WORD EQUATION $ABC = CDA$, $B \neq D$

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We find a new formula for solving $ABC = CDA$, $B \neq D$ for 4 nonempty words in a free semigroup. Properties of the solutions are derived.

1. Introduction.

In a free semigroup, let $Q$ be a quadruple $⟨A, B, C, D⟩$ of nonempty words satisfying $ABC = CDA$, $B \neq D$. Hmelevskii [1] gives a formula for $Q$ and proves that the solutions of $ABC = CDA$ cannot be represented by a finite set of formulas involving words and positive integer exponents. Hmelevskii also shows that such a representation does exist for equations in 3 variables. This paper contains a simpler formula for $Q$ and proofs of some of its properties. For more results about words, see [2], [3].

2. Terminology.

Fix an alphabet of letters. A word $W$ is a finite sequence of letters. $|W|$ is the length of $W$; the empty word is 1; $|1| = 0$. Word $X$ followed by word $Y$ is written as a product $XY$. $X \leq Y$ if $XZ = Y$ for some possibly empty word $Z$. A product of $k$ copies of $W$, written as $W^k$, is a power of $W$ if $k \geq 0$ with $W^0 = 1$ and a proper power if $W \neq 1$ and $k \geq 2$. Write $W$ backwards to get $W^*$. So $(XY)^* = Y^*X^*$. Word $W$ is periodic if $W = A(BA)^k$ for some $B \neq 1$, $A, k \geq 2$.

A solution is a quadruple $Q = ⟨A, B, C, D⟩$ of words with $ABC = CDA$, $B \neq D$ and $A, B, C, D \neq 1$. Also use the notation $Q = ⟨Q_1, Q_2, Q_3, Q_4⟩$. All such $Q$ form a set $Σ$. A quadruple $Q$ is unitary if $|A| = 1$. Define $σ(Q) = ABCD$. Using words $X, Y, Z$, define special quadruples:

$A_k = A_k(X, Y, Z) = ⟨X(YX)^k, YXZ, X(YX)^{k+1}, ZXY⟩$, $k \geq 0$.

$B_k = B_k(X, Y, Z) = ⟨X, YXZ, XYX(ZXYX)^k, ZXY⟩$, $k \geq 0$.

For any quadruple $U = ⟨A, B, C, D⟩$, define functions $p, q$:

$p(U) = ⟨ABC, B, C, D⟩$ and $q(U) = ⟨C, D, A, B⟩$.

If $U \in Σ$ then $p(U) \in Σ$ and $q(U) \in Σ$. Let $Γ$ be the set of all finite products of $p$’s and $q$’s. The identity function $i$ is in $Γ$ since $(qq)(U) = q(q(U)) = U = i(U)$.  

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Remark 2.1. \(A_k(X, Y, Z) \in \Sigma\) if and only if \(YXZ \neq ZXY\) and \(X \neq 1\) if \(k = 0\).

Remark 2.2. \(B_k(X, Y, Z) \in \Sigma\) if and only if \(YXZ \neq ZXY\) and \(X \neq 1\).

Remark 2.3. If \(\langle A, B, C, D \rangle\) is a solution then \(A \neq C, |A| \neq |C|, |B| = |D|\).

Remark 2.4. If \(\langle A, B, C, D \rangle\) is a solution and \(|A| = 1\) then \(|C| > 1\).

3. Summary of results.

(1) Each solution \(Q\) equals \(g(A_k)\) for some \(g \in \Gamma\) and \(A_k \in \Sigma\) (Theorem 5.1).

(2) If \(ABC = CDA, B \neq D\) then \(ABCD\) is not a proper power (Theorem 5.2).

(3) Each unitary solution equals some \(B_k \in \Sigma\) (Theorem 5.3).

(4) If \(\langle A, B, C, D \rangle = g(A_k(a, b, c)), k \geq 0, g \in \Gamma,\) letters \(a, b, c,\) then \(\{B, D\} = \{bac, cab\}\); \(A, C\) have odd lengths; \(A = A^*, C = C^*\) (Theorem 5.4).

(5) For each \(\langle A, B, C, D \rangle \in \Sigma,\) \(ABCD\) or \(CDAB\) is periodic and \(ABCD\) or \(CDAB\) equals \(\sigma(B_k)\) for some unitary \(B_k \in \Sigma, k = 0\) or 1 (Theorem 5.5).

4. Preliminaries.

Lemma 4.1. Let \(Q = A_k(X, Y, Z)\) be a unitary solution with \(k \geq 0\). Then \(Q\) equals some \(A_0\).

Proof. \(|X(YX)^k| = 1\) implies either \(k = 0\) or \(k = 1\). If \(k = 0\) we are done. If \(k = 1\) then \(X = 1, |Y| = 1, YZ \neq ZY, A_k = A_1(1, Y, Z) = A_0(Y, 1, Z)\). \(\square\)

Lemma 4.2. If \(V = g(U)\) with \(g \in \Gamma, U = A_k, k \geq 0\) then \(U_1 \leq V_1, U_3 \leq V_3\).

Proof. True if \(g\) is the identity function or if \(g\) is \(p\) or \(q\). Use an induction argument on the number of \(p\)'s and \(q\)'s in \(g\). \(\square\)

Lemma 4.3. Let \(V = g(A_k)\) be a unitary quadruple for some \(g \in \Gamma, k \geq 0\). Then \(g = qp^nq\) for some \(n \geq 0\).

Proof. Let \(U = A_k\). Since \(|V_1| = 1\), applying Lemma 4.2 to \(V\) yields \(|U_1| = 1\). By Remark 2.4, \(|U_3| > 1\). We can assume \(g\) is not the identity function because we can use \(n = 0\) in that case. Then \(g\) can be expressed as a reduced product of \(p\)'s and \(q\)'s with no 2 adjacent \(q\) terms.

We use the following observations:

(1) \(V = g(U)\) and \(g\) is not the identity function.

(2) \(|U_1| = 1, |U_3| > 1, |V_1| = 1\).

(3) \(q\) interchanges the first and third components of a quadruple.
(4) $p$ increases the length of the first component of a quadruple.
(5) $p$ preserves the length of the third component of a quadruple.

We conclude that a reduced product for $g$ equals $qp^nq$ for some $n \geq 1$. □

**Lemma 4.4.** If $YZX \neq ZXY$, $n \geq 2$ then $\sigma(B_n(X,Y,Z)) = \sigma(B_0(X,V,W))$ for some words $V$, $W$ such that $VXW \neq WXV$.

**Proof.** For any $t \geq 0$,

$$\sigma(B_{2t+2}) = (XYXZ)^{2t+4}XY, \quad \sigma(B_{2t+3}) = (XYXZ)^{2t+5}XY.$$  

$$\sigma(B_{2t+2}) = \sigma(B_0(X,V,W)) \text{ for } V = Y, W = Z(YXZ)^{t+1}.$$  

$$\sigma(B_{2t+3}) = \sigma(B_0(X,V,W)) \text{ for } V = YZX,Y, W = Z(YXZ)^{t+1}.$$  

In each case, $YXZ \leq VXW$ and $ZXY \leq WXV$. Therefore $VXW \neq WXV$. □

The following Lemma is Proposition 1.3.4 in [2].

**Lemma 4.5.** For words, $XZ = YX$ with $Y, Z \neq 1$ implies $Y = UV$, $X = U(VU)^k$, $Z = VU$ for some $U, V$ with $k \geq 0$. If $X \neq 1$ then we can choose $V \neq 1$.

5. Main results.

**Theorem 5.1.** Each solution $Q$ equals $g(A_k)$ for some $g \in \Gamma$ and $A_k \in \Sigma$.

**Proof.** Let $Q = \langle A, B, C, D \rangle$ be a solution. Then $|A| \neq |C|$. We may assume $|A| < |C|$ since the function $g$, applied to $Q$, interchanges $A$ and $C$.

Define $m = m(Q) = |ABC|$ so that $m \geq 3$. Therefore $m > 3$ since $|A| \neq |C|$. Suppose $m = 4$. $|A| = 1 = |B|$, $|C| = 2$ else $|B| = 2$ implies $|A| = 1 = |C|$. Thus $Q = \langle a, a, aa, ba \rangle$ for some letter $a$ and $B = a = D$, impossible. So $m \geq 5$. Assume $m = 5$. Then $\langle |A|, |B|, |C| \rangle$ equals $\langle 1, 1, 3 \rangle$ or $\langle 1, 2, 2 \rangle$.

In the first case, $Q = \langle a, b, aba, b \rangle$, contradicting $B \neq D$. In the second case, $Q = \langle a, ab, aa, ba \rangle = A_0(a,1,b)$ for letters $a \neq b$. Thus the theorem is true for $m = 5$.

Use induction on $m$. Assume $m > 5$. Suppose $|AB| < |C|$. Then $C = ABI = JDA$ for some $I,J \neq 1$. $ABJDA = ABC = CDA = ABIDA$ so $I = J$, $ABI = C = IDA$. Then $R = \langle I, D, A, B \rangle$ is a solution.

$$m(R) = |IDA| = |C| < |ABC| = m(Q).$$  

By an induction assumption, $R = h(A_k)$ for some $h \in \Gamma$, $A_k \in \Sigma$. Therefore $q(p(h(A_k))) = q(p(R)) = \langle A, B, IDA, D \rangle = Q$. Use $g = qp$. Now suppose $|C| \leq |AB|$. Using $ABC = CDA$, deduce $C = AI = JA$ for some $I,J \neq 1$. Then $|J| = |I| \leq |B| = |D|$ since $|AI| = |C| \leq |AB|$. Using
\(ABC = CDA,\)

\(ABC = AIDA \Rightarrow BC = IDA \Rightarrow B = IK\) for some \(K\) using \(|I| \leq |B|\).

\(ABJA = CDA \Rightarrow ABJ = CD \Rightarrow D = LJ\) for some \(L\) using \(|J| \leq |D|\).

Then \(AIKJA = ABC = CDA = AILJA\) implies \(K = L\). Apply Lemma 4.5 to \(AI = JA\).

We get \(A = X(YX)^k, I = YX, J = XY\) for some words \(Y \neq 1, X\) and \(k \geq 0\). So \(C = AI = X(YX)^{k+1}\), \(B = IK = YXK, D = KJ = KXY\). If \(k = 0\) then \(X = A \neq 1\). Thus

\[\langle A, B, C, D \rangle = \langle X(YX)^k, YXK, X(YX)^{k+1}, KXY \rangle = A_k(X, Y, K) \in \Sigma.\]

Thus \(Q = g(A_k(X, Y, Z))\) using the identity function for \(g\) and \(Z = K\). \(\square\)

**Theorem 5.2.** If \(ABC = CDA, B \neq D\) then \(ABCD\) is not a proper power.

**Proof.** It suffices to show that \(ABCD\) or \(CDAB\) is not a proper power. Assume \(|A| < |C|\). Suppose \(ABCD = U^k\) for \(k \geq 2\). So \(U \leq ABC\) and \(CDAB = V^k\) with \(V \leq CDA\). Since \(|U| = |V|\) and \(ABC = CDA\), it follows that \(U = V\) and \(ABCD = U^k = V^k = CDAB, D = B\), a contradiction. Now assume \(|C| < |A|\). Then a similar argument show that \(CDAB\) is not a proper power. \(\square\)

**Theorem 5.3.** Each unitary solution equals some \(B_n \in \Sigma\).

**Proof.** Let \(V\) be a unitary solution. Then \(V = g(A_k(X, Y, Z))\) with \(g \in \Gamma, A_k \in \Sigma\) by Theorem 5.1. \(U = A_k(X, Y, Z)\) is unitary by Lemma 4.2. By Lemma 4.1, \(U = A_0(R, S, T)\) for some \(R, S, T\) with \(SRT \neq TRS, |R| = 1, g = qp^nq\) for some \(n \geq 0\) by Lemma 4.3. So \(V = qp^nq(A_0(R, S, T)) = B_n(R, S, T)\). \(\square\)

**Theorem 5.4.** If \(U = g(A_k(a, b, c))\) with \(k \geq 0, g \in \Gamma, \) letters \(a, b\) and \(c,\) then:

(i) \(\{U_2, U_3\} = \{bac, cab\},\)
(ii) \(U_1, U_3\) have odd lengths,
(iii) \(U_1 = (U_1)^*, U_3 = (U_3)^*\).

**Proof.** Call \(U\) good if (i), (ii), (iii) are true for \(U\). It suffices to prove 3 statements:

(1) If \(U = A_k(a, b, c)\) then \(U\) is good.
(2) If \(U\) is good then so is \(g(U)\).
(3) If \(U\) is good then so is \(p(U)\).

Statements (1), (2) are easily verified. As for (3), assume \(U\) is good. Then \(U_1U_2U_3 = U_3U_4U_1, (U_2)^* = (U_4), (U_1)^* = U_1, (U_3)^* = (U_3)\). Let \(V = \langle U_1U_2U_3, U_2, U_3, U_4 \rangle = p(U)\). Properties (i), (ii) are easily verified for \(V\). To check (iii) for \(V, (V_2)^* = (U_3)^* = U_3 = V_3\) and \((V_1)^* = (U_1U_2U_3)^* = (U_3)^*(U_2)^*(U_1)^* = U_3U_4U_1 = U_1U_2U_3 = V_1\). \(\square\)
Theorem 5.5. If $(A,B,C,D) \in \Sigma$ then $ABCD$ or $CDAB$ is periodic. If $|A| < |C|$ then $ABCD = XY(XZXY)^{k+2} = \sigma(B_k)$ for some $X$, $Y$, $Z$, $YXZ \neq ZXY$, $|X| = 1$, $k = 0$ or 1. By symmetry, if $|C| < |A|$ then $CDAB$ equals such a product.

Proof. Assume $|A| < |C|$. $ABC = CDA$ implies $C = FA$, $F \neq 1$. So $ABFA = FADA$, $ABF = FAD$. Rewrite this: $EF = FG$, $E = AB$, $G = AD$. Apply Lemma 4.5 to $EF = FG$. Get $F = P(QP)^n$, $AB = PQ$, $AD = QP$, $Q \neq 1$, $n \geq 0$, $ABCD = P(QP)^{n+2}$. $P = 1$ cannot occur since then $AB = Q = AD$ and $B = D$, impossible. Thus $P \neq 1$, $Q \neq 1$, $AB = PQ$, $AD = QP$ imply $A$, $P$ and $Q$ all start with the same word $X$ of length 1. Therefore there exist $Y$, $Z$ with $P = XY$, $Q = XZ$. It follows that $ABCD = P(QP)^{n+2} = \sigma(B_n(X,Y,Z))$. By Theorem 5.2, $ABCD$ is not a proper power. Therefore $XYXZ = PQ \neq QP = XZXY$ implies $YXZ \neq ZXY$.

For $n = 0$ or 1, use $k = n$.

For $n > 1$, apply Lemma 4.4 to $\sigma(B_n(X,Y,Z))$ and use $k = 0$. □


Define solutions: $Q_k = (qp)^{2k}(S)$ with $S = \langle a, ba, aba, ab \rangle = A_0(a,b,1)$, letters $a \neq b$. Using a simplified version of a function $G$ found in [1] we have:

$$Q_k = \langle G(2k+2), ba, G(2k+3), ab \rangle, \ k \geq 0,$$

where $G(2) = a$, $G(3) = aba$, and $G(n) = G(n-1)Z(n-1)G(n-2)$, $n \geq 4$; $Z(n) = ba$ ($ab$) if $n$ even (odd).

For example, $Q_1$, $Q_2$, $Q_3$ are computed from $G(4), \ldots, G(9)$ where:

$$G(4), \ldots, G(7) = XX, XYX, X(YX)^2, (XY)^2(YX)^2,$$

$$G(8) = (XY)^2Y(YX)^2(YX)^2,$$

$$G(9) = (XY)^2Y(XY)^2YXYY(XY)^2(YX)^2,$$

using $X = aba$, $Y = ababa$.

References


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