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A closed Riemann surface X which can be realised as a p -sheeted covering of the Riemann sphere is called p -gonal, and such a covering is called a p -gonal morphism. A p -gonal Riemann surface is called real p -gonal if there is an anticonformal involution (symmetry) σ of X commuting with the p -gonal morphism. If the p -gonal morphism is a cyclic regular covering the Riemann surface is called real cyclic p -gonal, otherwise it is called real generic p -gonal. The species of the symmetry σ is the number of connected components of the fixed point set $\text{Fix}(\sigma)$ and the orientability of the Klein surface $X/\langle\sigma\rangle$. In this paper we find the species for the possible symmetries of real cyclic p -gonal Riemann surfaces by means of Fuchsian and NEC groups.

1. Introduction.

A closed Riemann surface X which can be realised as a p -sheeted covering of the Riemann sphere is called p -gonal, and such a covering is called a p -gonal morphism. The p -gonal Riemann surfaces have been extensively studied, see [1], [2], [6], [8], [9], [12] and [13]. A p -gonal Riemann surface is called real p -gonal if there is an anticonformal involution (symmetry) σ of X commuting with the p -gonal morphism.

Let X_g be a real p -gonal Riemann surface of genus $g \geq 2$. A symmetry σ of X_g is an anticonformal involution of X_g . The topological type of a symmetry is determined by the number of connected components, called ovals, of the fixed-point set $\text{Fix}(\sigma)$ and the orientability of the Klein surface $X/\langle\sigma\rangle$. We say that σ has species $\Sigma_\sigma = +k$ if $\text{Fix}(\sigma)$ consists of k ovals and $X/\langle\sigma\rangle$ is orientable, and $\Sigma_\sigma = -k$ if $\text{Fix}(\sigma)$ consists of k ovals and $X/\langle\sigma\rangle$ is non-orientable. The set $\text{Fix}(\sigma)$ corresponds to the real part of a complex algebraic curve representing X , which admits an equation with real coefficients.

If the p -gonal morphism is a cyclic regular covering, then the Riemann surface is called real cyclic p -gonal. When $p = 2$ the surface X_g is called hyperelliptic. A Riemann surface represented by an algebraic curve given

by an equation of the form

$$(1.1) \quad y^p = \prod (x - a_i) \prod (x - b_j)^2 \cdots \prod (x - m_j)^{p-1}$$

where the coefficients of the polynomial $\prod (x - a_i) \cdots \prod (x - m_j)^{p-1}$ are real is a real cyclic p -gonal Riemann surface. The complex conjugation induces a symmetry on the above curve. A natural problem is to study and classify all possible symmetries of such a Riemann surface up to conjugacy, as they will produce non-isomorphic real models of the complex algebraic curve.

In Section 2 we characterise real cyclic p -gonal Riemann surfaces, where p is an odd prime, in terms of signatures of Fuchsian and NEC groups. In Section 3 we determine all possible symmetries of a real cyclic p -gonal Riemann surface represented by an algebraic curve with equation (1.1).

2. Signatures of real cyclic p -gonal Riemann surfaces.

Let X_g be a compact Riemann surface of genus $g \geq 2$. The surface X_g can be represented as a quotient $X_g = \mathcal{H}/\Gamma$ of the upper half plane \mathcal{H} under the action of a surface Fuchsian group Γ , that is, a cocompact orientation-preserving subgroup of the group $\mathcal{G} = \text{Aut}(\mathcal{H})$ of conformal and anticonformal automorphisms of \mathcal{H} without elliptic elements. A discrete, cocompact subgroup Γ of $\text{Aut}(\mathcal{H})$ is called an NEC (*non-euclidean crystallographic group*). The subgroup of Γ consisting of the orientation-preserving elements is called the *canonical Fuchsian subgroup* of Γ , it is denoted by Γ^+ . The algebraic structure of an NEC group and the geometric structure of its quotient orbifold are given by the signature of Γ :

$$(2.1) \quad s(\Gamma) = (h, \pm, [m_1, \dots, m_r], \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\}).$$

The orbit space \mathcal{H}/Γ is an orbifold with underlying surface of genus h , having r cone points and k boundary components, each with $s_j \geq 0$ corner points. The signs "+" and "-" correspond to orientable and non-orientable orbifolds respectively. The integers m_i are called the proper periods of Γ and they are the orders of the cone points of \mathcal{H}/Γ . The brackets $(n_{i1}, \dots, n_{is_i})$ are the period cycles of Γ and the integers n_{ij} are the link periods of Γ and the orders of the corner points of \mathcal{H}/Γ . The group Γ is called the *fundamental group* of the orbifold \mathcal{H}/Γ .

A group Γ with signature (2.1) has a *canonical presentation* with generators:

$$(2.2) \quad x_1, \dots, x_r, e_1, \dots, e_k, c_{ij}, 1 \leq i \leq k, 1 \leq j \leq s_i + 1, \text{ and} \\ a_1, b_1, \dots, a_h, b_h$$

if \mathcal{H}/Γ is orientable, or

$$d_1, \dots, d_h$$

otherwise, and relators:

$$(2.3) \quad x_i^{m_i}, \quad i = 1, \dots, r,$$

$$c_{ij}^2, (c_{ij-1}c_{ij})^{n_{ij}}, c_{i0}e_i^{-1}c_{is_i}e_i, \quad i = 1, \dots, k, j = 2, \dots, s_i + 1$$

and $x_1 \cdots x_r e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_h b_h a_h^{-1} b_h^{-1}$ or $x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_h^2$ according to whether \mathcal{H}/Γ is orientable or not. This last relation is called the long relation.

The hyperbolic area of the orbifold \mathcal{H}/Γ coincides with the hyperbolic area of an arbitrary fundamental region of Γ and equals:

$$(2.4) \quad \mu(\Gamma) = 2\pi \left(\varepsilon h - 2 + k + \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right) + \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}} \right) \right),$$

where $\varepsilon = 2$ if there is a $''+$ sign and $\varepsilon = 1$ otherwise. If Γ' is a subgroup of Γ of finite index then Γ' is an NEC group and the following Riemann-Hurwitz formula holds:

$$(2.5) \quad [\Gamma : \Gamma'] = \mu(\Gamma')/\mu(\Gamma).$$

An NEC group Γ without elliptic elements is called a *surface group* and it has signature $(h; \pm; [-], \{(-), \cdot, \cdot, (-)\})$. In such a case \mathcal{H}/Γ is a *Klein surface*, i.e., a surface with a dianalytic structure of topological genus h , orientable or not according to the sign $''+$ or $''-$, and having k boundary components. Conversely, a Klein surface whose complex double has genus greater than one can be expressed as \mathcal{H}/Γ for some NEC surface group Γ . Furthermore, given a Riemann (resp. Klein) surface represented as the orbit space $X = \mathcal{H}/\Gamma$, with Γ a surface group, a finite group G is a group of automorphisms of X if and only if there exists an NEC group Δ and an epimorphism $\theta : \Delta \rightarrow G$ with $\ker(\theta) = \Gamma$ (see [5]). The NEC group Δ is the lifting of G to the universal covering $\pi : \mathcal{H} \rightarrow \mathcal{H}/\Gamma$ and is called *the universal covering transformation group* of (X, G) .

Definition 1. For a prime p , a real cyclic p -gonal Riemann surface is a triple (X, f, σ) where σ is a symmetry of X , f is a cyclic p -gonal morphism and $f \circ \sigma = c \circ f$, and c is the complex conjugation.

Notice that by Lemma 2.1 in [1] the condition $f \circ \sigma = c \circ f$ is automatically satisfied for genera $g \geq (p - 1)2 + 1$, since the p -gonal morphism is unique. From now on, the genera will satisfy the condition above. As a consequence of the assumption $g \geq (p - 1)2 + 1$ for the genera of the p -gonal surface X_g we have that the group C_p generated by the p -gonal morphism is a normal subgroup of $\text{Aut}^+(X_g)$. Notice the the classification method fails for surfaces with genera in the range $2 \leq g \leq (p - 1)2$. For instance, there are two 7-gonal surfaces of genus 3. One of them, X_3 , is the Klein's quartic with $\text{Aut}^+(X_3) \simeq PSL_2(7)$, in this case C_7 is non-normal in $PSL_2(7)$.

We give now a characterisation of real cyclic p -gonal Riemann surfaces represented by real equations via NEC groups.

Theorem 1 ([7]). *Let X be a Riemann surface with genus g . The surface X admits a symmetry σ and a meromorphic function f such that (X, f, σ) is a real cyclic p -gonal Riemann surface represented by a curve with real equation $y^p = \prod(x - a_i) \cdots \prod(x - m_j)^{p-1}$ if and only if there are an NEC group Δ with signature $(0, +, [p, \dots, p], \{(p, \dots, p)\})$ and an epimorphism $\theta : \Delta \rightarrow D_p$ such that X is conformally equivalent to $\mathcal{H}/\text{Ker } \theta$ and $\text{Ker } \theta$ is an NEC Fuchsian surface group.*

Let (X, f, σ) be a real cyclic p -gonal Riemann surface uniformised by a Fuchsian surface group Γ . Consider the automorphism $\varphi : X \rightarrow X$ such that $X/\langle\varphi\rangle$ is the Riemann sphere and φ is a deck-transformation of the covering f . Notice that the group Δ is the universal covering transformation group of (X, φ, σ) , that $D_p = \langle\varphi, \sigma\rangle$ and that the canonical Fuchsian subgroup Δ^+ is the universal covering transformation group of (X, φ) . Thus $X/\langle\varphi\rangle$ is a sphere with conic points of order p . Let $\bar{\sigma}$ be the symmetry in the Riemann sphere $X/\langle\varphi\rangle$ induced by σ . Since the triple (X, φ, σ) is represented by the equation $y^p = \prod(x - a_i) \cdots \prod(x - m_j)^{p-1}$, the symmetry σ is given by the map $\sigma : X \rightarrow X$ defined by $\sigma : (x, y) \rightarrow (\bar{x}, \bar{y})$. The set of real solutions of 1.1 is the set $\text{Fix}(\sigma)$. Thus $\bar{\sigma}$ is conjugated to the complex conjugation. Then $X/\langle\varphi, \sigma\rangle = X/\langle\varphi\rangle/\langle\bar{\sigma}\rangle$ is a disc with corner(s) and conic points of order p .

With the above notation:

Theorem 2. *Let X be a real cyclic p -gonal Riemann surface such that $\langle\varphi, \sigma\rangle$ is isomorphic to D_p . If G is the group of conformal and anticonformal automorphisms of X , then X/G is uniformised by an NEC group Λ such that there is a surface Fuchsian subgroup $\Gamma \leq \Lambda$ uniformising X and the group Λ has one of the following signatures:*

- (I) $(0, +, [\overbrace{p, \dots, p}^r, qp^\epsilon], \{(\overbrace{p, \dots, p}^s)\})$, where $\epsilon = 0$ or 1 and $2r + s = \frac{2g+2(1-\epsilon)(p-1)}{q(p-1)}$. $G/\langle\varphi\rangle = C_q \times C_2$.
- (II) $(0, +, [\overbrace{p, \dots, p}^r], \{(qp^{\epsilon_1}, \overbrace{p, \dots, p}^{s_1}, qp^{\epsilon_2}, \overbrace{p, \dots, p}^{s_2})\})$, where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 = \frac{2g+(2-\epsilon_1-\epsilon_2)(p-1)}{q(p-1)}$. $G/\langle\varphi\rangle = D_q$.
- (III) $(0, +, [\overbrace{p, \dots, p}^r, 2p^{\epsilon_1}], \{(qp^{\epsilon_2}, \overbrace{p, \dots, p}^s)\})$, where $\epsilon_i = 0$ or 1 and $2r + s = \frac{g+(1-q\epsilon_1-\epsilon_2)(p-1)}{q(p-1)}$. $G/\langle\varphi\rangle = D_q \times C_2$.
- (IV) $(0, +, [\overbrace{p, \dots, p}^r], \{(2p^{\epsilon_1}, \overbrace{p, \dots, p}^{s_1}, 2p^{\epsilon_2}, \overbrace{p, \dots, p}^{s_2}, qp^{\epsilon_3}, \overbrace{p, \dots, p}^{s_3})\})$, where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 + s_3 = \frac{2g+(2-q\epsilon_1-q\epsilon_2-2\epsilon_3)(p-1)}{2q(p-1)}$. $G/\langle\varphi\rangle = D_q \times C_2$.

- (V) $(0, +, [\overbrace{p, \dots, p}^r], \{(2p^{\epsilon_1}, \overbrace{p, \dots, p}^{s_1}, 3p^{\epsilon_2}, \overbrace{p, \dots, p}^{s_2}, 3p^{\epsilon_3}, \overbrace{p, \dots, p}^{s_3})\})$,
 where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 + s_3 = \frac{g+(1-3\epsilon_1-2\epsilon_2-2\epsilon_3)(p-1)}{6(p-1)}$.
 $G/\langle\varphi\rangle = S_4$.
- (VI) $(0, +, [\overbrace{p, \dots, p}^r, 3p^{\epsilon_1}], \{(2p^{\epsilon_2}, \overbrace{p, \dots, p}^s)\})$, where $\epsilon_i = 0$ or 1 and $2r + s = \frac{g+(1-4\epsilon_1-3\epsilon_2)(p-1)}{6(p-1)}$. $G/\langle\varphi\rangle = A_4 \times C_2$.
- (VII) $(0, +, [\overbrace{p, \dots, p}^r], \{(2p^{\epsilon_1}, \overbrace{p, \dots, p}^{s_1}, 3p^{\epsilon_2}, \overbrace{p, \dots, p}^{s_2}, 4p^{\epsilon_3}, \overbrace{p, \dots, p}^{s_3})\})$,
 where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 + s_3 = \frac{g+(1-6\epsilon_1-4\epsilon_2-3\epsilon_3)(p-1)}{12(p-1)}$.
 $G/\langle\varphi\rangle = S_4 \times C_2$.
- (VIII) $(0, +, [\overbrace{p, \dots, p}^r], \{(2p^{\epsilon_1}, \overbrace{p, \dots, p}^{s_1}, 3p^{\epsilon_2}, \overbrace{p, \dots, p}^{s_2}, 5p^{\epsilon_3}, \overbrace{p, \dots, p}^{s_3})\})$,
 where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 + s_3 = \frac{g+(1-15\epsilon_1-10\epsilon_2-6\epsilon_3)(p-1)}{30(p-1)}$.
 $G/\langle\varphi\rangle = A_5 \times C_2$.

Notice that in cases (VII) and (VIII) the factor group C_2 of $G/\langle\varphi\rangle$ is generated by the antipodal map.

Proof. Consider the chain of coverings $X = \mathcal{H}/\Gamma \rightarrow X/\langle\varphi\rangle = \mathcal{H}/\Delta^+ \rightarrow X/G = \mathcal{H}/\Lambda$ with uniformising groups $\Gamma \leq \Delta^+ \leq \Lambda$, where $s(\Delta^+) =$

$(0, +, [\overbrace{p, \dots, p}^p], \{\})$ and $s(\Lambda) = (h, \pm, [m_1, \dots, m_r], \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\})$. Furthermore by Lemma 2.1 in [1] the group $\langle\varphi\rangle$ is a normal subgroup of G . By Theorem 1 the factor group $\overline{G} = G/\langle\varphi\rangle$ is a finite group of conformal and anticonformal automorphisms of the Riemann sphere. (See also [12].)

In other words, we have an epimorphism $\theta : \Lambda \rightarrow \overline{G}$ with $\text{Ker } \theta = \Delta^+$. This yields the signature of the group Λ in terms of the signature of Δ^+ and the group \overline{G} . Let p_i and q_{ij} be the orders in \overline{G} of $\theta(x_i)$ and $\theta(c_{ij-1}c_{ij})$ respectively, where x_i, c_{ij} are generators in the canonical presentation of Λ associated to the signature (2.1). By [3] and [5] each proper period m_i induces $\frac{|\overline{G}|}{p_i}$ proper periods $\frac{m_i}{p_i}$ in $s(\Delta^+)$. Each link-period n_{ij} induces $\frac{|\overline{G}|}{2q_{ij}}$ proper periods $\frac{n_{ij}}{q_{ij}}$ in $s(\Delta^+)$. But $\frac{m_i}{p_i} = p$ or $\frac{m_i}{p_i} = 1$ and $\frac{n_{ij}}{q_{ij}} = p$ or $\frac{n_{ij}}{q_{ij}} = 1$, since Δ^+ is the group of the Riemann sphere with conic points of prime order p . We denote $K_1 = \{i \mid \frac{m_i}{p_i} = 1\}$, $K_p = \{i \mid \frac{m_i}{p_i} = p\}$, $H_1 = \{(i, j) \mid \frac{n_{ij}}{q_{ij}} = 1\}$ and $H_p = \{(i, j) \mid \frac{n_{ij}}{q_{ij}} = p\}$. Thus $\rho = \sum_{i \in K_p} \frac{|\overline{G}|}{p_i} + \sum_{(i,j) \in H_p} \frac{|\overline{G}|}{2q_{ij}}$

Using the Riemann-Hurwitz formula $|\overline{G}| = \mu(\Delta^+)/\mu(\Lambda)$ we obtain

$$\begin{aligned}
 (2.6) \quad & -2 + \left(\sum_{i \in K_p} \frac{|\overline{G}|}{p_i} + \sum_{(i,j) \in H_p} \frac{|\overline{G}|}{2q_{ij}} \right) \frac{(p-1)}{p} \\
 & = |\overline{G}|(\alpha h - 2 + k) + \sum_{i \in K_p} |\overline{G}| \left(1 - \frac{1}{pp_i} \right) + \sum_{i \in K_1} |\overline{G}| \left(1 - \frac{1}{p_i} \right) \\
 & \quad + \sum_{(i,j) \in H_p} \frac{|\overline{G}|}{2} \left(1 - \frac{1}{pq_{ij}} \right) + \sum_{(i,j) \in H_1} \frac{|\overline{G}|}{2} \left(1 - \frac{1}{q_{ij}} \right),
 \end{aligned}$$

therefore $h = 0, k = 1, s(\Lambda) = (0, +, [pp_1, \dots, pp_r], \{(pq_1, \dots, pq_s)\})$, where $p_i, q_j \in \{1, p\}$. By setting K_1, K_p, H_1 and H_p in Equation (2.6) we obtain that p_i, q_j satisfy the equation

$$(2.7) \quad |\overline{G}| - 2 = \sum_1^r |\overline{G}| \left(1 - \frac{1}{p_i} \right) + \sum_1^s \frac{|\overline{G}|}{2} \left(1 - \frac{1}{q_j} \right).$$

To find $s(\Lambda)$ it is enough to find the nontrivial solutions of (2.7). We divide the study of (2.7) in eight cases according to the factor group \overline{G} in the epimorphism $\theta : \Lambda \rightarrow \overline{G}$ with $\text{Ker}(\theta) = \Delta^+$:

(I) $\overline{G} = C_q \times C_2$, where $C_2 = \langle \overline{\sigma} \rangle$. The solution of Equation (2.7) is $p_1 = q$. Applying Riemann-Hurwitz formula to the covering $X \rightarrow X/G$ we obtain

the signature $(0, +, [\overbrace{p, \dots, p}^r, qp^\epsilon], \{(\overbrace{p, \dots, p}^s)\})$, where $\epsilon = 0$ or 1 and $2r + s = \frac{2g+2(1-\epsilon)(p-1)}{q(p-1)}$.

(II) $\overline{G} = D_q$. The solution of (2.7) is $q_{j_1} = q_{j_2} = q$. Therefore $s(\Lambda) = (0, +, [\overbrace{p, \dots, p}^r], \{(\overbrace{qp^{\epsilon_1}, p, \dots, p}^{s_1}, \overbrace{qp^{\epsilon_2}, p, \dots, p}^{s_2})\})$, where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 = \frac{2g+(2-\epsilon_1-\epsilon_2)(p-1)}{q(p-1)}$.

(III) $\overline{G} = D_q \times C_2$. The solution of (2.7) is $p_1 = 2$, and $q_1 = q$. Thus $s(\Lambda)$ becomes $(0, +, [\overbrace{p, \dots, p}^r, 2p^{\epsilon_1}], \{(\overbrace{qp^{\epsilon_2}, p, \dots, p}^s)\})$, where $\epsilon_i = 0$ or 1 and $2r + s = \frac{g+(1-q\epsilon_1-\epsilon_2)(p-1)}{q(p-1)}$.

(IV) $\overline{G} = D_q \times C_2$. The solution in this case is $q_{j_1} = q_{j_2} = 2$ and $q_{j_3} = q$. This yields $s(\Lambda) = (0, +, [\overbrace{p, \dots, p}^r], \{(\overbrace{2p^{\epsilon_1}, p, \dots, p}^{s_1}, \overbrace{2p^{\epsilon_2}, p, \dots, p}^{s_2}, \overbrace{qp^{\epsilon_3}, p, \dots, p}^{s_3})\})$, where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 + s_3 = \frac{2g+(2-q\epsilon_1-q\epsilon_2-2\epsilon_3)(p-1)}{2q(p-1)}$.

(V) $\overline{G} = S_4$. The solution of (2.7) is $q_{j_1} = 2, q_{j_2} = q_{j_3} = 3$. Then $s(\Lambda) = (0, +, [\overbrace{p, \dots, p}^r], \{(2p^{\epsilon_1}, \overbrace{p, \dots, p}^{s_1}, 3p^{\epsilon_2}, \overbrace{p, \dots, p}^{s_2}, 3p^{\epsilon_3}, \overbrace{p, \dots, p}^{s_3})\})$, where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 + s_3 = \frac{g+(1-3\epsilon_1-2\epsilon_2-2\epsilon_3)(p-1)}{6(p-1)}$.

(VI) $\overline{G} = A_4 \times C_2$. The solution of (2.7) is $p_1 = 3$, and $q_1 = 2$. Thus $s(\Lambda)$ becomes $(0, +, [\overbrace{p, \dots, p}^r, 3p^{\epsilon_1}], \{(2p^{\epsilon_2}, \overbrace{p, \dots, p}^s)\})$, where $\epsilon_i = 0$ or 1 and $2r + s = \frac{g+(1-4\epsilon_1-3\epsilon_2)(p-1)}{6(p-1)}$.

(VII) $\overline{G} = S_4 \times C_2$. The solution in this case is $q_{j_1} = 2, q_{j_2} = 3$ and $q_{j_3} = 4$. This yields $s(\Lambda) = (0, +, [\overbrace{p, \dots, p}^r], \{(2p^{\epsilon_1}, \overbrace{p, \dots, p}^{s_1}, 3p^{\epsilon_2}, \overbrace{p, \dots, p}^{s_2}, 4p^{\epsilon_3}, \overbrace{p, \dots, p}^{s_3})\})$, where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 + s_3 = \frac{g+(1-6\epsilon_1-4\epsilon_2-3\epsilon_3)(p-1)}{12(p-1)}$.

(VIII) $\overline{G} = A_5 \times C_2$. The solution now is $q_{j_1} = 2, q_{j_2} = 3$ and $q_{j_3} = 5$. This yields $s(\Lambda) = (0, +, [\overbrace{p, \dots, p}^r], \{(2p^{\epsilon_1}, \overbrace{p, \dots, p}^{s_1}, 3p^{\epsilon_2}, \overbrace{p, \dots, p}^{s_2}, 5p^{\epsilon_3}, \overbrace{p, \dots, p}^{s_3})\})$, where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 + s_3 = \frac{g+(1-15\epsilon_1-10\epsilon_2-6\epsilon_3)(p-1)}{30(p-1)}$. This finishes the proof.

3. Species of symmetries of real cyclic p -gonal Riemann surfaces.

Let X be a real cyclic p -gonal Riemann surface X with real equation. In the next theorem we study the topological types of the possible real forms of X .

Theorem 3. *Let X be a real cyclic p -gonal Riemann surface with p -gonal automorphism φ admitting a symmetry σ with fixed points and such that $\langle \sigma, \varphi \rangle = D_p, p$ prime. If τ is another symmetry of X , then possible species of τ are (and all cases occur):*

- (1) $s(\Lambda)$ as in (I).
 - a) $q \equiv 1 \pmod{2}$. $\Sigma_\sigma = \Sigma_\tau$. If $r + \epsilon > 0$, then $\Sigma_\sigma = -1$. If $r + \epsilon = 0$, then $\Sigma_\sigma \in \{-1, +1\}$.
 - b) $q \equiv 0 \pmod{2}$. $\Sigma_\tau = \Sigma_\sigma$ as in case (1a) or $\Sigma_\tau = 0$.
- (2) $s(\Lambda)$ as in (II).
 - a) $q \equiv 1 \pmod{2}$. $\Sigma_\sigma = \Sigma_\tau$ and $\Sigma_\sigma = -1$.
 - b) $q \equiv 0 \pmod{2}, q \neq 2$. $\Sigma_\sigma = -1$ and $\Sigma_\tau = -1$ or $\Sigma_\tau = +p, +1$.
- (3) $s(\Lambda)$ as in (III). $\Sigma_\tau = 0$ or $\Sigma_\sigma = \Sigma_\tau$, besides $\Sigma_\sigma = -1$.
- (4) $s(\Lambda)$ as in (IV).
 - a) $q \equiv 1 \pmod{2}$. $\{\Sigma_\sigma, \Sigma_\tau\} \subset \{\Sigma_1, \Sigma_2\}$, where $\Sigma_1 \in \{-1, +1, +p\}$. $\Sigma_2 \in \{-1, +1, +p\}$. In both cases $\Sigma_\sigma \neq +p$ and $\Sigma_\sigma \neq +1$ if σ is of the first type.

- b) $q \equiv 0 \pmod{2}$, $q \neq 2$. $\{\Sigma_\sigma, \Sigma_\tau\} \subset \{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}$, where $\Sigma_4 = 0$, and $\Sigma_2 \in \{-1, +1, +p\}$ and $\Sigma_1, \Sigma_3 \in \{-1, +1, +p\}$. In all cases $\Sigma_\sigma \neq +p$ and $\Sigma_\sigma = +1$ if σ is of the second type.
- $q = 2$. $\{\Sigma_\sigma, \Sigma_\tau\} \subset \{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}$, where $\Sigma_4 = 0$, and $\Sigma_i \in \{-1, +1, +p\}$ for $1 \leq i \leq 3$. In all cases $\Sigma_\sigma \neq +p$.
- (5) $s(\Lambda)$ as in (V). $\Sigma_\tau = 0$ or $\Sigma_\sigma = \Sigma_\tau$, with $\Sigma_\sigma = -1$.
- (6) $s(\Lambda)$ as in (VI). $\Sigma_\tau = 0$ or $\Sigma_\sigma = \Sigma_\tau$, with $\Sigma_\sigma = -1$.
- (7) $s(\Lambda)$ as in (VII). $\{\Sigma_\sigma, \Sigma_\tau\} \subset \{\Sigma_1, \Sigma_2, 0\}$. $\Sigma_1 \in \{-1, +1, +p\}$ and $\Sigma_2 \in \{-1, +1, +p\}$, but $\Sigma_\sigma = -1$.
- (8) $s(\Lambda)$ as in (VIII). $\Sigma_\tau = 0$ or $\Sigma_\sigma = \Sigma_\tau$, with $\Sigma_\sigma = -1$.

Proof. Consider the following chain of epimorphisms: $\theta : \Lambda \xrightarrow{\bar{\theta}} G \xrightarrow{\phi} \bar{G}$. The symmetries of X are symmetries τ in G which are lifts of symmetries $\bar{\tau}$ in \bar{G} . The species of τ is given by the conjugacy classes of reflections in $\bar{\Lambda} = \bar{\theta}^{-1}\langle\varphi, \tau\rangle = \bar{\theta}^{-1}\phi^{-1}\langle\bar{\tau}\rangle = \theta^{-1}\langle\bar{\tau}\rangle$ and the orientability of $\mathcal{H}/\bar{\Lambda}$. Notice that $\langle\varphi, \tau\rangle$ is either a cyclic group C_{2p} or a dihedral group D_p of order $2p$.

As in Theorem 2 we divide the proof in eight cases corresponding to the different types of groups \bar{G} of conformal and anticonformal automorphisms of the Riemann sphere. The signature of Λ in each case is given by the corresponding case in Theorem 2.

(1a) $\bar{G} = C_q \times C_2$, $q \equiv 1 \pmod{2}$. In this case \bar{G} contains just one conjugacy class of symmetries and so does G : The one represented by σ . Moreover $D_p = \langle\varphi, \sigma\rangle$ is a normal subgroup of index q in G . By [5] the signature of $\bar{\theta}^{-1}(\langle\varphi, \sigma\rangle)$ is $(0, +, [\overbrace{p, \dots, p}^{rq+\epsilon}], \{\overbrace{(p, \dots, p)}^{qs}\})$. By [14] (see also [4]) $\Sigma_\sigma = \pm 1$ as $D_p = \langle\varphi, \sigma\rangle$. The sign $+$ can only occur if $\bar{\theta}^{-1}(\langle\varphi, \sigma\rangle)$ has no proper periods, i.e., $r + \epsilon = 0$. If $s = 0$, then $r + \epsilon > 0$, the possible species is -1 .

(1b) $\bar{G} = C_q \times C_2 = \langle\hat{\rho}, \bar{\sigma} \mid \hat{\rho}^q, \bar{\sigma}^2, \hat{\rho}^{-1}\bar{\sigma}\hat{\rho}\bar{\sigma}\rangle$, with $q \equiv 0 \pmod{2}$. In this case \bar{G} contains two conjugacy classes of symmetries, with representatives namely $\bar{\sigma}$ and $\hat{\rho}^{q/2}\bar{\sigma} = \bar{\tau}$, and so does G . To find the species of the symmetries we have to consider the normal subgroups $\bar{\theta}^{-1}(\langle\varphi, \sigma\rangle)$ and $\bar{\theta}^{-1}(\langle\varphi, \tau\rangle)$ of Λ with factor group C_q . By [5] they have signatures $(0, +, [\overbrace{p, \dots, p}^{rq+\epsilon}], \{\overbrace{(p, \dots, p)}^{qs}\})$ and $(0, +, [\overbrace{p, \dots, p}^{rq+\epsilon+sq/2}], \{-\})$ respectively. So species Σ_σ is as in (1a) and $\Sigma_\tau = 0$.

(2a) $\bar{G} = D_q = \langle\hat{\rho}, \bar{\sigma} \mid \hat{\rho}^q, \bar{\sigma}^2, (\hat{\rho}\bar{\sigma})^2\rangle$, with $q \equiv 1 \pmod{2}$. The group \bar{G} contains one conjugacy class of symmetries and so does G . By the epimorphism $\theta : \Lambda \rightarrow D_q$ the images of reflections in Λ leave one fixed coset in D_q , so

we get that $\bar{\Lambda}_\sigma$ has signature $(0, +, [\overbrace{p, \dots, p}^{rq+s_1\frac{q-1}{2}+s_2\frac{q-1}{2}}, \{(\overbrace{p, \dots, p}^{s_1+s_2+\epsilon_1+\epsilon_2})\}])$. Now, $s_1 + s_2 + r > 0$ since Λ is a NEC group, then $\Sigma_\sigma = -1$ by [4] and [14].

(2b) $\bar{G} = D_q = \langle \hat{\rho}, \bar{\sigma} | \hat{\rho}^q, \bar{\sigma}^2, (\hat{\rho}\bar{\sigma})^2 \rangle$, with $q \equiv 0 \pmod{2}$. The group \bar{G} (and the group G) contains two conjugacy classes of symmetries, with representatives namely $\bar{\sigma}$ and $\hat{\rho}\bar{\sigma} = \bar{\tau}$. To find Σ_σ and Σ_τ we have to study the images of reflections by an epimorphism $\theta : \Lambda \rightarrow D_q$. Each of these images leaves either 2 $\bar{\sigma}$ -cosets fixed and none from $\bar{\tau}$ or the other way round.

Thus the signatures of $\bar{\Lambda}_\sigma$ and $\bar{\Lambda}_\tau$ are $(0, +, [\overbrace{p, \dots, p}^{rq+s_1\frac{q-2}{2}+s_2\frac{q}{2}}, \{(\overbrace{p, \dots, p}^{\epsilon_1+2s_1+\epsilon_2})\}])$ and $(0, +, [\overbrace{p, \dots, p}^{rq+s_1\frac{q}{2}+s_2\frac{q-2}{2}}, \{(\overbrace{p, \dots, p}^{\epsilon_1+2s_2+\epsilon_2})\}])$. Now σ has 1 oval and does not separate because $\bar{\theta}^{-1}(\langle \varphi, \sigma \rangle)$ contains proper periods since $s_1 + s_2 + r > 0$ and $q > 2$. If $\epsilon_1 + s_2 + \epsilon_2 > 0$, then $\langle \varphi, \tau \rangle = D_p$ and as before $\Sigma_\tau = -1$. If $\epsilon_1 + s_2 + \epsilon_2 = 0$ the signature of $\bar{\theta}^{-1}(\langle \varphi, \tau \rangle)$ becomes $(0, +, [\overbrace{p, \dots, p}^{(2r+s_1)\frac{q}{2}}, \{(-)\}])$. Thus $\Sigma_\tau = -1$, if $\langle \varphi, \tau \rangle = D_p$, and $\Sigma_\tau = +p, +1$ if $\langle \varphi, \tau \rangle = C_{2p}$.

(3) $\bar{G} = D_q \rtimes C_2 = \langle \bar{\rho}, \bar{\sigma}_1, \bar{\sigma}_2 | \bar{\rho}^2, \bar{\sigma}_1^2, \bar{\sigma}_2^2, (\bar{\sigma}_1\bar{\sigma}_2)^q, \bar{\rho}\bar{\sigma}_1\bar{\rho}\bar{\sigma}_2 \rangle$. The group \bar{G} (and G) contains two conjugacy classes of symmetries, with representatives namely $\bar{\sigma} = \bar{\sigma}_1$ and $\bar{\rho}\bar{\sigma} = \bar{\tau}$. The images of reflections in Λ are all mapped to conjugate reflections in \bar{G} . They are conjugate to $\bar{\sigma}$ as we know that σ has fixed points. Thus $\Sigma_\tau = 0$. On the other hand $\bar{\Lambda}_\sigma$ has always proper periods. Therefore $\Sigma_\sigma = -1$.

(4) $\bar{G} = D_q \times C_2 = \langle \bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3 | \bar{\sigma}_i^2, (\bar{\sigma}_1\bar{\sigma}_2)2, (\bar{\sigma}_2\bar{\sigma}_3)2, (\bar{\sigma}_3\bar{\sigma}_1)^q \rangle$, with $\bar{\sigma}_2$ central in \bar{G} . First of all the group $\langle \varphi, \sigma_2 \rangle$ is a normal subgroup of G with factor group $D_q = \langle \bar{\sigma}_1, \bar{\sigma}_3 \rangle$.

(4a) $q \equiv 1 \pmod{2}$. In this case G has two conjugacy classes of reflections with representatives with images $\bar{\sigma}_1$ and $\bar{\sigma}_2$. Then there are two possible species for a symmetry of X : $\Sigma_{\sigma_1}, \Sigma_{\sigma_2}$. The possible signatures for $\bar{\theta}^{-1}(\langle \varphi, \sigma_i \rangle)$ are given by the epimorphism $\Lambda \xrightarrow{\hat{\theta}} D_q$. By this epimorphism the images of c_0 and c_{s_1+i} , for $i \geq 2$, are conjugated to $\bar{\sigma}_1$ and the image of c_1, \dots, c_{s_1+1} is the identity (representing the central symmetry). Therefore c_0, c_1, c_{s_1+1} and c_{s_1+2} fixes q $\langle \bar{\sigma}_2 \rangle$ -cosets and one $\langle \bar{\sigma}_1 \rangle$ -coset each, each c_1, \dots, c_{s_1+1} fixes $2q$ $\langle \bar{\sigma}_2 \rangle$ -cosets (and none $\langle \bar{\sigma}_1 \rangle$ -coset), and finally each c_{s_1+i} , $i \geq 2$ fixes two $\langle \bar{\sigma}_1 \rangle$ -cosets (and none $\langle \bar{\sigma}_1 \rangle$ -coset) in \bar{G} . Thus $\bar{\Lambda}_1$ and

$\overline{\Lambda}_2$ have signatures

$$\begin{aligned}
 & (0, +, [\underbrace{p, \dots, p}_{2rq+qs_1+(q-1)(s_2+s_3)+\frac{q-1}{2}(\epsilon_1+\epsilon_2)}], \{ (\underbrace{p, \dots, p}_{\epsilon_2+2s_2+2\epsilon_3+\epsilon_1+2s_3}) \}) \quad \text{and} \\
 & (0, +, [\underbrace{p, \dots, p}_{2rq+qs_2+\epsilon_3+qs_3}], \{ (\underbrace{p, \dots, p}_{q\epsilon_1+2qs_1+q\epsilon_2}) \}), \quad \text{see [10]}.
 \end{aligned}$$

Altogether we have that Σ_1 is -1 if $s_2 + s_3 + \epsilon_1 + \epsilon_2 + \epsilon_3 > 0$, and Σ_1 is $+p, +1$ if $s_2 + s_3 + \epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ and $\langle \varphi, \sigma_1 \rangle$ is C_{2p} . On the other hand Σ_2 is -1 if $s_1 + \epsilon_1 + \epsilon_2 > 0$ and $r + s_2 + s_3 + \epsilon_3 > 0$, Σ_2 is $+1$ if $s_1 + \epsilon_1 + \epsilon_2 > 0$ and $r = s_2 = s_3 = \epsilon_3 = 0$, and finally Σ_2 is $+p, +1$ if $s_1 + \epsilon_1 + \epsilon_2 = 0$ and $\langle \varphi, \sigma_2 \rangle = C_{2p}$. In both cases $\Sigma_\sigma \neq +p$ since $\langle \varphi, \sigma \rangle = D_p$ and if σ is conjugate to σ_1 then again $\Sigma_\sigma \neq +1$. No further restrictions exist.

(4b) $q \equiv 0 \pmod{2}$. In this case G has four conjugacy classes of reflections with representatives with homomorphic images $\overline{\sigma}_1, \overline{\sigma}_2, \overline{\sigma}_3$ and $(\overline{\sigma}_3\overline{\sigma})^{q/2}\overline{\sigma}_2 = \overline{\sigma}_4$. Then there are four possible species for a symmetry of X : $\Sigma_i, 1 \leq i \leq 4$. The species are also given by the epimorphism $\Lambda \xrightarrow{\hat{\theta}} D_q$. By this epimorphism the images of c_0 and $c_{s_1+s_2+i}$, for $i \geq 3$, are conjugated to $\overline{\sigma}_1$, the image of c_1, \dots, c_{s_1+1} is the identity (representing the central symmetry), and the images of c_{s_1+i} , for $2 \leq i \leq s_2 + 2$, are conjugate to $\overline{\sigma}_3$. First of all $\Sigma_4 = 0$ since no images of reflections by $\hat{\theta}$ are conjugate to $\overline{\sigma}_4$.

If $q = 2$, then all the 3 symmetries are central and, as in (4a) the possible species for them are $-1, +1$ and $+p$.

If $q \neq 2$ then with the same procedure as in (4a) we get the following signatures for $\overline{\Lambda}_1, \overline{\Lambda}_2$ and $\overline{\Lambda}_3$:

$$\begin{aligned}
 & (0, +, [\underbrace{p, \dots, p}_{2rq+q(s_1+s_2)+\frac{q-2}{2}s_3+\frac{q-2}{2}\epsilon_1+\frac{q}{2}\epsilon_2}], \{ (\underbrace{p, \dots, p}_{2\epsilon_3+2\epsilon_1+4s_3}) \}), \\
 & (0, +, [\underbrace{p, \dots, p}_{2rq+qs_2+\epsilon_3+qs_3}], \{ (\underbrace{p, \dots, p}_{q\epsilon_1+2qs_1+q\epsilon_2}) \}), \\
 & (0, +, [\underbrace{p, \dots, p}_{2rq+q(s_1+s_3)+\frac{q-2}{2}s_2+\frac{q-2}{2}\epsilon_2+\frac{q}{2}\epsilon_1}], \{ (\underbrace{p, \dots, p}_{2\epsilon_3+2\epsilon_2+4s_2}) \}).
 \end{aligned}$$

Both $\overline{\Lambda}_1$ and $\overline{\Lambda}_3$ must have proper periods because otherwise all parameters in the signature of Λ except ϵ_3 are 0 and then Λ is a spherical group. Therefore Σ_1 is -1 if $s_3 + \epsilon_1 + \epsilon_3 > 0$, Σ_1 is $+p, +1$ if $s_3 + \epsilon_1 + \epsilon_3 = 0$ and $\langle \varphi, \sigma_1 \rangle = C_{2p}$. Σ_2 is -1 if $s_1 + \epsilon_1 + \epsilon_2 > 0$ and $r + s_2 + s_3 + \epsilon_3 > 0$, Σ_2 is $+1$ if $s_1 + \epsilon_1 + \epsilon_2 > 0$ and $r = s_2 = s_3 = \epsilon_3 = 0$, and finally Σ_2 is $+p, +1$ if $s_1 + \epsilon_1 + \epsilon_2 = 0$ and $\langle \varphi, \sigma_2 \rangle = C_{2p}$. Finally Σ_3 is -1 if $s_2 + \epsilon_2 + \epsilon_3 > 0$, Σ_3 is $+p, +1$ if $s_2 + \epsilon_2 + \epsilon_3 = 0$ and $\langle \varphi, \sigma_3 \rangle = C_{2p}$. In all cases $\Sigma_\sigma \neq +p$ since $\langle \varphi, \sigma \rangle = D_p$. Again $\Sigma_\sigma \neq +1$ if σ is conjugate to σ_1 or σ_3 . No further restrictions exist.

(5) and (8) $\bar{G} = \langle \bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3 | \bar{\sigma}_i^2, (\bar{\sigma}_1\bar{\sigma}_2)^2, (\bar{\sigma}_2\bar{\sigma}_3)^3, (\bar{\sigma}_3\bar{\sigma}_1)^q \rangle$, where $q = 3$ in (5) and $q = 5$ in (8). \bar{G} , and thus G , contains two conjugacy classes of symmetries, with representatives namely $\bar{\sigma} = \bar{\sigma}_1$ and $\bar{\tau}$, with $\bar{\tau}$ conjugated to the antipodal map. Then $\Sigma_\sigma = \Sigma_{\sigma_1}$ and $\Sigma_\tau = 0$. As in case (2a), by [10] and [14], given the epimorphism $\bar{\theta}$, all the generating reflections of Λ induce reflections in $\bar{\theta}^{-1}(\langle \sigma_1 \rangle)$. So $\Sigma_\sigma = -1$ as they induce also proper periods.

(6) $\bar{G} = A_4 \rtimes C_2 = \langle \bar{\rho}, \bar{\sigma}_1, \bar{\sigma}_2 | \bar{\rho}^3, \bar{\sigma}_1^2, \bar{\sigma}_2^2, (\bar{\sigma}_1\bar{\sigma}_2)^2, \bar{\rho}\bar{\sigma}_1\bar{\rho}\bar{\sigma}_2 \rangle$, where C_2 is generated by the antipodal map. With the same arguments as in (3) we obtain that G has two types of symmetries with representatives σ and τ where $\Sigma_\tau = 0$ and $\Sigma_\sigma = -1$.

(7) $\bar{G} = \langle \bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3 | \bar{\sigma}_i^2, (\bar{\sigma}_1\bar{\sigma}_2)^2, (\bar{\sigma}_2\bar{\sigma}_3)^3, (\bar{\sigma}_3\bar{\sigma}_1)4 \rangle = S_4 \times C_2$. This case is as case (4b) where the central symmetry is conjugated to the antipodal map and $\bar{\sigma}_1$ is conjugated to $\bar{\sigma}_2$. There are 3 conjugacy classes of symmetries with species 0, $\Sigma_1 = \Sigma_{\sigma_3}$ and $\Sigma_2 = \Sigma_{\sigma_2}$. Now Σ_1 is -1 if $s_3 + \epsilon_1 + \epsilon_3 > 0$, Σ_1 is $+p, +1$ if $s_3 + \epsilon_1 + \epsilon_3 = 0$ and $\langle \varphi, \sigma_3 \rangle$ is C_{2p} . On the other hand Σ_2 is -1 if $s_1 + s_2 + \epsilon_1 + \epsilon_2 + \epsilon_3 > 0$, Σ_2 is $+p$ if $s_1 + s_2 + \epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ and $\langle \varphi, \sigma_2 \rangle = C_{2p}$. $\Sigma_\sigma \neq +p, +1, 0$ since $\langle \varphi, \sigma \rangle = D_p$ and σ has fixed points.

To finish we show the existence of surfaces with the desired symmetries by listing appropriate groups G and epimorphisms θ . The p -gonal surfaces with the desired symmetries will be uniformised by the groups $\text{Ker}(\theta)$. We distinguish the same eight cases as in Theorem 2.

(1) Let $G = \langle \varphi, \rho, \sigma | \varphi^t, \rho^q, \sigma^2, (\varphi\sigma)^2, (\varphi\rho)^{pq}, \rho^{-1}\sigma\rho\sigma \rangle$ and let $\theta : \Lambda \rightarrow G$ be defined by $\theta(x_i) = \varphi^{v_i}$, $1 \leq i \leq r$, $\theta(x_{r+1}) = \rho\varphi^{\epsilon+v_{r+1}}$, $\theta(c_{2j-1}) = \sigma$, $\theta(c_{2j}) = \varphi\sigma$, $\theta(e) = \rho^{-1}\varphi^l$, where $j_1 + \dots + j_r + \epsilon + v_{r+1} + l \equiv 0 \pmod p$.

(2) Let $G = \langle \varphi, \tau, \sigma | \varphi^p, \sigma^2, \tau^2, (\varphi\sigma)^2, (\sigma\tau)^q, (\varphi\tau)^2 \rangle$. Let $\theta : \Lambda \rightarrow G$ be defined by $\theta(x_i) = \varphi^{v_i}$, $1 \leq i \leq r$, $\theta(c_0) = \tau$, $\theta(c_j) = \sigma\varphi^{u_j}$ for $1 \leq j \leq s_1 + 1$, where $u_1 = \epsilon_1$ and $u_j = 1 - u_{j-1}$, $\theta(c_j) = \tau\varphi^{u_j}$, with $s_1 + 2 \leq j \leq s_1 + s_2 + 2$ where $u_{s_1+2} = \epsilon_2 + 1 - u_{s_1+1}$ and $u_j = 1 - u_{j-1}$, $\theta(e) = \varphi^l$, where $v_1 + \dots + v_r + l \equiv 0 \pmod p$.

To obtain the species $\Sigma_\tau = +p, +1$ we consider groups G with presentation $G = \langle \varphi, \tau, \sigma | \varphi^p, \sigma^2, \tau^2, (\varphi\sigma)^2, (\sigma\tau)^q, \varphi^{-1}\tau\varphi\tau \rangle$.

(3) Let $G = \langle \varphi, \rho, \tau, \sigma | \varphi^p, \rho^2, \tau^2, \sigma^2, \rho\sigma\rho\tau, (\varphi\rho)^2(\varphi\sigma)^2, (\sigma\tau)^q, (\varphi\tau)^2 \rangle$. Let $\theta : \Lambda \rightarrow G$ be defined by $\theta(x_i) = \varphi^{v_i}$, $1 \leq i \leq r$, $\theta(x_{r+1}) = \rho\varphi^{\epsilon_1+v_{r+1}}$, $\theta(c_0) = \tau$, $\theta(c_j) = \sigma\varphi^{u_j}$ for $1 \leq j \leq s_1 + 1$, where $u_1 = \epsilon_1$ and $u_j = 1 - u_{j-1}$, $\theta(e) = \rho\varphi^l$, where $v_1 + \dots + v_{r+1} + \epsilon_1 + l \equiv 0 \pmod p$.

(4) $G = \langle \varphi, \sigma_1, \sigma_2, \sigma_3 | \sigma_i^2, (\sigma_1\sigma_2)^2, (\sigma_2\sigma_3)^2, (\sigma_3\sigma_1)^q, \varphi^p, (\varphi\sigma_i)^2 \rangle$. Let $\theta : \Lambda \rightarrow G$ be defined by $\theta(x_i) = \varphi^{v_i}$, $1 \leq i \leq r$, $\theta(c_0) = \sigma_1$, $\theta(c_j) = \sigma_2\varphi^{u_j}$ for $1 \leq j \leq s_1 + 1$, where $u_1 = \epsilon_1$ and $u_j = 1 - u_{j-1}$, $\theta(c_j) = \sigma_3\varphi^{u_j}$, with $s_1 + 2 \leq j \leq s_1 + s_2 + 2$ where $u_{s_1+2} = \epsilon_2 + 1 - u_{s_1+1}$ and $u_j = 1 - u_{j-1}$, $\theta(c_j) = \sigma_1\varphi^{u_j}$, with $s_1 + s_2 + 3 \leq j \leq s_1 + s_2 + s_3 + 3$ where $u_{s_1+s_2+3} =$

$\epsilon_3 + 1 - u_{s_1+s_2+2}$ and $u_j = 1 - u_{j-1}$, $\theta(e) = \varphi^l$, where $v_1 + \dots + v_r + l \equiv 0 \pmod p$.

To obtain the species $+p, +1$ one or two of the relations $(\varphi\sigma_i)^2$ in the presentation of G must be substituted by relations $\varphi^{-1}\sigma_i\varphi\sigma_i$.

(5) and (8) $G = \langle \varphi, \sigma_1, \sigma_2, \sigma_3 \mid \sigma_i^2, (\sigma_1\sigma_2)^2, (\sigma_2\sigma_3)^3, (\sigma_3\sigma_1)^q, \varphi^p, (\varphi\sigma_i)^2 \rangle$, where $q = 3$ in (5) and $q = 5$ in (8) and let $\theta : \Lambda \rightarrow G$ be defined as in (4).

(6) $G = \langle \varphi, \rho, \sigma_1, \sigma_2 \mid \varphi^p, \rho^3, \sigma_1^2, \sigma_2^2, (\sigma_1\sigma_2)^2, \rho^2\sigma_1\rho\sigma_2, (\varphi\sigma_1)^2, (\varphi\sigma_2)^2, (\varphi\rho)^{3p} \rangle$. Let $\theta : \Lambda \rightarrow G$ be defined as $\theta(x_i) = \varphi^{v_i}$, $1 \leq i \leq r$, $\theta(x_{r+1}) = \rho\varphi^{\epsilon_1+v_{r+1}}$, $\theta(c_0) = \sigma_1$, $\theta(c_j) = \sigma_2\varphi^{u_j}$ for $1 \leq j \leq s+1$, where $u_1 = \epsilon_2$ and $u_j = 1 - u_{j-1}$, $\theta(e) = \rho^2\varphi^l$, where $v_1 + \dots + v_{r+1} + \epsilon_1 + l \equiv 0 \pmod p$.

(7) $G = \langle \varphi\sigma_1, \sigma_2, \sigma_3 \mid \sigma_i^2, (\sigma_1\sigma_2)^2, (\sigma_2\sigma_3)^3, (\sigma_3\sigma_1)^4, \varphi^p, (\varphi\sigma_i)^2 \rangle$ and let $\theta : \Lambda \rightarrow G$ be defined as in (4). To obtain the species $+p, +1$ either the relations $(\varphi\sigma_1)^2$ and $(\varphi\sigma_2)^2$ or the relation $(\varphi\sigma_3)^2$ in the presentation of G must be changed to the corresponding commuting relation.

The kernels of the above epimorphisms will uniformise surfaces with a symmetry with species -1 for general groups Λ . The same epimorphisms yield the species $+1$ in cases 1 and 4 under the restrictions on Λ given in the first part of the theorem. Again, the same epimorphisms yield the species $+p, +1$ under the corresponding restrictions on Λ given in the first part of the theorem.

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