Hence we want to find a precise equivalent.

For this let introduce \( \Delta_{\rho} \) the Laplacian associated to the rescaled metrics \( g_{\rho} = 1/\rho^2(\delta_{\rho})^*g \), and for any function \( f \) from \( B_g(\rho) \) to \( \mathbb{R} \) lets associate a function \( f_{\rho} \) on \( B_{\rho}(1) \) by \( f_{\rho}(x) = f(\rho \cdot x) \). Then it is an easy computation to see that for any \( x \in B_{\rho}(1) \):
\[
\rho^2(\Delta f)(\rho \cdot x) = (\Delta_{\rho} f_{\rho})(x)
\]
hence the eigenvalues of \( \Delta_{\rho} \) on \( B_{\rho}(1) \) are exactly the eigenvalues of \( \Delta \) on \( B_g(\rho) \) multiplied by \( \rho^2 \) and our problems becomes the study of the spectrum of the Laplacian \( \Delta_{\rho} \) on \( B_{\rho}(1) \). In the light of what precedes we would like to show that there is some operator \( \Delta_{\infty} \) acting on \( B_{\infty}(1) \) such that, in some sense, the net of Laplacian \( (\Delta_{\rho}) \) converges towards \( \Delta_{\infty} \) such that the spectra also converge to the spectrum of \( \Delta_{\infty} \). The next section aims at giving a precise meaning to this.


This section adapts to our purpose some notion of convergences well-known for a fixed Hilbert space.

3.1. Convergence on a net of Hilbert spaces. Let \( (X_{\alpha}, d_{\alpha}, m_{\alpha})_{\alpha \in \mathcal{A}} \), where \( \mathcal{A} \) is a partially ordered set, be a net of compact measured metric spaces converging to \( (X_{\infty}, d_{\infty}, m_{\infty}) \) in the Gromov-Hausdorff measured topology. We will write \( L^2_{\alpha} = L^2(X_{\alpha}, m_{\alpha}) \) (resp. \( L^2_{\infty}(X_{\infty}, m_{\infty}) \)) for the square integrable function spaces. Their respective scalar product will be \( \langle \cdot, \cdot \rangle_{\alpha} \) (resp. \( \langle \cdot, \cdot \rangle_{\infty} \)) and \( \| \cdot \|_{\alpha} \) (resp. \( \| \cdot \|_{\infty} \)).

Furthermore we suppose that in every \( L^2_{\alpha} \) the continuous functions form a dense subset.
**Definition 11.** We say that a net \((u_\alpha)_{\alpha \in A}\) of functions \(u_\alpha \in L^2_\alpha\) strongly converges to \(u \in L^2_\infty\) if there exists a net \((v_\beta)_{\beta \in B} \subset C^0(X_\infty)\) converging to \(u\) in \(L^2_\infty\) such that
\[
\lim_{\beta} \limsup_{\alpha} \|f_\alpha^* v_\beta - u_\alpha\| = 0;
\]
where \((f_\alpha)\) is the net of Hausdorff approximations. We will also talk of strong convergence in \(L^2\).

**Definition 12.** We say that a net \((u_\alpha)_{\alpha \in A}\) of functions \(u_\alpha \in L^2_\alpha\) weakly converges to \(u \in L^2_\infty\) if and only if for every net \((v_\alpha)_{\alpha \in A}\) strongly converging to \(v \in L^2_\infty\) we have
\[
\lim_{\alpha} \langle u_\alpha, v_\alpha \rangle = \langle u, v \rangle_{\infty}.
\]
We will also talk of weak convergence in \(L^2\).

The following lemmas justify those two definitions:

**Lemma 13.** Let \((u_\alpha)_{\alpha \in A}\) be a net of functions \(u_\alpha \in L^2_\alpha\). If \((\|u_\alpha\|_\alpha)\) is uniformly bounded, then there exists a weakly converging subnet.

**Proof.** Let \((\phi_k)_{k \in \mathbb{N}}\) be a complete orthonormal basis of \(L^2_\infty\). Using the density of continuous functions in \(L^2_\infty\), for each \(k\) we can retrieve a net of continuous functions \((\varphi_k, \beta)_{\beta \in B}\) strongly converging to \(\phi_k\) in \(L^2_\infty\). Replacing by a subnet of \(A\) and \(B\) if necessarily, we can assume that the following limit exists:
\[
\lim_{\beta} \lim_{\alpha} \langle u_\alpha, f_\alpha^* \varphi_{1, \beta} \rangle = a_1 \in \mathbb{R}
\]
and from the uniform bound hypothesis it follows that \(a_1 \in \mathbb{R}\). Repeating this procedure we can assume that for every \(k \in \mathbb{N}\) the following limit exists:
\[
\lim_{\beta} \lim_{\alpha} \langle u_\alpha, f_\alpha^* \varphi_{k, \beta} \rangle = a_k \in \mathbb{R}.
\]
Let us fix an integer \(N\). For any \(\epsilon > 0\) there is a \(\beta_\epsilon \in B\) such that
\[
|\langle \varphi_{k, \beta}, \varphi_{l, \beta} \rangle_{\infty} - \delta_{kl}| < \epsilon
\]
for any \(\beta \geq \beta_\epsilon\) and \(k, l = 1, \ldots, N\). Moreover for any \(\beta \geq \beta_\epsilon\) there is an \(\alpha_{\epsilon, \beta} \in A\) such that
\[
|\langle f_\alpha^* \varphi_{k, \beta}, f_\alpha^* \varphi_{l, \beta} \rangle - \delta_{kl}| < 2\epsilon
\]
for any \(\alpha \geq \alpha_{\epsilon, \beta}\) and \(k, l = 1, \ldots, N\). Let \(L_{\alpha, \beta} = \text{Vect}\{f_\alpha^* \varphi_{k, \beta} \mid k = 1, \ldots, N\}\) and \(P_{\alpha, \beta} : L^2_\alpha \to L^2_{\alpha, \beta}\) be the projection to the linear subspace \(L_{\alpha, \beta} \subset L^2_\alpha\) we have
\[
\left| \sum_{k=1}^{N} |\langle u_\alpha, f_\alpha^* \varphi_{k, \beta} \rangle |^2 - \|P_{\alpha, \beta} u_\alpha\|_{\alpha}^2 \right| \leq \theta_N(\epsilon)
\]
for every $\alpha \geq \alpha_{e,\beta}$ and $\beta \geq \beta_{e}$, where $\theta_{N}$ is a function depending only of $N$ such that $\lim_{\epsilon \to 0} \theta_{N}(\epsilon) = 0$. This implies for every $N$

\[
\sum_{k=1}^{N} |a_k|^2 = \lim_{\beta} \lim_{\alpha} \sum_{k=1}^{N} |\langle u_\alpha, f_\alpha^* \phi_{k,\beta} \rangle_\alpha|^2 = \lim_{\beta} \lim_{\alpha} \|P_{\alpha,\beta} u_\alpha\|^2_{\alpha} \leq \lim_{\alpha} \sup \|u_\alpha\|_{\alpha}^2 < \infty
\]

thus

\[
u = \sum_{k=1}^{N} a_k \phi_k \in L^2_\infty.
\]

We shall prove that some subnet of $(u_\alpha)_\alpha$ weakly converges to $u$. Take any $v \in L^2_\infty$ and set $b_k = \langle v, \phi_k \rangle_\infty$. By the properties of the strong convergence it is enough to show (3) for a well chosen net. Let $v^N_\beta = \sum_{k=1}^{N} b_k \phi_{k,\beta}$. By construction $v^N_\beta \in C_0$ and $\lim_{N \to \infty} \lim_{\beta} v^N_\beta = v$ strongly. We have

\[
\lim_{\beta} \lim_{\alpha} \langle u_\alpha, f_\alpha^* v^N_\beta \rangle_\alpha = \lim_{\beta} \lim_{\alpha} \sum_{k=1}^{N} b_k \langle u_\alpha, f_\alpha^* \phi_{k,\beta} \rangle_\alpha = \sum_{k=1}^{N} a_kb_k
\]

which tends to $\langle u, v \rangle_\infty$ as $N \to \infty$. Thus, there exists a net of integers $(N_\beta)_\beta$ tending to $+\infty$ such that $v^N_\beta$ strongly converges to $v$ and

\[
\lim_{\beta} \lim_{\alpha} \langle u_\alpha, f_\alpha^* v^N_\beta \rangle_\alpha = \langle u, v \rangle_\infty.
\]

Lemma 14. Let $(u_\alpha)_\alpha \in A$ be a weakly converging net to $u \in L^2_\infty$. Then

\[
\sup_{\alpha} \|u_\alpha\|_\alpha < \infty \quad \text{and} \quad \|u\|_\infty \leq \lim_{\alpha} \inf \|u_\alpha\|_\alpha.
\]

Furthermore, the net strongly converges if and only if

\[
\|u\|_\infty = \lim_{\alpha} \|u_\alpha\|_\alpha.
\]

Proof. Let suppose that the net $(u_\alpha)$ is weakly converging and $\sup_{\alpha} \|u_\alpha\|_\alpha = +\infty$. We can extract a sequence such that $\|u_{\alpha_k}\|_\alpha_k > k$. Setting

\[
v_k = \frac{1}{k} \frac{u_{\alpha_k}}{\|u_{\alpha_k}\|_\alpha_k}
\]

one has $\|v_k\|_\alpha_k = 1/k \to 0$ thus $v_k$ strongly converges to $0$, which implies

\[
\langle u_{\alpha_k}, v_k \rangle_\alpha_k \to \langle u, 0 \rangle_\infty = 0
\]

but we also have

\[
\langle u_{\alpha_k}, v_k \rangle_\alpha_k = \frac{1}{k} \|u_{\alpha_k}\|_\alpha_k \geq 1
\]

this is a contradiction and thus we obtain $\sup_{\alpha} \|u_\alpha\|_\alpha < \infty$. 

\[\square\]
Let \((w_\alpha)\) be a strongly converging net to \(u\), then
\[
0 \leq \liminf_{\alpha} \|u_\alpha - w_\alpha\|_\alpha^2 \\
= \liminf_{\alpha} (\|u_\alpha\|^2_\alpha + \|w_\alpha\|^2_\alpha - 2\langle u_\alpha, w_\alpha \rangle_\alpha) \\
= \liminf_{\alpha} \|u_\alpha\|^2_\alpha - \|u\|^2_\infty.
\]
The final claim comes from the properties of the strong convergence and the following equality:
\[
\|u_\alpha - w_\alpha\|_\alpha^2 = \|u_\alpha\|^2_\alpha + \|w_\alpha\|^2_\alpha - 2\langle u_\alpha, w_\alpha \rangle_\alpha.
\]
\[\square\]

3.2. Convergence of bounded operators. Let \(\mathcal{L}(L^2_\alpha)\) bet the set of linear bounded operators acting on \(L^2_\alpha\) and \(\| \cdot \|_{\mathcal{L}_\alpha}\) their norm (for \(\alpha \in \mathcal{A} \cup \infty\)). Let \(B_\infty \in \mathcal{L}(L^2_\infty)\) and \(B_\alpha \in \mathcal{L}(L^2_\alpha)\) for every \(\alpha \in \mathcal{A}\).

**Theorem and Definition 15.** Let \(u, v \in L^2_\infty\) and \((u_\alpha)_{\alpha \in \mathcal{A}}, (v_\alpha)_{\alpha \in \mathcal{A}}\) two nets such that \(u_\alpha, v_\alpha \in L^2_\alpha\). We say that the net of operators \((B_\alpha)_{\alpha \in \mathcal{A}}\) strongly (resp. weakly, compactly) converges to \(B\) if \(B_\alpha u_\alpha \to Bu\) strongly (resp. weakly, strongly) for every net \((u_\alpha)\) strongly (resp. weakly, weakly) converging to \(u\) \iff
\[
\lim_{\alpha} \langle B_\alpha u_\alpha, v_\alpha \rangle_\alpha = \langle Bu, v \rangle_\infty
\]
for every \((u_\alpha), (v_\alpha)\), \(u\) and \(v\) such that \(u_\alpha \to u\) strongly (resp. weakly, weakly) and \(v_\alpha \to v\) weakly (resp. strongly, weakly).

**Proof.** The equivalence comes from the definition of the weak convergence and the fact that a net \((u_\alpha)\) strongly converges to \(u\) if and only if \(\langle u_\alpha, v_\alpha \rangle_\alpha \to \langle u, v \rangle_\infty\) for every net \((v_\alpha)_\alpha\) weakly converging to \(v \in L^2_\infty\). The “if” part is straightforward, for the “only if” we see that for every net \((v_\alpha)\) strongly converging to \(v\) we have \(\langle u_\alpha, v_\alpha \rangle_\alpha \to \langle u, v \rangle_\infty\), which implies the weak convergence of the net \((u_\alpha)\). Using now the hypothesis we get the convergence of the net \(\|u_\alpha\|_\alpha\) and thus the strong convergence of \((u_\alpha)\) by Lemma 14. \[\square\]

**Proposition 16.** Let \((B_\alpha)\) be a strongly converging net to \(B\) then
\[
\liminf_{\alpha} \|B_\alpha\|_{\mathcal{L}_\alpha} \geq \|B\|_{\mathcal{L}_\infty}
\]
and if the convergence is compact then it is an equality and \(B\) is a compact operator as is its adjoint \(B^*\).

**Proof.** Let \(\epsilon > 0\), there is \(u \in \mathcal{L}^2_\infty\) such that \(\|u\|_\infty = 1\) and \(\|Bu\|_\infty > \|B\|_{\mathcal{L}_\infty} - \epsilon\). Take \((u_\alpha)\) a net converging strongly to \(u\). Then \(\|u_\alpha\|_\alpha \to 1\), furthermore the strong convergence of \((B_\alpha)\) implies that \(\|B_\alpha u_\alpha\|_\alpha \to \|Bu\|_\infty\) thus
\[
\liminf_{\alpha} \|B_\alpha\|_{\mathcal{L}_\alpha} \geq \liminf_{\alpha} \frac{\|B_\alpha u_\alpha\|_\alpha}{\|u_\alpha\|_\alpha} = \|Bu\|_\infty > \|B\|_{\mathcal{L}_\infty} - \epsilon.
\]
Suppose now that the convergence is compact. Take a net \((u_\alpha)\) such that \(\|u_\alpha\|_\alpha = 1\) and
\[
\lim_\alpha \|B_\alpha\|_{\mathcal{L}_\alpha} - \|B_\alpha u_\alpha\|_\alpha = 0.
\]
Extracting a subnet if necessary we can suppose that the net \((u_\alpha)\) weakly converges to \(u\). By Lemma 14 we have \(\|u\|_\infty \leq 1\), furthermore the compact convergence implies the strong convergence of \(B_\alpha u_\alpha\) to \(Bu\) thus
\[
\|B\|_{\mathcal{L}_\infty} \geq \frac{\|Bu\|_\infty}{\|u\|_\infty} \geq \|Bu\|_\infty = \lim_\alpha \|B_\alpha u_\alpha\|_\alpha = \lim_\alpha \|B_\alpha\|_{\mathcal{L}_\alpha}.
\]
Now let us prove that in the latest case, \(B\) is compact. Let \((v_\beta)_{\beta \in B}\) a net weakly converging to \(v\) in \(\mathcal{L}_2^\infty\) then
\[
\langle u, Bv_\beta\rangle_\infty = \langle B^*u, v_\beta\rangle_\infty \to \langle B^*u, v\rangle_\infty = \langle u, Bv\rangle_\infty
\]
thus \(Bv_\beta\) weakly converges to \(Bv\). For every \(\beta\) let \((u_\alpha, \beta)\) be a strongly converging net such that \(\lim_\alpha u_\alpha, \beta = v_\beta\). For every \(\beta\) the compact convergence of \((B_\alpha)\) implies the strong convergence of \(B_\alpha u_\alpha, \beta\) to \(Bv_\beta\). Now let us take a net of positive numbers such that \(\lim_\beta \epsilon(\beta) = 0\), then there is \(\alpha(\beta)\) such that for every \(\alpha \geq \alpha(\beta)\) we have
\[
\|B_\alpha u_\alpha, \beta\|_\alpha - \|Bv_\beta\|_\infty \leq \epsilon(\beta).
\]
Set \(w_\beta = u_{\alpha(\beta)}, \beta\) then \(\lim_\beta w_\beta = v\) weakly and by the compact convergence we obtain the strong convergence of \((B_{\alpha(\beta)} w_\beta)\) to \(Bv\) but
\[
\lim_\beta \|B_{\alpha(\beta)} w_\beta\|_{\alpha(\beta)} - \|Bv_\beta\|_\infty = 0
\]
which implies \(\|Bv_\beta\|_\infty \to \|Bv\|_\infty\). We can conclude using Lemma 14. \(\square\)

3.3. Convergence of spectral structures. Here we see \(L_2^\alpha\) as a Hilbert space. Then \(A_\alpha\) and \(A\) will be self-adjoint operators, \(E_\alpha\) and \(E\) their respective spectral measure and \(R_\mu, R\) their resolvents for \(\mu\) in the resolvent space. We want to study the links between the convergence of \((A_\alpha)\), \((E_\alpha)\) and \((R_\mu^\alpha)\). The following theorem says that it is the same:

**Theorem 17.** Let \((A_\alpha)\) and \(A\) be self-adjoint operators \(E_\alpha\), \(E\) their spectral measures and \(R_\mu^\alpha\), \(R_\mu\) their resolvents for \(\mu\) in the resolvent space, then the following assertions are equivalent:

1. \(R_\mu^\alpha \to R_\mu\) strongly (resp. compactly) for \(\mu\) outside the union of the spectra of \(A_\alpha\) and \(A\).
2. \(\varphi(A_\alpha) \to \varphi(A)\) strongly (resp. compactly) for every continuous function, with compact support \(\varphi : \mathbb{R} \to \mathbb{C}\).
3. \(\varphi_\alpha(A_\alpha) \to \varphi(A)\) strongly (resp. compactly) for every net \(\{\varphi_\alpha : \mathbb{R} \to \mathbb{C}\}\) of continuous functions vanishing at infinity and uniformly converging to \(\varphi\) a continuous function vanishing at infinity.
4. \(E_\alpha(\{\lambda, \mu\}) \to E(\{\lambda, \mu\})\) strongly (resp. compactly) for every pair of real numbers outside the spectrum of \(A\).
Let us recall that a quadratic form $Q$ on a complex (resp. real) Hilbert space $\mathcal{H}$ comes from a sesquilinear (resp. bilinear) form, positive and symmetric $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{C}$ (resp. $\mathbb{R}$) where $D(E) \in \mathcal{H}$ is a linear subspace and $Q(u) = \mathcal{E}(u, u)$. Notice that $\mathcal{E}_1(u, v) = \langle u, v \rangle_{\mathcal{H}} + \mathcal{E}(u, v)$ for every $u$ and $v \in D(\mathcal{E})$ is also a sesquilinear (resp. bilinear), symmetric and positive form. Thus $(D(\mathcal{E}), \mathcal{E}_1)$ is a pre-Hilbert space. We say that $Q$ is closed if and only if $(D(\mathcal{E}), \mathcal{E}_1)$ is a Hilbert space. In what follows, we will not distinguish $Q$ and the functional $\mathcal{E}$ defined by $\mathcal{E}(u) = Q(u)$ on $D(\mathcal{E})$ and $\mathcal{E}(u) = \infty$ on $\mathcal{H}\setminus D(\mathcal{E})$. In this context, $Q$ is closed if and only if $\mathcal{E}$ is lower semi-continuous as a function $\mathcal{E} : \mathcal{H} \to \mathbb{R}$.

**Definition 18.** Let $(\mathcal{E}_\alpha)$ be a net of closed quadratic forms, where $\mathcal{E}_\alpha$ is a closed quadratic form on $L^2_\alpha$ for every $\alpha \in A$. We will say that this net is asymptotically compact if and only if for every net $(v_\alpha)_{\alpha \in A}$ such that
\[
\limsup_{\alpha} \mathcal{E}_\alpha(v_\alpha) + \|v_\alpha\|_\alpha^2 < \infty
\]
there is a strongly converging subnet.

Now a **spectral structure** on a Hilbert space $\mathcal{H}$ over $\mathbb{C}$ (resp. $\mathbb{R}$) is a family
\[
\Sigma = \{A, \mathcal{E}, E, (T_t), (R_\zeta)\}
\]
where $A$ is a self-adjoint operator seen as the infinitesimal generator of the densely defined quadratic form $\mathcal{E}$ (such that $D(\mathcal{E}) = D(\sqrt{A})$ and $\mathcal{E}(u, v) = \langle \sqrt{A}u, \sqrt{A}v \rangle_{\mathcal{H}}$ for every $u$ and $v$ in $D(\mathcal{E})$), $E$ is its spectral measure, $(T_t)_{t \geq 0}$ is a one parameter semi-group of strongly continuous contractions ($T_t = e^{-tA}$, $t \geq 0$) and $R_\zeta$ is a strongly continuous resolvent ($R_\zeta = (\zeta - A)^{-1}$ for $\zeta \in \rho(A)$, where $\rho(A)$ is the resolvent set of $A$). In what follows we will study a family of spectral structures $\Sigma_\alpha$ on $L^2_\alpha$, thus we will have
\[
\Sigma_\alpha = \{A_\alpha, \mathcal{E}_\alpha, E_\alpha, (T^\alpha_t), (R^\alpha_\zeta)\}.
\]

**Definition 19.** Let $(\Sigma_\alpha)_{\alpha \in A}$ be a net with $\Sigma_\alpha$ a spectral structure on $L^2_\alpha$ and $\Sigma$ a spectral structure on $L^2_\infty$, we will say that the net $(\Sigma_\alpha)_{\alpha}$ strongly (resp. compactly) converges to $\Sigma$ if and only if one of the conditions of Theorem 17 is satisfied.

**Proposition 20.** Let $(\Sigma_\alpha)_{\alpha \in A}$ of spectral structures strongly converging to $\Sigma$ then for any net $(v_\alpha)_{\alpha}$ weakly converging to $v$ we have
\[
\mathcal{E}(v) \leq \liminf_{\alpha} \mathcal{E}_\alpha(v_\alpha).
\]
Furthermore, if the net $(\Sigma_\alpha)_{\alpha \in A}$ converges compactly, then the net of quadratic forms $(\mathcal{E}_\alpha)_\alpha$ is asymptotically compact.
Proof. Assume that the net of resolvents \((R_\alpha^\lambda)\) is strongly convergent and write
\[
\alpha_\lambda^\lambda(u, v) = -\lambda(u - \lambda R_\alpha^\lambda u, v)_\alpha
\]
(the Deny-Yosida approximation of bilinear form associated to \(E_\alpha\)), then the net \((\alpha_\lambda^\lambda(u, u))\) converges to \(E_\alpha(u)\) increasing when \(\lambda \to -\infty\) (see Mosco [Mos94] 1.(i)). From the assumption it easy to see that for \((u_\alpha)\) and \((v_\alpha)\) converging strongly to \(u\) and weakly to \(v\) respectively
\[
\lim_\alpha \alpha_\lambda^\lambda(u_\alpha, v_\alpha) = -\lambda(u - \lambda R_\lambda u, v)_\infty = \alpha^\lambda(u, v)
\]
we recall that (see Dal Maso [Mas93] Proposition 12.12)
\[
\alpha_\lambda^\lambda(u, u) \geq \alpha_\lambda^\lambda(v, v) + 2\lambda \langle v - \lambda R_\lambda v, u - v \rangle_\infty
\]
hence for any net \(v_\alpha\) weakly converging to \(u\) and \(w_\alpha\) a strongly converging net to \(u\) we have
\[
E_\alpha(v_\alpha) \geq \alpha_\alpha^\lambda(v_\alpha, v_\alpha) \geq \alpha_\alpha^\lambda(w_\alpha, w_\alpha) + 2\lambda \langle w_\alpha - \lambda R_\alpha^\lambda w_\alpha, v_\alpha - w_\alpha \rangle
\]
thus \(\lim \inf_\alpha E_\alpha(v_\alpha) \geq \alpha^\lambda(u, u)\) for any \(\lambda < 0\), now taking \(\lambda \to -\infty\) we can conclude that \(\lim \inf_\alpha E_\alpha(v_\alpha) \geq E(u)\).

Now assume that \((\Sigma_\alpha)\) compactly converges and let \((u_\alpha)_{\alpha \in A}\) be a net such that
\[
\sup_\alpha (E_\alpha(u_\alpha) + \|u_\alpha\|^2_\alpha) \leq M < \infty.
\]
Taking a subnet if necessary we can suppose that \((u_\alpha)_{\alpha}\) weakly converges to \(u\). Let \(\rho > 0\) be out of \(A_\infty\)'s spectrum. As
\[
\int_{[\rho, \infty]} d\langle E_\alpha u_\alpha, u_\alpha \rangle_\alpha \leq \frac{1}{\rho} \int_{[\rho, \infty]} \lambda d\langle E_\alpha(\lambda) u_\alpha, u_\alpha \rangle_\alpha \leq \frac{E_\alpha(u_\alpha)}{\rho} \leq \frac{M}{\rho}
\]
we have
\[
\|u_\alpha\|^2_\alpha \leq \int_{[0, \rho]} d\langle E_\alpha u_\alpha, u_\alpha \rangle_\alpha + \frac{M}{\rho}
\]
and the compact convergence implies
\[
\lim_\alpha \int_{[0, \rho]} d\langle E_\alpha u_\alpha, u_\alpha \rangle_\alpha = \int_{[0, \rho]} d\langle Eu, u \rangle_\infty
\]
hence
\[
\limsup_\alpha \|u_\alpha\|^2_\alpha \leq \int_{[0, \rho]} d\langle Eu, u \rangle_\infty + \frac{M}{\rho} \leq \|u\|^2_\infty + \frac{M}{\rho}
\]
and taking \(\rho \to \infty\) we get
\[
\limsup_\alpha \|u_\alpha\|^2_\alpha \leq \|u\|^2_\infty
\]
finally we deduce the strongly convergence of the net \((u_\alpha)\) using Lemma 14. □
The main reason we introduced all these convergences is the following theorem, the proof of which we postpone to avoid drowning the reader in too many technical details.

**Theorem 21.** Let $\Sigma_\alpha \to \Sigma$ compactly and suppose that all resolvents $R_\alpha^2$ are compact. Let $\lambda_k$ (resp. $\lambda_\alpha^k$) be the $k^{\text{th}}$ eigenvalue of $A$ (resp. $A_\alpha$) with multiplicity. We take $\lambda_k = +\infty$ if $k > \dim L^2_\infty + 1$ when $\dim L^2_\infty < \infty$ and $\lambda^\alpha_k = +\infty$ if $k > \dim L^2_\alpha + 1$ when $\dim L^2_\alpha < \infty$. Then for every $k$

$$\lim_\alpha \lambda_k^\alpha = \lambda_k.$$  

Furthermore let $\{\varphi^\alpha_k \mid k = 1, \ldots, \dim L^2_\alpha\}$ be an orthonormal bases of $L^2_\alpha$ such that $\varphi^\alpha_k$ is an eigenfunction of $A_\alpha$ for $\lambda^\alpha_k$. Then there is a subnet such that for all $k \leq \dim L^2_\infty$ the net $(\varphi^\alpha_k)_\alpha$ strongly converges to the eigenfunction $\varphi_k$ of $A$ for the eigenvalue $\lambda_k$, and such that the family $\{\varphi_k \mid k = 1, \ldots, \dim L^2_\alpha\}$ is an orthonormal basis of $L^2_\infty$.

4. Proof of Theorem 1.

4.1. Homogenisation of the Laplacian. In this section we are going to built the operator $\Delta_\infty$ of Theorem 1. We remind the reader that $D_f$ is a fundamental domain, we then begin by taking $\chi^i$ as the unique periodic solution (up to an additive constant) of

$$\Delta \chi^i = \Delta x_i \text{ on } D_f.$$  

The operator $\Delta_\infty$ is then defined by

$$\Delta_\infty f = -\frac{1}{\text{Vol}(g)} \left( \int_{D_f} g^{ij} - g^{ik} \frac{\partial \chi^j}{\partial y_k} \ d\mu_g \right) \frac{\partial^2 f}{\partial x_i \partial x_j}. \tag{5}$$

Now let us write $\eta_j(x) = \chi^j(x) - x_j$ the induced harmonic function and

$$q^{ij} = d \frac{1}{\text{Vol}(g)} \left( \int_{D_f} g^{ij} - g^{ik} \frac{\partial \chi^j}{\partial y_k} \ d\mu_g \right)$$

we can notice that the $d\eta_i$ are harmonic 1-forms on the torus. It is not difficult now to show that:

**Proposition 22.** Let $\langle \cdot, \cdot \rangle_2$ be the scalar product induced on 1-forms by the Riemannian metric $g$. Then

$$q^{ij} = \frac{1}{\text{Vol}(g)} \langle d\eta_i, d\eta_j \rangle_2 = q^{ji}$$

thus $\Delta_\infty$ is an elliptic operator.

In fact we can say more, $(q^{ij})$ induces a scalar product on harmonic 1-forms (whose norm will be written $\| \cdot \|_2$) and then to $H^1(T, \mathbb{R})$. Indeed, as mentioned earlier, we can see the $(d\eta_i)$ as 1-forms over the torus. Being a free
family they can be seen as a basis of $H^1(\mathbb{T}, \mathbb{R})$ (Hodge’s theorem). Thus by duality this yields also a scalar product $(q_{ij})$ over $H_1(\mathbb{T}, \mathbb{R})$ (whose induced norm will be written $\| \cdot \|_2^\ast$). The question naturally arising is to know the link between this norm and the stable norm. To see this we have to go back on $H^1(\mathbb{T}, \mathbb{R})$. Indeed the stable norm is the dual of the norm obtained by quotient of the sup norm on 1-forms (see Pansu [Pan99] Lemma 17), which we write $\| \cdot \|_\infty$, and the norm $\| \cdot \|_2$ comes from the normalised $L^2$ norm. Thus mixing the Hölder inequality and the Hodge-de Rham theorem we get:

**Proposition 23.** For every 1-form $\alpha$ we have

$$\|\alpha\|_2 \leq \|\alpha\|_\infty^\ast$$

thus by duality, for every $\gamma \in H_1(\mathbb{T}, \mathbb{R})$ we have

$$\|\gamma\|_\infty \leq \|\gamma\|_2^\ast$$

in other words the unit ball of $\| \cdot \|_2^\ast$ is included in $B_\infty(1)$.

To finish this section, let us remark that the manifold $H_1(\mathbb{T}, \mathbb{R})/H_1(\mathbb{T}, \mathbb{Z})$ with the flat metric induced by $\| \cdot \|_2^\ast$ is usually called the Jacobi manifold or the Albanese torus of $(\mathbb{T}, g)$.

### 4.2. Asymptotic compactness.

Let us now define the various functional spaces involved. For $\rho \in \mathbb{R}$, $L^2(B_\rho(1), d\mu_\rho)$ will be the space of square integrable functions over the ball $B_\rho(1)$, which is a Hilbert space with the scalar product

$$(u, v)_\rho = \int_{B_\rho(1)} uv \, d\mu_\rho$$

whose norm will be $| \cdot |_\rho$. $H^1_{\rho,0}(B_\rho(1))$ will be the closure of $C^\infty(B_\rho(1))$ functions with compact support, in $H^1_{\rho}(B_\rho(1))$ for the norm $\| \cdot \|_\rho$ defined by

$$\|v\|_{\rho,0}^2 = |v|_\rho^2 + \sum_{i=1}^n \left| \frac{\partial v}{\partial x_i} \right|_{\rho}^2$$

and with

$$H^1_{\rho}(B_\rho(1)) = \left\{ v \left| v, \frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n} \in L^2(B_\rho(1), d\mu_\rho) \right. \right\}.$$

For all that follows, $V_\rho$ will be a closed sub-space such that

$$H^1_{\rho,0}(B_\rho(1)) \subset V_\rho \subset H^1_{\rho}(B_\rho(1)).$$

Thus we can define a spectral structure on $L^2_\rho$ by expanding the Laplacian defined on $V_\rho$ on $L^2_\rho$. If $V_\rho = H^1_{\rho,0}(B_\rho(1))$ we deal with the Dirichlet problem, and if $V_\rho = H^1_{\rho}(B_\rho(1))$ we then deal with the Neumann problem. We then put the following norm on $V_\rho$:

$$\|v\|_{\rho,0}^2 = |v|_\rho^2 + (v, \Delta_\rho v)_\rho$$
we then have:

**Lemma 24.** Let \((u_\rho)_\rho\) be a net with \(u_\rho \in V_\rho\) for every \(\rho\), if there is a constant \(C\) such that for all \(\rho > 0\) we have
\[
\|u_\rho\|_{\rho,0} \leq C
\]
then there is a strongly converging subnet in \(\mathcal{L}^2\).

**Proof.** Let \(B = \cup_\rho B_\rho(1)\) we are going to show that the strong convergence in \(L^2(B,\mu_\infty)\) implies the strong convergence in \(\mathcal{L}^2\). Then the compact embedding of \(H^1_\infty(B)\) in \(L^2(B,\mu_\infty)\) will conclude the proof.

Let us first notice that the periodicity gives the existence of two constant \(\alpha\) and \(\beta\) such that
\[
\alpha|v|_\infty \leq |v|_\rho \leq \beta|v|_\infty.
\]
Let us start by taking a net \((u_\rho)\) strongly converging in \(L^2(B,\mu_\infty)\) to \(u_\infty\) we also assume \(u_\rho \in V_\rho\) for every \(\rho\), because it is all we need. Now let \(c_\rho \in C_0^\infty(B_\rho(1))\) be a sequence of functions strongly converging to \(u_\infty\) and take \(p\) large enough for the support of \(u_\rho\) to be in \(B_\rho(1)\). We have
\[
|c_\rho - u_\rho|_\rho \leq \beta|c_\rho - u_\infty|_\infty + \beta|u_\infty - u_\rho|_\infty
\]
now let \(\varepsilon > 0\) then for \(p\) large enough \(\beta|c_\rho - u_\infty|_\infty \leq \varepsilon\). We fix \(p\) large enough and take \(\rho\) large enough for the second term to converge to 0.

In order to conclude observe that from the assumptions the net \((u_\rho)\) is bounded in \(H^1_\infty(B)\), hence using the compact embedding of \(H^1_\infty(B)\) in \(L^2(B,\mu_\infty)\) we can extract a strongly converging net in \(L^2(B,\mu_\infty)\) and by what we just did in \(\mathcal{L}^2\). \(\square\)

### 4.3. Compact convergence of the resolvents.

Let \(\lambda > 0\) and \(G_\lambda^0\) be the operator from \(L^2_\rho\) to \(V_\rho \subset L^2_\rho\) such that
\[
a_\lambda^0(G_\lambda^0 f, \phi) = (f, \phi)_\rho \quad \forall \phi \in V_\rho,
\]
where
\[
a_\lambda^0(u, v) = \int_{B_\rho(1)} g^{ij}_\rho \partial_i u \cdot \partial_j v \ d\mu_\rho + \lambda(u, v)_\rho.
\]
We want to show that the net of operators \((G_\lambda^0)\) converges compactly to \(G_\lambda\) the operator corresponding to the homogenised problem:
\[
a_\lambda^\infty(G_\lambda f, \phi) = (f, \phi)_\infty \quad \forall \phi \in V_\infty
\]
with \((f, \phi)_\infty = \int_{B_\infty(1)} f \phi \ d\mu_\infty\) and
\[
a_\lambda^\infty(u, v) = \int_{B_\infty(1)} q^{ij} \partial_i u \partial_j v \ d\mu_\infty + \lambda(u, v)_\infty
\]
in other words we want to show the following theorem:
Theorem 25. For every \( \lambda < 0 \), the net of resolvents \( (R^\rho_\lambda) \) of the Laplacian \( (\Delta_\rho) \) converges compactly to \( R^\infty_\lambda \), the resolvent of \( \Delta_\infty \) from the homogenised problem. Thus the net \( (\Sigma_\rho) \) compactly converges to \( \Sigma^\infty_\lambda \).

Proof. This comes from the fact that \( R^\rho_\lambda = -G^\rho_\lambda \) and \( R^\infty_\lambda = -G^-_\lambda \).

First step:
Let \( f_\rho \) be a weakly convergent net to \( f \) in \( L^2 \), thus from Lemma 14 this net is uniformly bounded in \( L^2 \) and in \( V'_\rho \), the dual space of \( V_\rho \).

Let \( f_\rho \in V_\rho \) then by (6) we have:

\[
\alpha \|G^\rho_\lambda f_\rho\|_{\rho,0}^2 \leq (f_\rho, G^\rho_\lambda f_\rho)_\rho \leq K \|f_\rho\|_{V'_\rho} \|G^\rho_\lambda f_\rho\|_{\rho,0}
\]

thus

\[
\|G^\rho_\lambda f_\rho\|_{\rho,0} \leq C \|f_\rho\|_{V'_\rho}
\]

the net \( (G^\rho_\lambda f_\rho) \) being uniformly bounded for the norms \( \|\cdot\|_{\rho,0} \), using Lemma 13 there is a subnet strongly converging in \( L^2 \), i.e.,

\[
(8) \quad u_\rho = G^\rho_\lambda f_\rho \to u^*_\lambda \text{ strongly in } L^2.
\]

Furthermore \( P_\rho = (g^ij_\rho)\nabla G^\rho_\lambda f_\rho \) is also bounded in \( L^2 \) thus there is a subnet of the net \( P_\rho \) weakly converging in \( L^2 \) to \( P^*_\lambda \in L^2_\infty \). For any \( \phi_\infty \in L^2_\infty \) let \( \phi_\rho \) be a strongly converging net to \( \phi_\infty \) in \( L^2 \) then

\[
(9) \quad \int_{B_\rho(1)} P_\rho \cdot \nabla \phi_\rho \, d\mu_\rho + \lambda (G^\rho_\lambda f_\rho, \phi_\rho)_\rho = (f_\rho, \phi_\rho)_\rho \to
\]

\[
\int_{B_\infty(1)} P^*_\lambda \cdot \nabla \phi_\infty \, d\mu_\infty + \lambda (u^*_\lambda, \phi_\infty)_\infty = (f, \phi_\infty)_\infty.
\]

Thus it is enough to show that \( P^*_\lambda = (q^ij) \nabla u^*_\lambda \) on \( B_\infty(1) \) because it induces \( u^*_\lambda = G^\lambda f \).

Second step:
We first take \( \chi^k(y) \) (see 4.1) such that \( M(\chi^k) = 0 \) and we define

\[
(10) \quad w_\rho(x) = x_k - \frac{1}{\rho} \chi^k(\rho x)
\]

for every \( k = 1, \ldots, d_1 \). Then

\[
(11) \quad w_\rho \to x_k \text{ strongly in } L^2,
\]

and by construction of \( \chi^k \) (see 4.1) we have

\[
(12) \quad -\partial_i (\det(g_\rho)^{1/2} g^ij_\rho \partial_j w_\rho) = 0 \text{ on } B_\rho(1).
\]

We multiply this equation by a test function \( \phi \in V_\rho \) and after an integration we get

\[
(13) \quad \int_{B_\rho(1)} g^ij_\rho \partial_j w_\rho \partial_i \phi \, d\mu_\rho = 0.
\]
Let $\varphi \in C^\infty_0(B_\infty(1))$ (notice that for $\rho$ large enough the support of $\varphi$ will be in $B_\rho(1)$) and $\phi = \varphi w_\rho$ which we put into Equation (6) and into Equation (13) we put $\phi = \varphi u_\rho$ (see (8)), and then we subtract the results:

$$
\int_{B_\rho(1)} g^{ij}_\rho \left( \partial_j u_\rho \partial_i \varphi w_\rho - \partial_j w_\rho \partial_i \varphi u_\rho \right) \, d\mu_\rho
= \int_{B_\rho(1)} f_\rho w_\rho \varphi \, d\mu_\rho - \lambda \int_{B_\rho(1)} \varphi u_\rho w_\rho \, d\mu_\rho.
$$

Now let $\rho \to \infty$ in (14), all terms converge because they are product of one strongly converging net and one weakly converging net in $L^2$. More precisely:

- $P_\rho$ defined $P_{\rho,i} = g^{ij}_\rho \partial_j u_\rho$ weakly converges to $P^*_\lambda$ in $L^2$ following (9).
- $\partial_i \varphi w_\rho$ strongly converges to $\partial_i \varphi u_\rho$ in $L^2$ from (11).
- $g^{ij}_\rho \partial_i w_\rho$ is $D_f/\rho$-periodic and weakly converges in $L^2$ towards its mean value
  $$q^{jk} = \mathcal{M} \left( g^{ij}(y) \left( \delta_{ik} - \partial_i \chi^k(y) \right) \right).$$
- $\partial_j \varphi u_\rho$ strongly converges to $\partial_j \varphi u^*_\lambda$ by (8), because $\varphi$ has compact support.
- Now for the right side, $w_\rho$ strongly converges as $u_\rho$ does and $f_\rho$ weakly converges to $f$.

To summarise (14) converges to (we write $P_{\lambda,i}^*$ the coordinates of $P^*_\lambda$)

$$
\int_{B_\infty(1)} \left( P_{\lambda,j}^* x_k - q^{jk} u^*_\lambda \right) \partial_j \varphi \, d\mu_\infty
= \int_{B_\infty(1)} f x_k \varphi \, d\mu_\infty - \lambda \int_{B_\infty(1)} \varphi u^*_\lambda x_k \, d\mu_\infty
$$

furthermore if we put into Equation (9), $\phi_\infty = \varphi x_k$ it gives

$$
\int_{B_\infty(1)} f x_k \varphi \, d\mu_\infty - \lambda \int_{B_\infty(1)} \varphi u^*_\lambda x_k \, d\mu_\infty = \int_{B_\infty(1)} P_{\lambda,j}^* \partial_j (\varphi x_k) \, d\mu_\infty
$$

and by mixing (15) and (16) we get for every $\varphi \in C^\infty_0(B_\infty(1))$ the following equality:

$$
\int_{B_\infty(1)} \left( P_{\lambda,j}^* x_k - q^{jk} u^*_\lambda \right) \partial_j \varphi \, d\mu_\infty = \int_{B_\infty(1)} P_{\lambda,j}^* \partial_j (\varphi x_k) \, d\mu_\infty
$$

which in terms of distribution can be translated into:

$$
- \sum_{j=1}^{d_1} \partial_j( P_{\lambda,j}^* x_k - q^{jk} u^*_\lambda) = - \sum_{j=1}^{d_1} \partial_j P_{\lambda,j}^* x_k \iff P_{\lambda,k}^* = \sum_{j=1}^{d_1} q^{jk} \partial_j u^*_\lambda
$$

which allow us to conclude that $u^*_\lambda = G_\lambda f$. \hfill \Box

It is now easy to finish the proof of Theorem 1, it comes from Theorem 25 and Theorem 21.
5. $\Gamma$-convergence of quadratic forms.

5.1. $\Gamma$ and Mosco-convergence of quadratic forms. We are now going to give a definition of $\Gamma$-convergence adapted to our problem.

**Definition 26 ($\Gamma$-convergence).** We say that a net $\{F_\alpha : L^2_\alpha \to \mathbb{R}\}_{\alpha \in A}$ of functions $\Gamma$-converges to $F : L^2_\infty \to \mathbb{R}$ if and only if the following assertions are satisfied:

1. **(F1)** For any net $(u_\alpha)_{\alpha \in A} \in L^2_\alpha$ strongly converging to $u \in L^2_\infty$ in $L^2$ we have
   \[ F(u) \leq \liminf_{\alpha} F_\alpha(u_\alpha). \]

2. **(F2)** For every $u \in L^2_\infty$ there is a net $(u_\alpha)_{\alpha \in A} \in L^2_\alpha$ strongly converging to $u$ in $L^2$ such that
   \[ F(u) = \lim_{\alpha} F_\alpha(u_\alpha). \]

**Remark.** This is slightly different from Definition 6, which is the usual one. By taking $F_\alpha$ infinite outside of $L^2_\alpha$ in $L^2$ we get back (in some way) the usual definition (see the introduction of [Mas93]).

Let us summarise some properties satisfied by this convergence.

**Lemma 27.**

(a) Let $\{F_\alpha : L^2_\alpha \to \mathbb{R}\}_{\alpha \in A}$ be a net of functions $\Gamma$-converging to a function $F : L^2_\infty \to \mathbb{R}$, then $F$ is lower semi-continuous.

(b) Let $(\mathcal{E}_\alpha)_{\alpha \in A}$ be a net of quadratic forms $\mathcal{E}_\alpha$ on $L^2_\alpha$ $\Gamma$-converging to a function $F : L^2_\infty \to \mathbb{R}$, then $F$ can be identified with a quadratic form on $L^2_\infty$.

There is also the following result, concerning compactness:

**Theorem 28.** From every net $(\mathcal{E}_\alpha)_{\alpha \in A}$ of quadratic forms $\mathcal{E}_\alpha$ on $L^2_\alpha$ we can extract a $\Gamma$-converging subnet, whose limit is a quadratic form on $L^2_\infty$.

**Remark.** This theorem is true for a wider variety of functions, with some restrictions on $\{L^2_\alpha\}_{\nu \in A}$. Of course the limit in that case is not always a quadratic form. Here it is Lemma 27 which gives information on the limit.

**Definition 29 (Mosco topology).** We say that a net $(\mathcal{E}_\alpha)_{\alpha \in A}$ of quadratic forms $\mathcal{E}_\alpha$ on $L^2_\alpha$ Mosco-converges to the quadratic form $\mathcal{E}$ on $L^2_\infty$ if condition (F2) of Definition 26 and (F1') are satisfied:

1. **(F1')** For any $(u_\alpha)_{\alpha \in A}$, $u_\alpha \in L^2_\alpha$ weakly converging net to $u \in L^2_\infty$ in $L^2$ we have
   \[ \mathcal{E}(u) \leq \liminf_{\alpha} \mathcal{E}_\alpha(u_\alpha). \]

The induced topology is called the Mosco topology.

It is obvious that the Mosco-convergence induces the $\Gamma$-convergence, thus this topology is stronger. Let us now define one last convergence:
Definition 30 (Compact Γ-convergence). We say that a net \((\mathcal{E}_\alpha)_{\alpha \in A}\) Γ-converges compactly to \(\mathcal{E}\) if \(\mathcal{E}_\alpha \to \mathcal{E}\) in the Mosco topology and if \((\mathcal{E}_\alpha)_{\alpha \in A}\) is asymptotically compact.

Let us show precisely how the Mosco and the Γ topologies are linked:

Lemma 31. Let us suppose \((\mathcal{E}_\alpha)_{\alpha \in A}\) asymptotically compact then \((\mathcal{E}_\alpha)_{\alpha \in A}\) Γ-converges to \(\mathcal{E}\) if and only if \((\mathcal{E}_\alpha)_{\alpha \in A}\) Mosco-converges to \(\mathcal{E}\).

Proof. We just need to show that the Γ-convergence implies the condition \((F1')\) from Definition 29. We proceed \textit{ad absurdum} and suppose that there is a weakly converging net \((u_\alpha)\) such that \(\liminf \alpha \mathcal{E}_\alpha(u_\alpha) < \mathcal{E}(u)\). Taking a subnet if necessarily we can suppose \(\lim \mathcal{E}_\alpha(u_\alpha) < \mathcal{E}(u)\) thus we also have \(\limsup_\alpha \mathcal{E}_\alpha(u_\alpha) + \|u_\alpha\|_2^2 < +\infty\). The asymptotic compactness is obviously inherited by a subnet thus we can extract a strongly converging subnet \(u_\alpha(\beta)\). The Γ-convergence being also inherited by a subnet of \(\mathcal{E}_\alpha\) we finally get

\[
\lim \mathcal{E}_\alpha(u_\alpha) = \lim_\alpha \mathcal{E}_\alpha(\beta)(u_\alpha(\beta)) \geq \mathcal{E}(u)
\]

which is absurd. □

5.2. Γ-convergence and spectral structures. The following theorem explains how the convergence of spectral structures and the Mosco-convergence are related:

Theorem 32. Let \((\Sigma_\alpha)\) be a net of spectral structures on \((L^2_\alpha)\) and \(\Sigma\) a spectral structure on \(L^2_\infty\) then \(\Sigma_\alpha \longrightarrow \Sigma\) strongly (resp. compactly) if and only if \(\mathcal{E}_\alpha\) Mosco-converges (resp. Γ-converges compactly) to \(\mathcal{E}\).

Proof. We are going to prove the equivalence between the strong (resp. compact) convergence of resolvents and the Mosco-convergence (resp. compact Γ-convergence) of the energies.

Let us begin by assuming the Mosco-convergence of the net \((\mathcal{E}_\alpha)\). We need to show that for every \(v \in L^2_\infty\) and any net \((z_\alpha)\) strongly converging to \(v\) the net \(u_\alpha = -R_\alpha^* z_\alpha\) strongly converges to \(u = -R_\lambda z\). First let us notice that the vector \(u\) is the unique minimiser of

\[
v \mapsto \mathcal{E}(v) - \lambda \|v\|_2^2 - 2\langle z, v\rangle_\infty
\]

we can characterise the same way \(u_\alpha\) for every \(\alpha\).

As an operator of \(L^2_\alpha\), \(R_\lambda^*\) is bounded by \(-\lambda^{-1}\). Thus the net \((u_\alpha)\) is bounded and we can extract a weakly converging subnet, still written \((u_\alpha)\), with limit \(\tilde{u}\). Now from condition \((F2)\) for every \(v \in L^2_\infty\) there is a net strongly converging to it such that \(\lim_\alpha \mathcal{E}_\alpha(v_\alpha) = \mathcal{E}(v)\). But for every \(\alpha\)

\[
(17) \quad \mathcal{E}_\alpha(u_\alpha) - \lambda \|u_\alpha\|_\alpha^2 - 2\langle z_\alpha, u_\alpha, v_\alpha\rangle_\alpha \leq \mathcal{E}_\alpha(v_\alpha) - \lambda \|v_\alpha\|_\alpha^2 - 2\langle z_\alpha, v_\alpha\rangle_\alpha
\]

thus taking the limit in \(\alpha \in A\) we get thanks to condition \((F1')\) of Definition 29 and the fact that for any weakly convergent net \(\|\tilde{u}\|_\infty \leq \liminf_\alpha \|u_\alpha\|_\alpha\)
(remember that $\lambda < 0$)
\[
E(\tilde{u}) - \lambda \|\tilde{u}\|_\infty^2 - 2\langle z, \tilde{u} \rangle_\infty \leq E(v) - \lambda \|v\|_\infty^2 - 2\langle z, v \rangle_\infty
\]
which implies $\tilde{u} = -R_\lambda z$. Due to $u$‘s unicity, we conclude that $(u_\alpha)$ weakly converges to $u$. Let us prove that $\|u_\alpha\|_\alpha$ converges to $\|u\|_\infty$. In that aim take a strongly convergent net $v_\alpha$ to $v$ such that $\lim_\alpha E_\alpha(v_\alpha) = E(u)$, and take a new look at inequality (17):
\[
E_\alpha(u_\alpha) - \lambda \|u_\alpha + z_\alpha/\lambda\|_\alpha^2 \leq E_\alpha(v_\alpha) - \lambda \|v_\alpha + z_\alpha/\lambda\|_\alpha^2
\]
using (F1') once again we find
\[
E(v) - \lambda \limsup_\alpha \|u_\alpha + z_\alpha/\lambda\|_\alpha^2 \leq E(v) - \lambda \|u + z/\lambda\|_\infty^2
\]
thus $\|u_\alpha + z_\alpha/\lambda\|_\alpha^2 \to \|u + z/\lambda\|_\infty^2$ which implies the strong convergence of $(u_\alpha + z_\alpha/\lambda)_\alpha$ and the strong convergence of $(z_\alpha)$ induces the strong convergence of $(u_\alpha)$.

We shall now study the compact $\Gamma$-convergence. Let us take a weakly convergent net $w_\alpha$ to $w$ and let $u_\alpha = -R_\lambda^\alpha w_\alpha$, then the net $u_\alpha$ is still bounded. Swapping $z_\alpha$ with $w_\alpha$ in (17) we get that $\limsup_\alpha E_\alpha(u_\alpha)$ is bounded, and thanks to the asymptotic compactness we can extract a strongly convergent subnet with $\tilde{u}$ its limit. Putting this in (17), with $z_\alpha = v_\alpha$ where $(v_\alpha)$ a strongly converging net to $v$ we get
\[
E(\tilde{u}) - \lambda \|\tilde{u}\|^2 - 2\langle w, \tilde{u} \rangle \leq E(v) - \lambda \|v\|^2 - 2\langle w, v \rangle
\]
thus $\tilde{u} = -R_\lambda w$. Once again, thanks to unicity, we conclude that $R_\lambda^\alpha w_\alpha$ strongly converges to $R_\lambda w$.

Reciprocally assume that for every $\lambda < 0$ the net $R_\lambda^\alpha$ strongly converges to $R_\lambda$. In what follows $(u_\alpha)$ will be a strong convergent net to $u$.

**Condition (F1')**: Already done, see Proposition 20.

**Condition (F2)**: Extract a subnet $\lambda_\alpha \to -\infty$ such that
\[
E(u, u) \geq \lim_\lambda \lim_\alpha a_\alpha^\lambda(u_\alpha, u_\alpha) \geq \lim_\alpha a_\alpha^\lambda(u_\alpha, u_\alpha)
\]
take $w_\alpha = \lambda_\alpha R_\lambda^\alpha u_\alpha$ for every $\alpha$ and notice that
\[
a_\alpha^\lambda(u_\alpha, u_\alpha) = -\langle u_\alpha - \lambda R_\lambda^\alpha u_\alpha, u_\alpha \rangle_\alpha - \lambda \langle u_\alpha - \lambda R_\lambda^\alpha u_\alpha, -\lambda R_\lambda^\alpha u_\alpha \rangle_\alpha
\]
\[
+ \lambda \langle u_\alpha - \lambda R_\lambda^\alpha u_\alpha, -\lambda R_\lambda^\alpha u_\alpha \rangle_\alpha
\]
\[
= -\langle u_\alpha - \lambda R_\lambda^\alpha u_\alpha \rangle_\alpha^2 + \lambda^2 \langle u_\alpha - \lambda R_\lambda^\alpha u_\alpha, -R_\lambda^\alpha u_\alpha \rangle_\alpha
\]
\[
= -\langle u_\alpha - \lambda R_\lambda^\alpha u_\alpha \rangle_\alpha^2 + E_\alpha(\lambda R_\lambda^\alpha u_\alpha)
\]
indeed if $a_\alpha$ is the bilinear form corresponding to $E_\alpha$ then $R_\lambda^\alpha u_\alpha$ can be seen as the sole element such that
\[
a_\alpha(-R_\lambda^\alpha u_\alpha, v_\alpha) - \lambda \langle -R_\lambda^\alpha u_\alpha, v_\alpha \rangle_\alpha = \langle u_\alpha, v_\alpha \rangle_\alpha, \quad \forall v_\alpha \in D(E_\alpha)
\]
hence
\[ a_\alpha^{\lambda_\alpha}(u_\alpha, w_\alpha) = \mathcal{E}_\alpha(w_\alpha, w_\alpha) - \lambda_\alpha \| u_\alpha - w_\alpha \|^2 \]
which implies \( w_\alpha \rightharpoonup u \) strongly in \( L^2 \) and
\[ \mathcal{E}(u, u) \geq \lim_{\alpha \to +\infty} \mathcal{E}_\alpha(w_\alpha, w_\alpha). \]

For the compact convergence case it suffices to prove the asymptotic compactness, but it has already been done in the proof of Proposition 20. □

6. Proof of Theorem 2.

The convergence of the eigenvalue is given by Theorem 1. Hence it remains to bound the asymptotic \( \lambda_1 \) (i.e., the limit) and characterise the equality. The proof we propose consists in finding an upper bound of \( \lambda_1(B_g(\rho)) \) for every \( \rho \) using a function depending of the distance from the centre of the ball. We then use the simple convergence of the distances \( (d_\rho) \) to the stable norm as seen in Section 2.2 and the measure part of Theorem 10.

Proof. Let \( f \) be a continuous function from \( \mathbb{R} \) to \( \mathbb{R} \) and define
\[ f_\rho : B_\rho(1) \to \mathbb{R} \]
\[ x \mapsto f(d_\rho(0, x)) \]
and \( f_\infty(x) = f(\|x\|_\infty) \) on \( B_\infty(1) \). We want to show that (remember that \( \delta_\rho(x) = \rho x \))
\[ \int f_\rho \cdot \chi_{B_g(\rho)} \circ \delta_\rho \, d\mu_\rho \xrightarrow{\rho \to \infty} \int_{B_\infty(1)} f_\infty \, d\mu_\infty. \] (18)
To obtain this we are going to cut the difference in three pieces, i.e.,
\[ \left| \int f_\rho \cdot \chi_{B_g(\rho)} \circ \delta_\rho \, d\mu_\rho - \int_{B_\infty(1)} f_\infty \, d\mu_\infty \right| \]
(19)
\[ \leq \left| \int f_\rho \cdot (\chi_{B_g(\rho)} \circ \delta_\rho - \chi_{B_\infty(1)}) \, d\mu_\rho \right| \]
(20)
\[ + \left| \int_{B_\infty(1)} (f_\rho - f_\infty) \, d\mu_\rho \right| \]
(21)
Now it suffices to notice that:
1) Part (19) goes to 0 because inside we have the product of \( \chi_{B_g(\rho)} \circ \delta_\rho - \chi_{B_\infty(1)} \), which is easily seen to simply converge to 0 thanks to Corollary 9, with bounded terms compactly supported.
2) Same reason for (20) because \( f_\rho - f_\infty \) simply converges to 0.
3) Finally the convergence to 0 of (21) is due once again to the measure part of Theorem 10.

As a conclusion we have (18). Injecting now $f_\rho$ into the Raleigh’s quotient we get:

$$\rho^2 \lambda_g(B_\rho(\rho)) \leq \frac{\iint (f')^2 \cdot \chi_{B_\rho(\rho)} \circ \delta_\rho \, d\mu_\rho}{\iint (f)^2 \cdot \chi_{B_\rho(\rho)} \circ \delta_\rho \, d\mu_\rho}.$$ 

We apply the limit (18) to obtain

$$\limsup_{\rho \to \infty} \rho^2 \lambda_g(B_\rho(\rho)) \leq \frac{\int_{B_\infty(1)} (f')^2 \, d\mu_\infty}{\int_{B_\infty(1)} f^2 \, d\mu_\infty}$$

and now taking for $f$ the right function (i.e., the solution of the differential equation $f'' + \frac{n-1}{x} f'(x) + \lambda_{e,n} f = 0$) we can conclude.

Let us now study the equality case. Take again the function $f$ which gives the eigenfunction of the Euclidean Laplacian on the Euclidean unit ball (i.e., the solution of $f'' + \frac{n-1}{x} f'(x) + \lambda_{e,n} f = 0$) and normalise it. The $\Gamma$-convergence theory allows to say, taking $E_\rho$ and $E_\infty$ as the energies of $\Delta_\rho$ and $\Delta_\infty$ on the balls $B_\rho(1)$ and $B_\infty(1)$ respectively for the adapted measures and thanks to Proposition 20 and Theorem 1

$$E_\infty(f_\infty) \leq \liminf_{\rho \to \infty} E_\rho(f_\rho) \leq \limsup_{\rho \to \infty} E_\rho(f_\rho) \leq \lambda_{e,n}.$$ 

(22) 

Now from the equality assumption we have

$$\lambda_{e,n} \leq E_\infty(f_\infty),$$ 

(23) 

thus (22) and (23) imply equality which in turn imply that $f_\infty$ is an eigenfunction for the first eigenvalue. Hence $f_\infty$ is smooth (at least in a neighbourhood of zero).

Now from the study of Bessel’s function (see [Bow58], §103-§105) we see that taking $p = (n-2)/2$ we have $f(x) = x^{-p}J_p(\sqrt{\lambda_e}x)$ with $J_p$ an analytic function defined by (see F. Bowman [Bow58] §84)

$$J_p(x) = \frac{x^p}{2^p p!} \left(1 - \frac{x^2}{2 \cdot 2n + 2} + \frac{x^4}{2 \cdot 4 \cdot 2n + 2 \cdot 2n + 4} + \cdots \right)$$

thus $f$ has the following shape:

$$f(x) = \frac{\lambda_e^p}{2^p p!} \left(1 - \frac{x^2 \lambda_e}{2 \cdot 2n + 2} + \frac{x^4 \lambda_e^2}{2 \cdot 4 \cdot 2n + 2 \cdot 2n + 4} + \cdots \right)$$

in other words $f$ has the following asymptotic expansion: $1 + \alpha_1 x^2 + \alpha_2 x^4 + \cdots$ (up to a multiplicative constant). Now notice that the function $1 + \alpha_1 x + \alpha_2 x^2 + \cdots$ admits an inverse $g \in C^\infty$ in a neighbourhood of zero, which
implies that \( g \circ f_\infty(x) = cst \cdot \|x\|_\infty^2 \) is \( C^2 \) in a neighbourhood of zero, thus the stable norm comes from a scalar products, which means that it is Euclidean.

In fact we have some more informations. Indeed in order for \( f_\infty \) to be an eigenfunction, the norme of the differential of the stable norm with respect to the Albanese metric (the scalar product giving the Laplacian \( \Delta_\infty \)) must be almost everywhere equal to one (a simple computation using the fact that the stable norm is Euclidean and the Cauchy-Schwartz inequality). Which implies that the unit ball of the Albanese metric must be inside the unit ball of the stable norm. Now the maximum principle and the monotony with respect to inclusion of the eigenvalues implies that equality holds if and only if the stable norm and the Albanese metric coincides. The stable norm being the Albanese metric we can now use Theorem 33 to conclude.

\[ \text{Theorem 33. Let } (\mathbb{T}^n, g) \text{ be a torus, its stable norm coincides with the Albanese metric if and only if the torus is flat.} \]

\[ \text{Proof. Let us take a base } \eta_1, \ldots, \eta_n \text{ of Harmonic one forms, any function } \alpha \text{ and any 2-form } \beta. \text{ We shall write } \langle \cdot, \cdot \rangle_g \text{ the pointwise scalar product induced by } g \text{ on forms } (\| \cdot \|_g \text{ the associated norm}) \text{ and } \langle \cdot, \cdot \rangle_g \text{ the integral scalar product normalized by the volume. Then by Hodge's theorem} \]

\[ \| \eta_i \|_\infty^2 = \inf_{\alpha,\beta} \sup_{x \in \mathbb{T}^n} \| \eta_i \|_g^2 + \| d\alpha \|_g^2 + \| d\beta \|_g^2 \geq \sup_{x \in \mathbb{T}^n} \| \eta_i \|_g^2 \]

and

\[ \langle \eta_i, \eta_i \rangle_g = \frac{1}{\text{Vol}_g(\mathbb{T}^n)} \int_{\mathbb{T}^n} \| \eta_i \|_g^2 d\text{vol}_g \leq \sup_{x \in \mathbb{T}^n} \| \eta_i \|_g^2 \]

the case of equality implies that \( \langle \eta_i, \eta_i \rangle_g = \langle \eta_i, \eta_i \rangle_g(x) \) for all \( x \in \mathbb{T}^n \). Now it suffices to see that the metric \( g \) can be written in the following way:

\[ \sum_{i,j} \lambda_{ij} \eta_i \circ \eta_j = g \]

where \( \eta_i \circ \eta_j = 1/2(\eta_i \otimes \eta_j + \eta_j \otimes \eta_i) \) and \( \Lambda = (\lambda_{ij}) \) is the matrice such that \( \Lambda^{-1} = (\langle \eta_i, \eta_j \rangle_g) \). Now taking local \( f_i \) such that \( df_i = \eta_i \) then the function \( F(x) = (f_1(x), \ldots, f_n(x)) \) is an isometry between an open set of \( \mathbb{T}^n \) and an euclidean space, thus the torus is flat.

\[ \text{7. Related topics.} \]

In that section we come back to the asymptotic volume, proving in the meantime a generalised Faber-Krahn inequality. Then we explain what can be deduced from our work for the heat kernel and how it is related to other's work. We finally state how Theorem 1 passes to graded nilmanifolds.
7.1. Asymptotic volume of tori.

7.1.1. Generalised Faber-Krahn inequality. We need some more definitions.

**Definition 34.** For a rectifiable submanifold \( N \) of \( \mathbb{R}^n \) (we can think of it of finite adapted Hausdorff measure) we will write \( I(N) \) the associated integral current. For an integral current \( C \), \( M(C) \) will be its mass as defined par H. Federer (see [Fed69] for example).

**Definition 35.** Let \( \mathbb{R}^n \), with the norm \( \| \cdot \| \) (\( \| \cdot \|_* \) will be the dual norm), we define

\[
\lambda_1(\Omega, \| \cdot \|) = \inf_f \frac{\int \| df \|^2_* d\mu}{\int \Omega f^2 d\mu}
\]

where \( \mu \) is the Lebesgue measure on \( \mathbb{R}^n \), and the infimum is taken over all Lipschitz functions vanishing on the border.

The following lemma holds:

**Lemma 36** (Faber-Krahn inequality for norms). Let \( D \) be a domain of \( \mathbb{R}^n \), with the norm \( \| \cdot \| \) and a measure \( \mu \) invariant by translation. Let \( D^* \) be the norm’s ball with same measure as \( D \), then

\[
\lambda_1(D^*, \| \cdot \|) \leq \lambda_1(D, \| \cdot \|)
\]

the equality case implying that \( D \) is a norm’s ball.

**Proof.** We need two ingredients for this proof. The first is an isoperimetric inequality, which is given by a result of Brunn (see a proof by M. Gromov in [MS86]). The second is a co-area formula, which can be found in Federer [Fed69] p. 438.

More specifically, let us write \( G_t = \{ x \mid |f(x)| = t \} \) then on one side we have

\[
\int_{G_t} h \alpha \wedge df = \int_0^{\sup f} \int_{G_t} h \alpha|_{G_t} dt = \int_0^{\sup f} I_{|f|=t}(h\alpha) dt
\]

and on the other

\[
\int_{\Omega} \| df \|_* d\mu = \int_0^{\sup f} M(I_{|f|=t}) dt
\]

(see P. Pansu [Pan99]) where \( d\mu \) is the translation invariant volume form on \( \mathbb{R}^n \) such that the norm’s ball of radius one has measure 1.

Take \( \alpha = \frac{1}{|df|^2} * df \) where * is the Hodge operator on differential forms over \( \mathbb{R}^n \) then we get

\[
\int_{\Omega} h d\mu = \int_0^{\sup f} \int_{G_t} h \alpha|_{G_t} dt.
\]
Now take a look at the same equality on $\Omega_t = \{ x \, | \, |f(x)| > t \}$ i.e.,

\( \int_{\Omega_t} h d\mu = \int_{t}^{\sup f} \int_{G_t} h\alpha_{|G_t} dt = \int_{t}^{\sup f} I_{|f|=t}(h\alpha) dt. \) (26)

Differentiating each member of equality (26) we get almost everywhere the following equality:

\( \int_{G_t} h\alpha_{|G_t} = I_{|f|=t}(h\alpha). \) (27)

Taking into account (27) and (25) we obtain

\( \int_{G_t} \| df \|_*^{\alpha_{|G_t}} = M(I_{|f|=t}). \) (28)

Applying the Cauchy-Schwartz inequality to the left side of (28) and making the appropriate identification thanks to (27) we finally have

\( \frac{M(I_{|f|=t})^2}{I_{|f|=t}(\alpha)} \leq I_{|f|=t}(\| df \|_*^{2\alpha}). \) (29)

The function $f^*$ associated to $f$ by symmetrisation is Lipschitz. Thus it satisfies a similar co-area formula. Hence we have for almost all $t$

\( I_{|f|=t}(\alpha) = - \frac{d}{dt} \operatorname{Vol}(\Omega_t) = - \frac{d}{dt} \operatorname{Vol}(\Omega_t^*) = I_{|f^*|=t}(\alpha^*). \) (30)

Now using Brunn’s isoperimetric inequality (see [MS86]) we have

\( M(I_{|f^*|=t}) \leq M(I_{|f|=t}). \) (31)

Injecting (30) and (31) in (29) and noticing that $\| df^* \|_*$ is constant on $\{|f| = t\}$, which implies that the equivalent of (29) for $f^*$ is an equality we get (for almost all $t$)

\( I_{|f^*|=t}(\| df^* \|_*^{2\alpha^*}) = \frac{M(I_{|f^*|=t})^2}{I_{|f^*|=t}(\alpha^*)} \leq \frac{M(I_{|f|=t})^2}{I_{|f|=t}(\alpha)} \leq I_{|f|=t}(\| df \|_*^{2\alpha}). \) (32)

Now we sum the extremal terms of (32) to obtain the desired inequality:

\[ \int_{\Omega^*} (\| df^* \|_*)^2 dv \leq \int_{\Omega} (\| df \|_*)^2 dv \]

which allows us to conclude the proof because

\[ \int_{\Omega^*} (f^*)^2 dv = \int_{\Omega} (f)^2 dv. \]

For the equality case, it suffices to see that it implies the equality case in Brunn’s isoperimetric inequality to conclude.

\[ \square \]

Let us notice that this lemma immediately implies:
Corollary 37. Let $D_1$ be the unit ball of the norm $\| \cdot \|$. Then
\[
\lambda_1(D_1) = \lambda_{e,n}.
\]
Thus
\[
\lambda_{e,n} \left( \frac{\mu(D_1)}{\mu(D)} \right)^\frac{2}{n} \leq \lambda_1(D, \| \cdot \|)
\]
where $\mu$ is a Haar measure on $\mathbb{R}^n$.

Proof. The symmetrisation from the previous theorem shows that the minimum of the Rayleigh’s quotient is obtained with functions depending on the distance from the centre of the ball. Hence we are led to the same calculations as in the Euclidean case. \hfill \Box

7.1.2. Lower bound for the asymptotic volume. We are now going to apply the generalised Faber-Krahn inequality to $\lambda_\infty$. With that aim in mind let us notice that $\lambda_\infty(B_\infty(1)) = \lambda_1(B_\infty(1), \| \cdot \|_2^\ast)$ with the dual norm of $\| \cdot \|_2^\ast$ defined by
\[
\| \xi \|_2^\ast = \sum_{ij} q_{ij} \xi_i \xi_j
\]
and let us write $B_{Al}$ the unit ball of $\| \cdot \|_2^\ast$. We now can apply the inequality of Lemma 36 and more precisely its Corollary 37:
\[
\lambda_\infty(B_\infty(1)) \geq \left( \frac{\mu(B_{Al})}{\mu(B_\infty(1))} \right)^{2/n} \lambda_{e,n}
\]
where $\mu$ is any Haar measure. Now applying Theorem 2 we get
\[
(33) \quad \left( \frac{\mu(B_{Al})}{\mu(B_\infty(1))} \right)^{2/n} \lambda_{e,n} \leq \lambda_{e,n}.
\]

We finally get the following proposition taking in (33) the Haar measure such that the measure of $B_\infty(1)$ is the asymptotic volume (i.e., the measure $\mu_\infty$) and transforming the other term in order to make the Albanese torus’s volume appear.

Proposition 3. Let $(\mathbb{T}^n, g)$ be a Riemannian torus, $B_g(\rho)$ the geodesic balls of radius $\rho$ centred on a fixed point and $\text{Vol}_g(B_g(\rho))$ their Riemannian volume induced on the universal cover, writing
\[
\text{Asvol}(g) = \lim_{\rho \to \infty} \frac{\text{Vol}_g(B_g(\rho))}{\rho^n}
\]
then:

1. $\text{Asvol}(g) \geq \frac{\text{Vol}_g(\mathbb{T}^n)}{\text{Vol}_{Al}(1^n)} \omega_n$.
2. In case of equality, the torus is flat.

Here $\omega_n$ is the unit Euclidean ball’s Euclidean volume.
Proof. There remains the equality case to be proved, which can be obtained using either the equality case of the Faber-Krahn inequality, which says that $B_\infty(1)$ is an ellipsoid either the equality case of Theorem 2 and then we conclude by using Theorem 33. □

We still have two remarks concerning this proposition, the first one is included in the following corollary:

**Corollary 4.** For $n = 2$ we have:

1. Asvol($g$) ≥ $\pi = \omega_2$.
2. In case of equality, the torus is flat.

In other words we obtain the theorem of D. Burago and S. Ivanov on the asymptotic volume of tori in the 2 dimensional case (see [BI95]). The second remark is that we can not do better this way. See [Ver01] Part Three for more details.

### 7.2. Long time asymptotics of the heat kernel.

Let $(\mathbb{T}^n, g)$ be a torus and $(\mathbb{R}^n, \tilde{g})$ its universal cover with the lifted metric. We remind the reader that $g_\rho = (1/\rho^2)^2 \delta_\rho \tilde{g}$ are the rescaled metrics and $\Delta_\rho$ their Laplacian, here it will be on $\mathbb{R}^n$.

We are going to study from the homogenisation point of view the long time asymptotic behaviour of the heat kernel i.e., we are interested in the behaviour as $t$ goes to infinity of a solution $u(t, x)$ of the following problem:

$$
\begin{cases}
\frac{\partial u}{\partial t} + \Delta u = 0 & \text{in } ]0, +\infty[ \times \mathbb{R}^n \\
u(0, x) = u_0(x).
\end{cases}
$$

(34)

For a probabilistic insight one could see M. Kotani and T. Sunada [KS00].

Let us introduce the rescaled functions

$$
u_\rho(t, x) = \rho^nu(\rho^2t, \delta_\rho x), \quad \rho > 0.
$$

It is straightforward that (see. Section 2.4) $u$ is a solution of (34) if and only if $u_\rho$ is a solution of

$$
\begin{cases}
\frac{\partial u_\rho}{\partial t} + \Delta_\rho u_\rho = 0 & \text{in } ]0, +\infty[ \times \mathbb{R}^n \\
u_\rho(0, x) = \rho^n\delta_\rho x)
\end{cases}
$$

(35)

hence studying $u(t, \cdot)$ as $t$ goes to infinity is the same as studying $u_\rho(1, \cdot)$ as $\rho \to \infty$. In other words we are once again lead to the study of the spectral structures $(\Delta_\rho)$ on $\mathbb{R}^n$. We have:

**Theorem 38.** The net of resolvents $(R_\rho^\lambda)$ weakly converges to the resolvent $(R_\infty^\lambda)$ of $\Delta_\infty$ in $L^2(\mathbb{R}^n)$.
Remark. The proof is the same as 25. In fact in that case we would rather talk of $G$-convergence. We now can apply the theorems from Chapter III of \cite{ZKON79}, more precisely Theorems 4 and 6.

**Theorem 39** (\cite{ZKON79} p. 136). The fundamental solution $k(t, x, y)$ of (34) has the following asymptotic expansion:

$$k(t, x, y) = k_{\infty}(t, x, y) + t^{-\frac{n}{2}} \theta(t, x, y)$$

where $k_{\infty}(t, x, y)$ is fundamental solution of

$$\frac{\partial u_{\infty}}{\partial t} + \Delta u_{\infty} = 0 \text{ in } [0, +\infty[ \times \mathbb{R}^n$$

and $\theta(t, x, y) \to 0$ uniformly as $t \to \infty$ on $|x|^2 + |y|^2 \leq at$, for any fixed constant $a > 0$.

Remark. This is slightly weaker than Theorem 1 of M. Kotani and T. Sunada in \cite{KS00}.

**Theorem 40** (\cite{ZKON79} p. 138). Let $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then $u(t, x)$ the solution of (34) has the following asymptotic expansion:

$$u(t, x) = c_0(4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} u_0(y)dy + t^{-\frac{n}{2}} \theta(t, x)$$

where $\theta(t, x)$ converges uniformly to 0 for $|x| < R$ where $R$ is a positive constant and $c_0$ is the determinant of the matrix associated to $\Delta_{\infty}$.

That last claim can be made precise by:

**Theorem 41** (Duro, Zuazua \cite{DZ00}). Let $u_0 \in L^1(\mathbb{R}^n)$. The sole solution of (34) satisfies for every $p \in [1, +\infty[$:

$$t^{n/2(1-1/p)} \|u(t) - u_{\infty}(t)\|_p \to 0, \text{ as } t \to +\infty$$

where $u_{\infty}$ is the unique solution of the homogenised problem (36). For $n = 1$ and $n = 2$ (37) is also true for $p = \infty$.

7.3. The macroscopical sound of graded nilmanifolds. In this part we want to emphasise the fact that Theorem 1 is still true for graded nilmanifolds, at least for the Dirichlet case, but it involves some sub-Riemannian geometry. We just give the statement. The details are to be found in \cite{Ver01} Chapter Two.

**Theorem 42**. Let $(M^n, g)$ be graded nilmanifold, $B_g(\rho)$ the induced Riemannian ball of radius $\rho$ on its universal cover and $\lambda_i(B_g(\rho))$ the $i$th eigenvalue of the Laplacian on $B_g(\rho)$ for the Dirichlet problem.

Then there exists an hypoelliptic operator $\Delta_{\infty}$ (the Kohn Laplacian of a left invariant metric), whose $i$th eigenvalue for the Dirichlet problem on the stable ball is $\lambda_i^\infty$ and such that

$$\lim_{\rho \to \infty} \rho^2 \lambda_i(B_g(\rho)) = \lambda_i^\infty.$$
Here the stable ball is the metric ball given by the Carnot-Caratheodory distance found in [Pan82] and arising from the stable norm.


Let \((\Sigma_\alpha)\) be a net of spectral structures and let us focus on the spectra. For a fixed operator \(\sigma(\cdot)\) will be its spectrum. Let us begin with the case of strong convergence.

**Proposition 43.** If \(\Sigma_\alpha \to \Sigma\) strongly, then for any \(\lambda \in \sigma(A)\) there is \(\lambda_\alpha \in \sigma(A_\alpha)\) such that the net \((\lambda_\alpha)\) converges to \(\lambda\), this is written

\[
\sigma(A) \subset \lim_\alpha \sigma(A_\alpha).
\]

**Proof.** Let \(\lambda \in \sigma(A)\) and \(\varepsilon > 0\) and take \(\zeta = \lambda + i \varepsilon\) then:

\[
\|R_\zeta\|_{\mathcal{L}_\alpha} = \frac{1}{\inf_{\rho \in \sigma(A_\alpha)} |\zeta - \rho|} \text{ and } \|R_\zeta\|_{\mathcal{L}_\infty} = \frac{1}{\inf_{\rho \in \sigma(A)} |\zeta - \rho|} = \frac{1}{\varepsilon}.
\]

From the assumption, the net of resolvents strongly converges hence by Proposition 16

\[
\limsup_\alpha \inf_{\rho \in \sigma(A_\alpha)} |\zeta - \rho| \leq \varepsilon
\]

and as it is true for any \(\varepsilon\), we can conclude. \(\square\)

**Lemma 44.** For any reals \(a,b\) out of the spectra of \(A\) such that \(-\infty \leq a < b \leq +\infty\) then

\[
a \leq E(u) \leq b \quad \text{for every } u \in E([a,b])L_\infty^2 \setminus \{0\},
\]

(where \(E([a,b]) = E([a, +\infty])\) if \(b = +\infty\)).

**Proof.** Let \(a < b\) two reals out of the spectra of \(A\) and

\[
u \in E([a,b])L_\infty^2 \setminus \{0\},
\]

then

\[
\int_{[a,b]} dEv = E([a,b])u = u = \int_{\mathbb{R}} dEv.
\]

Thus \(\langle Eu, u \rangle = 0\) on \(\mathbb{R} \setminus [a,b]\). Now if \(u \in D(A)\),

\[
E(u) = \langle Au, u \rangle = \int_{\mathbb{R}} \lambda d\langle E(\lambda)u, u \rangle = \int_{[a,b]} \lambda d\langle E(\lambda)u, u \rangle
\]

and the last term satisfies

\[
a \|u\|_\infty^2 = a \int_{[a,b]} d\langle E(\lambda)u, u \rangle
\]

\[
\leq \int_{[a,b]} \lambda d\langle E(\lambda)u, u \rangle \leq b \int_{[a,b]} d\langle E(\lambda)u, u \rangle = b \|u\|_\infty^2.
\]

\(\square\)
For any Borel set $I \subset \mathbb{R}$ we write $n(I) = \dim E(I)L^2_{\infty}$ and $n_\alpha(I) = \dim E_\alpha(I)L^2_{\infty}$.

**Proposition 45.** Let $a < b$ two reals out of the point spectrum of $A$. If $\Sigma_\alpha \rightarrow \Sigma$ strongly then

$$\liminf_\alpha n_\alpha([a, b]) \geq n([a, b])$$

and in particular,

$$\liminf_\alpha \dim L^2_\alpha \geq \dim L^2_{\infty}.$$  

**Proof.** Let us consider an orthonormal basis $\{\varphi_k \mid k = 1, \ldots, n([a, b])\}$ of $E([a, b])L^2_{\infty}$. Let $n \in \mathbb{N}$ be a fixed number if $n([a, b]) = \infty$ else $n = n([a, b])$. Then there are nets $\varphi_k^\alpha \in L^2_\alpha$ for $k = 1, \ldots, n$ such that $\lim_\alpha \varphi_k^\alpha = \varphi_k$. As $E_\alpha([a, b]) \rightarrow E([a, b])$ strongly, taking $\psi_k^\alpha = E_\alpha([a, b])\varphi_k^\alpha$ we get

$$\lim_\alpha \psi_k^\alpha = E([a, b])\varphi_k = \varphi_k$$

hence

$$\lim_\alpha (\psi_k^\alpha, \psi_j^\alpha)_\alpha = (\varphi_i, \varphi_j) = \delta_{ij}$$

from which we deduce that $(\psi_k^\alpha)_{k=1,\ldots,n}$ is a free family for $\alpha$ large enough and

$$\lim_\alpha n_\alpha([a, b]) \geq n.$$  

This proves the first assertion. For the second it comes from the fact that $n([a, b])$ converges to $\dim L^2_{\infty}$ as $a \rightarrow -\infty$ and $b \rightarrow +\infty$. \hfill \Box

Let us now have a look at the compact convergence case:

**Theorem 46.** If $\Sigma_\alpha \rightarrow \Sigma$ compactly converges, then for any $a, b$ out of the point spectrum of $A$ such that $a < b$ for $\alpha$ large enough we have $n_\alpha([a, b]) = n([a, b])$. In particular the limit of the sets $\sigma(A_\alpha)$ coincides with $\sigma(A)$.

**Proof.** The compact convergence implies that the operators $R_{\xi}$, $T_1$ and $E([\lambda, \mu])$ are compact (see Proposition 16). Thus the spectrum of $A$ is discrete and $n([a, b]) < \infty$ if $a < b < \infty$. Let $(0 \leq) \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the spectrum of $A$, where

$$\begin{cases} 
 n = 0 & \text{if the spectrum is empty,} \\
 n \in \mathbb{N} & \text{if the spectrum is finite, and} \\
 n = \infty & \text{if the spectrum is a sequence converging to infinity.} 
\end{cases}$$

**Step 1.** Fix $\varepsilon_0$ and let $\Lambda^0_1 = E([-\infty, \lambda_1 + \varepsilon_0])L^2_\alpha$ and $\Lambda_1 = L^2_{\infty}$, where $\lambda_1 = \lambda_1 + \varepsilon_0 = \infty$ if $n = 0$. Let

$$\mu_1 = \liminf_\alpha \inf \{ E_\alpha(u) \mid \|u\|_\alpha = 1, \ u \in \Lambda^0_1 \}.$$  

Lemma 44 allows us to say that $\lim_\alpha n_\alpha([-\infty, \mu_1]) = 0$ for any $\mu \in ]-\infty, \mu_1[$. Applying Proposition 45 we get $n([-\infty, \mu]) = 0$, in other words for any
\[ \mu \leq \mu_1 \text{ then } \mu \leq \lambda_1 \text{ thus } \mu_1 \leq \lambda_1. \] Hence if \( \mu_1 = +\infty, n = 0 \) and \( L^2_\alpha = 0 \) for \( \alpha \) large enough and the theorem is proved in that case.

Suppose that \( \mu_1 < +\infty \). For \( \alpha \) large enough we can find unit vectors \( \varphi^0_1 \in \Lambda^0_\alpha \) such that \( \liminf_\alpha \mathcal{E}_\alpha(\varphi^0_1) = \mu_1 \). From the asymptotic compactness of \( \mathcal{E}_\alpha \) we can extract a subnet \( (\varphi^0_1)_{\alpha \in \mathcal{A}} \) such that \( \varphi_1 = \lim_\alpha \varphi^0_1 \) strongly and thanks to Definition 20 \( \mathcal{E}(\varphi_1) \leq \mu_1 \). The strong convergence induces the convergence of the norms hence \( \|\varphi_1\| = 1 \) and

\[ \lambda_1 = \inf \{ \mathcal{E}(u) \mid \|u\| = 1, u \in \Lambda_1 \} \leq \mathcal{E}(\varphi_1) \leq \mu_1 < +\infty. \]

As a consequence \( n \geq 1, \lambda_1 = 0 = \mathcal{E}(\varphi_1) \) and \( \varphi_1 \) is eigenvector of \( \Lambda \) for \( \lambda_1 \).

Furthermore let us notice that as \( E_\alpha(\{\lambda_1 - \epsilon, \lambda_1 + \epsilon\}) \to E(\{\lambda_1 - \epsilon, \lambda_1 + \epsilon\}) \) strongly for any \( \epsilon > 0 \) fixed and \( E(\{\lambda_1 - \epsilon, \lambda_1 + \epsilon\}) \to E(\{\lambda_1\}) \) strongly when \( \epsilon \to 0 \) there is a net of positives numbers \( \epsilon_i^0 \to 0 \) such that \( E_\alpha(\{\lambda_1 - \epsilon_i^0, \lambda_1 + \epsilon_i^0\}) \to E(\{\lambda_1\}) \) strongly. From this we obtain a net

\[ \psi^0_1 = E_\alpha(\{\lambda_1 - \epsilon_i^0, \lambda_1 + \epsilon_i^0\}) \varphi^0_1 \to E(\{\lambda_1\}) \varphi_1 = \varphi_1. \]

**Step 2.** Let \( \Lambda^0_2 = E([-\infty, \lambda_2 + \epsilon_0]) L^2_\alpha \cap (\varphi^0_1)^\perp \), \( \Lambda_2 = \langle \varphi_1 \rangle^\perp \) and

\[ \mu_2 = \liminf_\alpha \inf \{ \mathcal{E}_\alpha(u) \mid \|u\|_\alpha = 1, u \in \Lambda^0_2 \}. \]

Again Lemma 44 allows us to say that \( \lim_\alpha n_\alpha([-\infty, \mu]) = 0 \) for any \( \mu \in ([\mu_1, \mu_2]) \) and Proposition 45 that \( \mu_2 \leq \lambda_2 \). Hence if \( \mu_2 = +\infty \), we have \( n = 1 \) and \( L^2_\alpha = \langle \psi^0 \rangle \) for \( \alpha \) large enough. Assume \( \mu_2 < \infty \). Take the unitary vectors \( \varphi^0_2 \in \Lambda^0_\alpha \) such that \( \liminf_\alpha \mathcal{E}_\alpha(\varphi^0_2) = \mu_2 \). Then the same discussion as Step 1 gives \( n \geq 2, \lambda_2 = \mu_2 \) and the strong convergence of a subnet of \( (\varphi^0_2) \) to \( \varphi_2 \) an eigenvector of \( \Lambda \) for the eigenvalue \( \lambda_2 \). We also find a net \( \epsilon_i^0 \to 0 \) such that \( \psi^0_2 = E_\alpha(\{\lambda_2 - \epsilon_i^0, \lambda_2 + \epsilon_i^0\}) L^2_\alpha \to \varphi_2 \). Now let us notice that for any \( \epsilon > 0 \) there is \( \alpha_\epsilon \in \mathcal{A} \) such that for all \( \alpha \geq \alpha_\epsilon \) we have:

1. \( \psi^0_i \in E_\alpha(\{\lambda_1 - \epsilon, \lambda_1 + \epsilon\}) L^2_\alpha \) for \( i = 1, 2 \);
2. if \( \lambda_1 + 2\epsilon < \lambda_2 \) then
   \[ E_\alpha(\{\lambda_1 - \epsilon, \lambda_1 + \epsilon\}) L^2_\alpha = \langle \psi^0_\alpha \rangle \quad \text{and} \quad E_\alpha(\{\lambda_1 + \epsilon, \lambda_2 - \epsilon\}) L^2_\alpha = 0. \]

**Step 3.** We repeat this procedure. Setting \( \Lambda^0_k = E([-\infty, \lambda_k + \epsilon_0]) L^2_\alpha \cap (\psi^0_1, \ldots, \psi^0_{k-1})^\perp \)

we have

\[ \lambda_k = \mu_k = \liminf_\alpha \inf \{ \mathcal{E}_\alpha(u) \mid \|u\|_\alpha = 1, u \in \Lambda^0_k \} \]

for \( k \leq n \). Let \( k \in \{1, 2, \ldots, n\} \) and \( \epsilon > 0 \) be sufficiently small compared with \( k \). Then, there exists \( \alpha_{k, \epsilon} \in \mathcal{A} \) such that for any \( \alpha \geq \alpha_{k, \epsilon} \):

1. For each \( \lambda \in \{\lambda_1, \ldots, \lambda_{k-1}\} \) with \( \lambda < \lambda_k \),
   \[ E_\alpha(\{\lambda - \epsilon, \lambda + \epsilon\}) L^2_\alpha = \langle \psi^0_i \mid p_\lambda \leq i \leq q_\lambda \rangle, \]
   where \( p_\lambda = \min\{i \in \mathbb{N} \mid \lambda_i = \lambda\} \) and \( q_\lambda = \max\{i \in \mathbb{N} \mid \lambda_i = \lambda\}; \]
(2) for each $i = 1, \ldots, k - 1$ with $\lambda_i < \lambda_{i+1}$,
\[ E_\alpha(|\lambda_i + \epsilon, \lambda_{i+1} - \epsilon)|L_\alpha^2 = \{0\}. \]

**Conclusion.** Let $a, b \in \mathbb{R}^+ \setminus \sigma(A)$ two given real numbers such that $a < b$, then from what precedes we have for $\alpha$ large enough
\[ E_\alpha([a, b])L_\alpha^2 = \langle \psi_k^\alpha \mid k = 1, \ldots, n \text{ with } a < \lambda_k \leq b \rangle. \]
Thus $n_\alpha([a, b])$ coincides with the number $k$ such that $a < \lambda_k \leq b$, in other words $n((a, b))$. \(\square\)

The proof of Theorem 21 is the same as above, but defining the $\Lambda_k^\alpha$ with the help of the $\varphi_k^\alpha$.

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WORD EQUATION $ABC = CDA, \ B \neq D$

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We find a new formula for solving $ABC = CDA, \ B \neq D$ for 4 nonempty words in a free semigroup. Properties of the solutions are derived.

1. Introduction.

In a free semigroup, let $Q$ be a quadruple $\langle A, B, C, D \rangle$ of nonempty words satisfying $ABC = CDA, \ B \neq D$. Hmelevskii [1] gives a formula for $Q$ and proves that the solutions of $ABC = CDA$ cannot be represented by a finite set of formulas involving words and positive integer exponents. Hmelevskii also shows that such a representation does exist for equations in 3 variables. This paper contains a simpler formula for $Q$ and proofs of some of its properties. For more results about words, see [2], [3].

2. Terminology.

Fix an alphabet of letters. A word $W$ is a finite sequence of letters. $|W|$ is the length of $W$; the empty word is 1; $|1| = 0$. Word $X$ followed by word $Y$ is written as a product $XY$. $X \leq Y$ if $XZ = Y$ for some possibly empty word $Z$. A product of $k$ copies of $W$, written as $W^k$, is a power of $W$ if $k \geq 0$ with $W^0 = 1$ and a proper power if $W \neq 1$ and $k \geq 2$. Write $W$ backwards to get $W^*$. So $(XY)^* = Y^*X^*$. Word $W$ is periodic if $W = A(BA)^k$ for some $B \neq 1, \ A, k \geq 2$.

A solution is a quadruple $Q = \langle A, B, C, D \rangle$ of words with $ABC = CDA, \ B \neq D$ and $A, B, C, D \neq 1$. Also use the notation $Q = \langle Q_1, Q_2, Q_3, Q_4 \rangle$. All such $Q$ form a set $\Sigma$. A quadruple $Q$ is unitary if $|A| = 1$. Define $\sigma(Q) = ABCD$. Using words $X, Y, Z$, define special quadruples:

$A_k = A_k(X, Y, Z) = \langle X(YX)^k, YXZ, X(YX)^{k+1}, ZXY \rangle, k \geq 0.$

$B_k = B_k(X, Y, Z) = \langle X, YXZ, XYX(ZXYX)^k, ZXY \rangle, k \geq 0.$

For any quadruple $U = \langle A, B, C, D \rangle$, define functions $p, q$:

$p(U) = \langle ABC, B, C, D \rangle$ \hspace{1cm} and \hspace{1cm} $q(U) = \langle C, D, A, B \rangle$.

If $U \in \Sigma$ then $p(U) \in \Sigma$ and $q(U) \in \Sigma$. Let $\Gamma$ be the set of all finite products of $p$’s and $q$’s. The identity function $i$ is in $\Gamma$ since $(qq)(U) = q(q(U)) = U = i(U)$. 

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Remark 2.1. \( A_k(X, Y, Z) \in \Sigma \) if and only if \( YXZ \neq ZXY \) and \( X \neq 1 \) if \( k = 0 \).

Remark 2.2. \( B_k(X, Y, Z) \in \Sigma \) if and only if \( YXZ \neq ZXY \) and \( X \neq 1 \).

Remark 2.3. If \( \langle A, B, C, D \rangle \) is a solution then \( A \neq C, |A| \neq |C|, |B| = |D| \).

Remark 2.4. If \( \langle A, B, C, D \rangle \) is a solution and \( |A| = 1 \) then \( |C| > 1 \).

3. Summary of results.

(1) Each solution \( Q \) equals \( g(A_k) \) for some \( g \in \Gamma \) and \( A_k \in \Sigma \) (Theorem 5.1).
(2) If \( ABC = CDA, B \neq D \) then \( ABCD \) is not a proper power (Theorem 5.2).
(3) Each unitary solution equals some \( B_k \in \Sigma \) (Theorem 5.3).
(4) If \( \langle A, B, C, D \rangle = g(A_k(a, b, c)) \), \( k \geq 0 \), \( g \in \Gamma \), letters \( a, b, c \), then \( \{B, D\} = \{bac, cab\} \); \( A, C \) have odd lengths; \( A = A^*, C = C^* \) (Theorem 5.4).
(5) For each \( \langle A, B, C, D \rangle \in \Sigma \), \( ABCD \) or \( CDAB \) is periodic and \( ABCD \) or \( CDAB \) equals \( \sigma(B_k) \) for some unitary \( B_k \in \Sigma, k = 0 \) or 1 (Theorem 5.5).

4. Preliminaries.

Lemma 4.1. Let \( Q = A_k(X, Y, Z) \) be a unitary solution with \( k \geq 0 \). Then \( Q \) equals some \( A_0 \).

Proof. \(|X(YX)^k| = 1 \) implies either \( k = 0 \) or \( k = 1 \). If \( k = 0 \) we are done. If \( k = 1 \) then \( X = 1, |Y| = 1, YZ \neq ZY, A_k = A_1(1, Y, Z) = A_0(Y, 1, Z). \)

Lemma 4.2. If \( V = g(U) \) with \( g \in \Gamma, U = A_k, k \geq 0 \) then \( U_1 \leq V_1, U_1 \leq V_3 \).

Proof. True if \( g \) is the identity function or if \( g \) is \( p \) or \( q \). Use an induction argument on the number of \( p \)'s and \( q \)'s in \( g \). \( \square \)

Lemma 4.3. Let \( V = g(A_k) \) be a unitary quadruple for some \( g \in \Gamma, k \geq 0 \). Then \( g = qp^nq \) for some \( n \geq 0 \).

Proof. Let \( U = A_k \). Since \( |V_1| = 1 \), applying Lemma 4.2 to \( V \) yields \( |U_1| = 1 \). By Remark 2.4, \( |U_3| > 1 \). We can assume \( g \) is not the identity function because we can use \( n = 0 \) in that case. Then \( g \) can be expressed as a reduced product of \( p \)'s and \( q \)'s with no 2 adjacent \( q \) terms.

We use the following observations:

(1) \( V = g(U) \) and \( g \) is not the identity function.
(2) \( |U_1| = 1, |U_3| > 1, |V_1| = 1 \).
(3) \( q \) interchanges the first and third components of a quadruple.
(4) $p$ increases the length of the first component of a quadruple.
(5) $p$ preserves the length of the third component of a quadruple.

We conclude that a reduced product for $g$ equals $qp^nq$ for some $n \geq 1$. □

**Lemma 4.4.** If $YXZ \neq ZXY$, $n \geq 2$ then $\sigma(B_n(X,Y,Z)) = \sigma(B_0(X,V,W))$ for some words $V$, $W$ such that $VXW \neq WXV$.

**Proof.** For any $t \geq 0$,

$$\sigma(B_{2t+2}) = \langle XYXZ \rangle^{2t+4}XY, \quad \sigma(B_{2t+3}) = \langle XYXZ \rangle^{2t+5}XY.$$  

$$\sigma(B_{2t+2}) = \sigma(B_0(X,V,W)) \text{ for } V = Y, W = Z(XYXZ)^{t+1}.$$  

$$\sigma(B_{2t+3}) = \sigma(B_0(X,V,W)) \text{ for } V = YXZXY, W = Z(XYXZ)^t.$$  

In each case, $YXZ \leq VXW$ and $ZXY \leq WXV$. Therefore $VXW \neq WXV$. □

The following Lemma is Proposition 1.3.4 in [2].

**Lemma 4.5.** For words, $XZ = XS$ with $Y$, $Z \neq 1$ implies $Y = UV$, $X = U(VU)^k$, $Z = VU$ for some $U$, $V$ with $k \geq 0$. If $X \neq 1$ then we can choose $V \neq 1$.

## 5. Main results.

**Theorem 5.1.** Each solution $Q$ equals $g(A_k)$ for some $g \in \Gamma$ and $A_k \in \Sigma$.

**Proof.** Let $Q = \langle A, B, C, D \rangle$ be a solution. Then $|A| \neq |C|$. We may assume $|A| < |C|$ since the function $q$, applied to $Q$, interchanges $A$ and $C$.

Define $m_m(Q) = |ABC|$ so that $m \geq 3$. Therefore $m > 3$ since $|A| \neq |C|$. Suppose $m = 4$. $|A| = 1 = |B|$, $|C| = 2$ else $|B| = 2$ implies $|A| = 1 = |C|$. Thus $Q = \langle a, a, aa, a \rangle$ for some letter $a$ and $B = a = D$, impossible. So $m \geq 5$. Assume $m = 5$. Then $|A|, |B|, |C|$ equals $\langle 1, 1, 3 \rangle$ or $\langle 1, 2, 2 \rangle$.

In the first case, $Q = \langle a, b, aba, b \rangle$, contradicting $B \neq D$. In the second case, $Q = \langle a, ab, aa, ba \rangle = A_0(a, 1, b)$ for letters $a \neq b$. Thus the theorem is true for $m = 5$.

Use induction on $m$. Assume $m > 5$. Suppose $|AB| < |C|$. Then $C = ABI = JDA$ for some $I, J \neq 1$. $ABJDA = ABC = CDA = ABIDA$ so $I = J$, $ABI = C = IDA$. Then $R = \langle I, D, A, B \rangle$ is a solution.

$$m(R) = |IDA| = |C| < |ABC| = m(Q).$$

By an induction assumption, $R = h(A_k)$ for some $h \in \Gamma$, $A_k \in \Sigma$. Therefore $q(p(h(A_k))) = q(p(R)) = \langle A, B, IDA, D \rangle = Q$. Use $g = qh$.

Now suppose $|C| \leq |AB|$. Using $ABC = CDA$, deduce $C = AI = JA$ for some $I, J \neq 1$. Then $|J| = |I| \leq |B| = |D|$ since $|AI| = |C| \leq |AB|$. Using
ABC = CDA,

\[ ABC = AIDA \implies BC = IDA \implies B = IK \] for some \( K \) using \( |I| \leq |B| \).

\[ ABJA = CDA \implies ABJ = CD \implies D = LJ \] for some \( L \) using \( |J| \leq |D| \).

Then \( AIKJA = ABC = CDA = AILJA \) implies \( K = L \). Apply Lemma 4.5 to \( AI = JA \).

We get \( A = X(YX)^k, I = YX, J = XY \) for some words \( Y \neq 1 \), \( X \) and \( k \geq 0 \). So \( C = AI = X(YX)^k, B = IK = YX, D = KJ = KXY \).

If \( k = 0 \) then \( X = A \neq 1 \). Thus

\[ \langle A, B, C, D \rangle = \langle X(YX)^k, YX, X(YX)^k, KXY \rangle = A_k(X, Y, K) \in \Sigma. \]

Thus \( Q = g(A_k(X, Y, Z)) \) using the identity function for \( g \) and \( Z = K \). □

**Theorem 5.2.** If \( ABC = CDA, B \neq D \) then \( ABCD \) is not a proper power.

**Proof.** It suffices to show that \( ABCD \) or \( CDAB \) is not a proper power. Assume \( |A| < |C| \). Suppose \( ABCD = U^k \) for \( k \geq 2 \). So \( U \leq ABC \) and \( CDAB = V^k \) with \( V \leq CDA \). Since \( |U| = |V| \) and \( ABC = CDA \), it follows that \( U = V \) and \( ABCD = U^k = V^k = CDAB \), \( D = B \), a contradiction. Now assume \( |C| < |A| \). Then a similar argument show that \( CDAB \) is not a proper power. □

**Theorem 5.3.** Each unitary solution equals some \( B_n \in \Sigma \).

**Proof.** Let \( V \) be a unitary solution. Then \( V = g(A_k(X, Y, Z)) \) with \( g \in \Gamma \), \( A_k \in \Sigma \) by Theorem 5.1. \( U = A_k(X, Y, Z) \) is unitary by Lemma 4.2. By Lemma 4.1, \( U = A_0(R, S, T) \) for some \( R, S, T \) with \( SRT \neq TRS \), \( |R| = 1 \).

\[ g = qnq \] for some \( n \geq 0 \) by Lemma 4.3. So \( V = q^2nq(A_0(R, S, T)) = B_n(R, S, T) \). □

**Theorem 5.4.** If \( U = g(A_k(a, b, c)) \) with \( k \geq 0 \), \( g \in \Gamma \), letters \( a, b \) and \( c \), then:

(i) \( \{U_2, U_4\} = \{bac, cab\} \),
(ii) \( U_1, U_3 \) have odd lengths,
(iii) \( U_1 = (U_1)^*, U_3 = (U_3)^* \).

**Proof.** Call \( U \) **good** if (i), (ii), (iii) are true for \( U \). It suffices to prove 3 statements:

1. If \( U = A_k(a, b, c) \) then \( U \) is good.
2. If \( U \) is good then so is \( g(U) \).
3. If \( U \) is good then so is \( p(U) \).

Statements (1), (2) are easily verified. As for (3), assume \( U \) is good. Then

\[ U_1U_2U_3 = U_3U_4U_1, (U_2)^* = (U_4), (U_1)^* = U_1, (U_3)^* = (U_3). \]

Let \( V = \langle U_1U_2U_3, U_2, U_3, U_4 \rangle = p(U) \). Properties (i), (ii) are easily verified for \( V \).

To check (iii) for \( V \), \( (V_3)^* = (U_3)^* = U_3 = V_3 \) and \( (V_1)^* = (U_1U_2U_3)^* = (U_3)^*(U_2)^*(U_1)^* = U_3U_4U_1 = U_1U_2U_3 = V_1 \). □
Theorem 5.5. If \( (A, B, C, D) \in \Sigma \) then \( ABCD \) or \( CDAB \) is periodic. If \( |A| < |C| \) then \( ABCD = XY(XZXY)^{k+2} = \sigma(B_k) \) for some \( X, Y, Z, YXZ \neq ZXY, |X| = 1, k = 0 \) or 1. By symmetry, if \( |C| < |A| \) then \( CDAB \) equals such a product.

Proof. Assume \( |A| < |C| \). \( ABC = CDA \) implies \( C = FA, F \neq 1 \). So \( ABFA = FADA, ABF = FAD \). Rewrite this: \( EF = FG, E = AB, G = AD \). Apply Lemma 4.5 to \( EF = FG \). Get \( F = P(QP)^n, AB = PQ, AD = QP, Q \neq 1, n \geq 0, ABCD = P(QP)^{n+2}. P = 1 \) cannot occur since then \( AB = Q = AD \) and \( B = D \), impossible. Thus \( P \neq 1, Q \neq 1, AB = PQ, AD = QP \) imply \( A, P \) and \( Q \) all start with the same word \( X \) of length 1. Therefore there exist \( Y, Z \) with \( P = XY, Q = XZ \). It follows that \( ABCD = P(QP)^{n+2} = \sigma(B_n(X, Y, Z)). \) By Theorem 5.2, \( ABCD \) is not a proper power. Therefore \( XYXZ = PQ \neq QP = XZXY \) implies \( YXZ \neq ZXY \).

For \( n = 0 \) or 1, use \( k = n \).

For \( n > 1 \), apply Lemma 4.4 to \( \sigma(B_n(X, Y, Z)) \) and use \( k = 0 \). □


Define solutions: \( Q_k = (qp)^{2k}(S) \) with \( S = \langle a, ba, aba, ab \rangle = A_0(a, b, 1) \), letters \( a \neq b \). Using a simplified version of a function \( G \) found in [1] we have:

\[ Q_k = \langle G(2k + 2), ba, G(2k + 3), ab \rangle, k \geq 0, \]

where \( G(2) = a, G(3) = aba \), and \( G(n) = G(n-1)Z(n-1)G(n-2), n \geq 4; \]
\( Z(n) = ba (ab) \) if \( n \) even (odd).

For example, \( Q_1, Q_2, Q_3 \) are computed from \( G(4), \ldots, G(9) \) where:

\[ G(4), \ldots, G(7) = XX, XYX, X(YX)^2, (XY)^2(YX)^2, \]
\[ G(8) = (XY)^2Y(YX)^2(YX)^2, \]
\[ G(9) = (XY)^2Y(YX)^2YXY(YX)^2(YX)^2, \]

using \( X = aba, Y = ababa \).

References


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POSITIVE SOLUTIONS TO $\Delta u - V u + W u^p = 0$
AND ITS PARABOLIC COUNTERPART
IN NONCOMPACT MANIFOLDS

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We consider the equation $\Delta u - V(x) u + W(x) u^p = 0$ and its parabolic counterpart in noncompact manifolds. Under some natural conditions on the positive functions $V$ and $W$, which may only have ‘slow’ or no decay near infinity, we establish existence of positive solutions in both the critical and the subcritical case. This leads to the solutions, in the difficult positive curvature case, of many scalar curvature equations in noncompact manifolds. The result is new even in the Euclidean space.

In the subcritical, parabolic case, we also prove the convergence of some global solutions to nontrivial stationary solutions.

1. Introduction.

In this paper we establish positive solutions to some semilinear elliptic equations in noncompact manifolds of dimension $n(\geq 3)$, which involve both the subcritical and critical exponent $(n + 2)/(n - 2)$. We will also prove the convergence of global solutions to nontrivial stationary solutions for some parabolic equations. For the sake of clarity we present these results in three subsections.

1.1. Results on the scalar curvature equations in noncompact complete manifolds. The scalar curvature equations

$$\Delta u - \frac{n - 2}{4(n - 1)} R(x) u + W u^{(n+2)/(n-2)} = 0,$$

are the targets of intensive study over the last decades. Here $M$ is a Riemannian manifold of dimension $n(\geq 3)$, $R$ is the scalar curvature, and $W$ is a function of $x$. In the case of compact manifolds, great progress have been made. For the Yamabe problem ($W$ is a constant), we refer to the survey paper [LP] and the book [Au2] for an account of this matter. In the 80’s, Yau [Yau] and Kazdan [Kaz] suggested the study of (1.1) in the noncompact setting. In the recent book [Au2], this study was proposed again by Aubin.
However the understanding in the noncompact case is still rather limited when the scalar curvature is nonnegative. In the negative scalar curvature case, we refer the reader to [AM], [LTY] and the references there. Some nonexistence and existence results in the positive scalar curvature case can be found in [Ji], [Zh3] (W = 1), [Ki2] (W = 1) and [Ni], [KN], [Ho] (R, W decay rapidly). In Theorem 1.1 below, we establish a general existence result on scalar curvature equations in the most difficult case, i.e., when the scalar curvature is positive and not necessarily decaying.

In order to state the result precisely, it is necessary to recall some well-known objects. We use M to denote a complete noncompact manifold with dimension n ≥ 3. We use 0 to denote a fixed point in M and write d(x) = d(x, 0), the distance from 0 to x ∈ M.

(1) Let R = R(x) be the scalar curvature of M, then the Yamabe invariant is

\[ Y(M) = \inf_{u \in C^0_c(M)} \int_M \left( |\nabla u|^2 + \frac{n-2}{4(n-1)} Ru^2 \right) dx / \|u\|^2_{L^{2n/(n-2)}(M)}. \]

(2) Given a function W = W(x) and a domain D ⊂ M, define

\[ Q(W, D) = \inf_{u \in C^0_c(D)} \left( \int_D |\nabla u|^2 + \frac{n-2}{4(n-1)} Ru^2 \right) dx / \left( \int_D Wu^{2n/(n-2)} dx \right)^{(n-2)/n}. \]

The quantities Y(M) and Q(W, D) have been used widely in the study of conformal properties of both compact and noncompact manifolds. For further properties see the papers [Au], [S], [ES], [E] and [Ki]. For instance, Condition (a) below is exactly the noncompact version of the main assumption in [ES]. Note also Q(1, M) = Y(M).

We also point out that the solutions in Theorem 1.1 below have finite energy in the sense that ∫M |∇u|^2 dx < ∞ and ∫M u^{n/(n-2)} dx < ∞. This will be clear from the construction of the solution.

We hope to find solutions with infinite energy in a future study.

**Theorem 1.1.** Suppose:

(a) M is a n(≥ 3) dimensional complete noncompact manifold with nonnegative scalar curvature and |B(x, r)| ≤ Cr^n for all x ∈ M and r ≥ 1; the Yamabe invariant Y(M) > 0; W(x) ≥ 0, 0 ≠ W ∈ L^∞(M) ∩ C^1(M);

(b) there is a compact exhaustion \{D_j\} of M such that

\[ \sup_j \left[ \left( \max_{x \in D_j} W(x) \right)^{(n-2)/n} Q(W, D_j) \right] < Q_0 = \frac{n(n-2)}{4} (\text{Vol}(S^n))^{2/n}; \]

here Q_0 is the constant in the sharp Sobolev inequality in \( \mathbb{R}^n \).
(c) there is a compact domain $D$ such that

$$\left( \sup_{D^c} W \right)^{(n-2)/n} Q(W, M) < Q(1, D^c).$$

(i) Then (1.1) has a positive solution $u \in L^{\frac{2n}{n-2}}(M)$ such that $u(x) \leq C/(1 + d(x)^{\frac{n-2}{2}})$.

(ii) If, in addition to (a) only, one assumes $M$ is nonparabolic, $\text{Ric} \geq 0$ and $R(x) \geq c_1 + d(x)^b > 0$ with $b < 2$. Let $u \geq 0$ be any solution to (1.1) such that $\int_M u^{2n/(n-2)} dx < \infty$. Then there exist $c_1, c_2 > 0$ and $a = a(b) > 0$ such that, for all $x \in M$,

$$u(x) \leq c_1 e^{-c_2 d(x)^a}.$$

Remark 1.1. Here we show that the assumptions in the theorem are quite natural and encompass large classes of manifolds.

Since $Q(1, B(0, r)^c) \geq Y(M) > 0$, Condition (c) is satisfied if $\lim_{d(x) \to \infty} W(x) = 0$. There are noncompact manifolds satisfying Condition (c) even if $W \equiv 1$ (see [Ki]).

In case $W = W(x)$ reaches absolute maximum at $x_0 \in M$, then $Q(W, D) \leq Q(W, D_1)$ if $x_0 \in D_1 \subset D$. Hence Condition (b) is satisfied if one can find just one compact domain $D_0$ containing $x_0$ so that

$$\left( \max_{x \in D_0} W(x) \right)^{(n-2)/n} Q(W, D_0) < Q_0.$$

The latter is the basic existence condition obtained in the compact case ([ES] (Proposition 1.1), [E]). See also [BN]. Ample examples of the function $W$ are provided in these two papers. Basically $W$ is required to satisfy some flatness condition at its maximum.

Another set of examples comes from compact perturbations of $S^2 \times \mathbb{R}^{n-2}$, $n \geq 6$. Choose a perturbed metric which is not locally conformal flat in $B(x_0, 1)$. Here $x_0$ is a point. By [Au], one can find a function $\phi \in C_0^\infty(B(x_0, 1))$ so that the Yamabe quotient involving $\phi$ is strictly less than $Q_0$. Therefore Condition (b) in Theorem 1.1 is satisfies if $W$ is a constant in $B(x_0, 1)$ provided the constant is the absolute maximum and $W$ converges to zero at infinity. In fact one only need some weaker flatness condition such as vanishing of certain derivatives at the maximum point (see [ES]). Other conditions are satisfied too since we can choose the perturbation so small that the scalar curvature is bounded away from zero.

Remark 1.2. In the papers [ES] and [E], Escobar and Schoen obtained important existence results concerning (1.1) in compact manifolds with and without boundaries.
In the paper \([\text{Ki}, \text{Ki}2]\), under similar assumptions, Kim obtained interesting existence result for (1.1) with \(W = 1\). However some clarification seems needed. In the last paragraph on p. 1987 [\text{Ki}], the quoted sharp Sobolev inequality of Aubin contains constants \(C(\epsilon)\) that may depend on the domains. This is because the domains (\(\Omega\) in [\text{Ki}]) may not be contained in a compact set even though their volume is finite. This complicates the claim that the ‘approximate solutions \(u_i\)’ are uniformly bounded in compact domains. In this paper we overcome the difficulty by proving a priori decay estimates for solutions, under merely an assumption on the volume of geodesic balls.

One can replace the volume assumption in (a) by assuming that the Ricci curvature is bounded from below and the injectivity radius is positive. Then the Sharp Sobolev inequality holds on the whole manifold (see [\text{He}] e.g.). In this case, the existence part of Theorem 1.1 (i) remains valid (see Remark 2.1 below).

**Remark 1.3.** Theorem 1.1 still holds if \(R\) is replaced by an ordinary function satisfying some similar assumptions. This can be proven by exploiting the results in [\text{E}] in the current setting. In both Theorem 1.1 and the Corollary 1 below, we can allow the function \(W\) to change sign. But we are not seeking the full generality this time.

It is well-known that the scalar curvature of ‘most’ manifolds with non-negative Ricci curvature decay slower than the inverse of distance square, as in Theorem 1.1 (ii). In the case, (finite energy) solutions given in Theorem 1.1 (ii) decay exponentially to zero near infinity. Therefore they do not produce complete conformal metrics. This result has two interesting implications. First, it snugly complements the existence result of complete conformal metric ([\text{Ki2}]), where the opposite assumption on the scalar \(R\) was made. Second it seems to reveal the limit of the direct variational approach, which requires the solution to have finite energy. Moreover it provides a method of conformal compactification. This can be regarded as a generalization of the stereographic projection between \(\mathbb{S}^n\) and \(\mathbb{R}^n\).

An immediate geometric application of the theorem is:

**Corollary 0.** Suppose \(M\) and \(W\) satisfy all conditions in Theorem 1.1 (i) and (ii). If \(M\) has only one end and it is topologically simple at infinity (finite type), then \(M\) is conformal to a closed compact manifold minus one point, with scalar curvature \(W\).

If \(W\) decays as \(c/d(x)^a\) with \(a > 0\), we can obtain existence result on (1.1) via a simpler proof without Condition (c).

**Corollary 1.** Let \(M\) be a complete noncompact manifold with bounded scalar curvature. Suppose the Yamabe invariant is positive and \(|B(x,r)| \leq cr^n\) for any \(x \in M\) and all \(r > 0\). Suppose also:
(a) There is a compact exhaustion \(\{D_j\}\) of \(M\) such that
\[
\sup_j \left( \max_{x \in D_j} W(x) \right)^{(n-2)/n} Q(W, D_j) < Q_0 = \frac{n(n-2)}{4} (\text{Vol}(S^n))^{2/n}.
\]
(b) \(0 \neq W \in L^\infty(M), W(x) \geq 0\) and \(W(x) \leq \frac{c}{1+d(x)^a}\) with \(a > 0\).

Then (1.1) has a positive solution.

The conclusion remains valid if \(M = \mathbb{R}^n, n \geq 3\), and \(R = R(x)\) is any bounded function satisfying (a) that is nonnegative outside a compact set.

Remark 1.4. Under further assumptions, one may be able to show that the metric \(u^{4/(n-2)}g\) is complete, using the idea \(\text{Ki2}\). But we are not able to construct an explicit example. See Remark 2.2 in the next section.

We construct an example of (1.1) covered by Corollary 1. Let \(W(x) > 0\) be a function satisfying (b) and achieving global maximum at \(x = 0\) \(\in \mathbb{R}^4\).

Let \(V(x) = \frac{n-2}{4(n-1)} R(x)\) be a bounded smooth function which is nonnegative outside a compact set. Suppose that the first eigenvalue of the operator \(-\Delta + V\) in \(B(0,1)\) is positive, i.e.,
\[
\int_{B(0,1)} (|\nabla u|^2 + V(x)u^2)dx \geq c \int_{B(0,1)} u^2dx, \quad u \in W^{1,2}_0(B(0,1)).
\]

Suppose also \(V(0) < 0\), then (1.1) in \(\mathbb{R}^4\) has a positive solution.

Let us verify that the conditions of Corollary 1 are met. According to Theorem 3.2 in \([E]\), for the above function \(V\) and \(W\), one has
\[
\left( \max_{x \in B(0,1)} W(x) \right)^{(n-2)/n} Q(W, B(0,1)) < Q_0.
\]

Since \(W\) achieves absolute maximum at \(x = 0\), the above holds when \(B(0,1)\) is replaced by any domain containing it. Hence Condition (a) is met. All other conditions follow easily.

1.2. Results on elliptic equations in the subcritical case. Let us describe the elliptic results in the subcritical case. Consider the equation
\[
(1.2) \quad \Delta u - V(x)u + W(x)u^p = 0 \quad \text{in} \quad \mathbb{R}^n.
\]

Here \(1 < p < \frac{n+2}{n-2}, n \geq 3\). In what follows, unless otherwise stated, we will assume that \(V = V(x)\) and \(W = W(x)\) are locally Hölder continuous, and bounded function.

This equation has a rich history. When \(V = W \equiv 1\), it is well-known that (1.2) has a so-called ground state solution, meaning a positive solution decaying exponentially to zero near infinity. In \([\text{Lio1-2}]\), P.L. Lions obtained existence of nontrivial solutions to (1.2) when \(V\) is a suitable perturbation of a positive constant near infinity and \(W = 1\). His approach is the famous concentration-compactness principle, which is variational in nature. Related
results are also obtained in Ding and Ni [DN] (when \( V = 1 \) and \( W \) satisfies various conditions). See also [NS]. When \( V \) and \( W \) have rapid decay, important existence results were achieved in Ni [Ni], Kenig and Ni [KN]. See also [Zh3]. Here and later a function is said to have rapid decay if it is smaller than \( C/|x|^b \) near infinity, where \( b > 2 \). Otherwise we say it has slow decay. In Stuart [St] existence result was obtained when \( V = 1 \) and \( W \) has slow decay. Subsequently many authors have taken up the study of the problem and produced numerous interesting results.

Despite these advances, the important middle range, i.e., when both \( V \) and \( W \) has slow decay, seems to be completely open in the nonradial case. Our next theorem largely fill this gap. Let us mention that this result is not a direct consequence of the variational approach. Since it is well-known that the concentrated compactness method requires that \( V \) converges to a positive constant at infinity in a special manner. Moreover if \( V \) has rapid decay and \( W \equiv 1 \), (1.2) does not have any positive solution if \( 1 < p < \frac{n}{n-2} \) (see [Zh3]). In this paper we will also introduce a dynamic approach to solve (1.2) (see Section 1.3).

The following table provides a glimpse of the current understanding on the existence of positive solutions to (1.2). It is not intended to be a complete account of the literature.

<table>
<thead>
<tr>
<th>( \Delta u - V(x)u + W(x)u^p = 0 )</th>
<th>existence results, ( 1 &lt; p &lt; \frac{n+2}{n-2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W \to 1 ) and ( V \to 1 ) at ( \infty )</td>
<td>under more assumptions of the convergence</td>
</tr>
<tr>
<td>( W ) and ( V ) decay rapidly</td>
<td>[Ni], [KN], [Lin], [Ka], [Zh3]</td>
</tr>
<tr>
<td>( V = 1 ), ( W ) decays</td>
<td>[St], [DN], [Li]</td>
</tr>
<tr>
<td>( V ) decays rapidly, ( W ) does not</td>
<td>there are nonexistence results [Ni], [Li], [Zh3]</td>
</tr>
<tr>
<td>( W ) and ( V ) have slow decay</td>
<td>current paper, with additional condition on ( W )</td>
</tr>
</tbody>
</table>

The second theorem of the paper is:

**Theorem 1.2.** Suppose \( \frac{a}{1+|x|^b} \leq V(x) \leq C_1 \) with \( b \in [0, 2] \), \( a > 0 \). Suppose \( 0 < W(x) \leq C_2/(1 + |x|^{2-(p-1)(n-2)/2}) \) and \( 1 < p < \frac{n+2}{n-2} \). Then Equation (1.2) has a positive solution such that \( \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \) and \( \int_{\mathbb{R}^n} W u^{p+1} \, dx \) are finite.

The conclusion still holds if \( \mathbb{R}^n \) is replaced by a complete manifold of nonnegative Ricci curvature and maximum volume growth, i.e., \( |B(x, r)| \geq cr^n > 0 \) for all \( x \) and \( r > 0 \).
Remark 1.5. When \( p \to \frac{n+2}{n-2} \) from below the number \( 2 - (p-1)(n-2)/2 \) converges to 0. Since it is well-known that (1.2) may not have any finite energy positive solution when \( V \) and \( W \) are just bounded functions. This indicates that the decay rate on \( W \) is close to optimal.

Note also that both Theorem 1.2 (and 1.3 below) may fail if \( V \) is allowed to decay faster than \( c/|x|^2 \) near infinity (see [Zh2]).

Remark 1.6. Theorems 1.2 and 1.3 continue to hold if the Laplacian in (1.2) and (1.3) is replaced by an uniformly elliptic divergence operator with bounded measurable coefficients depending on \( x \).

The strategy of the proof of Theorem 1.2 is the combined use of domain exhaustion method, Green’s function estimates and certain scaling arguments.

1.3. Results in the parabolic case. Next we present the parabolic results of the paper. One of the central questions in nonlinear analysis is whether or not a global solution to evolution equations would converge to a nontrivial equilibrium solution. This problem is relatively well studied if the ambient space is compact. However this is not the case in noncompact setting. In fact the understanding in this case is rather limited. To illustrate this shortcoming let us consider a model parabolic equation:

\[
\begin{aligned}
\Delta u - V(x)u + W(x)u^p - \partial_t u &= 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\
u(x,0) &= u_0(x).
\end{aligned}
\]

Here \( 1 < p < \frac{n+2}{n-2} \), \( n \geq 3 \). We assume that \( V = V(x) \) and \( W = W(x) \) are locally Hölder continuous, and bounded function.

Problems such as (1.3) also arises from many areas and are some of the central subjects in nonlinear analysis.

In the classical paper [NST], when \( \mathbb{R}^n \) is replaced by a bounded domain, interesting result on the convergence of solutions of (1.3) to that of (1.2) are obtained. However except a few exceptions, the corresponding results for \( \mathbb{R}^n \) and other noncompact domains have not been achieved in general. The papers [CDE] and [BJP] study (1.3) in the case \( V = W = 1 \). The paper [SZ] studies the case \( V, W \) and the solutions are radial. In the paper [Zh3] convergence results on (1.3) when \( V \) and \( W \) have rapid decay were obtained (see Section 1.2 for the meaning of rapid decay). As illustrated by many authors, the nature of the decay for \( V \) and \( W \) near infinity is a deciding factor for the existence and nonexistence of solutions to (1.2) and (1.3). Usually fundamental differences exist between the slower decay cases and the rapid decay ones.

Based on the study of global positive solutions of the linear part of the equation in (1.3) and a scaling argument, we will prove that Equation (1.3) has global positive solutions whose \( \omega \) limit set contains nontrivial positive solutions.
solutions to the elliptic equation (1.2). Up till now have not seen any comparable convergence results for Equation (1.3), except in the case when $V$ and $W$ are radial. We should mention that the types of equations we are studying require $W$ to decay at infinity. An interesting remaining problem is to obtain a similar convergence results when both $V$ and $W$ do not decay to zero. We hope to address it in future.

**Theorem 1.3.** Suppose $\frac{a}{1+|x|^b} \leq V(x) \leq C_1$ with $b \in [0,2)$, $a > 0$. Suppose $0 < W(x) \leq C_2/(1+|x|^{2-(2(p-1)(a-2)/2)})$ and $1 < p < 1 + \frac{4}{n}$. Then for any compactly supported nonnegative $f \neq 0$, there exists $\lambda > 0$ such that the problem

$$\begin{cases}
\Delta u - V(x)u + W(x)u^p - \partial_t u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\
u(x,0) = u_0(x) \equiv \lambda f(x),
\end{cases}$$

has a global positive solution. Moreover the $\omega-$ limit set contains a nontrivial equilibrium solution. The result continues to hold if $\mathbb{R}^n$ is replaced by a complete manifold of nonnegative Ricci curvature and maximum volume growth.

**Remark 1.7.** At this moment we do not know if the assumption $p < 1 + \frac{4}{n}$ can be improved to $p < \frac{n+2}{n-2}$ in the parabolic case.

The rest of the paper is organized as follows: In Section 2 we prove Theorem 1.1. In Section 3 we prove Theorem 1.2. In Section 4 we establish several preliminary estimates on the global solutions to (1.3). We will prove Theorem 1.3 in Section 5.

The proofs of different theorems are related but independent of each other.

### 2. Proof of Theorem 1.1.

The proof is divided into 5 steps. We will use the idea of finite domain exhaustion. A crucial step is to establish certain a priori decay of solutions of (1.1) near infinity.

**Step 1.** We begin by solving, for each $j > 0$, the variational problem

$$\inf_{u \in W^{1,2}_0(B(0,j))} \int_{B(0,j)} (|\nabla u|^2 + Vu^2)dx$$

subject to the constraint $\int_{B(0,j)} Wu^{p+1}dx = 1$. Here and throughout the section $p = (n+2)/(n-2)$ and $V = \frac{(n-2)}{4(n-1)}R(x)$. Following the standard arguments in ([Au], [S] and [ES]), for each $j > 0$, Problem (2.1) has a positive solution under our assumptions. In fact details of the proof are mirrored in Step 3 below where we will prove that these $u_j$ are uniformly bounded in any compact set.
Let \( u_j \geq 0 \) be a solution to (2.1). Then there exists a \( q_j > 0 \) such that
\[
\Delta u_j(x) - Vu_j(x) + q_j W u_j^{p/(n-2)}(x) = 0,
\]
\( x \in B(0,j); \quad u(x) = 0, \quad x \in \partial B(0,j). \)

In fact
\[
q_j = \inf_{u \in W^{1,2}_0(B(0,j))} \frac{\int_{B(0,j)} (|\nabla u|^2 + Vu^2)dx}{\left( \int_{B(0,j)} W u^{p+1}dx \right)^{2/(p+1)}}.
\]

Since the Yamabe invariant is positive and \( W \) is a bounded function. We know that, for any \( j > 0, \)
\[
\left( \int_{B(0,j)} W u^{p+1}dx \right)^{2/(p+1)} \leq (\sup W)^{2/(p+1)} \left( \int_{B(0,j)} u^{p+1}dx \right)^{2/(p+1)} \leq C (\sup W)^{2/(p+1)} \left( \int_{B(0,j)} (|\nabla u|^2 + Vu^2)dx \right).
\]

This shows that there exists a \( q > 0 \) such that \( q_j \) decreases to \( q \) when \( j \to \infty \). In fact \( q = Q(W,M) \), defined in Section 1.

We extend \( u_j \) to the whole manifold by defining \( u_j(x) = 0 \) when \( x \) is outside of \( B(0,j) \). The extended function, still denoted by \( u_j \) is a subsolution to Equation (2.2) in the whole manifold. Our goal is to show that a subsequence \( u_j \) converges to a positive solution to (1.1).

**Step 2.** In this step we will prove the following: There exists \( R_0 > 0 \) and \( C > 0 \) such that
\[
u_j(x) \leq \frac{C}{1 + d(x)^{(n-2)/2}}\]
for all \( j \) and \( x \) when \( d(x) \geq R_0 \). This estimate follows from an argument in [Zh4]. For simplicity we will drop the subscript \( j \) in this step.

For \( R > 1 \) and fixing \( x_0 \) such that \( d(x_0) = 2R^2 \). For each \( R > 1 \), let us introduce the scaled metric
\[g_1 = g/R^4.\]

Let \( M_1 \) be the manifold \( M \) with \( g \) replaced by \( g_1 \) and \( d_1, \Delta_1, \nabla_1 \) be the corresponding distance, Laplace-Beltrami operator, gradient respectively. Note that \( \Delta_1 = R^4 \Delta \).

Let us consider \( v \in C(M_1) \) defined by
\[v(x) = R^{n-2}u(x).
\]

Since \( \Delta u - Vu + q_j W u^{(n+2)/(n-2)} = 0 \) and \( R > 1 \), direct computation shows
\[
\Delta_1 v - V_1 v + q_j W v^{(n+2)/(n-2)} = R^{n+2} (\Delta u - Vu + q_j W u^{(n+2)/(n-2)}) \geq 0,
\]

Here we obtain
\[ \int_{d_1(x_0, x) \leq 1} v^{2n/(n-2)}(x) d_1 x = \int_{d(x_0, x) \leq R^2} u^{2n/(n-2)}(x) dx, \]
\[ \int_{d_1(x_0, x) \leq 1} |\nabla_1 v(x)|^2 d_1 x = \int_{d_1(x_0, x) \leq 1} g_1(\nabla_1 v(x), \nabla_1 v(x)) d_1 x \]
\[ = R^{-2} \int_{d(x_0, x) \leq R^2} |\nabla u(x)|^2 dx. \]

Estimate (2.3) will be proven once we can show that \( E_g \) that in \([\mathbf{Eg}]\) p. 44, which can be generalized to our case since the manifold \( M_1 \) has nonnegative Ricci curvature outside a compact set.

Take \( G(s) = s^\beta \) if \( s > 0 \), and zero otherwise, and put \( F(u) = \int_0^u G'(s)^2 ds = \beta^2 u^{2\beta-1}/(2\beta - 1) \). It is easy to verify that \( sF(s) \leq s^2 G'(s)^2 \leq \beta^2 G(s)^2 \) if \( \beta > 1 \).

Take \( \phi \in C^\infty[0, \infty) \) such that
\[ 0 \leq \phi \leq 1; \quad \phi(r) = 1, r \in [0, 1/2]; \quad \phi(r) = 0, r \in [1, \infty); \]
\[ -C \leq \phi'(r) \leq 0; \quad |\phi''(r)| \leq C. \]

Let \( \eta = \eta(x) = \phi(d_1(x_0, x)) \). Using \( \eta^2 F(v) \) as a test function on the inequality
\[ \Delta_1 v - V_1 v + q_j W v^{(n+2)/(n-2)} \geq 0, \]
we obtain
\[ \int |\nabla_1 v|^2 G'(v)^2 \eta^2 d_1 x + \int V_1 v F(v) \eta^2 d_1 x + 2 \int \nabla_1 v \nabla_1 \eta F(v) \eta d_1 x \]
\[ \leq \beta^2 \int q_j W v^{(n+2)/(n-2)} v^{-1} G(v)^2 \eta^2 d_1 x. \]

Using the inequality
\[ 2|\nabla_1 v \nabla_1 \eta F(v) \eta| \leq \epsilon |\nabla_1 v|^2 \eta^2 v^{-1} F(v) + \epsilon^{-1} |\nabla_1 \eta|^2 F(v) v \]
\[ \leq \epsilon |\nabla_1 v|^2 \eta^2 G'(v)^2 + \epsilon^{-1} \beta^2 |\nabla_1 \eta|^2 G(v)^2 \]
we obtain, for another \( \epsilon > 0 \),
\[ \|\nabla_1 (G(v) \eta)\|^2 + \frac{\beta^2}{2\beta - 1} \int V_1 [G(v) \eta]^2 \]
\[ \leq C \beta^2 \int |\nabla_1 \eta|^2 G(v)^2 d_1 x + (\beta^2 + \epsilon) q_j \int W v^{(n-2)} G(v)^2 \eta^2 d_1 x. \]

Here \( \epsilon \) can be chosen as any small positive number.
At this point we need to use Condition (c) which implies that $Q(1, B(0, r) ^c) > 0$ when $r$ is large. Indeed

\begin{equation}
\left[ \int_{B_1(x_0, r)} [G(v)\eta]^{2n/(n-2)} \right]^{(n-2)/n} \leq C_0 \| \nabla_1 (G(v)\eta) \|_2^2 + C_0 \int V_1 [G(v)\eta]^2.
\end{equation}

Here in fact, we can choose

\[ C_0 = 1/Q(1, B(0, R^2/2)^c) > 0. \]

Noting that $\beta^2 \geq 1$ if $\beta > 1$ and applying Hölder’s inequality, we find that

\begin{equation}
\|G(v)\eta\|_{2n/(n-2)}^2 \leq C C_0 \beta^2 \left( \int W^{n/2} u^{2n/(n-2)} d_1 x \right)^{2/n} \left( \int_{B_1(x_0, 1)} W^{n/2} v^{n/2} u^{2n/(n-2)} d_1 x \right)^{(n-2)/n} + C_0 q_j \left( \sup_{B_1(x_0, 1)} W \right)^{(n-2)/n} (\beta^2 + \epsilon) \|G(v)\eta\|_{2n/(n-2)}^2.
\end{equation}

Here all the norms are over the ball $B_1(x_0, 1)$.

Since

\[ \int_{B_1(x_0, 1)} W^{n/2} u^{2n/(n-2)} d_1 x \leq \int M W^{n/2} u^{2n/(n-2)} d_1 x = 1, \]

we have

\[ \|G(v)\eta\|_{2n/(n-2)}^2 \leq C C_0 \beta^2 \left( \int W^{n/2} v^{n/2} u^{2n/(n-2)} d_1 x \right)^{(n-2)/n} + C_0 q_j \left( \sup_{B_1(x_0, 1)} W \right)^{(n-2)/n} (\beta^2 + \epsilon) \|G(v)\eta\|_{2n/(n-2)}^2. \]
By Assumption (c) in the theorem, when $R$ is sufficiently large,

$$
\left( \sup_{B(0,R^2/2)^c} W \right)^{(n-2)/n} Q(W,M) < Q(1, B(0, R^2/2)^c).
$$

Since $q_j \to Q(W,M)$ when $j \to \infty$, we have

$$
\left( \sup_{B(0,R^2/2)^c} W \right)^{(n-2)/n} q_j < Q(1, B(0, R^2/2)^c) = \frac{1}{C_0}
$$

when $j$ is sufficiently large. Hence there exists a $\beta > 1$ such that

$$
\sup_{B_1(x_0,1)} W^{(n-2)/n} (\beta^2 + \epsilon)(C_0 q_j) < \lambda < 1.
$$

Using this and take $G(v) = v^{\beta_0}$, we see that, for a $\beta_0 > 1$ and sufficiently close to 1,

$$
(2.6) \quad \|v^{\beta_0}\|_{2n/(n-2)}^2 \leq 2C \int |\nabla_1 \eta|^2 v^{2\beta_0} d_1 x.
$$

Choose $\eta$ such that $\eta(x) = 1$ if $x \in B_1(x_0,1/2)$, $|\nabla_1 \eta| \leq 2$ and $\eta(x) = 0$ when $x \in B_1(x_0,1)^c$. Use Hölder’s inequality we see that

$$
\|v^{\beta_0}\|_{B_1(x_0,1/2)}^2 \leq C \int_{B_1(x_0,1)} v^{2\beta_0} d_1 x
$$

$$
\leq \|v\|_{L^{2n/(n-2)}(B_1(x_0,1))}^m \leq C < \infty.
$$

Here $m > 0$. Note this is the place where we are using Condition (a) on volume, which implies $|B_1(x_0,1)| = |B(x_0, R^2)|/R^{2n} \leq C$.

Now using standard method we immediately know that $v(x_0) \leq C_1$. For completeness we give a sketch of the proof.

Given $r_2, r_1$ such that $0 < r_2 < r_1 < 1/2$, we choose $\eta$ to be a radial function with support in $B_1(x_0,r_1)$ and such that $\eta = 1$ if $x \in B_1(x_0,r_1)$ and $|\nabla_1 \eta| \leq 2/(r_2 - r_1)$. Clearly (2.5) remains valid for such $\eta$ and any fixed $\beta \geq 1$. Let $\chi_x$ be the characteristic function of $B(x_0,r)$. By Hölder’s inequality, there exists a $\delta < 1$ and $m > 0$ such that

$$
\|G(v)\chi_{r_1}\|_2 \leq C \|G(v)\chi_{r_1}\|_{2n\delta/(n-2)},
$$

$$
\int_{B_1(x_0,r_1)} v^{4/(n-2)} G(v)^2 d_1 x \leq C \|v\chi_{r_1}\|_{2n\beta_0/(n-2)}^m \|G(v)\chi_{r_1}\|_{2n\delta/(n-2)}^2
$$

$$
\leq C \|G(v)\chi_{r_1}\|_{2n\delta/(n-2)}^2.
$$

Substituting the last two inequalities to the first inequality of (2.5), we obtain

$$
\|G(v)\chi_{r_2}\|_{2n/(n-2)} \leq \frac{C\beta}{r_1 - r_2} \|G(v)\chi_{r_1}\|_{2n\delta/(n-2)}.
$$
From this point, the standard Moser’s iteration of taking $\beta = \delta^{-m}$ and $r_m = r_1(2 + 2^{-m})/4$ shows that

$$v(x_0) \leq c < \infty$$

when $R$ is sufficiently large. Therefore, since $d(x_0) = 2R^2$,

$$u(x_0) = \frac{v(x_0)}{R^{n-2}} \leq \frac{c}{1 + d(x_0)^{(n-2)/2}}.$$ 

This completes Step 2.

**Step 3.** We prove that $u_j$ is uniformly bounded in any given finite domain. This follows from the ideas in [Au], [ES] and [Ki], together with the decay estimates in Step 2.

Let $u_j$ be the subsolution to (2.2), produced in Step 1.

For any fixed $j > 0$, let

$$D = \{ x \in M \mid u_j(x) \geq 1 \}.$$ 

By Step 2, $D$ is contained in a fixed ball of sufficiently large radius. This is a crucial observation for the subsequent proof.

Writing $w_j = u_j - 1$, we see that

$$-\Delta w_j + V(1 + w_j) \leq q_j (1 + w_j)^p.$$ 

For some $b > 0$ to be determined later, multiplying (2.8) by $w_j^{1+2b}$ and integrating, we obtain

$$\int_D \left[ \frac{1 + 2b}{(1 + b)^2} |\nabla w_j^{1+b}|^2 + V(1 + w_j)w_j^{1+2b} \right] dx \leq q_j \int_D W w_j^{1+2b}(w_j + 1)^p dx.$$ 

By virtue of the sharp Sobolev inequality of Aubin [Au], for any $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$\left( \int_D w_j^{(1+b)2n/(n-2)} dx \right)^{(n-2)/n} \leq (1 + \epsilon) \frac{1}{Q_0} \int_D |\nabla w_j^{1+b}|^2 dx + C(\epsilon) \int_D w_j^{2(1+b)} dx.$$ 

Here, as before, $Q_0$ is the best Sobolev constant in $\mathbb{R}^n$. 
Substituting (2.9) to the left-hand side of the above, we see that

\[(\int_D W_j^{(1+b)2n/(n-2)} dx)^{(n-2)/n} \leq (1 + \epsilon) \frac{1}{Q_0} \frac{(1 + b)^2}{(1 + 2b)} \cdot \int_D \left(q_j W_j^{1+2b}(1 + w_j)^{(n+2)/(n-2)} - V(w_j + 1)w_j^{1+2b}\right) dx + C(\epsilon) \int_D w_j^{2(1+b)} dx.\]

As in [K1], we write \(D_1 = \{x \in D \mid w_j(x) \geq 2\}\) and \(D_2 = D - D_1\). When \(x \in D_1\), it is clear that

\[(1 + w_j(x))^{(n+2)/(n-2)} \leq w_j^{(n+2)/(n-2)}(x) + c_1 w_j^{4/(n-2)}(x).\]

When \(x \in D_2\), \(w_j(x) \leq 2\). Hence

\[
\int_D W_j^{1+2b}(1 + w_j)^{(n+2)/(n-2)} dx \\
= \int_{D_1} W_j^{1+2b}(1 + w_j)^{(n+2)/(n-2)} dx \\
+ \int_{D_2} W_j^{1+2b}(1 + w_j)^{(n+2)/(n-2)} dx \\
\leq \int_{D_1} W_j^{2n/(n-2)+2b} dx + c_1 \int_{D_1} W_j^{(n+2)/(n-2)+2b} dx \\
+ \int_{D_2} W_j^{1+2b}(1 + w_j)^{(n+2)/(n-2)} dx \\
\leq \int_{D_1} W_j^{2n/(n-2)+2b} dx + c_2 \int_{D_1} W_j^{(n+2)/(n-2)+2b} dx \\
\leq \left(\int_{D_1} W_j^{(1+b)2n/(n-2)} dx\right)^{(n-2)/n} \left(\int_D W_j^{2n/(n-2)} dx\right)^{2/n} \\
+ c_2 \int_D W_j^{(n+2)/(n-2)+2b} dx \\
\leq \left(\int_{D_1} W_j^{(1+b)2n/(n-2)} dx\right)^{(n-2)/n} \left(\int_D W_j^{(n+2)/(n-2)+2b} dx\right).\]
Here we have used the Hölder’s inequality. Substituting the above into (2.10), we have

\[
\left( \int_{D} w_j^{(1+b)2n/(n-2)} \right)^{(n-2)/n} dx \\
\leq (1 + \epsilon) \frac{q_j (1 + b)^2}{Q_0 (1 + 2b)} \left( \int_{D} W_j^{(1+b)2n/(n-2)} \right)^{(n-2)/n} dx \\
+ C \int_{D_i} w_j^{(n+2)/(n-2)} + 2b \ dx + C(\epsilon) \int_{D} w_j^{2(1+b)} dx + c.
\]

By our assumption in Theorem 1.1 (\(\sup_{x \in B(0,j)} W(x)^{(n-2)/n} q_j < Q_0\)), we can choose \(\epsilon\) and \(b\) sufficiently small so that \((1 + \epsilon) \sup_{x \in B(0,j)} W(x)^{(n-2)/n} q_j < Q_0 (1 + 2b) \] \(< \lambda < 1\).

So (2.11) implies

\[
\left( \int_{D} w_j^{(1+b)2n/(n-2)} \right)^{(n-2)/n} dx \\
\leq \frac{C}{1 - \lambda} \left( \int_{D_i} w_j^{(n+2)/(n-2)} + 2b \ dx + C(\epsilon) \int_{D} w_j^{2(1+b)} dx + c \right).
\]

Choosing \(b\) sufficiently small, we know, for some \(l_i, l'_i > 0, i = 1, 2\),

\[
\int_{D} \left[ w_j^{2(1+b)} + w_j^{(n+2)/(n-2)} + 2b \right] dx \leq \Sigma_{i=1}^{2} \left( \int_{D} w_j^{2n/(n-2)} dx \right)^{l_i} |D|^{l'_i}.
\]

It follows that

\[
\int_{D} w_j^{(1+b)2n/(n-2)} dx \leq C,
\]

where \(C\) is independent of \(j\). From here standard regularity theory shows that \(u_j\) is uniformly bounded in any compact domain of \(M\). Step 3 is complete.

**Step 4.** We will show that a subsequence of \(u_j\) converges pointwise to a positive solution to (1.1), up to a constant multiple.

Since \(u_j\) is uniformly bounded in any compact domain, the standard elliptic theory shows that \(u_j\) is uniformly bounded in \(C^{2,\alpha}\) norm in any compact domain too. Hence a subsequence of \(u_j\), still denoted by \(u_j\), converges pointwise to a solution \(u\) to:

\[\Delta u(x) - Vu(x) + qWy^p(x) = 0, \ x \in M^n.\]

Here \(q\) is a positive constant. Now using a dilation of \(u\), we can obtain a solution to (1.2).
We need to prove that $u$ is positive. We will use the Concentrated Compactness Principle of P. L. Lions, as suggested in [Ki]. To this end we write

$$J(u) \equiv \int_M (|\nabla u|^2 + Vu^2)dx, \quad v_j \equiv u_j - u.$$ 

Next

$$J(u_j) = J(u + v_j) = J(u) + J(v_j) + 2\int_M (-\Delta u + Vu)v_jdx.$$ 

Clearly $v_j$ converges weakly to zero. Hence

$$J(u_j) - J(v_j) \to J(u). \quad (2.14)$$

Moreover, for any fixed $R > 0$,

$$J(v_j) = \int_{B(0,R)} (|\nabla v_j|^2 + Vv_j^2)dx + \int_{B(0,R)^c} (|\nabla v_j|^2 + Vv_j^2)dx$$

$$\geq Q(1, B(0, R)^c) \left( \int_{B(0,R)^c} |v_j|^{2n/(n-2)}dx \right)^{(n-2)/n} + o(1)$$

$$\geq Q(1, B(0, R)^c) \left( \int_{B(0,R)^c} W|v_j|^{2n/(n-2)}dx \right)^{(n-2)/n}$$

$$\cdot \left[ \sup_{B(0,R)^c} W \right]^{-(n-2)/n} + o(1)$$

as $R \to \infty$. Here we have used the fact that $v_j$ converges to 0 pointwise in any compact domain.

By the Fatou Lemma due to Brezis and Lieb,

$$\int_M W_{u_j}^{2n/(n-2)}dx - \int_M W_{v_j}^{2n/(n-2)}dx \to \int_M W^{2n/(n-2)}dx. \quad (2.16)$$

We claim that $u$ is not identically zero. Otherwise (2.16) shows that

$$\lim_{j \to \infty} \int_M W|v_j|^{2n/(n-2)}dx = 1, \quad (2.17)$$

since $\int_M W_{u_j}^{2n/(n-2)}dx = 1$. From (2.14), (2.15), (2.17) and the assumption that $u = 0$, we see that

$$\liminf_{j \to \infty} J(u_j) = \liminf_{j \to \infty} J(v_j) \geq Q(1, B(0, R)^c) \left[ \sup_{B(0,R)^c} W \right]^{-(n-2)/n} + o(1)$$

as $R \to \infty$. Recall that

$$\liminf_{j \to \infty} J(u_j) = Q(W, M).$$
Hence
\[
\left[ \sup_{B(0, R)^c} W \right]^{(n-2)/n} Q(W, M) \geq Q(1, B(0, R)^c) + o(1).
\]
This is a contradiction to Assumption (c) when \( R \) is sufficiently large.

Hence \( u \) is not identically zero. By the unique continuation property, we see that \( u(x) > 0 \) everywhere. A suitable dilation yields a positive solution to (1.1). This proves (i).

**Step 5.** We prove part (ii) of the theorem.

We need to prove an a priori estimate under weaker assumptions than (i).

First we note that Step 2 of the proof still applies in this case, with only a slight change. Indeed, (2.5) remains true. Now using the a priori assumption that \( \int_M u^{2n/(n-2)} dx < \infty \) and the scaling invariance of this integral, we see that
\[
\int_{B_1(x_0,1)} W u^{2n/(n-2)} d_1 x \to 0
\]
when \( x_0 \to \infty \). From this, (2.6) is an immediate consequence of (2.5).

Following the rest of the proof of Step 2, we know that \( u(x) \leq C/(1 + d(x)^{b}) \), with \( b < 2 \), we show that \( u \) actually has exponential decay near infinity.

Here is the proof. Since \( u \) is a solution to (1.1), we have
\[
\Delta u - (V - W u^{p-1}) u \geq 0.
\]
By the assumption that \( V(x) \geq c/(1 + d(x)^{b}) \) with \( b < 2 \) and the estimate \( u(x) \leq C/(1 + d(x)^{(n-2)/2}) \), we know that
\[
(V - W u^{p-1})(x) \geq \frac{c}{1 + d(x)^{b}} - \frac{C}{1 + d(x)^{2}} \geq \frac{c}{2(1 + d(x)^{b})}.
\]
Here we just used the equality \( p - 1 = 4/(n-2) \).

Let \( \Gamma_1 \) be the Green’s function of the operator
\[
\Delta - (V - W u^{p-1}).
\]
By the estimate of Green’s functions [Zh1], we have, when \( d(x) \geq 1 \) and for some \( a > 0 \),
\[
\Gamma_1(x,0) \leq C e^{-cd(x)^{a}}.
\]
For completeness, we sketch a proof here.

From the heat kernel estimates (Theorem 1.1 in [Zh1]), we have
\[
\Gamma_1(x,0) \leq \int_0^{\infty} \frac{C}{|B(x, \sqrt{t})|} e^{-c(\sqrt{t}/(1 + d(x)^{b}/2))^{(2-b)/2}} e^{-cd(x)^2/t} dt.
\]
By direct computation, the above shows

\[ \Gamma_1(x, 0) \leq \int_0^{\infty} \frac{C}{|B(x, \sqrt{t})|} e^{-cd(x)^2/(2t)} dt \ e^{-cd(x)^a/2} \]

where \( a > 0 \). Hence

\[ \Gamma_1(x, 0) \leq C \left[ \int_0^{d(x)^2} \frac{1}{|B(x, \sqrt{t})|} e^{-cd(x)^2/(2t)} dt \right. \\
+ \left. \int_{d(x)^2}^{\infty} \frac{1}{|B(x, \sqrt{t})|} e^{-cd(x)^2/(2t)} dt \right] e^{-cd(x)^a/2}. \]

By the doubling property of the balls, for \( t \leq d(x)^2 \) and \( d(x) \geq 1 \),

\[ |B(x, \sqrt{t})| \geq c(t/d(x)^2)^k |B(x, d(x))| \]

\[ \geq c'(t/d(x)^2)^k |B(0, d(x))| \leq c''(t/d(x)^2)^k. \]

When \( t \geq d(x)^2 \) and \( d(x) \geq 1 \),

\[ \int_{d(x)^2}^{\infty} \frac{1}{|B(x, \sqrt{t})|} e^{-cd(x)^2/(2t)} dt \leq C \int_{d(x)^2}^{\infty} \frac{1}{|B(0, \sqrt{t})|} dt \leq C, \]

by the extra assumption that \( M \) is nonparabolic. Hence

\[ \Gamma(x, 0) \leq C[1 + d(x)^2]e^{-cd(x)^a/2} \leq C'e^{-cd(x)^a/4}. \]

Note, for a large \( k > 0 \), \( u(z) \leq k \Gamma_1(z, 0) \) when \( d(z) \) is large, but fixed. By the maximum principle, we see that

\[ u(x) \leq c_1 e^{-c_2 d(x)^a} \]

for all \( x \). \( \square \)

**Remark 2.1.** If one assumes that the Ricci curvature is bounded from below and the injectivity radius is positive, then Aubin’s sharp Sobolev inequality (with an extra zero order term) holds on the whole manifold. Then the existence result can be obtained by carrying out Steps 2 to 4.

**Proof of Corollary 1.** The existence part is similar to the Proof of Theorem 1.1. We give the proof in several steps.

**Step 1.** This is identical to Step 1 in the Proof of Theorem 1.1
Step 2. This is again similar to Step 2 before. Using the same notations as in Step 2 before, one has:

\[(2.19)\]

\[
\|G(v)\eta\|_{2n/(n-2)}^2 \\
\leq CC_1\beta^2 \int |\nabla_1 \eta|^2 G(v)^2 d_1 x + C_1 q_j (\beta^2 + \epsilon) \int W v^{4/(n-2)} G(v)^2 \eta^2 d_1 x \\
\leq CC_1\beta^2 \int |\nabla_1 \eta|^2 G(v)^2 d_1 x + C_1 q_j (\beta^2 + \epsilon) \|G(v)\eta\|_{2n/(n-2)}^2 \\
\cdot \left( \int_{B_1(x_0,1)} W^{n/2} v^{2n/(n-2)} d_1 x \right)^{2/n} \\
\leq CC_1\beta^2 \int |\nabla_1 \eta|^2 G(v)^2 d_1 x \\
+ C_1 q_j \left( \sup_{B_1(x_0,1)} W \right)^{(n-2)/n} (\beta^2 + \epsilon) \|G(v)\eta\|_{2n/(n-2)}^2 \\
\cdot \left( \int_{B_1(x_0,1)} W v^{2n/(n-2)} d_1 x \right)^{2/n}.\]

Here all the norms are over the ball \(B_1(x_0,1)\). Since \(\int_{B_1(x_0,1)} W v^{2n/(n-2)} d_1 x \leq 1\) and \(W(x) \to 0\) as \(d(x) \to \infty\), the above shows that

\[
\|G(v)\eta\|_{2n/(n-2)}^2 \leq C_2 \beta^2 \int |\nabla_1 \eta|^2 G(v)^2 d_1 x.\]

From here we just follow the previous arguments step by step to conclude that \(v(x) \leq C\) and hence \(u_j(x) \leq C/d(x)^{(n-2)/2}\).

Step 3. This is identical to the previous Step 3 since we are working in a bounded domain.

Step 4. Since \(u_j(x) \leq C/(1 +d(x)^{(n-2)/2})\), by the decay condition on \(W\) and the volume growth assumption on \(M\), we see that

\[
\int_{B(0,r_0)^c} W u_j^{p+1} dx \leq \sum_{i=1}^{\infty} \int_{B(0,2^i r_0) - B(0,2^{i-1} r_0)} W u_j^{p+1} dx \\
\leq C \sum_{i=1}^{\infty} (2^{i-1} r_0)^{-a}.\]

Hence, for a sufficiently large \(r_0\), one has

\[(2.20)\]

\[
\int_{B(0,r_0)} W u_j^{p+1} dx = 1 - \int_{B(0,r_0)^c} W u_j^{p+1} dx \geq 1/2.\]

By the standard elliptic theory, \(u_j\) is uniformly bounded in \(C^{2,\alpha}\) norm for some \(\alpha > 0\). So a subsequence, still denoted by \(u_j\), converges pointwise to a
function $u$. By (2.20) and the unique continuation property, we know that $u(x) > 0$ for all $x$. This $u$ is a positive solution to

$$\Delta u(x) - Vu(x) + qWu^p(x) = 0, \quad x \in M^n.$$ 

Here $q$ is a positive constant. Now using an dilation of $u$, we can obtain a positive solution to (1.1). This proves the existence. \hfill \square

**Remark 2.2.** Let $u$ be a solution in Corollary 1. We show that $u^{4/(n-2)}g$ is a complete metric under the extra assumption: $r\Delta r \leq (n/2) - \delta$ for some $\delta > 0$ and, for a sufficiently small $\epsilon > 0$, $R(x) \leq \epsilon/(1 + d(x)^2)$ when $d(x)$ is large. However, it seems hard to find an example of such manifolds.

We will follow an idea in [Ki2]. It suffices to prove that

$$u(x) \geq c_0/(1 + d(x)^{(n-2)/2})$$

for some $c_0 > 0$.

Suppose this is not true. For any small $c > 0$, consider the set

$$D \equiv \left\{ x \in M \mid h(x) \equiv cr^{-(n-2)/2} - u(x) > 0 \right\}.$$ 

Here $r = d(x)$. The set $D$ is clearly nonempty by the above assumption. Moreover $D$ is outside any given compact set if $c$ is sufficiently small. This is due to the fact that $u$ is positive everywhere.

Let $D \subset B(0, r_0)^c$. Here we choose $r_0$ so large that $r\Delta r \leq (n/2) - \delta$ when $x \in D$. By direct computation, the following holds in the distribution sense:

$$\Delta (cr^{(2-n)/2} - u) \geq r^{1-(n/2)} \left( \frac{n-2}{2} - r\Delta r \right) c - \frac{(n-2)}{4(n-1)} R(x)u(x)r^{1+(n/2)}.$$ 

Using $u(x) \leq c/d(x)^{(n-2)/2}$ and our extra assumption in the remark, we see that

$$\Delta h \geq cr^{1-(n/2)} \left( \frac{n-2}{2} \delta - \frac{(n-2)}{4(n-1)} R(x)r^2 \right) > 0.$$ 

This shows that $h$ is subharmonic in $D$. However $h(x) > 0$ in $D$ and $h(x) = 0$ in $\partial D$. The contradiction implies that $D$ is empty when $c$ is sufficiently small.

### 3. Proof of Theorem 1.2.

We will only show the proof in the Euclidean case since the manifold case is similar.

We will use the method of domain exhaustion. We divide the proof into three steps.
Step 1. We prove an a priori decay estimate for certain sub solutions \( u \) solving

\[
\Delta u + Wu^p \geq 0
\]

in the weak sense. During this step, we assume that \( u \) satisfy \( \int_{\mathbb{R}^n} Wu^p+1 dx < \infty \) and \( \int_{\mathbb{R}^n} |\nabla u|^2 dx < \infty \).

Pick \( x \in \mathbb{R}^n \) and let \( R = |x|/2 \). Throughout the section we make a change of the variables \( y = x/R \).

Write \( u_1(y) = R^k u(Ry) \) with \( k = (n-2)/2 \), we know that \( u_1 \) satisfies

\[
\Delta u_1 + R^2 - (p-1)k Wu_1^p \geq 0.
\]

Here and later \( W_1(y) = W(Ry) \) and the \( \Delta \) in front \( u_1 \) is the Laplacian in \( y \) variable.

From (3.1), direct computation shows, for any \( l \geq 1 \)

\[
\Delta u_1^l + lR^2 - (p-1)k W_1 u_1^{l+p-1} \geq 0.
\]

Given any \( y_0 \) such that \(|y_0| = 1\) and \( s_0 > 0 \), we wish to show that \( u_1(y_0) \) is uniformly bounded when \( R \) is sufficiently large. Much of the remainder of the step is to prove this claim.

Let \( \Psi \) be a suitable cut-off function, by standard arguments we know that

\[
\int_{B(y_0, \sigma_1)} |\nabla u_1|^2 dy \leq C \tau^{-2} R^{2-(k-1)p} \int_{B(y_0, \sigma_2)} W_1 u_1^{2l+p-1} dy,
\]

where \( \tau = \sigma_2 - \sigma_1 \) and \( 0 < \sigma_1 < \sigma_2 < 1 \).

Using Sobolev embedding, it is easy to see that

\[
\int_{B_{\sigma_1}} u_1^{2\theta} dy \leq C \left[ R^{2-(p-1)k} \tau^{-2} \int_{B_{\sigma_2}} W_1 u_1^{2l+p-1} dy \right]^{\theta},
\]

where \( \theta = n/(n-2) \) here and throughout this section. We will modify (3.4) so that a Moser iteration can be carried out.

From scaling relation between \( u \) and \( u_1 \), it is easy to see that

\[
\int_{\mathbb{R}^n} u_1^{2n/(n-2)}(z) dz' = \int_{\mathbb{R}^n} u^{2n/(n-2)}(z) dz < \infty.
\]

Using the scaling \( x = Ry \), \( W_1(y) = W(Ry) \) and \( u_1(y) = R^k u(Ry) \), it is clear that

\[
\int_{B(y_0, \sigma_2)} W_1(y) u_1^{p+1}(y) dy = R^{k(p+1)-n} \int_{B(x_0, \sigma_2 R)} W(x) u^{p+1} dx.
\]
By the assumption at the beginning of the step

$$
(3.6) \quad \int_{B(y_0, \sigma_2)} W_1(y)u_1^{p+1}(y)dy \leq C R^{k(p+1) - n}.
$$

Since \( k = (n - 2)/2 \), one has \( k(p + 1) - n = -(2 - k(p - 1)) \).

Now let us go back to (3.4). Take

$$
q_1 = \frac{2n}{(n - 2)(p - 1)}, \quad q_1' = \frac{q_1}{q_1 - 1} = \frac{2n}{2n - (n - 2)(p - 1)},
$$

and use Hölder’s inequality, we know that

$$
\int_{B(y_0, \sigma_1)} u_1^{2q_1} dy \leq C \left[ R^{2-\frac{1}{q_1} - 1} \int_{B_{\sigma_2}} W_1 u_1^{2q_1} dy \right]^\theta \left[ \int_{B_{\sigma_2}} W_1 u_1^{(p-1)q_1} dy \right]^{1/q_1'}.
$$

Since \((p-1)q_1 = 2n/(n-2)\), by (3.5),

$$
(3.7) \quad \int_{B_{\sigma_1}} u_1^{2q_1} dy \leq C \left[ R^{2-\frac{1}{q_1} - 1} \left( \int_{B_{\sigma_2}} W_1 u_1^{2q_1} dy \right)^{1/q_1'} \right]^\theta \left[ \int_{B_{\sigma_2}} W_1 u_1^{(p-1)q_1} dy \right]^{1/q_1'}.
$$

Recall that \( W(x) \leq C/(1 + |x|^m) \). Hence \( W_1(y) \leq C R^{-m} \) when \(|y - y_0| \leq \sigma_2 \leq 1/2\). So (3.7) implies

$$
\int_{B_{\sigma_1}} u_1^{2q_1} dy \leq C \left[ R^{2-\frac{1}{q_1} - 1} \left( \int_{B_{\sigma_2}} W_1 u_1^{2q_1} dy \right)^{1/q_1'} \right]^\theta \left[ \int_{B_{\sigma_2}} W_1 u_1^{2q_1} dy \right]^{1/q_1'}.
$$

By choosing \( m = 2 - (p - 1)k = 2 - ((p - 1)(n - 2)/2) \), we see that

$$
(3.8) \quad \int_{B_{\sigma_1}} u_1^{2q_1} dy \leq C \left[ (\sigma_2 - \sigma_1)^{-2} \left( \int_{B_{\sigma_2}} u_1^{2q_1'} dy \right)^{1/q_1'} \right]^\theta.
$$

If \( p < \frac{n+2}{n-2} \), then

$$
q_1' = \frac{q_1}{q_1 - 1} = \frac{2n}{2n - (n - 2)(p - 1)} < \frac{n}{n - 2} = \theta.
$$

Therefore we can use Moser’s iteration on (3.8) to conclude that

$$
u_1(y_0) \leq C.
$$

This is so because

$$
\|u_1\|_{L^{2q_1'}(B_1)} \leq C.
$$
This shows

\[ u(x) \leq C / \left( 1 + |x|^{(n-2)/2} \right). \]

**Step 2.** We show that \( u \) has uniform exponential decay:

\[ u(x) \leq c_1 e^{-c_2 |x|^{(2-b)/2}} \]

for all \( x \).

From last step we know that \( u(x) \leq \frac{C}{1 + |x|^{(n-2)/2}} \). By the assumption on \( W \), we see that

\[ W(x) u^{p-1}(x,t) \leq \frac{C}{1 + |x|^{(p-1)(n-2)/2}} \frac{C}{1 + |x|^{(p-1)(n-2)/2}} \leq \frac{C}{1 + |x|^2}. \]

Since \( V(x) \geq \frac{a}{1 + |x|^b} \) with \( b < 2 \), we see that

\[
0 = \Delta u(x) - V(x) u(x) + W(x) u^p(x) \\
= \Delta u(x) - (V(x) - W(x) u^{p-1}(x)) u(x) \\
\leq \Delta u(x) - \frac{c_0}{1 + |x|^b} u(x)
\]

when \( |x| \geq r \) for a large \( r > 0 \). Here \( c_0 \) is a positive number.

Let \( \Gamma_1 \) be the Green’s function of the elliptic operator \( \Delta - \frac{c_0}{1 + |x|^b} \). Note that \( u(x,0) = 0 \) when \( |x| \) is large. It is also clear that \( \Gamma_1(x,0) \geq c(|x|) > 0 \). Applying the maximum principle on the exterior of a sufficiently large cylinder centered at the origin, we know that

\[ u(x) \leq C \Gamma_1(x,0) \]

when \( |x| \) and \( C \) are sufficiently large. By the upper bound of \( \Gamma_1 \) in [MU] (when the leading operator is the Laplacian), [ZH1] (general case), we have

\[ \Gamma_1(x,y) \leq C e^{-c(|x-y|/(1+|x|^{b/2}))^{(2-b)/2} / |x-y|^{n-2}}. \]

It follows that

\[ u(x) \leq c_1 e^{-c_2 |x|^{(2-b)/2}} \]

for all \( x \).

**Step 3.** We use the method of domain exhaustion.

We begin by solving, for each \( j > 0 \), the standard variational problem

\[
\inf_{u \in W^{1,2}_0(B(0,j))} \int_{B(0,j)} (|\nabla u|^2 + V u^2) dx
\]

(3.10)
Using the inequality $ab \leq C(a^m + b^{m'})$ with the exponents $m = 2/(2 - ((p - 1)(n - 2)/2))$ and $m' = m/(m - 1) = 4/((p - 1)(n - 2))$, we have

$$\int_{\mathbb{R}^n} \frac{1}{1 + |x|^{2-((p-1)(n-2)/2)}} u^{p+1}(x) dx$$

subject to the constraint $\int_{B(0,j)} W u^{p+1} dx = 1$. Let $u_j > 0$ be a solution to (3.10). Then there exists a $\lambda_j > 0$ such that

$$\Delta u_j(x) - Vu_j(x) + \lambda_j W u_j^p(x) = 0, \ x \in B(0,j); \ u(x) = 0, \ x \in \partial B(0,j).$$

We claim that there exists a $\lambda > 0$ such that $\lambda_j$ decreases to $\lambda$ when $j \to \infty$. Clearly $\lambda_j < \lambda_{j'}$ if $j' > j$. When $W$ satisfies the assumption in Theorem 1.2, we have, for any suitable $u$,

$$\int_{\mathbb{R}^n} W u^{p+1} dx \leq C \int_{\mathbb{R}^n} \frac{1}{1 + |x|^{2-((p-1)(n-2)/2)}} u^{p+1}(x) dx$$

$$= C \int_{\mathbb{R}^n} \frac{1}{1 + |x|^{2-((p-1)(n-2)/2)}} u^{2-((p-1)(n-2)/2)} u^{p-1+((p-1)(n-2)/2)} dx.$$

Using the inequality $ab \leq C(a^m + b^{m'})$ with the exponents $m = 2/(2 - ((p - 1)(n - 2)/2))$ and $m' = m/(m - 1) = 4/((p - 1)(n - 2))$, we have

$$\int_{\mathbb{R}^n} W u^{p+1} dx \leq C \int_{\mathbb{R}^n} \frac{1}{1 + |x|^2} u^2 dx + C \int_{\mathbb{R}^n} u^{p-1+((p-1)(n-2)/2)m'} dx$$

$$= C \int_{\mathbb{R}^n} \frac{1}{1 + |x|^2} u^2 dx + C \int_{\mathbb{R}^n} u^{2n/(n-2)} dx$$

$$\leq C \int_{\mathbb{R}^n} (|\nabla u|^2 + Vu^2) dx.$$ 

This shows that $\lambda_j$ is bounded away from zero. The claim is proven.

Let $u_j$ be a solution to (3.10). We extend the domain of $u_j$ to the whole space by setting $u_j(x) = 0$ when $x$ is outside of the ball $B(0,j)$. It is easy to verify that the extended function, still denoted by $u_j$, is a subsolution to (1.2) in the weak sense, i.e.,

$$\Delta u_j(x) - Vu_j(x) + \lambda_j W u_j^p(x) \geq 0, \ x \in \mathbb{R}^n,$$

in the weak sense. Since

$$\left( \int_{\mathbb{R}^n} u_j^{2n/(n-2)} \right)^{(n-2)/n} \leq C \int_{\mathbb{R}^n} |\nabla u_j|^2 = C \lambda_j \int_{\mathbb{R}^n} W u_j^{p+1} = C \lambda_j \leq C \lambda_1$$

for all $j$, we can use Step 2 to conclude that

$$u_j(x) \leq c_1 e^{-c_2|x|^{((2-b)/2)^2}}$$

for all $|x| \geq R_0$ and $j$. Here $R_0$ is a sufficiently large number. If $|x| \leq R_0$, by [GS1] or [GS2], $u_j(x)$ is uniformly bounded. Hence (3.13) actually holds for all $x$.

This shows that there exists $r_0 > 0$ such that

$$\int_{B(0,r_0)} W u_j^{p+1} dx = 1 - \int_{B(0,r_0)^c} W u_j^{p+1} dx \geq 1/2.$$
By the standard elliptic theory, $u_j$ is uniformly bounded in $C^{2,\alpha}$ norm for some $\alpha > 0$. So a subsequence, still denoted by $u_j$, converges pointwise to a function $u$. By (3.14) and the unique continuation property, we know that $u(x) > 0$ for all $x \in \mathbb{R}^n$. This $u$ is a positive solution to
\[
\Delta u(x) - Vu(x) + \lambda W u^p(x) = 0, \quad x \in \mathbb{R}^n.
\]
Now using an dilation of $u$, we can obtain a positive solution to (1.2). \qed

4. Existence of global solutions and energy estimates.

As indicated in [F] and [Zh2], the existence or nonexistence of global positive solutions to (1.3) is both strongly influenced by the exponent $p$ and the potentials $V$ and $W$. In this section we will show that (1.3) with bounded $W$ possesses global positive solutions under the condition that $V(x) \geq c/(1 + |x|^b)$ with $b < 2$ and $c > 0$. In fact, concerning the existence of global positive solutions, the condition on $V$ is sharp in general. This means that if $0 \leq V(x) \leq C/(1 + |x|^b)$ with $b > 2$, $W = 1$, then (1.3) has no global positive solutions if $1 < p < 1 + \frac{2}{n}$ (see [Zh2]).

We will also prove some energy estimate for global solutions which will be useful for the proof of Theorem 1.3. Those results in this section which overlap those in [SZ] or [Zh2] are presented here for completeness.

Let us also mention that all results in this section remain valid if $\Delta$ is replaced by an uniformly elliptic divergence operator with bounded measurable coefficients depending on $x$. In this more general case one needs the Green’s function estimates in [Zh1] to begin with. While in the special case some comparison methods are sufficient ([SZ]).

We denote by $e^{t(\Delta - V)}$ the semigroup (on $L^\infty$) associated with the linear part of the equation
\[
(4.1) \quad u_t - \Delta u + Vu = 0 \quad \text{in} \; \mathbb{R}^n \times (0, \infty).
\]
Namely, for all $\phi \in L^\infty$, $u(x, t) = (e^{t(\Delta - V)}\phi)(x)$ denotes the unique solution of (4.1) with initial data $\phi$. Also, we denote by $\Gamma$ the Green’s function of the operator $\Delta - V$ and for all suitable $f$, we put
\[
\Gamma * f(x) \equiv \int_{\mathbb{R}^n} \Gamma(x, y)f(y)dy = \int_0^\infty e^{t(\Delta - V)} f dt.
\]

Given $k > 0$, we introduce a weighted space $L_k^\infty$ defined as
\[
L_k^\infty = \{ u \mid u(\cdot) \in L^\infty(\mathbb{R}^n), (1 + |x|)^k u(x) < \infty \}.
\]
The norm of this space is given by $\|u\|_{\infty,k} = \sup_x (1 + |x|)^k |u(x)|$.

We will use $T(u_0)$ to denote the maximum time of existence of the solution to (1.3), which may also denoted by $u(\cdot, u_0)(\cdot)$. 
Proposition 4.1. Suppose $V(x) \geq \frac{a}{1 + |x|^b}$ with $b \in [0, 2)$, $a > 0$ and let $k \geq 0$.

There exists $C \geq 1$ such that for all $\phi \in L^\infty_k$,

\begin{equation}
\|e^{t(\Delta - V)}\phi\|_{\infty, k} \leq C\|\phi\|_{\infty, k}, \quad t \geq 0.
\end{equation}

Proof. By Theorem 1.1 in [Zh1], we have

\[ G(x, t; y, 0) \leq c_1 \frac{e^{-c_2[(1/2) + |x|^{1/2}]^\alpha} e^{-c_2[(1/2) + |y|^{1/2}]^\alpha}}{t^{n/2}} e^{-c_3|x-y|^2/t} \]

with $\alpha = (2 - b)/2$.

Given $f = f(x)$, we write

\[ G \ast f(x, t) = \int_{|y| \geq |x|/2} G(x, t; y, 0)f(y)dy + \int_{|y| \leq |x|/2} G(x, t; y, 0)f(y)dy \equiv J_1 + J_2. \]

Clearly

\[ J_1 \leq \frac{C}{1 + |x|^k} \int \frac{e^{-c_3|x-y|^2/t}}{t^{n/2}} dy \leq \frac{C}{1 + |x|^k}. \]

When $|y| \leq |x|/2$, one has $|x - y| \geq |x|/2 \geq |y|$. Therefore

\[ J_2 \leq \frac{C}{t^{n/2}} \int_{|y| \leq |x|/2} e^{-c_2[(1/2) + |x|^{1/2}]^\alpha} e^{-c_2|x|^2/t} e^{-c_3|x-y|^2/t} dy. \]

Here $c > 0$ is chosen sufficiently small. If $|x| \leq 1$, then obviously $J_2 \leq C$. So we can assume that $|x| \geq 1$. Direct computation shows that

\[ [t^{1/2}/(1 + |x|^{1/2}) + |x|^2/t] \geq |x|^\theta \]

for some $\theta > 0$ and all $t > 0$. Hence

\[ J_2 \leq Ce^{-|x|^\theta}. \]

Combining this with the estimate on $J_1$ completes the proof. \qed

Proposition 4.2. Suppose $V(x) \geq \frac{a}{1 + |x|^b}$ with $b \in [0, 2)$, $a > 0$. Suppose $0 \leq f(x) \leq 1/(1 + |x|^k)$ for some $k \geq 0$, then

\[ \Gamma \ast f(x) \equiv \int_{\mathbb{R}^n} \Gamma(x, y)f(y)dy \leq \frac{C(1 + |x|^b)}{1 + |x|^k}. \]

Proof. According to [Mu] or Corollary 1 in [Zh1], under the assumptions in the proposition, there exist positive constants $c_1, c_2$ such that, for all $x, y$ and $\alpha = (2 - b)/2$,

\begin{equation}
\Gamma(x, y) \leq c_1 e^{-c_2[(1+|x|^{1/2})^\alpha]} e^{-c_2[(1+|y|^{1/2})^\alpha]} \frac{1}{|x - y|^{n-2}}.
\end{equation}
When $|x| \leq 1$ and $|y| \geq 2$, we have
\[
\Gamma(x, y) \leq e^{-c|x|^{1-(b/2)}^2} \frac{C}{|x-y|^{n-2}}.
\]
Hence $\Gamma \ast f(x) \leq C$ when $|x| \leq 1$.

In order to estimate $\Gamma \ast f(x)$ when $|x| \geq 1$, let us write
\begin{equation}
(4.4)
\Gamma \ast f(x) = \int_{|y| \leq |x|/2} \Gamma(x, y)f(y)dy + \int_{|y| \geq |x|/2} \Gamma(x, y)f(y)dy + \int_{|y|/2 \leq |y| \leq 2|x|} \Gamma(x, y)f(y)dy \\
= I_1 + I_2 + I_3.
\end{equation}

When $|y| \leq |x|/2$, one has $|x-y| \geq |x| - |y| \geq |x|/2 \geq |y|$. Hence
\[
e^{-c_2|x-y|/(1+|x|^{b/2})^\alpha} \leq e^{-c_2|x|^{1-(b/2)}(2-b)^{2}/2},
\]
\[
e^{-c_2|x-y|/(1+|y|^{b/2})^\alpha} \leq e^{-c_2|y|^{1-(b/2)}(2-b)^{2}/2}.
\]
It follows that
\begin{equation}
(4.5)
I_1 \leq Ce^{-c_2|x|^{(1-(b/2))(2-b)^{2}/2}} \int_{\mathbb{R}^n} e^{-c_2|y|^{(1-(b/2))(2-b)^{2}/2}} |x-y|^{n-2}dy \\
\leq Ce^{-c_2|x|^{(1-(b/2))(2-b)^{2}/2}}.
\end{equation}

Similarly
\begin{equation}
(4.6)
I_2 \leq Ce^{-c_2|x|^{(1-(b/2))(2-b)^{2}/2}}.
\end{equation}

Next let us estimate $I_3$. Since $|x|/2 \leq |y| \leq 2|x|$ in this case, we have $f(y) \leq C/(1+|x|^k).$ Hence
\[
I_3 \leq \frac{C}{1+|x|^k} \int_{|y|/2 \leq |y| \leq 2|x|} \Gamma(x, y)dy \\
\leq \frac{C}{1+|x|^k} \int_{|y|/2 \leq |y| \leq 2|x|} e^{-c_2|x-y|/(1+|x|^{b/2})^\alpha} |x-y|^{n-2}dy.
\]
Take the substitution $y' = y/|x|$, we obtain, as $|x| \geq 1,$
\[
I_3 \leq \frac{C}{1+|x|^k} \int_{1/2 \leq |y'| \leq 2} e^{-c|x|^{1-(b/2)^2}}|x/|x||-y'|^{(2-b)^2/2} \frac{dy'}{||(x/|x|| - y'||^{n-2}}.
\]
Note that $|x/|x|| = 1$, so if we let $r = |x/|x|| - y'$, then
\[
I_3 \leq \frac{C}{1+|x|^k} \int_{0}^{1} e^{-c|x|^{1-(b/2)^2},r^{(2-b)^2/2}} dr |x/|x|^{1-(b/2)^2}.
\]
Letting $r_1 = |x|^{(1-(b/2))}r$, we have
\[
I_3 \leq \frac{C}{1+|x|^k} \int_{0}^{\infty} e^{-c r_1^{(2-b)^2/2}} r_1 dr_1 |x|^{1-(b/2)^2}.
\]
i.e.,

\[(4.7) \quad I_3 \leq \frac{C(1 + |x|^b)}{1 + |x|^k}.\]

The lemma is proven by combining (3.3)-(3.5). 

The result in the next proposition is known. But we give a sketch of the proof for the sake of completeness.

**Proposition 4.3.** Suppose \(\frac{a}{1+|x|^p} \leq V(x) \leq C_1\) with \(b \in [0, 2), a > 0\). Suppose \(W \in L^\infty(\mathbb{R}^n)\) and \(1 < p < \frac{n+2}{n-2}\). Then the following conclusions hold:

(i) For any compactly supported nonnegative \(f \neq 0\), there exists \(\lambda_0 > 0\) such that the problem

\[
\begin{aligned}
\Delta u - V(x)u + W(x)u^p - \partial_t u &= 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \\
u(x, 0) = u_0(x) \equiv \lambda f(x),
\end{aligned}
\]

has a global positive solution when \(\lambda \in (0, \lambda_0)\);

(ii) moreover the \(\omega-\) limit set contains a equilibrium solution;

(iii) all global positive solutions are bounded in \(D \times (c, \infty)\). Here \(c > 0\) and \(D\) is any compact domain.

**Proof.** (i) The existence of global solutions for small initial data is a simple consequence of Propositions 4.1 and 4.2 together with a fixed point argument. If one is restricted to (1.3) only, then a comparison method also yields the result (see [SZ]). However the current method has the advantage that it covers the case when \(\Delta\) in (1.3) is replaced by a uniformly elliptic operator with bounded measurable coefficients. We refer to [Zh2] for details of the proof.

(ii) Next us recall some well-known facts related to the existence of an energy functional for Equation (1.3).

For \(u_0 \in L^\infty \cap H^1\) it is well-known that \(u \in C([0, T(u_0)); H^1)\) and that the energy \(E(t)\), defined as

\[
E(t) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^n} V u^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^n} W u^{p+1} \, dx
\]

satisfies the identity

\[
E(0) - E(t) = \int_0^t \int_{\mathbb{R}^n} |u_t(x, s)|^2 \, dx \, ds.
\]

We will use the following two classical lemmas.
Lemma 4.1. Let \( u_0 \in L^\infty \cap H^1 \). If \( T(u_0) = \infty \), then \( E(t) \geq 0 \) for all \( t \geq 0 \), hence in particular
\[
\int_0^\infty \int |u_t(x,s)|^2 \, dx \, ds \leq E(0) \leq C\|u_0\|^2_{H^1}.
\]

Proof. This is a consequence of the classical concavity argument of Levine (see [Le1]).

Lemma 4.2. Let \( u_0 \in L^\infty \cap H^1 \). If \( T(u_0) = \infty \), then the \( \omega \)-limit set \( \omega(u_0) \) consists of equilibria (i.e., of solutions of the corresponding elliptic equation).

Proof. Assume \( u(t_j) \to v \) in \( L^\infty_k \) and fix \( t > 0 \). By continuous dependence of solutions of (1.3) over initial data in \( L^\infty \), it follows that \( u(t+t_j) \to S(t)v \) in \( L^\infty \). For each \( R > 0 \), we have
\[
\int_{|x| < R} |u(x,t+t_j) - u(x,t_j)|^2 \, dx \leq C(R) \int_{t_j}^{t+t_j} \int_{|x| < R} |u_t(x,s)|^2 \, dx \, ds \leq C(R) \int_t^\infty \int_{\mathbb{R}^n} |u_t(x,s)|^2 \, dx \, ds.
\]
Since the RHS goes to 0 as \( j \to \infty \) in view of Lemma 4.1, we deduce that
\[
\int_{|x| < R} |(S(t)v)(x) - v(x)|^2 \, dx = 0,
\]
hence \( S(t)v \equiv v \) for all \( t > 0 \), which means that \( v \) is an equilibrium. This proves the lemma and Part (ii) of the proposition.

(iii) This follows from the scaling argument in [Gi].

The following result, essentially given in [S], is important for the proof of Theorem 1.3:

Proposition 4.4. Let \( u_0 \in L^\infty \cap H^1 \) and assume that \( T(u_0) = \infty \). Then the following holds:
\[
\frac{1}{T} \int_h^{T+h} \int_{\mathbb{R}^n} W u^{p+1}(x,t) \, dx \, dt \leq C(\|u_0\|_{H^1}), \quad T \geq 1, \quad h \geq 0.
\]

Proof. Without loss of generality we take \( h = 0 \). We use an energy argument from [S, Theorem 2] (given there for \( V \equiv 0 \)). Let \( f(t) \equiv \int_{\mathbb{R}^n} u^2(x,t) \, dx \), then
by Lemma 4.1,
\[ f(t) - f(0) = 2 \int_0^t \int_{\mathbb{R}^n} uu_s \]
\[ \leq 2 \left( \int_0^t \int_{\mathbb{R}^n} u_s^2 \, dx \, ds \right)^{1/2} \left( \int_0^t \int_{\mathbb{R}^n} u^2 \, dx \, ds \right)^{1/2} \]
\[ \leq 2E(0)^{1/2} \left( \int_0^t f(s) \, ds \right)^{1/2}. \]
This easily implies
\[ f(t) \leq C(E(0))(f(0) + t) \leq C(\|u_0\|_{H^1})(t + 1), \quad t \geq 0. \]
Multiplying both sides of (1.3) by \( u \) and integrating, we obtain, for \( T > 0 \),
\[ \int_0^T \int_{\mathbb{R}^n} Wu^{p+1}(x,t) \, dx \, dt \]
\[ = \int_0^T \int_{\mathbb{R}^n} (|\nabla u(x,t)|^2 + V(x)u^2(x,t)) \, dx \, dt + \frac{1}{2} \int_{\mathbb{R}^n} (u^2(x,T) - u^2(x,0)) \, dx \]
\[ = 2 \int_0^T E(t) \, dt + \frac{2}{p+1} \int_0^T \int_{\mathbb{R}^n} Wu^{p+1}(x,t) \, dx \, dt + \frac{1}{2} (f(T) - f(0)). \]
Hence
\[ \frac{1}{T} \int_0^T \int_{\mathbb{R}^n} Wu^{p+1}(x,t) \, dx \, dt \]
\[ \leq \frac{p + 1}{p - 1} \left( \frac{f(T) - f(0)}{2T} + \frac{2}{T} \int_0^T E(t) \, dt \right) \]
\[ \leq \frac{p + 1}{p - 1} \left( \frac{f(T)}{2T} + 2E(0) \right) \leq C(\|u_0\|_{H^1}), \quad T \geq 1. \]

5. Proof of Theorem 1.3.

Again we will only give a proof of the Euclidean case, which is divided into several steps.

Step 1. Let \( u \) solves
\[ \Delta u + Wu^p - u_t \geq 0. \]
Pick \( x \in \mathbb{R}^n \) and let \( R = |x|/2 \). Throughout the section we make a change of the variables
\[ y = x/R, \quad s = t/R^2. \]
Write
\[ u_1(y, s) = R^k u(Ry, R^2 s) \]
with \( k = (n - 2)/2 \), we know that \( u_1 \) satisfies
\[
\Delta u_1 + R^{2-(p-1)k} w_1 u_1^p - \partial_s u_1 \geq 0. 
\]
Here and later \( W_1(y) = w(Ry) \) and the \( \Delta \) in front \( u_1 \) is the Laplacian in \( y \) variable.

From (5.1), direct computation shows, for any \( l \geq 1 \)
\[
\Delta u_1^l + lR^{2-(p-1)k} W_1 u_1^{l+p-1} - \partial_s u_1^l \geq 0. 
\]
Given any \( y_0 \) such that \( |y_0| = 1 \) and \( s_0 > 0 \), we wish to show that \( u_1(y_0, s_0) \) is uniformly bounded when \( R \) is sufficiently large. Much of the remainder of the step is to prove this claim.

For a \( \sigma \in (0, 1) \), write
\[
Q_\sigma = \{ y \mid |y - y_0| \leq \sigma \} \times [s_0 - \sigma^2, s_0].
\]
Since the support of \( u_0 \) is compact, the support of \( u_1(\cdot, 0) \) is contained in a ball centered at 0 with radius of the order \( c/R \). When \( s < s_0 \) and \( y \) is outside the support of \( u_1(\cdot, 0) \), we define \( u_1(y, s) = 0 \). In this way \( u_1 \) satisfies (5.2) in \( Q_\sigma \) when \( R \) is sufficiently large.

Let \( \Psi \) be a suitable cut-off function, by standard arguments we know that
\[
\sup_{s_0 - \sigma_2^2 \leq s \leq s_0} \int_{|y - y_0| \leq \sigma_1} u_1^{2l}(y, s) dy + c \int_{Q_{\sigma_1}} |\nabla u_1|^2 dy ds 
\leq C \tau^{-2} l^2 R^{2-(k-1)\theta} \int_{Q_{\sigma_2}} W_1 u_1^{2l+p-1} dy ds,
\]
where \( \tau = \sigma_2 - \sigma_1 \) and \( 0 < \sigma_1 < \sigma_2 < 1 \).

Using Sobolev embedding and Hölder’s inequality, it is easy to see that
\[
\int_{B(y_0, r)} f^{2(1 + (2/n))} dy \leq C \left( \int_{B(y_0, r)} f^2 dy \right)^{2/n} \left( \int_{B(y_0, r)} (|\nabla f|^2 + r^{-2} f^2) dy \right). 
\]
Using (5.3) and the above, we see that
\[
\int_{Q_{\sigma_1}} u_1^{2\theta} dy ds \leq C \left[ R^{2-(p-1)k} l^2 \tau^{-2} \int_{Q_{\sigma_2}} W_1 u_1^{2l+p-1} dy ds \right] 
\cdot \sup_{s_0 - \sigma_1^2 \leq s \leq s_0} \left( \int_{|y - y_0| \leq \sigma_1} u_1^{2l}(y, s) dy \right)^{2/n}. 
\]
Here and later \( \theta = 1 + (2/n) \). It follows that
\[
\int_{Q_{\sigma_1}} u_1^{2\theta} dy ds \leq C \left[ R^{2-(p-1)k} l^2 \tau^{-2} \int_{Q_{\sigma_2}} W_1 u_1^{2l+p-1} dy ds \right]^\theta. 
\]
We will modify (5.4) so that a Moser iteration can be carried out.
From the energy estimate in Section 4, Proposition 4.4, we know that
\[ \frac{1}{R^2} \int_{t-R^2}^{t} \int_{\mathbb{R}^n} W u^{p+1} \leq C. \]

Since \( E(t) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^n} V u^2 - \frac{1}{p+1} \int_{\mathbb{R}^n} W u^{p+1} \) and \( E(t) \) is non-increasing, we see that
\[ (5.5) \quad \frac{1}{R^2} \int_{t-R^2}^{t} \int_{\mathbb{R}^n} (|\nabla u|^2 + V u^2) \leq \frac{2}{p+1} \int_{t-R^2}^{t} \int_{\mathbb{R}^n} W u^{p+1} + 2E(t) \leq C(E(0)). \]

By Sobolev embedding, the above implies
\[ (5.6) \quad \frac{1}{R^2} \int_{t-R^2}^{t} \left( \int_{\mathbb{R}^n} u^{2n/(n-2)} \right)^{(n-2)/n} \leq C(E(0)). \]

From scaling relation between \( u \) and \( u_1 \), it is easy to see that
\[ \int_{\mathbb{R}^n} u^{2n/(n-2)}(z,t)dz = \int_{\mathbb{R}^n} u_1^{2n/(n-2)}(z',t/R^2)dz'. \]

Hence
\[ (5.7) \quad \int_{s-1}^{s} \left( \int_{\mathbb{R}^n} u_1^{2n/(n-2)} \right)^{(n-2)/n} \leq C(E(0)). \]

Next
\[ (5.8) \quad \int_{Q_{\sigma_1}} u_1^{2(1+(2/n))} \]
\[ \leq \int_{s_0-\sigma_1^2}^{s_0} \left( \int_{B(y_0,\sigma_1)} u_1^{2n/(n-2)} \right)^{(n-2)/n} \left( \int_{B(y_0,\sigma_1)} u_1^{2} \right)^{2/n} \]
\[ \leq \sup_{s\in(s_0-\sigma_1^2,s_0)} \left( \int_{B(y_0,\sigma_1)} u_1^{2} \right)^{2/n} \int_{s_0-\sigma_1^2}^{s_0} \left( \int_{B(y_0,\sigma_1)} u_1^{2n/(n-2)} \right)^{(n-2)/n}. \]

Combining (5.7) and (5.8) one obtains
\[ (5.9) \quad \int_{Q_{\sigma_1}} u_1^{2(1+(2/n))} \leq C(E(0)) \sup_{s\in(s_0-\sigma_1^2,s_0)} \left( \int_{B(y_0,\sigma_1)} u_1^{2} \right)^{2/n}. \]

We would like to find an upper bound for the right-hand side of (5.9). To this end we take \( l = 1 \) in (5.3) to get
\[ (5.10) \quad \sup_{s_0-\sigma_1^2 \leq s \leq s_0} \int_{|y-y_0|\leq \sigma_1} u_1^2(y,s)dy \leq C\tau^{-2} R^{2-(k-1)p} \int_{Q_{\sigma_2}} W_1 u_1^{p+1} dyds. \]
where $\tau = \sigma_2 - \sigma_1$. Using the scaling $t = sR^2$, $x = Ry$, $W_1(y) = W(Ry)$ and $u_1(y, s) = R^k u(Ry, R^2 s)$, it is clear that

$$
\int_{Q_{\sigma_2}} W_1(y) u_1^{p+1}(y, s) dy ds = R^{k(p+1)-n-2} \int_{t_0}^{t_2} \int_{B(x_0, \sigma_2 R)} W(x) u_1^{p+1} dx dt.
$$

By the energy estimate Proposition 4.4 again

$$
\int_{Q_{\sigma_2}} W_1(y) u_1^{p+1}(y, s) dy ds \leq R^{k(p+1)-n} C(E(0)).
$$

Since $k = (n - 2)/2$, one has $k(p + 1) - n = -(2 - k(p - 1))$. Taking $\sigma_2 = 2$ and $\sigma_1 \leq 1$, by (5.10) and (5.11), we deduce

$$
\sup_{s_0 - \sigma_1^2 \leq s \leq s_0} \int_{|y-y_0| \leq \sigma_1} u_1^2(y, s) dy \leq C(E(0))(2 - \sigma_1)^{-2} \leq C(E(0)).
$$

Substituting (5.12) to (5.9), we obtain

$$
\int_{Q_{\sigma_1}} u_1^{2(1+(2/n))} dy ds \leq C(E(0))
$$

for any $\sigma_1 \in (0, 1)$.

Now let us go back to (5.4). Take

$$
q_1 = \frac{2(n+2)}{n(p-1)}, \quad q_1' = \frac{q_1}{q_1 - 1} = \frac{2(n+2)}{3n - np + 4},
$$

and use Hölder’s inequality, we know that

$$
\int_{Q_{\sigma_1}} u_1^{2q_1} dy ds \leq C \left[ R^{2-(p-1)k} l^2 \tau^{-2} \int_{Q_{\sigma_2}} W_1 u_1^{2q_1} dy ds \right]^{\theta}
$$

$$
\leq C \left[ R^{2-(p-1)k} l^2 \tau^{-2} \left( \int_{Q_{\sigma_2}} W_1^{q_1'} u_1^{2q_1'} dy ds \right)^{1/q_1'} \left( \int_{Q_{\sigma_2}} u_1^{q_1} dy ds \right)^{1/q_1} \right]^{\theta}.
$$

Since $(p - 1)q_1 = 2(1 + \frac{2}{n})$, by (5.13),

$$
\int_{Q_{\sigma_1}} u_1^{2q_1} dy ds \leq C \left[ R^{2-(p-1)k} l^2 (\sigma_2 - \sigma_1)^{-2} \left( \int_{Q_{\sigma_2}} W_1^{q_1'} u_1^{2q_1'} dy ds \right)^{1/q_1'} \right]^{\theta}.
$$

Recall that $W(x) \leq C/(1 + |x|^m)$. Hence $W_1(y) \leq CR^{-m}$ when $|y-y_0| \leq \sigma_2 \leq 1/2$. So (5.14) implies

$$
\int_{Q_{\sigma_1}} u_1^{2q_1} dy ds \leq C \left[ R^{2-(p-1)k-m} l^2 (\sigma_2 - \sigma_1)^{-2} \left( \int_{Q_{\sigma_2}} u_1^{2q_1'} dy ds \right)^{1/q_1'} \right]^{\theta}.
$$
By choosing $m = 2 - (p - 1)k = 2 - ((p - 1)(n - 2)/2)$, we see that

$$\int_{Q_{\sigma_1}} u_1^{2\theta} dyds \leq C \left[ (\sigma_2 - \sigma_1)^{-2} \left( \int_{Q_{\sigma_2}} u_1^{2q_1'} \right)^{1/q_1'} \right]^\theta. \tag{5.15}$$

If $p < 1 + \frac{4}{n}$, then

$$q_1' < \frac{2(n + 2)}{3n - (n + 4) + 4} = \frac{n + 2}{n} = \theta.$$ 

Therefore we can use Moser’s iteration on (5.15) to conclude that $u_1(y_0, s) \leq C$. Using (5.7), this shows

$$u(x, t) \leq C/(1 + |x|^{(n-2)/2}). \tag{5.16}$$

**Step 2.** We show that $u$ has uniform exponential decay.

From last step we know that $u(x) \leq \frac{C}{1 + |x|^{(n-2)/2}}$. By the assumption on $W$, we see that $W(x) u^{p-1}(x, t) \leq \frac{C}{1 + |x|^{(n-2)/2}} \frac{C}{1 + |x|^{(n-2)/2}} \leq \frac{C}{1 + |x|^2}$.

Since $V(x) \geq \frac{a}{1 + |x|^b}$ with $b < 2$, we see that

$$0 = \Delta u(x, t) - V(x)u(x, t) + W(x, t)u^p(x, t) - u_t(x, t)$$

$$= \Delta u(x, t) - (V(x) - W(x)u^{p-1}(x, t))u(x, t) - u_t(x, t)$$

$$\leq \Delta u(x, t) - \frac{c_0}{1 + |x|^b} u(x, t) - u_t(x, t)$$

when $|x| \geq r$ for a large $r > 0$. Here $c_0$ is a positive number.

Let $\Gamma_1$ be the Green’s function of the elliptic operator $\Delta - \frac{c_0}{1 + |x|^b}$. Note that $u(x, 0) = 0$ when $|x|$ is large. It is also clear that $\Gamma_1(x, 0) \geq c(|x|) > 0$. Applying the maximum principle on the exterior of a sufficiently large cylinder centered at the origin, we know that

$$u(x, t) \leq C \Gamma_1(x, 0)$$

when $|x|$ and $C$ are sufficiently large. By the upper bound of $\Gamma_1$, as in Section 3, it follows that

$$u(x, t) \leq c_1 e^{-c_2 |x|^{(2-b)/2}}$$

for all $x, t$.

**Step 3.** Define

$$D_0 = \{ u_0 \in L^\infty_k; \ T(u_0) = \infty \text{ and } u(t; u_0) \to 0 \text{ in } L^\infty_k \text{ as } t \to \infty \}.$$
By the exponential decaying property of the fundamental solution of $\Delta - V$ (Proposition 4.3), it follows that $D_0$ contains an open neighborhood $W_0$ of $0$ in $L_k^\infty$ and that

$$D_0 = \{ u_0 \in L_k^\infty; \ T(u_0) = \infty \text{ and } 0 \in \omega_k(u_0) \}.$$ 

We claim that $D_0$ is open in $L_k^\infty$. Indeed, if $u_0 \in D_0$, there exists $t > 0$ such that $u(t; u_0) \in W_0$. But by continuous dependence of solutions of (1.1) in $L_k^\infty$, if $\|u_0 - u\|_{\infty,k}$ is sufficiently small, then $u(t; u_0) \in W_0 \subset D_0$, so that $u_0 \in D_0$. The claim follows.

Let now

$$\lambda^* = \sup \{ \lambda > 0; \ \lambda \phi \in D_0 \}.$$ 

We have just seen that $\lambda \phi \in D_0$ when $\lambda > 0$ is small, and it is well-known that $T(\lambda \phi) < \infty$ if $\lambda$ is large (see [Le2] for example). Therefore, $0 < \lambda^* < \infty$.

Let $\lambda_j \uparrow \lambda^*$ with $\lambda_j \phi \in D_0$. By standard scaling method, we have, for any bounded domain $D$,

$$\sup_{t \geq 0} \|u(t; \lambda_j \phi)\|_D \leq C(\lambda_j (\|\phi\|_{H^1} + \|\phi\|_\infty), D) \leq C(D), \quad j = 1, 2, \ldots.$$ 

Since by continuous dependence in $L_k^\infty$, we have, for each $t \in [0, T(\lambda^* \phi))$,

$$\|u(t; \lambda^* \phi)\|_D \leq \lim_j \|u(t; \lambda_j \phi)\|_D \leq C(D),$$

it follows that $T(\lambda^* \phi) = \infty$.

On the other hand, by the openness of $D_0$, $\lambda^* \phi \not\in D_0$. We claim that $\omega(\lambda^* \phi)$ contains a nontrivial nonnegative equilibrium $v$. Suppose the claim is false. Then by Step 2, there exists a sequence $\{t_j\}$ with $t_j \to \infty$, such that $\{u(t_j, \lambda^* \phi)\}$ is compact in $L_k^\infty$ norm. Moreover a subsequence would converge to 0 in $L_k^\infty$ norm. Hence $\|u(t_j, \lambda^* \phi)\|_{\infty,k}$ would be sufficiently small when $j$ is large. But this would imply that $\lambda^* \phi \in D_0$. This contradiction validates the claim. The strong maximum principle finally implies that $v > 0$ in $\mathbb{R}^n$. The proof is complete. \qed

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References


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[AM] P. Aviles and R. McOwen, Conformal deformations of complete manifolds with
negative curvature, J. Differential Geom., 21(2) (1985), 269-281, MR 87e:53058,
Zbl 0588.53028.

MR 84h:35054b, Zbl 0556.35046.

[BN] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations in-
volving critical Sobolev exponents, Comm. Pure Appl. Math., 36(4) (1983), 437-
477, MR 84h:35059, Zbl 0541.35029.

[BJP] J. Busca, M.A. Jendoubi and P. Polacik, Convergence to equilibrium for semilin-
ear parabolic problems in \( \mathbb{R}^n \). Preprint, 2000.

[CW] T. Cazenave and F.B. Weissler, Asymptotically self-similar global solutions of
the nonlinear Schrödinger and heat equations, Math. Z., 228 (1998), 83-120,
MR 99b:35149, Zbl 0916.35109.

[CDE] C. Cortézar, M. del Pino and M. Elgueta, The problem of uniqueness of the limit
in a semilinear heat equation, Comm. Partial Differential Equations, 24(11-12)

[DL] K. Deng and H.A. Levine, The role of critical exponents in blowup theo-
Zbl 0942.35025.

[DN] W.Y. Ding and W.M. Ni, On the existence of positive entire solutions of a semilin-
Zbl 0616.35029.

[Eg] H. Egneull, Asymptotic results for finite energy solutions of semilinear elliptic equa-

MR 88k:35013, Zbl 0635.35033.

[ES] J.F. Escobar and R.M. Schoen, Conformal metrics with prescribed scalar curva-

[FP] E. Feireisl and H. Petzeltova, Convergence to a ground state as a threshold phe-
nomenon in nonlinear parabolic equations, Differential Integral Equations, 10

[F] H. Fujita, On the blowing up of solutions of the Cauchy problem for \( u_t = \Delta u + u^{1+\alpha} \), J. Fac. Sci. Univ. Tokyo Sect. I, 13 (1966), 109-124, MR 35 #5761,
Zbl 0163.34002.

[GS1] B. Gidas and J. Spruck, Global and local behaviour of positive solutions of
MR 83f:35045, Zbl 0465.35003.

[GS2] _____, A priori bounds for positive solutions of nonlinear elliptic equations,
Comm. Partial Differential Equations, 6(8) (1981), 883-901, MR 82h:35033,
Zbl 0462.35041.


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