THE FINITE SIMPLE GROUPS HAVE COMPLEMENTED SUBGROUP LATTICES

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We prove that the lattice of subgroups of every finite simple group is a complemented lattice.

1. Introduction.

A group $G$ is called a $K$-group (a complemented group) if its subgroup lattice is a complemented lattice, i.e., for a given $H \leq G$ there exists a $X \leq G$ such that $\langle H, X \rangle = G$ and $H \wedge X = 1$. The main purpose of this Note is to answer a long-standing open question in finite group theory, by proving that:

Every finite simple group is a $K$-group.

In this context, it was known that the alternating groups, the projective special linear groups and the Suzuki groups are $K$-groups ([P]).

Our proof relies on the FSGC-theorem and on structural properties of the maximal subgroups in finite simple groups. The rest of this paper is divided into four sections. In Section 2 we collect some criteria for a subgroup of a group $G$ to have a complement and recall some useful known results. In Section 3 we deal with the classical groups, in 4 with the exceptional groups of Lie type and in Section 5 with the sporadic groups.

With reference to notation and terminology, we shall follow closely those in use in [P] and [S]. All groups are meant to be finite.

2. Preliminaries.

We begin with the following:

Proposition 2.1. Given the group $G$, let $T$, $X$ be subgroups of $G$ such that $T \leq X < G$. If the interval $[X/T]$ is a complemented lattice and if $X$ is contained in only one maximal subgroup $M$ of $G$, then every $H \leq G$ with $H \nleq M$ and $H \wedge T = 1$ has a complement in $G$.

Proof. Let $C$ be a complement of $\langle H, T \rangle \wedge X$ in $[X/T]$. Then $\langle H, C \rangle = \langle H, T, C \rangle \geq \langle \langle H, T \rangle \wedge X, C \rangle = X$. Since $H \nleq M$, we conclude that $\langle H, C \rangle = G$. Moreover $H \wedge C = H \wedge X \wedge C \leq \langle H, T \rangle \wedge X \wedge C = T$, hence $H \wedge C \leq H \wedge T = 1$. 

$\square$
The condition on \( M \) in Proposition 2.1 means that \([G/X]\) is a mono-coatomic interval with coatom \( M \).

**Corollary 2.2.** Let \( X \) be a \( K \)-subgroup and \([G/X]\) a mono-coatomic interval with coatom \( M \). Then every \( H \leq G \) not contained in \( M \)G has a complement in \( G \). In particular \( G/M \)G is a \( K \)-group.

**Proof.** There exists a \( g \in G \) such that \( H^g \not\leq M \). By Proposition 2.1 with \( T = 1 \), \( H^g \) has a complement. Hence also \( H \) has a complement \( C \) in \( G \). Moreover, if \( M_G < H \), then \( CM_G/M_G \) is a complement of \( H/M_G \) in \( G/M_G \).

**Proposition 2.3.** Let \( G \) be a simple group and \([G/X]\) a mono-coatomic interval with coatom \( M \). If \( N \) is a central subgroup of \( M \) of prime order with \( N \leq X \) and if \( X/N \) is a \( K \)-group, then \( G \) is a \( K \)-group.

**Proof.** Let \( H \) be a proper subgroup of \( G \). Since \( M_G = 1 \), without loss of generality we may assume \( H \not\leq M \). If now \( H \wedge N = 1 \), by Proposition 2.1 \( H \) has a complement in \( G \). Assume now \( N \leq H \); there exists a \( g \in G \) such that \( N^g \wedge H = 1 \). So if \( H \) has no complement in \( G \), by Proposition 2.1 we must have \( N^g \leq C(H) \). It follows that if \( F = \{ N^x \mid x \in G \} \) and \( F_1 = \{ N^x \mid N^x \not\leq H \} \), then \( N(H) \geq \langle H, F_1 \rangle \geq \langle F \rangle = G \), a contradiction.

We finally recall:

1. (2.1) The direct product of a family of groups is a \( K \)-group if and only if each factor is a \( K \)-group, see Corollary 3.1.5 in \([S]\).
2. (2.2) If \( G \) contains an abelian subgroup \( A \) generated by minimal normal subgroups of \( G \) and a complement \( K \) to \( A \) that is a \( K \)-group, then \( G \) is a \( K \)-group, see Lemma 3.1.9 in \([S]\).
3. (2.3) The symmetric and alternating groups, the projective special linear groups \( L_n(q) \) and the simple Suzuki groups \( ^2B_2(q) \) are \( K \)-groups, see \([P]\).

For our purpose it will be convenient to know which non-simple groups of Lie type \([C]\), p. 175, p. 268) are complemented.

**Proposition 2.4.** The following non-simple groups of Lie type are \( K \)-groups:

\( L_2(2), \ L_2(3), \ Sp_4(2), \ G_2(2), \ ^2G_2(3) \).

The following non-simple groups of Lie type are not \( K \)-groups:

\( ^2B_2(2), \ ^2F_4(2), \ U_3(2) \).

**Proof.** In fact \( L_2(2) \cong S_3, \ L_2(3) \cong A_4, \ Sp_4(2) \cong S_6 \), and we are done by (2.3). In \( G_2(2) \) there is a mono-coatomic interval \([G_2(2)/H]\) with \( H \cong L_3(2) \) and corefree coatom, by Theorem 2.5 in \([Co]\): Hence \( G_2(2) \) is a \( K \)-group by
The group $^2G_2(3)$ has a corefree maximal subgroup isomorphic to $Z_7 : Z_6$ ([K3]): Hence it is a $K$-group by (2.2). On the other hand, we have $^2B_2(2) \cong Z_5 : Z_4$ ([A]), $U_3(2) \cong 3^2 : Q_8$ ([KL], p. 43) and finally $| ^2F_4(2) : ^2F_4(2)' | = 2$, but all involutions of $^2F_4(2)$ are contained in $^2F_4(2)'$ ([AS], p. 75).

To prove the main theorem, we take a counterexample $L$ of minimal order and show that such a group $L$ does not exist.

3. The simple classical groups.

We are going to assume in this section that $L = G_0(n, q)$, a (simple) classical group as in [KL].

a) $G_0(n, q)$ is not of type $A_m$, $n = m + 1$, $m \geq 1$.

See (2.3).

b) $G_0(n, q)$ is not of type $C_m$, $n = 2m$, $m \geq 2$.

Proof. Let $r$ be a prime divisor of $m$, so that $m = rt$, $t \geq 1$. By Theorem 1 and Theorem 2 in [L], the interval $[PSp(2m, q)/PSp(2t, q^r)]$ is mono-coatomic. Moreover $PSp(2t, q^r)$ is simple, since $q^r \geq 4$, of order less than the order of $L$, hence a $K$-group. But then by Corollary 2.2, $L$ is a $K$-group, a contradiction.

d) $G_0(n, q)$ is not of type $B_m$, $n = 2m + 1$, $m \geq 3$, $q$ odd.
Proof. Assume \( q = p^f \), with \( f > 1 \) and let \( r \) be a prime divisor of \( f \). Then by Theorem 1 in [BGL], \( [PO_n(q)/PO_n(q^{1/r})] \) is monoatomic, a contradiction. Therefore we must have \( q = p \). Now, by §5 in [K1] and Proposition 4.2.15 in [KL], \( G_0(n, q) \) contains a maximal subgroup \( M \) which is a split extension of an irreducible elementary abelian 2-group by \( A_n \) or \( S_n \). Therefore \( M \) is a \( K \)-group by (2.2), and \( G_0(n, q) \) is a \( K \)-group, a contradiction. \( \square \)

\[ \text{e)} \quad G_0(n, q) \text{ is not of type } D_m, \quad n = 2m, \quad m \geq 4. \]

Proof. Let \( V = \mathbb{F}_q^n \) be the natural (projective) module for \( G_0(n, q) \), and let \( W \) be a nonsingular subspace of \( V \) of dimension 1. Since \( \Pi := G_0(n, q) \) is a counterexample of minimal order, the socle \( \text{soc} \ H_{\Pi} \) of the stabilizer \( H_{\Pi} \) of \( W \) in \( \Pi \), which is isomorphic to \( \Omega_{n-1}(q) \) if \( q \) is odd, and to \( Sp_{n-2}(q) \) if \( q \) is even, must be contained, by Corollary 2.2, in an element \( K_{\Pi} \) of \( \mathcal{C}(\Pi) \cup S \) different from \( H_{\Pi} \) (for the definition of the family \( \mathcal{C}(\Pi) \cup S \) we refer to §1.1 and §3.1 in [KL]).

By order considerations, one can prove that only condition (i) of Theorem 4.2 in [Li] applies: This means that \( K_{\Pi} \) must be an element of \( \mathcal{C}(\Pi) \). Since \( H_{\Pi} \in \mathcal{C}_1 \), one is left to show that there does not exist an element \( K_{\Pi} \) in \( \mathcal{C}_i \), for an \( i \neq 1 \), such that \( \text{soc} \ H_{\Pi} < K_{\Pi} < \Omega_{\Pi} \).

For \( q \) odd, the arguments used in the proof of Proposition 7.1.3 in [KL] show that such a \( K_{\Pi} \) does not exist, taking into account that in our situation \( n_2 = n - 1 \geq 7 \). To deal with the case when \( q \) is even, again one can proceed using arguments suggested in the proof of Lemma 7.1.4 in [KL]. \( \square \)

f) \( G_0(n, q) \) is not of type \( ^2D_m, \quad n = 2m, \quad m \geq 4. \)

Proof. Following the notation in [BGL], let \( G \) be the simple adjoint algebraic group over \( \mathbb{F}_q \) with associated Dynkin diagram of type \( D_m, \quad \lambda = \sigma_q \) and \( \mu = 2^\sigma_q \). Then \( O^\mu(G_\lambda) = PO_n^+(q), \quad O^\mu(G_\mu) = PO_n^-(q) = G_0(n, q), \)

\[ T := O^\mu(G_\mu \cap G_\lambda) = \begin{cases} \Omega_{n-1}(q) & \text{if } q \text{ is odd} \\ Sp_{n-2}(q) & \text{if } q \text{ is even.} \end{cases} \]

By Theorem 2 in [BGL], \( [G_0(n, q)/T] \) is monocoatomic. Since \( n \geq 8 \), \( T \) is simple, hence \( G_0(n, q) \) is a \( K \)-group, a contradiction. \( \square \)

We have therefore completed the proof that \( L \) is not a classical group.

4. The simple exceptional groups of Lie type.

Now we are going to show that the minimal counterexample \( L \) cannot be an exceptional group of Lie type \( G(q) \).

a) \( G(q) \) is not of type \( G_2, \quad ^2G_2. \)
Proof. If \( r \) is a prime divisor of \( f \), where \( q = p^f \), write \( q = q_0^n \). Then 
\( G(q_0) \leq G(q) \) ([Co], Theorem 2.3, 2.4, [K3], Theorem A, C). Hence by 
Proposition 2.4, we have \( L = G_2(p) \), for an odd prime \( p \). But then \( G_2(2) \) is 
maximal in \( G_2(p) \) by [K3], and we are done by Proposition 2.4. \( \square \)

b) \( G(q) \) is not of type \( F_4 \).

Proof. \( F_4(q) \) contains a quasisimple maximal subgroup \( M \) of type \( B_4(q) \), 
with \( |Z(M)| = (2, q - 1) \) ([LSS], p. 322). But then, by Proposition 2.3, 
\( F_4(q) \) is a \( K \)-group. \( \square \)

c) \( G(q) \) is not of type \( E_6, E_7, E_8 \).

Proof. We have \( F_4(q) \leq E_6(q) \) ([LS], Table 1), which excludes \( E_6 \).

If \( L \) is of type \( E_7 \), there exist subgroups \( H \leq M < \cdot G \) such that \( |M : H| = |Z(H)| = (2, q - 1) \) and \( H/Z(H) \cong L_2(q) \times PΩ^+_{12}(q) \) ([LS], Table 1). Hence 
\( H/Z(H) \) is a \( K \)-group by (2.1). We claim that \( [G/H] \) is monoatomic. 
Clear if \( q \) is even. For \( q \) odd, suppose \( H < M_1 < \cdot G \), with \( M_1 \neq M \). Since 
\( |M : H| = 2 \), we have \( |M_1| > |M| \geq q^{64} \). By the Theorem in [LS], \( M_1 \) 
either is a parabolic subgroup, or it appears in Table 1 in [LS]: However, 
both situations are excluded by rank or order considerations. So again by 
Proposition 2.3, \( G \) is a \( K \)-group, a contradiction.

Finally assume \( G \) is of type \( E_8 \). There exist subgroups \( H \leq M \cdot G \) such that 
\( |M : H| = |Z(H)| = (2, q - 1) \), with \( H/Z(H) \cong PΩ^+_{16}(q) \) ([I], p. 
286, [LS], Table 1), hence a \( K \)-group. Using the Theorem in [LS] again one 
shows that \( [G/H] \) is monoatomic, giving rise to a contradiction. \( \square \)

d) \( G(q) \) is not of type \( 2B_2 \).

See (2.3).

e) \( G(q) \) is not of type \( 2F_4 \).

Proof. The group \( 2F_4(2) \) is not simple, and we have seen that it is not a \( K \)-
group (Proposition 2.4). Its derived subgroup (the Tits group) is simple and 
it is a \( K \)-group, since it has a maximal subgroup isomorphic to \( L_2(25) \) ([A]). 
So now assume \( L = 2F_4(2^{2m+1}) \), with \( m \geq 1 \). By the Main Theorem in [M], 
there exist \( H < M < L \) such that \( |M : H| = 2 \) and \( H \cong Sp_4(2^{2m+1}) \). Since 
the nonabelian composition factors of maximal subgroups of \( L \) not conjugate 
to \( M \) are of type \( A_1(q), 2B_2(q), U_3(q) \) and \( 2F_4(q^{1/r}) \), \( r \) an odd prime, one 
concludes that \( [G/H] \) is monoatomic. \( \square \)

f) \( G(q) \) is not of type \( 2E_6 \).

Proof. In fact we have \( F_4(q) \leq 2E_6(q) \) from Table 1 in [LS]. \( \square \)
g) $G(q)$ is not of type $3^2D_4$.

**Proof.** From the Theorem in [K2], we have $G_2(q) < 3^2D_4(q)$. Since $G_2(q)$ is a $K$-group, we get a contradiction.

This concludes the proof that $L$ is not a group of Lie type.

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5. Sporadic simple groups.

We are left to deal with the sporadic groups: To this end, for each group we exhibit a maximal subgroup which is a $K$-group. From the tables in [A] we have:

$L_2(11) < M_{11}$, $L_2(11) < M_{12}$, $A_7 < M_{22}$, $M_{22} < M_{23}$, $M_{23} < M_{24}$, $L_2(11) < J_1$, $A_5 < J_2$, $L_2(19) < J_3$, $43 : 14 < J_4$, $M_{22} < HS$, $A_7 < Suz$, $M_{22} < McL$, $A_8 < Ra$, $S_4 \times L_3(2) < He$, $67 : 22 < L_9$, $A_7 < O'N$, $M_{23} < Co_2$, $M_{23} < Co_3$, $Co_3 < Co_1$, $S_{10} < Fi_{22}$, $S_{12} < Fi_{23}$, $Fi_{23} < Fi_{24}$, $A_{12} < HN$, $S_5 < Th$, $31 : 15 < BM$, $31 : 15 \times S_3 < M$.

We have thus completed the proof of the main theorem:

**Theorem.** Every finite simple group is a $K$-group.

**Acknowledgements.** We are grateful to E. Vdovin for helpful discussions.

**References**


Finite simple groups are K-groups


Received December 24, 2002 and revised February 24, 2003.