SYMMETRIES OF REAL CYCLIC \(p\)-GONAL RIEMANN SURFACES

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A closed Riemann surface \(X\) which can be realised as a \(p\)-sheeted covering of the Riemann sphere is called \(p\)-gonal, and such a covering is called a \(p\)-gonal morphism. A \(p\)-gonal Riemann surface is called real \(p\)-gonal if there is an anticonformal involution (symmetry) \(\sigma\) of \(X\) commuting with the \(p\)-gonal morphism. If the \(p\)-gonal morphism is a cyclic regular covering the Riemann surface is called real cyclic \(p\)-gonal; otherwise it is called real generic \(p\)-gonal. The species of the symmetry \(\sigma\) is the number of connected components of the fixed point set \(\text{Fix}(\sigma)\) and the orientability of the Klein surface \(X/\langle \sigma \rangle\). In this paper we find the species for the possible symmetries of real cyclic \(p\)-gonal Riemann surfaces by means of Fuchsian and NEC groups.

1. Introduction.

A closed Riemann surface \(X\) which can be realised as a \(p\)-sheeted covering of the Riemann sphere is called \(p\)-gonal, and such a covering is called a \(p\)-gonal morphism. The \(p\)-gonal Riemann surfaces have been extensively studied, see [1], [2], [6], [8], [9], [12] and [13]. A \(p\)-gonal Riemann surface is called real \(p\)-gonal if there is an anticonformal involution (symmetry) \(\sigma\) of \(X\) commuting with the \(p\)-gonal morphism.

Let \(X_g\) be a real \(p\)-gonal Riemann surface of genus \(g \geq 2\). A symmetry \(\sigma\) of \(X_g\) is an anticonformal involution of \(X_g\). The topological type of a symmetry is determined by the number of connected components, called ovals, of the fixed-point set \(\text{Fix}(\sigma)\) and the orientability of the Klein surface \(X/\langle \sigma \rangle\). We say that \(\sigma\) has species \(\Sigma_\sigma = +k\) if \(\text{Fix}(\sigma)\) consists of \(k\) ovals and \(X/\langle \sigma \rangle\) is orientable, and \(\Sigma_\sigma = -k\) if \(\text{Fix}(\sigma)\) consists of \(k\) ovals and \(X/\langle \sigma \rangle\) is non-orientable. The set \(\text{Fix}(\sigma)\) corresponds to the real part of a complex algebraic curve representing \(X\), which admits an equation with real coefficients.

If the \(p\)-gonal morphism is a cyclic regular covering, then the Riemann surface is called real cyclic \(p\)-gonal. When \(p = 2\) the surface \(X_g\) is called hyperelliptic. A Riemann surface represented by an algebraic curve given
by an equation of the form
\[ y^p = \prod (x - a_i) \prod (x - b_j)^2 \cdots \prod (x - m_j)^{p-1} \] (1.1)
where the coefficients of the polynomial \( \prod (x - a_i) \cdots \prod (x - m_j)^{p-1} \) are real
is a real cyclic \( p \)-gonal Riemann surface. The complex conjugation induces
a symmetry on the above curve. A natural problem is to study and classify
all possible symmetries of such a Riemann surface up to conjugacy, as they
will produce non-isomorphic real models of the complex algebraic curve.
In Section 2 we characterise real cyclic \( p \)-gonal Riemann surfaces, where
\( p \) is an odd prime, in terms of signatures of Fuchsian and NEC groups.
In Section 3 we determine all possible symmetries of a real cyclic \( p \)-gonal
Riemann surface represented by an algebraic curve with equation (1.1).

2. Signatures of real cyclic \( p \)-gonal Riemann surfaces.
Let \( X_g \) be a compact Riemann surface of genus \( g \geq 2 \). The surface \( X_g \) can
be represented as a quotient \( X_g = \mathcal{H}/\Gamma \) of the upper half plane \( \mathcal{H} \) under
the action of a surface Fuchsian group \( \Gamma \), that is, a cocompact orientation-
preserving subgroup of the group \( \mathcal{G} = \text{Aut}(\mathcal{H}) \) of conformal and anticonfor-
mal automorphisms of \( \mathcal{H} \) without elliptic elements. A discrete, cocompact
subgroup \( \Gamma \) of \( \text{Aut}(\mathcal{H}) \) is called an NEC (non-euclidean crystallographic)
group. The subgroup of \( \Gamma \) consisting of the orientation-preserving elements
is called the canonical Fuchsian subgroup of \( \Gamma \), it is denoted by \( \Gamma^+ \). The
algebraic structure of an NEC group and the geometric structure of its quo-
tient orbifold are given by the signature of \( \Gamma \):
\[ s(\Gamma) = (h, \pm, [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}) \] (2.1)
The orbit space \( \mathcal{H}/\Gamma \) is an orbifold with underlying surface of genus \( h \), having
\( r \) cone points and \( k \) boundary components, each with \( s_j \geq 0 \) corner points.
The signs \( "+" \) and \( "-" \) correspond to orientable and non-orientable orbifolds
respectively. The integers \( m_i \) are called the proper periods of \( \Gamma \) and they
are the orders of the cone points of \( \mathcal{H}/\Gamma \). The brackets \( (n_{i1}, \ldots, n_{is_i}) \) are
the period cycles of \( \Gamma \) and the integers \( n_{ij} \) are the link periods of \( \Gamma \) and the
orders of the corner points of \( \mathcal{H}/\Gamma \). The group \( \Gamma \) is called the fundamental
group of the orbifold \( \mathcal{H}/\Gamma \).
A group \( \Gamma \) with signature (2.1) has a canonical presentation with generators:
\[ x_1, \ldots, x_r, e_1, \ldots, e_k, c_{ij}, 1 \leq i \leq k, 1 \leq j \leq s_i + 1, \text{ and} \]
\[ a_1, b_1, \ldots, a_h, b_h \] if \( \mathcal{H}/\Gamma \) is orientable, or
\[ d_1, \ldots, d_h \]
otherwise, and relators:

\[(2.3) \quad x_i^{m_i}, \quad i = 1, \ldots, r, \]
\[c_{ij}^2, \ (c_{ij-1}c_{ij})^{n_{ij}}, \ c_{i0}e_i^{-1}c_{i0}e_i, \quad i = 1, \ldots, k,\]

and \(x_1 \cdots x_r e_1 \cdots e_k a_1 b_1 a_1^{-1}b_1^{-1} \cdots a_h b_h a_h^{-1}b_h^{-1}\) or \(x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_h^2\) according to whether \(\mathcal{H}/\Gamma\) is orientable or not. This last relation is called the long relation.

The hyperbolic area of the orbifold \(\mathcal{H}/\Gamma\) coincides with the hyperbolic area of an arbitrary fundamental region of \(\Gamma\) and equals:

\[(2.4) \quad \mu(\Gamma) = 2\pi \left( \varepsilon h - 2 + k + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left( 1 - \frac{1}{n_{ij}} \right) \right), \]

where \(\varepsilon = 2\) if there is a \(''^+''\) sign and \(\varepsilon = 1\) otherwise. If \(\Gamma'\) is a subgroup of \(\Gamma\) of finite index then \(\Gamma'\) is an NEC group and the following Riemann-Hurwitz formula holds:

\[(2.5) \quad [\Gamma : \Gamma'] = \mu(\Gamma')/\mu(\Gamma). \]

An NEC group \(\Gamma\) without elliptic elements is called a surface group and it has signature \((h; \pm; [-], \{(-), \ldots, (=-)\})\). In such a case \(\mathcal{H}/\Gamma\) is a Klein surface, i.e., a surface with a dianalytic structure of topological genus \(h\), orientable or not according to the sign \(''^+''\) or \(''^-''\), and having \(k\) boundary components. Conversely, a Klein surface whose complex double has genus greater than one can be expressed as \(\mathcal{H}/\Gamma\) for some NEC surface group \(\Gamma\).

Furthermore, given a Riemann (resp. Klein) surface represented as the orbit space \(X = \mathcal{H}/\Gamma\), with \(\Gamma\) a surface group, a finite group \(G\) is a group of automorphisms of \(X\) if and only if there exists an NEC group \(\Delta\) and an epimorphism \(\theta : \Delta \rightarrow G\) with \(\ker(\theta) = \Gamma\) (see [5]). The NEC group \(\Delta\) is the lifting of \(G\) to the universal covering \(\pi : \mathcal{H} \rightarrow \mathcal{H}/\Gamma\) and is called the universal covering transformation group of \((X,G)\).

**Definition 1.** For a prime \(p\), a real cyclic \(p\)-gonal Riemann surface is a triple \((X,f,\sigma)\) where \(\sigma\) is a symmetry of \(X\), \(f\) is a cyclic \(p\)-gonal morphism and \(f \circ \sigma = c \circ f\), and \(c\) is the complex conjugation.

Notice that by Lemma 2.1 in [1] the condition \(f \circ \sigma = c \circ f\) is automatically satisfied for genera \(g \geq (p - 1)2 + 1\), since the \(p\)-gonal morphism is unique. From now on, the genera will satisfy the condition above. As a consequence of the assumption \(g \geq (p - 1)2 + 1\) for the genera of the \(p\)-gonal surface \(X_g\) we have that the group \(C_p\) generated by the \(p\)-gonal morphism is a normal subgroup of \(\text{Aut}^+(X_g)\). Notice the the classification method fails for surfaces with genera in the range \(2 \leq g \leq (p - 1)2\). For instance, there are two 7-gonal surfaces of genus 3. One of them, \(X_3\), is the Klein’s quartic with \(\text{Aut}^+(X_3) \simeq PSL_2(7)\), in this case \(C_7\) is non-normal in \(PSL_2(7)\).
We give now a characterisation of real cyclic \( p \)-gonal Riemann surfaces represented by real equations via NEC groups.

**Theorem 1** ([7]). Let \( X \) be a Riemann surface with genus \( g \). The surface \( X \) admits a symmetry \( \sigma \) and a meromorphic function \( f \) such that \( (X, f, \sigma) \) is a real cyclic \( p \)-gonal Riemann surface represented by a curve with real equation \( y^p = \prod (x - a_i) \cdots \prod (x - m_j)^{p-1} \) if and only if there are an NEC group \( \Delta \) with signature \( (0, +, [p, \ldots, p], \{(p, \ldots, p)\}) \) and an epimorphism \( \theta : \Delta \to \mathbb{D}_p \) such that \( X \) is conformally equivalent to \( \mathcal{H}/\text{Ker} \theta \) and \( \text{Ker} \theta \) is an NEC Fuchsian surface group.

Let \( (X, f, \sigma) \) be a real cyclic \( p \)-gonal Riemann surface uniformised by a Fuchsian surface group \( \Gamma \). Consider the automorphism \( \varphi : X \to X \) such that \( X/\langle \varphi \rangle \) is the Riemann sphere and \( \varphi \) is a deck-transformation of the covering \( f \). Notice that the group \( \Delta \) is the universal covering transformation group of \( (X, \varphi, \sigma) \). Consider the automorphism \( \sigma \) map such that there is a surface Fuchsian subgroup \( \Gamma \). Theorem 2. \( \mathbb{D}_p \) admits a symmetry \( \mathbb{D}_p \), \( \mathbb{D}_p \) is the Riemann sphere and \( \mathbb{D}_p \) is conformally equivalent to \( \mathcal{H}/\text{Ker} \theta \) and \( \text{Ker} \theta \) is an NEC Fuchsian surface group.

With the above notation:

**Theorem 2.** Let \( X \) be a real cyclic \( p \)-gonal Riemann surface such that \( \langle \varphi, \sigma \rangle \) is isomorphic to \( \mathbb{D}_p \). If \( G \) is the group of conformal and anticonformal automorphisms of \( X \), then \( X/G \) is uniformised by an NEC group \( \Lambda \) such that there is a surface Fuchsian subgroup \( \Gamma \leq \Lambda \) uniformising \( X \) and the group \( \Lambda \) has one of the following signatures:

1. \( (0, +, [p, \ldots, p], q \mathbb{D}_q], \{(p, \ldots, p)\}), \) where \( \epsilon = 0 \) or 1 and \( 2r + s = \frac{2q + 2(1 - \epsilon)(p - 1)}{q(p - 1)} \), \( G/\langle \varphi \rangle = C_q \times C_2 \).
2. \( (0, +, [p, \ldots, p], \{(qp^{s_1}, p, \ldots, p, qp^{s_2}, p, \ldots, p)\}), \) where \( \epsilon_i = 0 \) or 1 and \( 2r + s_1 + s_2 = \frac{2q + 2(1 - \epsilon_1 - \epsilon_2)(p - 1)}{q(p - 1)} \), \( G/\langle \varphi \rangle = D_q \).
3. \( (0, +, [p, \ldots, p], 2p^{s_1}], \{(pq^{s_1}, p, \ldots, p)\}), \) where \( \epsilon_i = 0 \) or 1 and \( 2r + s = \frac{2q + 2(1 - \epsilon_1 - \epsilon_2)(p - 1)}{q(p - 1)} \), \( G/\langle \varphi \rangle = D_q \times C_2 \).
4. \( (0, +, [p, \ldots, p], \{(2p^{s_1}, p, \ldots, p, 2p^{s_2}, p, \ldots, p, qp^{s_3} p, \ldots, p)\}), \) where \( \epsilon_i = 0 \) or 1 and \( 2r + s_1 + s_2 + s_3 = \frac{2q + 2(1 - \epsilon_1 - \epsilon_2 - \epsilon_3)(p - 1)}{2q(p - 1)} \), \( G/\langle \varphi \rangle = D_q \times C_2 \).
(V) \(0, +, \{ [\hat{p}, \ldots, \hat{p}], \{(2p^{s_1}, \hat{p}, \ldots, \hat{p}, 3p^{s_2}, \hat{p}, \ldots, \hat{p}, 3p^{s_3}, \hat{p}, \ldots, \hat{p})\}\}, \) where \(\epsilon_i = 0\) or 1 and \(2r + s_1 + s_2 + s_3 = \frac{\varphi + (1 - 3\epsilon_1 - 2\epsilon_2 - 2\epsilon_3)(p-1)}{6(p-1)}. \)
\(G/\langle \varphi \rangle = S_4. \)

(VI) \(0, +, \{ [\hat{p}, \ldots, \hat{p}], \{(2p^{s_1}, \hat{p}, \ldots, \hat{p}, 3p^{s_2}, \hat{p}, \ldots, \hat{p}, 5p^{s_3}, \hat{p}, \ldots, \hat{p})\}\}, \) where \(\epsilon_i = 0\) or 1 and \(2r + s = \frac{\varphi + (1 - 4\epsilon_1 - 3\epsilon_2)(p-1)}{6(p-1)}\). \(G/\langle \varphi \rangle = A_4 \rtimes C_2. \)

(VII) \(0, +, \{ [\hat{p}, \ldots, \hat{p}], \{(2p^{s_1}, \hat{p}, \ldots, \hat{p}, 3p^{s_2}, \hat{p}, \ldots, \hat{p}, 4p^{s_3}, \hat{p}, \ldots, \hat{p})\}\}, \) where \(\epsilon_i = 0\) or 1 and \(2r + s_1 + s_2 + s_3 = \frac{\varphi + (1 - 6\epsilon_1 - 4\epsilon_2 - 3\epsilon_3)(p-1)}{12(p-1)}. \)
\(G/\langle \varphi \rangle = S_4 \times C_2. \)

(VIII) \(0, +, \{ [\hat{p}, \ldots, \hat{p}], \{(2p^{s_1}, \hat{p}, \ldots, \hat{p}, 3p^{s_2}, \hat{p}, \ldots, \hat{p}, 5p^{s_3}, \hat{p}, \ldots, \hat{p})\}\}, \) where \(\epsilon_i = 0\) or 1 and \(2r + s_1 + s_2 + s_3 = \frac{\varphi + (1 - 15\epsilon_1 - 10\epsilon_2 - 6\epsilon_3)(p-1)}{30(p-1)}. \)
\(G/\langle \varphi \rangle = A_5 \times C_2. \)

Notice that in cases (VII) and (VIII) the factor group \(C_2\) of \(G/\langle \varphi \rangle\) is generated by the antipodal map.

**Proof.** Consider the chain of coverings \(X = \mathcal{H}/\Gamma \rightarrow X/\langle \varphi \rangle = \mathcal{H}/\Delta^+ \rightarrow X/G = \mathcal{H}/\Lambda\) with uniformising groups \(\Gamma \leq \Delta^+ \leq \Lambda\), where \(s(\Delta^+) = (0, +, [\hat{p}, \ldots, \hat{p}], \{\}), \) and \(s(\Lambda) = (h, \pm, [m_1, \ldots, m_r], \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_1})\})\). Furthermore by Lemma 2.1 in [1] the group \(\langle \varphi \rangle\) is a normal subgroup of \(G\). By Theorem 1 the factor group \(G/\langle \varphi \rangle\) is a finite group of conformal and anticonformal automorphisms of the Riemann sphere. (See also [12].)

In other words, we have an epimorphism \(\theta : \Lambda \rightarrow G\) with \(\text{Ker} \; \theta = \Delta^+\). This yields the signature of the group \(\Lambda\) in terms of the signature of \(\Delta^+\) and the group \(G\). Let \(p_i\) and \(q_{ij}\) be the orders in \(G\) of \(\theta(x_i)\) and \(\theta(c_{ij}c_{ij}^{-1})\) respectively, where \(x_i, c_{ij}\) are generators in the canonical presentation of \(\Lambda\) associated to the signature (2.1). By [3] and [5] each proper period \(m_i\) induces \(\frac{[\Gamma]}{p_i}\) proper periods \(\frac{m_i}{p_i}\) in \(s(\Delta^+)\). Each link-period \(n_{ij}\) induces \(\frac{[\Gamma]}{2q_{ij}}\) proper periods \(\frac{n_{ij}}{q_{ij}}\) in \(s(\Delta^+)\). But \(\frac{m_i}{p_i} = p\) or \(\frac{m_i}{p_i} = 1\) and \(\frac{n_{ij}}{q_{ij}} = p\) or \(\frac{n_{ij}}{q_{ij}} = 1\), since \(\Delta^+\) is the group of the Riemann sphere with conic points of prime order \(p\). We denote \(K_1 = \{(i, j) \mid \frac{m_i}{p_i} = 1\}\), \(K_p = \{(i, j) \mid \frac{m_i}{p_i} = p\}\), \(H_1 = \{(i, j) \mid \frac{n_{ij}}{q_{ij}} = 1\}\), \(H_p = \{(i, j) \mid \frac{n_{ij}}{q_{ij}} = p\}\). Thus \(\rho = \sum_{i \in K_1} \frac{[\Gamma]}{p_i} + \sum_{(i, j) \in H_1} \frac{[\Gamma]}{2q_{ij}}\)
Using the Riemann-Hurwitz formula \(|G| = \mu(\Delta^+)/\mu(\Lambda)| we obtain

\[
2 + \left( \sum_{i \in K_p} \frac{|G|}{p_i} + \sum_{(i,j) \in H_p} \frac{|G|}{2q_{ij}} \right) \frac{(p - 1)}{p} \\
= |G|(\alpha h - 2 + k) + \sum_{i \in K_p} |G| \left( 1 - \frac{1}{pp_i} \right) + \sum_{i \in K_1} |G| \left( 1 - \frac{1}{p_i} \right) \\
+ \sum_{(i,j) \in H_p} \frac{|G|}{2} \left( 1 - \frac{1}{pq_{ij}} \right) + \sum_{(i,j) \in H_1} \frac{|G|}{2} \left( 1 - \frac{1}{q_{ij}} \right),
\]

therefore \( h = 0, k = 1, s(\Lambda) = (0, +, [pp_1, \ldots, pp_r], \{(pq_1, \ldots, pq_s)\}) \), where \( p_i, q_{ij} \in \{1, p\} \). By setting \( K_1, K_p, H_1 \) and \( H_p \) in Equation (2.6) we obtain that \( p_i, q_{ij} \) satisfy the equation

\[
|G| - 2 = \sum_{1}^{r} |G| \left( 1 - \frac{1}{p_i} \right) + \sum_{1}^{s} \frac{|G|}{2} \left( 1 - \frac{1}{q_{ij}} \right).
\]

To find \( s(\Lambda) \) it is enough to find the nontrivial solutions of (2.7). We divide the study of (2.7) in eight cases according to the factor group \( G \) in the epimorphism \( \theta : \Lambda \rightarrow G \) with \( \text{Ker}(\theta) = \Delta^+ \):

(I) \( G = C_q \times C_2 \), where \( C_2 = \langle \sigma \rangle \). The solution of Equation (2.7) is \( p_1 = q \). Applying Riemann-Hurwitz formula to the covering \( X \rightarrow X/G \) we obtain the signature \((0, +, [p_1, \ldots, p], \{(p_1, \ldots, p)\})\), where \( \epsilon = 0 \) or 1 and \( 2r + s = \frac{2g + 2(1 - \epsilon)(p - 1)}{q(p - 1)} \).

(II) \( G = D_q \). The solution of (2.7) is \( q_{j_1} = q_{j_2} = q \). Therefore \( s(\Lambda) = (0, +, [p_1, \ldots, p], \{(pq^1_1, p_1, \ldots, p), \{pq^2_1, p_1, \ldots, p\})\), where \( \epsilon_i = 0 \) or 1 and \( 2r + s_1 + s_2 = \frac{2g + (2 - \epsilon_1 - \epsilon_2)(p - 1)}{q(p - 1)} \).

(III) \( G = D_q \times C_2 \). The solution of (2.7) is \( p_1 = 2, \) and \( q_1 = q \). Thus \( s(\Lambda) \) becomes \((0, +, [p_1, \ldots, p, 2p^1], \{(qp^1_1, p_1, \ldots, p), \{qp^2, p_1, \ldots, p\})\), where \( \epsilon_i = 0 \) or 1 and \( 2r + s = \frac{2 + (1 - \epsilon_1 - \epsilon_2)(p - 1)}{q(p - 1)} \).

(IV) \( G = D_q \times C_2 \). The solution in this case is \( q_{j_1} = q_{j_2} = 2 \) and \( q_{j_3} = q \). This yields \( s(\Lambda) = (0, +, [p_1, \ldots, p], \{(2p^1_1, p_1, \ldots, p), \{2p^1_1, p_1, \ldots, p, q_{p_1}^{2}, p, \ldots, p\})\), where \( \epsilon_i = 0 \) or 1 and \( 2r + s_1 + s_2 + s_3 = \frac{2g + (2 - \epsilon_1 - \epsilon_2 - 2\epsilon_3)(p - 1)}{2q(p - 1)} \).
(V) $\overline{G} = S_4$. The solution of (2.7) is $q_{j_1} = 2$, $q_{j_2} = q_{j_3} = 3$. Then $s(\Lambda) = (0, +, \overline{p_1, p_2, ..., p_r}, \{2p_1^3, \overline{p_1, p_2, ..., p_r}, 3p_2^3, \overline{p_1, p_2, ..., p_r}, 3p_3^3, \overline{p_1, p_2, ..., p_r}\})$, where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 + s_3 = \frac{g+(1-3\epsilon_1-2\epsilon_2-2\epsilon_3)(p-1)}{6(p-1)}$.

(VI) $\overline{G} = A_4 \times C_2$. The solution of (2.7) is $p_1 = 3$, and $q_1 = 2$. Thus $s(\Lambda)$ becomes $(0, +, \overline{p_1, p_2, ..., p_r}, \{2p_2^3, \overline{p_1, p_2, ..., p_r}, 3p_3^3, \overline{p_1, p_2, ..., p_r}\})$, where $\epsilon_i = 0$ or 1 and $2r + s = \frac{g+(1-4\epsilon_1-3\epsilon_2)(p-1)}{6(p-1)}$.

(VII) $\overline{G} = S_4 \times C_2$. The solution in this case is $q_{j_1} = 2$, $q_{j_2} = 3$ and $q_{j_3} = 4$. This yields $s(\Lambda) = (0, +, \overline{p_1, p_2, ..., p_r}, \{(2p_1^3, \overline{p_1, p_2, ..., p_r}, 3p_2^3, \overline{p_1, p_2, ..., p_r}, 3p_3^3, \overline{p_1, p_2, ..., p_r}, 4p_4^3, \overline{p_1, p_2, ..., p_r}\})$, where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 + s_3 = \frac{g+(1-4\epsilon_1-3\epsilon_2-3\epsilon_3)(p-1)}{12(p-1)}$.

(VIII) $\overline{G} = A_5 \times C_2$. The solution now is $q_{j_1} = 2$, $q_{j_2} = 3$ and $q_{j_3} = 5$. This yields $s(\Lambda) = (0, +, \overline{p_1, p_2, ..., p_r}, \{(2p_1^3, \overline{p_1, p_2, ..., p_r}, 3p_2^3, \overline{p_1, p_2, ..., p_r}, 3p_3^3, \overline{p_1, p_2, ..., p_r}, 4p_4^3, \overline{p_1, p_2, ..., p_r}, 5p_5^3, \overline{p_1, p_2, ..., p_r}\})$, where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 + s_3 = \frac{g+(1-15\epsilon_1-10\epsilon_2-6\epsilon_3)(p-1)}{30(p-1)}$. This finishes the proof.


Let $X$ be a real cyclic $p$-gonal Riemann surface $X$ with real equation. In the next theorem we study the topological types of the possible real forms of $X$.

**Theorem 3.** Let $X$ be a real cyclic $p$-gonal Riemann surface with $p$-gonal automorphism $\varphi$ admitting a symmetry $\sigma$ with fixed points and such that $\langle \sigma, \varphi \rangle = D_p$, $p$ prime. If $\tau$ is another symmetry of $X$, then possible species of $\tau$ are (and all cases occur):

1. $s(\Lambda)$ as in (I).
   a) $q \equiv 1 \mod (2)$. $\Sigma_\sigma = \Sigma_\tau$. If $r + \epsilon > 0$, then $\Sigma_\sigma = -1$. If $r + \epsilon = 0$, then $\Sigma_\sigma \in \{-1, +1\}$.
   b) $q \equiv 0 \mod (2)$. $\Sigma_\tau = \Sigma_\sigma$ as in case (1a) or $\Sigma_\tau = 0$.

2. $s(\Lambda)$ as in (II).
   a) $q \equiv 1 \mod (2)$. $\Sigma_\sigma = \Sigma_\tau$ and $\Sigma_\sigma = -1$.
   b) $q \equiv 0 \mod (2)$, $\Sigma_\sigma \neq 2$. $\Sigma_\sigma = -1$ and $\Sigma_\tau = -1$ or $\Sigma_\tau = +p, +1$.

3. $s(\Lambda)$ as in (III). $\Sigma_\tau = 0$ or $\Sigma_\sigma = \Sigma_\tau$, besides $\Sigma_\sigma = -1$.

4. $s(\Lambda)$ as in (IV).
   a) $q \equiv 1 \mod (2)$. $\{\Sigma_\sigma, \Sigma_\tau\} \subset \{\Sigma_1, \Sigma_2\}$, where $\Sigma_1 \in \{-1, +1, +p\}$. $\Sigma_2 \in \{-1, +1, +p\}$. In both cases $\Sigma_\sigma \neq +p$ and $\Sigma_\sigma \neq +1$ if $\sigma$ is of the first type.
we divide the proof in eight cases corresponding to the different types of groups $\overline{G}$ of conformal and anticonformal automorphisms of the Riemann sphere. The signature of $\Lambda$ in each case is given by the corresponding case in Theorem 2.

(1a) $\overline{G} = C_q \times C_2$, $q \equiv 1 \mod (2)$. In this case $\overline{G}$ contains just one conjugacy class of symmetries and so does $G$: The one represented by $\sigma$. Moreover $D_p = \langle \varphi, \sigma \rangle$ is a normal subgroup of index $q$ in $G$. By [5] the signature of $\overline{\Theta}^{-1}(\langle \varphi, \sigma \rangle)$ is $\langle 0, +, \{ (\bar{p}, \ldots, \bar{p}) \} \rangle$. By [14] $\Sigma_\sigma = \pm 1$ as $D_p = \langle \varphi, \sigma \rangle$. The sign $+$ can only occur if $\overline{\Theta}^{-1}(\langle \varphi, \sigma \rangle)$ has no proper periods, i.e., $r + \epsilon = 0$. If $s = 0$, then $r + \epsilon > 0$, the possible species is $-1$.

(1b) $\overline{G} = C_q \times C_2 = \langle \rho, \bar{\sigma} \mid \rho^q, \bar{\sigma}^2, \rho^{-1} \bar{\sigma} \bar{\rho} \bar{\sigma} \rangle$, with $q \equiv 0 \mod (2)$. In this case $\overline{G}$ contains two conjugacy classes of symmetries, with representatives namely $\sigma$ and $\rho^{n/2} \bar{\sigma} = \bar{\tau}$, and so does $G$. To find the species of the symmetries we have to consider the normal subgroups $\overline{\Theta}^{-1}(\langle \varphi, \sigma \rangle)$ and $\overline{\Theta}^{-1}(\langle \varphi, \bar{\tau} \rangle)$ of $\Lambda$ with factor group $C_q$. By [5] they have signatures $\langle 0, +, \{ \bar{p}, \ldots, \bar{p} \}, \{ (\bar{p}, \ldots, \bar{p}) \} \rangle$ and $\langle 0, +, \{ \bar{p}, \ldots, \bar{p} \}, \{ - \} \rangle$ respectively. So species $\Sigma_\sigma$ is as in (1a) and $\Sigma_{\bar{\tau}} = 0$.

(2a) $\overline{G} = D_q = \langle \rho, \bar{\sigma} \mid \rho^q, \bar{\sigma}^2, (\rho \bar{\sigma})^2 \rangle$, with $q \equiv 1 \mod (2)$. The group $\overline{G}$ contains one conjugacy class of symmetries and so does $G$. By the epimorphism $\theta : \Lambda \rightarrow D_q$ the images of reflections in $\Lambda$ leave one fixed coset in $D_q$, so
we get that $\overline{\Lambda}_g$ has signature $(0,+,\{\underbrace{\bar{p},\ldots,\bar{p}}_r\},\{(\underbrace{\bar{p},\ldots,\bar{p}}_s)\})$. Now, $s_1 + s_2 + r > 0$ since $\Lambda$ is a NEC group, then $\Sigma_\sigma = -1$ by [4] and [14].

**(2b)** $\overline{G} = D_q = \langle \rho, \sigma | \rho^q, \sigma^2, (\rho^q \sigma^2)^2 \rangle$, with $q \equiv 0 \mod (2)$. The group $\overline{G}$ (and the group $G$) contains two conjugacy classes of symmetries, with representatives namely $\sigma$ and $\rho \sigma = \overline{\sigma}$. To find $\Sigma_\sigma$ and $\Sigma_\tau$ we have to study the images of reflections by an epimorphism $\theta : \Lambda \to D_q$. Each of these images leaves either $2\overline{\sigma}$-cosets fixed and none from $\overline{\tau}$ or the other way round.

Thus the signatures of $\Lambda_\sigma$ and $\Lambda_\tau$ are $(0,+,\{\underbrace{\bar{p},\ldots,\bar{p}}_r\},\{(\underbrace{\bar{p},\ldots,\bar{p}}_s)\})$ and $(0,+,\{\underbrace{\bar{p},\ldots,\bar{p}}_r\},\{(\underbrace{\bar{p},\ldots,\bar{p}}_s)\})$. Now $\sigma$ has 1 oval and does not separate because $\overline{\theta}^{-1}(\langle \varphi, \sigma \rangle)$ contains proper periods since $s_1 + s_2 + r > 0$ and $q > 2$. If $e_1 + s_2 + e_2 > 0$, then $\langle \varphi, \tau \rangle = D_p$ and as before $\Sigma_\tau = -1$. If $e_1 + s_2 + e_2 = 0$ the signature of $\overline{\theta}^{-1}(\langle \varphi, \tau \rangle)$ becomes $(0,+,\{\underbrace{\bar{p},\ldots,\bar{p}}_r\},\{-\})$. Thus $\Sigma_\tau = -1$, if $\langle \varphi, \tau \rangle = D_p$, and $\Sigma_\tau = +p_1 + 1$ if $\langle \varphi, \tau \rangle = C_{2p}$.

**(3)** $\overline{G} = D_q \rtimes C_2 = \langle \bar{p}, \overline{\sigma_1}, \overline{\sigma_2} | \bar{p}^2, \overline{\sigma_1}^2, \overline{\sigma_2}^2, (\overline{\sigma_1} \overline{\sigma_2})^q, \overline{\rho \sigma_1 \rho \sigma_2} \rangle$. The group $\overline{G}$ (and $G$) contains two conjugacy classes of symmetries, with representatives namely $\overline{\sigma} = \overline{\sigma_1}$ and $\overline{\rho \sigma} = \overline{\sigma}$. The images of reflections in $\Lambda$ are all mapped to conjugate reflections in $\overline{G}$. They are conjugate to $\overline{\sigma}$ as we know that $\sigma$ has fixed points. Thus $\Sigma_\sigma = 0$. On the other hand $\Lambda_\sigma$ has always proper periods. Therefore $\Sigma_\sigma = -1$.

**(4)** $\overline{G} = D_q \rtimes C_2 = \langle \overline{\sigma_1}, \overline{\sigma_2}, \overline{\bar{p}} | \overline{\sigma_1}^2, (\overline{\sigma_1} \overline{\sigma_2})^2, (\overline{\sigma_2} \overline{\sigma_1})^2, (\overline{\sigma_3} \overline{\sigma_1})^q \rangle$, with $\overline{\sigma_2}$ central in $\overline{G}$. First of all the group $\langle \varphi, \sigma_2 \rangle$ is a normal subgroup of $G$ with factor group $D_q = \langle \overline{\sigma_1}, \overline{\sigma_3} \rangle$.

**(4a)** $q \equiv 1 \mod (2)$. In this case $G$ has two conjugacy classes of reflections with representatives with images $\sigma_1$ and $\sigma_2$. Then there are two possible species for a symmetry of $X$: $\Sigma_{\sigma_1}, \Sigma_{\sigma_2}$. The possible signatures for $\overline{\theta}^{-1}(\langle \varphi, \sigma_1 \rangle)$ are given by the epimorphism $\Lambda \to D_q$. By this epimorphism the images of $c_0$ and $c_{s_1+i}$, for $i \geq 2$, are conjugated to $\overline{\sigma_1}$ and the image of $c_1, \ldots, c_{s_1+1}$ is the identity (representing the central symmetry). Therefore $c_0, c_1, c_{s_1+1}$ and $c_{s+2}$ fixes $q \langle \overline{\sigma_2} \rangle$-cosets and one $\langle \overline{\sigma} \rangle$-coset each, each $c_1, \ldots, c_{s_1+1}$ fixes $2q \langle \overline{\sigma_2} \rangle$-cosets (and none $\langle \overline{\sigma} \rangle$-coset), and finally each $c_{s_1+i}$, $i \geq 2$ fixes two $\langle \overline{\sigma_1} \rangle$-cosets (and none $\langle \overline{\sigma} \rangle$-coset) in $\overline{G}$. Thus $\overline{\Lambda}_1$ and
$\bar{\Lambda}_2$ have signatures

\[
\begin{align*}
&2rq+qs_1+(q-1)(s_2+s_3)+\frac{q-1}{2}(s_2+s_3) + \frac{q-1}{2}(s_2+s_3) + 2s_3 + 2s_2 + 2s_1 + 2s_3 \\
&(0, +, [p, \ldots, p], \{(p, \ldots, p)\}) \text{ and} \\
&2rq+qs_2+qs_3 + \frac{q-1}{2}(s_2+s_3) + \frac{q-1}{2}(s_2+s_3) + 2s_3 + 2s_2 + 2s_1 + 2s_3 \\
&(0, +, [p, \ldots, p], \{(p, \ldots, p)\})
\end{align*}
\]

Altogether we have that $\Sigma_1$ is $-1$ if $s_2 + s_3 + \epsilon_1 + \epsilon_2 + \epsilon_3 > 0$, and $\Sigma_1$ is $+p,+1$ if $s_2 + s_3 + \epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ and $\langle \varphi, \sigma_1 \rangle$ is $C_2p$. On the other hand $\Sigma_2$ is $-1$ if $s_1 + \epsilon_1 + \epsilon_2 > 0$ and and $r + s_2 + s_3 + \epsilon_3 > 0$, $\Sigma_2$ is $+1$ if $s_1 + \epsilon_1 + \epsilon_2 > 0$ and $r = s_2 = s_3 = \epsilon_3 = 0$, and finally $\Sigma_2$ is $+p,+1$ if $s_1 + \epsilon_1 + \epsilon_2 = 0$ and $\langle \varphi, \sigma_2 \rangle = C_2p$. In both cases $\Sigma_\sigma \neq +p$ since $\langle \varphi, \sigma \rangle = D_p$ and if $\sigma$ is conjugate to $\sigma_1$ then again $\Sigma_\sigma \neq +1$. No further restrictions exist.

(4b) $q \equiv 0 \mod (2)$. In this case $G$ has four conjugacy classes of reflections with representatives with homomorphic images $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3$ and $(\bar{\sigma}_3\bar{\sigma})^{q/2}\bar{\sigma}_2 = \bar{\sigma}_4$. Then there are four possible species for a symmetry of $X$: $\Sigma_1, 1 \leq i \leq 4$. The species are also given by the epimorphism $\Lambda \xrightarrow{\hat{\theta}} D_q$. By this epimorphism the images of $\epsilon_0$ and $c_{s_1+s_2+i}$, for $i \geq 3$, are conjugate to $\bar{\sigma}_1$, the image of $c_1, \ldots, c_{s_1+1}$ is the identity (representing the central symmetry), and the images of $c_{s_1+i}$, for $2 \leq i \leq s_2 + 2$, are conjugate to $\bar{\sigma}_3$. First of all $\Sigma_4 = 0$ since no images of reflections by $\hat{\theta}$ are conjugate to $\bar{\sigma}_4$.

If $q = 2$, then all the 3 symmetries are central and, as in (4a) the possible species for them are $-1, +1$ and $+p$.

If $q \neq 2$ then with the same procedure as in (4a) we get the following signatures for $\bar{\Lambda}_1, \bar{\Lambda}_2$ and $\bar{\Lambda}_3$:

\[
\begin{align*}
&2rq+qs_1+(q-1)(s_2+s_3)+\frac{q-1}{2}(s_2+s_3) + \frac{q-1}{2}(s_2+s_3) + 2s_3 + 2s_2 + 2s_1 + 2s_3 \\
&(0, +, [p, \ldots, p], \{(p, \ldots, p)\}), \\
&2rq+qs_2+qs_3 + \frac{q-1}{2}(s_2+s_3) + \frac{q-1}{2}(s_2+s_3) + 2s_3 + 2s_2 + 2s_1 + 2s_3 \\
&(0, +, [p, \ldots, p], \{(p, \ldots, p)\}), \\
&2rq+qs_1+(q-1)(s_2+s_3)+\frac{q-1}{2}(s_2+s_3) + \frac{q-1}{2}(s_2+s_3) + 2s_3 + 2s_2 + 2s_1 + 2s_3 \\
&(0, +, [p, \ldots, p], \{(p, \ldots, p)\})
\end{align*}
\]

Both $\bar{\Lambda}_1$ and $\bar{\Lambda}_3$ must have proper periods because otherwise all parameters in the signature of $\Lambda$ except $\epsilon_3$ are 0 and then $\Lambda$ is a spherical group. Therefore $\Sigma_1$ is $-1$ if $s_3 + \epsilon_1 + \epsilon_3 > 0$, $\Sigma_1$ is $+p,+1$ if $s_3 + \epsilon_1 + \epsilon_3 = 0$ and $\langle \varphi, \sigma_1 \rangle = C_2p$. $\Sigma_2$ is $-1$ if $s_1 + \epsilon_1 + \epsilon_2 > 0$ and $r + s_2 + s_3 + \epsilon_3 > 0$, $\Sigma_2$ is $+1$ if $s_1 + \epsilon_1 + \epsilon_2 > 0$ and $r = s_2 = s_3 = \epsilon_3 = 0$, and finally $\Sigma_2$ is $+p,+1$ if $s_1 + \epsilon_1 + \epsilon_2 = 0$ and $\langle \varphi, \sigma_2 \rangle = C_2p$. Finally $\Sigma_3$ is $-1$ if $s_2 + \epsilon_2 + \epsilon_3 > 0$, $\Sigma_3$ is $+p,+1$ if $s_2 + \epsilon_2 + \epsilon_3 = 0$ and $\langle \varphi, \sigma_3 \rangle = C_2p$. In all cases $\Sigma_\sigma \neq +p$ since $\langle \varphi, \sigma \rangle = D_p$. Again $\Sigma_\sigma \neq +1$ if $\sigma$ is conjugate to $\sigma_1$ or $\sigma_3$. No further restrictions exist.
and (5) $\overline{G} = \langle \sigma_1, \sigma_2, \sigma_3 | \sigma_1^2, (\sigma_1 \sigma_2)^2, (\sigma_2 \sigma_3)^3, (\sigma_3 \sigma_1)^q \rangle$, where $q = 3$ in (5) and $q = 5$ in (8). $\overline{G}$, and thus $G$, contains two conjugacy classes of symmetries, with representatives namely $\sigma = \sigma_1$ and $\tau$, with $\tau$ conjugated to the antipodal map. Then $\Sigma_\sigma = \Sigma_{\sigma_1}$ and $\Sigma_\tau = 0$. As in case (2a), by [10] and [14], given the epimorphism $\overline{\theta}$, all the generating reflections of $\Lambda$ induce reflections in $\overline{\theta}^{-1}(\langle \sigma_1 \rangle)$. So $\Sigma_\sigma = -1$ as they induce also proper periods.

(6) $\overline{G} = A_4 \times C_2 = \langle p, \sigma_1, \sigma_2 | p^3, \sigma_1^2, \sigma_2^2, (\sigma_1 \sigma_2)^2, (\sigma_1 \sigma_3)^3, (\sigma_2 \sigma_3)^3 \rangle$, where $C_2$ is generated by the antipodal map. With the same arguments as in (3) we obtain that $G$ has two types of symmetries with representatives $\sigma$ and $\tau$ where $\Sigma_\sigma = 0$ and $\Sigma_\tau = -1$.

(7) $\overline{G} = \langle \sigma_1, \sigma_2, \sigma_3 | \sigma_1^2, (\sigma_1 \sigma_2)^2, (\sigma_2 \sigma_3)^3, (\sigma_3 \sigma_1)^4 \rangle = S_4 \times C_2$. This case is as case (4b) where the central symmetry is conjugated to the antipodal map and $\overline{G}$ is conjugated to $\overline{\sigma_2}$. There are 3 conjugacy classes of symmetries with species $0$, $\Omega_1 = \Sigma_{\sigma_1}$ and $\Omega_2 = \Sigma_{\sigma_2}$. Now $\Omega_1$ is $-1$ if $s_3 + e_1 + e_3 > 0$, $\Omega_1$ is $+p, +1$ if $s_3 + e_1 + e_3 = 0$ and $\langle \varphi, \sigma_3 \rangle$ is $C_{2p}$. On the other hand $\Omega_2$ is $-1$ if $s_1 + s_2 + e_1 + e_2 + e_3 = 0$, $\Omega_2$ is $+p$ if $s_1 + s_2 + e_1 + e_2 + e_3 = 0$ and $\langle \varphi, \sigma_2 \rangle = C_{2p}$. $\Omega_\sigma \neq +p, +1$, since $\langle \varphi, \sigma \rangle = D_p$ and $\sigma$ has fixed points.

To finish we show the existence of surfaces with the desired symmetries by listing appropriate groups $G$ and epimorphisms $\theta$. The $p$-gonal surfaces with the desired symmetries will be uniformized by the groups Ker($\theta$). We distinguish the same eight cases as in Theorem 2.

(1) Let $G = \langle \varphi, \rho, \sigma | \varphi^i, \rho^q, \sigma^2, (\varphi \sigma)^2, (\varphi \rho)^q, \rho^{-1} \sigma \rho \sigma \rangle$ and let $\theta : \Lambda \to G$ be defined by $\theta(x_i) = \varphi^v_i$, $1 \leq i \leq r$, $\theta(x_{r+1}) = \rho \varphi^{u+1} \varphi^{-1}$, $\theta(c_{j-1}) = \sigma$, $\theta(c_{2j}) = \varphi \sigma$, $\theta(e) = \rho^{-1} \varphi^{l}$, where $j_1 + \cdots + j_r + \epsilon + v_{r+1} + l \equiv 0 \text{ mod } p$.

(2) Let $G = \langle \varphi, \tau, \sigma | \varphi^p, \tau^2, (\varphi \sigma)^2, (\varphi \sigma \tau)^2 \rangle$. Let $\theta : \Lambda \to G$ be defined by $\theta(x_i) = \varphi^v_i$, $1 \leq i \leq r$, $\theta(x_{r+1}) = \rho \varphi^{u+1} \varphi^{-1}$, $\theta(c_0) = \tau$, $\theta(c_j) = \sigma \varphi^{u_j}$ for $1 \leq j \leq s_1 + 1$, where $u_1 = e_1$ and $u_j = 1 - u_{j-1}$, $\theta(c_j) = \varphi^{u_j}$, with $s_1 + 2 \leq j \leq s_1 + s_2 + 2$ where $u_{s_1 + 2} = e_2 + 1 - u_{s_1 + 1}$ and $u_j = 1 - u_{j-1}$, $\theta(e) = \varphi^l$, where $v_1 + \cdots + v_r + l \equiv 0 \text{ mod } p$.

To obtain the species $\Sigma_\tau = +p, +1$ we consider groups $G$ with presentation $G = \langle \varphi, \tau, \sigma | \varphi^p, \tau^2, (\varphi \sigma)^2, (\varphi \sigma \tau)^2 \rangle$.

(3) Let $G = \langle \varphi, \rho, \sigma | \varphi^p, \rho^q, \sigma^2, \rho \sigma \rho \tau, (\varphi \rho)^2 (\varphi \sigma)^2, (\varphi \sigma \tau)^2 \rangle$. Let $\theta : \Lambda \to G$ be defined by $\theta(x_i) = \varphi^v_i$, $1 \leq i \leq r$, $\theta(x_{r+1}) = \rho \varphi^{u+1} \varphi^{-1}$, $\theta(c_0) = \tau$, $\theta(c_j) = \sigma \varphi^{u_j}$ for $1 \leq j \leq s_1 + 1$, where $u_1 = e_1$ and $u_j = 1 - u_{j-1}$, $\theta(e) = \rho \varphi^l$, where $v_1 + \cdots + v_{r+1} + e_1 + \epsilon + l \equiv 0 \text{ mod } p$.

(4) Let $G = \langle \varphi, \sigma_1, \sigma_2, \sigma_3 | \sigma_1^2, (\sigma_1 \sigma_2)^2, (\sigma_2 \sigma_3)^2, (\sigma_3 \sigma_1)^q, \varphi^p, (\varphi \sigma_1)^2 \rangle$. Let $\theta : \Lambda \to G$ be defined by $\theta(x_i) = \varphi^v_i$, $1 \leq i \leq r$, $\theta(c_0) = \sigma_1$, $\theta(c_j) = \sigma_2 \varphi^{u_j}$ for $1 \leq j \leq s_1 + 1$, where $u_1 = e_1$ and $u_j = 1 - u_{j-1}$, $\theta(c_j) = \sigma_1 \varphi^{u_j}$, with $s_1 + 2 \leq j \leq s_1 + s_2 + 2$ where $u_{s_1 + 2} = e_2 + 1 - u_{s_1 + 1}$ and $u_j = 1 - u_{j-1}$, $\theta(c_j) = \sigma_1 \varphi^{u_j}$, with $s_1 + s_2 + 3 \leq j \leq s_1 + s_2 + s_3 + 3$ where $u_{s_1 + s_2 + 3} =$
\( \epsilon_3 + 1 - u_{s_1 + s_2 + 2} \) and \( u_j = 1 - u_{j-1} \), \( \theta(e) = \varphi^j \), where \( v_1 + \cdots + v_r + l \equiv 0 \mod p \).

To obtain the species \(+p, +1\) one or two of the relations \((\varphi \sigma_i)^2\) in the presentation of \(G\) must be substituted by relations \(\varphi^{-1} \sigma_i \varphi \sigma_i\).

(5) and (8) \( G = \langle \varphi, \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^2, (\sigma_1 \sigma_2)^2, (\sigma_2 \sigma_3)^3, (\sigma_3 \sigma_1)^q, \varphi^p, (\varphi \sigma_i)^2 \rangle, \)

where \( q = 3 \) in (5) and \( q = 5 \) in (8) and let \( \theta : \Lambda \to G \) be defined as in (4).

(6) \( G = \langle \varphi, \rho, \sigma_1, \sigma_2 \mid \varphi^p, \rho^3, \sigma_1^2, \sigma_2^2, (\sigma_1 \sigma_2)^2, \rho^2 \sigma_1 \rho \sigma_2, (\varphi \sigma_1)^2, (\varphi \sigma_2)^2, (\varphi \rho)^{3q} \rangle. \)

Let \( \theta : \Lambda \to G \) be defined as \( \theta(x_i) = \varphi^{u_i} \), \( 1 \leq i \leq r \), \( \theta(x_{r+1}) = \rho \varphi^{u_0 + v_r + 1} \), \( \theta(c_0) = \sigma_1, \theta(c_j) = \sigma_2 \varphi^{v_j} \) for \( 1 \leq j \leq s + 1 \), where \( u_1 = \epsilon_2 \) and \( u_j = 1 - u_{j-1}, \theta(e) = \rho^2 \varphi^j \), where \( v_1 + \cdots + v_r + 1 + \epsilon_1 + l \equiv 0 \mod p \).

(7) \( G = \langle \varphi \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^2, (\sigma_1 \sigma_2)^2, (\sigma_2 \sigma_3)^3, (\sigma_3 \sigma_1)^4, \varphi^p, (\varphi \sigma_1)^2 \rangle \) and let \( \theta : \Lambda \to G \) be defined as in (4). To obtain the species \(+p, +1\) either the relations \((\varphi \sigma_1)^2\) and \((\varphi \sigma_2)^2\) or the relation \((\varphi \sigma_2)^2\) in the presentation of \(G\) must be changed to the corresponding commuting relation.

The kernels of the above epimorphisms will uniformise surfaces with a symmetry with species \(-1\) for general groups \(\Lambda\). The same epimorphisms yield the species \(+1\) in cases 1 and 4 under the restrictions on \(\Lambda\) given in the first part of the theorem. Again, the same epimorphisms yield the species \(+p, +1\) under the corresponding restrictions on \(\Lambda\) given in the first part of the theorem.

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References


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