REAL PALEY-WIENER THEOREMS FOR THE INVERSE FOURIER TRANSFORM ON A RIEMANNIAN SYMMETRIC SPACE

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We prove real Paley-Wiener theorems for the inverse Fourier transform on a semisimple Riemannian symmetric space $G/K$ of the noncompact type. The functions on $G/K$ whose Fourier transform has compact support are characterised by a $L^2$ growth condition. We also obtain real Paley-Wiener theorems for the inverse spherical transform.

1. Introduction.

The classical Fourier transform $\mathcal{F}_{cl}$ is an isomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^k)$ onto itself. The space $C_c^\infty(\mathbb{R}^k)$ of smooth functions with compact support is dense in $\mathcal{S}(\mathbb{R}^k)$, and the classical Paley-Wiener theorem characterises the image of $C_c^\infty(\mathbb{R}^k)$ under $\mathcal{F}_{cl}$ as rapidly decreasing functions having an holomorphic extension to $\mathbb{C}^k$ of exponential type. Since $\mathbb{R}^k$ is self-dual, the same theorem also applies to the inverse Fourier transform.

Let $G$ be a noncompact semisimple Lie group and $K$ a maximal compact subgroup of $G$. The Fourier transform $\mathcal{F}$ on the Riemannian symmetric space $X = G/K$ is an analogue of the classical Fourier transform on $\mathbb{R}^k$. A Paley-Wiener theorem for the Fourier transform $\mathcal{F}$, which characterises the image of $C_c^\infty(X)$ under $\mathcal{F}$ in terms of holomorphic extensions and growth behaviour, as in the classical case, was proved by Helgason, see [7]. Furthermore, the $L^2$-Schwartz space $\mathcal{S}^2(X)$ contains $C_c^\infty(X)$ as a dense subspace and $\mathcal{F}$ is an isomorphism of $\mathcal{S}^2(X)$ onto some generalised Schwartz space, see [4].

Unlike the classical case, however, we can not use a duality argument to deduce a Paley-Wiener theorem for the inverse Fourier transform. So how can we characterise the functions whose Fourier transform $\mathcal{F}$ has compact support?

The Fourier transform on $X$ reduces to the spherical transform $\mathcal{H}$ on $G$ when restricted to $K$-invariant functions. The paper [8] provides an answer to the above question for the spherical transform on Schwartz functions in the rank one and complex cases. The characterisation is in analogy with the classical Paley-Wiener theorem given in terms of meromorphic extensions and growth conditions.
In this paper we prove (real) Paley-Wiener theorems for the inverse Fourier transform for general Riemannian symmetric spaces, i.e., we characterise, as a subset of $L^2(X)$, the set of functions $f$ on $X$ whose Fourier transform $\mathcal{F}f$ has compact support. More precisely, $f \in C^\infty(X)$ has to satisfy

$$\lim_{n \to \infty} \|\Delta^n f\|_{L^2(X)}^{1/2n} < \infty,$$

where $\Delta$ is the Laplace-Beltrami operator (and $(1 + |\cdot|)^n f \in L^2(X)$ for all $n \in \mathbb{N} \cup \{0\}$ if we also want the Fourier image to be smooth). Specialising to bi-$K$-invariant functions yields (real) Paley-Wiener theorems for the inverse spherical transform for general noncompact semisimple Lie groups.

Our approach is based on real analysis techniques developed by H. H. Bang, see [2] and [3], and V.K. Tuan, see [10]. Also see [11] for a history and overview of (real) Paley-Wiener theorems for certain transforms (Fourier, Mellin, Hankel...) on $\mathbb{R}$. In particular we use Parseval’s formula, intertwining properties of $\mathcal{F}$, and the following characterisation of the radius of the support of a function $g$ on $\mathbb{R}^n$:

$$\sup_{\lambda \in \text{supp} \, g} \|\lambda\| = \lim_{n \to \infty} \left\{ \int_{\mathbb{R}^n} \|\lambda\|^{4n} |g(\lambda)|^2 d\lambda \right\}^{1/4n}.$$

For completeness and comparison, we first consider the Fourier transform on $\mathbb{R}^k$. The results here are originally due to H.H. Bang, see [2] and [3], and V.K. Tuan, see [10]. Notice the beautiful symmetry between the (statements of the) results for the various transforms.

2. The Fourier transform on $\mathbb{R}^k$.

For background and details, please see [9, Chapter 7]. Let $\mathcal{F}_{cl}$ denote the classical Fourier transform on $\mathbb{R}^k$:

$$\mathcal{F}_{cl}f(\lambda) := \int_{\mathbb{R}^k} f(x)e^{-i\lambda \cdot x} dx,$$

defined for nice functions $f$, for all $\lambda \in \mathbb{C}^k$ for which the above integral makes sense. Let $\Delta = \frac{d^2}{dx_1^2} + \cdots + \frac{d^2}{dx_k^2}$ denote the Laplacian on $\mathbb{R}^k$ and let $\mathcal{S}(\mathbb{R}^k)$ denote the Schwartz space of rapidly decreasing differentiable functions. Then $\mathcal{F}_{cl}(\Delta f)(\lambda) = -\|\lambda\|^2 \mathcal{F}_{cl}f(\lambda)$, $(\lambda \in \mathbb{R}^k)$, for all $f \in \mathcal{S}(\mathbb{R}^k)$, and the Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^k)$ onto itself, with inverse given by:

$$\mathcal{F}_{cl}^{-1}g(x) = (2\pi)^{-k} \int_{\mathbb{R}^k} g(\lambda)e^{i\lambda \cdot x} d\lambda, \quad (x \in \mathbb{R}^k)$$
for $g \in \mathcal{S}(\mathbb{R}^k)$. Parseval’s formula states that
\[
\langle f_1, f_2 \rangle := \int_{\mathbb{R}^k} f_1(x)\overline{f_2(x)}\,dx = (2\pi)^{-k} \int_{\mathbb{R}^k} \mathcal{F}_{cl} f_1(\lambda)\overline{\mathcal{F}_{cl} f_2(\lambda)}\,d\lambda =: \langle \mathcal{F}_{cl} f_1, \mathcal{F}_{cl} f_2 \rangle,
\]
for $f_1, f_2 \in \mathcal{S}(\mathbb{R}^k)$, which implies that $\|f\|_2 = \|\mathcal{F}_{cl} f\|_2$, for all $f \in \mathcal{S}(\mathbb{R}^k)$, and hence that the Fourier transform extends to an isometry from $L^2(\mathbb{R}^k)$ onto itself.

Let $f \in C^\infty(\mathbb{R}^k)$ such that $\Delta^n f \in L^2(\mathbb{R}^k)$ for all $n \in \mathbb{N} \cup \{0\}$ and let $f_2 \in C^\infty_c(\mathbb{R}^k)$. Then:
\[
\langle \mathcal{F}_{cl} (\Delta f), \mathcal{F}_{cl} f_2 \rangle = \langle \Delta f, f_2 \rangle = \langle f, \Delta f_2 \rangle = \langle \mathcal{F}_{cl} f, \mathcal{F}_{cl} (\Delta f_2) \rangle = \langle \mathcal{F}_{cl} f, -\|\lambda\|^2 \mathcal{F}_{cl} f_2 \rangle = \langle -\|\lambda\|^2 \mathcal{F}_{cl} f, \mathcal{F}_{cl} f_2 \rangle,
\]
and we conclude that $\mathcal{F}_{cl} (\Delta f)(\lambda) = -\|\lambda\|^2 \mathcal{F}_{cl} f(\lambda)$ a.e., by a density argument, whence $\mathcal{F}_{cl} (\Delta^n f)(\lambda) = (-1)^n \|\lambda\|^{2n} \mathcal{F}_{cl} f(\lambda)$ a.e., and
\[
\left(1\right) \quad \int_{\mathbb{R}^k} |\Delta^n f(x)|^2\,dx = (2\pi)^{-k} \int_{\mathbb{R}^k} \|\lambda\|^{4n} |\mathcal{F}_{cl} f(\lambda)|^2\,d\lambda,
\]
for all $n \in \mathbb{N} \cup \{0\}$.

We define the support, supp $g$, of $g \in L^2(\mathbb{R}^k)$ to be the smallest closed set, outside which the function $g$ vanishes almost everywhere, and $R_g := \sup_{\lambda \in \text{supp}\,g} \|\lambda\|$ to be the radius of the support of $g$; $R_g$ is finite if, and only if, $g$ has compact support.

**Lemma 2.1.** Let $g \in L^2(\mathbb{R}^k)$ such that $\|\lambda\|^{2n} g(\lambda) \in L^2(\mathbb{R}^k)$ for all $n \in \mathbb{N} \cup \{0\}$. Then
\[
R_g = \lim_{n \to \infty} \left( \int_{\mathbb{R}^k} \|\lambda\|^{4n} |g(\lambda)|^2\,d\lambda \right)^{1/4n}.
\]

**Proof.** Assume $g$ has compact support with $R_g > 0$. Then:
\[
\limsup_{n \to \infty} \left( \int_{\mathbb{R}^k} \|\lambda\|^{4n} |g(\lambda)|^2\,d\lambda \right)^{1/4n} \\
\leq R_g \limsup_{n \to \infty} \left( \int_{\|\lambda\| \leq R_g} |g(\lambda)|^2\,d\lambda \right)^{1/4n} = R_g.
\]
On the other hand,
\[
\int_{R_g - \varepsilon \leq \|\lambda\| \leq R_g} |g(\lambda)|^2\,d\lambda > 0,
\]

for any $\varepsilon > 0$, hence
\[
\liminf_{n \to \infty} \left\{ \int_{\mathbb{R}^k} \|\lambda\|^{4n} |g(\lambda)|^2 d\lambda \right\}^{1/4n} \\
\geq \liminf_{n \to \infty} \left\{ \int_{R_g - \varepsilon \leq \|\lambda\| \leq R_g} \|\lambda\|^{4n} |g(\lambda)|^2 d\lambda \right\}^{1/4n} \\
\geq (R_g - \varepsilon) \liminf_{n \to \infty} \left\{ \int_{R_g - \varepsilon \leq \|\lambda\| \leq R_g} |g(\lambda)|^2 d\lambda \right\}^{1/4n} = R_g - \varepsilon,
\]
and thus
\[
\lim_{n \to \infty} \left\{ \int_{\mathbb{R}^k} \|\lambda\|^{4n} |g(\lambda)|^2 d\lambda \right\}^{1/4n} = R_g.
\]

Now assume that $g$ has unbounded support. Then
\[
\int_{\|\lambda\| \geq N} |g(\lambda)|^2 d\lambda > 0,
\]
for any $N > 0$, so:
\[
\liminf_{n \to \infty} \left\{ \int_{\mathbb{R}^k} \|\lambda\|^{4n} |g(\lambda)|^2 d\lambda \right\}^{1/4n} \\
\geq \liminf_{n \to \infty} \left\{ \int_{\|\lambda\| \geq N} \|\lambda\|^{4n} |g(\lambda)|^2 d\lambda \right\}^{1/4n} \\
\geq N \liminf_{n \to \infty} \left\{ \int_{\|\lambda\| \geq N} |g(\lambda)|^2 d\lambda \right\}^{1/4n} = N,
\]
for arbitrary $N > 0$, and we conclude that
\[
\liminf_{n \to \infty} \left\{ \int_{\mathbb{R}^k} \|\lambda\|^{4n} |g(\lambda)|^2 d\lambda \right\}^{1/4n} = \infty.
\]

Let $L^2_c(\mathbb{R}^k)$ denote the subspace of $L^2(\mathbb{R}^k)$ of functions with compact support and let $L^2_{R_g}(\mathbb{R}^k) := \{ g \in L^2(\mathbb{R}^k) \mid R_g = R \}$. Let also $C^\infty_{R_g}(\mathbb{R}^k) := \{ g \in C^\infty_c(\mathbb{R}^k) \mid R_g = R \}$.

**Definition 2.2.** We define the $L^2$-Paley-Wiener space $PW^2(\mathbb{R}^k)$ to be the space of all functions $f \in C^\infty(\mathbb{R}^k)$ satisfying:
\begin{enumerate}
  \item $\Delta^n f \in L^2(\mathbb{R}^k)$ for all $n \in \mathbb{N} \cup \{0\}$.
  \item $R_f := \lim_{n \to \infty} \|\Delta^n f\|_2^{1/2n} < \infty$.
\end{enumerate}
Let also $\text{PW}_R^2(\mathbb{R}^k) := \{ f \in \text{PW}^2(\mathbb{R}^k) \mid \| f \|_R^2 = R \}$, for $R \geq 0$.

The proof of Theorem 2.3 below shows that the limit in (b) above is well-defined. The real version of the $L^2$-Paley-Wiener theorem for the inverse Fourier transform can now be formulated as follows:

**Theorem 2.3.** The inverse Fourier transform $\mathcal{F}_c^{-1}$ is a bijection of $L^2(\mathbb{R}^k)$ onto $\text{PW}^2(\mathbb{R}^k)$, mapping $L^2(\mathbb{R}^k)$ onto $\text{PW}^2(\mathbb{R}^k)$.

**Proof.** Let $g \in L^2_R(\mathbb{R}^k)$. Then $\| \lambda \|^n g(\lambda) \in L^1(\mathbb{R}^k)$ for all $n \in \mathbb{N} \cup \{ 0 \}$, and $\mathcal{F}_c^{-1} g \in C_c^\infty(\mathbb{R}^k)$. We also have $\Delta^n(\mathcal{F}_c^{-1} g) = \mathcal{F}_c^{-1}((-1)^n \| \lambda \|^{2n} g) \in L^2(\mathbb{R}^k)$ for all $n \in \mathbb{N} \cup \{ 0 \}$, by the formula for $\mathcal{F}_c^{-1}$, and (1) thus yields:

$$
\lim_{n \to \infty} \left\{ \int_{\mathbb{R}^k} |\Delta^n(\mathcal{F}_c^{-1} g)(x)|^2 \, dx \right\}^{1/4n} = \lim_{n \to \infty} \left\{ (2\pi)^{-k} \int_{\mathbb{R}^k} \| \lambda \|^n |g(\lambda)|^2 \, d\lambda \right\}^{1/4n} = R,
$$

whence $\mathcal{F}_c^{-1} g \in \text{PW}^2_R(\mathbb{R}^k)$.

Let now $f \in \text{PW}^2_R(\mathbb{R}^k)$. Then $\mathcal{F}_c(\Delta^n f)(\lambda) = (-1)^n \| \lambda \|^{2n} \mathcal{F}_c f(\lambda) \in L^2(\mathbb{R}^k)$ for all $n \in \mathbb{N}$, and another application of (1) shows that

$$
\lim_{n \to \infty} \left\{ (2\pi)^{-k} \int_{\mathbb{R}^k} \| \lambda \|^n |\mathcal{F}_c f(\lambda)|^2 \, d\lambda \right\}^{1/4n} = \lim_{n \to \infty} \left\{ \int_{\mathbb{R}^k} |\Delta^n f(x)|^2 \, dx \right\}^{1/4n} = R,
$$

and we conclude that $\mathcal{F}_c f$ has compact support with $R_{\mathcal{F}_c f} = R$. \hfill \Box

**Remark 2.4.** The classical (complex) $L^2$-Paley-Wiener theorem implies that $\text{PW}^2_R(\mathbb{R}^k)$ exactly consists of those $L^2(\mathbb{R}^k)$ functions that can be extended to holomorphic functions of exponential type $R$ on $\mathbb{C}^k$.

**Remark 2.5.** Let $f \in \text{PW}^2(\mathbb{R})$. Then $\frac{d^n}{dx^n} f \in L^p(\mathbb{R})$ for all $n \in \mathbb{N} \cup \{ 0 \}$, and:

$$
\lim_{n \to \infty} \left\| \frac{d^n}{dx^n} f \right\|_{L^p}^{1/n} = R_{\mathcal{F}_c f} = R_f^A,
$$

for all $1 \leq p \leq \infty$. This follows from [2, Theorem 1]. Similar results hold for $\mathbb{R}^k$, $k > 1$, see [3, Theorem 3] and [10, Theorem 4].

**Definition 2.6.** We define the Paley-Wiener space $\text{PW}(\mathbb{R}^k)$ as the space of all functions $f \in C^\infty(\mathbb{R}^k)$ satisfying:

(a) $(1 + |x|)^m \Delta^n f \in L^2(\mathbb{R}^k)$ for all $m, n \in \mathbb{N} \cup \{ 0 \}$.

(b) $R_f^A := \lim_{n \to \infty} \| \Delta^n f \|_2^{1/2n} < \infty$.

Let again $\text{PW}_R(\mathbb{R}^k) := \{ f \in \text{PW}(\mathbb{R}^k) \mid R_f^A = R \}$, for $R \geq 0$.
We notice that the only difference between $\text{PW}^2(\mathbb{R}^k)$ and $\text{PW}(\mathbb{R}^k)$ is an extra requirement of polynomial decay, to help ensure that $\mathcal{F}_{\text{cl}} f \in C^\infty(\mathbb{R}^k)$.

The real version of the Paley-Wiener theorem for the inverse Fourier transform is the following:

**Theorem 2.7.** The inverse Fourier transform $\mathcal{F}_{\text{cl}}^{-1}$ is a bijection of $C^\infty_c(\mathbb{R}^k)$ onto $\text{PW}(\mathbb{R}^k)$, mapping $C^\infty_R(\mathbb{R}^k)$ onto $\text{PW}_R(\mathbb{R}^k)$.

**Proof.** Let $g \in C^\infty_R(\mathbb{R}^k)$, then $\mathcal{F}_{\text{cl}}^{-1} g \in \mathcal{S}(\mathbb{R}^k)$, and $\mathcal{F}_{\text{cl}}^{-1} g \in \text{PW}_R(\mathbb{R}^k)$ by Theorem 2.3.

Let $f \in \text{PW}_R(\mathbb{R}^k) \subset \text{PW}_R^2(\mathbb{R}^k)$. Then $\mathcal{F}_{\text{cl}} f \in C^\infty(\mathbb{R}^k)$ since $f$ has polynomial decay, and $\mathcal{F}_{\text{cl}} f$ has compact support with $R_{\mathcal{F}_{\text{cl}}} f = R$ by Theorem 2.3.

\[\square\]

### 3. Lie group notation.

In this section we introduce the Lie group notation we need in the next sections. We refer to [5], [6] and [7] for further details.

Let $G$ be a real connected noncompact semisimple Lie group with finite center and let $\theta$ be a Cartan involution of $G$. Then the fixed point group $K := G^\theta$ is a maximal compact subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote their Lie algebras, and let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ into the $\pm 1$ eigenspaces of $\theta$. The Killing form on $\mathfrak{g}$ induces an $\text{Ad}K$-invariant scalar product on $\mathfrak{p}$ and hence a $G$-invariant Riemannian metric on $X := G/K$. With this structure, $X$ becomes a Riemannian globally symmetric space of the noncompact type.

Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Denote its real dual by $\mathfrak{a}^*$ and its complex dual by $\mathfrak{a}_C^*$. The Killing form of $\mathfrak{g}$ induces a scalar product $\langle \cdot, \cdot \rangle$ and hence a norm $\| \cdot \|$ on $\mathfrak{a}_C$ and $\mathfrak{a}_C^*$. Let $\Sigma \subset \mathfrak{a}^*$ denote the root system of $(\mathfrak{g}, \mathfrak{a})$ and let $W$ be the associated Weyl group. Choose a set $\Sigma_+ \subset \Sigma$ of positive roots, let $\mathfrak{n} := \bigoplus_{\alpha \in \Sigma_+} \mathfrak{g}_\alpha$ be the corresponding nilpotent subalgebra of $\mathfrak{g}$ and let $\mathfrak{a}_+ := \{ H \in \mathfrak{a} | \alpha(H) > 0 \forall \alpha \in \Sigma_+ \}$ be the positive Weyl chamber with $\overline{\mathfrak{a}_+}$ its closure. Denote by $\mathfrak{a}_C^+$ and $\overline{\mathfrak{a}_C^+}$ the similar cones in $\mathfrak{a}_C^*$, and define the element $\rho \in \mathfrak{a}^*$ by: $\rho(H) := \frac{1}{2} \sum_{\alpha \in \Sigma_+} m_\alpha \alpha(H)$, $H \in \mathfrak{a}$, where $m_\alpha = \dim \mathfrak{g}_\alpha$.

Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the Iwasawa decomposition of $\mathfrak{g}$ and $G = KAN$ the corresponding Iwasawa decompositions of $G$, where $A$ and $N$ are the Lie groups generated by $\mathfrak{a}$ and $\mathfrak{n}$ respectively. Every $g \in G$ can be represented as: $g = K \exp H(g)N = N \exp A(g)K$, where the projections onto the $A$-parts $A(g) \in \mathfrak{a}$ and $H(g) \in \mathfrak{a}$ are uniquely determined. We note that $A(g) = -H(g^{-1})$. Let $M := Z_K(\mathfrak{a})$, then $B := K/M$ is a compact homogeneous space. We define the vector $A(x, b) \in \mathfrak{a}$ as $A(x, b) := A(k^{-1} g)$, for $x = gK \in X$ and $b = kM \in B$.

Put $A_+ = \exp(\mathfrak{a}_+)$, then $\overline{A}_+ = \exp(\overline{\mathfrak{a}_C^+})$. The Cartan decomposition implies that the natural mapping from $K/M \times A_+ \times K$ into $G = K\overline{A}_+K$ is a
diffeomorphism onto its dense open image. We define the norm of an element $g \in G$ as: $|g| = |k_1 \exp(H)k_2| = \|H\|$, with $H \in \mathfrak{a}_+^*$; this is the $K$-invariant geodesic distance to the origin $eK$. We denote by $B_R := \{g \in G \mid |g| \leq R\}$ the $K$-invariant ball of radius $R$ around $e$.

We identify functions on $X$ with right-$K$-invariant functions on $G$. We normalise the invariant measure on $X$ as:

$$\int_X f(x)dx = \int_K \int_{\mathfrak{a}_+} \int_K f(k_1 \exp(H)k_2)J(H)dk_1dHdk_2,$$

for $f \in C_c^\infty(X)$, where the Jacobian $J$ is given by: $J(H) = \prod_{\alpha \in \Sigma_+} (e^{\alpha(H)} - e^{-\alpha(H)})^{m_\alpha}$, $dH$ is the Lebesgue measure on $\mathfrak{a}$ and $dk$ is the measure on $K$ such that $\int_K dk = 1$. We notice that $0 \leq J(H) \leq Ce^{2\rho(H)}$, for $H \in \mathfrak{a}_+$, where $C$ is a positive constant.

The spherical functions $\varphi_\lambda$, $\lambda \in \mathfrak{a}_C^*$, on $G$ are defined as:

$$\varphi_\lambda(g) := \int_K e^{(i\lambda + \rho)A(k^{-1}g)}dk = \int_K e^{-(i\lambda + \rho)H(g^{-1}k)}dk.$$

We note that $\varphi_\lambda$ is Weyl group invariant, $\varphi_{w\lambda} = \varphi_\lambda$, $w \in W$. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. We write $Df(g)$ for the action of $D \in U(\mathfrak{g})$ on $f \in C^\infty(G)$ from the left at $g \in G$. The $L^p$-Schwartz space $S^p(X)$, $0 < p \leq 2$, is defined as the space of all functions $f \in C^\infty(X)$ such that:

$$\sup_{g \in G} (1 + |g|)^m \varphi_o(g)^{-\frac{2}{p}} |Df(g)| < \infty,$$

for all $D \in U(\mathfrak{g})$ and $m \in \mathbb{N} \cup \{0\}$. We can also characterise $S^p(X)$ as the space of all functions $f \in C^\infty(X)$ satisfying:

$$(1 + |g|)^m Df(g) \in L^p(X),$$

for all $D \in U(\mathfrak{g})$ and $m \in \mathbb{N} \cup \{0\}$. We note that $S^p(X) \not\subset L^q(X)$ for $0 < q < p \leq 2$.

### 4. The Fourier transform.

In this section, we recall some facts and theorems for the Fourier transform on a noncompact semisimple Riemannian symmetric space, see [7, Chapter 3] for details and references.

The Fourier transform of a function $f$ on $X$ is defined as:

$$\mathcal{F}f(\lambda, b) := \int_X f(x)e^{-(i\lambda + \rho)(A(x,b))}dx,$$

for all $\lambda \in \mathfrak{a}_C^*$, $b \in B$ for which the integral exists. In particular, $\mathcal{F}f$ extends to a smooth function on $\mathfrak{a}_C^* \times B$, holomorphic in the first variable, for $f \in C_c^\infty(X)$, see also below.
The plane wave eigenfunction

\[ e_{\lambda,b}(x) := e^{(i\lambda + \rho)(A(x,b))}, \]

is a joint eigenfunction of \( \mathbb{D}(X) \), the commutative algebra of \( G \)-invariant differential operators on \( X \), for all \( \lambda \in a_C^* \), \( b \in B \), or, more precisely

\[ D e_{\lambda,b} = \Gamma(D)(i\lambda)e_{\lambda,b}, \quad \forall D \in \mathbb{D}(X), \quad (\lambda \in a_C^*, b \in B) \]

where \( \Gamma : \mathbb{D}(X) \to S(\mathfrak{a}^*)^W \) is the Harish-Chandra isomorphism. In particular,

\[ \Delta e_{\lambda,b} = -(\langle \lambda, \lambda \rangle + \|\rho\|^2)e_{\lambda,b}, \quad (\lambda \in a_C^*, b \in B) \]

for the Laplace-Beltrami operator \( \Delta \) on \( X \), and hence

\[ \mathcal{F}(\Delta f)(\lambda, b) = -(\langle \lambda, \lambda \rangle + \|\rho\|^2)\mathcal{F}f(\lambda, b), \quad (\lambda \in a_C^*, b \in B) \]

for all \( f \in C_c^\infty(X) \), by self-adjointness of \( \Delta \), see also (6).

A \( C^\infty \)-function \( \psi(\lambda, b) \) on \( a_C^* \times B \), holomorphic in \( \lambda \), is called a holomorphic function of uniform exponential type \( R \), if there exists a constant \( R \geq 0 \), such that, for each \( N \in \mathbb{N} \), we have:

\[ \sup_{\lambda \in a_C^*, b \in B} e^{-R|\lambda|}(1 + |\lambda|)^N|\psi(\lambda, b)| < \infty. \]

The space of holomorphic functions of uniform exponential type \( R \) will be denoted \( \mathcal{H}_R(a_C^* \times B) \) and we denote by \( \mathcal{H}(a_C^* \times B) \) their union over all \( R > 0 \). Let furthermore \( \mathcal{H}(a_C^* \times B)^W \) denote the space of all functions \( \psi \in \mathcal{H}(a_C^* \times B) \) satisfying the symmetry condition:

\[ \int_B e^{i(w\lambda + \rho)(A(x,b))} \psi(w\lambda, b) db = \int_B e^{i(\lambda + \rho)(A(x,b))} \psi(\lambda, b) db, \]

for \( w \in W \) and all \( \lambda \in a_C^*, x \in X \).

The Paley-Wiener theorem states that the Fourier transform is a bijection of the space \( C_c^\infty(X) \) onto the space \( \mathcal{H}(a_C^* \times B)^W \), with the following inversion formula:

\[ f(x) = \int_{a_C^*} \int_B e^{i(\lambda + \rho)(A(x,b))} \mathcal{F}f(\lambda, b)|c(\lambda)|^{-2} d\lambda db, \quad (x \in X) \]

where \( c(\lambda) \) is the Harish-Chandra \( c \)-function, for \( f \in C_c^\infty(X) \). Moreover, \( \mathcal{F}f \in \mathcal{H}(a_C^* \times B)^W \) if, and only if, supp \( f \subset B_R \). We note that \( |c(\lambda)|^{-2} \) is bounded by some polynomial for \( \lambda \in \mathfrak{a}^* \).

Let \( f_1, f_2 \in C_c^\infty(X) \), then Parseval’s formula for \( \mathcal{F} \) is given by:

\[ \int_X f_1(x)\overline{f_2(x)} dx = \int_{a_C^*} \int_B \mathcal{F}f_1(\lambda, b)\overline{\mathcal{F}f_2(\lambda, b)}|c(\lambda)|^{-2} d\lambda db. \]

We conclude that the Fourier transform extends to an isometry of \( L^2(X) \) onto \( L^2(a_C^* \times B, |c(\lambda)|^{-2} d\lambda db) \). In the following we adopt the convention \( L^2(a_C^* \times B) := L^2(a_C^* \times B, |c(\lambda)|^{-2} d\lambda db) \).
Let \( f \in C^\infty(X) \) such that \( \Delta^n f \in L^2(X) \) for all \( n \in \mathbb{N} \cup \{0\} \) and let \( f_2 \in C_c^\infty(X) \). Then self-adjointness of the Laplace-Beltrami operator \( \Delta \): \[
abla^n f(x)f_2(x)dx = \nabla^n f(x)\Delta^n f_2(x)dx,
\]
Parseval’s formula (5) and density of \( C_c^\infty(X) \) imply, as in the classical case, that
\[
\mathcal{F}(\Delta^n f)(\lambda, b) = (-1)^n(\|\lambda\|^2 + \|\rho\|^2)^n \mathcal{F} f(\lambda, b),
\]
a.e., for all \( n \in \mathbb{N} \cup \{0\} \).

5. The inverse Fourier transform.

We define the inverse Fourier transform \( \mathcal{F}^{-1} g \) of a function \( g \) on \( a^*_+ \times B \) via (4):
\[
\mathcal{F}^{-1} g(x) := \int_{a^*_+} \int_B e^{(i\lambda + \rho)(A(x,b))} g(\lambda,b)|e(\lambda)|^{-2}d\lambda db,
\]
for all \( x \in X \) for which the integral exists.

We define the support, \( \text{supp} g \), of \( g \in L^2(a^*_+ \times B) \) to be the smallest closed set in \( a^*_+ \times B \), outside which the function \( g \) vanishes almost everywhere, and \( R_g := \sup_{(\lambda, b) \in \text{supp} g} \|\lambda\| \) to be the ‘radius’ of the support of \( g \).

**Lemma 5.1.** Let \( g \in L^2(a^*_+ \times B) \) such that \( \|\lambda\|^{2n} g(\lambda, b) \in L^2(a^*_+ \times B) \) for all \( n \in \mathbb{N} \cup \{0\} \). Then
\[
R_g = \lim_{n \to \infty} \left\{ \int_{a^*_+} \int_B \|\lambda\|^{4n} |g(\lambda, b)|^2 |e(\lambda)|^{-2} d\lambda db \right\}^{1/4n}.
\]

**Proof.** As for Lemma 2.1.

Let \( L_c^2(a^*_+ \times B) \) denote the subspace of \( L^2(a^*_+ \times B) \) of functions with bounded support and let \( L^2_R(a^*_+ \times B) := \{ g \in L^2(a^*_+ \times B) \mid R_g = R \} \).

**Definition 5.2.** We define the \( L^2 \)-Paley-Wiener space \( \text{PW}^2(X) \) as the space of all functions \( f \in C^\infty(X) \) satisfying:
(a) \( \Delta^n f \in L^2(X) \) for all \( n \in \mathbb{N} \cup \{0\} \).
(b) \( R_f^2 := \lim_{n \to \infty} \|\Delta + \|\rho\|^2\|^n f\|_2^{1/2n} < \infty \).

Let also \( \text{PW}^2_R(X) := \{ f \in \text{PW}^2(X) \mid R_f^2 = R \} \), for \( R \geq 0 \).

The real \( L^2 \)-Paley-Wiener theorem for the inverse Fourier transform can now be formulated as follows:

**Theorem 5.3.** The inverse Fourier transform \( \mathcal{F}^{-1} \) is a bijection of \( L^2_c(a^*_+ \times B) \) onto \( \text{PW}^2(X) \), mapping \( L^2_R(a^*_+ \times B) \) onto \( \text{PW}^2_R(X) \).
Proof. Let \( g \in L^2_R(a^*_+ \times B) \). Then \( \mathcal{F}^{-1}g \in C^\infty(X) \) by Lebesgue’s dominated convergence theorem. Equation (2) gives, for \( D \in \mathbb{D}(X) \),

\[
D(\mathcal{F}^{-1}g)(x) = \int_{a^*_+} \int_B \Gamma(D)(i\lambda)e^{(i\lambda + \rho)(A(x,b))}g(\lambda, b)|c(\lambda)|^{-2}d\lambda db,
\]

which in particular shows that \( (\Delta + \|\rho\|^2)^n \mathcal{F}^{-1}g = \mathcal{F}^{-1}((-1)^n\|\lambda\|^{2n}) \) \( \in L^2(X) \) for all \( n \in \mathbb{N} \cup \{0\} \). Parseval’s formula (5) with

\[
f_1 = f_2 = \mathcal{F}^{-1}((-1)^n\|\lambda\|^{2n})
\]

yields:

\[
\lim_{n \to \infty} \left\{ \int_X |(\Delta + \|\rho\|^2)^n(\mathcal{F}^{-1}g)(x)|^2 dx \right\}^{1/4n} = \lim_{n \to \infty} \left\{ \int_{a^*_+} \int_B \|\lambda\|^4n|g(\lambda, b)|^2|c(\lambda)|^{-2}d\lambda db \right\}^{1/4n} = R,
\]

whence \( \mathcal{F}^{-1}g \in \text{PW}^2_R(X) \).

Let now \( f \in \text{PW}^2_R(X) \). Then \( \mathcal{F}((\Delta + \|\rho\|^2)^nf)(\lambda, b) = (-1)^n\|\lambda\|^2\mathcal{F}f(\lambda, b) \) \( \in L^2(a^*_+ \times B) \) for all \( n \in \mathbb{N} \) by (7). Another application of Parseval’s formula as above with \( f_1 = f_2 = (\Delta + \|\rho\|^2)^nf \) shows that \( R_{\mathcal{F}f} = R_{\mathcal{F}^2} = R \), and we conclude that \( \mathcal{F}f \) has bounded support. \( \square \)

**Corollary 5.4.** Let \( f \in C^\infty(X) \) be such that \( \Delta^nf \in L^2(X) \) for all \( n \in \mathbb{N} \cup \{0\} \). It then follows that \( \lim_{n \to \infty} \|\Delta^n f\|_2^{1/2n} < \infty \) if, and only if, \( \lim_{n \to \infty} \|\Delta + \|\rho\|^2\|^n f\|_2^{1/2n} < \infty \). Furthermore, \( \lim_{n \to \infty} \|\Delta^nf\|_2^{1/2n} = (R^2 + \|\rho\|^2)^{1/2} \), for \( f \in \text{PW}^2_R(X) \) with \( R > 0 \).

Proof. Let \( f \in \text{PW}^2_R(X) \), with \( R > 0 \), then \( \mathcal{F}f \in L^2_R(a^*_+ \times B) \). Parseval’s formula and an easy adaption of the proof of Lemma 2.1 shows that

\[
\lim_{n \to \infty} \|\Delta^nf\|_2^{1/2n} = \lim_{n \to \infty} \left\{ \int_{a^*_+} \int_B (\|\lambda\|^2 + \|\rho\|^2)^n|\mathcal{F}f(\lambda, b)|^2|c(\lambda)|^{-2}d\lambda db \right\}^{1/4n} = (R^2 + \|\rho\|^2)^{1/2}.
\]
Assume that \( \lim_{n \to \infty} \|\Delta^n f\|_2^{1/2n} < \infty \). Then \( \mathcal{F}(\Delta^n f)(\lambda, b) = (-1)^n(\|\lambda\|^2 + \|\rho\|^2)^n \mathcal{F}f(\lambda, b) \in L^2(\mathfrak{a}_+^* \times \mathcal{B}) \), for all \( n \in \mathbb{N} \), and

\[
\lim_{n \to \infty} \left\{ \int_{\mathfrak{a}_+^*} \int_{\mathcal{B}} \|\lambda\|^{4n} |\mathcal{F}f(\lambda, b)|^2 |c(\lambda)|^{-2} d\lambda db \right\}^{1/4n} \\
\leq \lim_{n \to \infty} \left\{ \int_{\mathfrak{a}_+^*} \int_{\mathcal{B}} (\|\lambda\|^2 + \|\rho\|^2)^{2n} |\mathcal{F}f(\lambda, b)|^2 |c(\lambda)|^{-2} d\lambda db \right\}^{1/4n} \\
= \lim_{n \to \infty} \|\Delta^n f\|_2^{1/2n} < \infty,
\]

that is, \( \mathcal{F}f \) has bounded support.

**Remark 5.5.** Assume that \( f \in \mathcal{S}^p(X) \), with \( 0 < p < 2 \), then \( \mathcal{F}f \) extends to an analytic function on a small tube domain around \( \mathfrak{a}_+^* \times \mathcal{B} \) in \( \mathfrak{a}_+^* \times \mathcal{B} \). Hence \( \mathcal{F}f \) cannot have compact support on \( \mathfrak{a}_+^* \times \mathcal{B} \) and we conclude that \( \mathcal{S}^p(X) \cap \text{PW}^2(X) = \{0\} \) for any \( 0 < p < 2 \).

**Definition 5.6.** We define the Paley-Wiener space \( \text{PW}(X) \) as the space of all functions \( f \in C^\infty(X) \) satisfying:

(a) \( (1 + |x|)^m \Delta^n f \in L^2(X) \) for all \( m, n \in \mathbb{N} \cup \{0\} \).

(b) \( R^\Delta_f = \lim_{n \to \infty} \|\Delta + \|\rho\|^2\|^n f\|_2^{1/2n} < \infty \).

Let also \( \text{PW}_R(X) := \{ f \in \text{PW}(X) | R^\Delta_f = R \} \), for \( R \geq 0 \).

Here \( |x| := |g| \), for \( x = gK \in X \). Again, the only difference between the Paley-Wiener spaces \( \text{PW}(X) \) and \( \text{PW}^2(X) \) is the polynomial decay condition (a), ensuring that \( \mathcal{F}f \in C^\infty(\mathfrak{a}_+^* \times \mathcal{B})^W \) (see below).

The space \( C^\infty_c(\mathfrak{a}_+^* \times \mathcal{B})^W \) is defined as the subspace of functions \( \psi \in C^\infty(\mathfrak{a}_+^* \times \mathcal{B}) \) satisfying the symmetry condition (3) for all \( w \in W \) and all \( \lambda \in \mathfrak{a}_+, x \in X \). Let finally \( C^\infty_c(\mathfrak{a}_+^* \times \mathcal{B}) := \{ f \in C^\infty(\mathfrak{a}_+^* \times \mathcal{B}) | R_g = R \} \).

The real Paley-Wiener theorem for the inverse Fourier transform then is:

**Theorem 5.7.** The inverse Fourier transform \( \mathcal{F}^{-1} \) is a bijection of \( C^\infty_c(\mathfrak{a}_+^* \times \mathcal{B})^W \) onto \( \text{PW}(X) \), mapping \( C^\infty_c(\mathfrak{a}_+^* \times \mathcal{B})^W \) onto \( \text{PW}_R(X) \).

**Proof.** Let \( g \in C^\infty_c(\mathfrak{a}_+^* \times \mathcal{B})^W \), then \( g \in L^2_R(\mathfrak{a}_+^* \times \mathcal{B}) \) and thus \( \mathcal{F}^{-1}g \in \text{PW}_R(X) \) by Theorem 5.3. We furthermore see that \( \mathcal{F}^{-1}g \in \mathcal{S}^2(X) \) by [4, Theorem 4.1.1], whence \( \mathcal{F}^{-1}g \) satisfies the polynomial decay condition (a).

Let now \( f \in \text{PW}_R(X) \). The basic estimate \( \|A(g)\| \leq C|g| \), for all \( g \in G \), gives us a polynomial estimate (in \( x \)) of the derivatives (with respect to \( \lambda \)) of the plane wave eigenfunctions \( e_{\lambda,b}(x) \). It is also well-known that \( (1 + |x|)^{-r}\varphi_0 \in L^2(X) \) for some large \( r \in \mathbb{N} \). All this, the polynomial decay condition (a), the Cauchy-Schwartz theorem and Lebesgue’s dominated convergence theorem imply that \( \mathcal{F}f \in C^\infty(\mathfrak{a}_+^* \times \mathcal{B})^W \). Furthermore \( \mathcal{F}f \) has the desired compact support by Theorem 5.3.
6. The inverse spherical transform.

In this section, we specialise our results to bi-$K$-invariant functions, that is, we consider the (inverse) spherical transform. We refer to [1], [5] and [6] for background concerning Paley-Wiener theorems for the spherical transform. Let $C^\infty(K\backslash G/K) \subset C^\infty(G)$ denote the subspace of bi-$K$-invariant differentiable functions on $G$. We will use similar notation for the $L^2$, Paley-Wiener and Schwartz spaces of $K$-invariant differentiable functions.

Let $f \in C_c^\infty(K\backslash G/K)$. The spherical transform $\mathcal{H}f$ of $f$ is defined as:

$$\mathcal{H}f(\lambda) := \int_G f(x) \varphi_{-\lambda}(x) dx,$$

for $\lambda \in \mathfrak{a}_C^*$. We note that $\mathcal{F}f(\lambda, b) = \mathcal{H}f(\lambda)$ for all $\lambda \in \mathfrak{a}_C^*$ and all $b \in B$. This follows from left-$K$-invariance of $f$, the identity $A(k \cdot x, b) = A(x, k^{-1} \cdot b)$ and integrating over $K$.

The spherical transform is an isomorphism of $\mathcal{S}^2(K\backslash G/K)$ onto $\mathcal{S}(\mathfrak{a}_C^*)^W$, the Weyl group invariant Schwartz functions on $\mathfrak{a}_C^*$. The inversion formula is given by:

$$f(x) = \frac{1}{|W|} \int_{\mathfrak{a}_C^*} \mathcal{H}f(\lambda) \varphi_{\lambda}(x) |c(\lambda)|^{-2} d\lambda, \quad (x \in G) \quad (8)$$

for $f \in \mathcal{S}^2(K\backslash G/K)$. We use (8) to define the inverse spherical transform $\mathcal{H}^{-1}g$ for a general function $g$ on $\mathfrak{a}_C^*$:

$$\mathcal{H}^{-1}g(x) := \frac{1}{|W|} \int_{\mathfrak{a}_C^*} g(\lambda) \varphi_{\lambda}(x) |c(\lambda)|^{-2} d\lambda.$$

Let $f \in C^\infty(K\backslash G/K)$ be such that $\Delta^n f \in L^2(K\backslash G/K)$ for all $n \in \mathbb{N} \cup \{0\}$. Then $\mathcal{H}((\Delta + |\rho|^2)^n f)(\lambda) = (-1)^n |\lambda|^2 \mathcal{H}f(\lambda)$ a.e., and Parseval’s formula for $\mathcal{H}$ gives:

$$\int_G |(\Delta + |\rho|^2)^n f(x)|^2 dx = \frac{1}{|W|} \int_{\mathfrak{a}_C^*} |\lambda|^{4n} |\mathcal{H}f(\lambda)|^2 |c(\lambda)|^{-2} d\lambda,$$

for all $n \in \mathbb{N} \cup \{0\}$. It also follows that the spherical transform extends to an isometry from $L^2(K\backslash G/K)$ onto $L^2(\mathfrak{a}_C^*, \frac{1}{|W|} |c(\lambda)|^{-2} d\lambda)^W$, where superscript $W$ denotes Weyl group invariance.

Let $L^2_c(\mathfrak{a}_C^*)^W$ denote the Weyl group invariant $L^2$-functions on $\mathfrak{a}_C^*$ with compact support and let subscript $R$ denote the radius of the support. The real versions of the Paley-Wiener theorems for the inverse spherical transform then becomes:

**Theorem 6.1.** The inverse spherical transform $\mathcal{H}^{-1}$ is a bijection of $L^2_c(\mathfrak{a}_C^*)^W$ onto $\text{PW}^2(K\backslash G/K)$, mapping $L^2_R(\mathfrak{a}_C^*)^W$ onto $\text{PW}_R^2(K\backslash G/K)$.

**Theorem 6.2.** The inverse spherical transform $\mathcal{H}^{-1}$ is a bijection of $C_c^\infty(\mathfrak{a}_C^*)^W$ onto $\text{PW}(K\backslash G/K)$, mapping $C^\infty_R(\mathfrak{a}_C^*)^W$ onto $\text{PW}_R(K\backslash G/K)$.
Proof. The above theorems are special cases of Theorem 5.3 and Theorem 5.7. We note, however, that we can prove them independently using Parseval’s formula and intertwining properties of $\mathcal{H}$. □

Remark 6.3. Let $f \in \text{PW}(K \setminus G/K)$ and consider $f$ as a function on $\mathfrak{a}$ by the application $H \mapsto f(\exp(H))$. Then $f$ does not extend to an entire function on $\mathfrak{a}_C$, due to the poles of the spherical function $\varphi_\lambda(\exp(H))$. There is, however, a description of the Paley-Wiener space $\text{PW}(K \setminus G/K)$ as functions having an explicit meromorphic extension and satisfying some exponential growth conditions for the rank 1 and the complex cases, see [8] for details.

References


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