SEPARATION OF GLOBAL SEMIANALYTIC SUBSETS OF 2-DIMENSIONAL ANALYTIC MANIFOLDS

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In this paper we prove that two global semianalytic subsets of a real analytic manifold of dimension two are separable if and only if there is no analytic component of the Zariski closure of the boundary which intersects the interior of one of the two sets and they are separable in a neighbourhood of each singular point of the boundary.

We show also that, unlike in the algebraic case, the obstructions at infinity are not relevant.

Introduction.

This paper is mainly concerned with the problem of separation for a special class of semianalytic subsets of an analytic surface $M$, namely the global semianalytic sets, i.e., semianalytic subsets admitting a description of the type

$$S = \bigcup_{i=1}^{p} \{ x \in M | f_i(x) = 0, g_{i1}(x) > 0, \ldots, g_{ik_i}(x) > 0 \}.$$ 

Two such semianalytic subsets $A$ and $B$ are said to be separable if there exists an analytic function $f \in \mathcal{O}(M)$ such that $f(A) > 0$ and $f(B) < 0$.

If there exists a nonzero analytic function $f \in \mathcal{O}(M)$ such that $f(A) \geq 0$ and $f(B) \leq 0$, $A$ and $B$ are said to be generically separable, equivalently $A$ and $B$ are generically separable if there exists a proper global analytic set $Y \subset M$ such that $A \setminus Y$ and $B \setminus Y$ are separable.

Of course $A$ and $B$ cannot be separated if $A \cap B \neq \emptyset$, exactly as they cannot be generically separated if $\hat{A} \cap \hat{B} \neq \emptyset$.

As in the algebraic case, it is easy to realize that two open sets, even if they are disjoint, are in general not separable, as for instance the open sets as in Figure 1.

The separation problem makes sense also for constructible subsets of the real spectrum of a ring and this problem has been solved by Bröcker in terms of finite spaces of orderings (see [ABR96]).

In the algebraic setting, in view of the Artin-Lang property, which acts as a translator between semialgebraic sets and constructible sets in the real
spectrum of the ring of regular functions, this result makes it possible to characterize completely the geometric obstructions to separation.

Unfortunately the Artin-Lang property does not hold in general for the ring $\mathcal{O}(M)$ of global analytic functions (see [AB90]) and it has been proved only for the field of meromorphic functions on an analytic manifold of dimension 2 (see [Cas94a]).

One among the reasons is the presence in $\text{Spec}_{\mathbb{C}} \mathcal{O}(M)$ of the so-called unbounded orderings, whose associated ultrafilter does not converge to a point.

In this paper we prove that these orderings (at least in dimension 2) have no role in such type of problems: In fact we prove that $A$ and $B$ can be separated in a surface $M$ if and only if they are separated in any compact set: This is essentially because of the fact that we can use in this setting Whitney approximation theorem.

This is one of main differences between the algebraic and the analytic cases: In fact it is easy to see that the last statement does not hold for semialgebraic sets, for instance, if we consider $A$ and $B$ as in Figure 2.

We handle the separation problem in a rather direct way; we find (see Theorem 2.3 and Theorem 4.2) that the obstructions lie in the boundary of
the sets $A$ and $B$ and that it is possible to list them in a similar way as in the algebraic case. Each one produces an obstruction for the separation of the associated constructible sets in the real spectrum which does not involve unbounded orderings.

The same methods apply to the basicness problem.

1. Generic equations for global analytic sets of codimension one.

The aim of this section is to prove the following theorem:

**Theorem 1.1.** Let $M \subset \mathbb{R}^n$ be a connected real analytic manifold and let $Y \subset M$ be a global analytic set such that its irreducible components have all codimension one. Then, there is a global analytic set $Y' \subset M$ such that the ideal $\mathcal{I}(Y \cup Y') = \{ f \in \mathcal{O}(M) | f|_{Y \cup Y'} = 0 \}$ is principal. Moreover we can assume $Y'$ to be smooth and $\dim Y \cap Y' < \dim Y$.

Before proving the theorem we recall that a global analytic set $Y$ admits coherent structures and admits complexifications, i.e., there exists a coherent ideal sheaf $\mathcal{F} \subset \mathcal{O}_M$ such that $Y = \text{Supp} \mathcal{O}_M / \mathcal{F}$ and there exists a complex analytic space $\tilde{Y}$ in a complexification $\tilde{M}$ of $M$ (in the sense of [Tog67], being $M$ a manifold) such that $\tilde{Y} \cap M = Y$; moreover this three properties (being global, admitting a coherent structure, being the real part of a complex analytic set) are equivalent.

One can prove that among those coherent sheaves there is a largest one, say still $\mathcal{F}$; also among those complex analytic sets there is a smallest one, say still $\tilde{Y}$. Moreover $\mathcal{I}_Y = \mathcal{F} \otimes \mathbb{C}$, i.e., they define on $Y$ the same structure, the so called well reduced structure (cf. [ABT75], [Gal76]).

**Lemma 1.2.** Let $\mathcal{I}$ be the sheaf generated by the germs of the elements in $\mathcal{I}(Y) = \{ f \in \mathcal{O}(M) | f|_Y = 0 \}$. $\mathcal{I}$ is a locally principal coherent sheaf of ideals.

**Proof.** The fact that $\mathcal{I}$ is coherent is known, cf. [Fri67], we want to prove that it defines on $Y$ the well reduced structure. Indeed, let $\mathcal{G} \subset \mathcal{O}_M$ be a coherent ideal sheaf such that $Y = \text{Supp} \mathcal{O}_M / \mathcal{G}$. By Cartan’s Theorem A, the stalk of $\mathcal{G}$ in each point is generated by global sections. Each $f \in \Gamma(M, \mathcal{G})$ vanishes on $Y$, and then $\mathcal{G} \subset \mathcal{I}$ as wanted.

So, we can find complexifications $\tilde{M}$, $\tilde{Y}$ of $M$ and $Y$ such that $\tilde{Y}$ has pure codimension one in $\tilde{M}$ and $\mathcal{I}_{\tilde{Y}} = \mathcal{I} \otimes \mathbb{C}$. Hence $\mathcal{I} \otimes \mathbb{C}$ is locally principal and this implies the same for $\mathcal{I}$. □

Let’s recall a general result for analytic vector bundles.

**Lemma 1.3.** Let $\xi = (E, \pi, M)$ be an analytic vector bundle of rank $k$ and let $\sigma : M \to E$ be a $C^\infty$ section. Consider $C^\infty(M, E)$ endowed with the Whitney topology. Then each neighbourhood $U$ of $\sigma$ in $C^\infty(M, E)$ contains a global analytic section.
Proof. Arguing as in [Tog80] we get finitely many global analytic sections, \( g^1, \ldots, g^N \), such that \( \sigma = \alpha_1 g^1 + \cdots + \alpha_N g^N \) where \( \alpha_1, \ldots, \alpha_N \in C^\infty(M, \mathbb{R}) \). Then, it is possible to find neighbourhoods \( U_i \) of \( \alpha_i \), \( i = 1, \ldots, N \), such that \( \beta_1 g^1 + \cdots + \beta_N g^N \in U \) for each \( \beta_i \in U_i \). Since \( \mathcal{O}(M) \) is dense in \( C^\infty(M, \mathbb{R}) \) (cf. [Hir94]) we can choose \( \beta_1, \ldots, \beta_N \) analytic and this proves the claim.

Proof of Theorem 1.1. By Lemma 1.2 it follows that there exist a locally finite open covering \( \{U_i\}_{i \in I} \) of \( M \) and \( f_i \in \mathcal{O}(U_i) \) such that \( \mathcal{I}_x = f_i \mathcal{O}_x \) for each \( x \in U_i \); then, \( f_i / f_j \in \mathcal{O}^*(U_i \cap U_j) \) for each \( i, j \) such that \( U_i \cap U_j \neq \emptyset \).

So, we can construct an analytic line bundle having \( \{U_i\}_{i \in I} \) as its neighbourhoods of trivialization and \( \{f_j / f_i\} \) as its transition functions. Observe that a global analytic section of this vector bundle is given by a collection \( \{g_i\}_{i \in I} \) of \( \mathcal{O}^*(U_i \cap U_j) \) for each \( i, j \) such that \( U_i \cap U_j \neq \emptyset \).

Note that the functions \( \{f_i\} \) give a global analytic section \( f \) such that \( Y = f^{-1}(0) = \cup_i V(f_i) \). Let \( \sigma = \{\sigma_i : U_i \to \mathbb{R}\}_{i \in I} \) be a \( C^\infty \) section transverse to the zero section and to the regular part of \( Y \). Such a section exists by general theory of \( C^\infty \) vector bundle, see for instance [Hir94].

By Lemma 1.3 we can approximate \( \sigma \) with a global analytic section \( g = \{g_i\}_{i \in I} \) with the same properties. This means that \( Y' = \cup_i V(g_i) \) is a smooth (global) analytic hypersurface of \( M \) and it intersects \( Y \) in codimension bigger than one. For each \( x \in Y' \cap U_i \), \( g_x \) is an irreducible germ such that \( \mathcal{J}(Y')_x = (g_x) \mathcal{O}_x \) and \( f_ix, g_x \) are coprime. We want to prove that \( Y' \) is the global analytic set we were looking for.

By taking on each open set \( U_i \) the function \( h_i = f_i g_i \), we have a section of an other line bundle whose transition functions are \( (f_j / f_i)^2 \). Now, the exponential map and the associated usual exact sequence

\[
0 \to \mathcal{O}_M \to \mathcal{O}_M' \to \mathcal{O}_M' / \mathcal{O}_M^+ \to \mathbb{Z}_2 \to 0,
\]

induce an isomorphism between \( H^1(M, \mathbb{Z}_2) \) and \( H^1(M, \mathcal{O}_M^*) \). Under this isomorphism the image of a line bundle is the cocycle of the signs of its transition functions, so, this cocycle is trivial and hence the line bundle is trivial. This means that there exists a zero cocycle \( \{\lambda_i\} \in H^0(M, \mathcal{O}_M^*) \) such that \( (f_j / f_i)^2 = \lambda_j^{-1} \lambda_i \). Then the collection of functions defined on \( U_i \) by \( f_i g_i \lambda_i \) defines a global analytic function \( h \) on \( M \) because \( f_i g_i \lambda_i |_{U_i \cap U_j} = f_j g_j \lambda_j |_{U_i \cap U_j} \). Moreover \( h \) generates \( \mathcal{J}(Y \cup Y') \). Indeed, let \( F \in \mathcal{O}(M) \) be in \( \mathcal{J}(Y \cup Y') \), Lemma 1.2 implies that \( f_x | F_x \) for each \( x \in U_i \) and by construction it follows that \( g_x | F_x \) for each \( x \in M \); since \( f_x \) and \( g_x \) are coprime, \( F \in (h) \mathcal{O}(M) \) as wanted.

Note that Theorem 1.1 proves that for each global analytic subset \( Y \subset M \) of codimension one the ideal \( \mathcal{J}(Y) \) is "generically" principal, i.e., there exists
a global analytic function, \( h \), vanishing on \( Y \), such that \( h\mathcal{O}_x = \mathcal{J}_x \) outside a global analytic set of dimension strictly lower, that is \( Y \cap Y' \).

Note also that, if \( H^1(M, \mathbb{Z}_2) = 0 \), the ideal \( \mathcal{J}(Y) \) is a principal ideal in \( \mathcal{O}(M) \).

As immediate corollaries we have the following propositions:

**Proposition 1.4.** Let \( Y \subset M \) be a global analytic set such that its irreducible components have all codimension one and let \( D \subset M \) be a discrete set of points. Then there exists an analytic hypersurface \( Y' \subset M \) such that \( Y' \cap D = \emptyset \) and \( \mathcal{J}(Y \cup Y') \subset \mathcal{O}(M) \) is principal.

**Proof.** Choose a \( C^\infty \) section \( \sigma \) such that \( \sigma(x) \neq 0 \) for each \( x \in D \), by Lemma 1.3 we can approximate it by an analytic section that still has this property. \( \square \)

**Proposition 1.5.** If \( Y \) is an irreducible global analytic set of codimension 1 then, the ring \( \mathcal{O}(M)\mathcal{J}(Y) \) is a rank one discrete valuation ring of the field \( \mathcal{M}(M) \).

**Proof.** Let \( Y' \subset M \) be as in Theorem 1.1, and let \( t \in \mathcal{O}(M) \) be a generator for the ideal \( \mathcal{J}(Y \cup Y') \). We want to prove that \( t \) generates the ideal \( \mathcal{J}(Y)\mathcal{O}(M)\mathcal{J}(Y) \) in \( \mathcal{O}(M)\mathcal{J}(Y) \), moreover each \( f/g \in \mathcal{J}(Y)\mathcal{O}(M)\mathcal{J}(Y) \) can be written as \( t^n u \) with \( u \in \mathcal{J}(Y)\mathcal{O}(M)\mathcal{J}(Y) \) and \( n \in \mathbb{N} \).

Indeed, let \( h \in \mathcal{O}(M) \) such that \( h \) vanishes on \( Y' \) and doesn’t vanish on \( Y \). We can write \( fh = ts \) for some \( s \in \mathcal{O}(M) \); then \( \frac{f}{g} = t \frac{s}{gh} \) with \( \frac{s}{gh} \in \mathcal{O}(M)\mathcal{J}(Y) \). If \( s \) doesn’t vanish on \( Y \) the claim holds with \( n = 1 \). Otherwise we can repeat the same arguments getting that \( sh = ts' \) for some \( s' \in \mathcal{O}(M) \), i.e., \( t^2 | fh^2 \). Let’s prove that there is a maximum integer \( n \) such that \( t^n | fh^n \). Indeed, choose \( x \in Y \) such that \( \mathcal{J}(Y_x) = (t_x) \) and \( h(x) \neq 0 \), since \( \mathcal{O}_{M,x} \) is noetherian, there exists a maximum integer \( n_x \) such that \( t_{x}^{n_{x}} | f_{x} \), in particular, being \( h_{x} \) a unit, \( t_{x}^{n_{x}} \) cannot divide \( f_{x}h_{x}^{n} \) for \( n > n_{x} \) and this implies our assertion.

Let \( \frac{f}{g} \in \mathcal{M}(M) \), then \( f = t^{m}f' \) and \( g = t^{m}g' \) where \( f', g' \) are units in \( \mathcal{O}(M)\mathcal{J}(Y) \). Clearly \( \frac{f}{g} \in \mathcal{O}(M)\mathcal{J}(Y) \) if and only if \( n \geq m \) and this proves our claim. \( \square \)

### 2. Separation in a neighbourhood of the boundary.

In this section we will prove some results which make it possible to pass from separation in a neighbourhood of the boundary to global separation.
An essential tool is Lojasiewicz’s inequality. It works for compact global semianalytic sets of any dimension, nevertheless this hypothesis of compactness is not needed in dimension two. The following Theorems 2.1 and 2.2 are proved in [DC99]:

**Theorem 2.1.** Let $M \subset \mathbb{R}^N$ be a two dimensional real connected analytic manifold and let $S \subset M$ be a closed global semianalytic subset. Given $f \in \mathcal{O}(M)$ there exists $h \in \mathcal{O}(M)$ such that:

1. $h \geq 0$, $\overline{V(h)} \cap S = \emptyset$,
2. $h \leq |f|$ on $S$,
3. $\frac{h}{f}$ extended to zero on $V(f) \cap S$ is continuous on $S$.

We shall often use Lojasiewicz’s inequality in the following formulation, compare [BCR87] for the algebraic case.

**Theorem 2.2.** Let $S$ be a closed global semianalytic subset of $M$, $\dim M = 2$, and let $f, g \in \mathcal{O}(M)$, then there exists a nonnegative function $\varepsilon \in \mathcal{O}(M)$ such that:

1. $(f + \varepsilon g)(x)$ has the same sign as $f(x)$, for any $x \in S$,
2. $\overline{V(\varepsilon)} \subseteq \overline{V(f)} \cap S$.

What follows is the main result of this section:

**Theorem 2.3.** Let $M \subset \mathbb{R}^N$ be a real connected analytic manifold, $\dim M = 2$, and let $A, B \subset M$ be closed global semianalytic sets such that $A \cap B \subset X = Y \cup D$ where $X$ is a global analytic set with $Y$ of pure dimension one and $D = \{x_n\}_{n \in \mathbb{N}}$ discrete.

Assume that there exist a neighbourhood $U$ of $Y$ and a global analytic function $f \in \mathcal{O}(M)$ such that

$$f(A \cap U \setminus Y) > 0 \quad f(B \cap U \setminus Y) < 0.$$ 

Moreover assume that for each $n$, the semianalytic set germs $A_n \setminus \{x_n\}$ and $B_n \setminus \{x_n\}$ are separable, i.e., there exist an open neighbourhood $U_n$ of $x_n$ and an analytic function $f_n \in \mathcal{O}(U_n)$ such that

$$f_n(A \cap U \setminus \{x_n\}) > 0 \quad f_n(B \cap U \setminus \{x_n\}) < 0.$$ 

Then there exists $F \in \mathcal{O}(M)$ that separates $A$ from $B$ outside $X$, meaning by this that

$$F(A \setminus X) > 0 \quad F(B \setminus X) < 0.$$ 

The proof will be done in several steps, the aim being to pass from several functions separating $A$ and $B$ in a neighbourhood of some piece of $X$ to a unique global function separating $A$ and $B$ in a neighbourhood of $X$. Then we shall pass from the neighbourhood of $X$ to the whole $M$. 
Lemma 2.4. Let $D = \{x_n\}_{n \in \mathbb{N}}$ be a discrete set of points such that $\dim(A \cap B)_x = 0$ for each $n$. Assume that for each $n$ the semianalytic set germs $A_{x_n} \setminus \{x_n\}$ and $B_{x_n} \setminus \{x_n\}$ are separable. Then, there exists a global analytic function $g \in \mathcal{O}(M)$ which separates $A$ from $B$ in a neighbourhood $V$ of $D$.

Proof. From the hypothesis, for each $n$, there exist an open neighbourhood $U_n$ of $x_n$ and an analytic function $f_n \in \mathcal{O}(U_n)$ such that $f_n(A \cap U_n \setminus \{x_n\}) > 0$ and $f_n(B \cap U_n \setminus \{x_n\}) < 0$.

Then, it follows that, on $A_{x_n} \cup B_{x_n}$, the germ defined by the zero set $V(f_n)$ is contained in $\{x_n\}$, which is the zero set of $\|x - x_n\|^2$. By the local Lojasiewicz inequality, being $A$ and $B$ closed, there exist an even integer $p_n > 0$ and a positive constant $c$ such that

$$|f_n(x)| \geq c\|x - x_n\|^{p_n} \quad \forall x \in A_{x_n} \cup B_{x_n}.$$ 

Up to take a bigger $p_n$ we can suppose $c = 1$. Denote by $m_{x_n}$ the maximal ideal of the local ring $\mathcal{O}_{M,x_n}$. By applying Cartan’s Theorem B we find a global analytic function $g \in \mathcal{O}(M)$, such that for all $x_n \in D$ we have $g - f_n \in m_{x_n}^{2p_n+2}$. Then, there exists an open neighbourhood $U_n$ of $x_n$ such that $g(x)$ has the same sign as $f_n(x)$ for any $x \in (A \cup B) \cap U_n$, so, $g$ separates $A$ from $B$ in a neighbourhood of $D$, as wanted. □

Lemma 2.5. Assume there exist a neighbourhood $W$ of $X$ and a global analytic function $q \in \mathcal{O}(M)$ such that

$$q(A \cap W \setminus X) > 0 \quad q(B \cap W \setminus X) < 0.$$ 

Then there exists a global analytic function $r \in \mathcal{O}(M)$ such that $q + r$ separates $A$ from $B$ outside $X$.

Proof. By hypothesis it follows that $V(q) \cap (A \cup B) \cap W \subset X$. Up to shrink it, we can assume $W$ to be closed. Then, by Theorem 2.1, we can construct a global analytic function $t \in \mathcal{O}(M)$ such that $t \geq 0$, $V(t) = X$ and $t \leq |q|$ on $(A \cup B) \cap W$. We can suppose $t < 1$ on $W$.

Let $V \subset \overline{V} \subset W$ be an open neighbourhood of $X$ in $M$. Since $A \setminus V$ and $B \setminus V$ are closed and disjoint, there exists $\varphi \in C^\infty(M)$ such that $\varphi(A \setminus V) > 0$ and $\varphi(B \setminus V) < 0$. Let $\sigma_1 : M \to \mathbb{R}$ be a $C^\infty$-function with $\sigma_1^{-1}(0) = M \setminus W$, $\sigma_1^{-1}(1) = \overline{V}$ and put $\sigma_2 = 1 - \sigma_1$. Then $\psi = \sigma_1 q + \sigma_2 \varphi$ is $C^\infty$ and $\psi(A \setminus X) > 0$ and $\psi(B \setminus X) < 0$.

Moreover, since $\psi = q$ on $V$ the function $\eta : M \to \mathbb{R}$ defined by

$$\eta(x) = \begin{cases} \frac{\psi(x) - q(x)}{t^2(x)} & \text{if } x \notin V \\ 0 & \text{if } x \in V \end{cases}$$

is $C^\infty$ and we are to approximate it by an analytic function.
Take \( \{K_n\}_n \) a sequence of compact sets in \( M \) such that \( K_0 = \emptyset, K_n \subset \text{Int}(K_{n+1}) \) and \( M = \bigcup_n K_n \).

Note that if \( A \cup B \subset V \) we can take \( r = 0 \), so we can suppose \( (A \cup B) \setminus V \neq \emptyset \).

Since \( M = \bigcup_n K_n \), \( (A \cup B) \setminus V \) intersects some \( K_{n_0} \), and then every \( K_m \) with \( m \geq n_0 \). So, after replacing \( \{K_n\}_n \) by \( \{\bar{K}_n\}_n \), \( \bar{K}_0 = \emptyset \) \( \bar{K}_n = K_{n_0+n-1} \), we can assume \( ((A \cup B) \setminus V) \cap K_{m+1} \neq \emptyset \) for each \( m \).

Set
\[
\begin{align*}
s_{m+1} &= \min\{|\psi(x)| \mid x \in ((A \cup B) \setminus V) \cap K_{m+1}\} \\
t_{m+1} &= \max\{|t^2(x)| \mid x \in (A \cup B) \cap K_{m+1}\}.
\end{align*}
\]

Note that \( s_{m+1} \) and \( t_{m+1} \) are well-defined and strictly positive constants. Let \( \varepsilon_{m+1} \in \mathbb{R} \) be a constant such that
\[
0 < \varepsilon_{m+1} < \min\left(\frac{s_{m+1}}{t_{m+1}}, 1\right).
\]

It is well-defined because \( t_{m+1} > 0 \).

According to Whitney’s approximation theorem, there exists a global analytic function \( r' \) such that, for each \( x \in K_{m+1} \setminus K_m \), \( |\eta(x) - r'(x)| < \varepsilon_{m+1} \).

We want to prove that \( r = r't^2 \) is the global analytic function we looked for.

We begin by showing that the analytic function \( q + r \) separates \( A \) and \( B \) outside \( V \).

For each \( x \in (A \cup B) \setminus V \) we have \( \psi(x) - q(x) = t^2(x)\eta(x) \) and then, if \( x \in K_{m+1} \setminus K_m \), we have
\[
|\psi(x) - (q + r)(x)| = t^2(x)|\eta - r'(x)| < t_{m+1}\varepsilon_{m+1} < s_{m+1}.
\]

So \( q + r \) has the same sign as \( \psi \) on \( (A \cup B) \setminus V \).

Consider now the sign of \( q + r \) on \( V \). By construction, being \( \eta = 0 \) on \( V \), we have, for each \( x \in V \), \( |r'(x)| < 1 \) (1 is bigger than \( \varepsilon_m \) for each \( m \)).

So if \( x \in ((A \cup B) \cap V) \setminus X \) we have the following sequence of inequalities:
\[
|q(x)| - |r'(x)t^2(x)| > |q(x)| - t^2(x) \geq t(x) - t^2(x) > 0.
\]

Note that the first and the last inequalities are strict because \( t(x) \neq 0 \) and \( t(x) < 1 \). It follows that \( q + r \) has the same sign as \( q \) on \( (A \cup B) \cap V \) and this completes the proof. \( \square \)

**Proof of Theorem 2.3.** By Lemma 2.4 there is a function \( g \) that separates \( A \) and \( B \) in a neighbourhood \( U \) of \( D \). Then, by Lemma 2.5 it is enough to glue together the function \( f \) of the statement with \( g \), to get a function \( q \) separating \( A \) and \( B \) in a neighbourhood of \( X \).

Let’s see that, up to shrink them, we can assume that \( U \) and \( V \) are closed global semianalytic sets. Since \( D \) is a discrete set, we can assume \( U \) to be
an union of disjoint balls. Let $U = \bigcup_{i \in I} B_i$ and let $f_i \in \mathcal{O}(M)$ be such that $B_i = \{ f_i > 0 \}$. Then the sheaf $\mathcal{I}_x = \prod_{i \in I} f_i \mathcal{O}_x$ is well-defined and coherent. We want to prove that $\mathcal{I}$ is principal, i.e., there exists a global analytic function $g$ such that $g_x \mathcal{O}_x = \mathcal{I}_x$ for each $x \in M$. Then we get $U = \{ g > 0 \}$ or $U = \{ g < 0 \}$. In order to prove that $\mathcal{I}$ is principal we have to find a locally finite open covering of $M$, $U_i$, and generators $g_i$ for $\mathcal{I}|_{U_i}$, such that $g_i/g_j$ is positive on $U_i \cap U_j$, when $U_i \cap U_j \neq \emptyset$. Choose $U_i$ and $g_i$ in this way: Each $U_i$ intersects at most a ball, say $B_j(i)$; if $U_i$ intersects $B_j(i)$ we take $g_i = f_j(i)$, if it does not intersect any ball and it is contained in one of them, take $g_i = 1$, else choose $g_i = -1$. The same argument holds for $V$ because $V$ can be written as a finite union of sets that are unions of disjoint balls.

Let $h$ be a positive equation for $Y$ such that $\frac{h}{f}$, extended to 0 on $V(f) \cap (A \cup B) \cap U$, is continuous on $(A \cup B) \cap U$. Such $h$ exists by Theorem 2.1. Similarly we can find $t \in \mathcal{O}(M)$, $t \geq 0$ and $V(t) = D$, with the same property with respect to $D$ and $V$. We want to prove that the global analytic function

$$q = th \left( \frac{f}{h} + \frac{g}{t} \right)$$

has the same sign as $\frac{f}{h}$ near $Y$. $ht$ being strictly positive outside $X$, it is enough to prove that $\frac{g}{t} < \frac{f}{h}$ near $Y$. It is true because we can clearly assume $Y \cap D = \emptyset$ so, for each $x_0 \in Y$, $\frac{g}{t}$ is bounded locally at $x_0$, while, by construction, $\lim_{x \to x_0} \left| \frac{f(x)}{h(x)} \right| = +\infty$. Similarly $q$ has the same sign of $g$ near $D$. □

**Remark 2.6.** Note that if $A \cap B = D$ is a discrete set one can remove the hypothesis on the dimension of $M$. Indeed, the thesis follows by Lemma 2.4 which holds without any dimension hypothesis and by Lemma 2.5 that, in this situation, can be proved using only the local Łojasiewicz inequality.

A consequence of these results is the following:

**Proposition 2.7.** Let $A, B$ be closed semianalytic subsets of $M$ such that $\dim A = 1$ and $\dim A \cap B = 0$. Then $A$ and $B$ can be separated outside $A \cap B$.

**Proof.** It is enough to prove that, for each $x \in A \cap B$, there exist a neighbourhood $U^x$ of $x$ and $f_x \in \mathcal{O}(U^x)$ such that $f_x(A \cap U^x \setminus \{x\}) > 0$ and $f_x(B \cap U^x \setminus \{x\}) < 0$ and this is true by [Rui84]. □
3. Generic separation versus separation.

In this section we prove that generic separation and separation are “almost” equivalent in dimension two, the proof uses essentially the same methods as in the algebraic and local analytic cases, cf. [ABF96], [ABR96, Chapter 3].

Obviously, separation implies generic separation but the converse is not true, as it is easily seen by taking for instance the sets in $\mathbb{R}^2$:

$$A = \{(x, y)|0 < x < 1, y > 0\} \cup \{(x, y)|0 < x < 1/2, y = 0\}$$

$$B = \{(x, y)|0 < x < 1, y < 0\} \cup \{(x, y)|1/2 < x < 1, y = 0\}.$$  

$A$ and $B$ are disjoint global semianalytic sets, they are obviously generically separable by the function $f = y$ but any function generically separating $A$ and $B$ must vanish at some points lying in $A \cup B$ and therefore they cannot be separated.

Note that any function $f$ which generically separates two global semianalytic sets, $A$ and $B$, must vanish identically on $A \cap B$ and therefore on $A \cap B^Z$. Then a necessary condition for $A$ and $B$ to be separated is that $A \cap B^Z \cap (A \cup B)$ is empty.

We shall prove that this condition is also sufficient. This result follows from next theorem which shows that, given two generically separable sets, there exists a “minimal” set outside which they are separable.

**Theorem 3.1.** Let $A, B$ be global semianalytic subsets of $M$, $\dim M = 2$. Assume they are generically separable, then, there exists $f \in \mathcal{O}(M)$ such that

$$f(A \setminus \overline{A \cap B}^Z) > 0 \quad f(B \setminus \overline{A \cap B}^Z) < 0.$$  

**Proof.** Suppose that $f \in \mathcal{O}(M)$ separates generically $A$ and $B$, that is, there is a proper analytic subset $W \subset M$ such that $f(A \setminus W) > 0$ and $f(B \setminus W) < 0$. After replacing $W$ by $\{f = 0\} \cap (A \cup B)^Z$, we may assume that $W = \overline{W} \cap (A \cup B)^Z$ and $f$ vanishes on $W$.

We write $W = \overline{A \cap B}^Z \cup W' \cup D$ where $W'$ and $D$ are respectively the union of all the irreducible global components of dimension one and zero not lying in $\overline{A \cap B}^Z$. Suppose first $W' \neq \emptyset$. Then, $\overline{A \cap W'}$ and $\overline{B \cap W'}$ are global semianalytic sets of dimension one which intersect each other in dimension zero. By Proposition 2.7, there exists $h \in \mathcal{O}(M)$ separating them. Consider the closed set $S = (\overline{A \cap \{h \leq 0\}}) \cup (\overline{B \cap \{h \geq 0\}})$. $S$ is global semianalytic because the closure of a global semianalytic set of $M$ is global, ([CA96]). By Theorem 2.2 there exists a nonnegative analytic function $\varepsilon \in \mathcal{O}(M)$ such that $g = f + \varepsilon h$ has the same sign as $f$ on $S$, and with the zero set of $\varepsilon$ contained in $\overline{V(f) \cap S}^Z$. Thus we get that $g$ separates $A \setminus \overline{A \cap B}^Z$ and
$B \setminus \overline{A \cap B}$ up to the discrete set $D$. In order to remove the set $D$ it is enough to apply two more times Theorem 2.2. We split $D$ as $D_1 \cup D_2$ where $D_1 = D \cap A$ and $D_2 = D \cap B$. In order to remove $D_1$ apply Theorem 2.2 to the functions $g$ and $1$ with respect to $B$ to obtain a function $\eta \in \mathcal{O}(M)$ such that $(g + \eta)(A \setminus \overline{A \cap B}) > 0$ and $(g + \eta)(B \setminus (D_2 \cup \overline{A \cap B})) < 0$. A second application of the same theorem to $g + \eta$ and $-1$ removes $D_2$, giving a function $\xi$ such that $g + \eta - \xi$ separates $A$ and $B$ outside $\overline{A \cap B}$.

This last argument can be used also when $W' = \emptyset$. \hfill \Box

**Corollary 3.2.** Let $A, B$ be global semianalytic subsets of $M$, $\dim M = 2$, assume that $A \cap B \cap (A \cup B) = \emptyset$. Then, $A$ and $B$ are separable if and only if they are generically separable. In particular if $A$ and $B$ are open they are separable if and only if they are generically separable.

Proof. Let’s prove the last statement. One implication is trivial, for the other we have to prove that $\overline{A \cap B}$ does not intersect $A \cup B$. Since $A$ and $B$ are open, an analytic component of $\overline{A \cap B}$, say $W$, intersecting one of the two sets has to intersect it in dimension one. This contradicts the hypothesis that $A$ and $B$ are not generically separable. Indeed, let $t \in \mathcal{O}(M)$ be an uniformizer for $\mathcal{O}(M)_{j(W)}$ (cf. Theorem 1.5), then any possible function $f$ generically separating $A$ and $B$ can be written as $f = t^m u$, with $u \notin j(W) \mathcal{O}(M)_{j(W)}$. Since $W \subset \overline{A \cap B}$, $m$ has to be odd, since $W$ intersects the interior of one of the two sets $m$ has to be even. \hfill \Box

4. Separation and walls.

Let $A, B$ be global semianalytic subsets in $M$, $\dim M = 2$, such that $\overline{A \cap B} = \emptyset$.

We recall that the boundary of a global semianalytic subset $S \subset M$, $\partial S = \overline{S} \setminus \mathring{S}$, is a semianalytic set of dimension $\leq 1$ contained in the zero set of the product of the functions appearing in any description of $S$. Therefore, $\partial S$ is global by \cite{Cas94b}.

Set $Y$ the Zariski closure of $\partial A \cup \partial B$. Note that $\partial A \cup \partial B$ being global, $Y$ is a proper analytic subset of $M$.

**Definition 4.1.** We will call a wall any irreducible component of $Y$ of dimension one. We say that a wall $W$ is odd if there is a 1-dimensional subset $W' \subset W$ which is contained in $\overline{A \cap B}$. We say that a wall $W$ is even if there is a 1-dimensional subset $W' \subset W$ which is contained in $\overline{A}$ or $\overline{B}$.
Note that walls may be neither odd nor even, also they can be both odd and even, as for instance in the proof of Corollary 3.2.

For any odd (resp. even) wall $W$, let $t \in \mathcal{O}(M)$ be an uniformizer for $\mathcal{O}(M)\mathfrak{j}(W)$, then any possible function $f$ generically separating $A$ and $B$ can be written as $f = t^m u$, with $u \notin \mathfrak{j}(W)\mathcal{O}(M)\mathfrak{j}(W)$ and $m$ odd (resp. even, possibly zero). It is clear that the parity of $m$ doesn’t depend on the choice of the generator.

In the right-hand of Figure 1 there is an example of an odd and even wall. Obviously, if some wall $W$ is simultaneously odd and even $A$ and $B$ can not be generically separable, we want to show that the converse is “almost” true.

**Theorem 4.2.** Let $A, B$ be open global disjoint semianalytic subsets of $M$, $\dim M = 2$, set $Y = \partial A^Z \cup \partial B^Z$ and $Z = A \cap B^Z$. Then $A$ and $B$ can be separated if and only if the following conditions hold:

1. No wall of $A$ and $B$ is simultaneously odd and even.
2. For every $x \in \text{Sing } Y$ the semianalytic set germs $A_x$ and $B_x$ are (generically) separated.

Denote by $Y^c$ the union of odd walls of $Y$. Assuming Conditions 1 and 2 of Theorem 4.2 let’s prove the following:

**Lemma 4.3.** Let $X \subset M$ be an analytic set such that $\mathfrak{j}(X \cup Y^c)$ is principal, $X \cap \text{Sing } Y$ is empty, $X \cap Y$ is discrete. Let $g \in \mathcal{O}(M)$ be a generator for $\mathfrak{j}(Y^c \cup X)$ and denote by $A^g$ and $B^g$ the sets

$$
A^g = \{ x \in A | g(x) > 0 \} \cup \{ x \in B | g(x) < 0 \}
$$

$$
B^g = \{ x \in A | g(x) < 0 \} \cup \{ x \in B | g(x) > 0 \}.
$$

Set $Y^g = \partial A^g \cap B^g$, $Z^g = \overline{A^g} \cap B^g$, then the following assertions hold:

1. $Y^g \subset X \cup Y$.
2. $Z^g \subset X \cup \text{Sing } Y$, in particular for any $x \notin X$, $\dim Z^g_x \leq 0$.
3. For each $x \in \text{Sing } Y$ the semianalytic sets germs $\overline{A^g}_x \setminus \{x\}$ and $\overline{B^g}_x \setminus \{x\}$ are separable.

**Proof.** 1. Since $X \cup Y$ is a global analytic set, it is enough to prove that $\partial A^g \cup \partial B^g$ is contained in it.

Fix $x \in \partial A^g = \overline{A^g} \setminus A^g$. Since $A^g \subset A \cup B$, $x \in \overline{A} \cup \overline{B}$. Suppose $x \in \overline{A}$. If $x \notin A$, $x \in \partial A$, then $x \in Y$. So we can assume $x \in A$ and this implies $g(x) \leq 0$. We want to prove that $g(x) = 0$. Suppose $g(x) < 0$, being $A$ open, we can find a neighbourhood $U_x$ of $x$ contained in $A \cap \{g < 0\}$. Since $A \cap B = \emptyset$, we have that $U_x \cap A^g = \emptyset$, which contradicts the hypothesis $x \in \overline{A^g}$. We argue similarly for $x \in \overline{B}$ and for $x \in \partial B^g$. 
2. Note that $Z^g \subset (Y^g)^c \cup \text{Sing } Y^g$, since $\text{Sing } Y^g$ is contained in $X \cup \text{Sing } Y$, it is enough to prove that $(Y^g)^c$ is contained in $X$, i.e., no wall in $Y$ is odd with respect to $A^g$ and $B^g$.

Take a wall $W \subset Y$. If it is odd with respect to $A$ and $B$ then $W \subset V(g)$, hence it is even with respect to $A^g$ and $B^g$. It cannot be odd, because if so the same argument shows that $W$ is even with respect to $(A^g)^g = A$ and $(B^g)^g = B$; this is not the case by hypothesis. If $W$ is not odd then $g$ does not change sign through $W$, so $W$ is not odd with respect to $A^g$ and $B^g$.

The second statement in 2) is clear.

3. Fix $x \in \text{Sing } Y$. Since $X \cap \text{Sing } Y = \emptyset$, $x \notin X$ hence $\dim(\overline{A} \cap \overline{B})_x \leq 0$.

By hypothesis $A_x$ and $B_x$ are generically separable, say by $f_x \in \mathcal{O}_x$. Then, $f_x g$ generically separates $\overline{A}^g_x$ and $\overline{B}^g_x$. This implies that $\overline{A}^g/\{x\}$ and $\overline{B}^g/\{x\}$ are separable, (cf. [ABR96]).

\[\square\]

Remark 4.4. Note that $A$ and $B$ are separable if and only if $A^g$ and $B^g$ are so. Note also that, if the ideal of $Y^c$ is principal, i.e., $X = \emptyset$, $A^g$ and $B^g$ intersect each other only in a discrete set of points contained in $\text{Sing } Y$.

Proof of Theorem 4.2. One implication is trivial, we want to prove the other.

Denote by $D$ the discrete set $\overline{A} \cap \overline{B}^c$, $D \subset \text{Sing } Y$. By Theorem 2.3 we know that it is enough to separate $\overline{A} \cap \overline{B}/Y^c$ and $\overline{B} \cap \overline{B}/Y^c$ where $U$ is a neighbourhood of $Y^c$, since $\overline{A}_x/\{x\}$ and $\overline{B}_x/\{x\}$ are separated for any $x \in D \subset \text{Sing } Y$ by hypothesis. By Theorem 3.1, it is enough to separate $A \cap U$ and $B \cap U$ (we can always assume $U \cap D = \emptyset$). Let $X_1$ and $X_2$ be analytic sets as in Lemma 4.3, i.e., $I(Y^c \cup X_i) = (g_i)\mathcal{O}_M$, $X_i \cap Y$ is a discrete set, $i = 1, 2$ and $X_1 \cap X_2 \cap \text{Sing } Y = \emptyset$. Then we have the sets $A^i = A^{g_i}$, $B^i = B^{g_i}$, $Y^i = Y^{g_i}$ and $Z^i = Z^{g_i}$, $i = 1, 2$. We have that $A$ and $B$ are separable in $U$ iff $A^i$ and $B^i$ are separable in $U$, $i = 1, 2$. As above, by Theorem 2.3 and Theorem 3.1, it is enough to prove that $\overline{A^i \cap U}/x/\{x\}$ and $\overline{B^i \cap U}/x/\{x\}$ are separable for each $x \in \overline{A}^i \cap \overline{B}^i/(Y^1)^c$ and that $A^i$ and $B^i$ are separable in $U \cap V_i$, where $V_i$ is a neighbourhood of $X_i$, $i = 1, 2$.

Since the first assertion is verified by Lemma 4.3, $A^i$ and $B^i$ are separable in $U$ if and only if the second one holds. More precisely, $A$ and $B$ are separable in $U$, iff $A^1$ and $B^1$ are separable in $U \cap V_1$, iff $A^2$ and $B^2$ are separable in $U \cap V_1$. But $\overline{A^2} \cap \overline{B^2} \cap U \cap V_1$ is empty, up to shrink $U$ and $V_1$, so $A^2$ and $B^2$ are actually separated in $U \cap V_1$ and this in turn implies the thesis. \[\square\]

As a consequence of our criterion we obtain:

Theorem 4.5. $A$ and $B$ are separable if and only if they are separable in any compact set.
This is completely different of what happens in the algebraic case where the behaviour at infinity is decisive for the separation of semialgebraic sets, as the example in Figure 2 shows. It remains true that the important information is at the boundary but they may be hidden and appear only after compactifying and doing some blow ups, more precisely in a compact model of the variety where the semialgebraic sets are at normal crossings, see [AAB99]. (The field of meromorphic functions on an analytic manifold is not preserved under blowing ups, so there is no special model of $M$ suitable to study $A$ and $B$.)

Remark 4.6. We want to justify the assertion in the introduction that unbounded orderings are not useful. This is clear if the obstruction is a point. If $W$ is a wall simultaneously odd and even, take two points $x_1, x_2 \in W$ such that $W$ is odd in a neighbourhood of $x_1$ and even in a neighbourhood of $x_2$ and take two orderings in $\text{Spec}_r \frac{O(M)_I(W)/J(W)}{J(W)}$ centered respectively in $x_1$ and $x_2$. Since $O(M)_I(W)$ is a discrete valuation ring, cf. Proposition 1.5, arguing as in [ABV94], it is easy to lift these two orderings to a four element fan in $\text{Spec}_r M$ where the corresponding constructible sets $\tilde{A}$ and $\tilde{B}$ are not separable. By construction the four orderings in this fan are bounded. Bearing in mind the Artin-Lang property, which holds in $\text{Spec}_r M$, we can resume all this in the following:

Theorem 4.7. Two global semianalytic sets $A$ and $B$ in a 2-dimensional analytic manifold $M$ can be separated if and only if the associated constructible sets $\tilde{A}, \tilde{B} \subset \text{Spec}_r M$ are separable in any 4-elements fan made of bounded orderings.

5. Basicness for global semianalytic sets.

We can use the criterion for separation of the above section to prove a similar result for another kind of problems: Basicness and principality of global semianalytic sets.

Theorem 5.1. Let $S \subset M$ be an open global semianalytic set. Then, $S$ is basic open (resp. principal open) if and only if the following conditions hold:

1. If a wall is simultaneously odd and even with respect to $S$ and $M \setminus \overline{S}$ then, it does not intersect $\overline{S}$.

2. For each $x \in \text{Sing} \overline{S}$, the semianalytic set germ $S_x$ is basic open.

Resp.

3. No wall is simultaneously odd and even with respect to $S$ and $M \setminus \overline{S}$.

Proof. Let’s prove the statement about basicness. One implication is trivial, we want to prove the other. Since $M \setminus \overline{S}$ is an open global semianalytic set, it is a finite union of basic open sets, say $B_1 \cup \cdots \cup B_r$, such a description
is possible by [Cas94b]. We want to prove that $S$ can be separated from each $B_i$.

Fix $i \in \{1, \ldots, r\}$ and set $Y_i = \partial S \cup \partial B_i$. It is enough to prove that no wall in $Y_i$ is simultaneously odd and even with respect to $S$ and $B_i$ and that, for each $x \in \operatorname{Sing} Y_i$, the semianalytic set germs $S_x$ and $B_{i,x}$ are separable. Let $W \subset Y_i$ be an odd wall, this means that there exists an arc contained in $S \cap B_i$. If $W$ is even, we can find another arc contained in $S$ or in $B_i$. The first assertion contradicts Hypothesis 1, the second the basicness of $B_i$. This proves that no odd wall can be even.

Since two open basic semianalytic set germs in dimension 2 can always be separated, cf. [ABR96], it is enough to prove that $B_{i,x}$ and $S_x$ are basic for each $x \in \operatorname{Sing} Y_i$. This is true by Hypothesis 2 (note that $S_x$ is principal if $x \not\in \overline{\partial S}$).

Hence, we can find $f_i \in \mathcal{O}(M)$ such that $f_i(S) > 0$ and $f_i(B_i) < 0$. Then, $S = \{f_1 > 0, \ldots, f_r > 0\}$. Indeed, if $x \in \overline{B_i}$ for some $i \in \{1, \ldots, r\}$ and this implies that $f_i(x) \leq 0$ for some $i \in \{1, \ldots, r\}$.

As it concerne the statement about principality, with the same argument we can separate $S$ from $M \setminus S$ proving that $S$ is principal.

Arguing as in Remark 4.6, we see that unbounded orderings are again useless for the basicness and principality properties, more precisely:

**Theorem 5.2.** An open global semianalytic set $S$ is basic open (principal) if and only if $\# \overline{\partial S} \cap F = 3$ ($\# \overline{\partial S} \cap F \neq 1, 3$) for any 4-elements fan $F$ made of bounded orderings.

**Acknowledgements.** We would like to thank F. Acquistapace for her constant help in this work.

**References**


Received June 7, 2002 and revised December 3, 2002. Both authors are members of GNSAGA, partially supported by MURST and EC through its IHP programme (HPRN-CT-00271).

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ISOMORPHISM THEOREM ON LOW DIMENSIONAL LIE ALGEBRAS

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Let \( g \) (resp. \( g' \)) be a Lie algebra of dimension \( d \leq 3 \) (resp. of finite dimension) over a field \( k \) of characteristic \( \neq 2 \). We prove that \( g \) is isomorphic to \( g' \) as Lie algebras over \( k \) if and only if the enveloping algebra \( U(g) \) of \( g \) is isomorphic to \( U(g') \) as \( k \)-algebras.

1. Introduction.

In this article, we study the isomorphism theorem on Lie algebras of dimension \( \leq 3 \). Our goal is the following theorem:

**Theorem 1.1.** Let \( g \) (resp. \( g' \)) be a Lie algebra of dimension \( d \leq 3 \) (resp. of finite dimension) over a field \( k \) (of characteristic not equal to 2). Then \( g \) is isomorphic to \( g' \) if and only if the universal enveloping algebra \( U(g) \) of \( g \) is isomorphic to the one \( U(g') \) of \( g' \).

For a Lie algebra of dimension 1 or 2, the theorem is clear by classification of low dimensional Lie algebras [3, I.4]. Malcolmson [4] proved the isomorphism theorem for 3-dimensional simple Lie algebras by using their Killing forms. We describe the simplicity of a 3-dimensional Lie algebra in terms of its enveloping algebra. To complete the isomorphism theorem on 3-dimensional Lie algebras, we prove the theorem for non-simple Lie algebras of dimension 3.

**Notation.** We denote by \( \sigma = \sigma_\mathfrak{g} : \mathfrak{g} \to U(\mathfrak{g}) \) a canonical map from a Lie algebra to its enveloping algebra \( U(\mathfrak{g}) \).

2. Preliminaries on enveloping algebras.

We prove some preliminary properties on the enveloping algebra \( U(\mathfrak{g}) \).

**Proposition 2.1.** The two-sided ideal \( I_{\text{com}} \) generated by \( \{ [a, b] := ab - ba \in U(\mathfrak{g}) ; a, b \in U(\mathfrak{g}) \} \) is equal to the one \( I_{[\mathfrak{g}, \mathfrak{g}]} \) generated by \( \sigma([\mathfrak{g}, \mathfrak{g}]) \).

**Proof.** We have only to verify \( I_{\text{com}} \subset I_{[\mathfrak{g}, \mathfrak{g}]} \). Since \( \sigma(g_1)\sigma(g_2)\cdots\sigma(g_s) \) \( (g_i \in \mathfrak{g}) \) generate \( U(\mathfrak{g}) \) as a \( k \)-vector space, it is enough to show that

\[
[\sigma(g_1) \cdots \sigma(g_s), \sigma(h_1) \cdots \sigma(h_r)] \in I_{[\mathfrak{g}, \mathfrak{g}]}
\]
for \(g_i, h_j \in \mathfrak{g}\). It follows from the formula
\[
[g, hh'] = [g, h]h' + h[g, h'] \quad \text{for } g, h, h' \in \mathcal{U}(\mathfrak{g}).
\]

**Proposition 2.2.** In the notation of Proposition 2.1, we have a canonical isomorphism \(\mathcal{U}(\mathfrak{g})/I_{\text{com}} = \mathcal{U}(\mathfrak{g})/I_{[\mathfrak{g}, \mathfrak{g}]} \rightarrow \mathcal{U}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])\) as \(k\)-algebras.

*Proof.* See [2, 2.2.14, p. 72]. By the functoriality of \(\mathcal{U}(\mathfrak{g})\) with respect to \(\mathfrak{g}\), we have a canonical \(k\)-algebra homomorphism \(\varphi: \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])\). Since \(\sigma(\mathfrak{g})\) generates \(\mathcal{U}(\mathfrak{g})\) as \(k\)-algebra, the homomorphism \(\varphi\) is surjective. On the other hand, every (Lie algebra) homomorphism from \(\mathfrak{g}\) to the Lie algebra associated to a commutative ring factors through \(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]\). Since \(\mathcal{U}(\mathfrak{g})/I_{\text{com}}\) is commutative, we have a \(k\)-algebra homomorphism \(\psi: \mathcal{U}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) \rightarrow \mathcal{U}(\mathfrak{g})/I_{\text{com}}\). Hence we can prove the kernel of \(\varphi\) is equal to \(I_{\text{com}}\) by the fact that the composite \(\psi \varphi\) is the canonical projection \(\mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})/I_{\text{com}}\). □

**Proposition 2.3.** Let \(\text{GK-dim}_k \mathcal{U}(\mathfrak{g})\) be the Gelfand-Kirillov dimension of \(\mathcal{U}(\mathfrak{g})\). Then we have \(\text{GK-dim}_k \mathcal{U}(\mathfrak{g}) = \dim_k \mathfrak{g}\).

*Proof.* See [5, 8.1.15 (iii)]. □

**Proposition 2.4.** Let \(\mathfrak{h}\) be an ideal of \(\mathfrak{g}\) which is abelian. Let \(I_{\mathfrak{h}}\) be the right ideal of \(\mathcal{U}(\mathfrak{g})\) generated by \(\sigma(\mathfrak{h})\), which is a two-sided ideal (cf. [2, 2.2.14]). Then, for any two-sided maximal ideal \(\mathfrak{m}\) with \(\mathcal{U}(\mathfrak{g})/\mathfrak{m} \cong k\) which contains \(I_{\mathfrak{h}}\), we have a Lie algebra isomorphism \(\mathfrak{h} \rightarrow I_{\mathfrak{h}}/I_{\mathfrak{h}}\mathfrak{m}\) via \(\sigma\). Here the Lie algebra structure of \(I_{\mathfrak{h}}/I_{\mathfrak{h}}\mathfrak{m}\) is defined by that of \(\mathcal{U}(\mathfrak{g})\).

*Proof.* First, we prove the proposition in the case \(\mathfrak{m} = \langle \sigma(\mathfrak{g}) \rangle\). Let \(g_1, \ldots, g_d\) be a basis of \(\mathfrak{g}\) such that \(g_1, \ldots, g_l\) is a basis of \(\mathfrak{h}\). By Poincaré-Birkhoff-Witt theorem, we have
\[
\mathcal{U}(\mathfrak{g}) = k \oplus \bigoplus_{\substack{s \geq 1 \\text{ and} \\ 1 \leq i_1 \leq \cdots \leq i_s \leq d}} k \sigma(g_{i_1}) \cdots \sigma(g_{i_s}).
\]
Here \(\sigma(g_{i_1}) \cdots \sigma(g_{i_s}) (1 \leq i_1 \leq \cdots \leq i_s \leq d)\) form a \(k\)-basis of \(U\). Since \(\mathfrak{h}\) is abelian, we have similar decompositions:
\[
I_{\mathfrak{h}} = \bigoplus_{\substack{s \geq 1 \\text{ and} \\ 1 \leq i_1 \leq \cdots \leq i_s \leq d \\text{ and} \\ i_1 \leq l}} k \sigma(g_{i_1}) \cdots \sigma(g_{i_s})
\]
\[
I_{\mathfrak{h}}\mathfrak{m} = \bigoplus_{\substack{s \geq 2 \\text{ and} \\ 1 \leq i_1 \leq \cdots \leq i_s \leq d \\text{ and} \\ i_1 \leq l}} k \sigma(g_{i_1}) \cdots \sigma(g_{i_s})
\]
as \( k \)-vector spaces. Hence we have an isomorphism
\[
\mathfrak{h} \xrightarrow{\cong} I_\mathfrak{h}/I_\mathfrak{h}m = \bigoplus_{1 \leq i_1 \leq l} k\langle g_{i_1} \rangle
\]
via \( \sigma \). Using \( I_\mathfrak{h} \subset m \) and \( I_\mathfrak{h}^2 \subset I_\mathfrak{h}m \), one can verify that the Lie algebra structure of \( I_\mathfrak{h}/I_\mathfrak{h}m \) is well-defined and abelian.

Next we show the proposition in the general case. Let \( \alpha : U(\mathfrak{g}) \to k \) be a surjective \( k \)-algebra homomorphism with kernel \( m \). Using \( \alpha \), we have an automorphism \( i : U(\mathfrak{g}) \to U(\mathfrak{g}) \) with \( i_{\sigma} = \sigma \) for all \( g \in \mathfrak{g} \).

Since \( m \) contains \( I_\mathfrak{h} \), the restriction of \( i \) to \( \sigma(\mathfrak{h}) \) is the identity of \( \sigma(\mathfrak{h}) \). One can easily verify \( i_{\langle \sigma(g) \rangle} = m \). Hence we have an isomorphism \( \mathfrak{h} \to I_\mathfrak{h}/I_\mathfrak{h}m \) using the isomorphism (1) and a commutative diagram
\[
\begin{array}{ccc}
\mathfrak{h} & \xrightarrow{i} & I_\mathfrak{h}/I_\mathfrak{h}\langle \sigma(g) \rangle \\
\downarrow \cong & & \downarrow i \\
\mathfrak{h} & \to & I_\mathfrak{h}/I_\mathfrak{h}m.
\end{array}
\]

**Corollary 2.5.** In the notation of Proposition 2.1, we regard the ideal \( I := I_{[\mathfrak{g}, \mathfrak{g}]} \) as ideal of the underlying Lie algebra \( U(\mathfrak{g}) \). Assume that \( [\mathfrak{g}, \mathfrak{g}] \) is abelian. Then, for any maximal ideal \( m \) with \( U(\mathfrak{g})/m \cong k \), the composite \( \mathfrak{g} \to I_{\mathfrak{g}} \to I/Im \) is an isomorphism of Lie algebras.

**Remark 2.6.** The composition \( \mathfrak{g} \to I_{\mathfrak{g}} \to I/Im \) is surjective for any Lie algebra, but not necessarily injective if \( [\mathfrak{g}, \mathfrak{g}] \) is not abelian. For example, consider a simple Lie algebra.

**Proposition 2.7.** Let \( \mathfrak{g}_0 \) be an ideal of \( \mathfrak{g} \). Suppose that there exists a subalgebra \( \mathfrak{g}_1 \) of \( \mathfrak{g} \) such that \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) as \( k \)-vector spaces. Then \( \mathfrak{g} \) is isomorphic to the semidirect product \( \mathfrak{g}_0 \rtimes \mathfrak{g}_1 \).

**Proof.** It is straightforward to show that the \( k \)-linear map \( \mathfrak{g}_0 \rtimes \mathfrak{g}_1 \to \mathfrak{g} \) defined by \( (g_0, g_1) \mapsto g_0 + g_1 \) is a Lie algebra isomorphism.

**Proposition 2.8.** Let \( \mathfrak{g}_i \) and \( \mathfrak{g}'_i \) \((i = 0, 1)\) be Lie algebras and \( \mathfrak{g}_1 \overset{d}{\to} \text{Der}_k \mathfrak{g}_0 \) \((\text{resp. } \mathfrak{g}'_1 \overset{d'}{\to} \text{Der}_k \mathfrak{g}_0')\) a derivation of \( \mathfrak{g}_0 \) \((\text{resp. } \mathfrak{g}_0')\). Assume that there exist Lie algebra isomorphisms \( \varphi_0 : \mathfrak{g}_0 \to \mathfrak{g}_0' \) and \( \varphi_1 : \mathfrak{g}_1 \to \mathfrak{g}_1' \) with a commutative diagram
\[
\begin{array}{ccc}
\mathfrak{g}_1 & \overset{d}{\to} & \text{Der}_k \mathfrak{g}_0 \\
\varphi_1 \downarrow & & \varphi_0^* \downarrow \\
\mathfrak{g}_1' & \overset{d'}{\to} & \text{Der}_k \mathfrak{g}_0'.
\end{array}
\]
Here $\varphi'_0$ is the induced homomorphism by $\varphi_0$. Then the semidirect product $\mathfrak{g}_0 \rtimes \mathfrak{g}_1$ is isomorphic to $\mathfrak{g}'_0 \rtimes \mathfrak{g}'_1$ by $(g_0, g_1) \mapsto (\varphi_0(g_0), \varphi_1(g_1))$.

Proof. Straightforward. See [1, Chapitre 1 §7].

3. Proof of Theorem 1.1.

We have only to show that, if $U(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g}')$, the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{g}'$.

Assume that $U(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g}')$. We remark that $\dim_k \mathfrak{g} = \dim_k \mathfrak{g}'$ and $\dim_k [\mathfrak{g}, \mathfrak{g}] = \dim_k [\mathfrak{g}', \mathfrak{g}']$ by Propositions 2.2 and 2.3.

In the case of $\dim_k \mathfrak{g} = 1, 2$, the theorem follows from the classification of Lie algebras (e.g., [3, I.4]).

By Propositions 2.2 and Corollary 2.5, we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] & \xrightarrow{\sigma} & U(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) \\
\downarrow & & \downarrow \bar{\psi} \\
\text{Der}_k([\mathfrak{g}, \mathfrak{g}]) & \xrightarrow{\cong} & \text{Der}_k(I/Iim) \\
\end{array}
\begin{array}{ccc}
& \sigma' & \\
\rho & \downarrow \rho' & \\
\text{Der}_k(I/Iim) & \xrightarrow{\cong} & \text{Der}_k([\mathfrak{g}', \mathfrak{g}']).
\end{array}
$$

Here $\rho$, $\rho'$ are Lie homomorphisms defined by inner derivation as usual, and $\bar{\psi}$ (resp. $\psi^*$) is the isomorphism induced by $\psi$.

Suppose $\dim_k [\mathfrak{g}, \mathfrak{g}] = 1$. Then there are just two isomorphism classes of 3-dimensional Lie algebras [3, I.4]: One is nilpotent; the other is not nilpotent. In this case, a Lie algebra $\mathfrak{g}$ is nilpotent if and only if its center contains $[\mathfrak{g}, \mathfrak{g}]$, i.e., the above $\rho$ is trivial for a maximal ideal $m$ with $U/m \cong k$. The theorem follows from the above diagram.

Next, we suppose $\dim_k [\mathfrak{g}, \mathfrak{g}] = 2$. Take elements $z \in \mathfrak{g} \setminus [\mathfrak{g}, \mathfrak{g}]$ and $z' \in \mathfrak{g}' \setminus [\mathfrak{g}', \mathfrak{g}']$. We denote by $\mathfrak{g}_1$ (resp. $\mathfrak{g}'_1$) the subalgebra of $\mathfrak{g}$ (resp. $\mathfrak{g}'$) generated by $z$ (resp. $z'$). Since

$$
\bar{\psi}(\sigma_0(z \mod [\mathfrak{g}, \mathfrak{g}]]) = a\sigma_{\mathfrak{g}'}(z' \mod [\mathfrak{g}', \mathfrak{g}']) + b \text{ for some } a \in k^*, b \in k,
$$
we have the following commutative diagram of Lie algebras:

\[
g_1 \xrightarrow{\psi_1} g'_1 \\
\downarrow \quad \downarrow \\
\text{Der}_k([g, g]) \xrightarrow{\psi^*} \text{Der}_k([g', g'])
\]

where \(\psi_1\) maps \(z\) to \(az'\), and \(\psi^*\) is the composite of the lower horizontal maps in the above diagram. The theorem follows from Propositions 2.7 and 2.8.

**Acknowledgements.** The second author is grateful to the third author for introducing this subject and encouraging him. The third author wishes to acknowledge the financial support of Korea Research Foundation made in the program Year 1997.

**References**


A PIERI RULE FOR HERMITIAN SYMMETRIC PAIRS I

THOMAS J. ENRIGHT, MARKUS HUNZIKER, AND NOLAN R. WALLACH

Let \((G, K)\) be a Hermitian symmetric pair and let \(g\) and \(k\) denote the corresponding complexified Lie algebras. Let \(g = k \oplus p^+ \oplus p^-\) be the usual decomposition of \(g\) as a \(k\)-module. \(K\) acts on the symmetric algebra \(S(p^-)\). We determine the \(K\)-structure of all \(K\)-stable ideals of the algebra. Our results resemble the Pieri rule for Young diagrams. The result implies a branching rule for a class of finite dimensional representations that appear in the work of Enright and Willenbring (preprint, 2001) and Enright and Hunziker (preprint, 2002) on Hilbert series for unitarizable highest weight modules.

1. Introduction.

The Pieri rule for the unitary group \(U(m)\) gives the decomposition of the tensor product of an irreducible representation \(F\) and a symmetric power of \(\mathbb{C}^m\). In the simplest case, if \((n_1, n_2, \ldots, n_m)\) with \(n_1 \geq n_2 \geq \cdots \geq n_m \geq 0, n_i \in \mathbb{Z}\), is the highest weight of \(F\) then \(F \otimes \mathbb{C}^m\) decomposes multiplicy free with highest weights \((n_1+1, n_2, \ldots, n_m)\) and \((n_1, n_2, \ldots, n_{i-1}, n_i+1, \ldots, n_m)\) for \(2 \leq i \leq m\) with \(n_{i-1} > n_i\). Here we consider a related question about the module structure of ideals in the context of Hermitian symmetric pairs.

Let \((G, K)\) be an irreducible symmetric pair of Hermitian type. That is, if \(g = \text{Lie}(G) \otimes \mathbb{C}\) then it is a simple Lie algebra over \(\mathbb{C}\) and if \(k = \text{Lie}(K) \otimes \mathbb{C}\) then \(k = CH \oplus [k, k]\) with \(ad(H)\) having eigenvalues \(0, 1, -1\) on \(g\). We write \(p^\pm = \{X \in g \mid [H, X] = \pm X\}\). Then \(g = p^- \oplus k \oplus p^+\) is a direct sum of \(k\)-modules. Let \(B\) denote the Killing form of \(g\). Then \(B\) induces a perfect pairing between \(p^+\) and \(p^-\). Thus we can look upon the symmetric algebra \(S(p^-)\) as polynomials on \(p^+\) and \(S(p^+)\) as differential operators with constant coefficients on \(p^-\).

The main result of this article describes the \(k\)-module structure of any \(K\)-stable ideal in the symmetric algebra \(S(p^-)\). This result is a consequence of Theorem 2.1 whose form resembles the Pieri rule described above and indicates that \(K\)-invariant ideals are similar in \(K\)-structure to monomomial ideals. This question regarding \(K\) structure of ideals in \(S(p^-)\) arose from the study of the the Hilbert series of unitarizable highest weight modules (with respect to the grading induced by \(H\)). By the Transfer Theorem of
[EW] and [EH], many unitarizable highest weight representations \(L\) have Hilbert series of the form:

\[ h_L(t) = R \frac{f(t)}{(1-t)^d}, \]

where \(d\) equals the Gelfand Kirillov dimension of \(L\), \(f(t)\) is the (polynomial) Hilbert series of a finite dimensional representation \(E_L\) of the reduced Hermitian symmetric pair \((G_L, K_L)\) and \(R\) is the ratio of the dimensions of the zero grade in \(L\) and \(E_L\). For the Wallach representations [EW] this ratio is \(1/f(0)\). The connection with this article comes when we note that for a Wallach representation \(L\) of \(SU(p,q) Sp(n,\mathbb{R})\) or \(SO(2,m)\), the \(G_L\)-representation \(E_L\) has a \(K_L\)-stable highest weight space (i.e., \(f(0) = 1\)). Therefore \(E_L\) is isomorphic to the quotient of \(S(p^-)\) by a \(K_L\)-invariant ideal. For these cases the results proved here are used in [EH] to offer explicit formulas for the Hilbert series of the Wallach representations.

In the literature there has been considerable interest in the question of the \(K\)-structure of ideals in \(S(p^-)\), as pointed out to us by the referee. Roughly speaking half of the Hermitian symmetric cases have been treated in case by case studies: The case corresponding to \(U(p,q)\) is treated in [DEP], the case corresponding to \(SO^*(2n)\) in [AD] and the case corresponding to \(Sp(n,\mathbb{R})\) in [A]. A related example not covered in the Hermitian symmetric setting is the action of \(GL(n)\) on the skew symmetric \((n+1) \times (n+1)\) matrices which is treated in [D].

### 2. \(K\)-invariant ideals.

We continue with the Hermitian symmetric setting and notation of the introduction. We fix a Cartan subalgebra \(\mathfrak{h}\) of \(\mathfrak{k}\) and a system of positive roots \(\Phi^+\) such that if \(\alpha \in \Phi^+\) then \(\alpha(H) \geq 0\). We set \(\Phi^+_n = \{\alpha \in \Phi^- | \alpha(H) = 1\}\). Let \(\Phi^+_c = \Phi^+ - \Phi^-_n\) then \(\Phi^+_c\) is a system of positive roots for \(\mathfrak{k}\) on \(\mathfrak{h}\). Let \(\gamma_1 < \gamma_2 < \cdots < \gamma_r\) denote Harish-Chandra’s strongly orthogonal roots. Schmid [S] has shown that as a \(K\)-module under the restriction of the adjoint representation \(S(p^-)\) is multiplicity free and the irreducible constituents are exactly the \(K\)-modules with highest weights of the form

\[-(n_1\gamma_1 + \cdots + n_r\gamma_r), \quad n_1 \geq n_2 \geq \cdots \geq n_r \geq 0, \quad n_i \in \mathbb{Z}.\]

We will write \(S(p^-)[n_1, \ldots, n_r]\) for the corresponding isotypic component. We also note that if \(d = \sum n_i\) then \(S(p^-)[n_1, \ldots, n_r] \subset S^d(p^-)\) (the homogeneous elements of degree \(d\)). The main result is:
Theorem 2.1. Let $n_1 \geq n_2 \geq \cdots \geq n_r \geq 0$, $n_i \in \mathbb{Z}$. Then

$$S(p^-)[n_1, \ldots, n_r]p^- =$$

$$S(p^-)[n_1 + 1, n_2, \ldots, n_r] \oplus \bigoplus_{2 \leq i \leq r, n_i < n_{i-1}} S(p^-)[n_1, \ldots, n_{i-1}, n_i + 1, \ldots, n_r],$$

with multiplication in $S(p^-)$.

Before turning to the proof we first recall four well-known results on the strongly orthogonal roots (mostly due to C. Moore). Let $(.,.)$ denote the dual form to $B_{\mathfrak{h}}$.

1. $(\gamma_i, \gamma_i) = (\gamma_j, \gamma_j)$ for all $i, j$.
2. If $\alpha \in \Phi^+$ then $(\alpha, \alpha) \leq (\gamma_1, \gamma_1)$.

Let $\mathfrak{h}^-$ denote the linear span of the coroots, $\gamma_i^\vee$, of the $\gamma_i$. Then dim $\mathfrak{h}^- = r$ and the $\gamma_i^\vee$ form an orthogonal basis of $\mathfrak{h}^-$. If $\alpha \in \Phi^+_n$ and $\alpha \neq \gamma_i$ for any $i$, then $\alpha|_{\mathfrak{h}^-}$ is either of the form

$$\frac{1}{2}(\gamma_i + \gamma_j) \quad \text{with} \quad i < j \quad \text{or} \quad \frac{1}{2}\gamma_i.$$

If $\alpha \in \Phi^+_c$ then $\alpha|_{\mathfrak{h}^-}$ is either of the form

$$-\frac{1}{2}(\gamma_i - \gamma_j) \quad \text{with} \quad i < j \quad \text{or} \quad -\frac{1}{2}\gamma_i.$$

With this notation in place we can prove:

Lemma 2.2. Let $n_1 \geq n_2 \geq \cdots \geq n_r \geq 0$, $n_i \in \mathbb{Z}$. Then

$$S(p^-)[n_1, \ldots, n_r]p^- \subset$$

$$S(p^-)[n_1 + 1, \ldots, n_r] \oplus \bigoplus_{2 \leq i \leq r, n_i < n_{i-1}} S(p^-)[n_1, \ldots, n_{i-1}, n_i + 1, \ldots, n_r].$$

Proof. Let $\lambda = -\sum n_i \gamma_i$ then it is standard that the possible highest weights that can occur in $S(p^-)[n_1, \ldots, n_r] \otimes p^-$ are of the form $\lambda - \alpha$ with $\alpha \in \Phi^+_n$. Schmid's result implies that if this highest weight occurs in $S(p^-)[n_1, \ldots, n_r]p^-$, then

$$\lambda - \alpha = -\sum m_i \gamma_i \quad \text{with} \quad m_1 \geq m_2 \geq \cdots \geq m_r \geq 0, m_i \in \mathbb{Z}.$$

Thus, in particular, we have

$$\lambda - \alpha|_{\mathfrak{h}^-} = -\sum m_i \gamma_i$$

with the conditions above satisfied. Using the forms of $\alpha|_{\mathfrak{h}^-}$ in (3) above we see that since the coefficients on the right-hand side of the equation are
all integers the only possibility for $\alpha$ is one of the $\gamma_i$. Now the Schmid conditions on the highest weights imply the lemma.

This simple result reduces the proof to showing that the predicted components actually occur. For our proof of this assertion (and hence of the theorem) we recall several results from [W]. Let $n_0^+$ denote the sum of the root spaces for the elements of $\Phi^+$. We choose a nonzero element $u_i$ in $S(p^-)^n_0 \cap S(p^-)[n_1, \ldots, n_r]$ with $n_j = 1$ for $j \leq i$ and $n_j = 0$ for $j > i$. Then one can easily see from Schmid’s result that:

(5) $S(p^-)^n_0$ is the polynomial ring on the algebraically independent elements $u_1, \ldots, u_r$.

We will now analyze these covariants in further detail. For each $1 \leq j \leq r$, let $\Phi^+_0$ denote the set of elements, $\alpha$, of $\Phi^+$ such that $\alpha|_{h^0}$ is of the form $\frac{1}{2}(\gamma_p \pm \gamma_q)$ with $q \leq p \leq j$. Then $\Phi^+_0 = \Phi^+_0 \cup -\Phi^+_0$ is a subrootsystem of $\Phi$. Let $g_0, j$ denote the subalgebra of $g$ generated by the root spaces for the roots in $\Phi^+_0$. If $u_j$ is the sum of all ideals of $g_0, j$ contained in $g_0, j \cap \mathfrak{z}$ then $(g_0, j/u_j, (\mathfrak{z} \cap g_0, j)/u_j)$ is also an irreducible symmetric pair of Hermitian type. Set $p_0^+ = g_0, j \cap p^+$. One has (see [W])

(6) $u_j \in S(p_{0, j}^-)$.

We denote by $x \mapsto \pi$ the conjugation of $g$ with respect to $Lie(G)$. Then $p^- = p^+$. If $x \in p^+$ then we denote by $\partial(x)$ the derivation of $S(p^-)$ defined by $\partial(x)y = B(x, y)$ for $y \in p^-$. We will also denote the extension of $\partial$ to $S(p^+)$ by $\partial$. In addition we will use the notation $u \mapsto u(0)$ for the augmentation map of $S(p^-)$ to $\mathbb{C}$ given as the extension to a homomorphism of $y \mapsto 0$ for $y \in p^-$. We define for $u, v \in S(p^-)$, $\langle u, v \rangle = \langle \partial(\pi)u(0) \rangle$. The following observation is well-known and easily checked:

(7) The Hermitian form $\langle \cdot, \cdot \rangle$ is positive definite and $K$-invariant.

Furthermore, if $u, v, w \in S(p^-)$ then $\langle uv, w \rangle = \langle v, \partial(\pi)w \rangle$.

We now begin the proof that the predicted representations in Theorem 2.1 actually occur. If $n_1 \geq \cdots \geq n_r \geq 0$ then we note that

$$u_1^{n_1-n_2}u_2^{n_2-n_3} \cdots u_{r-1}^{n_{r-1}-n_r}u_r^{n_r}$$

is a basis of the highest weight space of $S(p^-)[n_1, \ldots, n_r]$. Thus to prove the result we must show that

(8) $u_1^{n_1-n_2+1}u_2^{n_2-n_3} \cdots u_{r-1}^{n_{r-1}-n_r}u_r^{n_r} \in S(p^-)[n_1, \ldots, n_r]p^-$
and if $r \geq j > 1$ and $n_{j-1} > n_j$ then

\begin{equation}
\begin{aligned}
&u_1^{n_1-n_2}u_2^{n_2-n_3} \cdots u_{j-1}^{n_{j-1}-n_j-1} u_j^{n_j-n_{j+1}+1} \\
&\quad \cdots u_r^{n_r-n_{r-1}} u_r^{n_r} \in S(p^-)[n_1, \ldots, n_r][p^-].
\end{aligned}
\end{equation}

Equation (8) is obvious since $u_1 = X_{-\gamma_1}$. This proves the theorem if $r = 1$. We proceed by induction on $r$. If $r = 1$ then we have already observed that the result is true. Assume that the result is true for $r - 1 \geq 1$. If $n_r = 0$ and if $n_{j-1} > n_j$ with $j < r$ then the inductive hypothesis and (6) above implies that

\begin{equation}
\begin{aligned}
&u_1^{n_1-n_2}u_2^{n_2-n_3} \cdots u_{j-1}^{n_{j-1}-n_j-1} u_j^{n_j-n_{j+1}+1} \\
&\quad \cdots u_r^{n_r-1-n_{r-1}} u_r^{n_r} \in S(p^-)[n_1, \ldots, n_{r-1}][p^-_{0,r-1}]
\end{aligned}
\end{equation}

and so (9) is true in this case. By (6), the highest weights although not the full $\mathfrak{g}$-modules are contained in $S(p^-_{0,r})$ and thus we may reduce to the case $\mathfrak{g} = \mathfrak{g}_{0,r}$ which we now assume. This implies that the $\mathfrak{g}$-module $S(p^-)[n, n, \ldots, n]$ is one dimensional. We have

\begin{equation}
S(p^-)[n_1, \ldots, n_r] = S(p^-)[n_1 - n_r, \ldots, n_{r-1} - n_r, 0]u_r^{n_r}.
\end{equation}

So the result for $j < r$ and $n_r > 0$ follows from the case when $n_r = 0$. To complete the proof we look at the remaining case; when $n_{r-1} > n_r \geq 0$. As before we are reduced to proving (9) for $j = r$, $n_r = 0$ and $n_{r-1} > 0$ to complete the induction. Let $D = \partial(u_r)$. Then

\begin{equation}
\begin{aligned}
D : S(p^-)[n_1, \ldots, n_r] &\rightarrow S(p^-)[n_1 - 1, \ldots, n_r - 1].
\end{aligned}
\end{equation}

Here if $n_r < 0$ then we write $S(p^-)[n_1, \ldots, n_r] = \{0\}$. Suppose in this last case, that (9) is not true. Then we would have:

\begin{equation}
\begin{aligned}
&u_1^{n_1-n_2}u_2^{n_2-n_3} \cdots u_r^{n_r-1} u_r \notin S(p^-)[n_1, \ldots, n_{r-1}, 0][p^-].
\end{aligned}
\end{equation}

Now (10) implies that $D(S(p^-)[n_1, \ldots, n_{r-1}, 0][p^-]) = 0$ and hence

\begin{equation}
\begin{aligned}
0 &= \langle D S(p^-)[n_1, \ldots, n_{r-1}, 0][p^-], S(p^-) \rangle \\
&= \langle S(p^-)[n_1, \ldots, n_{r-1}, 0], \partial(u_r) S(p^-) \rangle \\
&= \langle S(p^-)[n_1, \ldots, n_{r-1}, 0], (\partial(u_r^+) S(p^-) \rangle.
\end{aligned}
\end{equation}

Again since we have reduced to the case where $\mathfrak{g} = \mathfrak{g}_{0,r}$, $u_r$ spans a one dimensional $\mathfrak{g}$-module and so

\begin{equation}
\begin{aligned}
\partial(u_r^+) S(p^-) &= S(p^-)[1, 1, \ldots, 1, 0].
\end{aligned}
\end{equation}

Combining these last two identities gives

\begin{equation}
\begin{aligned}
0 &= \langle S(p^-)[n_1, \ldots, n_{r-1}, 0], u_{r-1} S(p^-) \rangle.
\end{aligned}
\end{equation}

By induction on rank we conclude that

\begin{equation}
\begin{aligned}
0 &= \langle S(p^-)[n_1, \ldots, n_{r-1}, 0], S(p^-)[n_1, \ldots, n_{r-1}, 0] \rangle,
\end{aligned}
\end{equation}
for all \( n_1 \geq n_2 \geq \cdots \geq n_{r-1} \geq 1 \). This is a contradiction and so (10) does not hold. This completes the induction and the proof of the theorem. \( \square \)

As a consequence of this result we can describe the ideals generated by isotypic components in \( S(p^-) \).

**Corollary 2.3.** Let \( n_1 \geq n_2 \geq \cdots \geq n_r \) then

\[
S(p^-)(S(p^-)[n_1,\ldots,n_r]) = \bigoplus_{m_i \geq n_i, m_1 \geq \cdots \geq m_r \geq 0} S(p^-)[m_1,\ldots,m_r].
\]

**Proof.** Let \( I \) denote the ideal on the left and let \( I^d \) denote the \( d \)-th level in the grading. Multiplication by elements of \( p^- \) shifts the grade by one. Set \( d_0 = \sum n_i \). Then \( d_0 \) is the minimal value for \( I^d \neq 0 \). The corollary holds when restricted to this level of the grade. Now suppose \( d > d_0 \) and the corollary holds for \( I^{d-1} \):

\[
I^{d-1} = \bigoplus_{m_i \geq n_i, m_1 \geq \cdots \geq m_r \geq 0} S(p^-)[m_1,\ldots,m_r]. \tag{11}
\]

Now multiply by \( p^- \). Then \( I^d = p^- I^{d-1} \) which by the theorem gives

\[
I^d \subset \bigoplus_{m_i \geq n_i, m_1 \geq \cdots \geq m_r \geq 0} S(p^-)[m_1,\ldots,m_r]. \tag{12}
\]

For the opposite inclusion suppose the indices \( m_1,\ldots,m_r \) satisfy the conditions in (12). Since \( d > d_0 \) choose \( i, 1 \leq i \leq r \), maximal with \( m_i > n_i \). Then, by the induction hypothesis,

\[
S(p^-)[m_1,\ldots,m_i-1,m_{i+1},\ldots,m_r] \subset I^{d-1}.
\]

Multiplying by \( p^- \) and applying Theorem 2.1 implies \( S(p^-)[m_1,\ldots,m_r] \subset I^d \). This proves equality in (12) and completes the induction.

Let \( I \) be a general \( K \)-invariant ideal in \( S(p^-) \). Then \( I \) is generated as an ideal by a finite set of \( K \)-isotypic subspaces, say \( S(p^-[n^j]) \), where \( n^j = [n^j_1,\ldots,n^j_n] \), \( 1 \leq j \leq t \). If \( n,m \in \mathbb{Z}^r \) then we write \( n \succ m \) if \( n_i \geq m_i \) for all \( i, 1 \leq i \leq n \). Let \( L_j = \{ n \in \mathbb{Z}^r | n \succ n^j \} \) and \( L = \bigcup_j L_j \). With this notation in place we have:

**Corollary 2.4.** The ideal \( I \) is multiplicity free with \( K \)-decomposition:

\[
I = \bigoplus_{m \in L} S(p^-)[m_1,\ldots,m_r].
\]
3. A branching formula.

Let $\omega$ be the unique fundamental weight orthogonal to the compact roots. For any $\Phi^{+}_{c}$-dominant integral weight $\xi$, let $F_{\xi}$ be the irreducible finite dimensional $\frak{g}$-module with highest weight $\xi$. Let $N(\xi + \rho)$ denote the generalized Verma module $U(\frak{g}) \otimes_{U(\frak{g} \oplus \frak{p}^{+})} F_{\xi}$.

**Theorem 3.1.** For $m \in \mathbb{N}$ let $E_{m\omega}$ be the irreducible finite dimensional $\frak{g}$-module with highest weight $m\omega$. Then as a $\frak{t}$-module,

$$E_{m\omega} \cong \bigoplus_{m \geq n_{1} \geq \cdots \geq n_{r} \geq 0} F_{-n_{1}\gamma_{1} - n_{2}\gamma_{2} - \cdots - n_{r}\gamma_{r}} \otimes F_{m\omega}.$$

**Proof.** Let $\alpha$ denote the unique simple noncompact root. Then $\alpha = \gamma_{1}$. From the BGG resolution we obtain the exact sequence:

$$N(s_{\alpha}(m\omega + \rho)) \to N(m\omega + \rho) \to E_{m\omega} \to 0.$$

The $\frak{t}$-module $F_{m\omega}$ is one dimensional, so the image of the left term has the form $I \otimes F_{m\omega}$ where $I$ is the ideal in $S(\frak{p}^{-})$ generated by $S(\frak{p}^{-})[m+1,0,\ldots,0]$. The result now follows from Corollary 2.3. \qed

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PRODUCTS OF NON-STATIONARY RANDOM MATRICES
AND MULTIPERIODIC EQUATIONS OF SEVERAL
SCALING FACTORS

Ai-Hua Fan, Benoît Saussol, and Jörg Schmeling

Let $\beta > 1$ be a real number and $M : \mathbb{R} \to \text{GL}(\mathbb{C}^d)$ be a uniformly almost periodic matrix-valued function. We study the asymptotic behavior of the product

$$P_n(x) = M(\beta^{n-1}x) \ldots M(\beta x)M(x).$$

Under some conditions we prove a theorem of Furstenberg-Kesten type for such products of non-stationary random matrices. Theorems of Kingman and Oseledec type are also proved. The obtained results are applied to multiplicative functions defined by commensurable scaling factors. We get a positive answer to a Strichartz conjecture on the asymptotic behavior of such multiperiodic functions. The case where $\beta$ is a Pisot–Vijayaraghavan number is well studied.

1. Introduction.

Kingman’s subadditive ergodic theorem was originally proved in 1968 ([Ki1] and [Ki2]). A more recent proof was given by Katznelson and Weiss in 1982 [KW]. It is one of the most important results in ergodic theory. In this paper we consider the following set-up which resembles a dynamical system without invariant measure and try to get results similar to Kingman’s theorem. Let $\beta > 1$ be a positive real number. Let $\{f_n\}$ be a sequence of uniformly almost periodic functions (i.e., in the sense of Bohr, see Section 2.1) defined on the real line $\mathbb{R}$. Suppose the following subadditivity condition is fulfilled:

$$f_{n+m}(x) \leq f_n(x) + f_m(\beta^n x) \text{ for a.e. } x \text{ and all } n, m,$$

where a.e. refers to the Lebesgue measure. We would like to study the almost everywhere convergence of $n^{-1}f_n(x)$. The Kingman theorem applies in the special case where $\beta > 1$ is an integer and the $f_n$’s are periodic. The typical case in our mind is

$$f_n(x) = \log \|M(\beta^{n-1}x) \ldots M(\beta x)M(x)\|$$

where $M : \mathbb{R} \to \text{GL}(\mathbb{C}^d)$ is a matrix-valued uniformly almost periodic function. We will prove that the limit $\lim_{n \to \infty} n^{-1}f_n(x)$ exists almost everywhere
(a.e. for short) with respect to the Lebesgue measure under the condition that the $n^{-1} f_n(x)$ have joint periods (see Theorem 2.5). As a consequence, an Oseledec type theorem is proved for the matrix products involved in (1.1) (see Theorem 2.9). It is proved that the condition on the existence of joint periods is satisfied when $\beta$ is a PV-number (see Section 3).

Our consideration is partially motivated by the study of multiperiodic functions, already investigated by Strichartz et al. [JRS], Fan and Lau [FL], and Fan [F]. By a Multiperiodic function of one real variable we mean any function $F : \mathbb{R} \to \mathbb{R}$ which is a solution of a functional equation of the following form:

$$F(\xi) = f_1 \left( \frac{\xi}{\rho_1} \right) F \left( \frac{\xi}{\rho_1} \right) + \cdots + f_d \left( \frac{\xi}{\rho_d} \right) F \left( \frac{\xi}{\rho_d} \right)$$

where $d \geq 1$ is an integer; $\rho_1 > 1, \ldots, \rho_d > 1$ are $d$ real numbers, called scaling factors; $f_1, \ldots, f_d$ are $d$ complex valued functions defined on the real line, called determining functions. The equation will be called a multiperiodic equation.

We will assume that the determining functions $f_j$ are periodic or almost periodic in the sense of Bohr, as is the case in most applications. We will also assume that the scaling factors $\rho_j$ are commensurable in the sense that $\rho_j$ are powers of some real number $\beta > 1$. Without loss of generality, we assume that $\rho_j = \beta^j$ for $1 \leq j \leq d$. Then the multiperiodic equation becomes

$$(1.2) \quad F(\xi) = f_1 \left( \frac{\xi}{\beta} \right) F \left( \frac{\xi}{\beta} \right) + \cdots + f_d \left( \frac{\xi}{\beta^d} \right) F \left( \frac{\xi}{\beta^d} \right).$$

As far as we know, there is few work done for the non-commensurable case which is much more difficult.

In the literature, the case where $d = 1$ and $\beta = 2$ (or an arbitrary integer) has been studied, especially in the theory of wavelets [D]. In fact, the scaling function $\varphi$ of a wavelet satisfies a scaling equation

$$\varphi(x) = \sum a_n \varphi(2x - n).$$

The Fourier transform of $\varphi$ satisfies a multiperiodic equation of the form (1.2) with only one scaling factor $\beta = 2$ and only one determining function $f_1(x) = f(x) = \frac{1}{2} \sum_n a_n e^{inx}$.

The scaling functions in wavelets constitute a class of functions sharing a kind of similarity. More generally, multiperiodic functions arise as Fourier transforms of self-similar objects such as Bernoulli convolution measures ($d = 1$, $\beta > 1$ being a real number and $f$ being a trigonometric polynomial), inhomogeneous Cantor measures ($d$ may be greater than 1) or more general self-similar measures produced by iterated function systems (see [S]). In the case of one scaling factor (i.e., $d = 1$, then we write $f_1(x) = f(x)$), the existence of the solution of the multiperiodic Equation (1.2) is
simple and is assured by the consistency condition \( f(0) = 1 \) and a regularity condition, say \( f \) is Lipschitz continuous. Actually the solution can be written as an infinite product
\[
F(x) = \prod_{n=1}^{\infty} f\left(\frac{x}{\beta^n}\right).
\]

For the existence of the general Equation (1.2), we have:

**Theorem A.** Let \( d \geq 1 \). Suppose that the determining functions \( f_1, \ldots, f_d \) are Lipschitz continuous, and satisfy the consistency condition
\[
f_1(0) + \cdots + f_d(0) = 1.
\]
Suppose furthermore that \( f_1(0), \ldots, f_d(0) \in [0, +\infty) \). Then Equation (1.2) admits a unique continuous solution \( F \) such that \( F(0) = 1 \).

The proof of this theorem is postponed to Section 4.2.

Our study of Equation (1.2) is converted to that of vector valued equations of the form
\[
G(x) = M \left( \frac{x}{\beta} \right) G \left( \frac{x}{\beta} \right)
\]
where \( M : \mathbb{R} \to \mathcal{M}_{d \times d}(\mathbb{C}) \) is a matrix valued determining function and \( G : \mathbb{R} \to \mathbb{R}^d \) is a vector valued unknown function. Matrix products will be involved in the study of Equation (1.3), which produces some difficulties. However, Equation (1.3) is a simple recursive relation because it contains only one scaling factor. Equation (1.2) is equivalent to Equation (1.3) with \( M(x) \) and \( G(x) \) equal respectively to
\[
\begin{pmatrix}
  f_1(x) & f_2\left(\frac{x}{\beta}\right) & \cdots & f_{d-1}\left(\frac{x}{\beta^{d-2}}\right) & f_d\left(\frac{x}{\beta^{d-1}}\right) \\
  1 & 0 & \cdots & 0 & 0 \\
  0 & \ddots & \ddots & \vdots & \vdots \\
  \vdots & \ddots & 0 & 0 & 0 \\
  0 & \cdots & 0 & 1 & 0
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
  F(x) \\
  F(\frac{x}{\beta}) \\
  \vdots \\
  F(\frac{x}{\beta^{d-1}})
\end{pmatrix}
\]

We would like to know the asymptotic behavior at infinity of the solution \( G \). This is a natural question because \( G \) often represents Fourier transform of a function (a measure or a distribution) and the asymptotic behavior at infinity describes quantitatively the regularity of the solution. Unfortunately, there is no closed form formula for \( G \) in general and the behavior of \( G \) is rather complicated, as is shown by the Fourier transform of the Cantor measure \( (d = 1, \beta = 3 \) and \( f(\xi) = \cos \xi) \).

Following [JRS], we will study the pointwise asymptotic behaviors of
\[
h_n(x) := \frac{1}{n} \log |F(\beta^n x)|
\]
as $n \to \infty$. We will prove that, under some conditions, the limit $\lim_{n \to \infty} h_n(x)$ exists and is equal to a constant almost everywhere with respect to Lebesgue measure. This answers partially a question in [JRS]. More precisely, we have the following results, whose proofs are postponed to Section 4.3:

**Theorem B.** Suppose that the conditions in Theorem A are satisfied. Furthermore, suppose that the determining functions $f_1, \ldots, f_d$ are either identically zero or strictly positive and 1-periodic, and that $\beta > 1$ is a Pisot number. Let $F$ be the solution of Equation (1.2). Then there is a constant $L$ such that

$$\lim_{n \to \infty} \frac{1}{n} \log F(\beta^n x) = L \quad \text{a.e.}$$

The constant $L$ in the theorem is the leading Liapunov exponent of the matrix $M(x)$ above, defined by

$$L(M) = \inf_{n \geq 1} \frac{1}{n} \log \|M(\beta^{n-1} x)M(\beta^{n-2} x) \ldots M(\beta x)M(x)\|,$$

where $Mf$ denotes the Bohr mean of an almost periodic function $f$ (see Section 2.1 below).

**Theorem C.** Suppose that the conditions in Theorem A are satisfied. Furthermore, suppose that the determining functions $f_1, \ldots, f_d$ are 1-periodic, Lipschitz, and that $\beta > 1$ is a Pisot number with maximal conjugate of modulus $\rho$. Let $F$ be the solution of Equation (1.2). If

$$\sup_x \frac{(1 + |f_1(x)| + \cdots + |f_d(x/\beta^{d-2})|)(|f_1(x)| + \cdots + |f_d(x/\beta^{d-1})|)}{|f_d(x/\beta^{d-1})|} < \rho^{-1}$$

then there is a constant $\lambda \in \mathbb{R}$ such that

$$\lim_{n \to \infty} \frac{1}{n} \log \sum_{j=0}^{d-1} |F(\beta^{n-j} x)| = \lambda \quad \text{a.e.}$$

In this case we do not know if the constant $\lambda$ equals the leading Lyapunov exponent of the matrix.

Theorem A will be proved in Section 4.2 as a special case of a more general result (Theorem 4.1). Theorem B and Theorem C will be proved in Section 4.3. Both Theorem B and Theorem C are consequences of our Kingman’s Theorem (Theorem 2.5) and Oseledec’s Theorem (Theorem 2.9) which are discussed in Section 2. In Section 3, we prove that the joint period condition required in both Kingman’s Theorem and Oseledec Theorem is satisfied when $\beta > 1$ is a Pisot number.
2. Kingman theorem and Oseledec theorem.

2.1. Total Bohr ergodicity and joint $\epsilon$-period. Let us first recall the definition of uniformly almost periodic functions and some of their properties (see [Bo]). Next we will introduce the notions of total Bohr ergodicity and a joint $\epsilon$-period.

Let $f$ be a real or complex valued function defined on the real line. A number $\tau$ is called a translation number of $f$ belonging to $\epsilon \geq 0$ (or an $\epsilon$-period) if

$$\sup_{x \in \mathbb{R}} |f(x + \tau) - f(x)| \leq \epsilon.$$ 

We say that $f$ is a uniformly almost periodic (u.a.p.) function if it is continuous and if for any $\epsilon > 0$ the set of its translation numbers belonging to $\epsilon$ is relatively dense (i.e., there exists a number $\ell > 0$ such that any interval of length $\ell$ contains at least one such translation number). H. Bohr proved that the space of all u.a.p. functions is a closed sub-algebra of the Banach algebra $C_b(\mathbb{R})$ of bounded continuous functions equipped with the uniform norm and that it is the closure of the space of all (generalized) trigonometric polynomials of the form

$$\sum_{\text{finite}} A_n e^{i \Lambda_n x} \quad (A_n \in \mathbb{C}, \Lambda_n \in \mathbb{R}).$$

For any u.a.p. function $f$, as is proved by Bohr, the following limit exists:

$$Mf = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x) dx.$$ 

It is called the Bohr mean of $f$. For any locally integrable not necessarily u.a.p. $f$, we define $Mf$ as the limsup instead of the limit.

**Definition 2.1.** A sequence of real numbers $(u_n)_{n \geq 0}$ is said to be totally Bohr ergodic if for any arithmetic subsequence $(u_{am+b})_{m \geq 0}$ ($a \geq 1, b \geq 0$ being fixed) and for any real $p > 0$, the sequence $(u_{am+b}x)_{m \geq 0}$ is uniformly distributed (modulo $p$) for almost every $x \in \mathbb{R}$ with respect to the Lebesgue measure.

The following is the main property of totally Bohr ergodic sequences that we will use:

**Lemma 2.2.** Suppose that $(u_n)_{n \geq 0}$ is a totally Bohr ergodic sequence. Then for any u.a.p. function $f$ and any integers $a \geq 1, b \geq 0$, we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} f(u_{an+b}x) = Mf \quad \text{a.e.}$$
Proof. It is a consequence of the fact that \( u_{an+bx} \) is uniformly distributed (mod \( p \)) for almost all \( x \), for any real \( p > 0 \), the fact that \( f \) can be uniformly approximated by trigonometric polynomials and the Weyl criterion. \( \square \)

Remark 2.3. Suppose that \((u_n)_{n \geq 0}\) is a sequence such that \( \inf_{n \neq m} |u_n - u_m| > 0 \), then the sequence \((u_n x)\) is uniformly distributed for almost every point \( x \) [KL]. Consequently, the sequence \((u_n)\) is totally Bohr ergodic. A more special case is \( u_n = \beta^n \) with \( \beta > 1 \). This is the most interesting case for us. On the other hand, no bounded sequence can be totally Bohr ergodic.

Definition 2.4. Let \((F_n)_{n \geq 0}\) be a sequence of u.a.p. functions. Let \( \epsilon > 0 \) and \( N \in \mathbb{N} \). A real number \( \tau \) is called a joint \( \epsilon \)-translation number for \((F_n)_{n \geq N}\) if
\[
\sup_{n \geq N} \sup_{x \in \mathbb{R}} |F_n(x + \tau) - F_n(x)| \leq \epsilon.
\]
If for any \( \epsilon > 0 \) there exists \( N(\epsilon) \) such that such joint \( \epsilon \)-translation numbers for \((F_n)_{n \geq N(\epsilon)}\) are relatively dense, we say that \((F_n)_{n \geq 0}\) has joint periods.

2.2. Kingman’s theorem. Following ideas of Katznelson and Weiss [KW] we prove the following version of Kingman’s Theorem. The difficulty in our case is that we have to deal with an infinite measure space. We are also dealing with non-stationary sequences.

Theorem 2.5. Let \((u_n)_{n \geq 0}\) be a totally Bohr ergodic sequence of real numbers and \((f_n)_{n \geq 0}\) be a sequence of uniformly almost periodic functions. Suppose:

(i) The sequence \( (n^{-1}f_n) \) has joint periods.

(ii) The following subadditivity is fulfilled:
\[
f_{n+m}(x) \leq f_n(x) + f_m(u_n x) \quad \text{for a.e. } x \text{ and all } n,m.
\]

(iii) For any \( n \geq 1 \)
\[
\sup_m (f_m(u_n x) - f_m(x)) < \infty \quad \text{for a.e. } x.
\]

Then the following limit exists and is a constant:
\[
\lim_{n \to \infty} \frac{1}{n} f_n(x) = \inf \frac{1}{n} \mathbb{M} f_n \quad \text{for a.e. } x.
\]

Proof. The proof is a modification of Katznelson-Weiss’ proof [KW]. Without loss of generality we assume that \( u_0 = 1 \).

Let \( \gamma = \inf_n \frac{1}{n} \mathbb{M} f_n \). Let us put
\[
f^-(x) = \lim_{n \to \infty} \frac{1}{n} f_n(x), \quad f^+(x) = \lim_{n \to \infty} \frac{1}{n} f_n(x).
\]
We remark that the subadditivity implies that \( f^\pm(x) \leq f^\pm(u_n x) \) for all \( n \in \mathbb{N} \) and a.e. \( x \in \mathbb{R} \). In a finite measure space this would imply the
invariance a.e. In our case it is the boundedness (2.1) which implies that
\( f^\pm(u_n x) \leq f^\pm(x) \) a.e., what makes the functions \( f^\pm \) invariant in the sense that
\( f^\pm(x) = f^\pm(u_n x) \) a.e. (\( \forall n \)).

The first part of the proof, i.e., \( f^+ \leq \gamma \) a.e., is simple. We just exploit the
fact that any (infinite) arithmetical subsequence of \( (u_n x) \) is Bohr-uniform
distributed. This provides us with a kind of ergodic theorem. In fact, fix an
integer \( N \). For any integer \( n \) write \( n = mN + r \) with \( 0 \leq r < N \). We have
\[
    f_n(x) \leq \sum_{k=0}^{m-1} f_N(u_k N x) + f_r(u_m N x).
\]
The \( N \) functions \( f_r \) \( (r = 0, 1, \ldots, N - 1) \) being bounded, by Lemma 2.2, this
readily implies by the Bohr-uniform distribution of \( (u_k N x)_{k \geq 0} \) that
\[
    f^+(x) \leq \lim_{m \to \infty} \frac{1}{mN + r} \sum_{k=0}^{m-1} f_N(u_k N (x)) = \frac{1}{N} M^\gamma f_N
\]
for a.e. \( x \). Hence \( f^+(x) \leq \gamma \) for a.e. \( x \).

Next, we want to prove that \( f^-(x) \geq \gamma \) for a.e. \( x \). For this we assume
that \( \gamma > -\infty \), otherwise it is trivially true. Adding to each \( f_n \) the constant
value \(-n \|f_1\|\) creates a new subadditive sequence \( \tilde{f}_n := f_n - n \|f_1\| \) with
\( \tilde{f}_n \leq 0 \) \((n \geq 1)\), \( f^- = \tilde{f}^- + \|f_1\|\) and \( \gamma_f = \gamma + \|f_1\|\). So, we may assume
that \( f_n \leq 0 \) \((n \geq 1)\). We can furthermore set \( f_1 = 0 \) (observe that this will
not affect the subadditivity condition since \( f_n \leq 0 \)). Then for any \( \Delta > 0 \)
we truncate the function \( f_n \) in the way
\[
    f_{n, \Delta} = \max(f_n, -n \Delta).
\]
Note that the sequence \( f_{n, \Delta} \) fulfills the assumptions of the theorem. Note
that in this case \( \gamma_\Delta \geq -\Delta \) and also \( f^-_{\Delta}(x) \geq -\Delta \) for all \( x \). It is clear that
\[
    f^-_{\Delta}(x) = \max(f^-(x), -\Delta).
\]
Assume that we proved the theorem for the sequence \( f_{n, \Delta} \) for any \( \Delta \). Then
we claim that \( f^-_{\Delta}(x) \geq f^-(x) \) for a.e. \( x \) as \( \Delta \) goes to \( \infty \). In fact, if \( f^-_{\Delta}(x) > -\Delta \) for some \( \Delta \) then \( f^-_{\Delta}(x) = f^-(x) \) and \( \gamma_\Delta = \gamma_f > -\infty \). On the other hand
if \( f^-_{\Delta}(x) = -\Delta \) for all \( \Delta \) then obviously \( f^-(x) = -\infty \) and \( \gamma_\Delta = \gamma_f = -\infty \).
This proves the theorem for the sequence \( f_n \).

From now on we assume that \( f_1 = 0, f_n \leq 0 \) and \( f_n \) is truncated and we
skip the subscript \( \Delta \). Let \( \epsilon > 0 \). By Hypothesis (i) on the joint periods, there
is an integer \( N(\epsilon) \) such that the joint \( \epsilon \)-translation numbers are relatively
dense. For these numbers \( \tau \) we have
\[
    \left| \frac{f_n(x + \tau)}{n} - \frac{f_n(x)}{n} \right| \leq \epsilon \quad (\forall x \in \mathbb{R}, \forall n \geq N(\epsilon)).
\]
Notice that there is no loss of generality to suppose that $N(\epsilon)$ increases as $\epsilon$ decreases to 0. We define

$$n_\epsilon(x) = \min \left\{ n \geq N(\epsilon): \frac{1}{n} f_n(x) \leq f^-(x) + \epsilon \right\}.$$ 

Let $A_\epsilon^K = \{ x: n_\epsilon(x) > K \}$. Notice that if $\epsilon' < \epsilon''$, we have $N(\epsilon') \geq N(\epsilon'')$ and $n_{\epsilon'}(x) \geq n_{\epsilon''}(x)$, so that $A_{\epsilon''}^K \subset A_{\epsilon'}^K$ for $K > N(\epsilon)$.

We claim that:

**Lemma 2.6.** For any $\epsilon > 0$, we have

$$\lim_{K \to \infty} \hat{M}(A_{\epsilon}^K) = 0$$

where $\hat{M}A = \hat{M}1_A$ denotes the Bohr mean of the characteristic function of the set $A$ (defined if necessary with the limsup).

In order to prove this Lemma 2.6 we need the following lemma which says that $A_{\epsilon}^K$ is to some extent periodic:

**Lemma 2.7.** For any joint $\epsilon$-translation number $\tau$ of $(f_n/n)_{n \geq N(\epsilon)}$, we have $A_{2\epsilon}^K + \tau \subset A_{\epsilon}^K$.

Let us first prove Lemma 2.7. Suppose $x \in A_{2\epsilon}^K + \tau$, i.e., $x - \tau \in A_{2\epsilon}^K$, then

$$\frac{f_n(x - \tau)}{n} > f^-(x - \tau) + 2\epsilon \quad (N(2\epsilon) \leq n \leq K).$$

This, together with the fact that $\tau$ is a joint $\epsilon$-translation number for all $f_n/n$ with $n \geq N(\epsilon)(\geq N(2\epsilon))$ (see (2.2)), implies

$$\frac{f_n(x)}{n} \geq \frac{f_n(x - \tau)}{n} - \epsilon > f^-(x - \tau) + \epsilon \quad (N(\epsilon) \leq n \leq K).$$

That means $x \in A_{\epsilon}^K$. Thus we have finished the proof of Lemma 2.7.

Now let us prove Lemma 2.6. Since joint $\frac{\epsilon}{2}$-translation numbers are relatively dense there exists $L = L(\frac{\epsilon}{2}) > 0$ such that any interval of length $L$ contains such a joint $\frac{\epsilon}{2}$-translation number. Since $\cap_K A_{2\epsilon}^K = \emptyset$, for any $\eta > 0$ there exists $K_0 > 0$ such that

$$\left| A_{\frac{\epsilon}{2}}^K \cap [-L, L] \right| < L\eta \quad (\forall K \geq K_0)$$

(2.3)

where $| \cdot |$ denotes the Lebesgue measure (see the definition of $n_\epsilon(x)$). We claim that $\hat{M}(A_{\epsilon}^K) \leq \eta \ (\forall K \geq K_0)$. Otherwise $\hat{M}(A_{\epsilon}^K) > \eta$ for some $K \geq K_0$. Then by the definition of $\hat{M}(A_{\epsilon}^K)$ there exists $x_0 \in \mathbb{R}$ such that

$$\int_{x_0}^{x_0+L} \chi_{A_{\epsilon}^K}(x) dx \geq L\eta.$$
Take a joint $\frac{\varepsilon}{2}$-translation number $\tau \in [-x_0 - L, -x_0]$, i.e., $-L \leq x_0 + \tau \leq 0$. Then by Lemma 2.7, we have

\[
\left| A^\frac{\varepsilon}{2}_K \cap [-L, L] \right| \geq \int_{x_0 + \tau}^{x_0 + \tau + L} \chi_{A^\frac{\varepsilon}{2}_K}(x) \, dx = \int_{x_0}^{x_0 + L} \chi_{A^\frac{\varepsilon}{2}_K}(y + \tau) \, dy \geq \int_{x_0}^{x_0 + L} \chi_{A^\varepsilon_K}(y) \, dy.
\]

For the first inequality we have used the fact that $[x_0 + \tau, x_0 + \tau + L] \subset [-L, L]$ and for the last inequality we have used Lemma 2.7. What we have deduced contradicts (2.3). Thus Lemma 2.6 is proved.

We continue our proof of Theorem 2.5. Let $S := \|f^-\|_\infty < \infty$. Let $K$ be such that $\hat{M}(A^\varepsilon_K) \leq \varepsilon/S$. We define first

\[
g(x) = \begin{cases} f^-(x) & \text{if } x \notin A^\varepsilon_K \\ 0 & \text{if } x \in A^\varepsilon_K \end{cases}
\]

and

\[
m(x) = \begin{cases} n^\varepsilon(x) & \text{if } x \notin A^\varepsilon_K \\ 1 & \text{if } x \in A^\varepsilon_K. \end{cases}
\]

Lemma 2.6 implies that

\[
\mathbb{M}g \leq \mathbb{M}f^- + \varepsilon \quad \text{a.e.}
\]

Moreover by the invariance of $f^-$ we have (remember that $u_0 = 1$)

\[
g(x) \leq g(u_kx) \quad \text{for a.e. } x \text{ and all } 0 \leq k \leq m(x) - 1.
\]

Then we have

\[
f_{m(x)}(x) \leq (g(x) + \varepsilon)m(x) \leq \sum_{k=0}^{m(x)-1} g(u_kx) + \varepsilon m(x).
\]

We define inductively $m_0(x) = 0$ and

\[
m_k(x) = m_{k-1}(x) + m(u_{m_{k-1}(x)}x).
\]

Now choose $R > K$ and let $k(x)$ be the maximal $k$ for which $m_k(x) \leq R$. Note that $R - m_{k(x)}(x) < K$. Now we get by the subadditivity and Equation

\[
\text{(2.6)}
\]

\[
f_{m(x)}(x) \leq (g(x) + \varepsilon)m(x) \leq \sum_{k=0}^{m(x)-1} g(u_kx) + \varepsilon m(x).
\]
\[ f_R(x) \leq \sum_{k=0}^{k(x)-1} f_{m_k(x)}(u_{m_k(x)}x) + f_{R-m_k(x)}(u_{m_k(x)}x) \leq 0 \]
\[ \leq \sum_{k=0}^{k(x)-1} m_k(x) \sum_{j=m_k-1}^{m_k(x)} g(u_jx) + (m_k(x) - m_k-1(x))\epsilon \]
\[ \leq \sum_{j=0}^{R-1} g(u_jx) + m_k(x)\epsilon \]
\[ \leq \sum_{j=0}^{R-1} g(u_jx) + R\epsilon + KS. \]

Taking the Bohr mean, using that \( \bar{M}g = \bar{M}(g \circ u_j) \) and dividing by \( R \) gives
\[
\frac{1}{R} \bar{M}f_R \leq \bar{M}g + \epsilon + \frac{KS}{R} \leq \bar{M}f^- + 2\epsilon + \frac{KS}{R}
\]
by Equation (2.4). Now we let \( R \to \infty \) and we get
\[ \gamma \leq \bar{M}f^-. \]

We claim that \( f^- \leq \gamma \) implies \( f^- = \gamma \) for a.e. \( x \). Suppose this was not the case, then one could find \( \epsilon > 0, \delta > 0 \) and an interval \( J = (0, L) \) of length \( |J| = L = L_\epsilon \) such that \( |A \cap J| = \delta > 0 \), where
\[ A = \{x: f^-(x) < \gamma - \epsilon\}. \]
By the invariance of \( f^- \) we have \( u_kA = A \) for all \( k \in \mathbb{N} \). Hence, for all \( k \in \mathbb{N} \)
\[
\frac{1}{u_kL} \int_0^{u_kL} \chi_A dx = \frac{1}{L} \int_0^L \chi_A dx > \frac{\delta}{L} > 0.
\]
Since \( \limsup_k u_k = +\infty \) (see Remark 2.3), we have \( \bar{M}A > \frac{\delta}{L} \), and thus
\[
\bar{M}f^- < \gamma \left(1 - \frac{\delta}{L}\right) + (\gamma - \epsilon)\frac{\delta}{L} < \gamma.
\]
\[ \square \]

**Remark 2.8.** One can prove a similar theorem for more general sequences \((u_n(x))\). In this case it seems to be necessary to assume \( L^1 \) Bohr-uniform distribution.
2.3. Oseledec theorem. Kingman’s theorem implies the following Oseledec type theorem (see Ruelle [Ru]):

**Theorem 2.9.** Let $\beta > 1$ be a real number. Let $M : \mathbb{R} \to GL_d(\mathbb{C})$ be a uniformly almost periodic function. Write

$$M^n_x = M(\beta^{n-1}x) \ldots M(\beta x)M(x).$$

Suppose the $q$-exterior products $\frac{1}{n} \log \| (M^n_x)^\wedge q \|$ have joint periods, for $q = 1, \ldots, d$. Then there is $\Gamma \subset \mathbb{R}$ with $\beta \Gamma \subset \Gamma$ of full Lebesgue measure (in the sense that $\mathbb{R} \setminus \Gamma$ has 0 measure) such that if $x \in \Gamma$ then:

a) $\lim_{n \to \infty} (M^n_x \cdot M^n_x)^\frac{1}{2n} = \Lambda_x$ exists.

b) Let $\exp \lambda^{(1)}_x < \cdots < \exp \lambda^{(s)}_x$ be the eigenvalues of $\Lambda_x$ (where $s = s(x)$ and the $\lambda^{(r)}_x$ are reals), and $V^{(1)}_x, \ldots, V^{(s)}_x$ the corresponding eigenspaces. Let $m^{(r)}_x = \dim V^{(r)}_x$. We have $\lambda^{(r)}_{\beta x} = \lambda^{(r)}_x$ and $m^{(r)}_{\beta x} = m^{(r)}_x$ and

$$\lim_{n \to \infty} \frac{1}{n} \log \| M^n_x v \| = \lambda^{(r)}_x \quad \text{when} \quad v \in V^{(r)}_x \setminus V^{(r-1)}_x$$

for $r = 0, \ldots, s$ where $V^{(0)}_x = \{0\}$ and $V^{(s)}_x = U^{(1)}_x + \cdots + U^{(r)}_x$.

c) Moreover $V^{(r)}_x$ depends measurably on $x$ and $M_x V^{(r)}_x = V^{(r)}_x$.

d) In addition the functions $\lambda^{(r)}_x$ and $m^{(r)}_x$ are constant a.e.

**Proof.** This theorem follows in the standard way from the a.e. convergence of

$$\lim_{n \to \infty} \frac{1}{n} \log \| (M^n_x)^\wedge q \|$$

for $1 \leq q \leq d$, which in the classical case is insured by Kingman’s theorem.

So we only need to check that for the functions $f^{(q)}_n(x) = \log \| (M^n_x)^\wedge q \|$ and the sequence $u_n = \beta^n$ the assumptions of Theorem 2.5 hold.

First we note that $M^{-1}$ is uniformly almost periodic because $M$ is uniformly almost periodic and $M^{-1}(x) \in GL_d(\mathbb{C})$. Second we note that $M^\wedge q$ and $(M^\wedge q)^{-1}$ are again uniformly almost periodic, since each entry is a rational function of the entries of $M$ and $M^{-1}$, respectively. Hence,

$$\sup_{x \in \mathbb{R}} \| M(x)^\wedge q \| = \sup_{x \in \mathbb{R}} \| (M^\wedge q)^{-1} \| = W_q < \infty.$$ 

Subadditivity is obviously fulfilled since $(M^{n+m})^\wedge q = (M^{m}_{\beta^n x})^\wedge q (M^n_x)^\wedge q$.

Condition (2.1) follows from

$$\|(M^n_x)^\wedge q\| \leq \| M(\beta^n x)^\wedge q (M^n_x)^\wedge q (M(x)^\wedge q)^{-1}\| \leq \|(M^n_x)^\wedge q\| + W_q.$$ 

Finally by Remark 2.3 the sequence $\beta^n$ is totally Bohr ergodic, so Theorem 2.5 applies. Assertions (a), (b) and (c) follow from Proposition 1.3 (see also the proof of Theorem 1.6) in [Ru].
Now we prove (d). By Kingman’s theorem (Theorem 2.5), we have for almost all \( x \)

\[
\lim_{n \to \infty} \frac{1}{n} \log \|M_n \wedge^k x\| = \inf_{n \geq 1} \frac{1}{n} \log \|M_n \wedge^k x\| \quad (1 \leq k \leq s).
\]

(2.7)

On the other hand, by the properties of exterior product, if we write \( k = \sum_{i=1}^{j-1} m_x(s-i) + \ell \) with \( 0 \leq j < s \) and \( 0 \leq \ell \leq m_x(s-j) \) we have

\[
\lim_{n \to \infty} \frac{1}{n} \log \|M_n \wedge^k x\| = m_x(s) \lambda_x(s) + \cdots + m_x(s-j+1) \lambda_x(s-j+1) + \ell \lambda_x(s-j).
\]

(2.8)

We can solve \( \lambda_x(r) \) \((1 \leq r \leq s)\) from the system (2.7)-(2.8). The solution is independent of \( x \) since it depends only on the right-hand side terms in (2.7). Consequently \( m_x(r) \) is also independent of \( x \). \( \square \)

3. When \( \beta \) is a Pisot-Vijayaraghavan number.

We restrict our attention to the special case where \( \beta > 1 \) is a Pisot-Vijayaraghavan (PV) number and \( f_n \) are defined by (1.1). We will prove that, under some extra conditions, the sequence \( n^{-1}f_n \) has joint periods and the Kingman theorem and the Oseledec theorem apply. To do this, we need a distortion lemma and some properties of PV-numbers.

3.1. Distortion lemmas.

**Lemma 3.1.** Let \( M : \mathbb{R} \to GL_d(\mathbb{C}) \) such that

\[
D := \sup_{x \in \mathbb{R}} \|M(x)\| \|M(x)^{-1}\| < \infty.
\]

(3.1)

Let \( (x_k) \) and \( (y_k) \) be two sequences in \( \mathbb{R} \) and let \( \theta_k = \|M(x_k) - M(y_k)\| \). Then for all \( n \in \mathbb{N} \) and \( 0 \neq v \in \mathbb{C}^d \) we have

\[
\frac{|M(x_1) \ldots M(x_{n-1})M(x_n)v|}{|M(y_1) \ldots M(y_{n-1})M(y_n)v|} \leq \exp \left( C \sum_{k=1}^{n} D^k \theta_k \right),
\]

where \( C = (\sup_{x \in \mathbb{R}} \|M(x)\|)^{-1} \).

**Proof.** Let

\[
Q_1(x) = I, \quad Q_k(x) = M(x_1)M(x_2) \ldots M(x_{k-1}) \quad (1 < k \leq n)
\]

\[
Q^n(x) = I, \quad Q^k(x) = M(x_{k+1})M(x_{k+2}) \ldots M(x_n) \quad (1 \leq k < n).
\]

We can write

\[
\frac{|Q_n(x)v|}{|Q_n(y)v|} = \prod_{k=1}^{n} \frac{|Q_k(y)M(x_k)Q^k(x)v|}{|Q_k(y)M(y_k)Q^k(x)v|}.
\]

(3.2)
Proof. We may write
\[ |Q_k(y)M(x)Q(x)v| = |Q_k(x)(I + E_k)M(y)Q(x)v| \]
\[ = |Q_k(x)(I + E_k)Q_k^{-1}Q_k(x)M(y)Q(x)v| \]
\[ = \|I + \tilde{E}_k\|Q_k(x)M(y)Q(x)v| \]
where \( \tilde{E}_k = Q_k(x)E_kQ_k(x)^{-1} \). It follows that
\[ |Q_k(y)M(x)Q(x)v| \leq \|I + \tilde{E}_k\|. \tag{3.3} \]
It is obvious that
\[ \|\tilde{E}_k\| \leq D^{-1}\|E_k\|. \tag{3.4} \]
On the other hand
\[ \|E_k\| \leq \sup_x \|M(x)^{-1}\|\theta_k. \tag{3.5} \]
By combining (3.2), (3.3), (3.4) and (3.5), we obtain
\[ \frac{|M(x_1) \ldots M(x_{n-1})M(x_n)v|}{|M(y_1) \ldots M(y_{n-1})M(y_n)v|} \leq \prod_{k=0}^{n-1} \left( 1 + CD\theta_k \right). \]

If \( M(x) \) is nonnegative, the next lemma shows that Condition (3.1) is not needed for positive vectors:

Lemma 3.2. Let \( M : \mathbb{R} \to GL_d(\mathbb{R}) \) be such that the entries of \( M(x) \) are either identically zero or bounded from below by a positive number \( \delta > 0 \) (independent of entries). Then for any sequences \( (x_k) \) and \( (y_k) \) in \( \mathbb{R} \) and for any nonnegative vector \( v \) we have
\[ \frac{\left| M(x_1) \ldots M(x_{n-1})M(x_n)v \right|}{\left| M(y_1) \ldots M(y_{n-1})M(y_n)v \right|} \leq \exp \left( \frac{1}{\delta} \sum_{k=1}^{n} \theta_k \right), \]
where \( \theta_k = \|M(x_k) - M(y_k)\| \) and the norm \( |v| = \sum_{i=1}^{d} |v_i| \) on \( \mathbb{R}^d \) is chosen.

Proof. We may write
\[ |M(x_1) \ldots M(x_{n-1})M(x_n)v| = \sum_{i_0,i_1,\ldots,i_n} M(x_1)_{i_0,i_1} M(x_2)_{i_1,i_2} \ldots M(x_n)_{i_{n-1},i_n} v_i. \]
We have a similar expression for \( |M(y_1) \ldots M(y_{n-1})M(y_n)v| \). Now compare the two expressions term by term. By the hypothesis, both quantities
$M(x_i)_{i_0,i_1}$ and $M(y_i)_{i_0,i_1}$ are either zero or larger than $\delta$. So, using the trivial inequality $x/y \leq e^{x/y-1}$ we have

$$M(x_i)_{i_0,i_1} \leq M(y_i)_{i_0,i_1} e^{\delta^{-1}y_i}.$$ 

The same estimates hold for other pairs $M(x_k)_{i_{k-1},i_k}$ and $M(y_k)_{i_{k-1},i_k}$. The desired inequality follows. □

### 3.2. Two properties of PV-numbers.

Let $\beta > 1$ be a PV-number of order $r$. We denote its conjugates by $\beta_1', \ldots, \beta_{r-1}'. $ Then for $n \geq 1$, denote

$$F_n = \beta^n + \beta_1'^n + \cdots + \beta_{r-1}'^n.$$ 

#### Lemma 3.3.

The number $F_n$ is an integer and we have

$$|\beta^n - F_n| \leq (r-1)\rho^n \quad (\forall n \geq 1)$$

where $\rho = \max_{1 \leq j \leq r-1} |\beta_j'| < 1$.

Given any real number $\beta > 1$ (not necessarily integral), we can expand each number $x \in [0, 1)$ in a canonical way into its $\beta$-expansion $[\text{Re}]$ (see also [P] and [Bl]):

$$x = \sum_{n=1}^{\infty} \frac{\epsilon_n(x)}{\beta^n}$$

where $(\epsilon_n(x))_{n \geq 1}$ is a uniquely determined sequence in $\{0, 1, \ldots, [\beta]\}^{\infty}$. We may also call $(\epsilon_n(x))_{n \geq 1}$ the $\beta$-expansion of $x$. We note that not all sequences in $\{0, 1, \ldots, [\beta]\}^{\infty}$ are $\beta$-expansions. Let $D_\beta$ be the set of all possible $\beta$-expansions of numbers in $[0, 1)$. A finite sequence $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ (of length $n$) in $\{0, 1, \ldots, [\beta]\}^n$ is said to be admissible if it is the prefix of the $\beta$-expansion of some number $x$. For such an admissible sequence, we define

$$I(\epsilon_1, \ldots, \epsilon_n) = \{x \in [0, 1) : \epsilon_1(x) = \epsilon_1, \ldots, \epsilon_n(x) = \epsilon_n\}.$$ 

It is known that if $D_\beta$ is endowed with the lexicographical order, the map which associates $x$ to its $\beta$-expansion is strictly increasing. The set $I(\epsilon) = I(\epsilon_1, \ldots, \epsilon_n)$ is an interval, called a $\beta$-interval of level $n$. Its length is denoted by $|I(\epsilon)|$.

#### Lemma 3.4.

Suppose $\beta > 1$ is a PV-number. There is a constant $C > 0$ such that

$$C^{-1}\beta^{-n} \leq |I(\epsilon_1, \ldots, \epsilon_n)| \leq C\beta^{-n}$$

for any integer $n \geq 1$ and any $\beta$-interval $I(\epsilon_1, \ldots, \epsilon_n)$.

See [F] for proofs of Lemma 3.3 and Lemma 3.4.
3.3. Existence of joint periods.

**Definition 3.5.** Let $\beta > 1$ be a positive real number and let $M : \mathbb{R} \rightarrow M_{d \times d}(\mathbb{C})$. If the entries of $M$ are functions of the form $f(\beta^n x)$ ($n \in \mathbb{Z}$) where $f$ is 1-periodic continuous, we say that $M$ is $\beta$-adapted u.a.p.

**Remark 3.6.** The matrix $M(x)$ defined by (1.4) associated to a multiperiodic function is $\beta$-adapted.

**Proposition 3.7.** Let $\beta > 1$ be a PV-number and let $M : \mathbb{R} \rightarrow GL_d(\mathbb{C})$ be $\beta$-adapted and $\alpha$-Hölder continuous. Suppose that

$$D \rho^\alpha < 1$$

where $\rho$ is the maximal modulus of the conjugates of $\beta$ and $D$ is the same as in the distortion lemma (Lemma 3.1). Then for any $1 \leq q \leq d$ the sequence $n^{-1} f_n^{(q)}(x)$ has joint periods, where

$$f_n^{(q)}(x) = \log \| (M(\beta^{n-1} x) \ldots M(\beta x) M(x))^{\lambda_q} \|.$$

**Proof.** Since $M$ is $\beta$-adapted, the entries of $M(\beta^k x)$ are all of the form $h_{i,j}(\beta^{k} x)$ with 1-periodic function $h_{i,j}(x)$ and integer $\ell_{i,j} \geq 0$ for sufficiently large $k$. So, if necessary, we consider $\log \| (M(\beta^{n-1} x) \ldots M(\beta^k x))^{\lambda_q} \|$ for some sufficiently large but fixed $k_0 \geq 0$.

Consider $\tau = \beta^m \eta_m + \cdots + \beta \eta_1 + \eta_0$ where $m \geq 1$ and $0 \leq \eta_i \leq \beta$ are integers. We are going to show that all such $\tau$ are joint $\epsilon$-translation numbers for $n^{-1} f_n^{(\alpha)}(x)$ with $n \geq N(\epsilon)$, where $N(\epsilon)$ depending on $\epsilon$ is an integer to be determined.

By Lemma 3.3, we have

$$\inf_{j \in \mathbb{Z}} |\beta^k \tau - j| \leq C' \rho^k$$

for all $k$ and some constant $C'$ independent of $k$ and $\tau$. For $k \geq 0$ each entry of $M^{\lambda_q}(\beta^{k+k_0} x)$ is a degree $q$ polynomial in $d^2$ variables of the form $h(\beta^{k+k_0} x)$, with $h$ $\alpha$-Hölder, 1-periodic, and $\ell \geq 0$. Notice that we have

$$h(\beta^{\ell+k}(x + \tau)) = h(\beta^{\ell+k} x) + O(\rho^{k_0}),$$

hence

$$\| M(\beta^{k+k_0}(x + \tau))^{\lambda_q} - M(\beta^{k+k_0}(x))^{\lambda_q} \| = C_q \rho^{k_0}$$

for some constant $C_q$. By the distortion Lemma 3.1 and the above estimate we have

$$|f_n^{(q)}(x + \tau) - f_n^{(q)}(x)|$$

$$= \left| \log \frac{\| (M(\beta^{n-1} x + \beta^{n-1} \tau) \ldots M(\beta x + \beta \tau) M(x + \tau))^{\lambda_q} \|}{\| (M(\beta^{n-1} x) \ldots M(\beta x) M(x))^{\lambda_q} \|} \right|$$

$$= C_q C' \sum_{k=1}^{n} D^k \rho^{(k-k_0)\alpha} \leq \frac{CC' D \rho^{-k_0}}{1 - D \rho^\alpha} =: c.$$
So, we may choose $N(\epsilon) = C/\epsilon$. In order to finish the proof, it suffices to notice that Lemma 3.4 implies that all these $\tau$ form a subset with bounded gap in $\mathbb{R}$.

**Proposition 3.8.** Let $\beta > 1$ be a PV-number and let $M : \mathbb{R} \to GL_d(\mathbb{C})$ be $\beta$-adapted and $\alpha$-Hölder continuous. Suppose that the entries of $M(x)$ are either identically zero or larger than a constant $\delta > 0$. Then $n^{-1}f_n(x)$ has joint periods, where

$$f_n^{(q)}(x) = \log \| (M(\beta^{n-1}x) \ldots M(\beta x)M(x))^{\wedge q} \|.$$ 

**Proof.** The proof is the same as the last proposition. But we use Lemma 3.2 instead of Lemma 3.1. □

4. Multiperiodic functions.

As we pointed out in the introduction and as we will see in Section 4.2, our scalar Equation (1.2) can be converted to the vector Equation (1.3). So, we first study the vector Equation (1.3).

**4.1. Equation** $G(x) = M(x/\beta)G(x/\beta)$. Let $M : \mathbb{R} \to M_{d \times d}(\mathbb{C})$ be a matrix valued function. We consider the following vector valued equation:

$$G(x) = M \left( \frac{x}{\beta} \right) G \left( \frac{x}{\beta} \right)$$

where the unknown $G : \mathbb{R} \to \mathbb{C}^d$ is a vector valued function.

**Theorem 4.1.** Let $\beta > 1$ be a real number and $M(x)$ be a complex matrix valued function. Suppose that $M$ is Lipschitzian and that $M(0)$ is nonnegative and has 1 as a simple eigenvalue with a corresponding strictly positive eigenvector $v$. Then there exists, up to a multiplicative constant, a unique continuous solution $G(0) \neq 0$ of the equation $G(x) = M(x/\beta)G(x/\beta)$. The solution can be defined by

$$G(x) = \lim_{n \to \infty} M \left( \frac{x}{\beta} \right) M \left( \frac{x}{\beta^2} \right) \ldots M \left( \frac{x}{\beta^n} \right) v$$

where the convergence is uniform on every compact subset in $\mathbb{R}$.

**Proof.** Write $v = (v_1, \ldots, v_d)^t$. We introduce the following norm for $\mathbb{C}^d$:

$$\|z\| = \max_{1 \leq j \leq d} \frac{|z_j|}{v_j} \quad (z = (z_j)_{1 \leq j \leq d} \in \mathbb{C}^d).$$

Then a matrix $A = (a_{i,j}) \in M_{d \times d}(\mathbb{C})$, considered as an operator on the normed space $(\mathbb{C}^d, \| \cdot \|)$, admits its operator norm

$$\|A\| = \max_{1 \leq i \leq d} \frac{1}{v_i} \sum_{j=1}^d |a_{i,j}| v_j.$$
Notice that \(\|M(0)\| = 1\) because \(M(0)v = v\).

Since the eigenvalue 1 of \(M(0)\) is simple (and isolated), and \(M(x)\) is Lipschitz continuous, by the perturbation theory of matrices, there is a neighborhood of 0, say \([-\delta, \delta]\) (\(\delta > 0\)), such that for any \(x \in [-\delta, \delta]\), \(M(x)\) has a simple eigenvalue \(\lambda(x)\) and a corresponding eigenvector \(v(x)\) satisfying

\[
|\lambda(x) - 1| \leq C|x|, \quad \|v(x) - v\| \leq C|x| \quad (x \in [-\delta, \delta])
\]

for some constant \(C > 0\). We claim that the limit

\[
G(x) = \lim_{n \to \infty} M\left(\frac{x}{\beta}\right) M\left(\frac{x}{\beta^2}\right) \ldots M\left(\frac{x}{\beta^n}\right) v
\]

exists (uniformly on any compact set). It is clear that the limit function is a solution.

Denote

\[
Q_n(x) = M\left(\frac{x}{\beta}\right) M\left(\frac{x}{\beta^2}\right) \ldots M\left(\frac{x}{\beta^n}\right).
\]

The proof of the existence of the limit in (4.2) is based on the following lemma:

**Lemma 4.2.** For any \(\delta > 0\), there exists a constant \(D > 0\) such that for any \(n \geq 1\) and any \(x \in [-\delta, \delta]\) we have

\[
\|Q_n(x)\| \leq D \quad \|Q_n(x)v - v\| \leq D|x|.
\]

To get the boundedness of \(\|Q_n(x)\|\), it suffices to notice that

\[
\|Q_n(x)\| \leq \prod_{j=1}^{n} f\left(\frac{x}{\beta^j}\right)
\]

where the scalar function \(f(x) = \|M(x)\|\) is Lipschitz and \(f(0) = 1\) (we have used our choice of the norm of \(C^d\)), and that the products converge uniformly on \([-\delta, \delta]\) to a continuous function \([FL]\). Now we prove that

\[
\|Q_n(x)v - Q_{n-1}(x)v\| \leq C'\frac{|x|}{\beta^n}
\]

where \(C' > 0\) is some constant. In fact, since \(M(x)v(x) = \lambda(x)v(x)\), we have

\[
M\left(\frac{x}{\beta^n}\right) v - v = M\left(\frac{x}{\beta^n}\right) \left[v - v\left(\frac{x}{\beta^n}\right)\right] + \left[\lambda\left(\frac{x}{\beta^n}\right) v\left(\frac{x}{\beta^n}\right) - v\right].
\]

Multiplying both sides by \(Q_{n-1}(x)\), we get

\[
\|Q_n(x)v - Q_{n-1}(x)v\| \\
\leq \left\|Q_n(x)\left(v - v\left(\frac{x}{\beta^n}\right)\right)\right\| + \left\|Q_{n-1}(x)\left(v - \lambda\left(\frac{x}{\beta^n}\right) v\left(\frac{x}{\beta^n}\right)\right)\right\|.
\]
Notice that
\[ \| \lambda(x)v(x) - v \| \leq \| \lambda(x) - 1 \| \| v(x) \| + \| v(x) - v \|. \]
Using the last inequality, the estimates in (4.1) and that we have just proved \( \| Q_n(x) \| \leq D \), we obtain (4.3). Then for \( n > m \)
\[ \| Q_n(x)v - Q_m(x)v \| \leq \sum_{k=m+1}^{n} \| Q_k(x)v - Q_{k-1}(x)v \| \leq \frac{C'|x|}{\beta^{m-1}(\beta - 1)}. \]
That means \( Q_n(x)v \) is a Cauchy sequence in the space \( C([-\delta, \delta]) \) of continuous functions equipped with uniform norm. Since for any fixed integer \( n_0 \), we have
\[ \lim_{n \to \infty} Q_n(x)v = Q_{n_0}(x) \cdot \lim_{n \to \infty} Q_n \left( \frac{x}{\beta^n} \right) v, \]
it follows that the uniform convergence of \( Q_n(x)v \) on \([-\delta, \delta]\) implies its uniform convergence on any compact set.

The uniqueness of solution is easy. Let \( G \neq 0 \) be a solution. First notice that \( G(0) \) is an eigenvector of \( M(0) \) associated to the simple eigenvalue 1. Hence we may assume that \( G(0) = v \). By iterating the equation, we get
\[ G(x) = Q_n(x)G \left( \frac{x}{\beta^n} \right) = Q_n(x)v + Q_n(x) \left( G \left( \frac{x}{\beta^n} \right) - v \right). \]
The last term converges to zero (uniformly on any compact set) because of \( \| Q_n(x)v \| \leq D \). So, \( G(x) \) must be the limit of \( Q_n(x)v \).

**Remark 4.3.** In the theorem, neither the almost periodicity of \( M(x) \) nor the positivity of \( M(x) \) is required, but only the positivity of \( M(0) \). That 1 is an eigenvalue of \( M(0) \) is necessary for Equation (1.3) to have a solution \( G(x) \) such that \( G(0) \neq 0 \).

**Remark 4.4.** Lipschitz continuity is not really necessary. Hölder continuity or even Dini continuity is sufficient.

**Remark 4.5.** If the entries of \( M \) are (real) analytic, then the solution \( G \) is also analytic. Because, for any \( x_0 \in \mathbb{R} \), there is a disk on the complex plane centered at \( x_0 \) on which \( Q_n(x)v \) (as functions of complex variable \( x \)) uniformly converges.

### 4.2. Existence of multiperiodic functions

Here we give a proof of Theorem A based on Theorem 4.1.

Let \( M(x) \) be as in (1.4). It is easy to see that the characteristic polynomial of \( M(0) \) takes the form
\[ P(u) = u^d - f_1(0)u^{d-1} - f_2(0)u^{d-2} - \cdots - f_{d-1}(0)u - f_d(0). \]
The consistency condition implies that 1 is an eigenvalue of \( M(0) \). Notice that
\[ P'(1) = d - ((d - 1)f_1(0) + (d - 2)f_2(0) + \cdots + 2f_{d-2} + f_{d-1}(0)) > 0. \]
So, the eigenvalue 1 is simple. By Theorem 4.1, there is a unique solution of \( G(x) = M(x/\beta)G(x/\beta) \). Let
\[
G(x) = (G_1(x), G_2(x), \ldots, G_d(x))^t.
\]
Then \( G_1(x) \) is a solution of (1.2). If \( F \) is a solution of (1.2). Let
\[
\tilde{G}_1(x) = F(x), \tilde{G}_2(x) = F(x/\beta), \ldots, \tilde{G}_d(x) = F(x/\beta^{d-1}).
\]
Then \( \tilde{G} = (\tilde{G}_1, \ldots, \tilde{G}_d)^t \) is a solution of \( G(x) = M(x/\beta)G(x/\beta) \). Thus the uniqueness of the solution of Equation (1.3) implies that of Equation (1.2).

4.3. Asymptotic behavior of multiperiodic functions. Let us consider the asymptotic behavior of a multiperiodic function, or more generally the asymptotic behavior of a solution \( G \) of Equation (1.3) provided it exists (the existence may be guaranteed by Theorem 4.1).

**Theorem 4.6.** Let \( \beta > 1 \) be a PV-number whose maximal conjugate has modulus \( \rho \). Let \( M : \mathbb{R} \to \text{GL}_d(\mathbb{C}) \) be a \( \beta \)-adapted u.a.p. Hölder function of order \( \alpha > 0 \). Suppose that \( G \) is a solution of \( G(x) = M(x/\beta)G(x/\beta) \).

Suppose furthermore that one of the following conditions is satisfied:

(i) \( D\rho^{\alpha} < 1 \) where \( D = \sup_{x \in \mathbb{R}} \|M(x)\|\|M(x)^{-1}\| \). (N.b. \( \beta \) must be Pisot.)

(ii) The entries of \( M(x) \) are either identically zero or larger than a constant \( \delta > 0 \).

Then for a.e. \( x \in \mathbb{R} \) the limit
\[
h(x) = \lim_{n \to \infty} \frac{1}{n} \log |G(\beta^n x)|
\]
exists and is independent of \( x \).

**Proof.** We first consider Case (i). By Proposition 3.7, Theorem 2.9 applies. Hence for a.e. \( x \), if we denote by \( r(x) \) the integer such that the vector \( G(x) \in V_x^{(r)} \setminus V_x^{(r-1)} \) we get
\[
\lim_{n \to \infty} \frac{1}{n} \log |G(\beta^n x)| = \lim_{n \to \infty} \frac{1}{n} \log |M_x^n G(x)| = \lambda_x^{(r(x))}.
\]
But \( G(\beta x) = M(x)G(x) \) hence \( G(\beta x) \in V_{\beta x}^{(r)} \setminus V_{\beta x}^{(r-1)} \), from what follows \( r(\beta x) = r(x) \), i.e., \( r \) is invariant. Hence constant a.e. because of the total Bohr ergodicity of the sequence \( \beta^n \).

Case (ii). We use the notation of Proposition 3.8. Since \( n^{-1}f_n \) has joint periods, Theorem 2.5 applies (see the proof of Theorem 2.9 for details). Hence the following limit exists a.e.:
\[
\lim_{n \to \infty} \frac{1}{n} f_n(x) = L,
\]
where $L = \inf_n \frac{1}{n} M(f_n)$. In view of $G(\beta^n x) = M(\beta^n x) \cdots M(x)G(x)$ the positivity of $M$ and $G$ gives
\[
\lim_{n \to \infty} \frac{1}{n} \log |G(\beta^n x)| = L \quad \text{a.e.}
\]

Note that when $G$ is the solution of Equation (1.3) with $M$ and $G$ given by (1.4) we have
\[
|G(x)| = |F(x)| + |F(x/\beta)| + \cdots + |F(x/\beta^{d-1})|,
\]
thus the asymptotic behavior of $\frac{1}{n} \log |G(\beta^n x)|$ and $\frac{1}{n} \log \sum_{j=0}^{d-1} |F(\beta^n x)|$ are the same. Thus Theorem C follows as an immediate corollary of Theorem 4.6. This partially answers a question in [JRS] (Conjecture 4.1., p. 263).

We now prove Theorem B. By the primitivity of $M(0)$ and the hypothesis, there exists an integer $\tau \geq 1$ such that $\widetilde{M}(x) := M(x/\beta^{\tau-1}) \cdots M(x/\beta)M(x)$ has all its entries strictly positive (even larger than $c\delta^\tau$ for some constant $c > 0$). Consider the equation
\[
G(x) = \widetilde{M}(x/\beta)G(x/\beta^\tau).
\]
We examine the first entries of both sides. We can find two constants $0 < c_1 < c_2$ such that we get
\[
c_1 F(x) \leq F(x/\beta^{\tau+1}) + \cdots + F(x/\beta^{\tau+d}) \leq c_2 F(x).
\]
Thus Theorem B follows from Theorem 4.6.

**Theorem 4.7.** Under the same conditions as Theorem 4.6, for any $q \in \mathbb{R}^+$, the following limit exists:
\[
\lim_{n \to \infty} \frac{1}{n} \log \int_0^1 \|M(\beta^{n-1} x) \cdots M(\beta x)M(x)\|^q dx.
\]

**Proof.** Write
\[
Z_n = \int_0^1 P_n(x)^q dx \quad \text{with} \quad P_n(x) = \|M(\beta^{n-1} x) \cdots M(\beta x)M(x)\|.
\]
It suffices to show that there is a constant $C > 0$ such that
\[
Z_{n+m} \leq CZ_n Z_m \quad (n \geq 1, m \geq 1).
\]
We assume that $q = 1$, just for simplicity. We will use the fact that there is a constant $L > 0$ such that
\[
\|M(x) - M(y)\|_2 \leq L|x - y|^\alpha \quad (\forall x, y \in \mathbb{R}).
\]
We use the notation \( \prod_{i=0}^{n} M_i = M_n M_{n-1} \ldots M_1 M_0 \) for the (noncommutative) product of the matrices \( M_0, \ldots, M_n \). Write

\[
Z_{n+m} = \sum_{\epsilon} \int_{I(\epsilon)} \left\| \prod_{k=0}^{m-1} M(\beta^{n+k}x) \cdot \prod_{j=0}^{n-1} M(\beta^j x) \right\| dx 
\leq \sum_{\epsilon} \int_{I(\epsilon)} \left\| \prod_{j=0}^{n-1} M(\beta^j x) \right\| \cdot \left\| \prod_{k=0}^{m-1} M(\beta^{n+k}x) \right\| dx
\]

where the sum is taken over all \( \beta \)-intervals \( I(\epsilon) \) of level \( n \) (see Lemma 3.4). Let \( a_\epsilon \) be the left endpoint of \( I(\epsilon) \). The integral in the last sum, after the change of variables \( \beta x = y \), becomes

\[
\beta^{-n} \int_0^{\beta^n I(\epsilon)} \left\| \prod_{j=0}^{n-1} M(\beta^j a_\epsilon + \beta^{-n+j} y) \right\| \cdot \left\| \prod_{k=0}^{m-1} M(\beta^k y + \beta^{n+k} a_\epsilon) \right\| dy.
\]

Notice that if \( \epsilon = (\epsilon_1, \ldots, \epsilon_n) \), then

\[
\beta^{n+k} a_\epsilon = \beta^{n+k} \left( \frac{\epsilon_1}{\beta} + \cdots + \frac{\epsilon_n}{\beta^n} \right) = \beta^{n+k-1}\epsilon_1 + \cdots + \beta^k \epsilon_n.
\]

So, by Lemma 3.3, there is an integer \( n_\epsilon \) such that

\[
|\beta^{n+k} a_\epsilon - n_\epsilon| = O(\rho^k + \rho^k + \cdots + \rho^{n+k-1}) = O(\rho^k).
\]

By the distortion lemma (Lemma 3.1, Lemma 3.2), we have

\[
\left\| \prod_{j=0}^{n-1} M(\beta^j a_\epsilon + \beta^{-n+j} y) \right\| \leq C \left\| \prod_{j=0}^{n-1} M(\beta^j a_\epsilon) \right\| 
\leq \left\| \prod_{k=0}^{m-1} M(\beta^k y + \beta^{n+k} a_\epsilon) \right\| \leq C \left\| \prod_{k=0}^{m-1} M(\beta^k y) \right\|.
\]

Therefore, we get

\[
Z_{n+m} \leq C \beta^{-n} \sum_{\epsilon} P_n(a_\epsilon) \leq C' Z_n Z_m.
\]

Corollary 4.8. Let \( F \) be the multiperiodic function defined by (1.2). Suppose that \( \beta > 1 \) is a PV-number and that \( f_1, \ldots, f_d \) are either identically zero or larger than a constant \( \delta > 0 \). Suppose further that \( M^\ell_x := M(\beta^{\ell-1}x) \ldots M(\beta x) M(x) \) has strictly positive entries for some integer \( \ell > 0 \). Then for any \( q \in \mathbb{R}^+ \), the following limit exists:

\[
\lim_{T \to \infty} \frac{1}{\log T} \int_0^T F(x)^q dx.
\]
Proof. Without loss of generality, we may only consider the subsequence $T_n = \beta^n$. Since $|G(x)| = \sum_{j=0}^{d-1} |F(x/\beta^j)|$ where $G$ is the solution of the associated vector Equation (1.3), we have only to show the existence of the limit

$$\lim_{n \to \infty} \frac{1}{n} \int_0^{\beta^n} |G(x)|^q dx.$$ 

Making the change of variables $x = \beta^n y$, we are led to prove the existence of the limit

$$\lim_{n \to \infty} \frac{1}{n} \int_0^1 |G(\beta^n x)|^q dx.$$ 

Notice that $G(\beta x) = M_x^n G(x)$. Notice also that $G(x)$ has strictly positive entries by the hypothesis on $M$. So, for the nonnegative matrix $M_x^n$ we have

$$C^{-1} \|M_x^n\| \leq |G(\beta^n x)| \leq C \|M_x^n\| \quad (\forall x \in [0,1])$$

for some constant $C > 0$. By the proof of the last theorem, $\log \int_0^1 |G(\beta^n x)|^q dx$ is subadditive. □

Remark 4.9. Let $M(x)$ be the matrix defined by (1.4). Let $\widetilde{M}$ be the numerical matrix obtained by replacing $f_j(x)$ in $M(x)$ by 0 or 1 according to $f_j(x) \equiv 0$ or not. Then $\widetilde{M}^\ell > 0$ implies $M_x^\ell > 0$. In particular, $\widetilde{M}^d > 0$ if $f_j(x)$ are all strictly positive.

Example 4.10. Let $\beta > 1$ be a PV-number. Let $f_1(x)$ and $f_2(x)$ be two strictly positive 1-periodic Hölder continuous functions such that $f_1(0) + f_2(0) = 1$. There is a unique multiperiodic function $F$ defined by

$$F(x) = f_1 \left( \frac{x}{\beta} \right) F \left( \frac{x}{\beta} \right) + f_2 \left( \frac{x}{\beta^2} \right) F \left( \frac{x}{\beta^2} \right).$$

For almost every $x \in \mathbb{R}$, $n^{-1} \log F(\beta^n x)$ has a limit as $n \to \infty$; for any $q \in \mathbb{R}^+$, $(\log T)^{-1} \int_0^T F(x)^q dx$ has a limit as $T \to \infty$.

Example 4.11. Let $\beta > 1$ and $a, b \in \mathbb{Z}$. Consider the contractive transformations on $\mathbb{R}$ defined by

$$S_1x = \frac{x + a}{\beta}, \quad S_1x = \frac{x + b}{\beta^2}.$$ 

For any $0 < p < 1$, there exists a unique probability measure $\mu$ with compact support such that

$$\mu = p \mu \circ S_1^{-1} + (1 - p) \mu \circ S_2^{-1}.$$ 

It is a self-similar measure. Its Fourier transform satisfies the equation

$$\hat{\mu}(x) = f_1(x/\beta)\hat{\mu}(x/\beta) + f_2(x/\beta^2)\hat{\mu}(x/\beta^2).$$
with $f_1(x) = p e^{2\pi i a x}$ and $f_2(x) = q e^{2\pi i b x}$ with $q = 1 - p$. This is a special case of Equation (1.2). The corresponding matrix defined by (1.4) and its inverse are respectively equal to

$$M(x) = \begin{pmatrix} p e^{2\pi i a x} & q e^{2\pi i b x}/\beta \\ 1 & 0 \end{pmatrix}, \quad M(x)^{-1} = q^{-1} e^{-2\pi i b x/\beta} \begin{pmatrix} 0 & q e^{2\pi i b x}/\beta \\ 1 & -p e^{2\pi i a x} \end{pmatrix}.$$  

If we take the norm $|v| = \max(|v_1|, |v_2|)$ on $\mathbb{C}^2$, the operator norms for $M(x)$ and $M(x)^{-1}$ are respectively $\|M(x)\| = 1$ and $\|M(x)^{-1}\| = 1 + \frac{1-p}{1-p}$. So, when $\beta$ is a PV-number, under the condition $1 + \frac{1-p}{1-p} < \frac{1}{\rho}$, for almost all $x \in \mathbb{R}$ the following limit exists and does not depend on $x$:

$$\lim_{n \to \infty} \frac{1}{n} \log \left( |\hat{\mu}(\beta^n x)| + |\hat{\mu}(\beta^{n-1} x)| \right).$$  

Acknowledgements. This work was partially done during the first author’s visit to Lund University, Sweden and the last author’s visit to WIPM of the Academy of Sciences of China and Wuhan University, China.

References


PROPERLY EMBEDDED MINIMAL DISKS BOUNDED BY NONCOMPACT POLYGONAL LINES

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In this paper we give a uniqueness and existence result for minimal disks with some noncompact, U-shaped boundaries in a slab of $\mathbb{R}^3$.

1. Introduction and preliminaries.

Minimal surfaces containing straight lines have special properties that distinguish them from the rest of minimal surfaces. In this article, we emphasize Schwarz’s reflection principle. Examples of this type were well studied during the last two centuries.

Recently, in [11], F.J. López and F. Wei obtained an existence and uniqueness theorem for properly immersed minimal disks whose boundaries consist of two disjoint straight lines and a segment which meets the lines orthogonally.

Following this, López and the second author of this paper have constructed a deformation of López-Wei disks which consists of properly embedded minimal disks bounded by straight lines and contained in a wedge of a slab (see [9] and [10]). Essentially, the deformation modifies the angle formed by the two halfplanes containing the connected components of the boundary. The surfaces that appear in this deformation for angle zero correspond to some Jenkins-Serrin graphs (see [6]). The López-Martín examples have nice geometric properties such as the convex hull property. These examples are a solution to Plateau’s problem for a polygonal noncompact boundary consisting of a double U shaped contour (see Figure 1). These surfaces can be used as a new type of barrier for the maximum principle application ([8] and [9]). Examples of this kind are also closely related to minimal surfaces with helicoidal ends ([15]).

In this paper, we obtain all the solutions to the aforementioned Plateau problem with noncompact polygonal boundary, which are contained in the slab, but not lie necessarily in the convex hull of their boundary (see Figure 2). To be more precise, we deal with the study of properly embedded minimal surfaces whose boundary $\Gamma_{\theta,d}$ consists of the following configuration of straight lines:

Fix $\theta \in [0, \pi]$ and $d \geq 0$, and consider two half-lines $r_1^+$ and $r_2^-$ in $\mathbb{R}^3$, meeting at an angle of $\theta$. If $\theta = 0$ this means that the straight lines are
parallel. Let $q_1^+$ and $q_1^-$ be two points in $r_1^+$ and $r_1^-$, respectively, such that they are symmetric with respect the inner bisector of these half-lines. We choose $q_1^+$ and $q_1^-$ in such a way that either $q_1^+ = q_1^-$ or the half-lines $\ell_1^+$ and $\ell_1^-$ on $r_1^+$ and $r_1^-$ starting at $q_1^+$ and $q_1^-$, respectively, do not intersect. Write $d = \operatorname{dist}(q_1^+, q_1^-)$.

Let $\Pi_1$ be the plane determined by $\ell_1^+$ and $\ell_1^-$, $\Pi_2$ a plane parallel and distinct to $\Pi_1$ and let $S$ denote the slab determined by $\Pi_1$ and $\Pi_2$. Let $\ell_2^+$ and $\ell_2^-$ be the orthogonal projections to $\Pi_2$ of $\ell_1^+$ and $\ell_1^-$, respectively. Denote $q_2^+$ (resp. $q_2^-$) as the orthogonal projection to $\Pi_2$ of $q_1^+$ (resp. $q_1^-$), and label $\ell_0^+$ (resp. $\ell_0^-$) as the segment $[q_1^+, q_2^+]$ (resp. $[q_1^-, q_2^-]$). Finally, we define

$$\Gamma_{\theta d}^+ = \bigcup_{i=0}^{2} (\ell_i^+), \quad \Gamma_{\theta d}^- = \bigcup_{i=0}^{2} (\ell_i^-), \quad \Gamma_{\theta d} = \Gamma_{\theta d}^+ \cup \Gamma_{\theta d}^-.$$

We consider the following generalized Plateau problem:

**Problem 1.** Determine a properly immersed minimal surface $X : M \to \mathbb{R}^3$ satisfying:

1. $M$ is homeomorphic to the closed unit disk $\bar{D}$ minus two boundary points $E_1$ and $E_2$, that we call the ends of $M$.
2. $X(\partial(M)) = \Gamma_{\theta d}$.
3. If $d > 0$, $X$ is an embedding.
4. In the limit case $\ell_0^+ = \ell_0^-$ (i.e., $d = 0$), the maps $X|_{M-\gamma^+}$ and $X|_{M-\gamma^-}$ are injective, where $\gamma^+$ and $\gamma^-$ are the two connected components of $\partial(M)$.
5. $X(M)$ lies in a slab that contains $S$.

Observe that if (5) is satisfied then it is easy to prove (see Lemma 2.1 in [12]) that $X(M)$ lies in the slab $S$. Then Condition (5) is equivalent to

1. $X(M)$ lies in $S$.

We prove the following:
Main Theorem. If $0 < \theta < \pi$ there exist $d_\theta$ and $d'_\theta$ with $0 < d'_\theta < d_\theta$ such that:

(i) If $d > d_\theta$ there are no solutions of Problem 1.
(ii) If $d = d_\theta$, Problem 1 has a unique solution.
(iii) If $d \in ]d'_\theta, d_\theta[$ or $d = 0$, Problem 1 has two solutions.
(iv) If $d = d'_\theta$, Problem 1 has three solutions.
(v) If $d \in [0, d'_\theta[$, Problem 1 has four solutions.

If $\theta = \pi$ there exist $d_\pi$ with $0 < d_\pi$ such that:

(i) If $d > d_\pi$ there are no solutions of Problem 1.
(ii) If $d = 0$ or $d = d_\pi$, Problem 1 has a unique solution.
(iii) If $d \in ]0, d_\pi[$, Problem 1 has two solutions.

If $\theta = 0$ there exist $d'_0$ with $0 < d'_0 < \text{dist}(\Pi_1, \Pi_2)$ such that:

(i) If $d \geq \text{dist}(\Pi_1, \Pi_2)$ there are no solutions of Problem 1.
(ii) If $d = 0$ or $d \in ]d'_0, \text{dist}(\Pi_1, \Pi_2)[$, Problem 1 has a unique solution.
(iii) If $d = d'_0$, Problem 1 has two solutions.
(iv) If $d \in ]0, d'_0[$, Problem 1 has three solutions.

López and Wei proved in [11] that there exists a unique solution of Problem 1 when $\theta = \pi$ and $d = 0$. Therefore, we always omit this case in our discussions.

![Figure 2](image_url)

**Figure 2.** The four solutions in case $\theta = \pi$, $d = \frac{\text{dist}(\Pi_1, \Pi_2)}{2}$.
The first and second one on the left corresponds to López-Martín examples.

The aim of this paper is to prove the uniqueness and existence of the solutions stated in the Main Theorem. The paper is set out as follows:

In Section 2, we obtain the uniqueness result stated in the above theorem. For the sake of clarity we divide the proof in several subsections. In the first one, we shall see that if $M$ is a solution of our Plateau problem then $M$ is conformally equivalent to a twice punctured closed disk with piecewise analytic boundary and its meromorphic data extend to the closed disk.
Among the results obtained in this subsection, we emphasize the following proposition:

**Proposition 1.** Assume \( X : M \rightarrow \mathbb{R}^3 \) is a solution of Problem 1 for \( 0 \leq \theta < \pi \) and denote by \( \mathcal{E}(\Gamma_{\theta d}) \) the convex hull of \( \Gamma_{\theta d} \). Then \( X(M) \) lies either in \( \mathcal{E}(\Gamma_{\theta d}) \) or in \((S \setminus \mathcal{E}(\Gamma_{\theta d})) \cup \Gamma_{\theta d} \). If \( \theta = \pi \), then \( X(M) \) lies in one of the half-slabs determined by the strip \( \mathcal{E}(\Gamma_{\pi d}) \).

Roughly speaking, the above proposition asserts that if \( 0 \leq \theta < \pi \), then the solutions of our problem lie either in the interior of the convex hull of the boundary or in the exterior of it. Subsection 2.2 is devoted to proving that \( M \) inherits the horizontal symmetry of its boundary and also the vertical symmetry in case \( d = 0 \). Finally, in Subsection 2.3, taking into account the preceding steps, we determine a model of the complex structure and Weierstrass representation of any solution of Plateau’s problem above. As a consequence, we obtain that, in the general case, a solution of our Plateau problem also inherits the vertical symmetry of its boundary.

The existence part of the Main Theorem can be found in Section 3. We prove that the Weierstrass data obtained in Section 2 really correspond to solutions of our problem.

As we mentioned before, López-Martín examples can be used as barriers in order to prove nonexistence results for minimal surfaces with planar boundaries in a wedge of a slab. Furthermore, they extended the family of minimal surfaces satisfying the convex hull property. To state these results we need some notation. Define \( L = \{(0,0,t) \mid -\frac{1}{2} < t < \frac{1}{2}\} \) and \( W = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid -\frac{1}{2} \leq x_3 \leq \frac{1}{2}\} \). For \( \theta \in [0,\pi] \), we also write \( W_\theta = L \cup \{(x_1,x_2,x_3) \in W \setminus L \mid \text{Arg}((x_1,x_2)) \in [0,\theta]\} \) and \( \Sigma_\theta = L \cup \{(x_1,x_2,x_3) \in W \setminus L \mid \text{Arg}((x_1,x_2)) = \theta\} \). Using this notation López and the second author have proved the following:

**Theorem 1 ([10]).** Let \( M \) be a connected properly immersed minimal surface in a wedge \( W_{2\pi - \epsilon} \) for some \( 0 < \epsilon < 2\pi \). Then one has:

(i) If \( \partial(M) \subset \Sigma_0 \), then \( M \) is a planar region in \( \Sigma_0 \).

(ii) If \( \partial(M) \subset W_\theta \), for \( \theta \in [0,\pi] \), then \( M \) lies in the convex hull of its boundary.

Finally, we would like to mention that the Main Theorem and Proposition 1 are announced in the proceedings of the conference Differential Geometry Valencia 2001 ([5]).

2. Conformal structure and Weierstrass representation.

As we mentioned before, this section is devoted to study the underlying complex structure and Weierstrass data of the solutions of our problem.

Throughout this paper \((x_1,x_2,x_3)\) denotes a set of Cartesian coordinates such that:
• $\ell_0^+$ and $\ell_0^-$ have the direction of $x_3$-axis,
• $\{x_3 = \frac{1}{2}\}$ and $\{x_3 = -\frac{1}{2}\}$ are the equations of planes $\Pi_1$ and $\Pi_2$, respectively,
• the origin is the middle point between $\frac{q_1^++q_2^+}{2}$ and $\frac{q_1^-+q_2^-}{2}$,
• the $x_2$-axis is the inner bisector of the orthogonal projection of $\ell_i^+$ and $\ell_i^-$ to the plane $x_3 = 0$, $i = 1, 2$, and
• $\Gamma_{\theta, d}^+ \subset \{x_1 \geq 0\}$.

Along this section, $X : M \to \mathbb{R}^3$ denote a proper conformal minimal immersion satisfying Conditions (1)-(5) of Problem 1. For the sake of simplicity, we use $\Gamma$, $\Gamma^+$ and $\Gamma^-$ instead of $\Gamma_{\theta, d}^+$, $\Gamma_{\theta, d}^-$ and $\Gamma_{\theta, d}$. Taking into account that $X(M) \subset S$ and the maximum principle, we deduce that $\Pi_i \cap X(M) \subset \Pi_i \cap \Gamma$, for $i = 1, 2$.

As we announced, we shall divide the study of conformal structure in several subsections.

2.1. Conformal type of $M$. The conformal type of $M$ can be easily determined using a global result on conformal structure of properly immersed minimal surfaces by P. Collin, R. Kusner, W.H. Meeks and H. Rosenberg (see [4]). From Theorem 3.1 of [4] we obtain that $M$ is parabolic and hence, taking into account the topological type of $M$, $M$ is conformally equivalent to the closed unit disk $\mathbb{D}$ minus two boundary points $E_1$ and $E_2$, where the biholomorphism extends piecewise analytically to the boundary.

Next, we prove that the Gauss map and Weierstrass data extend continuously to the ends. To obtain this, we need some additional results.

Let $U(E_i)$, $i = 1, 2$, be two open disjoint neighbourhoods of the ends of $M$ and let $C_a$ denote the catenoid given by the equation $x_1^2 + x_2^2 = a^2 \cosh^2 (\frac{x_3}{a})$, where $a \in \mathbb{R}^+$. Define $\sigma_a = X^{-1}(X(M) \cap C_a)$, for $a > 0$. With this notation we shall prove the following:

Lemma 1. There exists $a_0 > 0$ such that for $a \geq a_0$, $\sigma_a = \sigma_a^1 \cup \sigma_a^2$, where $\sigma_a^1$ and $\sigma_a^2$ are two disjoint simple compact analytic curves such that $\sigma_a^i \subset U(E_i)$, for $i = 1, 2$.

Proof. Clearly, since $X(M) \cap C_a$ is compact and $X$ is proper, we have that $\sigma_a$ is compact for $a > 0$. Furthermore, $\sigma_a$ is a set of properly immersed analytic lines, because it is the intersection of distinct minimal surfaces. Denote by $\text{Int}(C_a)$ and $\text{Ext}(C_a)$, the interior and exterior connected component of $\mathbb{R}^3 \setminus C_a$, respectively.

Note that we can consider $a_1$ sufficiently large to insure the following: $\ell_0^+ \cup \ell_0^- \subset \text{Int}(C_a)$ and $S \cap \text{Int}(C_a) \subset S \cap \text{Int}(C_{a'}, \Gamma)$ if $a_1 \leq a \leq a'$. As $C_a$ and $X(M)$ are transverse along $\bigcup_{i=1}^2 \ell_i^+ \cup \ell_i^-$, for $a \geq a_1$, we can assert that only one curve lying in $\sigma_a$ approaches to each one of the four points in $\sigma_a \cap \partial(M)$. 
Moreover, since $σ_{a_1}$ is compact, we can find $U'(E_i)$, connected neighbourhoods of $E_i$ such that $U'(E_i) \subset U(E_i)$ and $X(U'(E_i)) \subset \operatorname{Ext}(C_{a_1})$. Consider now $a_0 > a_1$ such that $σ_{a_0} \subset U'(E_1) \cup U'(E_2)$.

Therefore, if $a \geq a_0$ we deduce that $σ_a = σ_a^1 \cup σ_a^2$, with $σ_a^i \subset U'(E_i)$, $i = 1, 2$ and $σ_a^1 \cap σ_a^2 = \emptyset$. Now, we must prove that $σ_a^i$ are simple curves for $i = 1, 2$. Suppose that there exists an arc $a$ of $σ_a^i$. In this case, either $X(Ω) \subset \operatorname{Ext}(C_a)$ or $X(Ω) \subset \operatorname{Int}(C_a)$. In the first case, we have that $X(Ω) \subset \operatorname{Ext}(C_a) \cap \operatorname{Int}(C_{a_2})$, for some $a \leq a_2$. Hence, using the family of catenoids $\{C_t\}_{a \leq t \leq a_2}$ and the maximum principle, we obtain that $X(Ω)$ is contained in the catenoid $C_a$, which is contrary to our assumptions. Moreover, since $a_0 \leq a$ we can assert that $X(Ω) \subset \operatorname{Ext}(C_{a_1})$. Consequently, if $X(Ω) \subset \operatorname{Int}(C_a)$ we may consider the family of catenoids $\{C_t\}_{a \leq t \leq a_0}$. The maximum principle gives again a contradiction.

Label $γ_i^+ = X^{-1}(Γ_i^+)$ and $γ_i^- = X^{-1}(Γ_i^-)$, for $i = 1, 2$. Consider also $γ_0^+ = X^{-1}(ℓ_0^+ \cup ℓ_0^-) \cap γ^+$ and $γ_0^- = X^{-1}(ℓ_0^+ \cup ℓ_0^-) \cap γ^−$.

Concerning the boundary behaviour we have, up to relabellings, three possibilities:

Case 1. $X(γ^+) = Γ^+$, $X(γ^-) = Γ^−$ and $γ_i^+ \cup γ_i^-$ diverges to $E_i$, for $i = 1, 2$ (see Figure 3(1)).

Case 2. $X(γ^+) = Γ^+$, $X(γ^-) = Γ^−$, $γ_0^+ \cup γ_0^-$ diverges to $E_1$ and $γ_2^+ \cup γ_2^−$ diverges to $E_2$ (see Figure 3(2)).

Case 3. When $d = 0$ we have also the case $X(γ^+) = ℓ_1^+ \cup ℓ_0 \cup ℓ_2^-$, $X(γ^-) = ℓ_1^- \cup ℓ_0 \cup ℓ_2^+$ and $γ_i^+ \cup γ_i^−$ diverges to $E_i$, for $i = 1, 2$, where $ℓ_0 = ℓ_0^− = ℓ_0^+$ (see Figure 3(3)).

![Figure 3](image-url)

Now, we shall prove that if $d = 0$ then Case 2 and Case 3 do not occur.

**Lemma 2.** Assume $d = 0$. Then the boundary of $X : M \rightarrow \mathbb{R}^3$ is as in Case 1.

**Proof.** Note that if $d = 0$ and the boundary is either as in Case 2 or as in Case 3, $X(M)$ contains a Möbius strip. Let us define $ℓ_ρ$ as the translation of vector $(0, 0, ρ)$ and consider $N$ the topological surface of $\mathbb{R}^3$ given by
\[ N = \bigcup_{n \in \mathbb{Z}} t_n(X(M)). \] Since \( X(M) \subset S \) we have that \( N \) is a connected embedded topological surface in \( \mathbb{R}^3 \). A well-known topological result asserts that then \( N \) must be orientable, but this is absurd because \( N \) contains Möbius strips.

Denote \( \vec{a}_2 = (0, 1, 0) \). Observe that \( \vec{a}_2 \) is the unitary vector in the direction of the inner bisector of \( \ell_2^+ \) and \( \ell_2^- \) pointing to \( \mathcal{E}(\Gamma) \). At this point we can prove a restatement of Proposition 1 above:

**Proposition 1.** If \( 0 \leq \theta < \pi \), then \( X(M) \) lies either in \( \mathcal{E}(\Gamma) \) or in \( (S \setminus \mathcal{E}(\Gamma)) \cup \Gamma \). If \( \theta = \pi \), then \( X(M) \) lies in one of the half-slabs determined by the strip \( \mathcal{E}(\Gamma) \).

**Proof.** Assume \( 0 \leq \theta < \pi \). In accordance to Lemma 2 we have that the boundary behaviour is either as in Case 1 or as in Case 2. Consider \( \beta = X^{-1}(X(M) \cap \{x_2 = 0\}) \). Since \( \beta \) is a nodal set of an harmonic function we have that \( \beta \) is a set of properly immersed analytic lines. Using the maximum principle we obtain that there are no compact connected regions of \( M \) bounded by curves in \( \beta \). Furthermore, as we are assuming \( 0 \leq \theta < \pi \), the theorem of the order of contact (see [13, §437]) gives us that there are no curves in \( \beta \) approaching to either \( \gamma_i^+ \) or \( \gamma_i^- \), for \( i = 1, 2 \). Now, we consider the following half-strips:

\[ B^+ = \{(x_1, 0, x_3) \mid x_1 \geq \frac{d}{2}, -\frac{1}{2} \leq x_3 \leq \frac{1}{2}\}, \]
\[ B^- = \{(x_1, 0, x_3) \mid x_1 \leq -\frac{d}{2}, -\frac{1}{2} \leq x_3 \leq \frac{1}{2}\}. \]

First, we shall prove that if there exists a curve in \( \beta \) starting at either \( \gamma_0^+ \) or \( \gamma_0^- \) and diverging to one end, then there are no curves starting at the same vertical segment and diverging to the same end. Assume that \( \beta' \) and \( \beta'' \) are two curves starting at \( \gamma_0^+ \) and diverging to \( E_1 \). The other cases can be treated in the same way. Clearly, if \( d > 0 \) we have \( X(\beta') \cup X(\beta'') \subset B^+ \). Suppose that \( d = 0 \) and \( X(\beta') \) and \( X(\beta'') \) are contained in different half-strips. Then we can consider, taking a piece of \( \gamma_0^+ \) if necessary, a piecewise analytic curve \( \hat{\beta} \) that diverge to \( E_1 \) and contains \( \beta' \) and \( \beta'' \). It is not difficult to see that there is an angle between the curves in \( \hat{\beta}, \Theta \), that goes by \( X \) to an angle greater or equal than \( 2\Theta \). Since \( X : M \longrightarrow \mathbb{R}^3 \) is conformal this is a contradiction. Then we conclude that in both cases \( X(\beta') \) and \( X(\beta'') \) are contained in the same half-strip. Therefore, we can find a connected component, \( \Omega \), of \( M \setminus \beta \) such that \( X(\Omega) \) is contained in one of the half-slabs determined by \( \{x_2 = 0\} \) and \( X(\partial(\Omega)) \) is in a half-strip. Consequently, applying Statement (i) in Theorem 1 we obtain that \( X(\Omega) \) is a planar domain of \( \{x_2 = 0\} \) which contradicts our assumptions.

Moreover, we shall prove that there are no compact curves in \( \beta \setminus (\gamma_0^+ \cup \gamma_0^-) \) starting at \( \gamma_0^+ \) and ending at \( \gamma_0^- \). Assume there exists \( \tau \) such a curve. As
there are no compact regions of \( M \setminus \beta \), we infer that \( \tau \) is the unique curve that starts at \( \gamma_0^+ \) and ends at \( \gamma_0^- \). Taking into account the above paragraph and the fact that \( \Gamma^+ \) and \( \Gamma^- \) are in the same half-slab of \( S \) determined by \( \{x_2 = 0\} \) we conclude that must exist a pair of curves, \( \tau_1 \) and \( \tau_2 \) starting at \( \gamma_0^+ \) and \( \gamma_0^- \), respectively and diverging to either \( E_1 \) or \( E_2 \). We assert that both curves must diverge to the same end. Indeed, if \( \tau_1 \) and \( \tau_2 \) diverge each one to one different end, then there exists a curve \( \tau_3 \) diverging to both ends. But this curve \( \tau_3 \) intersects \( \tau \) transversally in a odd number of points while \( X(\tau_3) \) intersects \( X(\tau) \) transversally in a even number of points.

Without loss of generality, we can assume that \( \tau_1 \) and \( \tau_2 \) diverge to \( E_1 \). Now, we may consider, taking pieces of \( \gamma_0^+ \) and \( \gamma_0^- \) if necessary, a piecewise analytic curve \( \tau' \) from \( E_1 \) to \( E_1 \) that encloses a disk \( \Omega \) of \( M \setminus \beta \). If \( X(\tau_1) \) and \( X(\tau_2) \) are contained in the same half-strip, the domain \( \Omega \) verifies the conditions of statement (i) in Theorem 1 and we obtain a contradiction. Assume that \( X(\tau_1) \) and \( X(\tau_2) \) are contained in different half-strips. Note that then there is an angle between the curves in \( \tau' \), \( \Theta \), that goes by \( X \) to an angle greater or equal than \( 2\Theta \). Using again that \( X : M \rightarrow \mathbb{R}^3 \) is conformal we get a contradiction.

Consequently, \( \beta \setminus (\gamma_0^+ \cup \gamma_0^-) \) consists of curves starting at \( \gamma_0^+ \cup \gamma_0^- \) and diverging to one end and divergent curves. Next we prove that there are no curves diverging to only one end. As before we have that one of these curves would be contained either in \( B^+ \) or in \( B^- \). Otherwise, in each of these cases it is possible to find a connected component, \( \Omega \), of \( M \setminus \beta \) such that \( X(\Omega) \) is contained in a half-slab of \( S \) and \( X(\partial(\Omega)) \) is contained in a half-strip of \( \{x_2 = 0\} \). Consequently, applying Statement (i) in Theorem 1 we obtain that \( X(\Omega) \) is a planar domain of \( \{x_2 = 0\} \) which contradicts our assumptions. Furthermore, using again that \( \Gamma^+ \) and \( \Gamma^- \) are in one of the half-slabs determined by \( \{x_2 = 0\} \) we deduce that if there exists a curve that starts at \( \gamma_0^+ \) and diverge to one end, then there exists a curve that starts at \( \gamma_0^- \) and diverge to the other end. All these facts allows us to assert that in \( \beta \setminus (\gamma_0^+ \cup \gamma_0^-) \) either there are no curves starting at \( \gamma_0^+ \cup \gamma_0^- \), or there are a pair starting at \( \gamma_0^+ \) or \( \gamma_0^- \) and diverging to different ends or there are four curves, a pair starting at \( \gamma_0^+ \) and diverging to different ends and another pair starting at \( \gamma_0^- \) and diverging to different ends. Moreover, we may find curves in \( \beta \setminus (\gamma_0^+ \cup \gamma_0^-) \) diverging to the two ends. Note that then the number of curves diverging to \( E_1 \) is the same as the number of curves diverging to \( E_2 \). It is not hard to see, using Statement (i) in Theorem 1, that two consecutive curves diverging to the same end have to be in different half-strips, it is to say, if one is in \( B^+ \) the other one is in \( B^- \) and that all divergent curves are disjoint. Assume that there are more than two curves in \( \beta \) diverging to \( E_1 \) and consider the compact curves \( \sigma^i = \sigma^i_a \), for \( i = 1, 2 \) and \( a \geq a_0 \) given in Lemma 1.
Now, we analyze each of the possibilities for the boundary separately.

**Case 1.** Denote $p_1^+ = \gamma_1^+ \cap \sigma^1$ and $p_1^- = \gamma_1^- \cap \sigma^1$. Then, denoting $p_3 : \mathbb{R}^3 \rightarrow \{x_3 = 0\}$ as the orthogonal projection over the plane $\{x_3 = 0\}$ we deduce that $p_3(X(\sigma^1))$ is a curve in $\{x_3 = 0\}$ such that $|\arg(p_3(X(p_1^+))) - \arg(p_3(X(p_1^-)))| > 2\pi$. Since $X_3(p_1^+) = X_3(p_1^-) = \frac{1}{2}$ we infer that $X(\sigma^1)$ has self-intersections, which is contrary to our assumptions. As a consequence, there is at most two curves in $\beta$ diverging to $E_1$ and the same for $E_2$.

**Case 2.** Denote $p_2^+ = \gamma_2^+ \cap \sigma^1$, $p_2^- = \gamma_2^- \cap \sigma^1$, $p_3^+ = \gamma_3^+ \cap \sigma^2$ and $p_3^- = \gamma_3^- \cap \sigma^2$ and suppose that $\sigma^1$ has been parametrized so that it starts at $p_2^+$ and ends at $p_2^-$. Using the same notation as above we can see that $p_3(X(\sigma^i))$ are curves in $\{x_3 = 0\}$ satisfying $|\arg(p_3(X(p_2^+))) - \arg(p_3(X(p_2^-)))| > 2\pi$ and $|\arg(p_3(X(p_3^+))) - \arg(p_3(X(p_3^-)))| > 2\pi$. Moreover, $p_3(X(\sigma^1))$ and $p_3(X(\sigma^2))$ rotates around $(0,0,0)$ in reverse sense, it is to say, if $p_3(X(\sigma^1))$ rotates clockwise then $p_3(X(\sigma^2))$ rotates counterclockwise, and vice versa. Since $X_3(p_i^+) = X_3(p_i^-)$ for $i = 1, 2$ we infer that $X(\sigma^1)$ and $X(\sigma^2)$ intersect each other. This contradicts our assumptions and therefore there is at most two curves diverging to each end in $\beta$.

The same argument used in both cases proves that if $\beta \setminus (\gamma_0^+ \cup \gamma_0^-)$ consists of two curves, $\tau_1$ and $\tau_2$, diverging to the two ends such that $X(\tau_i) \subset B^+$ and $X(\tau_j) \subset B^-$, then the boundaries of the three connected components of $M \setminus \beta$ are $\gamma^+ \cup \tau_1$, $\tau_1 \cup \tau_2$ and $\gamma^- \cup \tau_2$.

Taking into account this and the fact that $\Gamma^+ \cup \Gamma^-$ is in one of the half-slabs determined by $\{x_2 = 0\}$ we have that either $\beta \setminus (\gamma_0^+ \cup \gamma_0^-)$ is empty or it consists of:

1. Two curves diverging to the two ends (see Figure 4(1)),
2. a curve starting at $\gamma_0^+$ and diverging to $E_1$, a curve starting at $\gamma_0^+$ and diverging to $E_2$ and a curve diverging to the two ends (see Figure 4(2)),
3. a curve starting at $\gamma_0^-$ and diverging to $E_1$, a curve starting at $\gamma_0^-$ and diverging to $E_2$ and a curve diverging to the two ends (see Figure 4(3)),
4. a curve starting at $\gamma_0^+$ and diverging to $E_1$, a curve starting at $\gamma_0^+$ and diverging to $E_2$, a curve starting at $\gamma_0^-$ and diverging to $E_1$ and a curve starting at $\gamma_0^-$ and diverging to $E_2$ (see Figure 4(4)).

Clearly, if $\beta \setminus (\gamma_0^+ \cup \gamma_0^-)$ is empty we obtain that $X(M)$ is contained in the half-slab $\{x_2 \geq 0\}$. Therefore $X(M)$ satisfies the conditions of Statement (ii) in Theorem 1 and so $X(M) \subset \mathcal{E}(\Gamma)$.

Assume that we have one of the other possibilities. Then we shall prove that $X(M) \subset (S\setminus\mathcal{E}(\Gamma)) \cup \Gamma$. Note that it is sufficient to study the connected components of $M \setminus \beta$ whose image is contained in the half-slab $\{x_2 \geq 0\}$. Note that these connected components are those whose contains any of the curves $\gamma_i^+$ or $\gamma_i^-$, for $i = 1, 2$. At this point, it can be easily check that each
of these connected components have the boundary contained in one of the following wedges:

\[(S \setminus \overset{\circ}{E}(\Gamma)) \cap \{(x_1, x_2, x_3) \mid x_1 \geq 0, x_2 \geq 0, \frac{-1}{2} \leq x_3 \leq \frac{1}{2}\},\]

\[(S \setminus \overset{\circ}{E}(\Gamma)) \cap \{(x_1, x_2, x_3) \mid x_1 \leq 0, x_2 \geq 0, \frac{-1}{2} \leq x_3 \leq \frac{1}{2}\}.\]

Using again Assertion (ii) of Theorem 1, we conclude that the image of these connected components is contained entirely in the correspondent wedge.

Summarizing, we have proved that \(X(M) \subset (S \setminus \overset{\circ}{E}(\Gamma))\). Now the Proposition is an easy consequence of the maximum principle.

Next, we analyze the case \(\theta = \pi, \ d > 0\). Let us define \(\Delta = X^{-1}(X(M) \cap \{x_1 = \frac{d^2}{2}\})\). It is well-known that \(\Delta\) is a nodal set of an harmonic function and so it is a set of properly immersed analytic lines. Using the maximum principle we obtain that there are no compact connected regions of \(M\) bounded by curves in \(\Delta\). Then, \(\Delta \setminus \gamma_0^+\) consists of a set of divergent curves. Since \(\Gamma^+ \subset \{x_1 \geq \frac{d^2}{2}\}\) and \(\Gamma^- \subset \{x_1 \leq \frac{d^2}{2}\}\), we infer that if there exist a curve in \(\Delta\) starting at \(\gamma_0^+\) and diverging to one end, then another curve starting at \(\gamma_0^+\) and diverging to the other end must exist. Reasoning as in case \(0 \leq \theta < \pi, \ d = 0\) we can see that the image of such a pair of curves is contained in one of the following half-strips:

\[C^+ = \{(\frac{d^2}{2}, x_2, x_3) \mid x_2 \geq 0, \frac{-1}{2} \leq x_3 \leq \frac{1}{2}\},\]

\[C^- = \{(\frac{d^2}{2}, x_2, x_3) \mid x_2 \leq 0, \frac{-1}{2} \leq x_3 \leq \frac{1}{2}\},\]

and if two curves in \(\Delta\) diverge to the same end there must be one of them with the image contained in \(C^+\) and the other one with the image in \(C^-\). Therefore, adapting to this situation the argument presented above for the two different possibilities of the boundary, it is not hard to see that there is at most a curve diverging to each end. And then \(\Delta \setminus \gamma_0^+\) consists of either a curve diverging to the two ends or a pair of curves starting at \(\gamma_0^+\) and diverging to different ends. Note that in both cases \(X(\Delta)\) is contained either in \(C^+\) or \(C^-\). In order to conclude the proposition it is sufficient...
to apply Statement (ii) in Theorem 1 to each of connected components of
\( M \setminus \Delta \).

**Remark 1.** Assume that \( T \) is a plane in \( \mathbb{R}^3 \) and that the divergent curves in \( X^{-1}(X(M) \cap T) \) verifies that two consecutive divergent curves are in different half-strips of \( T \). Then reasoning as in the proof of Proposition 1 we can see that there are at most two curves diverging to each end.

**Corollary 1.** The boundary of the immersion \( X \) is as in Case 1.

**Proof.** Assuming that \( 0 \leq \theta < \pi \) and taking into account Proposition 1 we have that either \( X(M) \subset \mathcal{E}(M) \) or \( X(M) \subset (S \setminus \mathcal{E}(\Gamma)) \cup \Gamma \). Suppose that the boundary behaviour is as in Case 2 and consider the compact curves \( \sigma^i = \sigma^i_a, \ i = 1, 2 \) given in Lemma 1 for some \( a \geq a_0 \). Clearly, \( X(\sigma^1) \) starts at \( \ell_1^+ \) and ends at \( \ell_2^- \) and \( X(\sigma^2) \) is a curve starting at \( \ell_1^- \) and ending at \( \ell_2^+ \). Since both curves lie either in \( C_a \cap \mathcal{E}(\Gamma) \) or in \( C_a \cap ((S \setminus \mathcal{E}(\Gamma)) \cup \Gamma) \), they intersect, which contradicts our assumptions.

Taking the above corollary into account and the fact that \( X_3 \) is a bounded harmonic function one has the following:

**Corollary 2.** The function \( X_3 : M \to \mathbb{R}^3 \) extends continuously to the ends.

Let us consider \( \delta_t = X^{-1}(X(M) \cap \{x_3 = t\}) \) for \( t \geq -\frac{1}{2} \). Concerning \( \delta_t \) we can prove:

**Corollary 3.** The set \( \delta_t \) is compact and consists of a simple arc, for all \( t \in ]-\frac{1}{2}, \frac{1}{2}[ \). Moreover, the Gauss map \( g \) of \( X \) omits the points 0 and \( \infty \).

**Proof.** Clearly, from Corollary 2, we deduce that \( \delta_t \) is compact. Since \( \delta_t \) is the nodal set of a harmonic function we have that \( \delta_t \) is a one-dimensional proper real analytic subvariety of \( M \). Then, taking into account the maximum principle we deduce that there are no regions in \( M \) bounded by curves in \( X^{-1}(\delta_t) \). Therefore, \( \delta_t \) is a regular simple curve in \( M \) starting at \( \ell_0^+ \) and ending at \( \ell_0^- \). Moreover, the theorem of the order of contact (see [13, §437]) gives that there are no points in \( M \) with vertical normal vector.

In the case \( 0 \leq \theta < \pi \) the uniqueness of solutions \( X : M \to \mathbb{R}^3 \) of Problem 1 satisfying \( X(M) \subset \mathcal{E}(M) \) were completely studied by F.J. López and F. Martín in [9]. Henceforth, in the remainder of the section we assume that \( X(M) \subset (S \setminus \mathcal{E}(\Gamma)) \cup \Gamma \). Furthermore, we always assume that \( X(M) \subset \{x_2 \leq 0\} \) in the case \( \theta = \pi \). With this assumptions, we can prove:

**Lemma 3.** We have the following possibilities for the set \( \tau_0 = X^{-1}(X(M) \cap \{x_1 = 0\}) \):

i) If \( d > 0 \), \( \tau_0 \) consists of a curve diverging to both ends, \( E_1 \) and \( E_2 \).
ii) If \( d = 0 \), \( \tau_0 \setminus (\gamma_0^+ \cup \gamma_0^-) \) either consists of a curve diverging to both ends or \( \tau_0 \setminus (\gamma_0^+ \cup \gamma_0^-) = \tau_0^1 \cup \tau_0^2 \), where \( \tau_0^i \) are curves starting at \( \gamma_0^+ \) or \( \gamma_0^- \) and diverging to \( E_i \), for \( i = 1, 2 \).

**Proof.** Since \( \tau_0 \) is the nodal set of a harmonic function we have that \( \tau_0 \) is a set of properly immersed analytic curves in \( M \). Observe that \( \tau_0 \neq \emptyset \). If not, applying Statement (ii) in Theorem 1 we obtain that \( X(M) \) are two planar domains. Moreover, by the maximum principle, there are no compact connected regions in \( M \setminus \tau_0 \) bounded by curves in \( \tau_0 \). Clearly, taking into account that \( X(M) \cap \{ x_1 = 0 \} \subset \{ (x_1, x_2, x_3) \mid x_1 = 0, x_2 \leq 0, -\frac{1}{2} \leq x_3 \leq \frac{1}{2} \} \) and Statement (i) in Theorem 1 we obtain i).

Assume now that \( d = 0 \). In this case we have \( \gamma_0^+ \cup \gamma_0^- \subset \tau_0 \). Since \( \Gamma^+ \subset \{ x_1 \geq 0 \} \) and \( \Gamma^- \subset \{ x_1 \leq 0 \} \), we infer that if there exists a curve in \( \tau_0 \) starting at \( \gamma_0 \) and diverging to one end, then there exist another curve which starts at \( \gamma_0^\pm \) and diverges to the other end. Then reasoning as in the above paragraph we obtain ii). \( \square \)

**Proposition 2.** Counting multiplicities \( \sharp[ N^{-1}(a, \bar{a})] \leq 5 \) and
\[
\sharp[ N^{-1}(a, \bar{a}) \cap (M \setminus \partial(M))] \leq 3,
\]
where \( N : M \rightarrow S^2 \) is the Gauss map of \( X \). Furthermore, if \( 0 < \theta < \pi \) we have
\[
\sharp[ N^{-1}(a, \bar{a})] \leq 5, \forall a \in S^2 \cap \{ x_3 = 0 \}.
\]

**Proof.** We prove the first assertion in Proposition 2. The second assertion can be proved using similar arguments. Let us consider \( \beta_t = X^{-1}(X(M) \cap \{ x_2 = t \}) \), for \( t \in \mathbb{R} \). Note that \( \beta_t \) is a set of properly immersed analytic lines, because it is the nodal set of a harmonic function. Hence, using the maximum principle, we infer that there are no compact domains in \( M \) bounded by curves in \( \beta_t \), for all \( t \in \mathbb{R} \). Therefore, any two curves in \( \beta_t \) do not intersect in more than one point. If not, we can find a compact domain of \( M \setminus \beta_t \) bounded by curves in \( \beta_t \).

We start with the case \( t < 0 \). Observe that in this case \( \beta_t \) is a nonempty set of divergent curves, converging to a unique end or to the two ends. If \( \beta_t = \emptyset \) for some \( t < 0 \) we deduce that \( X(M) \subset \{ x_2 \geq 0 \} \) and applying Statement (ii) in Theorem 1 we obtain \( X(M) \subset \mathcal{E}(\Gamma) \) which contradicts our assumption.

Let \( \alpha_1 \) and \( \alpha_2 \) be a pair of arcs in \( \beta_t \) diverging to \( E_i \) such that \( \alpha_1 \cap \alpha_2 \neq \emptyset \). Therefore, there exists \( U_i \), a neighbourhood of \( E_i \), verifying that:
- \( U_i \setminus (U_i \cap \tau_0) \) has two connected components, \( U_i^+ \) and \( U_i^- \), where \( \tau_0 \) was defined in Lemma 3.
- \( \alpha_1 \cap U_i \subset U_i^+ \) and \( \alpha_2 \cap U_i \subset U_i^- \).
If not, one can find a neighbourhood of \( E_i \), \( U_i \), verifying the first condition and such that either \( (\alpha_1 \cup \alpha_2) \cap U_i \subset U_i^+ \) or \( (\alpha_1 \cup \alpha_2) \cap U_i \subset U_i^- \). Hence,
we deduce that $X((\tilde{\alpha}_1 \cup \tilde{\alpha}_2) \cap U_i)$, $i = 1,2$ are contained in a half strip of $\{x_2 = t\}$ and so there is a connected component in $M \setminus \beta_t$, $\Omega$, whose boundary is in a half-strip of $\{x_2 = t\}$. Therefore, we can apply Statement (i) in Theorem 1 to conclude that $X(\Omega)$ is a planar domain in $\{x_2 = t\}$, which is a contradiction.

Let us prove that if $\alpha_1$ and $\alpha_2$ are two curves in $\beta_t$ diverging to $E_t$, then they are disjoint. Indeed, if $\alpha_1$ and $\alpha_2$ intersect then we have four arcs $\{\tilde{\alpha}_i\}_{i=1}^4$ in $\beta_t$ diverging to $E_i$ and $\tilde{\alpha}_j \cap \tilde{\alpha}_l \neq \emptyset$. But this contradicts the above result.

Assume now that $\alpha_1$ is a curve diverging to $E_i$ and that $\alpha_2$ is a curve diverging to the two ends. If $\alpha_1$ and $\alpha_2$ intersect each other then we have three arcs $\{\tilde{\alpha}_i\}_{i=1}^3$ in $\beta_t$ diverging to $E_i$ and $\tilde{\alpha}_j \cap \tilde{\alpha}_l \neq \emptyset$. But, again this contradicts the above result.

As a consequence, only curves diverging to the two ends can intersect. Now, we shall prove that there are at most two of these curves whose intersection is not empty. Assume there exist $\alpha_i$ for $i = 1,2,3$ curves in $\beta_t$ diverging to the two ends such that $\alpha_1 \cap \alpha_i \neq \emptyset$, for $i = 2,3$. Then, by considering appropriate arcs in $\alpha_i$, for $i = 1, 2, 3$ and using the assertion proved above about arcs diverging to one end, we can find a connected component of $M \setminus \beta_t$ that satisfies the conditions of Statement (i) in Theorem 1 and then $X(\Omega)$ must be a planar domain in $\{x_2 = t\}$, which contradicts our assumptions. Moreover, it is clear that if $\alpha_1$ and $\alpha_2$ are two curves in $\beta_t$ diverging to the two ends whose intersection is not empty then $\alpha_1 \cap \alpha_2$ is a unique point.

Lastly, if we have two curves in $\beta_t$ diverging to the two ends whose intersection is not empty, then there are no more intersections in $\beta_{t'}$ for any $t' < 0$, $t' \neq t$. If not, using again the above assertion, we deduce that the pair of divergent curves in $\beta_t$ intersect the pair of divergent curves in $\beta_{t'}$. Since $\beta_t$ and $\beta_{t'}$ are contained in parallel planes, this is a contradiction.

Now, we tackle the case $t > 0$. Observe that this case only has sense if $0 \leq \theta < \pi$ and that $\beta_t \cap \partial(M) = p_i^+ \cup p_i^- \cup p_i^- \cup p_i^-$, where $p_i^+ \in \gamma_i^+$ and $p_i^- \in \gamma_i^-$, for $i = 1,2$. Since $X(M) \subset (S \setminus \mathcal{E}(M)) \cup \Gamma$ we deduce that connected curves in $\beta_t$ are contained in a half-strip of $\{x_2 = t\}$. Therefore, there are no curves in $\beta_t$ diverging to one end. Indeed, we have a connected component of $M \setminus \beta_t$ satisfying the conditions in Statement (i) in Theorem 1 and so we get a contradiction. Hence, it is clear that curves in $\beta_t$ diverging to two ends are disjoint and moreover a divergent curve starting at $\partial(M)$ and a curve diverging to the two ends can not intersect each other. Then, $\alpha_1 \cap \alpha_2 \neq \emptyset$ only in two situations:

i) When $\alpha_1$ is a curve starting at $\gamma_1^+$ and diverging to $E_2$ and $\alpha_2$ is a curve starting at $\gamma_2^+$ and diverging to $E_1$. 
Moreover, we observe that if there exist a pair of curves as in i) for $t_0 > 0$, this pair is unique. It is to say, $\{x_2 = t\} \cap \{x_1 > 0\} \cap X(N^{-1}(\{t_2, t_2\})) = \emptyset$ for $t > 0$, $t \neq t_0$. And the same occurs for a pair of curves as in ii). Therefore, we have at most two points in $\delta_t \cap N^{-1}(\{t_2, t_2\})$ for $t > 0$.

We recall that the set $\beta_0 = \beta$ was studied in the proof of Proposition 1. Since we are assuming $X(M) \subset (S \setminus E(\Gamma)) \cup \Gamma$ the possibilities for $\beta_0$ are those described in 1, 2, 3 and 4. We also point out that Case 2 is not compatible with Case i) analyzed in the case $t > 0$, Case 3 is not compatible with Case ii) analyzed in the case $t > 0$ and so Case 4 is not compatible with either i) or ii). Therefore, there exist at most five points of ordinary contact in $M$ and only three of them can lie in $M \setminus \partial(M)$. 

**Lemma 4.** For any $p \in \gamma_1^+ \cup \gamma_1^- \cup \gamma_2^+ \cup \gamma_2^-$, counting multiplicities, one has $\sharp [g^{-1}(g(p)) \cap \partial(M)] < 6$.

**Proof.** Assume $X(p) \in \ell_1^+$. The proofs of the other possibilities are similar.

Label $\Sigma$ as the tangent plane to $X(M)$ at $X(p)$ and let us consider $\Lambda = X^{-1}(\Sigma \cap X(M))$. Since $\Lambda$ is the nodal set of a harmonic function, then $\Lambda$ is a set of properly immersed analytic curves. Using the interior maximum principle we also deduce that there are no compact simply connected region of $M$ bounded by curves in $\Lambda$.

First, we study the case $0 < \theta < \pi$. In this case $\ell_1^+ \subset \Sigma$ and since $X(M) \subset (S \setminus E(\Gamma)) \cup \Gamma$, it is straightforward to prove that $\Lambda \cap X^{-1}(\Gamma - \ell_1^+) = \emptyset$. Hence, taking into account the reasoning at the beginning of the proof and that $M$ is simply connected, we have that if $\lambda$ is a curve in $\Lambda$ starting at $\gamma_1^+$, then $\lambda$ diverges to an end and when two curves in $\Lambda$ start at $\gamma_1^+$, they do not intersect.

If $\sharp [g^{-1}(g(p)) \cap \partial(M)] \geq 6$ then, using once again the theorem of the order of contact (see [13, §437]), there are at least 6 curves in $\Lambda$ starting at $\gamma_1^+$. Observe that then there exist at least three curves diverging to the same end. Consider this set of diverging curves. If there is a pair of consecutive curves in this set contained in the same half-strip of $\Sigma$ then the connected component between them, that we call $\Omega$, satisfies the conditions of Statement (i) in Theorem 1 and then $X(\Omega)$ must be a planar domain in $\Sigma$, which contradicts our assumptions. On the contrary, if each pair of consecutive curves are in different half-strips, we can use Remark 1 and obtain a new contradiction.

Finally, we analyze the case $\theta = \pi$. Observe that in this case $\Lambda \cap \partial(M) = \gamma_1^+ \cup \gamma_1^-$. Suppose $\sharp [g^{-1}(g(p)) \cap \partial(M)] \geq 6$. We note that in this case compact curves starting at $\gamma_1^+$ and ending at $\gamma_1^-$ can appear in $\Lambda$. Otherwise, we can only have one of these curves, because if there exist two or more curves of this type in $\Lambda$ we would get a compact domain, $\Omega$, in $M \setminus \Lambda$ satisfying
X(Ω) ⊂ Σ, and this is a contradiction. Either there exists a compact curve in Λ or Λ consists only of divergent curves starting at γ₁⁺ ∪ γ₁⁻, it is not hard to see, using Statement (i) in Theorem 1, that there are at least a set of three curves (counting the curve γ₁⁺) diverging to the same end in Λ such that two consecutive curves in this set are in different half-strips of Σ. Then we can conclude as in the former case. □

Using the above lemmas we can now prove:

**Proposition 3.** The map g extends continuously to the ends. In particular, the total curvature of M is finite. Furthermore, the limit tangent plane to M at Eᵢ is πᵢ, i = 1, 2.

**Proof.** We shall prove that the map g extends to E₁. The same argument can be used for E₂. Taking into account Lemma 4 it is not difficult to prove that the following limits exist:

\[
\lim_{p \to E₁} g(p), \quad \lim_{p \to E₁} g(p), \quad i = 1, 2.
\]

(1)

For the proof of this fact see Claim 3.15 in [9].

Since M is conformally equivalent to a sector \( S_{θ₁} = \{ r e^{iΘ} | r > 0, Θ \in [0, θ₁] \} \), a truncated sector \( S_{θ₁}(R) = S_{θ₁} \setminus D(0, R) \) can be seen as a neighbourhood of E₁ in M. Furthermore, we can assume that R is sufficiently large so that \( X(r e^{iθ₁}) ∈ ℓ₁⁺ \) for \( r > R \). According to Schwarz Principle we can consider the reflection respect to \( ℓ₁⁺ \) of \( X(S_{θ₁}(R)) \). Taking into account Proposition 2 we deduce that the Gauss map N on the truncated sector \( S_{2θ₁}(R) = \{ r e^{iΘ} | r > 0, Θ ∈ [0, 2θ₁] \} \setminus D(0, R) \), assumes the values \( \tilde{a}_2 \) and \( -\tilde{a}_2 \) a finite number of times. Then it is possible to choose \( R' > R \) sufficiently large so that g restricted to \( S_{2θ₁}(R') \) omits the values \(-1, 1\).

At this point, we need the following technical result:

**Claim 3.15:** Let \( α < γ < β \), \( S(R) = \{ r e^{iΘ} | r > 0, Θ ∈ [α, β] \} \setminus D(0, R) \), for \( R > 0 \) and let \( f \) be holomorphic in \( S(R) \) and for some complex c satisfy \( \lim_{r \to +∞} f(r e^{iγ}) = c \). Suppose that there are two distinct complex number absent from the range of f. Then \( \lim_{r \to +∞} f(r e^{iθ}) = c \) for every \( Θ ∈ [α, β] \).

We refer to reference book [1, pp. 441-445].

This theorem and (1) imply that g extends continuously to E₁. Finally, since \( M ⊂ S \) we have that the limit tangent plane at Eᵢ coincides with \( πᵢ \), i = 1, 2. □

Consider again the compact curves \( σᵢ^a \) given in Lemma 1 for \( i = 1, 2 \) and \( a ≥ a₀ \) sufficiently large. Let us denote by \( Uₐ^i \) the connected component of \( M \setminus (σ₁^a ∪ σ₂^a) \) that contains the end Eᵢ, \( i = 1, 2 \). With this notation we can prove the following useful result:
Corollary 4. For a sufficiently large, \( X(U^i_a) \) is a graph over the plane \( \pi_i \), \( i = 1, 2 \).

Proof. Take \( a \geq a_0 \) sufficiently large so that \( g(U^i_a) \) does not intersect the equator \( \{ z \in \mathbb{C} \mid |z| = 1 \} \).

Taking into account that \( X(M) \subset (S \setminus \mathcal{E}(\Gamma)) \cup \Gamma \) and the definition of \( U^i_a \), it is not hard to see that \( p_3|_{X(U^i_a)} \) is a local diffeomorphism onto \( \Omega^i_a \), where \( \Omega^i_a \) is the exterior unbounded domain in the plane \( \{ x_3 = 0 \} \) determined by the curve \( p_3(\ell^+_i) \cup p_3(\ell^-_i) \cup p_3(\sigma^i_a) \). As \( X \) is proper, the same occurs for the map \( p_3 \circ X|_{U^i_a} \). So, \( p_3 \circ X|_{U^i_a} \) is a covering map, and taking into account that \( \Omega^i_a \) is simply connected we deduce that \( p_3 \circ X|_{U^i_a} \) is one-to-one. This concludes the proof. \( \square \)

2.2. The symmetries of the surface. The method for proving that \( \{ x_3 = 0 \} \) is a plane of symmetry of \( X(M) \) is based on the well-known Alexandrov’s reflection method and consists of a generalization of Schoen’s ideas (see [14]) to our particular case of noncompact boundary. For a precise presentation of our result the following notation is required. Recall that \( \delta_t = X^{-1}(X(M) \cap \{ x_3 = t \}) \). We also denote for \( t \geq -1/2 \):

\[
M_+(t) = \{(x_1, x_2, x_3) \in X(M) / x_3 \geq t\},
\]
\[
M_-(t) = \{(x_1, x_2, x_3) \in X(M) / x_3 \leq t\}.
\]

A thorough reading of the paragraph 3.2.2 of [9] will convince the readers that, sharpening some arguments, the proof of Theorem 3.24 still works in the case \( X(M) \subset (S \setminus \mathcal{E}(\Gamma)) \cup \Gamma \). Then, we have:

**Proposition 4.** \( X(M) \) is symmetric with respect the plane \( \{ x_3 = 0 \} \). Furthermore, \( M_+(0) \setminus (\ell^+_0 \cup \ell^-_0) \) and \( M_-(0) \setminus (\ell^+_0 \cup \ell^-_0) \) are graphs over \( \{ x_3 = 0 \} \).

Now, we recover two consequences of the above proposition that we need in what follows.

**Corollary 5.** There are only two branch points \( R^+_0 \in \gamma^+ \) and \( R^-_0 \in \gamma^- \) of \( g \) along \( \gamma^+_0 \cup \gamma^-_0 \), \( g \) has multiplicity two at these points and \( R^+_0, R^-_0 \in \delta_0 \). Furthermore, the set \( \mathcal{G} = \{ p \in M / |g(p)| = 1 \} \) consists of \( \gamma^+_0 \cup \gamma^-_0 \cup \delta_0 \).

**Corollary 6.** The limit normal vectors at the ends are opposite.

In the remainder of the paper and without loss of generality, we assume that

\( g(E_1) = 0 \), \( g(E_2) = \infty \).

Next we prove that if \( d = 0 \) then \( \{ x_1 = 0 \} \) is a plane of symmetry of \( X(M) \). As in the horizontal symmetry case, the proof is inspired on Alexandrov’s reflection method. However, the argument exhibited here is slightly different from classical Alexandrov’s technique which uses a family of parallel planes.
In this case we use the pencil of vertical planes that contains the vertical segment \( \ell_0 = \ell_0^+ = \ell_0^- \).

For the sake of simplicity, in this paragraph we consider a new set of Cartesian coordinates obtained from the old one by a rotation of \(-\frac{\pi}{2}\) around the \(x_3\)-axis. Observe that in the new coordinate the \(x_1\)-axis is the inner bisector of the orthogonal projection of \( \ell_i^+ \) and \( \ell_i^- \) to the plane \( \{x_3 = 0\}, \ i = 1, 2, \) and \( \Gamma^+ \subset \{x_2 \geq 0\} \). Moreover, we need to introduce some notation.

For \( \xi \in [0, \pi - \frac{\theta}{2}] \), \( t \in ]-\infty, 0] \) and a set \( A \subset \mathbb{R}^3 \) we define:

\[
H_\xi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \text{Arg}(x_1 + ix_2) = \xi\}, \quad H_{\xi,t} = H_\xi + (t, 0, 0),
\]

\[
P_\xi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \text{Arg}(x_1 + ix_2) = \xi - \frac{\theta}{2}\}, \quad P_{\xi,t} = P_\xi + (t, 0, 0),
\]

\[
H^+_\xi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \text{Arg}(x_1 + ix_2) \in [\xi, \pi - \frac{\theta}{2}]\},
\]

\[
H^-_\xi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \text{Arg}(x_1 + ix_2) \in [-\pi + \frac{\theta}{2}, \xi]\},
\]

\[
\delta(\xi) = X(M) \cap H_\xi, \quad A_+(\xi) = A \cap H^+_\xi, \quad A_-(\xi) = A \cap H^-_\xi,
\]

\[
\delta(\xi, t) = X(M) \cap H_{\xi,t}, \quad A_+(\xi, t) = A \cap (H^+_\xi + (t, 0, 0)),
\]

\[
A_-(\xi, t) = A \cap (H^-_\xi + (t, 0, 0)),
\]

where \( \text{Arg} : \mathbb{C}\setminus ]-\infty, 0] \longrightarrow \mathbb{R} \) denotes the principal argument. Note that \( H_\xi \perp P_\xi \). In addition we label \( s_\xi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \) and \( s_{\xi,t} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \) as the orthogonal symmetries with respect to the planes containing \( H_\xi \) and \( H_{\xi,t} \), respectively. In the same way, we label \( p_\xi : \mathbb{R}^3 \longrightarrow H_\xi \) as the orthogonal projection. With these definitions we denote \( A_+^*(\xi) = s_\xi(A_+(\xi)) \) and \( A_+^*(\xi, t) = s_{\xi,t}(A_+(\xi, t)) \). In particular we denote \( M_+(\xi) = X(M)_+(\xi) \), \( M_+(\xi, t) = X(M)_+(\xi, t) \), \( M_-(\xi) = X(M)_-(\xi) \), \( M_-(\xi, t) = X(M)_-(\xi, t) \) and \( \Delta_\xi = M^+_\xi(\xi) \cap M^-_\xi(\xi) \).

Since the following argument is valid for all \( \xi \in [\frac{\pi}{2}, \pi - \frac{\theta}{2}] \) and \( t \in ]-\infty, 0] \) we omit the parameters \( \xi \) and \( t \) in the description of the different sets.

With the above notations, it is not difficult to see that

\[
(3) \quad \Delta \cap (\Gamma^+_+ \cup \Gamma^-_-) = \delta \cap \Gamma.
\]

From Proposition 3 and Corollary 4 we can also consider \( a_1 \) sufficiently large so that \( a_1 \geq a_0 \), \( X(U^i) = X(U_{a_1}^i) \) is a graph over the plane \( \{x_3 = 0\}, \ i = 1, 2 \) and \( x_3|X(U^1) > 0 \) and \( x_3|X(U^2) < 0 \), where \( a_0 \) is as in Lemma 1 and \( U_{a_1}^i \) is defined in Subsection 2.1.

Now, we can prove the following assertion:

**Claim 1.** If \( X^{-1}(\Delta \setminus \delta) \subset U^1 \cup U^2 \), then \( \Delta = \delta \).

The proof of this claim is similar to the proof of Claim 3.19 in [9]. We refer the reader to [9] for details.
Now we define the set
\[ I = \left\{ \xi_1 \in [0, \pi - \frac{\theta}{2}] \mid [M_+^*(\xi) \cap \{ x_3 \geq 0 \}] \supseteq [M_-^*(\xi) \cap \{ x_3 \geq 0 \}], \right. \]
\[ \pi - \frac{\theta}{2} \leq \xi \leq \xi_1 \right\} . \]

Our objective is to prove that \( I = [0, \pi - \frac{\theta}{2}] \). We divide the proof of this fact into several points:

**Claim 2.** \( \frac{\pi}{2}, \pi - \frac{\theta}{2} \) \( \subseteq I \).

If \( A, B \subset \mathbb{R}^3 \), we say that \( A \geq_\xi B \) provided for every \( x \in \mathbb{R}^3 \) for which \( p_{\xi}^{-1}(\{x\}) \cap A \neq \emptyset \) and \( p_{\xi}^{-1}(\{x\}) \cap B \neq \emptyset \), we have that the orthogonal coordinate to \( H_{\xi,t} \) of any point in \( p_{\xi}^{-1}(\{x\}) \cap A \) is equal to or greater than the respective orthogonal coordinate of any point in \( p_{\xi}^{-1}(\{x\}) \cap B \).

Given \( \xi \in \left[ \frac{\pi}{2}, \pi - \frac{\theta}{2} \right] \), we define the set
\[ I_\xi = \left\{ t \in ]-\infty, 0] \mid M_+^*(\xi, t) \text{ is a graph over } H_{\xi,t} \right. \]
\[ \text{and } M_+^*(\xi, t) \leq_\xi M_-^*(\xi, t) \right\} . \]

Our purpose is to show that \( I_\xi = ]-\infty, 0] \), for all \( \xi \in ]\frac{\pi}{2}, \pi - \frac{\theta}{2}] \). Note that this fact implies \( \frac{\pi}{2}, \pi - \frac{\theta}{2} \) \( \subset I \).

First, we are going to see that \( I_\xi \neq \emptyset \). To do this, let \( t' < 0 \) such that \( X^{-1}(M_+^*(\xi, t)) \subset U^1 \cup U^2 \), where \( U^i \) are defined as above, \( \forall t \leq t' \) (observe that \( X^{-1}(\Delta_\xi,t) \subset U^1 \cup U^2 \)). Hence from Claim 1, \( \Delta_{\xi,t} = \delta_{\xi,t}, t \leq t' \).

Then, it is clear that \( X^{-1}(M_+^*(\xi, t)) \) consists of two simply connected components, one of them in \( U^1 \) and the other one in \( U^2 \), \( \forall t \geq t' \), and thus \( M_+^*(\xi, t) \) is the union of two disjoint graphs \( G_1^*(\xi, t) \) and \( G_2^*(\xi, t) \) over the same simply connected domain \( G^*(\xi, t) \) in the plane \( \{ x_3 = 0 \} \).

From the definition of \( U^1 \) and \( U^2 \), we have that \( p_{\xi}(G_1^*(t')) \cap p_{\xi}(G_2^*(t')) = \emptyset \).

Let us see that \( M_+^*(\xi, t) \) is a graph over the halfplane \( H_{\xi,t}, t \leq t' \).

First, observe that \( p_{\xi} \) is injective on \( \delta_{\xi,t}, t \leq t' \). Indeed, note that \( \delta_{\xi,t} \subset H_{\xi,t} \) is a graph over a connected piece of a straight line, and so the \( p_{\xi} \) is injective.

Moreover, a similar argument gives that the set \( P_{\xi,s} \cap M_+^*(\xi, t') \) is a connected curve, for sufficiently large \( s \). Furthermore, the function \( x_3 \) is monotone over \( P_{\xi,s} \cap M_+^*(\xi, t') \). Otherwise, there would exist some points in \( M_+^*(\xi, t) \) whose normal vector lie in \( \{ x_1 = 0 \} \cap S^2 \). Thus, we could take \( t \leq t' \) in such a way that \( \delta_t \) contains a point with normal vector in \( P_{\xi} \cap S^2 \).

Hence, from the theorem of the order of contact (see [13, §437]) and taking into account that \( X(U^i) \) are graphs over the plane \( x_3 = 0 \), \( i = 1, 2 \), we deduce that \( \Delta_{\xi,t} - \delta_{\xi,t} \neq \emptyset \), which is contrary to Claim 1.
This proves that $M_+(\xi, t')$ is a graph over the plane $H_\xi$, and so the same holds for $M_+(\xi, t)$, $t \leq t'$.

Taking into account that $\Delta_{\xi,t} = \delta_{\xi,t}$, for $t \leq t'$ and $X(M) \in (S \setminus E(M)) \cup \Gamma$, we deduce that $M_+^*(\xi, t) \leq \xi$ $M_-(\xi, t)$, $t \leq t'$. Thus, $]-\infty, t'] \subset \mathcal{I}_\xi$.

It is obvious that $\mathcal{I}_\xi$ is closed in $]-\infty, 0]$. Next, we shall see that $0 = \text{Maximum}(\mathcal{I}_\xi)$. We proceed by contradiction. Assume $t_0 = \text{Maximum}(\mathcal{I}_\xi) < 0$. Let $K = X(M \setminus (U^1 \cup U^2))$. Since $t_0 \in \mathcal{I}_\xi$, $K_+(\xi, t_0)$ is a graph over the plane $H_\xi$. Using the interior maximum principle and the maximum principle at the boundary, it is not hard to see that there exists $\varepsilon > 0$ such that $K_+(\xi, t)$ is a graph over $H_\xi$ and $\Delta_{\xi,t} \cap K = \delta_{\xi,t} \cap K$ for $t \in ]0, t_0 + \varepsilon[$. Hence, using Claim 1, we deduce that $\Delta_{\xi,t} = \delta_{\xi,t}$. However, the maximality of $t_0$ leads us to $\Delta_{\xi,t} \setminus \delta_{\xi,t} \neq \emptyset$, which is absurd. This finishes the proof of the claim.

**Claim 3.** The set $\mathcal{I}$ is closed in $[0, \pi - \theta_2]$ and its minimum is 0.

Obviously, $\mathcal{I}$ is closed. To prove $\text{Minimum}(\mathcal{I}) = 0$ we proceed by contradiction. Let us assume that $\text{Minimum}(\mathcal{I}) = \xi_0 > 0$. As in the preceding claim, we consider $K = X(M \setminus (U^1 \cup U^2))$. Taking into account that $\xi_0 \in \mathcal{I}$ we have

\[(4) \quad [M_+^*(\xi_0) \cap \{x_3 \geq 0\}] \geq [M_-(\xi_0) \cap \{x_3 \geq 0\}].\]

Hence, since $K_+(\xi_0)$ is compact, there exists $\varepsilon_0 > 0$ sufficiently small so that $[K_+(\xi_0 - \varepsilon) \cap \{x_3 \geq 0\}] \geq [K_-(\xi_0 - \varepsilon) \cap \{x_3 \geq 0\}]$, for all $0 \leq \varepsilon \leq \varepsilon_0$. Otherwise, we could find sequences $\{\xi_n\} \nearrow \xi_0$, with $\xi_n \in [0, \xi_0[$, and $\{x_n\}, \{y_n\}$ in $K$, fulfilling the following conditions:

i) $x_n \in K_+(\xi_n), y_n \in K_-(\xi_n)$ and $\delta_{\xi_0}(x_n) = y_n, \forall n \in \mathbb{N}$.

ii) $\{x_n\} \to x \in K_+(\xi_0), \{y_n\} \to y \in K_-(\xi_0)$.

From i) and ii) we deduce that $s_{\xi_0}(x) = y$. On the other hand, (3) implies that any point lying in $\Delta_{\xi_0} \setminus \delta_{\xi_0}$ is an interior point of contact between $M_+^*(\xi_0)$ and $M_-(\xi_0)$. Assume $\Delta_{\xi_0} \setminus \delta_{\xi_0} \neq \emptyset$. Then, making use of the interior maximum principle we deduce $M_+^*(\xi_0) = M_-(\xi_0)$, which is absurd because, since $\xi_0 > 0$, $\Gamma$ is not symmetric with respect to the plane $H_{\xi_0}$.

Therefore $x = y \in K \cap \delta_{\xi_0}$. Hence, taking into account i) and ii), we have that $N(X^{-1}(x)) = s_{\xi_0}(N(X^{-1}(x)))$ and so $N(X^{-1}(x))$ is parallel to $H_{\xi_0}$. Therefore, by the theorem of the order of contact (see [13, §437]) we have that $x \in K \cap (\delta_{\xi_0} \setminus (\delta_{\xi_0} \cap \ell_{\xi_0}))$ and $x \not\in \{q_1^+, q_2^-, q_2^+, q_2^\}$ and $q_2^+ \not\in \delta_{\xi_0}$). From this fact and taking into account (4), the maximum principle at the boundary can be applied to a neighbourhood of the point $x$. We get $M_+^*(\xi_0) = M_-(\xi_0)$, which is as above a contradiction.

By the preceding reasoning, we have $\Delta_{\xi} \setminus \delta_{\xi} \subset X(U^1 \cup U^2)$ for $\xi \geq \xi_0 - \varepsilon_0$. From Claim 1 we conclude that $\Delta_{\xi} = \delta_{\xi}$ for $\xi \geq \xi_0 - \varepsilon_0$. Clearly this implies $[M_+^*(\xi) \cap \{x_3 \geq 0\}] \geq [M_-(\xi) \cap \{x_3 \geq 0\}]$ for $\xi \geq \xi_0 - \varepsilon_0$ and so $\xi_0 - \varepsilon_0 \in \mathcal{I}$, which contradicts that $\xi_0$ is a minimum.
Summarizing we have \([M^*_+ (0) \cap \{x_3 \geq 0\}] \geq [M_-(0) \cap \{x_3 \geq 0\}]\). We can repeat the above argument starting from \(\Gamma^-\) instead of \(\Gamma^+\) and obtain \([M^*_+ (0) \cap \{x_3 \geq 0\}] \geq [M_+ (0) \cap \{x_3 \geq 0\}]\) and so \(X(M) \cap \{x_3 \geq 0\}\) is symmetric with respect to the plane \(\{x_2 = 0\}\). Finally, by the horizontal symmetry mentioned in 2.2 we have that \(X(M)\) is symmetric with respect to the plane \(\{x_2 = 0\}\).

### 2.3. Determination of conformal structure and Weierstrass data of \(M\).

This subsection is devoted to determining the Weierstrass data associated to the minimal immersion \(X : M \rightarrow \mathbb{R}^3\). We define \(n = \frac{2\pi}{3j - 2}\).

Observe that \(n \in [\frac{2}{3}, 1]\). As \(M\) is simply-connected, the map \((-i)^\frac{n}{2}\) has a well-defined branch on \(M\). Let \(f\) be the branch of \((-i)^\frac{n}{2}\) such that \(\text{Arg}(f(p)) = 0\), whenever \(-i \text{arg}(p) \in \mathbb{R}^+\).

As before, \(Q_j^+ = \gamma_0^+ \cap \gamma_j^+\), \(Q_j^- = \gamma_0^- \cap \gamma_j^-, j = 1, 2\). Observe that \(f(Q_j^+) = i\) and \(f(Q_j^-) = -i\), \(j = 1, 2\). Moreover, since \(X(M) \subset (S \setminus \mathcal{E}(\Gamma)) \cup \Gamma\) it is easy to see that \(f(p) \neq f(Q_j^+)\) for all \(p \in \gamma_j^+, j = 1, 2\). Then, taking into account that \(g(E_1) = 0\) and \(g(E_2) = \infty\), one has:

\[
\begin{align*}
    f(\gamma_1^-) &= s_1^- = \{\lambda i \mid \lambda \in [-1, 0]\}, & f(\gamma_2^-) &= s_2^- = \{\lambda i \mid \lambda \in (-\infty, -1]\}, \\
    f(\gamma_1^+) &= s_1^+ = \{\lambda i \mid \lambda \in [0, 1]\}, & f(\gamma_2^+) &= s_2^+ = \{\lambda i \mid \lambda \in [1, \infty]\}.
\end{align*}
\]

Using Corollary 5, we deduce that \(f|_{Q_j^+ R_0^-}\) are injective for \(j = 1, 2\). Hence, if we write \(f(R_0^+) = e^{i\frac{j\pi}{2}}\) and \(f(R_0^-) = e^{i\frac{j\pi}{2}}\), it is not hard to check that

\[
    f(\gamma_0^-) = s_0^- = \left\{e^{i\frac{j\pi}{2}} \mid \lambda \in [-\pi, \pi]\right\}, \quad f(\gamma_0^+) = s_0^+ = \left\{e^{i\frac{j\pi}{2}} \mid t \in [t_0, \pi]\right\}.
\]

Using again that \(X(M) \subset (S \setminus \mathcal{E}(\Gamma)) \cup \Gamma\), the fact that there are at most two points on \(\gamma_0^+\) where the Gauss map achieves the values \(\pm \tilde{a}_2\) (see the proof of Proposition 1) and Corollary 5, one deduces that \(-\pi < t_1 < 0 < t_0 < \pi\).

Let \(\Lambda\) denote the connected component of \(\mathbb{C} \setminus (f(\partial(M)) \cup \{0\})\) containing the point \(\{1\}\) (see Figure 5). We have the following result:

**Lemma 5.** The map \(f : M \rightarrow \mathbb{C}\) fulfills that:

(i) \(f(M \setminus \partial(M)) = \Lambda\), and
(ii) \(f|_{M \setminus \partial(M)} : M \setminus \partial M \rightarrow \Lambda\) is a biholomorphism.

**Proof.** In order to prove (i) we note that \(f\) is holomorphic and nonconstant, and so \(f(M \setminus \partial(M))\) is an open subset of \(\mathbb{C}^\times\) (note that no points in \(M\) have vertical normal vector). On the other hand, taking into account that \(\overline{M} = M \cup \{E_1, E_2\}\) is compact, \(f(\overline{M})\) is a closed subset of \(\mathbb{C}\). Therefore the set \(W = f(M \setminus \partial(M)) \cap (\mathbb{C} \setminus (f(\partial(M)) \cup \{0\})) = f(M) \cap (\mathbb{C} \setminus (f(\partial(M)) \cup \{0\}))\) is a closed subset of \(\mathbb{C} \setminus (f(\partial(M)) \cup \{0\})\). Then, either \(W = \mathbb{C} \setminus (f(\partial(M)) \cup \{0\})\) or \(W\) is a connected component of \(\mathbb{C} \setminus (f(\partial(M)) \cup \{0\})\).
To see that the first possibility does not occur we proceed by contradiction. Assume \( W = \mathbb{C} \setminus (f(\partial(M)) \cup \{0\}) \). Observe that \( f(p) \in S^1 \) if and only if \( g(p) \in S^1 \). Hence, taking into account Corollary 5 and (5) we infer that \( \{e^{\frac{it}{2}}, t \in [t_1, t_0] \cup [\frac{\pi}{2}, \frac{3\pi}{2}]\} \subset f(\delta_0) \). Moreover, since there are no ramification points in \( \delta_0 \setminus \{R_0^+, R_0^-, R_1^+, R_1^-\} \), we deduce that \( \text{Arg}(f) \) is a monotone function in \( \delta_0 \) and \( f(\delta_0) = S^1 \). Since \( \{1, e^{2n\pi i}, e^{n\pi i}, e^{2n\pi i}\} \subset S^1 \), we have that there are at least four points in \( \delta_0 \cap N^{-1}(\{a_2, -a_2\}) \). But it is contrary to Proposition 2. Since \( f(\gamma_0^+) \) and \( f(\gamma_0^-) \) lie in the boundary of \( W \), we easily obtain that \( W = \Lambda \) and so \( f(M \setminus \partial(M)) = \Lambda \).

The same argument presented above gives us that \( f(\delta_0) = \{e^{\frac{it}{2}} \mid t \in [t_1, t_0]\} \) and \( f|_{\delta_0} : \delta_0 \longrightarrow \{e^{\frac{it}{2}} \mid t \in [t_1, t_0]\} \) is a one-to-one function. To finish the proof, we define \( \gamma = \partial(M) \). Since \( M \) is conformally a closed disk with piecewise analytic boundary, then \( \gamma \) is a piecewise analytic curve homeomorphic to \( S^1 \). Note that since \( g|_M \) has no poles and \( g \) extends continuously to \( M \), we can assert the same for \( f \). Then, we know that for any \( w \in \Lambda = f(M \setminus \partial(M)) \), \( \sharp(\{f^{-1}(w)\}) = \frac{1}{2\pi i} \int_\gamma \frac{df}{f-w} \in \mathbb{Z} \). Thus, if we define \( h : \Lambda \longrightarrow \mathbb{Z} \) by \( h(w) = \sharp(\{f^{-1}(w)\}) \), the function \( h \) is continuous on \( \Lambda \), and so it is constant. From the above arguments we have that \( h(w) = 1 \), \( \forall w \in f(\delta_0 \setminus \{R_0^+, R_0^-\}) \) and this concludes the proof.

Let \( N_{t_0}^{t_1} \) be the following four-punctured torus:
\[
N_{t_0}^{t_1} = \{(u, v) \in \mathbb{C}^+ \times \mathbb{C} \mid v^2 = (u-e^{\frac{it_0}{2}})(u+e^{\frac{it_0}{2}})(u-e^{\frac{it_1}{2}})(u+e^{-\frac{it_1}{2}})\}.
\]
Consider \( U_{t_0}^{t_1} \subset N_{t_0}^{t_1} \) as the connected component of \( u^{-1}(\mathbb{C} \setminus f(\partial(M))) \) containing the point \( P_0 = \left(1, 2\sqrt{-\sin\left(\frac{t_0}{2}\right) \sin\left(\frac{t_1}{2}\right)}\right) \). Define \( \mathcal{N}' = \overline{U_{t_0}^{t_1}} \). At this point we prove the following proposition:

**Proposition 5.** \( M \) is biholomorphic to \( N' \). Furthermore, the Weierstrass data are given on \( N' \) by
\[
g(u) = iu^{\frac{2}{\lambda}}, \quad \Phi_3 = \lambda \frac{du}{v},
\]
where \( \lambda \in \mathbb{R}^+ \) and we choose the branch of \( u^{\frac{2}{\lambda}} \) satisfying \( 1^{\frac{2}{\lambda}} = 1 \).

**Proof.** Consider \( f : M \setminus \partial(M) \longrightarrow \Lambda \), the biholomorphism defined in Lemma 5. Observe that the \( u \)-projection is a biholomorphism from \( N' \setminus \partial(N') \) onto \( \Lambda \) and so \( F = f^{-1} \circ u : N' \setminus \partial(N') \longrightarrow M \setminus \partial(M) \) is a biholomorphism. Since \( N' \setminus \partial(N') \) and \( M \setminus \partial(M) \) are conformally equivalent to Jordan regions in \( \mathbb{C} \), a well-known result of complex analysis asserts that \( F \) can be extended to a biholomorphism from \( N' \) onto \( M \). For the sake of simplicity, in what follows we identify \( M \) with \( N' \).

Now, we consider the holomorphic function \( \omega = \frac{\Phi_3}{\Phi_1} \) on \( N' \). Let \( p_0 \in s_0^+ \cup s_0^- \) and \( \{p_0^1, p_0^2\} = u^{-1}(p_0) \). Since the surface is symmetric with respect
to the plane \( \{ x_3 = 0 \} \) (see 2.2), we have that \( \Phi_3(p_0^1) = -\Phi_3(p_0^2) \). But this is also the behaviour of \( v \). Consequently, \( \bar{\omega} = \omega \circ u^{-1} \) is a holomorphic function in \( \Lambda \) that can be extended to \( s_0^+ \cup s_0^- \).

As \( \Phi_3 \) and \( v \) extend to the punctures in a natural way, \( \bar{\omega} \) can be seen as a holomorphic function on \( \{ z \in \mathbb{C} \mid \text{Re}(z) \geq 0 \} \). Furthermore, \( \bar{\omega} \) is real on \( \{ z \in \mathbb{C} \mid \text{Re}(z) = 0 \} \) and so, using the Schwarz Principle, \( \bar{\omega} \) extends to the whole \( \mathbb{C} \). Then, \( \bar{\omega} \) is a holomorphic function on \( \mathbb{C} \) without zeroes or poles and therefore \( \bar{\omega} = \lambda \in \mathbb{C}^* \). Finally, using again that \( \bar{\omega} \) is real on \( \{ z \in \mathbb{C} \mid \text{Re}(z) = 0 \} \) we deduce \( \lambda \in \mathbb{R}^+ \). Observe that, up to a rigid motion, we can assume that \( \lambda \in \mathbb{R}^+ \).

As we announced, we now prove that \( X(M) \) also inherits the vertical symmetry of its boundary when \( d > 0 \).

**Proposition 6.** *In the above setting, \( t_1 = -t_0 \).*

**Proof.** First of all, observe that the result about the vertical symmetry proved in Paragraph 2.2 implies \( t_1 = -t_0 \) if \( d = 0 \). We shall see that this fact suffices to prove the general case.

Since \( X : M \rightarrow \mathbb{R}^3 \) is a solution of Problem 1 we have that \( X_1(R_0^-) = -\frac{d}{2} \) and \( X_1(R_0^+) = \frac{d}{2} \). Thus

\[
d = X_1(R_0^+) - X_1(R_0^-) = \text{Re} \int_{\tilde{\delta}} \Phi_1,
\]

where \( \tilde{\delta} \) is the lift to \( M \) of the curve \( e^{it_1^2} \), \( t_1 \leq t \leq t_0 \), in the \( u \)-plane. Taking into account the expressions for \( g \) and \( \Phi_3 \) given in Proposition 5, it is not difficult to obtain that

\[
d = \frac{\lambda}{4} f_1(t_0, t_1) = \frac{\lambda}{4} \int_{t_1}^{t_0} \frac{\cos(\frac{t}{n})}{v(t, t_0, t_1)} dt,
\]

where \( v(t, t_0, t_1) = \sqrt{(\sin(\frac{t_1}{n}) - \sin(\frac{t}{n}))(\sin(\frac{t}{n}) - \sin(\frac{t_0}{n}))} \). Furthermore, we have that \( X_2(R_0^+) = X_2(R_0^-) \). Thus

\[
X_2(R_0^+) - X_2(R_0^-) = \text{Re} \int_{\tilde{\delta}} \Phi_2 = 0.
\]

A direct computation using again the expressions of \( g \) and \( \Phi_3 \) given in Proposition 5 gives

\[
\text{Re} \int_{\tilde{\delta}} \Phi_2 = \frac{\lambda}{4} f_2(t_0, t_1) = \frac{\lambda}{4} \int_{t_1}^{t_0} \frac{\sin(\frac{t}{n})}{v(t, t_0, t_1)} dt.
\]

From the definitions of the functions \( f_1 \) and \( f_2 \) we have

\[
f_1(t_0, t_1) = f_1(-t_1, -t_0), \quad f_2(t_0, t_1) = -f_2(-t_1, -t_0),
\]

for all \( (t_0, t_1) \in [0, \pi] \times [-\pi, 0] \). Observe that if \( X : M \rightarrow \mathbb{R}^3 \) is the solution of Problem 1 given in Proposition 5, then \( (t_0, t_1) \) must satisfy \( f_1(t_0, t_1) \geq \ldots \)
where \( a \) and \( f_2(t_0, t_1) = 0 \). From the properties of \( f_1 \) and \( f_2 \) given in (6), it suffices to study the zeros of the functions \( f_1 \) and \( f_2 \) in the triangle given by 

\[ T = \{(t_0, t_1) \in [0, \pi] \times [-\pi, 0] \mid t_0 \leq -t_1\}. \]

Let us denote by \( L_1 = \{0\} \times [-\pi, 0] \), \( L_2 = \{(t_0, t_1) \in [0, \pi] \times [-\pi, 0] \mid t_0 = -t_1\} \) and \( L_3 = [0, \pi] \times \{-\pi\} \) the sides of the triangle \( T \). We also define the sets \( C_1 = \{(t_0, t_1) \in T \mid f_1(t_0, t_1) = 0\} \), \( C_2 = \{(t_0, t_1) \in T \mid f_2(t_0, t_1) = 0\}. \) It is clear from (6) that \( L_2 \subset C_2 \). Furthermore, by the vertical symmetry proved in Paragraph 2.2 we deduce that \( C_1 \) can only intersect \( C_2 \) in points of \( L_2 \). For the sake of clarity, we divide the rest of the proof in several steps.

**Step 1.** The objective of this step is to show that \( C_i \) is a set of analytic curves in \( T \), for \( i = 1, 2 \).

Consider the meromorphic 1-form given by \( \Phi = \frac{u_{\phi}}{v} du \). Note that \(-f_2\) and \( f_1 \) are the real and imaginary part of \( 4f_\phi \Phi \). By deriving, one has

\[
(7) \quad \mathcal{L}(\Phi) = \frac{\partial^2 \Phi}{\partial t_0^2} + \frac{\partial^2 \Phi}{\partial t_1^2} + a_1 \frac{\partial \Phi}{\partial t_0} + a_2 \frac{\partial \Phi}{\partial t_1} + a_0 \Phi = d(F),
\]

where

\[
a_1 = \cos(t_0) + \sin(t_0) \sin(t_1) - \frac{2 \cos(t_0)\sin(t_0) - \sin(t_0)}{2}, \]

\[
a_2 = -\cos(t_1) + \sin(t_1) \sin(t_2) - \frac{2 \cos(t_1)\sin(t_1) - \sin(t_1)}{2}, \]

\[
a_0 = \frac{4 - n^2}{4n^2},
\]

and \( F \) is the following meromorphic function:

\[
F = \frac{u_{\phi}^{n+1}(au^2 + bu^3 + cu^2 + du + e)}{4n^2v^3},
\]

where \( a = 2 + n, b = -i(4 + n)(\sin(t_0) + \sin(t_0)), c = -4(1 + 2 \sin(t_0) \sin(t_0)), d = -i(4 + n)(\sin(t_0) + \sin(t_0)) \) and \( e = 2 - n \).

Integrating by parts in (7), we have that \( f_1 \) and \( f_2 \) are zeroes of the second order elliptic operator given by \( \mathcal{L} \). As \( C_1 \) and \( C_2 \) are the nodal sets of \( f_1 \) and \( f_2 \), respectively, we can assert (see [2]) that \( C_i \) is a set of regular curves and the critical points on the nodal lines are isolated. Furthermore, when the nodal lines meet, they form an equiangular system. Moreover, by the Maximum Principle for elliptic operators, \( C_i \) cannot contain closed curves.

**Step 2.** The purpose of the present step is to study the behaviour of the curves in \( C_1 \) at the boundary of the triangle \( T \). We shall see that \( C_1 \cap L_1 = (0, t_1), C_1 \cap L_2 = (t_0, -t_0) \) and that there are no curves in \( C_1 \) approaching \( L_3 \).
First, we shall prove that $\frac{\partial f_1}{\partial t_1}(t_0, t_1) > 0$, for all $(t_0, t_1) \in T$. It is not difficult to see that
\[
\frac{\partial}{\partial t_1} \left( \frac{\cos \left( \frac{\pi}{n} t \right)}{v(t, t_0, t_1)} \right) = \frac{\cos \left( \frac{\pi}{n} t \right) \cos \left( \frac{\xi}{2} \right) - \cos \left( \frac{\xi}{2} \right) \cos \left( \frac{\pi}{n} t \right)}{4v(t, t_0, t_1)(\sin \left( \frac{\xi}{2} \right) - \sin \left( \frac{\xi}{2} \right))} \leq \frac{\cos \left( \frac{\xi}{n} \right)}{\sin \left( \frac{\xi}{n} \right) - \sin \left( \frac{\xi}{n} \right)} \frac{\sin \left( \frac{\xi}{2} \right) - \sin \left( \frac{\xi}{2} \right)}{v(t, t_0, t_1)}.
\]

Thus integrating by parts in the above equality we obtain
\[
\frac{\partial f_1}{\partial t_1}(t_0, t_1) = \int_{t_1}^{t_0} \frac{\cos \left( \frac{\xi}{n} \right) \cos \left( \frac{\xi}{2} \right) - \cos \left( \frac{\xi}{2} \right) \cos \left( \frac{\xi}{n} \right)}{4v(t, t_0, t_1)(\sin \left( \frac{\xi}{2} \right) - \sin \left( \frac{\xi}{2} \right))} dt.
\]

In order to prove $\frac{\partial f_1}{\partial t_1}(t_0, t_1) > 0$ we shall see that the function
\[
h(t, n) = \cos \left( \frac{\xi}{n} \right) \cos \left( \frac{\xi}{2} \right) - \cos \left( \frac{\xi}{2} \right) \cos \left( \frac{\xi}{n} \right) \geq 0,
\]
for $t \in [t_1, t_0]$. Since $h(-t, n) = h(t, n)$ and $t_0 \leq -t_1$, it suffices to prove that $h(t, n) \geq 0$ for $t \in [t_1, 0]$. Moreover, taking into account $h(t_1, n) = 0$, it is enough to see that $\frac{\partial h}{\partial t}(t, n) = -\frac{1}{n} \sin \left( \frac{\xi}{n} \right) \cos \left( \frac{\xi}{2} \right) + \frac{1}{2} \sin \left( \frac{\xi}{2} \right) \cos \left( \frac{\xi}{n} \right) \geq 0$ for $t \in [t_1, 0]$.

Assume first $t_1 \in [-\frac{\pi}{n}, 0]$. As $n \in [\frac{\pi}{2}, 1]$ we have $0 \leq -\sin \left( \frac{\xi}{n} \right) \leq -\sin \left( \frac{\xi}{n} \right)$ and $0 \leq \cos \left( \frac{\xi}{n} \right) \leq \cos \left( \frac{\xi}{n} \right)$. Hence $\frac{\partial h}{\partial t}(t, n) \geq 0$.

If $t_1 \in [-n\pi, -\frac{\pi}{n}]$, the study of the signs in the expression of $\frac{\partial h}{\partial t}$ gives directly that this partial is nonnegative.

Finally, we consider the case $t_1 \in [-\pi, -n\pi]$. As in the former case, studying the signs in the expression of $\frac{\partial h}{\partial t}$ we obtain that $\frac{\partial h}{\partial t}(t, n) \geq 0$ for $t \in [-n\pi, 0]$. Otherwise, it is not difficult to see that $\frac{\partial^2 h}{\partial t \partial n}(t, n) \geq 0$ for $t \in [t_1, -n\pi]$. As $\frac{\partial h}{\partial t}(t, \frac{\pi}{2}) = 2 \cos \left( \frac{\xi}{2} \right) \sin \left( \frac{\xi}{2} \right) \left( -2 \cos \left( \frac{\xi}{2} \right) + \cos \left( \frac{\xi}{n} \right) \right) \geq 0$ for $t \in [t_1, -n\pi]$, we conclude that $\frac{\partial h}{\partial t}(t, n) \geq 0$.

Our next objective is to prove that $f_1(0, -n\pi) < 0$. Indeed, making the change of variable $s = t + n\pi/2$ one has:
\[
f_1(0, -n\pi) = \int_{-n\pi/2}^{n\pi/2} \sin \left( \frac{\xi}{n} \right) \sqrt{\sin \left( \frac{n\pi}{2} \right) \sin \left( -\frac{n\pi}{2} + 2\pi \right)} ds = 
\int_{0}^{n\pi/2} \sin \left( \frac{\xi}{n} \right) \sqrt{\sin \left( \frac{n\pi}{2} \right) \sin \left( \frac{n\pi}{2} - 2\pi \right)} ds.
\]

An easy computation gives us that the numerator in the last integral is always nonpositive, and then $f_1(0, -n\pi) < 0$. From the definition of $f_1$ we also have $f_1(0, -\frac{n\pi}{2}) > 0$. Then, taking into account that $\frac{\partial f_1}{\partial t_1}(0, t_1) > 0$, we have that there exists a unique $\tilde{t}_1 \in ] -n\pi, -\frac{n\pi}{2} [ \text{ such that } f_1(0, \tilde{t}_1) = 0$. 

We now consider the function $\tilde{f}_1(t_0) = f_1(t_0, -t_0)$, that is the function $f_1$ restricted to the side $L_2$. Taking into account (6) one has

$$\frac{\partial \tilde{f}_1}{\partial t_0}(t_0) = \frac{\partial f_1}{\partial t_0}(t_0, -t_0) - \frac{\partial f_1}{\partial t_1}(t_0, -t_0) = -2 \frac{\partial f_1}{\partial t_1}(t_0, -t_0) < 0. \tag{9}$$

According to the definition of $f_1$ we have $\tilde{f}_1\left(\frac{n\pi}{2}\right) = f_1\left(\frac{n\pi}{2}, -\frac{n\pi}{2}\right) > 0$. Our next purpose is to see that $\tilde{f}_1(n\pi) = f_1(n\pi, -n\pi) < 0$. Note that

$$\tilde{f}_1(n\pi) = 2\sqrt{2} \int_{-\pi}^{0} \frac{\cos\left(\frac{t}{n}\right)}{\sqrt{\cos(t) - \cos(n\pi)}}. \tag{10}$$

A direct computation gives

$$\tilde{f}_1(n\pi) = 2\sqrt{2} \int_{0}^{\frac{\pi}{n}} \sin\left(\frac{s}{n}\right) \frac{\sqrt{\cos(s + \frac{n\pi}{2}) - \cos(n\pi)} - \sqrt{\cos(s - \frac{n\pi}{2}) - \cos(n\pi)}}{\sqrt{\cos(s + \frac{n\pi}{2}) - \cos(n\pi)} \sqrt{\cos(s - \frac{n\pi}{2}) - \cos(n\pi)}}. \tag{11}$$

It is not hard to see that the numerator in the above integrand is nonpositive, in particular $\tilde{f}_1(n\pi) < 0$.

Therefore there exists $t_0 \in \left[\frac{n\pi}{2}, n\pi\right]$ such that $f_1(t_0, -t_0) = 0$, $f_1$ is positive in $\{(t_0, -t_0) \in T \mid 0 < t_0 < t_0\}$ and negative in $\{(t_0, -t_0) \in T \mid t_0 < t_0 < \pi\}$.

Now, we prove that $\lim_{t_1 \to -\pi} f_1(t_0, t_1) = -\infty$. In order to do this we consider a new set of parameters

$$s(t_0, t_1) = \sin\left(\frac{t_0}{2}\right), \quad r(t_0, t_1) = \sin\left(\frac{t_1}{2}\right). \tag{10}$$

Note that $r(t_0, -\pi) = -1$. Our next objective is to see that $\lim_{r \to -1} f_1(s, r) = -\infty$. In order to do this, we derive again the 1-form $\Phi$ defined in Step 1 and we obtain the following equality:

$$\frac{\partial^4 \Phi}{\partial r^4} + b_3 \frac{\partial^3 \Phi}{\partial r^3} + b_2 \frac{\partial^2 \Phi}{\partial r^2} + b_1 \frac{\partial \Phi}{\partial r} + b_0 \Phi = d(\varphi), \tag{11}$$

where

$$b_3 = \frac{2(5r^3 + 2s - 8r^2s + r(-2 + 3s^2))}{(-1 + r^2)(r - s)^2},$$

$$b_2 = \frac{-16(r - s)^2 + n^2(-8 + 99r^2 - 116rs + 25s^2)}{4n^2(-1 + r^2)(r - s)^2},$$

$$b_1 = \frac{(-48 + 57n^2)r + (48 - 39n^2)s}{4n^2(-1 + r^2)(r - s)^2},$$

$$b_0 = \frac{3(n^2 - 4)}{4n^2(-1 + r^2)(r - s)^2}.$$
and \( \varphi \) is the following meromorphic function:

\[
\varphi(u) = \frac{u^{\frac{2}{n}+1}(a'u^4 + b'u^3 + c'u^2 + d'u + e')(u^2 - 1 - 2\sinu)^\frac{1}{2}}{4n^2(-1 + r^2)(r - s)^2(u^2 - 1 - 2\ri u)^\frac{1}{2}},
\]

with \( a' = -3n(2 + n), \ b' = -3\sin((-4 + 3n)r - (4 + 5n)s), \ c' = 12n(1 + 2rs), \ d' = -3\sin((4 + 3n)r + (4 - 5n)s) \) and \( e' = 3n(n - 2) \). Integrating by parts in (11), we have that \( f_1 \) is a solution of the fourth order ordinary differential equation given by

\[
\frac{\partial^4 f_1}{\partial r^4} + b_3 \frac{\partial^3 f_1}{\partial r^3} + b_2 \frac{\partial^2 f_1}{\partial r^2} + b_1 \frac{\partial f_1}{\partial r} + b_0 f_1 = 0.
\]

Observe that this equation presents a regular singular point in \( r = -1 \) and then we can use the Frobenius method to compute the limit of \( f_1 \) when \( r \) tends to \( -1 \) (see §4.8 in [3]). Taking into account the coefficients of Equation (12) and the aforementioned method, we deduce that

\[
f_1(s, r) = c_1 \log(1 + r)\phi_1(s, r) + c_2 \phi_2(s, r) + c_3(r + 1)\phi_3(s, r) + c_4(r + 1)^2\phi_4(s, r),
\]

where \( c_i \in \mathbb{R}, \ \phi_1(s, r) \neq 0 \) and \( \phi_i \) are analytic at the points \((s, -1), \ i \in \{1, 2, 3, 4\}. \) A direct computation using (8) and (10) proves that \( \lim_{r \to -1} \frac{\partial f_0}{\partial r} = +\infty \). Thus \( c_1 \neq 0 \) and \( \lim_{r \to -1} f_1(s, r) = -\infty. \)

**Step 3.** With regard to \( C_2 \), we shall check that \( \{0, t_1\} \subset (\tilde{t}_0, -\tilde{t}_0) \) is a critical point of \( C_2 \) in \( L_2 \), then \( \tilde{t}_0 > t_0. \)

Clearly, from the definition of \( f_2 \) one has \( f_2(0, t_1) < 0 \) for \( t_1 \in [-n\pi, 0[. \) Thus we obtain the first assertion in the present step.

In order to prove the second one we need an appropriate expression for \( \frac{\partial f_2}{\partial t_0}. \) Observe that

\[
\frac{\partial \Phi}{\partial t_0} = u^{\frac{2}{n}} \tau - \frac{2a}{\nu u^{\frac{2}{n} - 1}} \psi du + a \left( u^{\frac{2}{n}} \psi \right),
\]

where

\[
\tau = \frac{-i \left( -1 + e^{i\frac{\nu}{2}} u \right)}{4 \left( 1 + e^{i\frac{\nu}{2}} u \right)} du, \quad v, \quad \psi = \left( u + e^{-i\frac{\nu}{2}} \right) \left( u - e^{i\frac{\nu}{2}} \right), \quad a = \frac{1}{4 \left( \sin \left( \frac{\nu}{2} \right) - \sin \left( \frac{\nu}{2} \right) \right)}.
\]
Integrating by parts once again in the above equality and computing the real part, we get:

\[
\frac{\partial f_2}{\partial t_0}(t_0, t_1) = \int_{t_1}^{t_0} \left( \frac{\cos\left(\frac{t_0}{n}\right) \sin\left(\frac{t_1}{2}\right)}{nv(t, t_0, t_1)} + \frac{\sin\left(\frac{t_0}{n}\right) \tan\left(\frac{t_1 + t_0}{4}\right)}{4v(t, t_0, t_1)} \right) dt \\
- \frac{\sin\left(\frac{t_0}{2}\right)}{n \sin\left(\frac{t_0}{n}\right) - \sin\left(\frac{t_1}{2}\right)} f_1(t_0, t_1).
\]

In particular, if \( t_1 = -t_0 \) one has that the integral of the first summand vanishes, and so

\[
\frac{\partial f_2}{\partial t_0}(t_0, -t_0) = \int_{-t_0}^{t_0} \frac{\sin\left(\frac{t_0}{n}\right) \tan\left(\frac{t_1 + t_0}{4}\right)}{4v(t, t_0, t_1)} dt + \frac{1}{2n} f_1(t_0, -t_0).
\]

Taking into account that \( \tilde{t}_0 < n\pi \), it is not difficult to see that

\[
\int_{-t_0}^{t_0} \frac{\sin\left(\frac{t_0}{n}\right) \tan\left(\frac{t_1 + t_0}{4}\right)}{4v(t, t_0, t_1)} dt > 0,
\]

for \( 0 < t_0 \leq \tilde{t}_0 \), and so \( \frac{\partial f_2}{\partial t_0}(t_0, -t_0) \) is positive in the points where the function \( f_1 \) is nonnegative. This concludes the assertion.

By Steps 1, 2 and 3 we deduce that there are no points \((t_0, t_1)\) in \( T \) with \( f_1(t_0, t_1) \geq 0 \) and \( f_2(t_0, t_1) = 0 \) apart from the points \( \{ (t_0, -t_0) \mid 0 \leq t_0 \leq \tilde{t}_0 \} \).

**Corollary 7.** \( M \) is invariant under the antiholomorphic involution \( S_\sigma(u) = \bar{u} \) which corresponds to the reflection in the plane \( \{ x_1 = 0 \} \).

### 3. The existence results.

In the former section we have seen that if \( X: M \to \mathbb{R}^3 \) is a solution of Problem 1 satisfying \( X(M) \subset (S\setminus \mathcal{E}(\Gamma_{\theta d})) \cup \Gamma_{\theta d} \) then \( M \) is biholomorphic to \( N' = \mathcal{U}_0^{-\tilde{t}_0} \) and their Weierstrass data are given by

\[
g = iu^{\frac{2}{n}}, \quad \Phi_3 = \lambda \frac{du}{v}, \quad v = \sqrt{u^4 - 2u^2 \cos(t_0) + 1}, \quad n \in \left[ \frac{2}{3}, 1 \right].
\]

At this point, we observe that, up to an easy conformal transformation, the above Weierstrass data for \( n \in [1, 2] \) correspond to a López-Martín example, it is to say, solutions of Problem 1 verifying \( X(M) \subset \mathcal{E}(\Gamma_{\theta d}) \). For the existence of this examples we refer the reader to [10].

Moreover, if \( n = 2 \) then the surface is a Jenkins-Serrin graph. Extension by Schwarz reflection of these surfaces gives embedded doubly periodic examples with two orthogonal planes of symmetry between adjacent saddle towers. These examples were studied by H. Karcher in [7].

Therefore, to complete the existence part of the Main Theorem it suffices to prove that indeed for \( n \in \left[ \frac{2}{3}, 1 \right] \) the above conformal representation leads
to a solution of Problem 1 which lies in the \textit{exterior} of the convex hull of its boundary and this is the purpose of the present section.

In what follows we denote \( M = N' \) and \( \gamma^+_i = u^{-1}(s^+_i) \), \( \gamma^-_i = u^{-1}(s^-_i) \), \( i = 0, 1, 2 \), where \( s^+_i \) and \( s^-_i \) are those defined in Section 2 corresponding to \( t_1 = -t_0 \) for \( i = 0, 1, 2 \) (see Figure 5). Moreover, we label \( \gamma^+_i = \bigcup_{i=0}^{2} \gamma^+_i \) and \( \gamma^-_i = \bigcup_{i=0}^{2} \gamma^-_i \). We use the notation \( \ell^\pm_i \) introduced in Section 1 for the half-lines in the polygonal \( \Gamma_{\theta^d} \). Furthermore, recall that the set of Cartesian coordinates was introduced at the beginning of Section 2.

Now, we consider the curve \( \gamma^+_0 \) which consists of two copies, \( \delta^+_1 \) and \( \delta^+_2 \), of \( s^+_0 \). We can assume that \( \delta^+_1(t) \) and \( \delta^+_2(t) \) are the two lifts to \( M \) of the curve \( e^{it^2} \), \( t \in [t_0, \pi] \), in the \( u \)-plane, satisfying \( \delta^+_1(\pi) \in \gamma^+_1 \) and \( \delta^+_2(\pi) \in \gamma^+_2 \), respectively. Define \( \tilde{h} : [0, \pi] \to \mathbb{R} \) as \( \tilde{h}(t_0) = 2\text{Re} \left( \int_{\delta^+_2(t)} \right) \), where \( \tau = \frac{du}{v} \).

A direct computation gives \( \tilde{h}(t_0) = \frac{\sqrt{2}}{2} \int_{t_0}^{\pi} \frac{dt}{\sqrt{\cos(t_0) - \cos(t)}} > 0 \). As we are assuming that the immersion \( X : M \to \mathbb{R}^3 \) is normalized so that the distance between the planes \( \pi_1 \) and \( \pi_2 \) is 1, we have \( \lambda = \frac{1}{\tilde{h}(t_0)} \).

As usual, we define \( (\Phi_1, \Phi_2, \Phi_3) = \frac{1}{2} \left( -i(u^{-\frac{2}{n}} + u^2), u^{-\frac{2}{n}} - u^2, 1 \right) \Phi_3 \).

Since \( M \) is homeomorphic to a closed disk minus two boundary points, we have that \( X : M \to \mathbb{R}^3 \) given by \( X(p) = \int_{p_0}^{p} (\Phi_1, \Phi_2, \Phi_3) \) is a well-defined conformal minimal immersion verifying Condition (1) in the statement of Problem 1. Let us see that \( (M, g, \Phi_3) \) fulfill also the other conditions.

Denote \( S_h, S_v \) the antiholomorphic transformations on \( M \) given by \( S_h((u, v)) = (1/\overline{v}, \overline{v}/\overline{u^2}) \), \( S_v((u, v)) = (\overline{v}, \overline{u}) \). Observe that the point \( P_0 = (1, 2\sin(t_0/2)) \) is invariant under \( S_h \) and \( S_v \). Moreover, we have

\begin{equation}
(13) \quad g \circ S_h = 1/\overline{g} \quad g \circ S_v = -\overline{g} \quad S^*_h(\phi_3) = -\overline{\phi_3} \quad S^*_v(\phi_3) = \overline{\phi_3} .
\end{equation}

Hence elementary arguments imply that \( S_h \) (resp. \( S_v \)) induces on \( X(M) \) a symmetry with respect to the plane \( \{x_3 = 0\} \) (resp. \( \{x_1 = 0\} \)).
Moreover, it is clear that $0 \leq \gamma_i^{-}$. Lemma 7. The function we shall see: \[ d \]

A thoughtful study of the function \[ d \] will be very useful in order to prove the rest of the conditions on the immersion $X : M \rightarrow \mathbb{R}^3$. In this context we shall see:

Lemma 6. The maps $X|_{\gamma^+}$, $X|_{\gamma^-}$ are injective, $X(\gamma_i^+) = \ell_i^+$ and $X(\gamma_i^-) = \ell_i^-$, for $i = 0, 1, 2$, it is to say, $X(\partial(M)) = \Gamma_{\theta d}$ for $\theta = (3n - 2)\pi/n$.

Taking into account Lemma 6, the expression of the oriented distance $d : ]0, \pi[ \rightarrow \mathbb{R}$ between $\ell_0^+$ and $\ell_0^-$ is $d(t_0) = \text{Re} \left( \int_{\delta} \Phi \right)$, where now $\delta$ is the lift of the oriented curve $e^{it}, t \in [-t_0, t_0]$, in the $u$-plane. Since $(S_h)_*(\delta) = \hat{\delta}$, $(S_v)_*(\hat{\delta}) = -\hat{\delta}$ and taking into account (13) we deduce that $\int_{\delta} \Phi = \int_{\delta} \hat{\Phi}_1 = \frac{\lambda}{4} \bar{f}_1(t_0)$, where, as in Section 2, $\Phi = \frac{v^2}{v} du$ and $\bar{f}_1(t_0) = f_1(t_0, -t_0)$. Hence we have

\[ d(t_0) = \frac{1}{4} \frac{\bar{f}_1(t_0)}{h(t_0)}. \]

A thoughtful study of the function $d$ will be very useful in order to prove the rest of the conditions on the immersion $X : M \rightarrow \mathbb{R}^3$. In this context we shall see:

Lemma 7. The function $d : ]0, \pi[ \rightarrow \mathbb{R}$ satisfies:

1. It vanishes at only one point $\tilde{t}_0 \in ]\frac{n\pi}{2}, n\pi[$. Furthermore, $d$ is positive in $]0, \tilde{t}_0[$ and negative in $]\tilde{t}_0, \pi[$. 
2. $\lim_{t_0 \to 0} d(t_0) = 0$. In particular, $d$ is bounded in $]0, \tilde{t}_0[$.
3. It has only a critical point $t'_0 \in ]0, \tilde{t}_0[$ which is a maximum. In particular, $\# \left[ d^{-1}(\left\{ x \right\}) \right] = 2$, $\forall x \in ]0, d(\tilde{t}_0)[$.

Proof. We had seen (see Step 2 in Proposition 6) that there exists a unique $\tilde{t}_0 \in ]\frac{n\pi}{2}, n\pi[ \] such that $\bar{f}_1(t_0) = 0$, $\bar{f}_1(t_0) > 0$ in $]0, \tilde{t}_0[$ and $\bar{f}_1(t_0) < 0$ in $]\tilde{t}_0, \pi[$. Note that this proves the first assertion.

In order to prove the second statement, observe that

\[ \lim_{t_0 \to 0} \bar{f}_1(t_0) = \lim_{t_0 \to 0} \sqrt{2} \int_{-1}^{1} \frac{t_0 \cos(t_0 s)}{\sqrt{\cos(t_0 s) - \cos(t_0)}} ds = 2 \int_{-1}^{1} \frac{ds}{\sqrt{1 - s^2}} = 2\pi. \]

Moreover, it is clear that $0 \leq \sqrt{\cos(t_0) - \cos(t)} \leq \sqrt{1 - \cos(t)}$, $t \in [t_0, \pi]$, then $h(t_0) \geq \frac{\sqrt{2}}{2} \int_{t_0}^{\pi} \frac{dt}{\sqrt{1 - \cos(t)}} = - \log \left[ \tan \left( \frac{t_0}{4} \right) \right]$ , and so

\[ \lim_{t_0 \to 0} h(t_0) \geq \lim_{t_0 \to 0} \left( - \log \left[ \tan \left( \frac{t_0}{4} \right) \right] \right) = +\infty. \]
Both (17) and (18) give Assertion 2. Concerning Assertion 3, we shall prove that the functions $\tilde{f}_1, \tilde{h}$ satisfy the following differential equations:

$$\begin{align*}
\text{(19)} & \quad \tilde{f}_1''(t_0) + \cot(t_0)\tilde{f}_1'(t_0) + \frac{4-n^2}{4n^2} \tilde{f}_1(t_0) = 0, \\
\text{(20)} & \quad \tilde{h}''(t_0) + \cot(t_0)\tilde{h}'(t_0) - \frac{1}{4}\tilde{h}(t_0) = 0.
\end{align*}$$

The above ordinary differential equations can be obtained from the following equalities integrating by parts:

$$\begin{align*}
\frac{\partial^2 \Phi}{\partial t_0^2} + \cot(t_0) \frac{\partial \Phi}{\partial t_0} + \frac{4-n^2}{4n^2} \Phi &= d(\tilde{G}), \\
\frac{\partial^2 \tau}{\partial t_0^2} + \cot(t_0) \frac{\partial \tau}{\partial t_0} - \frac{1}{4} \tau &= d(\tilde{H}),
\end{align*}$$

where

$$\tilde{G}(u) = \frac{u^{2+n}}{n} \left( n \left( -1 + u^4 \right) + 2 \left( 1 + u^4 \right) - 4 u^2 \cos(t_0) \right)$$

and

$$\tilde{H}(u) = \frac{u (-1 + u^4)}{4 (1 + u^4 - 2 u^2 \cos(t_0))^{\frac{3}{2}}}.$$

Let $t'_0$ a critical point of $d$ in $]0, \tilde{t}_0[$. This implies that $d'(t'_0) = 0$ and so $(\tilde{f}_1 \tilde{h} - \tilde{f}_1 \tilde{h}')(t'_0) = 0$. Therefore we have the following expression for the second derivative of $d$ at the point $t'_0$:

$$d''(t'_0) = \frac{1}{4} \frac{\tilde{h}'' - \tilde{f}_1 \tilde{h}''(t'_0)}{\tilde{f}_1^2}.$$

Hence, using (19) and (20), we obtain that $d''(t'_0) = \frac{4-n^2}{4n^2} d(t'_0) < 0$. Consequently, there exists only one critical point of $d$, $t'_0$, in $]0, \tilde{t}_0[$ and it is a maximum. Obviously, $d(t'_0) = \text{Maximum}\{d(t_0) \mid t_0 \in ]0, \tilde{t}_0[\}$. Hence, it is clear that $\mathbb{I} \left[ d^{-1}(\{x\}) \right] \geq 2$, $\forall x \in ]0, d(t'_0)[$. If $\mathbb{I} \left[ d^{-1}(\{x\}) \right] > 2$, for some $x \in ]0, d(t'_0)[$, then it implies the existence of a local minimum of $d$ in $]0, \tilde{t}_0[$, which is absurd. This concludes the proof.

**Remark 2.** For each $n \in \left[ \frac{2}{3}, 1 \right]$, we denote either by $d_n$ or by $d'_n$ the maximum of the distance function $d(t_0)$, $t_0 \in ]0, \tilde{t}_0[$. Observe that the function $d(n, t_0)$ is a differentiable function on $\left[ \frac{2}{3}, 1 \right] \times ]0, \pi[$. Let us check that $\frac{\partial d}{\partial n} > 0$.

Taking into account that $\tilde{t}_0 < n\pi$ and the definition of the function $\tilde{f}_1$, we obtain

$$\frac{\partial d}{\partial n} = \frac{1}{4h} \frac{\partial \tilde{f}_1}{\partial n} = \frac{\sqrt{2}}{4n^2 h} \int_{-\tilde{t}_0}^{\tilde{t}_0} t \sin\left( \frac{t}{n} \right) > 0.$$

Then $n \to d_n$ is a continuous increasing function in $\left[ \frac{2}{3}, 1 \right]$, equivalently the function $\theta \to d'_n$ is increasing in $]0, \pi[$. Therefore, $d'_{\theta_0} < d'_{\theta_1}$ for $0 \leq \theta_0 < \theta_1 \leq \pi$. 
At this point, we recall that the distance function for the examples which lie in the convex hull of their boundary coincides with our function $d$ for $n \in [1, 2]$ (see [10]). As $\frac{d \rho}{d \theta} > 0$ we have that $d_{n_0} < d_{n_1}$ for $\frac{2}{3} \leq n_0 < n_1 \leq 2$. Hence, we infer that $d''_\theta < d_\theta$, for $\theta \in [0, \pi]$, where $d_\theta$ is the maximum of the distance function for the López-Martín examples (see Remark 4 in [10]).

From the asymptotic expansion of the Weierstrass data we have that $X$ can be expressed locally around $E_1$ as

$$X(u) = (X_1(u), X_2(u), X_3(u))$$

$$= \left(-i\frac{n\lambda}{2(2-n)}\pi^{1-\frac{2}{n}}(1 + O_1(u) + iO_2(u)), O_3(u)\right),$$

where $O_i(u)/|u|$ is a bounded function in a neighbourhood of $E_1$, $i = 1, 2, 3$. Using this fact and Lemma 2.1 in [12] it is not difficult to prove the following lemma:

**Lemma 8.** The minimal immersion $X : M \to \mathbb{R}^3$ is proper and $X(M)$ is contained in the slab $S$.

Let us consider $M_1 = \{(u, v) \in M \mid |u| \leq 1\}$ and $M_2 = \{(u, v) \in M \mid |u| \geq 1\}$. We recall that we had denoted by $\hat{\delta}$ and $\delta_i^+$ the lifts to $M$ of the curves of $u(M)$ given by $e^{i\frac{t}{2}}$, for $t \in [-t_0, t_0]$ and $t \in [t_0, \pi]$, respectively satisfying $\delta_i^+(\pi) \in \gamma_i^+$, $i = 1, 2$. Clearly, the surfaces $M_1$ and $M_2$ are topologically a closed disk minus one boundary point. Moreover, $M = M_1 \cup M_2$, $M_1 \cap M_2 = \hat{\delta}$ and $\partial(M_1) = \hat{\delta} \cup \gamma_1^+ \cup \gamma_1^- \cup \delta_1^+ \cup \delta_1^-$, $\partial(M_2) = \hat{\delta} \cup \gamma_2^+ \cup \gamma_2^- \cup \delta_2^+ \cup \delta_2^-$, where $\delta_i^- = S_i(\delta_i^+)$ for $i = 1, 2$.

Our next objective is to prove the following assertion:

**Claim 4.** $X|_{\hat{\delta}}$ is injective and $X(\hat{\delta}) \subset \{x_2 \leq 0, x_3 = 0\}$.

**Proof.** To see this we observe that $X_2(\hat{\delta}(t)) = X_2(\hat{\delta}(t)) - X_2(\hat{\delta}(-t_0)) = -\sqrt{2} \int_{-t_0}^{t} \frac{\sin(s/u)}{\sqrt{\cos(s) - \cos(t_0)}} ds$. Since $0 < t_0 \leq n\pi$ we have that $X_2 \circ \hat{\delta}$ is a nonpositive decreasing function for $t \in [-t_0, 0]$. This fact and the vertical symmetry $S_\nu$ prove the assertion.

Let us denote $M^+ = \{(u, v) \in M \mid \text{Im}(u) \geq 0\}$ and $M^- = \{(u, v) \in M \mid \text{Im}(u) \leq 0\}$, and define $\rho$ as the lift to $M$ of the divergent curve $[0, +\infty]$ in the $u$-plane. We parametrize $\rho$ as follows: $\rho(t) = u^{-1}(t), \quad t \in [0, +\infty]$.

Obviously, the surfaces $M^+$ and $M^-$ are topologically a closed disk minus two boundary points. Furthermore, $M = M^+ \cup M^-, \quad M^+ \cap M^- = \rho$ and $\partial(M^+) = \rho \cup \gamma_1^+ \cup \gamma_2^+ \cup \gamma_3^+$, $\partial(M^-) = \rho \cup \gamma_1^- \cup \gamma_2^- \cup \gamma_3^-$. Next we prove:

**Claim 5.** $X(\rho(t)) \subset \{(x_1, x_2, x_3) \mid x_1 = 0, x_2 \leq -\varepsilon, -\frac{1}{2} < x_3 < \frac{1}{2}\}$, where $\varepsilon = -X_2(P_0) > 0$. 
Proof. Taking the symmetry $S_h$ into account, it suffices to prove the assertion for $X(\rho([0,1]))$. From the Weierstrass data we deduce that

$$X_2(\rho(t)) = -\varepsilon - \int_t^1 \frac{s^{-\frac{2}{n}} - s_{\rho}^{-\frac{2}{n}}}{\sqrt{s^4 - 2 \cos(t_0)s^2 + 1}} \, ds.$$ 

Using Claim 4 we have $\varepsilon = -X_2(P_0) > 0$. So $X_2 \circ \rho$ is decreasing and nonpositive in $[0,1]$. □

Moreover, taking (21) and the symmetry $S_h$ into account, we obtain that

$$\lim_{t \to 0} X_2(\rho(t)) = \lim_{t \to +\infty} X_2(\rho(t)) = -\infty.$$ 

At this point we can prove the following lemma:

**Lemma 9.** The minimal immersion $X : M \to \mathbb{R}^3$ verifies:

1. $X(M) \subset ((S \setminus \mathcal{E}(\Gamma_{\theta d})) \cup \Gamma_{\theta d})$.
2. The surfaces $X(M_1 \setminus (\delta_1^+ \cup \delta_1^-))$ and $X(M_2 \setminus (\delta_2^+ \cup \delta_2^-))$ are graphs on the plane $\{x_3 = 0\}$.
3. $d > 0$ implies that $X$ is an embedding. If $d = 0$, then $X|_{M \setminus \gamma^+}$ and $X|_{M \setminus \gamma^-}$ are injective.

**Proof.** Denote by $p_3$ the orthogonal projection on the plane $\{x_3 = 0\}$. Using (21) once again and the symmetries, it is not hard to see that $X(M^+)$ and $X(M^-)$ are contained in a wedge of the slab $S$. Then we can apply Statement (ii) in Theorem 1 to conclude that $X(M^+)$ and $X(M^-)$ lies in the convex hull of their boundary.

In case $d = 0$, taking into account Lemma 6, Claim 5 and the interior maximum principle we have the proof of Assertion 1. Moreover, in this case the interior maximum principle also gives us that $X(M^+) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0, -\varepsilon < x_2 < 0\} = \emptyset$, and so, taking into account the symmetry $S_v$, we deduce that

$$p_3(X(M)) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0, -\varepsilon < x_2 < 0\} = \emptyset. \tag{22}$$

In case $d > 0$ the above reasoning implies

$$X(M) \subset S \setminus \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0, x_2 > 0\}. \tag{23}$$

Next, we prove Assertion 1 for $d > 0$ and Assertion 2. In what follows we denote $\alpha = p_3(\partial(X(M_1)))$. We also introduce the following notation:

(A) If $d = 0$, $W_0$ will denote the bounded connected component of $\mathbb{R}^2 \setminus \alpha$, whereas $W_1$ will denote the unbounded connected component which is disjoint from $p_3(\mathcal{E}(\Gamma_{\theta d}))$.

(B) If $d > 0$, observe that $\alpha$ is a connected simple curve. In this case we denote $W_0$ as the connected component of $\mathbb{R}^2 \setminus \alpha$ containing the point $(0,0)$ and $W_1$ the other one.
In both cases $p_3(M_1) \subset W_0 \cup W_1 \cup \alpha$. Since $p_3(X(\partial(M_1))) = \alpha$ and $p_3 \circ X$ is proper on $M_1$ (see Lemma 8), then $p_3 \circ X(M_1 \setminus \partial(M_1)) \cap W_0$ is closed in $W_0$. Furthermore, $g(M_1 \setminus \partial(M_1)) \cap S^1 = \emptyset$ and so $(p_3 \circ X)|_{M_1 \setminus \partial(M_1)}$ is a local diffeomorphism. In particular, $p_3 \circ X(M_1 \setminus \partial(M_1)) \cap W_0$ is an open set of $W_0$. Hence we deduce that either $p_3 \circ X(M_1 \setminus \partial(M_1)) \cap W_0 = \emptyset$ or $p_3 \circ X(M_1 \setminus \partial(M_1)) \cap W_0 = W_0$. According to (22) and (23) we have that $p_3 \circ X(M_1 \setminus \partial(M_1)) \cap W_0 = \emptyset$. A similar argument yields $p_3(X(M_1)) \cap W_1 = W_1$, i.e., $p_3(X(M_1)) = W_1 \cup \alpha$. Hence, using the symmetry $S_h$ and the interior maximum principle we conclude the proof of Assertion 1.

From the above reasoning we have that $p_3 \circ X : M_1 \setminus (\delta^+ \cup \delta^-) \longrightarrow W_1 \cup \alpha$ is a proper local diffeomorphism and so $p_3 \circ X$ is a covering map. Since $(p_3 \circ X)|_{\gamma^+_1}$ is one-to-one we obtain that $X(M_1 \setminus (\delta^+ \cup \delta^-))$ is a graph on the plane $\{x_3 = 0\}$.

Using that $\partial(X(M_1)) \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leq x_3 \leq \frac{1}{2}\}$ and Lemma 2.1 in [12] we infer that

$$ (24) \quad X(M_1) \subset \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leq x_3 \leq \frac{1}{2} \right\}. $$

Then, taking into account the symmetry $S_h$, we obtain Assertion 2. Finally, Assertion 2 and (24) give us Assertion 3.

Acknowledgements. We would like to thank Professor F.J. López for helpful conversations.

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Received November 6, 2002. This research was partially supported by MCYT-FEDER Grant number BFM2001-3489.

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PARTIAL REGULARITY FOR WEAK SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH SUPERCritical EXPONENTS

ZONGMING GUO AND JIAYU LI

Let $\Omega$ be an open subset in $\mathbb{R}^n$ ($n \geq 3$). In this paper, we study the partial regularity for stationary positive weak solutions of the equation

\begin{equation}
\Delta u + h_1(x)u + h_2(x)u^\alpha = 0 \quad \text{in } \Omega.
\end{equation}

We prove that if $\alpha > \frac{n+2}{n-2}$, and $u \in H^1(\Omega) \cap L^{\alpha+1}(\Omega)$ is a stationary positive weak solution of (1.1), then the Hausdorff dimension of the singular set of $u$ is less than $n - \frac{2\alpha+1}{\alpha-1}$, which generalizes the main results in Pacard 1993 and Pacard 1994.

1. Introduction.

Let $\Omega$ be an open subset in $\mathbb{R}^n$ ($n \geq 3$). In this paper, we prove a partial regularity result for positive weak solutions of the equation

\begin{equation}
\Delta u + h_1(x)u + h_2(x)u^\alpha = 0 \quad \text{in } \Omega,
\end{equation}

where $\alpha > \frac{n+2}{n-2}$, $h_i \in C^1(\Omega)$, $a_i \leq h_i(x) \leq b_i$, $0 < a_i < b_i$ and $|\log h_i(x)| \leq \beta$ ($i = 1, 2$) for $x \in \Omega$. As we know, there is not much known about the properties of the weak solutions of (1.1).

We say that $u$ is a positive weak solution of (1.1) in $\Omega$ if $u(x) \geq 0$ for a.e. $x \in \Omega$ and for all $\phi \in C^\infty(\Omega)$ with compact support in $\Omega$,

\begin{equation}
-\int_\Omega u\Delta \phi \, dx = \int_\Omega [h_1(x)u + h_2(x)u^\alpha] \phi(x) \, dx.
\end{equation}

We say that a weak solution $u$ is stationary, if it satisfies

\begin{equation}
\int_\Omega \left[ \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial \phi^i}{\partial x_i} - \frac{1}{2} |\nabla u|^2 \frac{\partial \phi^i}{\partial x_i} + \frac{1}{2} u^2 \frac{\partial h_1}{\partial x_i} \phi^i + \frac{1}{2} h_1u^2 \frac{\partial \phi^i}{\partial x_i} 
+ \frac{1}{\alpha+1} u^{\alpha+1} \frac{\partial h_2}{\partial x_i} \phi^i + \frac{1}{\alpha+1} h_2 u^{\alpha+1} \frac{\partial \phi^i}{\partial x_i} \right] \, dx = 0
\end{equation}

for all regular vector field $\phi$ with compact support in $\Omega$ (summation over $i$ and $j$ is understood).
For weak solutions in $H^1(\Omega) \cap L^{\alpha+1}(\Omega)$ this identity is obtained by assuming that the functional $E(u)$ is stationary with respect to domain variations, that is,

$$\frac{d}{dt} E(u_t)|_{t=0} = 0$$

where

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{1}{2} \int_\Omega h_1 u^2 - \frac{1}{\alpha + 1} \int_\Omega h_2 u^{\alpha+1} dx$$

and $u_t(x) = u(x + t\phi(x))$.

Let $u \in H^1(\Omega) \cap L^{\alpha+1}(\Omega)$ be a positive weak solution of (1.1). We denote by $\Sigma$ the set of points $x \in \Omega$ such that $u$ is not bounded in any neighborhood $W$ of $x$ in $\Omega$. If $u$ is bounded in a neighborhood of $x$ then the classical regularity theory ensures that $u$ is regular in the neighborhood of $x$. Therefore $\Sigma$ is the singular set of $u$. Moreover, $\Sigma$ is a closed subset of $\Omega$.

If $\alpha < \frac{n}{n-2}$, a simple bootstrap argument shows that all positive weak solutions of (1.1) are regular. It is well-known that the singular set may not be empty if $\alpha \geq \frac{n}{n-2}$. Pacard [Pa2] constructed solutions with singular sets of Hausdorff dimension $d < n - \frac{2\alpha}{\alpha-1}$. Schoen and Yau proved in [SY] that the singular set of a positive weak solution of (1.1) is not always as simple as in the examples given in [Pa2].

In [Pa1] and [Pa3], Pacard showed that the Hausdorff dimension of the singular set of a stationary positive weak solution $u$ of the equation $-\Delta u = u^\alpha$ in $\Omega$ is less than $n - \frac{2\alpha+1}{n-1}$.

In a recent paper [GL], we considered the compactness for positive solutions of Equation (1.1). Using the ideas in [LT1] and [LT2], we obtained the measure estimate of the blow up set of a sequence of positive smooth solutions $\{u_i\}$ of (1.1) with $\{\|u_i\|_{H^1(\Omega)} + \|u_i\|_{L^{\alpha+1}(\Omega)}\}$ bounded. We applied such result to a semilinear eigenvalue problem

$$-\Delta u = \lambda(u + u^\alpha) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

when $\Omega$ is a smooth star-shaped domain and obtained that any branch of positive solutions $(\lambda(s), u(s))$ of (1.4) must converge to a (singular) positive solution $u_0$ of the equation

$$-\Delta u = \lambda_0(u + u^\alpha) \text{ in } \Omega$$

as $\lambda(s) + \|u(s)\|_{L^\infty(\Omega)} \to \infty$, $s \to \infty$, where $\lambda_0 = \lim_{s \to \infty} \lambda(s)$ and $0 < \lambda_0 < \infty$. The existence of such branches of positive solutions is obtained by Rabinowitz. It was proved in [BDT] and [Da] that some branches are simple curves.

In this paper, we shall prove a partial regularity theorem for a stationary positive weak solution of (1.1) with $\alpha > \frac{n+2}{n-2}$. 

Theorem A. Let $\alpha > \frac{n+2}{n-2}$. If $u \in H^1(\Omega) \cap L^{\alpha+1}(\Omega)$ is a stationary positive weak solution of (1.1), then the Hausdorff dimension of the singular set of $u$ is less than $n - 2\frac{\alpha+1}{\alpha-1}$.

Our result covers the main results in [Pa1] and [Pa3]. The proof is quite different, we used the duality of a weighted Hardy space and a weighted BMO, which was used in [CLL] to get a partial regularity result for a weak heat flow.

When $h_1 = 0$ and $h_2$ is a constant, it is not hard to construct solutions of (1.1) which are singular (see [Lin] and [Re]). However, when $h_2$ is not a constant, the problem is much harder. A singular solution was given in this case by Johnson-Pan-Yi [JPY]. Let $\Omega = B_R$, here $B_R \subset \mathbb{R}^n$ ($n \geq 3$) is a ball with center at 0 and radius of $R > 0$. Consider the equation

\begin{equation}
\Delta u + K(|x|)u^\alpha = 0 \text{ in } B_R
\end{equation}

with $K(|x|)$ satisfying the following conditions in [JPY]:

(K1) $K \in C^1[0,\infty)$, $K'(0) = 0$, $K(r) > 0$ for $r \geq 0$, and $\lim_{r \to \infty} K(r) = K(\infty) > 0$;

(K2) There is a $\delta > 0$ such that $\lim_{r \to \infty} r^\delta (K(r) - K(\infty)) = 0$, $\lim_{r \to \infty} r^{1+\delta} K'(r) = 0$;

(K3) $K'(r) \leq 0$ for $r > 0$.

It is proved in [JPY] (Theorem 1) that the equation

\begin{equation}
\Delta u + K(|x|)u^\alpha = 0 \text{ in } \mathbb{R}^n
\end{equation}

has a singular solution $U_0(r)$ with $r = |x|$, which satisfies

$$
\lim_{r \to 0} r^{\frac{2}{\alpha-1}} U_0(r) = \left[ \frac{1}{K(0)} \cdot \frac{2}{\alpha-1} \left( n - 2 - \frac{2}{\alpha-2} \right) \right]^{\frac{1}{\alpha-1}},
$$

$$
\lim_{r \to 0} r^{\frac{2}{\alpha-1}+1} U'_0(r) = \frac{2}{\alpha-1} \left[ \frac{1}{K(0)} \cdot \frac{2}{\alpha-1} \left( n - 2 - \frac{2}{\alpha-2} \right) \right]^{\frac{1}{\alpha-1}}.
$$

It is clear that $U_0(|x|)$ for $x \in B_R$ is a singular solution of Equation (1.6).

Throughout this paper, $C$ will denote a universal constant depending only on $\alpha$, $\beta$, $n$ and $a_i, b_i$ ($i = 1, 2$), unless it is explicitly stated.

2. $H^1_w(\mathbb{R}^n)$ and $M^2_{1, \nu}g(x)$.

In this section we review definitions and properties of the space $H^1_w(\mathbb{R}^n)$ and the function $M^2_{1, \nu}g(x)$. See Strömberg & Torchinsky [ST] for more details.

Let $\mu$ be the Lebesgue measure in $\mathbb{R}^n$ and $d\mu(x) = dx$. Let $\nu$ be a weighted measure with respect to the Lebesgue measure in $\mathbb{R}^n$ with weight $w(x)$. Then

$$
H^1_w(\mathbb{R}^n) = \{ f \in S'(\mathbb{R}^n) : M_1(F_\phi) \in L^1_w(\mathbb{R}^n) \},
$$

where $\phi \in C^\infty(\mathbb{R}^n)$, $F_\phi \in L^1(\mathbb{R}^n)$, and $M_1(F_\phi)$ is the Hardy-Littlewood maximal function of $F_\phi$. The norm is defined by

$$
\| f \|_{H^1_w} = \| M_1(F_\phi) \|_{L^1_w}.
$$
where

\[ F_\phi(x) = \frac{1}{t^n} \int_{\mathbb{R}^n} f(y) \phi \left( \frac{y - x}{t} \right) \, dy, \]

\( \phi \) is any smooth function with support in the unit ball and \( M_1(F_\phi(x)) = \sup_{t>0} F_\phi(x) \).

For \( g \in L^1_{\text{loc}}(\mathbb{R}^n) \), define

\[ M^\sharp_{1,\nu} g(x) = \sup_{t>0} \frac{1}{\nu(B(x,t))} \int_{B(x,t)} |g(y) - (g)_{x,t}| \, dy, \]

where

\[ (g)_{x,t} \equiv \frac{1}{B(x,t)} \int_{B(x,t)} g \, dy, \]

and \( B(x,t) \subset \mathbb{R}^n \) is the ball centered at \( x \) with radius \( t \). It follows from Theorem 2 in Chapter IX in [ST] that for \( f \in \hat{D}_0 \), \( g \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( \nu \in D_d \) for some \( d > 0 \) (see Doubling \( D_d \) condition in Chapter I in [ST]), there exists \( C > 0 \) independent of \( f \) and \( g \) such that

\[ \int_{\mathbb{R}^n} f(x)g(x) \, dx \leq C \left( \int_{\mathbb{R}^n} M_1(F_\phi(x))M^\sharp_{1,\nu} g(x)w(x) \, dx \right). \] (2.1)

Since \( \hat{D}_0 \) is dense in \( H^1_w(\mathbb{R}^n) \) (see Theorem 1 of Chapter VII in [ST]), we conclude that (2.1) holds for \( f \in H^1_w(\mathbb{R}^n) \) and \( g \in L^1_{\text{loc}}(\mathbb{R}^n) \).

In this paper, we define \( w(x) = |x|^{-2/(\alpha - 1)} \) and \( d\nu(x) = |x|^{-2/(\alpha - 1)} \, dx \). Then \( \nu \) is a doubling weighted measure with respect to the Lebesgue measure of \( \mathbb{R}^n \) with weight \( |x|^{-2/(\alpha - 1)} \) and \( \nu \in D_{n,\frac{2}{(\alpha - 1)}} \). Moreover,

\[ \nu(B(x,t)) = \frac{(\alpha - 1)\omega_n}{n(\alpha - 1) - 2} t^{n-\frac{2}{\alpha - 1}}, \]

where \( \omega_n \) is the area of the \((n-1)\)-dimensional unit sphere in \( \mathbb{R}^n \).

### 3. A monotonicity inequality and blow up.

In this section, we first recall a monotonicity inequality for stationary positive weak solutions of (1.1) established in [GL], using this monotonicity inequality and a blow up argument, we then obtain a decay property of the scaled energy. Assume henceforth that \( u \in H^1(\Omega) \cap L^{\alpha+1}(\Omega) \) is a stationary positive solution of (1.1).
For any \( x_0 \in \Omega \) and \( r > 0 \), define
\[
E_u(x_0, r) \equiv \frac{(\alpha - 1)}{2(\alpha + 1)} e^{Cr r^{-\mu}} \int_{B(x_0, r)} h_2 u^{\alpha + 1} \, dx \\
+ \frac{1}{4} \left( e^{Cr} \frac{d}{dr} \left( r^{-\mu} \int_{\partial B(x_0, r)} u^2 \, ds \right) \right) \\
+ \frac{1}{4} \left( e^{Cr r^{-\mu - 1}} (-1 + Cr) \int_{\partial B(x_0, r)} u^2 \, ds \right) \\
+ C \int_0^r e^{C\xi \xi^{(\alpha +1)/\alpha - 1}} \, d\xi,
\]
where \( \mu = n - \frac{2a + 1}{\alpha - 1} \) and \( C \) depends only upon \( \alpha, \beta, n \) and \( a_i, b_i \) \( (i = 1, 2) \).

It is proved in [GL] that \( E_u(x_0, r) \) can be written to the equivalent forms:
\[
E_u(x_0, r) \equiv \frac{1}{2} e^{Cr r^{-\mu}} \int_{B(x_0, r)} |\nabla u|^2 \, dx - \frac{1}{2} e^{Cr r^{-\mu}} \int_{B(x_0, r)} h_1 u^2 \, dx \\
- \frac{1}{2} e^{Cr r^{-\mu}} \int_{B(x_0, r)} h_2 u^{\alpha + 1} \, dx \\
+ \frac{1}{(\alpha - 1)} e^{Cr r^{-\mu - 1}} \int_{\partial B(x_0, r)} u^2 \, ds \\
+ \frac{C}{4} e^{Cr r^{-\mu}} \int_{\partial B(x_0, r)} u^2 \, ds + C \int_0^r e^{C\xi \xi^{(\alpha +1)/\alpha - 1}} \, d\xi
\]
and
\[
E_u(x_0, r) \equiv \left( \frac{\alpha - 1}{\alpha + 3} \right) e^{Cr r^{-\mu}} \left[ \int_{B(x_0, r)} \frac{1}{(\alpha + 1)} h_2 u^{\alpha + 1} \, dx \right] \\
+ \frac{1}{2} \int_{B(x_0, r)} |\nabla u|^2 \, dx - \frac{1}{2} \int_{B(x_0, r)} h_1 u^2 \, dx \\
+ \frac{1}{(\alpha + 3)} \frac{d}{dr} \left( e^{Cr r^{-\mu}} \int_{\partial B(x_0, r)} u^2 \, ds \right) \\
+ \left( \frac{C}{4} - \frac{C}{(\alpha + 3)} \right) e^{Cr r^{-\mu}} \int_{\partial B(x_0, r)} u^2 \, ds \\
+ C \int_0^r e^{C\xi \xi^{(\alpha +1)/\alpha - 1}} \, d\xi.
\]

All the derivatives in the above expressions are to be understood in the sense of distributions. Lemma 3.1 and Lemma 3.2 below are proved in [GL].

**Lemma 3.1.** If \( u \in H^1(\Omega) \cap L^{\alpha + 1}(\Omega) \) is a stationary positive weak solution of (1.1), then \( E_u(x_0, r) \), defined above, is an increasing function of \( r \).

**Lemma 3.2.** \( E_u(x_0, r) \) is a continuous function of \( x_0 \in \Omega \) and \( r > 0 \).
Now we show the following lemma:

**Lemma 3.3.** There exist $0 < r_0 < 1$ independent of $x_0 \in \Omega$ and some constant $C > 0$ depending only upon $\alpha$, $n$, such that the following inequality holds:

\[
(3.1) \quad r^{-\mu} \int_{B(x_0,r)} |\nabla u|^2 dx \leq CE_u(x_0,2r) \leq CE_u(x_0,r_0) \text{ for } r < r_0/2.
\]

**Proof.** We consider the last one of the three equivalent formulations of $E_u(x_0,r)$ given above. By Lemma 2.3 in [GL] we know that there exists $0 < r_0 < 1$ such that

\[
(3.2) \quad E_u(x_0,r) \geq 0 \text{ for all } x_0 \in \Omega, \ 0 < r < r_0,
\]

and for $r < r_0$,

\[
(3.3) \quad \frac{1}{2} \left( \frac{\alpha - 1}{\alpha + 3} \right) e^{Cr_{r^{-\mu}}} \int_{B(x_0,r)} h_1 u^2 dx \leq \frac{1}{2(\alpha + 1)} \left( \frac{\alpha - 1}{\alpha + 3} \right) e^{Cr_{r^{-\mu}}} \int_{B(x_0,r)} h_2 u^{\alpha+1} dx.
\]

We denote by $\phi(r) = r^{-\mu} \int_{B(x_0,r)} |\nabla u|^2 dx$. Since $E_u(x_0,r)$ is an increasing function of $r$, we integrate it from $0$ to $r < r_0$ and obtain that for almost every $x_0 \in \Omega$, (note that $e^{Cr} > 1$)

\[
\frac{\alpha - 1}{2} \int_0^r \phi(\rho) d\rho + e^{Cr_{r^{-\mu}}} \int_{\partial B(x_0,r)} u^2 ds \leq (\alpha + 3)E_u(x_0,r) r \text{ for } r < r_0.
\]

(Here we have used $\lim_{r \to 0} r^{-\mu} \int_{\partial B(x_0,r)} u^2 ds = 0$ a.e. $x_0 \in \Omega$. This fact is proved in [GL]..) Now we use Remark 2 in [Pa1] and we see that there exists some $\sigma \in [r/2,r]$ such that

\[
\phi(\sigma) \leq \frac{8}{r} \int_0^r \phi(\rho) d\rho \leq CE_u(x_0,r),
\]

for some constant $C > 0$ depending only upon $\alpha$, $\beta$ and $n$. In addition we have $\phi(r/2) \leq 2^\mu \phi(\sigma)$, if $\sigma \in [r/2,r]$. This gives us the desired result for almost every $x_0$ and, by continuity, for every $x_0$.

**Proposition 3.4.** Assume that there exist $x_0 \in \Omega$ and $0 < r_1 < r_0$ such that $E_u(x_0,r_1) \leq \delta$. Then

\[
(3.4) \quad r^{-\mu} \int_{B(y,r)} |\nabla u|^2 dx \leq C\delta,
\]

for all $y \in B(x_0,r_1/8)$ and $0 < r < r_1/4$, where $C$ only depends upon $n$, $\alpha$, $\beta$. 
Proof. Let $0 < r < r_1$. We know that for any $y \in B(x_0, r/2)$, $B(y, r/2) \subset B(x_0, r) \subset B(x_0, r_1)$. Thus,

$$\int_{B(y, r/2)} |\nabla u|^2 dx \leq \int_{B(x_0, r)} |\nabla u|^2 dx.$$ 

Thus, (note that $e^{Cr} > 1$)

$$E_u(x_0, r) \geq 2^{-\mu} \left( \frac{\alpha - 1}{2(\alpha + 1)(\alpha + 3)} \right) \left( \frac{r}{2} \right)^{-\mu} \int_{B(x_0, r)} h_2 u^{\alpha + 1} dx$$

$$+ \left( \frac{\alpha - 1}{\alpha + 3} \right) e^{Cr} r^{-\mu} \left( \frac{1}{2} \int_{B(y, r/2)} |\nabla u|^2 dx - \tilde{C} r^n \right)$$

$$+ \frac{1}{(\alpha + 3)} \frac{d}{dr} \left( e^{Cr} r^{-\mu} \int_{\partial B(x_0, r)} u^2 ds \right)$$

$$+ C e^{Cr} r^{-\mu} \left( \frac{1}{4} - \frac{1}{(\alpha + 3)} \right) \int_{\partial B(x_0, r)} u^2 ds + C \int_0^r e^{C\xi \xi^{\alpha + 1}/(\alpha - 1)} d\xi.$$ 

Define $\psi(r) = \left( \frac{r}{2} \right)^{-\mu} \int_{B(y, r/2)} |\nabla u|^2 dx$. By the argument similar to that in the proof of Lemma 2.3 in [GL], we have, for almost every $x_0 \in \Omega$,

$$2^{\mu - 1} (\alpha - 1) \int_0^r \psi(\rho) d\rho \leq (\alpha + 3) E_u(x_0, r_1) r.$$ 

Using Remark 2 in [Pa1] again, we see that there exists some $\sigma \in [r/2, r]$ such that

$$\psi(\sigma) \leq \frac{8}{r} \int_0^r \psi(s) ds \leq C E_u(x_0, r_1),$$ 

for some constant $C > 0$ only depending upon $\alpha$, $\beta$ and $n$. It is clear that $\psi(r/2) \leq 2^\mu \psi(\sigma)$. Since

$$\psi(r/2) = \left( \frac{r}{4} \right)^{-\mu} \int_{B(y, r/4)} |\nabla u|^2 dx,$$ 

we have the desired result for almost every $y \in B(x_0, r_1/8)$. By continuity, we see that it holds for every $y \in B(x_0, r_1/8)$.

Define

$$F_u(x_0, r) = r^{-\mu} \int_{B(x_0, r)} |\nabla u|^2 dx + C \int_0^r e^{C\xi \xi^{(\alpha + 1)/(\alpha - 1)}} d\xi,$$

where $C \int_0^r e^{C\xi \xi^{(\alpha + 1)/(\alpha - 1)}} d\xi$ is the function in the formulations of $E_u(x_0, r)$. Then we have the following lemma:

**Lemma 3.5.** We have that

$$r^{-\mu} \int_{B(x_0, r)} h_2 u^{\alpha + 1} dx \leq CF_u(x_0, r) \text{ for all } x_0 \in \Omega \text{ and } 0 < r < r_0$$
and
\[ r^{-\mu} \int_{B(x_0, r)} h_1 u^2 \, dx \leq CF_u(x_0, r) \quad \text{for all } x_0 \in \Omega \text{ and } 0 < r < r_0, \]
where \( C \) depends only upon \( \alpha, n, a_i \) and \( b_i \) \((i = 1, 2)\).

Proof. We only show the first inequality, the second can be obtained by a similar argument. Since \( E_u(x_0, r) \geq 0 \) for all \( x_0 \in \Omega \) and \( 0 < r < r_0 \), it can be seen from the second of the three equivalent formulations given above that
\[ r^{-\mu} \int_{B(x_0, r)} h_1 u^2 \, dx \leq C \left( F_u(x_0, r) + r^{-\mu-1} \int_{\partial B(x_0, r)} u^2 \, ds \right), \]
for some constant \( C \) depending only upon \( \alpha, n, a_i \) and \( b_i \) \((i = 1, 2)\). On the other hand, the trace embedding theorem gives
\[ H^1(B(x_0, r)) \hookrightarrow W^{1,2}(\partial B(x_0, r)) \hookrightarrow L^{2(\frac{n-1}{n-2})}(\partial B(x_0, r)). \]
Therefore,
\[ \|u\|_{L^{2(\frac{n-1}{2})}(\partial B(x_0, r))} \leq C \|u\|_{H^1(B(x_0, r))}. \]
By Hölder inequality,
\[ r^{-1} \int_{\partial B(x_0, r)} u^2 \, ds \leq C \left( \int_{\partial B(x_0, r)} u^{\frac{2(\frac{n-1}{n-2})}{n-2}} \right)^{\frac{n-2}{n-1}} \leq C \|u\|_{H^1(B(x_0, r))}^2, \]
so we obtain
\[ r^{-\mu-1} \int_{\partial B(x_0, r)} u^2 \, ds \leq CF_u(x_0, r). \]
This implies that the first inequality in the lemma holds.

**Theorem 3.6.** There exist constants \( 0 < \epsilon_0, \tau < 1, 0 < r_2 < r_0/4 \), such that
\[ E_u(x_0, r) \leq \epsilon_0 \quad \text{(3.8)} \]
implies
\[ F_u(x_0, \tau r) \leq \frac{1}{2} F_u(x_0, r) \quad \text{for all } x_0 \in \Omega \text{ and } 0 < r < r_2. \quad \text{(3.9)} \]

Proof. It follows from Proposition 3.4 that if \( E_u(x_0, r) \leq \epsilon_0 \), then for \( \eta < r/4 \),
\[ \eta^{-\mu} \int_{B(x_0, \eta)} |\nabla u|^2 \, dx \leq C \epsilon_0. \]
This implies \( \lim_{\eta \to 0} \eta^{-\mu} \int_{B(x_0, \eta)} |\nabla u|^2 \, dx = 0 \). (Otherwise we can choose \( \epsilon_0 \) smaller to deduce a contradiction.)
If the result were false, there would exist balls $B(x_k, r_k) \subset \Omega$ with $r_k \to 0$ as $k \to \infty$ such that
\begin{equation}
F_u(x_k, r_k) \equiv \lambda_k^2 \to 0,
\end{equation}
whereas
\begin{equation}
F_u(x_k, \tau r_k) > \frac{1}{2} \lambda_k^2,
\end{equation}
for $\tau > 0$ selected as below. We rescale our variables to the unit ball $B(0, 1) \subset \mathbb{R}^n$ as follows: For $z \in B(0, 1)$, we set
\begin{equation}
v_k(z) \equiv r_k^{2/(\alpha - 1)} \left( \frac{u(x_k + r_k z) - a_k}{\lambda_k} \right),
\end{equation}
where
\begin{equation}
a_k \equiv \frac{1}{|B(x_k, r_k)|} \int_{B(x_k, r_k)} u dy = (u)_{x_k, r_k},
\end{equation}
$|B(x_k, r_k)| = \text{Vol}(B(x_k, r_k))$ denotes the average of $u$ over $B(x_k, r_k)$, $k = 1, 2, \ldots$.

Using (3.10), (3.11) and (3.12) we have
\begin{align*}
sup_k \int_{B(0, 1)} |v_k|^2 dz < \infty, \\
sup_k \int_{B(0, 1)} |\nabla v_k|^2 dz < \infty,
\end{align*}
but
\begin{equation}
\frac{1}{\tau^\mu} \int_{B(0, \tau)} |\nabla v_k|^2 dz > \frac{1}{2} - e^{C \tau^{2/\alpha}} \geq 1/4 \quad (k = 1, 2, \ldots),
\end{equation}
if we choose $\tau < \left( \frac{1}{4} e^{-C} \right)^{\frac{\alpha - 1}{2\alpha}}$. In fact, we know that
\begin{equation}
C \int_0^{\tau r_k} e^{C \xi^{(\alpha + 1)/(\alpha - 1)}} d\xi \leq C e^{C(\alpha - 1)} \frac{2 \alpha}{2\alpha - 1} (\tau r_k)^{\frac{2\alpha}{\alpha - 1}}
\end{equation}
and since $C \int_0^{\tau r_k} e^{C \xi^{(\alpha + 1)/(\alpha - 1)}} d\xi < \lambda_k^2$, it holds that
\begin{equation}
\frac{C(\alpha - 1)}{2\alpha} r_k^{\frac{2\alpha}{\alpha - 1}} < \lambda_k^2.
\end{equation}
Thus, it follows from (3.11) that
\begin{equation}
\lambda_k^2 \left( \tau^{-\mu} \int_{B(0, \tau)} |\nabla v_k|^2 dz \right) \geq \left( \frac{1}{2} - e^{C \tau^{2/\alpha}} \right) \lambda_k^2.
\end{equation}
The sequence $\{v_k\}_{k=1}^\infty$ is thus bounded in $H^1(B(0, 1))$, so there exists a subsequence (still denoted by $\{v_k\}$) such that
\begin{align*}
v_k \to v & \text{ strongly in } L^2(B(0, 1)) \\
\nabla v_k \to \nabla v & \text{ weakly in } L^2(B(0, 1)).
\end{align*}
Choose any function \( w \in C^\infty_0(B(0,1)) \). Define
\[
w_k(y) \equiv w \left( \frac{y-x_k}{r_k} \right), \quad (y \in B(x_k, r_k)),
\]
\[
h_1(y) \equiv \bar{h}_1 \left( \frac{y-x_k}{r_k} \right),
\]
\[
h_2(y) \equiv \bar{h}_2 \left( \frac{y-x_k}{r_k} \right).
\]
Since \( u \) is a weak solution of (1.1), we have
\[
\int_{B(x_k, r_k)} \nabla u \nabla w_k dy = \int_{B(x_k, r_k)} (h_1(y)u + h_2(y)u^\alpha) w_k(y) dy.
\]
Thus,
\[
\int_{B(0,1)} \nabla v_k \nabla w dz = \int_{B(0,1)} \left[ r_k^2 \bar{h}_1(z) \left( v_k(z) + \frac{a_k}{A_k} \right) \right. \\
+ \lambda_k^{-1} \bar{h}_2(z) \left( v_k(z) + \frac{a_k}{A_k} \right) \alpha \left] w(z) dz,
\]
where \( A_k = \lambda_k r_k^{-2/(\alpha-1)} \). Since
\[
I_k^1 := r_k^2 \int_{B(0,1)} \bar{h}_1 \left( v_k + \frac{a_k}{A_k} \right) w dz \\
\leq r_k^2 \left( \int_{B(0,1)} \bar{h}_1 \left( v_k + \frac{a_k}{A_k} \right)^2 dz \right)^{1/2} \left( \int_{B(0,1)} \bar{h}_1 w^2 dz \right)^{1/2} \\
\leq r_k^2 \lambda_k^{-2} r_k^{-2} \left( r_k^\mu \int_{B(x_k, r_k)} h_1 u^2 dx \right)^{1/2} \left\| \bar{h}_1 w^2 \right\|_{L^2(B(0,1))} \\
\leq Cr_k \left\| \bar{h}_1 u^2 \right\|_{L^2(B(0,1))} \to 0
\]
(here we used Lemma 3.5) as \( k \to \infty \) and
\[
I_k^2 := \lambda_k^{-1} \int_{B(0,1)} \bar{h}_2 \left( v_k + \frac{a_k}{A_k} \right)^\alpha w dz \\
\leq \lambda_k^{-1} \left( \int_{B(0,1)} \bar{h}_2 \left( v_k + \frac{a_k}{A_k} \right)^{\alpha+1} dz \right)^{\alpha/(\alpha+1)} \\
\cdot \left( \int_{B(0,1)} \bar{h}_2 |w|^{\alpha+1} dz \right)^{1/(\alpha+1)} \\
\leq \lambda_k^{-1} \left( \lambda_k^{-(\alpha+1)} r_k^\mu \int_{B(x_k, r_k)} h_2 u^{\alpha+1} dx \right)^{\alpha/(\alpha+1)} \left\| \bar{h}_2^{1/(\alpha+1)} w \right\|_{L^{\alpha+1}(B(0,1))}
\[ \le C_{\lambda}^{(\alpha-1)/\alpha(\alpha+1)} \| \tilde{h}_2^{1/(\alpha+1)} w \|_{L^{\alpha+1}(B(0,1))} \rightarrow 0 \]

(here we used Lemma 3.5) as \( k \rightarrow \infty \).

Letting \( k \rightarrow \infty \) in (3.17), we get

\[ \int_{B(0,1)} \nabla v \nabla w dz = 0. \tag{3.18} \]

Hence \( v \) is harmonic function, and hence smooth, and we have the bound

\[ \| \nabla v \|_{L^\infty(B(0,1/2))} \le \frac{C}{|B(0,1)|} \int_{B(0,1)} v^2 dz < \infty, \tag{3.19} \]

where \( |B(0,1)| = \text{Vol}(B(0,1)) \). We will show in next section that

\[ \nabla v_k \to \nabla v \text{ strongly in } L^2 \left( B \left( 0, \frac{1}{4} \right) \right) \tag{3.20} \]

then we have,

\[ \frac{1}{\tau^\mu} \int_{B(0,\tau)} |\nabla v|^2 dz \le C \tau^{n-\mu} < \frac{1}{4} \tag{3.21} \]

provided \( 0 < \tau < \min \left\{ \left( \frac{1}{4C} \right)^{\frac{\alpha-1}{\alpha(\alpha+1)}}, \left( \frac{e^{-C}}{4} \right)^{(\alpha+1)/(2\alpha)}, \frac{1}{4} \right\} \), which contradicts (3.13).

4. Compactness.

In this section we turn our attention to (3.20). We choose a smooth cut-off function \( \zeta : \mathbb{R}^n \to \mathbb{R} \) satisfying

\[ 0 \le \zeta \le 1, \]

\[ \zeta \equiv 1 \text{ on } B \left( 0, \frac{1}{4} \right), \]

\[ \zeta \equiv 0 \text{ on } \mathbb{R}^n \setminus B \left( 0, \frac{5}{16} \right). \]

Lemma 4.1. The sequence \( \{ \zeta v_k \}_{k=1}^\infty \) is bounded in \( M^2_{1,\nu}(\mathbb{R}^n, \mathcal{R}) \).

Proof. We first show that for \( 0 < r < r_0 < 1 \),

\[ E_u(x_0,r) \le CF_u(x_0,r) \text{ for all } x_0 \in \Omega. \tag{4.1} \]
In fact, it follows from the second of the three formulations of $E_u(x_0, r)$ given above that

\begin{equation}
E_u(x_0, r) \leq \frac{1}{2} e^{Cr-r-\mu} \int_{B(x_0, r)} |\nabla u|^2 \, dx + \frac{1}{(\alpha - 1)} e^{Cr-r-\mu} \int_{\partial B(x_0, r)} u^2 \, ds
+ \frac{C}{4} e^{Cr-r-\mu} \int_{\partial B(x_0, r)} u^2 \, ds + C \int_0^r e^{C \xi (\alpha + 1)/(\alpha - 1)} \, d\xi.
\end{equation}

By the trace embedding theorem and the argument similar to the one used in the proof of Lemma 3.5, we obtain

\begin{equation}
r^{-\mu - 1} \int_{\partial B(x_0, r)} u^2 \, ds \leq CF_u(x_0, r).
\end{equation}

Note that $e^{Cr} < e^C$. Our claim can be obtained from (4.2) and (4.3).

Fix any point $z_0 \in B(0, \frac{3}{4})$ and any radius $0 < r < \frac{1}{8}$, set

\[ y_k = x_k + r_k z_0 \in B(x_k, \frac{3}{4} r_k). \]

By the claim above and an argument similar to the one used in the proof of Lemma 3.3, we obtain that

\[ \frac{1}{(rr_k)^\mu} \int_{B(y_k, rr_k)} |\nabla u|^2 \, dy \leq CE_u \left( y_k, \frac{1}{4} r_k \right) \]

\[ \leq C \left( r_k^{-\mu} \int_{B(y_k, \frac{1}{4} r_k)} |\nabla u|^2 \, dy + C \int_0^{\frac{1}{4} r_k} e^{C \xi (\alpha + 1)/(\alpha - 1)} \, d\xi \right) \]

\[ \leq CF_u(x_k, r_k) = C \lambda_k^2. \]

Rescaling this estimate we obtain,

\begin{equation}
r^{-\mu} \int_{B(z_0, r)} |\nabla v_k|^2 \, dz \leq C
\end{equation}

for $k = 1, 2, \ldots$, all $0 < r < \frac{1}{8}$ and $z_0 \in B(0, \frac{3}{4})$. This implies that

\begin{equation}
\frac{1}{r^{n-\frac{2}{\alpha - 1}}} \int_{B(z_0, r)} |v_k - (v_k)_{z_0,r}| \, dz \leq C < \infty
\end{equation}

for $k$, $r$ and $z_0$ as above. This implies $v_k \in L^{1,n-\frac{2}{\alpha - 1}}(B(0, \frac{3}{4}))$ and $L^{1,n-\frac{2}{\alpha - 1}}(B(0, 3/4))$ is a Campanato space (see [Gi]). Since $B(0, \frac{3}{4})$ is type
On the other hand, if \( z \) for \( w \) Subtracting (3.18) from (3.17) we obtain

Proof. (4.8)

(A) \([\text{[Gi]}, \text{Chapter III, Definition 1.3}]\), Proposition 1.2 in Chapter III in \([\text{Gi}]\) implies that

\[
(4.6) \quad \sup_{z_0 \in B(0,3/4), \ 0 < r < 1/8} \frac{1}{r^{n-2}} \int_{B(z_0,r)} |v_k|dz \\
\leq C \sup_{z_0 \in B(0,3/4), \ 0 < r < 1/8} \frac{1}{r^{n-2}} \int_{B(z_0,r)} |v_k - (v_k)_{z_0,r}|dz \leq C.
\]

Since \( \zeta \) is smooth, then

\[
(4.7) \quad |(\zeta v_k)_{z_0,r} - \zeta(v_k)_{z_0,r}| \leq \frac{Cr}{|B(z_0,r)|} \int_{B(z_0,r)} |v_k|dz \quad \text{on} \ B(z_0,r)
\]

for any ball \( B(z_0,r) \). Thus, if \( z_0 \in B(0,\frac{3}{4}) \), \( 0 < r < \frac{1}{8} \), we have,

\[
(4.8) \quad \frac{1}{|B(z_0,r)|} \int_{B(z_0,r)} |\zeta v_k - (\zeta v_k)_{z_0,r}|dz \\
\leq \frac{1}{|B(z_0,r)|} \int_{B(z_0,r)} |v_k - (v_k)_{z_0,r}|dz + \frac{Cr}{|B(z_0,r)|} \int_{B(z_0,r)} |v_k|dz.
\]

On the other hand, if \( z_0 \in \mathbb{R}^n \setminus B(0,\frac{3}{4}) \), \( 0 < r < \frac{1}{8} \), we have

\[
(4.9) \quad \int_{B(z_0,r)} |\zeta v_k - (\zeta v_k)_{z_0,r}|dz = 0.
\]

It follows from (4.6), (4.8) and (4.9) that

\[
(4.10) \quad \zeta v_k \in \mathcal{L}^{1,n-\frac{2}{n}}(\mathbb{R}^n).
\]

This also implies that \( \{\zeta v_k\}_{k=1}^{\infty} \) is bounded in \( M^2_{1,\nu}(\mathbb{R}^n, \mathbb{R}) \) for \( k = 1, 2, \ldots \).

**Proposition 4.2.** The rescaled functions \( \{\nabla v_k\}_{k=1}^{\infty} \) converge strongly in \( L^2(B(0,\frac{1}{4})) \).

**Proof.** Subtracting (3.18) from (3.17) we obtain

\[
(4.11) \quad \int_{B(0,1)} (\nabla v_k - \nabla v) \nabla w dz \\
= r_k^2 \int_{B(0,1)} \tilde{h}_1 \left( v_k + \frac{a_k}{A_k} \right) w + \lambda_k^{a-1} \int_{B(0,1)} \tilde{h}_2 \left( v_k + \frac{a_k}{A_k} \right)^{a} w dz
\]

for \( w \in C_0^\infty(B(0,1)) \). Hence it holds for \( w \in H_0^1(B(0,1)) \cap L^\infty(B(0,1)) \). We now insert \( w \equiv \zeta^2(v_k - v) \) into (4.11). The left-hand side of (4.11) is

\[
L_k \equiv \int_{B(0,1)} \zeta^2 |\nabla v_k - \nabla v|^2 dz + 2 \int_{B(0,1)} \zeta(v_k - v) (\nabla v_k - \nabla v) \nabla \zeta dz \\
\geq \int_{B(0,\frac{1}{4})} |\nabla v_k - \nabla v|^2 dz + o(1)
\]

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as \( k \to \infty \), in view of (3.14) and (3.15). The right-hand side of (4.11) reads
\[
R_k \equiv r_k^2 \int_{B(0,1)} \tilde{h}_1 \left( v_k + \frac{a_k}{A_k} \right) \zeta^2 (v_k - v) \, dz \\
+ \lambda_k^{\alpha - 1} \int_{B(0,1)} \tilde{h}_2 \left( v_k + \frac{a_k}{A_k} \right)^\alpha \zeta^2 (v_k - v) \, dz \\
= R_k^1 + R_k^2.
\]
\[
R_k^1 \leq r_k^2 \left( \int_{B(0,1)} \tilde{h}_1 \left( v_k + \frac{a_k}{A_k} \right)^2 \right)^{1/2} \left( \int_{B(0,1)} \tilde{h}_1 \zeta^4 (v_k - v)^2 \, dz \right)^{1/2} \\
= Cr_k^2 \left( \lambda_k^{-\alpha - n} \int_{B(x_k, r_k)} u^2 \, dx \right)^{1/2} \\
\leq Cr_k^2 (r_k^{-2})^{1/2} = Cr_k \to 0
\]
as \( k \to \infty \).

Now we show that
\[
\zeta \left( v_k + \frac{a_k}{A_k} \right)^\alpha \in H_{w}^1 (\mathbb{R}^n)
\]
for \( k = 1, 2, \ldots \). We first consider
\[
M_1 \left( \zeta \left( v_k + \frac{a_k}{A_k} \right)^\alpha \right) (z) := \sup_{t > 0} \frac{1}{t^n} \int_{\mathbb{R}^n} \left( \zeta^{1/\alpha} f_k \right)^\alpha (y) \phi \left( \frac{y - z}{t} \right) \, dy
\]
where \( f_k(y) := v_k(y) + \frac{a_k}{A_k} \), \( \phi \) is a Schwartz function with nonvanishing integral (see [ST]).

If \( t \geq 1 + \frac{|z|}{t} \), we have
\[
\frac{1}{t^n} \int_{\mathbb{R}^n} \left( \zeta^{1/\alpha} f_k \right)^\alpha (y) \phi \left( \frac{y - z}{t} \right) \, dy \\
\leq \frac{1}{t^n} \int_{B(0,1)} \left( \zeta^{1/\alpha} f_k \right)^\alpha (y) \phi_t \, dy \\
\leq \frac{1}{t^n} \left( \int_{B(0,1)} \left( \zeta^{1/\alpha} f_k \right)^{\alpha + 1} \, dy \right)^{\alpha/(\alpha + 1)} \left( \int_{B(0,1)} \phi_t^{\alpha + 1} \, dy \right)^{1/(\alpha + 1)} \\
\leq C \frac{1}{t^n} \lambda_k^{-(\alpha + 1)} r_k^{-\mu} \int_{B(x_k, r_k)} u^{\alpha + 1} \, dx \\
\leq C \frac{\alpha \lambda_k^{1-\alpha}}{t^n \lambda_k^{\alpha + \mu}} \\
\leq \frac{C}{(4 + |z|)^n} \lambda_k^{\alpha (1-\alpha)/(\alpha + \mu)}.
\]
Therefore,
\[
\int_{\mathbb{R}^n} M_1 \left( \zeta \left( v_k(z) + \frac{a_k}{A_k} \right)^\alpha \right) w(z) \, dz
= \int_{\mathbb{R}^n} M_1 \left( \zeta \left( v_k(z) + \frac{a_k}{A_k} \right)^\alpha \right) |z|^{-2/(\alpha - 1)} \, dz
\leq C \lambda_k^{\frac{(1-\alpha)}{\alpha+1}} \int_{\mathbb{R}^n} |z|^{-2/(\alpha - 1)} (4 + |z|)^{-n} \, dz
\leq C \lambda_k^{\frac{(1-\alpha)}{\alpha+1}}.
\]

If \( t < 1 + \frac{|z|}{4} \), we have, if \( |y - z| < t \), then \( |y - z| < 1 + \frac{|z|}{4} \) and \( |y| > \frac{3}{4} |z| - 1 \).
Therefore, if \( |z| > 8/3 \), then \( |y| > 1 \). Thus, for \( 0 < \epsilon < 1 \) and \( z \in B(0, 3) \),
\[
\frac{1}{t^n} \int_{\mathbb{R}^n} \left( \zeta^{1/\alpha} f_k \right)^\alpha \phi \left( \frac{y - z}{t} \right) \, dy
\leq \frac{1}{t^n} \left( \int_{B(z, t)} \left( \zeta^{1/\alpha} f_k \right)^{\alpha+1-\epsilon} \, dy \right)^{\alpha/(\alpha+1-\epsilon)}
\cdot \left( \int_{B(z, t)} \phi_t^{(\alpha+1-\epsilon)/(1-\epsilon)} \, dy \right)^{(1-\epsilon)/(\alpha+1-\epsilon)}
\leq C \left( M \left( \left( \zeta^{1/\alpha} f_k \right)^{\alpha+1-\epsilon} \right) \right)^{\alpha/(\alpha+1-\epsilon)},
\]
where \( M(\cdot) \) is the Hardy-Littlewood maximal function. If \( z \in \mathbb{R}^n \setminus B(0, 3) \),
\[
\frac{1}{t^n} \int_{\mathbb{R}^n} \left( \zeta^{1/\alpha} f_k \right)^\alpha \phi \, dy = 0.
\]
Therefore,
\[
\int_{\mathbb{R}^n} M_1 \left( \zeta(z) \left( v_k(z) + \frac{a_k}{M_k} \right)^\alpha \right) |z|^{-2/(\alpha - 1)} \, dz
\leq C \int_{B(0, 3)} \left[ M \left( \left( \zeta^{1/\alpha} f_k \right)^{\alpha+1-\epsilon} \right) \right]^{\alpha/(\alpha+1-\epsilon)} |z|^{-2/(\alpha - 1)} \, dz
\leq C \left( \int_{B(0, 3)} \left[ M \left( \left( \zeta^{1/\alpha} f_k \right)^{\alpha+1-\epsilon} \right) \right]^{(\alpha+1)/(\alpha+1-\epsilon)} \, dz \right)^{\alpha/(\alpha+1)}
\cdot \left( \int_{B(0, 3)} |z|^{-\frac{2+\epsilon}{\alpha-1}} \, dz \right)^{1/(\alpha+1)}
\leq C \left( \int_{B(0, 3)} \left( \zeta^{1/\alpha} f_k \right)^{\alpha+1} \, dz \right)^{\alpha/(\alpha+1)} \left( \int_0^3 r^{\mu-1} \, dr \right)^{1/(\alpha+1)}
\]
where we used the facts that $\mu > 0$ if $\alpha > \frac{n+2}{n-2}$, and
\[
\int_{B(0, 3)} \left[ M\left(\left(\zeta^{1/\alpha} f_k\right)^{\alpha+1} - \varepsilon\right)\right]^{(\alpha+1)/(\alpha+1-\epsilon)} dz
\]
\[
= \int_{\mathbb{R}^n} \left[ M\left(\left(\zeta^{1/\alpha} f_k\right)^{\alpha+1} - \varepsilon\right)\right]^{(\alpha+1)/(\alpha+1-\epsilon)} dz
\]
\[
\leq \int_{\mathbb{R}^n} \left(\zeta^{1/\alpha} f_k\right)^{\alpha+1} dz,
\]
because
\[
M\left(\left(\zeta^{1/\alpha} f_k\right)^{\alpha+1} - \varepsilon\right)\equiv 0 \quad \text{for} \quad z \in \mathbb{R}^n \setminus B(0, 3).
\]
It concludes that $\left(\zeta^{1/\alpha} f_k\right)^{\alpha} \in H^1_{\omega}(\mathbb{R}^n)$. Therefore, it follows from (2.1) that
\[
R_k^2 \leq \lambda^{-1} \int_{\mathbb{R}^n} M_1\left(\left(\zeta^{1/\alpha} f_k\right)^{\alpha} \right) M_{1,\nu}^{\alpha} (\zeta(v_k - v)) |z|^{-2/(\alpha-1)} dz
\]
\[
\leq C \lambda^{-1} \int_{\mathbb{R}^n} M_1\left(\left(\zeta^{1/\alpha} f_k\right)^{\alpha} \right) |z|^{-2/(\alpha-1)} dz
\]
\[
\leq C \lambda^{-1} \lambda_{\alpha+1}^{\alpha(1-\alpha)}
\]
\[
= C \lambda_{\alpha+1}^{\alpha-1} \rightarrow 0
\]
as $k \rightarrow \infty$.

5. Proof of Theorem A.

In this section we shall prove Theorem A. We recall the definition of function space $L^{p,q}(\Omega)$:
\[
L^{p,q}(\Omega) = \left\{ v \in L^p(\Omega) : \sup_{x \in \Omega, r > 0} r^{-q} \int_{B(x, r) \cap \Omega} u^p dx < +\infty \right\}.
\]
This is called Morrey space (see [Gi]). Now we recall a theorem in [Pa2].

**Theorem 5.1.** Let $u$ be a positive weak solution of (1.1), assume that $u \in L^{\alpha,\lambda+\theta}(\Omega)$ for $\lambda = n - \frac{2\alpha}{\alpha-1}$ and some $\theta > 0$ then $u$ is regular in $\Omega$.

Note that Pacard [Pa2] proved this theorem only for the case that $h_1 \equiv 0$ and $h_2 \equiv 1$ in $\Omega$, but we can easily see from his proof that this theorem still holds in our case.

Set
\[
V \equiv \{ x \in \Omega : E_u(x, r) < \epsilon_0 \quad \text{for some} \quad 0 < r < r_2 \},
\]
where $\epsilon_0$ and $r_2$ are constants in Theorem 3.6. Furthermore, using Theorem 3.6, we can show that (cf. [Gi]), if $x \in V$, there exists $r^* > 0$ sufficiently small such that
\begin{equation}
F_u(y, r) \leq Cr^\gamma
\end{equation}
for some $0 < \gamma < \frac{2\alpha}{\alpha-1}$, $C > 0$, all $y$ near $x$, and all sufficiently small radii $0 < r < r^*$. It follows from Lemma 3.5 that
\begin{equation}
lm^\mu \left( \int_{B(x, r)} |\nabla u|^2 + u^{\alpha+1} \right) \leq C r^\gamma
\end{equation}
for all $y$ near $x$, and $0 < r < r^*$. Now we show that $u \in L^{\alpha, \lambda} \left( B(x, r^*/2) \right)$ for some $\theta_0 > 0$. In fact, choosing $\theta_0 = \frac{\alpha \gamma}{\alpha+1}$, we have, for $0 < r < r^*$, $r^{-(n+\theta_0)} \int_{B(x, r)} u^{\alpha} dy \leq r^{-(n+\theta_0)} \left( \int_{B(x, r)} u^{\alpha+1} dy \right)^{\alpha/(\alpha+1)} r^{1/(\alpha+1)} \leq C.$
This implies (5.4) and therefore, by Theorem 5.1, $u$ is regular at $x$. Hence $u$ is regular in $V$.
Define $\Sigma = \Omega \setminus V$. Then
\begin{equation}
\Sigma \supset \cap_{r>0} \left\{ x \in \Omega : E_u(x, r) \geq \epsilon_0 \right\}
\end{equation}
It is proved in [GL] that
\begin{equation}
\Sigma \subset \cap_{r>0} \left\{ x \in \Omega : \int_{B(x, r)} \left( u^{\alpha+1} + |\nabla u|^2 \right) dy \geq C \epsilon_0 r^\mu \right\}
\end{equation}
Thus, standard covering arguments imply that the Hausdorff dimension of $\Sigma$ is less than $n - 2\frac{n+1}{\alpha-1}$. This completes the proof of Theorem A.

**Remark.** The conclusion of Theorem A still holds for the stationary positive weak solutions of the equation $\Delta u + h_1(x)u^\kappa + h_2(x)u^\alpha = 0$ in $\Omega$ where $0 < \kappa < \alpha$, $\alpha > \frac{n+2}{n-2}$. It should be very interesting to know whether our partial regularity theorem holds for the equation $\Delta u + h(x)f(u) = 0$ in $\Omega$ where $f(s)$ satisfies that $f(s) > 0$ for $s > 0$ and $f$ has the growth rate $\alpha > \frac{n+2}{n-2}$. The main difficulty is how to establish the monotonicity inequality.
Acknowledgements. Part of this work was done while the first author was visiting the Institute of Mathematics, Chinese Academy of Sciences, he would like to thank the Institute for their hospitality. The authors wish to thank Professor Ding Weiyue for bringing this problem to their attention. They also thank the referee for valuable comments. This research was partially supported by the Outstanding Young Scientists Grants in China and the National Key Basic Research Fund of China.

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Received January 30, 2001 and revised June 24, 2002.

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A HODGE DECOMPOSITION FOR THE COMPLEX OF INJECTIVE WORDS

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Reiner and Webb (preprint, 2002) compute the $S_n$-module structure for the complex of injective words. This paper refines their formula by providing a Hodge type decomposition. Along the way, this paper proves that the simplicial boundary map interacts in a nice fashion with the Eulerian idempotents.

The Laplacian acting on the top chain group in the complex of injective words is also shown to equal the signed random to random shuffle operator. Uyemura-Reyes, 2002, conjectured that the (unsigned) random to random shuffle operator has integral spectrum. We prove that this conjecture would imply that the Laplacian on (each chain group in) the complex of injective words has integral spectrum.

1. Introduction.

Let $V = \langle v_1, \ldots, v_n \rangle$ be an $n$-dimensional Euclidean space. For each $r$, let $\Gamma_r = V^\otimes r$ and let $\partial_r : \Gamma_r \to \Gamma_{r-1}$ be the map given by:

$$\partial_r(a_1 \otimes \cdots \otimes a_r) = \sum_{j=1}^{r} (-1)^{j-1}(a_1 \otimes \cdots \otimes a_{j-1} \otimes a_{j+1} \otimes \cdots \otimes a_r).$$

It is well-known that the $\partial_r$ are boundary maps, i.e., that $\partial_r \cdot \partial_{r+1} = 0$.

Let $M_r$ be the multilinear part of $\Gamma_r$. So, $M_r = 0$ if $r > n$, and for $r \leq n$,

$$M_r = \langle v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} : i_1, \ldots, i_r \text{ are distinct} \rangle.$$

Note that $\dim(M_r) = n(n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$. Also, it is clear that $\partial_r(M_r) \subset M_{r-1}$ and so

$$0 \to M_n \to M_{n-1} \to \cdots \to M_0 \to 0$$

is a subcomplex of $(\Gamma_*, \partial_*)$. This paper will concern the homology of this subcomplex.

The complex $(M_*, \partial_*)$ appears in earlier work on the subword order of injective words on the alphabet $\{1, 2, \ldots, n\}$. This poset is the face poset of a regular CW complex $K_n$ whose homology agrees with the homology of $(M_*, \partial_*)$. In [F], Farmer proves that $K_n$ is homotopy equivalent to a wedge of $(n-1)$-spheres thus showing that the homology of $(M_*, \partial_*)$ vanishes except
at top degree. Bjorner and Wachs [BW] prove a stronger result – that the lexicographic order on permutations induces a recursive coatom ordering on the poset of injective words. This in turn gives a dual CL-shelling of \( K_n \).

Reiner and Webb [RW] study \((M_*, \partial_*)\) as a subcomplex of \((\Gamma_*, \partial_*)\). The natural action of \( S_n \) on \( \{v_1, \ldots, v_n\} \) extends to an action of \( S_n \) on \((M_*, \partial_*)\) which preserves \((M_*, \partial_*)\). Reiner and Webb compute the homology of \((M_*, \partial_*)\) as an \( S_n \)-module.

**Theorem 1.1** (Reiner-Webb). As an \( S_n \)-module, the top homology of \((M_*, \partial_*)\) is

\[
\bigoplus_{k=0}^{n} (-1)^{n-k} \text{ind}_{S_{n-k}}^{S_n} (\varepsilon_{n-k}) = \bigoplus_{k=0}^{n} (-1)^{n-k} \text{ind}_{S_{n-k} \times S_k} (\varepsilon_{n-k} \otimes \text{Reg}_k)
\]

where \( \varepsilon_{n-k} \) denotes the trivial representation of \( S_{n-k} \). Furthermore, the multiplicity of an irreducible \( S^\lambda \) of \( S_{n-k} \) in the top homology is equal to the number of standard Young tableaux of shape \( \lambda \) which have their smallest descent even.

In this paper, we will do two things. First, we will show that there is a natural Hodge decomposition of the homology of \((M_*, \partial_*)\). This decomposition will split \( H_n(M) \) into \( n \) components

\[
H_n(M) = \bigoplus_{j=1}^{n} H_n^{(j)}(M).
\]

We will show that the dimension of \( H_n^{(j)}(M) \) is equal to the number of derangements with exactly \( j \) cycles. More specifically, we will show that each \( H_n^{(j)}(M) \) is invariant under the action of \( S_n \) and prove that \( H_n^{(j)}(M) \) is a sum of linear characters induced from centralizers of permutations with exactly \( j \) cycles.

Second, we will study the Laplacian \( \Lambda_* \) associated to the complex \((M_*, \partial_*)\). We will show that \( \Lambda_n \) is closely connected to the transition matrix for random to random shuffling. Random to random shuffling has been studied by Uyemura-Reyes in [Uy]. In [Uy], the author makes a conjecture about the spectrum of the transition matrix for random to random shuffling which together with our results imply the conjecture that the spectrum of \( \Lambda_n \) is integral. We go on to compute \( \Lambda_r \), for \( r < n \) in terms of \( \Lambda_n \). This computation shows that \( \Lambda_r \) is positive definite for \( r < n \) thus giving another proof that \( H_r(M) = 0 \) for \( 0 \leq r < n \). This computation also shows that the spectrum of \( \Lambda_r \) is integral if the spectrum of \( \Lambda_n \) is integral. Thus, if Uyemura-Reyes' conjecture on the spectrum of random to random shuffling is correct, then the spectra of all \( \Lambda_r \) are integral.
2. A Hodge type decomposition of $H_*(M)$.

We begin by recalling the definition of the Eulerian idempotents $e^{(j)}_n$ in $CS_n$. For each $r$ and $k$, let $S(r; k)$ denote the set of permutations in $S_r$ with exactly $k - 1$ descents. Following Loday [Lo], define elements $l^{(k)}_r$ and $\lambda^{(k)}_r$ in $CS_r$ according to the following formulae:

\begin{align*}
(2.1) \quad l^{(k)}_r &= (-1)^{k-1} \sum_{\sigma \in S(r; k)} \text{sgn}(\sigma) \sigma, \\
(2.2) \quad \lambda^{(k)}_r &= \sum_{i=0}^{k-1} (-1)^i \binom{n+i}{i} l^{(k-i)}_r, \\
(2.3) \quad (-1)^{k-1} \lambda^{(k)}_r &= \sum_{j=1}^{n} k^j e^{(j)}_n.
\end{align*}

It is worth noting that the first two equations define the $\lambda^{(k)}_r$ explicitly. The third equation then determines the $e^{(j)}_n$ in terms of the $\lambda^{(k)}_r$ because the transition matrix $(k^j)_{k,j}$ is a Vandermonde and hence invertible.

There is a significant literature on the Eulerian idempotents and their many remarkable properties. We will need two of these properties. The first is the well-known fact that $e^{(1)}_n, e^{(2)}_n, \ldots, e^{(n)}_n$ form a set of pairwise orthogonal idempotents in $CS_n$ that decompose the identity. In other words, $e^{(j)}_n \cdot e^{(\ell)}_n = 0$ if $j \neq \ell$ and $e^{(1)}_n + e^{(2)}_n + \cdots + e^{(n)}_n = \text{id}$. This implies that if $X$ is any $S_n$-module, then

$$X = \bigoplus_j e^{(j)}_n \cdot X.$$ 

The second fact we will need describes the relationship between the $e^{(j)}_n$ and the boundary map $\partial$. To state this result, it will be helpful to write permutations in one-line notation. Let $i \in \{1, 2, \ldots, n\}$ and let $S_n \setminus \{i\}$ denote permutations of $\{1, 2, \ldots, n\} \setminus \{i\}$. There is a natural identification of $S_n \setminus \{i\}$ with $S_{n-1}$ which comes about by changing each occurrence of $i+j$ to $i+j-1$. Via this identification, we can think of $e^{(\ell)}_{n-1}$ as sitting inside of the group algebra of $S_n \setminus \{i\}$.

As in [RW], we will think of $\partial$ as acting on linear combinations of injective words on the alphabet $\{1, 2, \ldots, n\}$. For $i \in \{1, 2, \ldots, n\}$, let $\partial[i]$ denote the part of $\partial$ which removes the number $i$. The next theorem presents a surprisingly elegant outcome to the computation of $\partial[i] e^{(k)}_n$.

Before stating and proving the result, we will give an example to be sure that the notation is clear. Let $n = 3$. In the example that follows, we will use $a, b, c$ in place of $1, 2, 3$ so as to avoid confusion with coefficients. The
Eulerian idempotents are given by:

\[ e_3^{(1)} = \frac{1}{6}(2 \cdot abc + bac + acb - bca - cab - 2 \cdot cba) \]

\[ e_3^{(2)} = \frac{1}{2}(abc + cba) \]

\[ e_3^{(3)} = \frac{1}{6}(abc - bac - acb + bca + cab - cba). \]

For the purposes of this example, we will apply \( \partial [b] \) to each of these. Doing so, we get

\[ \partial [b] e_3^{(1)} = \frac{1}{6}(-2ac + ac + ac - ca - ca + 2ca) = 0 \]

\[ \partial [b] e_3^{(2)} = \frac{1}{2}(-ac - ca) = -e_2^{(1)} \]

\[ \partial [b] e_3^{(3)} = \frac{1}{6}(-ac - ac - ac + ca + ca + ca) = -e_2^{(2)}. \]

Part 2 of Theorem 2.1 is needed later to show that the complex of injective words has a Hodge decomposition.

**Theorem 2.1.** Fix \( n \) and \( i \in \{1, 2, \ldots, n\} \). Then:

1. \( \partial [i] \lambda_n^{(k)} = (-1)^{i-1}k \lambda_{n-1}^{(k)} \)
2. \( \partial [i] e_n^{(k)} = (-1)^{i-1}e_{n-1}^{(k-1)} \)

Before proving this theorem, let us verify two lemmas to be used in its proof.

**Lemma 2.1.** Among the \( n \) places the letter \( i \) could be inserted into a permutation \( \sigma \in S_{[n]} \setminus \{i\} \) which has \( j \) descents, \( j + 1 \) choices yield permutations with \( j \) descents, while the other \( n - j - 1 \) choices all yield permutations with \( j + 1 \) descents.

**Proof.** First we consider the case \( i = n \), then use a graph for a permutation to generalize to all \( i \). Notice that inserting \( n \) between two letters descending letters preserves the number of descents, while inserting \( n \) between two ascending letters increases the number of descents. Thus, there are \( n - 2 - j \) ways to increase the number of descents by one by inserting \( n \) between two ascending letters. In addition, inserting \( n \) before the first letter gives one more way to increase the number of descents by one.

For \( i \neq n \), the analysis will also need to consider whether or not \( i \) is intermediate in value to the pair of consecutive labels where it is to be inserted. To this end, we define the graph of a permutation as follows:

**Definition 2.1.** For each \( \pi \in S_{[n]} \setminus \{i\} \), define the related function \( \pi' : [0, n] \to [0, n+1] \) by \( \pi'(j) = \pi(j) \) for \( 1 \leq j < i \) and \( \pi'(j) = \pi(j + 1) \) for \( n - 1 \geq j > i \). In addition, let \( \pi'(0) = n + 1 \) and \( \pi'(n) = 0 \). Then the
graph of $\pi$ is obtained by plotting the points $(j, \pi'(j))$ for each $j \in [0, n]$, and for each $j \in [0, n - 1]$ connecting the point $(j, \pi'(j))$ to $(j + 1, \pi'(j + 1))$ by a straight line segment.

This graph has negative slope at each descent and positive slope at each ascent. Furthermore, it crosses the line $y = i$ with negative slope one more time than it does with positive slope, because it represents a continuous function which begins above the line $y = i$ and ends below the line $y = i$ (and which has nonzero slope everywhere it touches the line $y = i$).

We claim that the number of places to insert $i$ which will increase the number of descents by one is equal to sum of the number of ascents that do not cross the line together with the number of descents which do cross the line. This is clear except at the endpoints. By letting $\pi'(0) = n + 1$ and $\pi'(n + 1) = 0$, we created descents at the initial and final positions in $\pi$, which are only counted above when the graph crosses the line $y = i$ at these points, namely when $\pi(1) < i$ and when $\pi(n) > i$, respectively. These are exactly the situations where inserting $i$ at the initial or final positions will indeed increase the number of descents by one.

We already observed that the number of descents crossing the line is one more than the number of ascents crossing the line. Thus, the total number of ways to increase the number of descents by one is one more than the total number of ascents, i.e., it is $n - j - 1$, as desired. A similar argument shows that all of the remaining $j + 1$ options will preserve the number of descents.

**Lemma 2.2.** If $\tau \in S_n$ is obtained from $\sigma \in S_n\setminus\{i\}$ by inserting $i$ after the $(d - 1)$-st letter of $\sigma$, then $\text{sgn}(\tau) = (-1)^i - d \text{sgn}(\sigma)$.

**Proof.** If $d = 1$, so $i = \tau_1$, then inserting $i$ created $i - 1$ new descents, because the values $1, \ldots, i - 1$ all appear later than the value $i$. Thus, $\text{sgn}(\tau) = (-1)^{i - 1} \text{sgn}(\sigma)$ in this case, as desired. Now we proceed by induction on $d$. Moving the letter $i$ from position $r$ to position $r + 1$ in $\tau$ by an adjacent transposition reverses the sign of $\tau$. Likewise, increasing $d$ from $r$ to $r + 1$ reverses the sign of $(-1)^{i - d} \text{sgn}(\sigma)$ from $(-1)^{i - r} \text{sgn}(\sigma)$ to $(-1)^{i - (r + 1)} \text{sgn}(\sigma)$, so $\text{sgn}(\tau)$ continues to agree with $(-1)^{i - 1} \text{sgn}(\sigma)$ as $d$ increases. \qed

**Proof of Theorem 2.1.** We first prove identity (1). Note that

$$\lambda_n^{(k)} = \sum_{i=0}^{k-1} (-1)^i \binom{n+i}{i} (-1)^{k-i-1} \sum_{\sigma} \text{sgn}(\sigma) \sigma$$
where the sum is over $\sigma$ with $k - i - 1$ descents. Replacing $k - i - 1$ by $j$ yields

$$
\lambda_n^{(k)} = \sum_{j=0}^{k-1} \binom{n + k - j - 1}{k - j - 1} (-1)^j \sum_{\sigma} \text{sgn}(\sigma)\sigma
$$

which simplifies to

$$
\lambda_n^{(k)} = (-1)^{k-1} \sum_{j=0}^{k-1} \binom{n + k - j - 1}{n} \sum_{\sigma} \text{sgn}(\sigma)\sigma.
$$

In each of the last two equations, the sum is over $\sigma$ with $j$ descents. Similarly,

$$
\lambda_{n-1}^{(k)} = (-1)^{k-1} \sum_{j=0}^{k-1} \binom{n + k - j - 2}{n - 1} \sum_{\sigma} \text{sgn}(\sigma)\sigma.
$$

Lemma 2.1 shows that for each $\sigma \in S_n\setminus\{i\}$ with $j$ descents, there are $j + 1$ permutations $\tau \in S_n$ with $j$ descents such that $\partial[i]\tau = \pm\sigma$ and there are $n - j - 1$ permutations $\tau \in S_n$ with $j + 1$ descents such that $\partial[i]\tau = \pm\sigma$. When our boundary operator $\partial[i]$ deletes $\tau_d$ from $\tau$ to obtain $\sigma \in S_{n\setminus\{i\}}$, we have $\partial[i]\tau = (-1)^{d-1}\sigma$, but Lemma 2.2 proves that in this case, $\text{sgn}(\tau) = (-1)^{i-d}\text{sgn}(\sigma)$. Combining these signs, observe that

$$
\partial[i]\text{sgn}(\sigma)\tau = (-1)^{i-1}\text{sgn}(\sigma)\sigma,
$$

independent of $d$.

Hence, the coefficient of $\sigma$ in $\partial[i]\lambda_n^{(k)}$ will be:

$$
(-1)^{k-1} \cdot (-1)^{i-1}\text{sgn}(\sigma) \left( \binom{n + k - j - 1}{n} \cdot (j + 1) \right. 
+ \binom{n + k - j - 2}{n} \cdot (n - j - 1)
\left. \right)
$$

which is equal to

$$
(-1)^{k+i-2}\text{sgn}(\sigma) \left( \binom{n + k - j - 2}{n} \left( \frac{n + k - j - 1}{k - j - 1} \cdot (j + 1) \right. 
+ \frac{k - j - 1}{k - j - 1} \cdot (n - j - 1) \left. \right) \right).
$$

This simplifies to

$$
(-1)^{k+i-2}\text{sgn}(\sigma) \left( \binom{n + k - j - 2}{n} \cdot \frac{kn}{k - j - 1} \right. 
= (-1)^{k+i-2}\text{sgn}(\sigma) \left( \binom{n + k - j - 2}{n - 1} \cdot k. \right).
$$
This latter expression is the coefficient of $\sigma$ in $\lambda_{n-1}^{(k)}$ multiplied by $(-1)^{i-1} \cdot k$. This holds for each $\sigma$, regardless of the number of descents in $\sigma$, so we get

$$\partial[i] \cdot \lambda_{n}^{(k)} = (-1)^{i-1} \cdot k \cdot \lambda_{n-1}^{(k)},$$

confirming identity (1).

We next prove that identity (1) implies identity (2). Applying $\partial[i]$ to both sides of (2.3) gives:

$$(-1)^{k-1}(-1)^{i-1} \cdot k \cdot \lambda_{n-1}^{(k)} = \partial[i] \sum_{j=1}^{n} k^j \epsilon^{(j)}_{n-1}.$$

Applying (2.3) again, to the left-hand side yields,

$$(-1)^{i+k-2} \cdot k \cdot (-1)^{k-1} \sum_{j=1}^{n-1} k^j \epsilon^{(j)}_{n-1} = \sum_{j=1}^{n} k^j \partial[i] \epsilon^{(j)}_{n}.$$

Hence,

$$(-1)^{k} \sum_{j=1}^{n-1} k^j \epsilon^{(j)}_{n-1} = \sum_{j=1}^{n} k^j \partial[i] \epsilon^{(j)}_{n}.$$

So,

$$0 = k \partial[i] \epsilon^{(1)}_{n} + \sum_{j=2}^{n} k^j \left( \partial[i] \epsilon^{(j)}_{n} - (-1)^{i-1} \epsilon^{(j-1)}_{n-1} \right).$$

The fact that this holds for all $k$ implies that each coefficient of the polynomial in $k$ is 0, and so we get $\partial[i] \epsilon^{(1)}_{n} = 0$ and $\partial[i] \epsilon^{(j)}_{n} = (-1)^{i-1} \epsilon^{(j-1)}_{n-1}$.

The second statement in Theorem 2.1 is particularly interesting when compared to a result that appears in the work of Gerstenhaber and Schack [GS]. In that work, the authors show that for the boundary $\delta$ in the usual complex for computing Hochschild homology of a commutative algebra,

$$\delta e^{(k)}_{n} = e^{(k)}_{n-1} \delta$$

for all $n$ and $k$. Note that this bears some similarity to the result we prove in Theorem 2.1 for the simplicial case although in the simplicial case the boundary is applied on only one side and the Hodge index decreases by one rather than being constant.

For each $S \subseteq n$, let $V_{S}$ denote the span of the $v_{i}$ for $i \in S$, and let $M_{S}$ denote the multilinear part of $V_{S} \otimes |S|$. We will continue using the Reiner-Webb point of view so that the injective words on the set $S$ form a basis for $M_{S}$. Note that $\partial(M_{S}) \subseteq \bigoplus_{i \in S} M_{S \setminus \{i\}}$, which means we can decompose $\partial$ as a sum of the operators $\partial[i]$ for $i \in S$.

Suppose $|S| = r$. Then the symmetric group $S_{r}$ acts on $M_{S}$ by permuting the positions in which letters appear in the injective words on $S$. For each $k$ with $1 \leq k \leq r$, let $M_{S}^{(k)}$ denote the image of $e^{(k)}_{r}$ under this action. By
Theorem 2.1 (2), \( \partial[i] \left( M_S^{(k)} \right) \subseteq M_{S \setminus \{i\}} \) for all \( i \in S \). So if we let \( M_r^{(k)} \) denote \( \bigoplus_{|S|=r} M_S^{(k)} \), then
\[
\partial(M_r^{(k)}) \subseteq M_{r-1}^{(k-1)}.
\]
Thus, the complex \((M_*, \partial_*)\) splits as a direct sum of the sub-complexes \( C^{(k)} \) where \( C^{(k)} \) is
\[
0 \to M_n^{(k)} \to M_{n-1}^{(k-1)} \to \cdots \to M_{n-k+1}^{(1)} \to 0.
\]
Let \( H_r^{(k)}(M) \) denote the homology of the subcomplex \( C^{(k)} \). We recall that \( H_r^{(k)}(M) = 0 \) unless \( r = n \).

Notice that the \( S_n \)-action on values which gives rise to the \( S_n \)-module structure studied in [RW] commutes with the \( S_r \) action on positions in injective words in \( M_r \). Thus, it makes sense to study the \( S_n \)-module structure of \( M_r^{(k)} \) for each \( r \) and for \( H_r^{(k)}(M) \), with \( S_n \) acting on values, despite the fact that the Eulerian idempotents act on positions. Our next result determines \( H_r^{(k)}(M) \) as an \( S_n \)-module. To state this result, we will need some notation and results from [Ha].

For each \( \sigma \in S_n \), let \( Z_\sigma \) denote the centralizer of \( \sigma \). In [Ha], a character \( \chi_\sigma \) is defined as the induction of a linear character \( \Psi_\sigma \) from \( Z_\sigma \) to \( S_n \). To describe \( \Psi_\sigma \), we first need a description of the \( Z_\sigma \). Suppose \( \sigma \) consists of \( m_\ell \ell \)-cycles for each \( \ell \). Then \( Z_\sigma \) is the direct product of \( C_\ell \wr S_{m_\ell} \) where \( C_\ell \) is the cyclic group of order \( \ell \) and \( \wr \) denotes wreath product.

Let \( \tau = \prod_\ell (\alpha_\ell; \beta_1, \beta_2, \ldots, \beta_{m_\ell}) \) be an element of \( Z_\sigma \) where \( \alpha_\ell \in S_{m_\ell} \) and each \( \beta_i \) is in \( C_\ell \). Then
\[
\Psi_\sigma(\tau) = \prod_\ell \prod_{i=1}^{m_\ell} \gamma_\ell(\beta_i)
\]
where \( \gamma_\ell \) is the linear character on \( C_\ell \) which assigns \( e^{2\pi i/\ell} \) to the generator of \( C_\ell \).

The following theorem from [Ha] will help us understand \( M_r^{(k)} \):

**Theorem 2.2** (Hanlon). For each \( n \) and \( k \), let \( I_n^{(k)} \) denote the left ideal in \( CS_n \) generated by \( e_n^{(k)} \). As an \( S_n \)-module,
\[
\text{sgn} * I_n^{(k)} = \bigoplus_\sigma \chi_\sigma
\]
where the sum is over a choice of representative from each conjugacy class that consists of permutations with exactly \( k \) cycles, and \(*\) denotes internal product.

The \( S_n \)-modules \( \bigoplus_\sigma \chi_\sigma \) in Theorem 2.2 have also appeared in a completely different context, in work of Bergeron, Bergeron and Garsia on the free Lie algebra [BBG]. In contrast to Theorem 2.2 and [BBG], we will
study \(S_n\)-modules in which we sum over conjugacy classes of derangements rather than conjugacy classes of permutations.

Our next result determines each \(H_n^{(k)}(M)\) as an \(S_n\)-module, thereby providing a refinement of the theorem of Reiner and Webb which gives the \(S_n\)-module structure of \(H_n(M)\).

**Theorem 2.3.** For each \(n\) and \(k\),

\[
\text{sgn} \ast H_n^{(k)}(M) = \bigoplus_{\sigma} \chi_{\sigma}
\]

where \(\ast\) denotes the internal product, and the sum is over a choice of representative from each conjugacy class consisting of derangements with exactly \(k\) cycles. In particular, \(\dim(H_n^{(k)}(M))\) equals the number of derangements with exactly \(k\) cycles.

We will use cycle indices to prove Theorem 2.3. For each \(\sigma \in S_n\), let \(j_i(\sigma)\) denote the number of \(i\)-cycles of \(\sigma\). Let \(a_1, a_2, \ldots\) be a set of commuting indeterminates. Define \(Z(\sigma)\), the cycle indicator of \(\sigma\), to be

\[
Z(\sigma) = a_1^{j_1(\sigma)}a_2^{j_2(\sigma)}a_3^{j_3(\sigma)}\ldots
\]

So \(Z(\sigma)\) is a monomial which identifies the cycle type of \(\sigma\). Thus, \(\sigma\) and \(\tau\) are conjugate in \(S_n\) iff \(Z(\sigma) = Z(\tau)\).

Let \(\Psi\) be a class function on \(C_{S_n}\). The cycle index of \(\Psi\) is

\[
Z(\Psi) = \frac{1}{n!} \sum_{\sigma \in S_n} \Psi(\sigma)Z(\sigma).
\]

Since the monomial \(Z(\sigma)\) uniquely identifies the conjugacy class of \(\sigma\), two class functions are identical if and only if they have the same cycle index.

We will need two results about cycle indices from [Ha]. In the results below, \(\varepsilon_t\) denotes the trivial representation of \(S_t\) (so \(Z(\varepsilon_1) = \frac{1}{t!} \sum_{\sigma \in S_t} Z(\sigma)\)) and \([\cdot]\) denotes the composition product on \(C[[a_1, a_2, \ldots]]^*\), i.e., for \(A, B \in C[[a_1, a_2, \ldots]]\),

\[
A[B] = A(a_i \leftarrow B(a_j \leftarrow a_{ij}))
\]

where \(\leftarrow\) denotes substitution. Recall that \(\mu\) denotes the ordinary number theoretic Möbius function.

The following two results are proved in [Ha]:

**Theorem 2.4** (Hanlon). Let \(\sigma \in S_n\) with \(Z(\sigma) = a_1^{j_1}a_2^{j_2}\ldots a_n^{j_n}\). Then

\[
Z(\chi_{\sigma}) = \prod_{\ell=1}^n Z(\varepsilon_{j_\ell})\left[\frac{1}{\ell} \sum_{d|\ell} \mu(d)\frac{a_{\ell/d}}{d}\right].
\]
Theorem 2.5 (Hanlon). Let $I_n^{(k)}$ denote the left ideal in $CS_n$ generated by $e_n^{(k)}$. Then
\[
\sum_{n,k} Z(I_n^{(k)}) \lambda^k = \prod_{\ell} (1 + (-1)^{\ell} a_{\ell}) \frac{1}{\ell} \sum_{d|\ell} \mu(d) \lambda^{\ell/d}.
\]

We are now ready to give:

Proof of Theorem 2.3. Recall that the Euler characteristic of a chain complex is the alternating sum of the ranks of its homology groups, and the Hopf Trace Formula refines this to a statement about module structure. Since the homology of $(M_\ast, \partial_\ast)$ vanishes except at the top degree, we deduce that
\[
Z(H_n^{(k)}(M)) = \sum_{r=n-k+1}^n Z(M_r^{(k-(n-r))})(-1)^{n-r}.
\]

Note that
\[
M_r = \bigoplus_{|S|=r} M_S = \text{ind}_{S_r \times S_{n-r}} S_r (\text{Reg}_r \otimes \varepsilon_{n-r})
\]
where $\text{Reg}_r$ denotes the right-regular representation of $S_r$. It follows that
\[
M_r^{k-n+r} = \text{ind}_{S_r \times S_{n-r}} S_r (I_r^{(k-n+r)} \otimes \varepsilon_{n-r}).
\]

We will use one other well-known fact about cycle indices, namely that for any virtual characters $\Psi$ of $S_r$ and $\Theta$ of $S_{n-r}$,
\[
Z\left(\text{ind}_{S_r \times S_{n-r}} (\Psi \otimes \Theta)\right) = Z(\Psi)Z(\Theta).
\]

Combining these facts we have:
\[
\sum_n \sum_{k=1}^n Z(H_n^{(k)}(M)) \lambda^k
\]
\[
= \sum_n \sum_{k=1}^n \sum_{r=n-k+1}^n Z(M_r^{(k-(n-r))})(-1)^{n-r} \lambda^k
\]
\[
= \sum_n \sum_{k=1}^n \sum_{r=n-k+1}^n Z(I_r^{(k-n+r)})(\lambda^{k-n+r})(-\lambda^{n-r})Z(\varepsilon_{n-r})
\]
\[
= \left(\sum_{r,t} Z(I_r^{(t)}) \lambda^t\right) \cdot \left(\sum_{m=0}^{\infty} (-\lambda)^m Z(\varepsilon_m)\right)
\]
\[
= \left(\prod_{\ell} (1 + (-1)^{\ell} a_{\ell}) \frac{1}{\ell} \sum_{d|\ell} \mu(d) \lambda^{\ell/d}\right) \cdot \exp\left(\sum_p \frac{(-\lambda)^p a_p}{p}\right)
\]
in the last step using the well-known fact that
\[ \sum_m Z(\varepsilon_m) = \exp \left( \sum_i \frac{a_i}{i} \right). \]
Thus,
\[ \sum_n \sum_{k=1}^n Z(\text{sgn} * H_n^{(k)}(M)) \lambda^k = F_1 \cdot F_2 \]
where
\[ F_1 = \prod_\ell (1 - a_\ell)^{-\frac{1}{\ell} \sum_{d|\ell} \mu(d) \lambda^{\ell/d}} \]
and
\[ F_2 = \exp \left( - \sum_p \frac{\lambda^p a_p}{p} \right). \]
We can rewrite \( F_1 \) as
\[ F_1 = \prod_\ell \exp \left( \ln(1 - a_\ell) \left( \frac{-1}{\ell} \sum_{d|\ell} \mu(d) \lambda^{\ell/d} \right) \right) \]
Letting \( p \) denote \( \ell/d \) and \( n \) denote \( md \), we have
\[ F_1 = \exp \left( \sum_{p,d,m} \frac{1}{mpd} \mu(d) \lambda^p a_d^m \right) \]
\[ = \exp \left( \sum_p \frac{\lambda^p a_p}{p} \right) \left[ \sum_{d,m} \frac{1}{md} \mu(d) a_d^m \right] \]
\[ = \exp \left( \sum_p \frac{\lambda^p a_p}{p} \right) \left[ \sum_n \frac{1}{n} \sum_{d|n} \mu(d) a_d^{n/d} \right] \]
\[ = \exp \left( \sum_p \frac{\lambda^p a_p}{p} \right) \exp \left( \sum_p \frac{\lambda^p a_p}{p} \right) \left[ \sum_{\ell \geq 2} \frac{1}{\ell} \sum_{d|\ell} \mu(d) a_d^{\ell/d} \right]. \]
So,
\[ \sum_{n,k} Z(\text{sgn} * H_n^{(k)}(M)) \lambda^k = \exp \left( \sum_p \frac{\lambda^p a_p}{p} \right) \left[ \sum_{\ell \geq 2} \frac{1}{\ell} \sum_{d|\ell} \mu(d) a_d^{\ell/d} \right], \]
which proves the result. \( \square \)
We conclude this section by showing how to recover Theorem 1.1 from Theorem 2.3. Setting $\lambda = 1$ in Theorem 2.3, we obtain

$$
\sum_n Z(\text{sgn} \ast H_n(M)) = \left( \sum_{n,k} Z(\text{sgn} \ast H_n^{(k)}(M)) \right)
$$

$$
= \exp \left( \sum_p \frac{a_p}{p} \left[ \sum_{\ell \geq 2} \frac{1}{\ell^2} \sum_{d | \ell} \mu(d) a_{d/\ell}^{\ell/d} \right] \right)
$$

$$
= \exp \left( \sum_p \sum_{\ell \geq 2} \frac{1}{\ell^2} \sum_{d | \ell} \mu(d) a_{d/\ell}^{\ell/d} \right).
$$

Letting $u = \ell/d$, we have

$$
\sum_n Z(\text{sgn} \ast H_n(M)) = \exp \left( \sum_{p,d,u} \frac{1}{pd} \mu(d) a_u^{d/p} \right) \cdot \exp \left( - \sum_p \frac{a_p}{p} \right)
$$

where the latter factor accounts for the provision that $\ell$ cannot equal 1. So, substituting $v$ for $pd$ yields

$$
\sum_n Z(\text{sgn} \ast H_n(M)) = \exp \left( \sum_{u,v} \frac{a_v}{uv} \left( \sum_{d | v} \mu(d) \right) \right) \cdot \exp \left( - \sum_p \frac{a_p}{p} \right)
$$

$$
= \exp \left( \sum_u \frac{1}{u} a_u \right) \cdot \exp \left( - \sum_p \frac{a_p}{p} \right).
$$

Thus,

$$
\sum_n Z(H_n(M)) = \left( \frac{1}{1 - a_1} \right) \cdot \exp \left( \sum_p \frac{(-1)^p a_p}{p} \right)
$$

$$
= \left( \sum_k Z(\text{Reg}_k) \right) \cdot \left( \sum_{m=0}^{\infty} (-1)^m Z(\varepsilon_m) \right)
$$

$$
= \sum_n \sum_k Z\left( \text{ind}_{S_{n-k} \times S_k}^{S_n} (\varepsilon_{n-k} \otimes \text{Reg}_k) \right) (-1)^{n-k}
$$

which is the Reiner-Webb Theorem.

3. Signed random to random shuffles.

In recent work, Uyemura-Reyes [Uy] considers random to random shuffling of a deck of $n$ cards and conjectures that the transition matrix, when normalized to have integer entries, also has integer spectrum. For small values of $n$, he notes that the nullspace of the transition matrix has dimension equal to
the number of derangements of \( n \) and that the nullspace, as an \( S_n \)-module, carries the same representation, up to a sign twist, as the representation that appears on the right-hand side of Theorem 1.1. In this section, we explain this phenomenon by studying the Laplacian \( L \) of the complex \((M_*, \partial_*)\). We show that if the normalized random-to-random shuffle operator has integral spectrum (as conjectured in [Uy]), then the Laplacian on each chain group in the complex of injective words will also have integral spectrum.

**Definition 3.1.** For each \( r \), let \( \upsilon_r \) and \( \Upsilon_r \) be the elements of the group algebra \( \mathbb{C}S_r \) given by:

\[
\upsilon_r = r \cdot \text{id} + \sum_{u < v}(v, u, u + 1, \ldots, v - 1) + \sum_{u > v}(v, u, u - 1, \ldots, v + 1)
\]

and

\[
\Upsilon_r = r \cdot \text{id} + \sum_{u < v}(-1)^{v-u}(v, u, u + 1, \ldots, v - 1) + \sum_{u > v}(-1)^{u-v}(v, u, u - 1, \ldots, v + 1).
\]

If we think of \( S_r \) as acting on a deck of \( r \) cards by permuting the positions of the cards, then \( \upsilon_r \) sums permutations which pick at random two positions \( v \) and \( u \) and move the card in position \( v \) to position \( u \). Thus \( \upsilon_r \) is \( r^2 \) times the transition matrix for random to random shuffling. Note that \( \Upsilon_r \) is simply \( \upsilon_r \) twisted by the sign. Thus we will refer to \( \Upsilon_r \) as the *signed random to random shuffle* element in \( \mathbb{C}S_r \).

The following conjecture appears in the dissertation of Uyemura-Reyes:

**Conjecture 3.1 (Uyemura-Reyes).** The eigenvalues of \( \upsilon_n \) are (rational) integers.

As in Section 2, we will use the collection \( B_r \) of injective words of length \( r \) on the alphabet \( \{1, 2, \ldots, n\} \) as a basis for \( M_r \). Let \( \delta_r : M_r \to M_{r+1} \) be the transpose of \( \partial_{r+1} \) with respect to the inner products on \( M_r \) and \( M_{r+1} \) which have \( B_r \) and \( B_{r+1} \) as orthonormal bases. So if \( D \) is the matrix for \( \partial_{r+1} \) with respect to the bases \( B_r \) and \( B_{r+1} \), then \( D^t \) is the matrix for \( \delta_r \) with respect to the same bases. Note that \( \delta_s \) is a coboundary on \( M_s \). We let \( H^s(M) \) denote the cohomology with respect to this coboundary.

Let \( \Lambda_r : M_r \to M_r \) be the Laplacian

\[
\Lambda_r = \delta_{r-1} \cdot \partial_r + \partial_{r+1} \cdot \delta_r.
\]

We recall the well-known fact that a basis for the kernel of the Laplacian \( \Lambda_r \) gives a simultaneous basis for \( H_r(M) \) and \( H^r(M) \).

**Theorem 3.1.** The Laplacian on the top-dimensional chain group satisfies \( \Lambda_n = \Upsilon_n \).
Proof. To apply the coboundary \( \delta_{n-1} \) to a basis element \( j_1j_2\ldots j_{n-1} \), we must sum over all sequences \( i_1i_2\ldots i_n \) with coefficient being the \( j_r \hat{j}_r \) entry from \( \partial_n \). Since \( \partial_n(i_1\ldots i_n) \) is a sum of terms of the form \( \pm j_1\ldots j_{n-1} \) where \( j_1\ldots j_{n-1} \) is obtained by deleting an entry from \( i_1\ldots i_n \), the \( j_r \hat{j}_r \) entry of \( \partial_n \) is 0 unless \( j \) is a subsequence of \( \hat{j} \). It follows that if \( v \) is the single number in \( \{1, 2, \ldots, n\} \) which is missing from \( \{j_1, \ldots, j_{n-1}\} \) then

\[
\delta_{n-1}(j_1\ldots j_{n-1}) = (v_1j_1\ldots j_{n-1}) - (j_1v_2j_2\ldots j_{n-1}) + (j_1j_2v_3\ldots j_{n-1}) - \ldots
\]

So \( \delta_{n-1}\partial_n \) is the operator which acts on a sequence \( i_1i_2\ldots i_n \) by removing an element and re-inserting it in all possible ways. Moreover, if the removed element occupies position \( u \) and it is re-inserted in position \( v \) then the sign of that operation is \( (-1)^{(u-1)+(v-1)} = (-1)^{v-u} \). On the other hand, \( \partial_{n+1}\delta_n = 0 \). It follows that \( \Lambda_n \) is equal to \( \Upsilon_n \) which proves the result. \( \square \)

As noted above, Uyemura-Reyes conjectures that the spectrum of \( v_n \) is integral from which it would follow that the spectrum of \( \Lambda_n \) is integral. We end this section by relating \( \Lambda_r \) to \( \Lambda_n \). From this relationship one can deduce that if Conjecture 3.1 holds, then \( \Lambda_r \) has integral spectrum for all \( r \).

**Theorem 3.2.** Let \( i_1\ldots i_r \) be a basis element of \( M_r \). Let \( A \) denote \( \{i_1, \ldots, i_r\} \) and let \( \overline{A} \) denote the complement of \( A \) in \( \{1, 2, \ldots, n\} \). Then,

\[
\Lambda_r(i_1\ldots i_r) = (r+1)(n-r)I + \Upsilon_r + \sum_{a \in A, b \in \overline{A}} (a,b) (i_1\ldots i_r)
\]

where \( \Upsilon_r \) is acting by permutation of positions on \( i_1\ldots i_r \) whereas \( (a,b) \) in the last summation is acting by permuting the values of the \( i_j \) within the set \( \{1, 2, \ldots, n\} \).

**Proof.** We write \( \Lambda_r(i_1\ldots i_r) \) as a sum of three expressions

\[
\Lambda_r(i_1\ldots i_r) = X + Y + Z
\]

where \( X \) is the sum of all terms in \( \partial_{r+1}\delta_r(i_1\ldots i_r) \) in which \( \delta_r \) inserts an element \( j \) of \( \overline{A} \) in some position \( u \) and then \( \partial_{r+1} \) removes the same number \( j \), where \( Y \) is the sum of all terms in \( \delta_{r+1}\partial_r(i_1\ldots i_r) \) in which \( \partial_r \) removes an element \( j \in A \) and \( \delta_{r+1} \) re-inserts that same element \( j \) and where \( Z \) is the remaining terms in \( \Lambda_r(i_1\ldots i_r) \).

Note that:

\[
X = (r+1)(n-r)
\]

and that:

\[
Y = \Upsilon_r.
\]

It will take some considerable effort now to analyze \( Z \).

The terms \( \tau \) in \( \partial_{r+1}\delta_r(i_1\ldots i_r) \) that contribute to \( Z \) are those in which an element \( j \) from \( \overline{A} \) is inserted into \( i_1\ldots i_r \) at some position \( u \) by \( \delta_r \) and then one of the \( i_\ell \) is removed by \( \partial_{r+1} \). For each such \( \tau \), there is a corresponding
term $\hat{\tau}$ in $\delta_{r-1}\partial_r(i_1\ldots i_r)$ where $i_\ell$ is removed first by $\partial_r$ and then $j$ is inserted in position corresponding to $u$ by $\delta_{r-1}$. It is straightforward to check that $\tau = -\hat{\tau}$ and so these terms cancel.

There is one circumstance in which this cancellation does not eliminate every term. These are the terms $\tau$ where $j$ is inserted immediately behind $i_\ell$, i.e., where $u = \ell + 1$. In this case, the term $\hat{\tau}$ which should cancel $\tau$ is already committed to cancel the term $\tau'$ in which $j$ is inserted immediately in front of $i_\ell$.

For $j \in A$ and $i_\ell \in A$, the term in which $j$ is inserted immediately behind $i_\ell$ and then $i_\ell$ is deleted has sign $+1$ and is obtained by acting on $i_1i_2\ldots i_r$ with the transposition $(j,i_\ell) \in S_n$. The result follows. \[\square\]

It is worth noting that the operator $\Upsilon_r$, acting on positions, commutes with the action of $S_n$ on words of length $r$ in $1,2,\ldots,n$. The first part of Theorem 3.3 alternatively follows from [F] or from the shelling for $K_n$ in [BW].

**Theorem 3.3.** For $r < n$:

1. $\Lambda_r$ is positive definite.
2. If Conjecture 3.1 holds, then the spectrum of $\Lambda_r$ is integral.

**Proof.** For this argument, it will be helpful to reconceptualize $M_r$. Let $i_1i_2\ldots i_r$ be a basis element of $M_r$ and let $\{j_1,\ldots,j_{n-r}\} = A$. We will identify $i_1\ldots i_r$ with $[i_1\ldots i_r] = \frac{1}{(n-r)!} \sum_{\sigma \in S_{n-r}} i_1\ldots i_r j_\sigma_1 j_\sigma_2 \ldots j_\sigma_{n-r} \in M_n$.

The advantage this has is that the operator $\sum_{a \in A, b \in \overline{A}} (a,b)$ whose action seemed to depend on the actual set $A$ can be redefined as the operator:

$$\Gamma = \sum_{a \in \{1,\ldots,r\}, b \in \{r+1,\ldots,n\}} (a,b)$$

where the permutation $(a,b)$ is acting now by permutation of positions. So, $\Lambda_r = ((r+1)(n-r)I) + \Upsilon_r + \Gamma$.

Let $\Omega = ((r+1)(n-r)I) + \Gamma$. Note that $\Omega$ can be written as:

$$\Omega = ((r+1)(n-r)I) + T(1,n) - T(1,r) - T(r+1,n)$$

where $T(u,v) = \sum_{a \leq u < b \leq v} (a,b)$. Recall that

$$M_r = (\text{Reg}_r \otimes \varepsilon_{n-r}) \uparrow_{S_r \times S_{n-r}}^{S_n}$$

where $\text{Reg}_r$ denotes the regular representation of $S_r$. Therefore,

$$M_r = \bigoplus_{\alpha \vdash r} f_\alpha(S^\alpha \otimes \varepsilon_{n-r}) \uparrow_{S_r \times S_{n-r}}^{S_n}$$
where $S^\alpha$ denotes the Specht module indexed by $\alpha$ and $f_\alpha$ is the number of standard Young tableaux of shape $\alpha$.

For $x$ a square in row $i$ and column $j$ of a Ferrer’s diagram of $\alpha$, recall that $c_x$, the content of $x$, is $j - i$. A well-known result from the representation theory of $S_n$ states that for a Specht module $S^\lambda$ with $\lambda \vdash n$, $T_n$ acts as the scalar $\sum_{x \in \lambda} c_x$. It follows that for every $\lambda \vdash n$ which occurs in $(S^\alpha \otimes \varepsilon_{n-r}) \uparrow^{S_n}_{S_r \times S_{n-r}}$, the operator $\Omega$ acts as the scalar:

$$(r + 1)(n - r) + \sum_{x \in \lambda} c_x - \sum_{x \in \alpha} c_x - \sum_{x \in (n-r)} c_x$$

which simplifies to expression (3.2):

$$(3.2) \quad (r + 1)(n - r) + \sum_{x \in \lambda/\alpha} c_x - \left(\frac{n-r}{2}\right).$$

We will make two observations based on this formula. The first is that the eigenvalues of $\Omega$ are integral. Also, both $\Omega$ and $\Upsilon_r$ are easily seen to be diagonalizable. Moreover, they commute. It follows that the eigenvalues of $\Lambda_r = \Upsilon_r + \Omega$ can be written as sums of eigenvalues of $\Upsilon_r$ and $\Omega$. However, $\Upsilon_r$ is conjugate to $\upsilon_r$ and hence has the same spectrum. Thus, if Conjecture 3.1 holds, then all eigenvalues of $\Lambda_r$ are sums of integers. This proves Part (2) of the theorem.

To prove that $\Lambda_r$ is positive definite, first note that it is enough to show that $\Omega$ is positive definite since $\Upsilon_r$ is positive semi-definite, being a direct sum of $\binom{n}{r}$ copies of the Laplacian in top degree for the case with $n = r$. To see that $\Omega$ is positive definite, we start with the expression for the action of $\Omega$ on copies of $S_n$ irreducibles given in (3.2) above. The first observation follows from the fact that $S^\lambda$ has nonzero multiplicity in $(S^\alpha \otimes \varepsilon_{n-r}) \uparrow^{S_n}_{S_r \times S_{n-r}}$ if and only if $\lambda/\alpha$ is a horizontal strip. Let $(\rho_1, \gamma_1), (\rho_2, \gamma_2), \ldots, (\rho_{n-r}, \gamma_{n-r})$ be the coordinates of the squares in $\lambda/\alpha$. The fact that $\lambda/\alpha$ is a horizontal strip implies that $1 \leq \gamma_1 < \gamma_2 \cdots < \gamma_{n-r}$. Thus,

$$\sum_s \gamma_s \geq \left(\frac{n-r}{2}\right) + (n-r).$$

Also, observe that if a square $x$ of the Ferrer’s diagram of $\lambda/\alpha$ is in row $i$, then there are $(i-1)$ squares of $\alpha$ in the rows above it. So, $\sum(\rho_s - 1) \leq r$, i.e., $\sum \rho_s \leq r + (n-r)$.

Putting these bounds together gives that the eigenvalue $\omega$ given in formula (3.2) satisfies

$$\omega \geq (r + 1)(n - r) + \left(\frac{n-r}{2}\right) + (n-r) - (r + (n-r)) - \left(\frac{n-r}{2}\right)$$

which simplifies to:

$$\omega \geq (r + 1)(n - r) - r > 0.$$
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Received January 2, 2003 and revised April 22, 2003. The second author was supported by an NSF postdoctoral research fellowship.

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DENSITY OF TUBE PACKINGS IN HYPERBOLIC SPACE

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Given a hyperbolic manifold $M$ and an embedded tube of radius $r$ about some geodesic, we determine an upper bound on the percentage of the volume of $M$ occupied by the tube.

1. Introduction.

Packing problems have long been a topic of interest. Traditionally, efforts had been focused on Euclidean space, but as interest in hyperbolic space has grown, many of the Euclidean problems have been translated into the hyperbolic arena, in which the problems are almost always vastly more complicated.

The particular packing problem of interest here is a hyperbolic version of packing congruent right circular cylinders in Euclidean space. In Euclidean space, two equivalent ways to define a right circular cylinder are as the set of all points within a fixed distance of a given line or as the union of all lines passing perpendicularly through a given disk. In hyperbolic space, these two concepts are different. We will use the word tube in the former situation and the phrase right circular cylinder in the latter situation. Using this terminology, we are then investigating packings of congruent tubes in hyperbolic space.

Density is perhaps the primary focus in any investigation of packings. Unfortunately, density can be somewhat difficult to define in hyperbolic space, especially when one is dealing with objects of infinite volume. We will simplify the issue by dealing with only a certain class of packings, although the result would likely follow in more general settings, assuming one defined density properly.

Our main result is an upper bound on the density of symmetric packings of congruent tubes of radius $r$ in hyperbolic space. We produce a means of computing the upper bound in arbitrary dimensions, and develop an explicit formula in dimension three. There is no reason to believe that our bounds are sharp, as we make a number of estimates along the way. We note that for the corresponding problem in three-dimensional Euclidean space, there is a sharp bound of $\frac{\pi}{\sqrt{12}}$ [BK90]. In $\mathbb{H}^3$, there is a prior result [MM00a], which provides an upper bound for very large radius tubes and is asymptotically sharp. The result we develop here works well for moderate radius tubes.
However, for small radius tubes, our upper bound on density approaches 1, not the Euclidean upper bound $\frac{\pi}{\sqrt{12}}$ or the suspected hyperbolic limiting case of zero density. A further result [Prz02] deals with the small radius case. This later paper also includes an analysis of densities in a large number of known manifolds.

In Section 7, we also produce various applications to the study of small volume hyperbolic three-manifolds.

2. Dirichlet domains for tube packings.

Defining the density of a packing is often complicated. Since our applications for tube packings all concern tubes in finite volume manifolds, we will simply ignore the complications by dealing with only symmetric packings.

**Definition 2.1.** A symmetric packing of tubes in $\mathbb{H}^n$ is a collection of nonoverlapping congruent tubes subject to the condition that the collection of tubes is preserved by the action of some discrete group $\Gamma \subset \text{Isom}(\mathbb{H}^n)$ where $\mathbb{H}^n/\Gamma$ is a finite volume manifold. The density of the packing is the percentage of the volume of this manifold which is occupied by the projection of the tubes.

Although our definition of density involves a manifold, we do not want to have to determine the manifold to determine density. For our purposes, it will be easier to deal with regions lying in hyperbolic space. Thus we consider one specific fundamental domain for the manifold.

**Definition 2.2.** The Dirichlet domain of a tube $T$ in a symmetric packing is the set of all points which are closer to the axis of $T$ than to the axis of any other tube in the packing.

As in the case of Dirichlet domains for sphere packings, the boundary of the Dirichlet domain will consist of $n - 1$ dimensional manifolds (called faces) which are equidistant from two tubes. The point on a given face which lies on the common perpendicular to the two corresponding tubes is referred to as the center of the face. We note that some faces might not contain a center.

The Dirichlet domain of a tube will, of course, not be a finite volume object since it will contain the tube itself and the tube is of infinite length. Again, resorting to the symmetry, there is some action by translation along the tube, and this action will preserve the Dirichlet domain. This allows us to consider not the entire Dirichlet domain, but just some finite portion of it.

**Definition 2.3.** A fundamental Dirichlet domain for a packing is a fundamental domain for the action of $\Gamma \cap \text{Stab}(T)$ on a Dirichlet domain. We
require that a fundamental Dirichlet domain have a limited type of convexity, specifically, any line segment perpendicular to the axis of $T$ with one endpoint on the axis of $T$ and the other on the boundary of the Dirichlet domain will either lie entirely within the fundamental Dirichlet domain or intersect it in at most a point on the axis of $T$.

A fundamental Dirichlet domain, of course, will have the same volume as the quotient manifold $\mathbb{H}^n/\Gamma$.

Our approach to placing an upper bound on the density of the tube packing will be to place a lower bound on the volume of the region lying within the fundamental Dirichlet domain but outside the tube. We do this in two steps. First, we locate volume which lies near the center of a face and then we locate volume which lies far from the center of a face. In order to determine the density based on the effects of these two contributions, we will have to use a more localized concept of density.

**Definition 2.4.** Let $\Omega$ be a finite area region lying on the boundary of a tube $T$. Let $D$ be a Dirichlet domain for $T$. Consider the set $X_{\Omega} \subset D$ consisting of the union of all line segments which:

i) Have one endpoint on the axis of $T$,
ii) are perpendicular to the axis of $T$,
iii) have one endpoint on $\partial D$, and
iv) pass through $\Omega$.

The density over the region $\Omega$ is defined to be the percentage of the volume of $X_{\Omega}$ which lies in $T$.

There is a simple relationship between the volume of $X_{\Omega} \cap T$ and $\text{Area}(\Omega)$, where $\text{Area}$ is meant to be $n - 1$ dimensional volume. We have three-dimensional applications in mind, so will use terminology that is well-suited there. In $\mathbb{H}^3$ the relationship is $\text{Vol}(X_{\Omega} \cap T) = \frac{1}{2} \tanh r \cdot \text{Area}(\Omega)$.

If one takes the portion of $\partial T$ which lies in a fundamental Dirichlet domain and divides it into various regions $\Omega_i$ then the density of the tube packing will be a weighted average of the densities over the $\Omega_i$, with the weighting given by the areas of the $\Omega_i$. In particular, we shall divide $\partial T$ into regions corresponding to the faces of the Dirichlet domain and then subdivide each of those regions into points near the center of the face and points far from the center of the face. We will then establish upper bounds on the density over those regions. This will establish an upper bound on the density of the packing.

3. Cones in hyperbolic space.

Our effort to develop an upper bound on density for tube packings will start by generalizing a result in [Prz01] which allows us to locate some volume that lies outside of the tubes. First, we define the region in question.
Definition 3.1. Given two nonoverlapping tubes $T_1$ and $T_2$ of radius $r$, we take a ball $B_i$ of radius $r$ lying in $T_i$ with center on the common perpendicular to the axes of $T_1$ and $T_2$. We define the region $W$ to be the set of points which are closer to both $B_1$ and $B_2$ than to any other radius $r$ ball which is disjoint from both $B_1$ and $B_2$.

This construction parallels what was done in [Prz01]. As there, we see that the region $W$ is a union of two right circular cones (when $W$ is nonempty).

Proposition 3.2. Let the distance between the axes of $T_1$ and $T_2$ be $2r+2d$. If $\tanh^2(r+d) < \tanh r \tanh 2r$, then $W$ is nonempty and is the union of two right circular cones.

Proof. Since the argument presented here is essentially identical to the one in [Prz01], we shall omit many of the details. Choose a point $p \in W$. Let $B_3$ be a radius $r$ ball which is disjoint from $B_1$ and $B_2$. It is sufficient to consider the case in which $B_3$ is as close to $p$ as possible. Note that since $p$ lies in $W$, $B_3$ cannot contain $p$. We claim that the optimal position for $B_3$ is for it to be adjacent to both $B_1$ and $B_2$ with its center coplanar with $p$ and the centers of $B_1$ and $B_2$. By taking a cross section in this plane, it is easy to complete the rest of the proof. □

Our interest is in tubes not balls, so we state a similar result involving tubes. From this point, we always assume that $\tanh^2(r+d) < \tanh r \tanh 2r$.

Proposition 3.3. The points in the region $W$ are closer to $T_1$ and $T_2$ than to any other radius $r$ tube which is disjoint from $T_1$ and $T_2$.

Proof. Choose a point $p \in W$. Let $T_3$ be a radius $r$ tube which is disjoint from $T_1$ and $T_2$. Let $B_1$ and $B_2$ be as before and let $B_3$ be the radius $r$ ball in $T_3$ which is closest to $p$. Since $p$ is closer to $B_1$ and $B_2$ than to $B_3$, it is closer to $T_1$ and $T_2$ than to $B_3$. As the point in $T_3$ which is closest to $p$ will lie on the boundary of $B_3$, we see that $p$ is closer to $T_1$ and $T_2$ than to $T_3$. □

Finally, we consider the (nonempty) regions $W_{ij}$ corresponding to all possible pairs of tubes $T_i$ and $T_j$.

Proposition 3.4. The interiors of the regions $W_{ij}$ do not overlap each other.

Proof. Choose a point $p$ in $W_{ij}$. Determine the two tubes which are closest to $p$. These tubes must be $T_i$ and $T_j$. This rules out the possibility that $p$ also lies in $W_{kl}$ where $\{i, j\} \neq \{k, l\}$. □
4. Points near face centers.

The main result of [Prz01] can be used to determine a lower bound on the volume lying outside of a tube and near the center of a face which touches the boundary of the tube. We wish to generalize this to arbitrary faces and also modify it a little to make it easier to estimate density. We start by making some definitions.

**Definition 4.1.** Given a face $f$ of the Dirichlet domain for a tube $T_1$, construct the corresponding region $W$ as in Definition 3.1 where $T_2$ is the tube on the opposite side of $f$. Let $\Sigma$ be the intersection of $\partial W$ with $\partial B_1$. The region $\Sigma$ (if nonempty) will be an $n-2$ sphere (see Figure 1). Project $\Sigma$ orthogonally onto the $n-1$ dimensional hyperplane $\Pi$ passing through $p_1$ perpendicular to the altitude of the cones in $W$. This projection will also be an $n-2$ sphere. Let its radius be $R$.

![Figure 1.](image)

We now construct the region which we shall use to determine an upper bound on density.

**Definition 4.2.** Let $C_1$ be the right circular cylinder whose:

i) Base is the $n-1$ ball bounded by the projection of $\Sigma$ onto $\Pi$,

ii) altitude lies on the (extended) altitude of $W$ and is of length $r+d$.

Let $C_2$ be the corresponding cylinder constructed by exchanging the roles of $T_1$ and $T_2$.

We will show that the set $C = (C_1 \cup C_2) \setminus (T_1 \cup T_2)$ has the desired properties for a density computation. There are several things we need to verify.

**Proposition 4.3.** $C \subset W$. As a result, the only Dirichlet domains that $C$ intersects are the ones for $T_1$ and $T_2$.

**Proof.** Because of the rotational symmetry of $C_i$ about the altitude of $W$, it is sufficient to check this in a two dimensional cross section. We note that we need only verify that $C_1 \cap C \subset W$. 
In two dimensions, we are dealing with the situation illustrated by Figure 2.

Figure 2.

Here $W$ reduces to a union of two isosceles triangles and $C_1$ is a quadrilateral with two right angles. Since the point at which $\partial C_1$ intersects $W$ is at a distance of $r$ from $p_1$ (by definition), we see that $C_1 \setminus T_1 \subset C \setminus B_1 \subset W$.

Because $C \subset W$, the points of $C$ are closer to $T_1$ and $T_2$ than to any other tubes. Thus each point in $C$ lies in either the Dirichlet domain for $T_1$ or the Dirichlet domain for $T_2$. □

At this point, we may partition the face $f$ into regions near its center and regions far from its center. Let $\Omega = C_1 \cap \partial T_1$ and let $X_{\Omega}$ be the corresponding region as in Definition 2.4.

**Proposition 4.4.** $(C \cap D) \subset (X_{\Omega} \setminus T_1)$.

**Proof.** If we extend $C_1$ to a semi-infinite cylinder $C_1^\infty$, it will contain $C_2$. Further, we claim that $((C_1^\infty \cap D) \setminus T_1) \subset (X_{\Omega} \setminus T_1)$. To see this, take a cross section along any 2 dimensional hyperplane perpendicular to the axis of $T_1$. Such a cross section is indicated in Figure 3.

Figure 3.

It is clear that within this cross section, $C \cap D \subset ((C_1^\infty \cap D) \setminus T_1) \subset (X_{\Omega} \setminus T_1)$. □
Of course, we could prove the same thing about $T_2$ and its Dirichlet domain. It is important to note the symmetry of $C$, $W$, $T_1 \cup T_2$, and $f$ under the isometry which swaps $T_1$ and $T_2$, and thus that the portions of $C$ lying in the two Dirichlet domains are congruent, so in particular have the same volume.

**Proposition 4.5.** $\text{Vol}(X_\Omega \setminus T_1) \geq \text{Vol}(C_1 \setminus T_1)$.

**Proof.** Since $(C \cap D) \subset (X_\Omega \setminus T_1)$, $\text{Vol}(X_\Omega \setminus T_1) \geq \text{Vol}(C \cap D) = \frac{1}{2} \text{Vol}(C) = \text{Vol}(C_1 \setminus T_1)$. □

**Proposition 4.6.** The density over the region $\Omega$ is at most

$$\left(1 + \frac{\text{Vol}(C_1 \setminus T_1)}{\text{Vol}(X_\Omega \cap T)}\right)^{-1}.$$

**Proof.**

$$\frac{\text{Vol}(X_\Omega)}{\text{Vol}(X_\Omega \cap T_1)} = 1 + \frac{\text{Vol}(X_\Omega \setminus T_1)}{\text{Vol}(X_\Omega \cap T_1)} \geq 1 + \frac{\text{Vol}(C_1 \setminus T_1)}{\text{Vol}(X_\Omega \cap T)}.$$

The density over $\Omega$ is $\frac{\text{Vol}(X_\Omega \cap T_1)}{\text{Vol}(X_\Omega)}$, yielding the desired result. □

5. Points far from face centers.

In the previous section, we determined an upper bound on the density contributed by points near face centers. We now need to deal with points which are not near face centers. First, we should be specific about which points are under consideration here.

**Definition 5.1.** Given a face $f$, let $\Omega$ be defined as it was in the previous section. Let $\Omega^C$ be the set of points in $\partial T_1 \setminus \Omega$ through which we can produce a line segment which has one endpoint on the axis of $T_1$, has one endpoint in $f$, and is perpendicular to the axis of $T_1$. Denote the union of all such line segments $X_{\Omega^C}$.

As before, we will produce an upper bound on the density of $T_1 \cap X_{\Omega^C}$ within $X_{\Omega^C}$. This will be achieved by placing a lower bound on $\text{Vol}(X_{\Omega^C})$. Specifically, we shall determine a lower bound on the distance from the axis of $T_1$ to points in $f \cap X_{\Omega^C}$. By removing any part of $X_{\Omega^C}$ whose distance from the axis of $T_1$ is greater than this lower bound, we will have reduced $X_{\Omega^C}$ to its intersection with some tube which is coaxial with $T_1$ and of larger radius. It is then easy to compute the relevant volumes.

However, we’d prefer to avoid having to actually compute a distance function on $f$, so we take a somewhat less direct approach. We’ll need to deal with the axes of the tubes here, so let $l_i$ be the axis of $T_i$. 
Proposition 5.2. To determine a lower bound on $\inf_{p \in f \setminus C} \text{dist}(p, l_1)$ it is sufficient to assume that $l_1$ and $l_2$ are coplanar. Let $g_{\text{min}}$ denote the minimum distance in this situation.

Proof. Let $g(p) = \max(\text{dist}(p, l_1), \text{dist}(p, l_2))$. For points $p$ on the face $f$, $g(p) = \text{dist}(p, l_1) = \text{dist}(p, l_2)$. Then

$$\inf_{p \in f \setminus C} \text{dist}(p, l_1) \geq \inf_{p \in \mathbb{R}^n \setminus (T_1 \cup T_2 \cup C)} g(p).$$

If we were to rotate $l_1$ and $l_2$ about their intersections with their common perpendicular then the value of $g(p)$ will be at least the value achieved when $l_1, l_2$ and $p$ are coplanar. Thus, it is sufficient to consider $p$ to be coplanar with $l_1$ and $l_2$. \qed

Proposition 5.3. When $l_1$ and $l_2$ are coplanar, $\inf_{p \in \mathbb{R}^n \setminus (T_1 \cup T_2 \cup C)} g(p)$ occurs at a point on $\partial C$.

Proof. It is sufficient to work within the plane containing $l_1$ and $l_2$. By moving $p$ if necessary, we can reduce $g(p)$ unless $p$ is equidistant from $l_1$ and $l_2$ or $p \in \partial C$. Within this two dimensional setting, the set of points equidistant from $l_1$ and $l_2$ is just a line midway between them. Along this line, $g(p)$ will decrease as $p$ moves closer to the common perpendicular of $l_1$ and $l_2$. Hence, the exceptional case in which $p$ is equidistant from $l_1$ and $l_2$ can be reduced to $p \in \partial C$. \qed

Now, we relate this to a density estimate.

Proposition 5.4. The density of $D$ over $\Omega^C$ is at most the ratio of the volumes of tubes of radius $r$ and $g_{\text{min}}$.

Proof. The density of $D$ over $\Omega^C$ is $\frac{\text{Vol}(X_{\Omega^C \cap T_1})}{\text{Vol}(X_{\Omega^C})}$. Since $X_{\Omega^C}$ is a union of line segments all of length at least $g_{\text{min}}$, its volume is at least as great as the volume of the portion of $X_{\Omega^C}$ which lies within $g_{\text{min}}$ of $l_1$. Since $X_{\Omega^C} \cap T_1$ is the portion of $X_{\Omega^C}$ which lies within $r$ of $l_1$, the ratio of the two volumes is at most the ratio of the volumes of a tube of radius $r$ with a tube of radius $g_{\text{min}}$. \qed

6. Computations.

An upper bound on density is described in the previous sections, but unless we can actually compute the upper bound, it is of little use. Here, we embark upon an effort to evaluate the many expressions involved. Some of the expressions are sufficiently complicated that we approximate them. The resulting upper bound on density is thus not as strong as possible.

As a start, we simply determine the value of $R$, the radius used in constructing the cylinder $C_1$. In order to do this, we will introduce some intermediate variables which we have not yet mentioned.
Let us introduce these variables as we recall how $R$ was produced. Given two balls $B_1$ and $B_2$ of radius $r$ whose centers $p_1$ and $p_2$ are separated by a distance $2r + 2d$, we situated a third radius $r$ ball $B_3$ (center $p_3$) so as to have it tangent to each of the first two. Because of the rotational symmetry involved, we take a cross section along the plane containing the centers of the three balls. Consider the triangle $p_1p_2p_3$, and let $\gamma$ be the angle $p_3p_2p_1$ (which is congruent to angle $p_3p_1p_2$). See Figure 4. Within this triangle, the cross section of the region $W$ is the set of points lying closer to both $p_1$ and $p_2$ than to $p_3$. Of course, this region (if nonempty) will be bounded by the perpendicular bisectors of the segments $p_1p_3$ and $p_2p_3$. Along the bisector of $p_2p_3$ we locate the point $q$ within $W$ (if there is one) which is at a distance of $r$ from $p_1$. Let $\beta$ be the angle $qp_1p_2$. If we project $q$ perpendicularly onto the line perpendicular to $p_1p_2$ through $p_1$ then $R$ is the distance from the projection to $p_1$. If for any reason this construction fails, we set $R = 0$.

Figure 4.

**Proposition 6.1.** If $\tanh r \tanh 2r \geq \tanh(r + d) \tanh(r + 2d)$, then $\beta$ is determined by

$$
\cosh r \cosh 2r - \cos(\gamma - \beta) \sinh r \sinh 2r = \cosh r \cosh(2r + 2d) - \cos \beta \sinh r \sinh(2r + 2d)
$$

and $R$ is determined by $\tanh R = \tanh r \sin \beta$. Otherwise, there is no point $q$ so $R = 0$.

**Proof.** The point $q$ is equidistant from $p_3$ and $p_2$. Using the law of cosines, we can determine the length of the segments $qp_3$ and $qp_2$. Equating these yields the desired expression.

As long as the perpendicular bisector of $p_2p_3$ intersects $p_1p_2$ at a point within $r$ of $p_1$, there will be a point $q$. Constructing a right triangle using the bisector as one leg, half of $p_2p_3$ as the other and a portion of $p_1p_2$ as
the hypotenuse, we find that the point $q$ exists as long as the hypotenuse has length at least $r + 2d$. Using hyperbolic trigonometry, this requires that $\tanh r \cos \gamma \geq \tanh(r + 2d)$. We readily compute that $\cos \gamma = \frac{\tanh(r + d)}{\tanh 2r}$ and thus that $q$ exists as long as $\tanh r \tan h(2r) \geq \tanh(r + d) \tanh(r + 2d)$.

We then determine $R$ by using hyperbolic trigonometry.

(Note: In much of what follows, we shall assume that $R \neq 0$. The results are still true in the case in which $R = 0$, but they are often meaningless. When it matters, we will deal with the $R = 0$ case. Also, $R$ is a function of $r$ and $d$, although we will suppress that in the notation.)

We will occasionally need an upper bound on $\beta$.

**Proposition 6.2.** $\beta \leq \frac{\gamma}{2}$. Equality is achieved only when $d = 0$.

**Proof.** Since $p_1p_2$ is at least as long as $p_1p_3$, if the angle $qp_1p_2$ were larger than the angle $qp_1p_3$, it would follow that $qp_2$ would be longer than $qp_3$. Since $qp_2$ and $qp_3$ have the same length, we see that $\gamma - \beta \geq \beta$. The only case in which $p_1p_2$ and $p_1p_3$ have the same length is $d = 0$.

Although this is not the order in which we worked earlier, it is quicker to determine $g_{\text{min}}$ than to deal with the density over $\Omega$.

**Proposition 6.3.** $\tanh g_{\text{min}} = \cosh R \tanh(r + d)$.

**Proof.** It will be helpful in this argument to refer to Figure 5.

![Figure 5](image-url)

As was shown earlier, determining $g_{\text{min}}$ reduces to a two dimensional computation. We have a pentagon with four right angles, with two unknown but equal sides forming the non-right angle at a vertex we shall call $A$. Across from this angle is the side $p_1p_2$ of length $2r + 2d$. The remaining two sides
have length \( R \). We need to find the point(s) on the two unknown sides which minimize the function \( g \), the larger of the distances to \( p_1 \) and \( p_2 \).

Given the nature of the function \( g \), there are only two types of locations for the minimizing point(s). Either the point is equidistant from \( p_1 \) and \( p_2 \) or it is locally the closest point to either \( p_1 \) or \( p_2 \). We wish to eliminate the second possibility.

Clearly, on one of the sides, \( s_1 \), the right angled vertex minimizes the distance to \( p_1 \) and on the other \( s_2 \), the right angled vertex minimizes the distance to \( p_2 \). However, it should be equally clear that these points are not minima of \( g \). If there is a point on \( s_1 \) which is a local minimum of the distance to \( p_2 \), then the line joining this point to \( p_2 \) would form a right angle with \( s_1 \). That would force the (produced) angle \( p_1Ap_2 \) to be acute.

We shall show that this can’t happen.

The point \( q \) lies on \( s_1 \). Extend the line segment \( p_1q \) until it hits the (extended) bisector of the angle \( A \). This produces a right triangle with \( \beta \) as one angle and the adjacent side of length \( r + d \). The other angle \( \alpha \) will be smaller than half of the angle \( p_1Ap_2 \). Thus it is sufficient to show that \( \alpha \geq \frac{\pi}{4} \).

Form an isosceles triangle by adjoining another copy of this triangle along the leg opposite \( \beta \). This triangle has base of length \( 2r + 2d \), two angles of size \( \beta \) (at \( p_1 \) and \( p_2 \)), and one angle of size \( 2\alpha \). By the law of cosines,

\[
\cos 2\alpha = -\cos^2 \beta + \sin^2 \beta \cosh(2r + 2d)
\]
\[
= -1 + 2 \sin^2 \beta \cosh^2(r + d)
\]
\[
\leq -1 + 2 \sin^2 \frac{\gamma}{2} \cosh^2(r + d)
\]
\[
= -1 + (1 - \cos \gamma) \cosh^2(r + d)
\]
\[
= \sinh^2(r + d) - \frac{\sinh(r + d) \cosh(r + d)}{\tanh 2r}
\]
\[
\leq \sinh^2(r + d) - \sinh(r + d) \cosh(r + d) < 0.
\]

Thus \( \alpha \geq \frac{\pi}{4} \) so we have shown that \( g_{\min} \) can be determined by considering points on \( s_1 \) which are equidistant from \( p_1 \) and \( p_2 \). Of course, the only such point is the vertex \( A \). It is easy then to determine that \( \tanh g_{\min} = \cosh R \tanh(r + d) \).

So far, we have not needed to know the dimension in which we are working. The arguments in the previous sections worked regardless of dimension and the computations have so far been independent of dimension. Unfortunately, the remaining computations involve volumes, which will, of course depend on the dimension. While we do not believe that it would be much more difficult to develop formulas which work in all dimensions, the computations are already fairly complicated in dimension three. Since we produce
no applications of the result in higher dimensions, we restrict ourselves to
dimension three from this point on.

The remaining work involves computing \( \text{Vol}(C_1 \setminus T_1) \) and \( \text{Area}(\Omega) \). It is
not particularly difficult to determine these expressions, but they both end
up being integrals which likely can’t be evaluated in closed form. To simplify
the computations, we shall approximate these expressions. We start with
\( \text{Vol}(C_1 \setminus T_1) \).

Recall that \( C_1 \) is a right circular cylinder of radius \( R \) and height \( r + d \)
and that \( T_1 \) is the set of points which are within \( r \) of some specific line in
the base of \( C_1 \).

**Proposition 6.4.**

\[
\text{Vol}(C_1 \setminus T_1) \geq \int_0^{2\pi} \int_0^R \int_r \tanh^{-1}(\cosh \rho \tanh(r + d)) \sinh \rho \cosh^2 z \ dz \ d\rho \ d\theta.
\]

**Proof.** We perform the computations in a cylindrical coordinate system
\((\rho, \theta, z)\). Specifically, we choose a particular plane in \( \mathbb{H}^3 \) and establish a
polar coordinate system \((\rho, \theta)\) on the plane. For an arbitrary point, \( z \) is the
distance to the plane, and \((\rho, \theta)\) are the coordinates of the perpendicular
projection of the point onto the plane. It is not too difficult to see that the
volume element in this coordinate system is \( \sinh \rho \cosh^2 z \ dz \ d\rho \ d\theta \).

We now take the \( z = 0 \) plane to be the base of \( C_1 \) and the line \( \rho = 0 \)
to be the altitude of \( C_1 \). The “top” of \( C_1 \) is a plane parallel to \( z = 0 \) at a
distance of \( r + d \). We note that this is not the set \( z = r + d \), which is not a
plane. Rather, the top is the set \( z = \tanh^{-1}(\cosh \rho \tanh(r + d)) \) as is easy
to verify. Finally, we need to compute the lower bound on \( z \). Since points
of \( C_1 \setminus T_1 \) are all at least \( r \) from some line in the \( z = 0 \) plane, using \( z = r \)
as a lower bound will only decrease the volume.

The bounds on \( \rho \) and \( \theta \) should be obvious. \( \square \)

We note that this integral can be evaluated in closed form.

Lastly, we must determine the area of \( \Omega \). Before we can do this, we’ll
have to find a parametrization for \( \partial \Omega \), which will, of course, require a choice
of a coordinate system. Since \( \Omega \) lies on \( \partial T_1 \) which bears a natural Euclidean
structure, we shall use that coordinate system. However, some of the inter-
mediate computations will require coordinates on all of \( \mathbb{H}^3 \). We choose to
work in the upper half space model.

**Proposition 6.5.** In the natural Euclidean coordinates on \( \partial T_1 \), the bound-
dary of \( \Omega \) is the parametrized curve

\[
\left( \cosh r \ln \sqrt{\cosh 2R + \cos t \sinh 2R}, \sinh r \sin^{-1} \frac{\coth r \sin t \sinh R}{\sqrt{\cosh 2R + \cos t \sinh 2R}} \right)
\]

where \( t \in [0, 2\pi] \).
Proof. In the upper half space, we shall place the axis of $T_1$ along the positive $x_3$ axis and place the base of $C_1$ in the plane $x_1 = 0$ with its center at $(0, 0, 1)$. It is then easy to see that the boundary of the base is the parametrized curve
$$(0, \sin t \sinh R, \cosh R + \cos t \sinh R)$$
for $t \in [0, 2\pi]$.

The “sides” of $C_1$ are surfaces consisting of line segments passing through this curve perpendicular to the base. In the upper half space model, these lines will be (Euclidean) circles. Because the circles are perpendicular to the $x_1 = 0$ plane (and the $x_3 = 0$ plane), they will be cross sections of (Euclidean) spheres centered at $(0, 0, 1)$. As a function of $t$, the radius of the sphere will be $\sqrt{\cosh 2R + \cos t \sinh 2R}$. Further, on a given circle, the $x_2$ coordinate will be fixed at $\sin t \sinh R$.

We must determine where $C_1$ meets $\partial T_1$. In the upper half space model, $\partial T_1$ will be a (Euclidean) cone with vertex at the origin and vertex angle $\phi = \cos^{-1} \sech r$. Thus, we must find the set of points which satisfy $x_2 = \sin t \sinh R$, are at a distance of $\sqrt{\cosh 2R + \cos t \sinh 2R}$ from $(0, 0, 0)$, and at an angle of $\phi$ from the $x_3$ axis. A simple trigonometric computation shows that the curve
$$(\sqrt{\sin^2 \phi (\cosh 2R + \cos t \sinh 2R) - \sin^2 t \sin^2 R},
\sin t \sinh R, \cos \phi \sqrt{\cosh 2R + \cos t \sinh 2R})$$

is the desired set. Actually, there would be a second copy with a negative $x_1$ value, but we have discarded that as $C_1$ exists on only one side of $x_1 = 0$. We have chosen that to be the positive side.

It is now easy to transfer to the Euclidean coordinates on $\partial T_1$ yielding the indicated curve. \(\square\)

Unfortunately, using this parametrization to compute the area of $\Omega$ would be complicated. We instead approximate the area with the area of a suitably sized ellipse.

**Proposition 6.6.** $\text{Area}(\Omega) \leq \pi R \cosh r \sinh r \sin^{-1} \frac{\tanh R}{\tanh r}$.

Proof. First, we notice that performing a linear transformation on the coordinate system for $\partial T$ will affect $\text{Area}(\Omega)$ only by scaling it. Thus, we scale by $R \cosh r$ in the direction parallel to the axis of $T_1$ and by $\sinh r \sin^{-1} \frac{\tanh R}{\tanh r}$ in the perpendicular direction. The image of $\Omega$ is then bounded by
$$
\left(\frac{1}{R} \ln \sqrt{\cosh 2R + \cos t \sinh 2R}, \frac{\sin^{-1} \frac{\coth r \sin t \sinh R}{\sqrt{\cosh 2R + \cos t \sinh 2R}}}{\sin^{-1} \frac{\tanh R}{\tanh r}}\right)
$$

where $t \in [0, 2\pi]$. 
Letting $x$ be the first coordinate and $y$ the second, we have that

$$\sin^2 t = \left[ \frac{\tanh r}{\sinh R} e^{Rx} \sin \left( y \sin^{-1} \frac{\tanh R}{\tan r} \right) \right]^2$$

$$1 - \cos^2 t = \left[ \frac{\tanh r}{\sinh R} e^{Rx} \sin \left( y \sin^{-1} \frac{\tanh R}{\tan r} \right) \right]^2$$

$$1 - \left( \frac{e^{2Rx} - \cosh 2R}{\sinh 2R} \right)^2 = \left[ \frac{\tanh r}{\sinh R} e^{Rx} \sin \left( y \sin^{-1} \frac{\tanh R}{\tan r} \right) \right]^2$$

$$2e^{2Rx} \cosh 2R - e^{4Rx} - 1 = 4 \left[ \cosh R \tanh r e^{Rx} \sin \left( y \sin^{-1} \frac{\tanh R}{\tan r} \right) \right]^2$$

$$\cosh 2R - \cosh 2Rx = 2 \left[ \cosh R \tanh r \sin \left( y \sin^{-1} \frac{\tanh R}{\tan r} \right) \right]^2$$

$$\sinh^2 R - \sinh^2 Rx = \left[ \cosh R \tanh r \sin \left( y \sin^{-1} \frac{\tanh R}{\tan r} \right) \right]^2$$

$$1 - \frac{\sinh^2 Rx}{\sinh^2 R} = \left[ \frac{\sin \left( y \sin^{-1} \frac{\tanh R}{\tan r} \right)}{\tan R \tanh r} \right]^2 .$$

Thus, the image of $\Omega$ is bounded by a curve of the form $\sinh^2 ax + \sinh^2 by = 1$. One can check that under certain circumstances, including $\sin b \geq \sinh a$, this curve bounds a region whose area is at most $\pi$. Thereafter, one need only check that $\frac{\tanh R}{\tan R} \geq \sinh R$. This places the desired bound on $\text{Area}(\Omega)$.

We are finally in a position to start making specific claims about tube density.

**Proposition 6.7.** The density of a symmetric packing of tubes of radius $r$ in $\mathbb{H}^3$ is at most the larger of

$$\sup_d \left( 1 + \frac{2 \int_{0}^{2\pi} \int_{0}^{R} \int_{r}^{\tanh^{-1}(\cosh \rho \tanh(r+d))} \sinh \rho \cosh^2 z \, dz \, d\rho \, d\theta}{\pi R \sinh^2 r \sin^{-1} \frac{\tanh R}{\tan r}} \right)^{-1}$$

and

$$\sup_d \left( \frac{\sinh^2 r}{\sinh^2 \tanh^{-1}(\cosh R \tanh(r+d))} \right).$$

**Proof.** The density of the tube packing is at most the larger of the density over $\Omega$ and the density over $\Omega^C$. The latter of these should be fairly simple to compute, giving the second of the two functions in the statement of this proposition.

The density over $\Omega$ can be bounded above by the first function by incorporating the various results concerning the volume of $C_1 \setminus T_1$ and the area of $\Omega$. 

If $d$ is large enough that $\tanh r \tanh 2r < \tanh(r + d) \tanh(r + 2d)$, then $R = 0$ so $\Omega$ is empty, making the first expression irrelevant (and incomputable). The second expression simplifies to just $\frac{\sinh^2 r}{\sinh^2 (r + d)}$. \hfill \Box

**Proposition 6.8.** Both of the suprema in Proposition 6.7 are achieved when $d = 0$.

**Proof.** This proof is a long and rather unpleasant computation. Presumably, one could also verify this statement numerically. Rather than reproduce the entire argument here, we shall indicate some of the key steps and leave the rest to the interested reader.

To start, we perform a change of variable $\rho = Ru$ in the triple integral, yielding:

$$
\left(1 + 2 \int_0^{2\pi} \int_0^1 \int_{tanh(\frac{r}{R})}^{tanh(r + d)} \frac{\sinh Ru \cosh^2 z dz du d\theta}{\pi \sinh^2 r \sin^{-1} \frac{\tanh R}{\tanh r}}\right)^{-1}.
$$

To establish that this is maximized when $d = 0$, it would be sufficient to show that

$$
\frac{(\sinh Ru) \int_{\tanh^{-1}(\cosh Ru \tanh(r + d))}^{\tanh^{-1}(\cosh Ru \tanh(r + d))} \cosh^2 z dz}{\sin^{-1} \frac{\tanh R}{\tanh r}}
$$

is minimized when $d = 0$.

This can be evaluated rather easily to give, after rearrangement,

$$
\left(\frac{\sinh Ru}{\sinh R} \left(\frac{\tanh R}{\sin^{-1} \frac{\tanh R}{\tanh r}}\right)\tanh r \cdot \frac{\cosh Ru \tanh(r + d)}{2} \left[\frac{\cosh Ru \tanh(r + d)}{1 - \cosh^2 Ru \tanh^2(r + d)} - \sinh r \cosh r - r\right] + \tanh^{-1}(\cosh Ru \tanh(r + d)) - \sinh r \cosh r - r\right).
$$

After proving that $R$ is a decreasing function of $d$, one sees that most of the factors in the above expression are easily dealt with, with the exception of $\cosh R$ and the bracketed expression. The negative terms in the bracketed expression can be ignored, leaving $\cosh R$ multiplied by a function of $v = \tanh^{-1}(\cosh Ru \tanh(r + d))$. We then factor $\sinh v$ out of the bracketed expression, yielding the product of $\cosh R \sinh v$ and an increasing function of $v$. Showing that $\cosh R \sinh v$ is an increasing function of $d$ then shows that $v$ is also an increasing function of $d$, finishing the proof.

To show that $\cosh R \sinh \tanh^{-1}(\cosh Ru \tanh(r + d))$ is increasing as a function of $d$, we first show that it’s sufficient to assume that $u = 1$. With some fairly minimal computations, one then sees that it is sufficient to show that $\sech^2 R - \tanh^2 (r + d)$ is a decreasing function of $d$. This computation is rather involved so we will stop here. \hfill \Box
Theorem 6.9. The density of a symmetric packing of tubes of radius $r$ in $\mathbb{H}^3$ is at most the larger of
\[
\left(1 + \frac{2[cosh R \tanh^{-1}(cosh R \tanh r) - -(cosh R - 1)\left(\frac{1}{2} \sinh 2r + r\right)]}{R \sinh^2 r \sin^{-1} \frac{\tanh R}{\tanh r}}\right)^{-1}
\]
and
\[
\frac{\sinh^2 r}{\sinh^2 \tanh^{-1}(cosh R \tanh r)}
\]
where $\tanh R = \frac{\sinh r}{2 \cosh^2 r}$. Let $\rho(r)$ denote the value of the larger of these two functions.

Proof. By substituting $d = 0$ in Proposition 6.7 and then evaluating the integral, we get the indicated expression. \qed

It appears to be the case that the former expression is always the larger, although we did not attempt to verify this, beyond plotting the two graphs. We also note that for large $r$, (roughly 7.1 or more), Marshall and Martin’s asymptotic result [MM00a] is better than ours. Figure 6 is a graph of $\rho(r)$ for $r < 3$.

7. Applications.

There are various results concerning tubes in hyperbolic 3-manifolds and at the moment, Agol’s [Ago02] is one of the strongest.

Theorem 7.1 ([Ago02]). Let $M$ be a hyperbolic 3-manifold and let $\gamma$ be a geodesic link in $M$ with an embedded open tubular neighborhood $T$ of radius $r$. Let $M_\gamma$ denote $M \setminus \gamma$ in a complete hyperbolic metric. Then
\[
\text{Vol}(M_\gamma) \leq (\coth r \coth 2r)^{\frac{3}{2}} \left(\text{Vol}(M) + \left(\frac{\coth r}{\coth 2r} - 1\right) \text{Vol}(T)\right).
\]

Agol proceeds by noting that $\text{Vol}(T) \leq \text{Vol}(M)$, thereby producing a relationship between $r$ and the volumes of $M$ and $M_\gamma$. We may now improve this estimate.

Corollary 7.2. Let $M$ be a hyperbolic 3-manifold and let $\gamma$ be a geodesic link in $M$ with an embedded open tubular neighborhood $T$ of radius $r$. Let $M_\gamma$ denote $M \setminus \gamma$ in a complete hyperbolic metric. Then
\[
\text{Vol}(M) \geq (\tanh r \tanh 2r)^{\frac{3}{2}} \text{Vol}(M_\gamma) \left(1 + \rho(r)\left(\frac{\coth r}{\coth 2r} - 1\right)\right)^{-1}.
\]

Proof. $\text{Vol}(T) \leq \rho(r)\text{Vol}(M)$. Then one need only rearrange the terms. \qed

We now use this to improve estimates concerning small volume hyperbolic 3-manifolds.
Figure 6.

**Proposition 7.3.** All orientable hyperbolic 3-manifolds have volume at least 0.324.

*Proof.* Cao and Meyerhoff have shown [CM01] that the minimal volume noncompact orientable hyperbolic 3-manifold has volume 2.0298... The minimal volume orientable hyperbolic 3-manifold is known, by a result of Gabai, Meyerhoff, and Thurston [GMT03], to contain an embedded tube of radius at least $\frac{\log 3}{2}$ about its shortest geodesic. Using our improved version of Agol's result, we have that

$$\text{Vol}(M) \geq \text{Vol}(M_\gamma)(\tanh r \tanh 2r)^\frac{1}{2} \left(1 + \rho(r) \left(\frac{\coth r}{\coth 2r} - 1\right)\right)^{-1}$$

$$\geq 2.0298(\tanh \frac{\log 3}{2} \tanh \log 3)^\frac{1}{2} \left(1 + \rho \left(\frac{\log 3}{2}\right) \left(\frac{\coth \frac{\log 3}{2}}{\coth \log 3} - 1\right)\right)^{-1}$$

$$\geq 0.324.$$
Agol had already established a lower bound of 0.32, so our result represents only a very small improvement. This is in part because our density estimate is weaker for small tube radii. One can see a larger improvement in results concerning large tubes.

**Proposition 7.4.** The shortest geodesic in the smallest volume orientable hyperbolic 3-manifold has length at least $0.184$ and has an embedded tube about it of radius at most $0.946$.

**Proof.** Again, using our modified version of Agol’s result, we can see that if $r > 0.946$ then $\text{Vol}(M) \geq 0.943$, which is greater than the volume of the Weeks manifold. With this knowledge, we then resort to a result of Marshall and Martin [MM00b] which produces a lower bound on geodesic length, given tube radius. For tubes of radius between $\frac{\log 3}{2}$ and $0.946$, we see that geodesic length is at least $0.184$. □

The lower bound on geodesic length has been growing at a rapid pace, but as of now, the previous best known lower bound is $0.162$ [HK02].

**References**


Received August 30, 2002 and revised March 24, 2003.
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SUPERLINEAR PROBLEMS

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We solve elliptic semilinear boundary value problems in which the nonlinear term is superlinear. By weakening the hypotheses, we are able to include more equations than hitherto permitted. In particular, we do not require the superquadracity condition imposed by most authors, and it is not assumed that the region is bounded.

1. Introduction.

Consider the problem
\[ -\Delta u = f(x, u), \ x \in \Omega; \quad u = 0 \text{ on } \partial \Omega, \]
where \( \Omega \subset \mathbb{R}^n \) is a bounded domain whose boundary is a smooth manifold, and \( f(x, t) \) is a continuous function on \( \overline{\Omega} \times \mathbb{R} \). This semilinear Dirichlet problem has been studied by many authors. It is called sublinear if there is a constant \( C \) such that
\[ |f(x, t)| \leq C(|t| + 1), \quad x \in \Omega, \ t \in \mathbb{R}. \]
Otherwise, it is called superlinear. Beginning in [1], almost all researchers studying the superlinear problem assumed:

(a1) There are constants \( c_1, c_2 \geq 0 \) such that
\[ |f(x, t)| \leq c_1 + c_2|t|^s, \]
where \( 0 \leq s < (n + 2)/(n - 2) \) if \( n > 2 \).
(a2) \( f(x, t) = o(|t|) \) as \( t \to 0 \).
(a3) There are constants \( \mu > 2, \ r \geq 0 \) such that
\[ 0 < \mu F(x, t) \leq tf(x, t), \quad |t| \geq r, \]
where \( F(x, t) = \int_0^t f(x, s) ds \).

They proved:

**Theorem 1.1.** Under Hypotheses (a1)-(a3), Problem (1) has a nontrivial weak solution.
Condition \((a_3)\) is convenient, but it is very restrictive. In particular, it implies that there exist positive constants \(c_3, c_4\) such that
\[
F(x, t) \geq c_3 |t|^{\mu} - c_4, \quad x \in \Omega, \ t \in \mathbb{R}.
\]
Although this condition is weaker, it still eliminates many superlinear problems.

A much weaker condition that implies superlinearity is
\((a_3')\) Either
\[
F(x, t)/t^2 \to \infty \quad \text{as} \quad t \to \infty
\]
or
\[
F(x, t)/t^2 \to \infty \quad \text{as} \quad t \to -\infty.
\]

The purpose of the present paper is to explore what happens when \((a_3)\) is replaced with \((a_3')\). Surprisingly, we find the following to be true:

**Theorem 1.2.** Under Hypotheses \((a_1), (a_2)\) and \((a_3')\) the boundary value problem
\[
-\Delta u = \beta f(x, u), \ x \in \Omega; \quad u = 0 \quad \text{on} \quad \partial \Omega,
\]
has a nontrivial solution for almost every positive \(\beta\).

Unfortunately, this theorem does not give any information for any specific \(\beta\). We then turned our attention to solving \((1)\) under a weaker assumption than \((a_3)\). For this purpose we introduced:

\((a_3'')\) There are constants \(\mu > 2, \ r \geq 0\) such that
\[
\mu F(x, t) - tf(x, t) \leq C(t^2 + 1), \quad |t| \geq r.
\]

Note that \((a_3)\) implies both \((a_3')\) and \((a_3'')\), but they are much weaker. We prove:

**Theorem 1.3.** Under Hypotheses \((a_1), (a_2), (a_3')\) and \((a_3'')\) Problem \((1)\) has a nontrivial solution.

It should be noted that the first option in \((a_3')\) together with \((5)\) implies \((3)\).

We also have:

**Theorem 1.4.** If we replace Hypothesis \((a_3'')\) with:
\((a_3''')\) The function
\[
H(x, t) := tf(x, t) - 2F(x, t)
\]
is convex in \(t\),
then Problem \((1)\) has at least one nontrivial solution.
Costa-Magalhães [2] solved (1) under the following assumptions:

\[ |f(x,t)| \leq a_0 |t|^{p-1} + b_0, \quad x \in \Omega, \; t \in \mathbb{R}, \]

\[ \limsup_{|t| \to \infty} \frac{F(x,t)}{|t|^q} \leq b < \infty \quad \text{uniformly for a.e. } x \in \Omega, \]

\[ \liminf_{|t| \to \infty} \frac{H(x,t)}{|t|^{\mu}} \geq a > 0 \quad \text{uniformly for a.e. } x \in \Omega, \]

\[ \limsup_{t \to 0} \frac{2F(x,t)}{t^2} \leq \alpha < \lambda_0 \quad \text{uniformly for a.e. } x \in \Omega, \]

\[ F(x,t)/t^2 \to \infty \quad \text{as } |t| \to \infty, \]

where \( 1 \leq p < 2n/(n-2), \; \mu > n(q-2)/2. \)

Willem-Zou [11] proved a weaker form of Theorem 1.2 for a special case. They do not require Hypothesis (a_3), but they do assume

\[ tf(x,t) \geq 0, \quad t \in \mathbb{R} \]

and

\[ tf(x,t) \geq c_0 |t|^{\mu}, \quad |t| \geq r \]

for some constants \( c_0 > 0, \; r \geq 0, \; \mu > 2. \) Some authors have replaced Hypothesis (a_3) with (3). Although (3) is a more natural assumption, it is still too restrictive to be desirable. It is for this reason that we introduced assumptions (a_3') and (a_3'').

Stronger versions of Theorems 1.2-1.4 will be given in the next section. In them we are not restricted to any particular boundary value problem, and it is not assumed that the region \( \Omega \) is bounded.

### 2. The main theorems.

Many elliptic semilinear problems can be described in the following way: Let \( \Omega \) be a domain in \( \mathbb{R}^n \), and let \( A \) be a selfadjoint operator on \( L^2(\Omega) \). We assume that \( A \geq \lambda_0 > 0 \) and that

(7) \[ C_0^\infty(\Omega) \subset D := D(A^{1/2}) \subset H^{m,2}(\Omega) \]

for some \( m > 0 \), where \( C_0^\infty(\Omega) \) denotes the set of test functions in \( \Omega \) (i.e., infinitely differentiable functions with compact supports in \( \Omega \)) and \( H^{m,2}(\Omega) \) denotes the Sobolev space. If \( m \) is an integer, the norm in \( H^{m,2}(\Omega) \) is given by

(8) \[ \|u\|_{m,2} := \left( \sum_{|\mu| \leq m} \|D^\mu u\|^2 \right)^{1/2}. \]

Here \( D^\mu \) represents the generic derivative of order \( |\mu| \) and the norm on the right-hand side of (8) is that of \( L^2(\Omega) \). If \( m \) is not an integer, there are
several ways of defining the space $H^{m,2}(\Omega)$, all of which are equivalent. We shall not assume that $m$ is an integer.

A typical example of an operator $A$ satisfying these hypotheses is a second order elliptic operator with smooth coefficients applied to functions satisfying zero Dirichlet boundary conditions on a smooth bounded domain in $\mathbb{R}^n$. Only the abstract properties listed above are relevant to our analysis.

Let $q$ be a number satisfying
\begin{align}
2 < q &\leq 2n/(n - 2m), \\
2m < n &\quad 2 < q < \infty, \\
2m < n &\quad n \leq 2m,
\end{align}
and let $f(x, t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$. This means that $f(x, t)$ is continuous in $t$ for a.e. $x \in \Omega$ and measurable in $x$ for every $t \in \mathbb{R}$.

We consider the problem
\begin{align}
Au = f(x, u), \quad u \in D.
\end{align}
By a solution of (10) we shall mean a function $u \in D$ such that
\begin{align}
(u, v)_D = (f(\cdot, u), v), \quad v \in D.
\end{align}
If $u$ is a solution of (11) and $f(x, u)$ is in $L^2(\Omega)$, then $u$ is in $D(A)$ and solves (10) in the classical sense. Otherwise we call it a weak (or semistrong) solution.

We make the following assumptions:

(A) The function $f(x, t)$ satisfies
\begin{align}
|f(x, t)| &\leq V(x)^q(|t|^{q-1} + 1)
\end{align}
and
\begin{align}
f(x, t)/V(x)^q = o(|t|^{q-1}) \quad &\text{as } |t| \to \infty,
\end{align}
where $V(x) > 0$ is a function in $L^q(\Omega)$ such that
\begin{align}
\|Vu\|_q &\leq C\|u\|_D, \quad u \in D.
\end{align}
Here
\begin{align}
\|u\|_q &:= \left(\int_{\Omega} |u(x)|^q dx\right)^{1/q}, \\
\|u\|_D &:= \|A^{1/2}u\|,
\end{align}
and $q' = q/(q - 1)$. If $\Omega$ and $V(x)$ are bounded, then (14) will hold automatically by the Sobolev inequality. However, there are functions $V(x)$ which are unbounded and such that (14) holds even on
unbounded regions $\Omega$ (cf., e.g., [4]). With the norm (16), $D$ becomes a Hilbert space.

(B) The point $\lambda_0$ is an isolated simple eigenvalue with a bounded eigenfunction $\varphi_0(x) \neq 0$ a.e. in $\Omega$.

(C) There is a $\delta > 0$ such that

$$2F(x, t) \leq \lambda_0 t^2, \quad |t| \leq \delta, \quad x \in \Omega,$$

where

$$F(x, t) := \int_0^t f(x, s) ds.$$

(D) There is a function $W(x) \in L^1(\Omega)$ such that either

$$W(x) \leq F(x, t)/t^2 \to \infty \quad \text{as} \quad t \to \infty, \quad x \in \Omega$$

or

$$W(x) \leq F(x, t)/t^2 \to \infty \quad \text{as} \quad t \to -\infty, \quad x \in \Omega.$$

(The function $W(x)$ need not be positive.)

(E) There are constants $\mu > 2, C \geq 0$ such that

$$[\mu F(x, t) - tf(x, t)]/(t^2 + 1) \leq C, \quad t \in \mathbb{R}, \quad x \in \Omega.$$

We shall prove:

**Theorem 2.1.** Under the above hypotheses, the problem

$$Au = f(x, u), \ u \in D$$

has at least one nontrivial solution.

We also have:

**Theorem 2.2.** Replace Hypothesis (E) with:

(E') The function

$$H(x, t) := tf(x, t) - 2F(x, t)$$

is convex in $t$.

Then Problem (18) has at least one nontrivial solution.

Problem (18) is called sublinear if $f(x, t)$ satisfies

$$|f(x, t)| \leq C(|t| + 1), \quad x \in \Omega, \ t \in \mathbb{R}.\quad (20)$$

Otherwise it is called superlinear. Hypothesis (D) requires (18) to be superlinear.

Problem (18) has been studied by many people. The vast majority of results obtained concern sublinear problems. Much less has been proved for the superlinear case. In [1] the basic assumption was

$$0 < \mu F(x, t) \leq tf(x, t), \quad |t| \geq r\quad (21)$$
for some $\mu > 2$ and $r \geq 0$. This is a very convenient hypothesis since it readily achieves mountain pass geometry as well as satisfaction of the Palais-Smale condition. However it is a severe restriction; it strictly controls the growth of $f(x,t)$ as $|t| \to \infty$. Almost every author discussing superlinear problems has made this assumption. We have been able to weaken this assumption considerably, but not to our complete satisfaction. We assume either that

$$\mu F(x,t) - tf(x,t) \leq C(t^2 + 1), \quad |t| \geq r$$

for some $\mu > 2$ and $r \geq 0$ or that (19) is convex in $t$. These allow much more freedom for the function $f(x,t)$. But they do not allow as much freedom as we would like.

If we drop Hypothesis (E) completely, then we are able to prove the following theorems:

**Theorem 2.3.** If we replace Hypotheses (C) and (D) with:

(C') There are a $\delta > 0$ and a $\tilde{\lambda} > \lambda_0$ such that

$$2F(x,t) \geq \tilde{\lambda}t^2, \quad |t| \leq \delta, \quad x \in \Omega,$$

and

(D') there is a function $W(x) \in L^1(\Omega)$ such that

$$W(x) \geq P(x,t) \to -\infty \quad \text{as} \quad |t| \to \infty, \quad x \in \Omega,$$

where

$$P(x,t) := F(x,t) - \frac{1}{2}\lambda_0 t^2,$$

(22)

and drop Hypothesis (E), then Problem (18) has at least one nontrivial solution.

We also have:

**Theorem 2.4.** Assume that (A)-(D) hold. Then for almost every $\beta \in (0,1)$, the equation

$$Au = \beta f(x,u)$$

(23)

has a nontrivial solution. In particular, the eigenvalue problem (23) has infinitely many solutions.

**Theorem 2.5.** Replace Hypothesis (C) in Theorem 2.4 with:

(C'') There are $\delta > 0$ and $\tilde{\lambda} \leq \lambda_0$ such that

$$2F(x,t) \leq \tilde{\lambda}t^2, \quad |t| \leq \delta, \quad x \in \Omega,$$

and (D) with:
(D′′) Either
\[ \int_{\Omega} F(x, R\varphi_0) \, dx / R^2 \to \infty \text{ as } R \to \infty \]
or
\[ \int_{\Omega} F(x, -R\varphi_0) \, dx / R^2 \to \infty \text{ as } R \to \infty. \]

Then (23) has a nontrivial solution for almost every \( \beta \in (0, \lambda_0 / \widetilde{\lambda}) \).

**Corollary 2.6.** Replace Hypothesis (C′′) in Theorem 2.5 with:

(C′′′) \( F(x, t) / t^2 \to 0 \) uniformly as \( t \to 0 \).

Then (23) has a nontrivial solution for almost every \( \beta \in (0, \infty) \).

The method (called the monotonicity trick) which allows one to solve (23) for almost all values of \( \beta \) in some interval was first introduced by Struwe [8] for minimization problems. It was applied by Jeanjean [3] and others for various types of problems.

### 3. Preliminaries.

Define
\begin{equation}
G(u) := \|u\|^2_D - 2 \int_{\Omega} F(x,u) \, dx.
\end{equation}

Under Hypothesis (A), it is known that \( G \) is a continuously differentiable functional on the whole of \( D \). In fact, the following were proved in [5, pp. 56-58]:

**Proposition 3.1.** Under Hypothesis (A), \( F(x,u(x)) \) and \( v(x)f(x,u(x)) \) are in \( L^1(\Omega) \) whenever \( u, v \in D \).

**Proposition 3.2.** \( G(u) \) has a Fréchet derivative \( G'(u) \) on \( D \) given by
\begin{equation}
(G'(u), v)_D = 2(u, v)_D - 2(f(\cdot, u), v).
\end{equation}

**Proposition 3.3.** The derivative \( G'(u) \) given by (25) is continuous in \( u \).

**Theorem 3.4.** Under Hypotheses (A)-(C), the following alternative holds:

Either:

(a) There is an infinite number of \( y(x) \in D(A) \setminus \{0\} \) such that
\begin{equation}
Ay = f(x, y) = \lambda_0 y,
\end{equation}
or
(b) for each \( \rho > 0 \) sufficiently small, there is an \( \varepsilon > 0 \) such that
\begin{equation}
G(u) \geq \varepsilon, \quad \|u\|_D = \rho.
\end{equation}
4. Proofs.

Proof of Theorem 2.1. We take

\[ G(u) = \|u\|^2_D - 2 \int \Omega F(x, u) dx. \] (28)

Under our hypotheses, Propositions 3.1-3.3 apply, and

\[ (G'(u), v) = 2(u, v)_D - 2(f(\cdot, u), v), \quad u, v \in D. \] (29)

By Theorem 3.4 we see that there are positive constants \( \varepsilon, \rho \) such that

\[ G(u) \geq \varepsilon, \quad \|u\|_D = \rho \] (30)

unless

\[ Au = \lambda_0 u = f(x, u), \quad u \in D \setminus \{0\} \] (31)

has a solution. This would give a nontrivial solution of (18). We may therefore assume that (30) holds. Next we note that

\[ G(\pm R\phi_0)/R^2 = \|\phi_0\|^2_D - 2 \int \Omega \{F(x, \pm R\phi_0)/R^2\} \phi_0^2 dx \to -\infty \quad \text{as} \quad R \to \infty \]

by Hypothesis (D), since \( \phi_0 \neq 0 \) a.e. Since \( G(0) = 0 \) and (30) holds, we can now apply the usual mountain pass theorem (cf., e.g., [5, p. 22]) to conclude that there is a sequence \( \{u_k\} \subset D \) such that

\[ G(u_k) \to c \geq \varepsilon, \quad G'(u_k) \to 0. \] (32)

Then

\[ G(u_k) = \rho_k^2 - 2 \int F(x, u_k) dx \to c \] (32)

and

\[ (G'(u_k), u_k) = 2\rho_k^2 - 2(f(\cdot, u_k), u_k) = o(\rho_k), \] (33)

where \( \rho_k = \|u_k\|_D \). Assume that \( \rho_k \to \infty \), and let \( \tilde{u}_k = u_k/\rho_k \). Since \( \|\tilde{u}_k\|_D = 1 \), there is a renamed subsequence such that \( \tilde{u}_k \to \tilde{u} \) weakly in \( D \), strongly in \( L^2_{loc}(\Omega) \) and a.e. in \( \Omega \). By (32),

\[ \int \Omega \frac{2F(x, u_k)}{u_k^2} \tilde{u}_k^2 dx \to 1. \]

Let

\[ \Omega_1 = \{x \in \Omega : \tilde{u}(x) \neq 0\}, \quad \Omega_2 = \Omega \setminus \Omega_1. \]

Then

\[ \frac{2F(x, u_k)}{u_k^2} \tilde{u}_k^2 \to \infty, \quad x \in \Omega_1 \]

by Hypothesis (D). If \( \Omega_1 \) has positive measure, then

\[ \int \Omega \frac{2F(x, u_k)}{u_k^2} \tilde{u}_k^2 dx \geq \int_{\Omega_1} \frac{2F(x, u_k)}{u_k^2} \tilde{u}_k^2 dx + \int_{\Omega_2} W(x) dx \to \infty. \]
Thus, the measure of $\Omega_1$ must be 0, i.e., we must have $\tilde{u} \equiv 0$ a.e. Moreover,

$$\int_{\Omega} \frac{\mu F(x,u_k) - u_k f(x,u_k)}{u_k^2} \tilde{u}_k^2 \, dx \to \frac{\mu}{2} - 1.$$  

But by Hypothesis (E),

$$\limsup \frac{\mu F(x,u_k) - u_k f(x,u_k)}{u_k^2} \tilde{u}_k^2 \leq \limsup Cu_k^2 + 1 \tilde{u}_k^2 = 0,$$

which implies that $(\mu/2) - 1 \leq 0$, contrary to assumption. Hence, the $\rho_k$ are bounded. We can now follow the usual procedures to obtain a weak solution of (18) satisfying $G(u) = c \geq \varepsilon$ (cf., e.g., [5, p. 64]). Since $G(0) = 0$, we see that $u \neq 0$. This completes the proof. □

We postpone the proof of Theorem 2.2 until the next section.

In proving Theorem 2.3, we shall make use of:

**Lemma 4.1.** Under Hypothesis (C'), there is an $\alpha \neq 0$ such that $G(\alpha \varphi_0) < 0$.

**Proof.** We can assume that

$$\|\varphi_0\|_D = 1.$$  

Thus,

$$G(\alpha \varphi_0) = \alpha^2 - 2 \int_{\Omega} F(x,\alpha \varphi_0) \, dx$$

$$\leq \alpha^2 - \tilde{\lambda} \alpha^2 \int_{|\alpha \varphi_0(x)| < \delta} \varphi_0(x)^2 \, dx$$

$$+ \int_{|\alpha \varphi_0(x)| \geq \delta} V'(|\alpha \varphi_0|^q + |\alpha \varphi_0|)$$

$$\leq \alpha^2 - \tilde{\lambda} \alpha^2 \|\varphi_0\|^2 + C|\alpha|^q \|V \varphi_0\|_q^q$$

$$\leq \alpha^2[1 - (\tilde{\lambda}/\lambda_0) + C'|\alpha|^{q-2}].$$

This can be made negative by taking $\alpha$ sufficiently small. □

**Lemma 4.2.** Under Hypothesis (D'),

$$G(u) \to \infty \text{ as } \|u\|_D \to \infty.$$  

**Proof.** Suppose there is a sequence $\{u_k\} \subset D$ such that $\rho_k = \|u_k\| \to \infty$ and

$$G(u_k) \leq K.$$  

Write

$$u_k = w_k + \alpha_k \varphi_0, \quad \tilde{u}_k = u_k/\rho_k, \quad \tilde{w}_k = w_k/\rho_k, \quad \tilde{\alpha}_k = \alpha_k/\rho_k,$$
where $w_k \perp \varphi_0$. If $\lambda_1 > \lambda_0$ is the next point in the spectrum of $A$, then
\[ \lambda_1 \| w \|^2 \leq \| w \|_D^2, \quad w \perp \varphi_0. \]
Thus
\[
G(u_k) = \| u_k \|_D^2 - \lambda_0 \| u_k \|^2 - 2 \int_{\Omega} P(x, u_k) \, dx \\
\geq \left( 1 - \frac{\lambda_0}{\lambda_1} \right) \| w_k \|_D^2 - 2 \int_{\Omega} P(x, u_k) \, dx \\
\geq \left( 1 - \frac{\lambda_0}{\lambda_1} \right) \| w_k \|_D^2 - 2 \int_{\Omega} W(x) \, dx.
\]
The only way this would not converge to $\infty$ is if $\| w_k \|_D$ is bounded. But then $\| \tilde{w}_k \|_D \to 0$, and $|\tilde{\alpha}_k| \to 1$. Since $\| \tilde{u}_k \|_D = 1$, there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in $D$, strongly in $L^2_{\text{loc}}(\Omega)$ and a.e. in $\Omega$. Since $\tilde{w} = 0$ and $|\tilde{\alpha}| = 1$, we have $\tilde{u}(x) = \tilde{\alpha} \varphi_0(x) \neq 0$ a.e. Hence, $|u_k(x)| = \rho_k |\tilde{u}_k(x)| \to \infty$ a.e. Consequently,
\[
\int_{\Omega} P(x, u_k) \, dx \to -\infty,
\]
showing that $G(u_k) \to \infty$. This completes the proof. \(\square\)

We can now give:

**Proof of Theorem 2.3.** Let
\[
m = \inf_D G.
\]
Then there is a sequence $\{u_k\} \subset D$ such that $G(u_k) \to m$. In view of Lemma 4.2, we must have $\| u_k \|_D \leq C$. Thus, there is a renamed subsequence such that $u_k \to u$ weakly in $D$, strongly in $L^2_{\text{loc}}(\Omega)$ and a.e. in $\Omega$. Now,
\[
G(u) = \| u \|_D^2 - 2 \int_{\Omega} F(x, u) \, dx \\
= \| u_k \|_D^2 - 2([u_k - u]_D, u)_D - \| u_k - u \|_D^2 \\
- 2 \int_{\Omega} F(x, u_k) \, dx + 2 \int_{\Omega} [F(x, u_k) - F(x, u)] \, dx \\
\leq G(u_k) - 2([u_k - u]_D, u)_D + 2 \int_{\Omega} [F(x, u_k) - F(x, u)] \, dx.
\]
From our hypotheses, it follows that
\[
\int_{\Omega} F(x, u_k) \, dx \to \int_{\Omega} F(x, u) \, dx
\]
(cf., e.g., [5, p. 64]). We therefore have in the limit $G(u) \leq m$, from which we conclude that $G(u) = m$ and $G'(u) = 0$. Hence, $u$ is a weak solution of (10). We see from Lemma 4.1 that $m < 0$. Since $G(0) = 0$, we see that $u \neq 0$. This completes the proof. \(\square\)
5. The eigenvalue problem.

In this section we shall give the proofs of Theorems 2.4, 2.5 and 2.2. They will be based on the following result given in [7]. Let $E$ be a reflexive Banach space with norm $\|\cdot\|$, and let $A, B$ be two closed subsets of $E$. Suppose that $G \in C^1(E, \mathbb{R})$ is of the form: 

$$G(u) := I(u) - J(u), \quad u \in E,$$

where $I, J \in C^1(E, \mathbb{R})$ map bounded sets to bounded sets. Define

$$G_\lambda(u) = \lambda I(u) - J(u), \quad \lambda \in \Lambda,$$

where $\Lambda$ is an open interval contained in $(0, +\infty)$. Assume one of the following alternatives holds:

(H$_1$) $I(u) \geq 0$ for all $u \in E$ and either $I(u) \to \infty$ or $|J(u)| \to \infty$ as $\|u\| \to \infty$.

(H$_2$) $I(u) \leq 0$ for all $u \in E$ and either $I(u) \to -\infty$ or $|J(u)| \to \infty$ as $\|u\| \to \infty$.

Furthermore, we suppose that:

(H$_3$) $a_0(\lambda) := \sup_A G_\lambda \leq b_0(\lambda) := \inf_B G_\lambda$, for any $\lambda \in \Lambda$.

We let $\Phi$ be the set of mappings $\Gamma(t) \in C(E \times [0, 1], E)$ with the following properties:

a) for each $t \in [0, 1], \Gamma(t)$ is a homeomorphism of $E$ onto itself and $\Gamma(t)^{-1}$ is continuous on $E \times [0, 1]$;

b) $\Gamma(0) = I$;

c) for each $\Gamma(t) \in \Phi$ there is a $u_0 \in E$ such that $\Gamma(1)u = u_0$ for all $u \in E$

and $\Gamma(t)u \to u_0$ as $t \to 1$ uniformly on bounded subsets of $E$.

A subset $A$ of $E$ links a subset $B$ of $E$ if $A \cap B = \emptyset$ and, for each $\Gamma(t) \in \Phi$, there is a $t \in (0, 1]$ such that $\Gamma(t)A \cap B \neq \emptyset$.

We have:

**Theorem 5.1.** Assume that (H$_1$) (or (H$_2$)) and (H$_3$) hold.

1. If $A$ links $B$ and $A$ is bounded, then for almost all $\lambda \in \Lambda$ there exists $u_k(\lambda) \in E$ such that $\sup_k \|u_k(\lambda)\| < \infty, G_\lambda'(u_k(\lambda)) \to 0$ and

$$G_\lambda(u_k(\lambda)) \to a(\lambda) := \inf_{\Gamma \in \Phi} \sup_{s \in [0, 1], u \in A} G_\lambda(\Gamma(s, u)), \quad k \to \infty.$$

Furthermore, if $a(\lambda) = b_0(\lambda)$, then $\operatorname{dist}(u_k(\lambda), B) \to 0, \quad k \to \infty$.

2. If $B$ links $A$ and $B$ is bounded, then for almost all $\lambda \in \Lambda$ there exists $v_k(\lambda) \in E$ such that $\sup_k \|v_k(\lambda)\| < \infty, G_\lambda'(v_k(\lambda)) \to 0$ and

$$G_\lambda(v_k(\lambda)) \to b(\lambda) := \sup_{\Gamma \in \Phi} \inf_{s \in [0, 1], v \in B} G_\lambda(\Gamma(s, v)), \quad k \to \infty.$$

Furthermore, if $a_0(\lambda) = b(\lambda)$, then $\operatorname{dist}(v_k(\lambda), A) \to 0, \quad k \to \infty$.

We shall also need the following extension of Theorem 3.4:
Theorem 5.2. Let \( \lambda \) be a parameter satisfying \( 1 < \lambda \leq K < \infty \). Under Hypotheses (A)-(D), for each \( \rho > 0 \) sufficiently small (not depending on \( \lambda \)), we have
\[
G_\lambda(u) := \lambda \|u\|_D^2 - 2 \int_{\Omega} F(x,u)dx \geq (\lambda - 1)\rho^2, \quad \|u\|_D = \rho.
\]
If we replace Hypothesis (C) with Hypothesis (C''), assuming \( 1 < \tilde{\lambda}/\lambda_0 < \lambda \leq K < \infty \), then we have
\[
G_\lambda(u) \geq \left( \lambda - \frac{\tilde{\lambda}}{\lambda_0} \right) \rho^2, \quad \|u\|_D = \rho.
\]
Proof. Let \( \lambda_1 > \lambda_0 \) be the next point in the spectrum of \( A \), and let \( N_0 \) denote the eigenspace of \( \lambda_0 \). We take \( M = N_0^\perp \cap D \). By Hypothesis (B), there is a \( \rho > 0 \) such that
\[
\|y\|_D \leq \rho \Rightarrow |y(x)| \leq \delta/2, \quad y \in N_0.
\]
Now suppose \( u \in D \) satisfies
\[
|u|_D \leq \rho \text{ and } |u(x)| \geq \delta
\]
for some \( x \in \Omega \). We write
\[
u = w + y, \quad w \in M, \quad y \in N_0.
\]
Then for those \( x \in \Omega \) satisfying (38) we have
\[
\delta \leq |u(x)| \leq |w(x)| + |y(x)| \leq |w(x)| + (\delta/2).
\]
Hence
\[
|y(x)| \leq \delta/2 \leq |w(x)|,
\]
and consequently,
\[
|u(x)| \leq 2|w(x)|
\]
for all such \( x \). Now we have by (12) and (14)
\[
G_\lambda(u) \geq \lambda \|u\|_D^2 - \lambda_0 \int_{|u|<\delta} u^2dx - C \int_{|u|>\delta} (|Vu|^q + V^q|u|)dx
\]
\[
\geq \lambda \|u\|_D^2 - \lambda_0 \|u\|^2 - C' \int_{|u|>\delta} |Vu|^qdx
\]
\[
\geq (\lambda - 1)\|y\|_D^2 + \lambda \|w\|_D^2 - \lambda_0 \|w\|^2 - C'' \int_{|w|>\delta} |Vw|^qdx
\]
in view of the fact that \( \|y\|_D^2 = \lambda_0 \|y\|^2 \) and (41) holds. Thus, by (14),
\[
G_\lambda(u) \geq (\lambda - 1)\|y\|_D^2 + \left( \frac{\lambda_0}{\lambda_1} - C''\|w\|_D^{q-2} \right) \|w\|_D^2, \quad \|u\|_D \leq \rho.
\]
We take $\rho > 0$ to satisfy

$$1 - \frac{\lambda_0}{\lambda_1} > C''' \rho^{q-2}.$$  

This gives

$$G_\lambda(u) \geq (\lambda - 1)\rho^2 + \left(\lambda - \frac{\lambda_0}{\lambda_1} - C''' \rho^{q-2} - \lambda + 1\right) \|w\|_D^2$$

$$\geq (\lambda - 1)\rho^2, \quad \|u\|_D = \rho.$$  

Hence, (36) holds.

To prove (37) under Hypothesis $(C''')$, let $\eta = \tilde{\lambda}/\lambda_0$ and $A = (\eta, K)$. Under Hypothesis $(C''')$ we have in place of (42)

$$G_\lambda(u) \geq (\lambda - \eta)\|y\|_D^2 + \left(\lambda - \frac{\tilde{\lambda}}{\lambda_1} - C''' \|w\|_D^{q-2}\right) \|w\|_D^2, \quad \|u\|_D \leq \rho.$$  

We take $\rho > 0$ to satisfy

$$\eta - \frac{\tilde{\lambda}}{\lambda_1} > C''' \rho^{q-2}.$$  

Consequently,

$$G_\lambda(u) \geq (\lambda - \eta)\rho^2 + \left(\lambda - \frac{\tilde{\lambda}}{\lambda_1} - C''' \rho^{q-2} - \lambda + \eta\right) \|w\|_D^2$$

$$\geq (\lambda - \eta)\rho^2, \quad \|u\|_D = \rho.$$  

This gives (37), and the proof is complete. \qed

We now turn to the proofs of Theorems 2.4 and 2.5. We prove the latter first. We shall prove Theorem 2.5 by applying Theorems 5.1 and 5.2.

Proof. We take $E = D$, $A = (\eta, K)$, where $\eta = \tilde{\lambda}/\lambda_0$, $K > 1$ is a finite number, and

$$I(u) = \|u\|_D^2, \quad J(u) = 2 \int_{\Omega} F(x, u) \, dx.$$  

For the purpose of this application, it is sufficient to know that the sets

$$A_{\pm} = [0, \pm R\varphi_0], \quad B = \{x \in D : \|x\|_D = \rho\}$$

link each other if $R > \rho$ (cf., e.g., [5]). In our case Hypothesis $(H_1)$ is satisfied. We now check that $(H_3)$ holds. We observe that

$$G_\lambda'(u) = 0$$

is equivalent to (23) with $\beta = 1/\lambda$. Now, at least one of the expressions

$$J(\pm R\varphi_0)/R^2 = 2 \int_{\Omega} F(x, \pm R\varphi_0) \, dx/R^2 \rightarrow \infty \text{ as } R \rightarrow \infty$$

is satisfied, completing the proof. \qed
by Hypothesis (D′′). Hence, for $R$ sufficiently large, one of the inequalities

$$G_{\lambda}(\pm R\varphi_0)/R^2 \leq K\|\varphi_0\|^2_D - 2 \int_{\Omega} F(x, \pm R\varphi_0)/R^2 \leq 0$$

holds. Thus,

$$a_0(\lambda) \leq 0, \quad \lambda \in \Lambda.$$  

Moreover, it follows from Theorem 5.2 that (37) holds. Hence,

$$b_0(\lambda) \geq (\lambda - \eta)\rho^2, \quad \lambda \in \Lambda.$$  

This shows that Hypothesis (H_3) holds. We can now apply Theorem 5.1 to conclude that for almost all $\lambda \in \Lambda$,

$$\text{sup}_{k} \|u_k(\lambda)\| < \infty, \quad G'_{\lambda}(u_k(\lambda)) \rightarrow 0 \quad \text{and} \quad G_{\lambda}(u_k(\lambda)) = a(\lambda) \geq b_0(\lambda).$$

Once it is known that the sequence $\{u_k\}$ is bounded, we can apply the usual theory to conclude that there is a solution of

$$G'_{\lambda}(u) = 0, \quad G_{\lambda}(u) = a(\lambda)$$

(cf., e.g., [5, p. 64]). Moreover, from the definition, we see that $a(\lambda) \geq (\lambda - \eta)\rho^2$. Hence, the equation $G'_{\lambda}(u) = 0$ has a nontrivial solution for almost every $\lambda \in \Lambda$. This is equivalent to (23) having a nontrivial solution for almost every $\beta \in (K^{-1}, \eta^{-1})$. Since $K$ was arbitrary, the result follows. \hfill \Box

To prove Theorem 2.4, it suffices to take $\bar{\lambda} = \lambda_0$ and show that Hypothesis (D) implies Hypothesis (D′′). To see this, we note that

$$\int_{\Omega} F(x, \pm R\varphi_0) \, dx/R^2 = \int_{\Omega} \frac{F(x, \pm R\varphi_0)}{R^2\varphi_0^2} \varphi_0^2 \, dx \rightarrow \infty$$

by Hypothesis (D) and the fact that $\varphi_0(x) \neq 0$ a.e.

To prove Corollary 2.6, we let $\varepsilon$ be any positive number. By Hypothesis (C′′′), there is a $\delta > 0$ such that

$$F(x, t)/t^2 \leq \varepsilon, \quad |t| \leq \delta, \quad x \in \Omega.$$  

By Theorem 2.5, Equation (23) has a nontrivial solution for a.e. $\beta \in (0, \lambda_0/\varepsilon)$. Since $\varepsilon$ was arbitrary, the result follows.

We now give the proof of Theorem 2.2.

**Proof.** By Theorem 2.4, for each arbitrary $K > 1$, and a.e. $\lambda \in (1, K)$, there exists $u_\lambda$ such that $G'_{\lambda}(u_\lambda) = 0$, $G_\lambda(u_\lambda) = a(\lambda) \geq (\lambda - 1)\rho^2$. Choose $\lambda_n \rightarrow 1$, $\lambda_n > 1$. Then there exists $u_n$ such that

$$G'_{\lambda_n}(u_n) = 0, \quad G_\lambda(u_n) = a(\lambda_n) \geq a(1) \geq b_0(1).$$
By Theorem 3.4, we may assume \( b_0(1) \geq \varepsilon > 0 \). Therefore,
\[
\int_{\Omega} \frac{2F(x, u_n)}{\|u_n\|^2_D} \, dx \leq c.
\]
Now we prove that \( \{u_n\} \) is bounded. If \( \|u_n\| \to \infty \), let \( w_n = u_n/\|u_n\| \), then \( w_n \to w \) weakly in \( D \), strongly in \( L^2_{\text{loc}}(\Omega) \) and a.e. in \( \Omega \).

**Case 1:** \( w \neq 0 \) in \( D \). We get a contradiction as follows:
\[
c \geq \int_{\Omega} \frac{2F(x, u_n)}{\|u_n\|^2_D} \, dx = \int_{\Omega} \frac{2F(x, u_n)}{u_n^2} |w_n|^2 \, dx 
\geq \int_{w \neq 0} \frac{2F(x, u_n)}{u_n^2} |w_n|^2 \, dx - \int_{w = 0} W_1(x) \, dx \to \infty.
\]

**Case 2:** \( w = 0 \) in \( D \). We define \( t_n \in [0, 1] \) by
\[
G_{\lambda_n}(t_n u_n) = \max_{t \in [0, 1]} G_{\lambda_n}(t u_n).
\]
For any \( c > 0 \) and \( \bar{w}_n = c w_n \), we have
\[
\int_{\Omega} F(x, \bar{w}_n) \, dx \to 0
\]
(cf., e.g., [5, p. 64]). Thus,
\[
G_{\lambda_n}(t_n u_n) \geq G_{\lambda_n}(c w_n) = c^2 \lambda_n - 2 \int_{\Omega} F(x, \bar{w}_n) \, dx \geq c^2/2
\]
for \( n \) large enough. That is, \( \lim_{n \to \infty} G_{\lambda_n}(t_n u_n) = \infty \) and \( (G'_{\lambda_n}(t_n u_n), u_n) = 0 \). Therefore,
\[
G_{\lambda_n}(t_n u_n) = \int_{\Omega} \left( f(x, t_n u_n)t_n u_n - 2F(x, t_n u_n) \right) \, dx 
= \int_{\Omega} H(x, t_n u_n) \, dx \to \infty.
\]
By Hypothesis (E'),
\[
G_{\lambda_n}(u_n) = \int_{\Omega} H(x, u_n) \, dx \geq \int_{\Omega} H(x, t_n u_n) \, dx \to \infty.
\]
But
\[
G_{\lambda_n}(u_n) = a(\lambda_n) \leq \sup_{s \in [0,1], u \in A} G_{\lambda_n}((1 - s)u) 
\leq \sup_{s \in [0,1], u \in A} G_K((1 - s)u) 
< c,
\]
a contradiction. Thus, \( \|u_n\|_D \leq C \). It now follows that
\[
G'(u_n) \to 0, \quad G(u_n) \to a(1) \geq b_0(1).
\]
We can now apply Theorem 3.4.1 in [5, p. 64] to obtain the desired solution.

\[\square\]

References


Received November 10, 2002. The first author was supported in part by an NSF grant and the second author was supported by NSFC grant (GP10001019). Zou thanks the Mathematics Department of the University of California at Irvine for an appointment to the department for the years 2001-2004.

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SPECIALIZATION OF POLYNOMIAL COVERS OF PRIME DEGREE

Leonardo Zapponi

Let $K$ be a complete discrete valued field of unequal characteristic $(0,p)$. The aim of this paper is to describe the semi-stable models for covers $\mathbb{P}^1_K \to \mathbb{P}^1_K$ of degree $p$, unramified outside $r \leq p$ points and totally ramified above one of them, under the assumption that the ramification locus has a particular reduction type (which always occurs if $r \leq 4$). We are principally concerned with the minimal semi-stable models which separate the ramified fibers.

1. Introduction.

The study of the stable models for covers between algebraic curves defined over a complete discrete valuation ring is a subject which has been intensively developed during these last years. Roughly speaking, the problem can be summarized as follows: Starting from a ramified cover $\beta : C \to D$ between nonsingular projective curves defined over a complete discrete valued field $(K,v)$ of unequal characteristic, one would like to construct a semi-stable model $\mathcal{C} \to \mathcal{D}$ of this cover over its valuation ring $R$. The general theory, and in particular the Semi-Stable Reduction Theorem, asserts that such a model always exists, up to a finite extension of the base field (cf. [1] and [4]). Its uniqueness is not ensured. If we require that the model separates the ramified locus $\mathcal{R}$ (that is, the ramification points specialize to pairwise distinct points) and that the curve $C - \mathcal{R}$ is hyperbolic, then there exists a minimal semi-stable model for the cover, which is unique, up to isomorphism. Moreover, this model behaves well under (finite) base change, so that it can be considered as an important birational invariant attached to the cover. In particular, there is a well-defined notion of reduction type: The number of irreducible components of the special fiber, their intersection graph and the thicknesses of the singular points (cf. Section 2).

Historically, the investigation started with the study of prime to $p$ covers i.e., covers such that the residual characteristic $p$ of $R$ does not divide the order of their monodromy group $G$. Even if this is the simplest situation, there are no complete results. For example, there are no general criteria which can decide whether or not the cover has good reduction. The current research trend is concerned with the case where $p$ divides some ramification
indices. M. Raynaud [5] treats the case where $G$ has a $p$-Sylow subgroup of order $p$ and the cover is unramified outside three points. These results are sharpened by S. Wewers in [7]. Finally, C. Lehr [2] and M. Saïdi [6], study the semi-stable model for a $p$-cyclic cover in full generality. In order to have a more complete reference on the subject, see also the results of B. Green, M. Matignon and Y. Henrio.

In this paper, we concentrate on covers $\mathbb{P}^1 \to \mathbb{P}^1$ of prime degree $p$. The only restrictions we make are the following: There is at least one totally ramified point (wild ramification) and the branch locus has a particular reduction type (cf. Section 6). Note that these covers need not to be Galois. It turns out that the semi-stable models essentially depend only on the ramification data and on the $v$-adic distance between the branch points. There are four cases: If the branch locus has good reduction, we essentially recover Wewers’ results. The real interesting case is where the branch points have bad reduction. We choose two branch points $P$ and $Q$ and consider the semi-stable model $C \to D$ of the cover which separates these points. There are three cases: $P$ and $Q$ are $v$-adically far from each other, $P$ and $Q$ have medium distance from each other (the critical case) and $P$ and $Q$ are $v$-adically close to each other. In the first case, the specializations of $P$ and $Q$ belong to two different irreducible components $D_1$ and $D_2$ of the special fiber of $D$ and there is a unique irreducible component lying above each $D_i$. In the second case, $P$ and $Q$ specialize in the same irreducible component and there is one component above it. In the last case, $P$ and $Q$ specialize again in the same component but many irreducible component can lie over it. This last case is more delicate and the knowledge of the ramification data and of the $p$-adic distances between the branch points is not sufficient to completely determine the semi-stable model. Nevertheless, all the possible cases occurring are classified.

The strategy is to start with a natural model for the cover $\mathbb{P}^1 \to \mathbb{P}^1$ over $R$ and then to blow it up several times until the ramification points are separated. This construction can be carried out explicitly because the covering curve is the projective line. This technique can be extended to the case where the degree of the cover is composite and even to the case where the curve $C$ has (potentially) good reduction, but we do not do it in this paper (see [8] for more details).

2. Some notation and basic facts.

Throughout this paper, $(K, v)$ denotes a complete valued field of unequal characteristic $(0, p)$. Its valuation ring $R$ is a complete discrete valuation ring with maximal ideal $\mathfrak{p}$ and residue field $k = \mathcal{O}_K/\mathfrak{p}$. The valuation $v : K^* \to \mathbb{Q}$ is normalized by the condition $v(p) = 1$, so that it extends
the usual $p$-adic valuation. In particular $v(K^*) = e^{-1} \mathbb{Z}$, where the positive integer $e$ is the absolute ramification index of $K$, that is $pR = p^e$.

Let $C$ be a projective, geometrically irreducible, nonsingular curve of genus $g$ defined over $K$ and consider a finite subset $S$ of $C(K)$ of cardinality $n \geq 0$ (for $n = 0$, we simply set $S = \emptyset$). The pair $(C, S)$ is called a pointed curve of type $(g, n)$. We say that $(C, S)$ is hyperbolic if the inequality $2g - 2 + n > 0$ holds. A semi-stable model for $(C, S)$ over $R$ is a proper and flat scheme $\mathcal{C}/R$ satisfying the following conditions:

(1) The generic fiber $\mathcal{C}_K = \mathcal{C} \otimes_R K$ is isomorphic to $C$.
(2) The special fiber $\mathcal{C}_k = \mathcal{C} \otimes_R k$ is reduced and has only ordinary double points as singularities.
(3) The elements of $S$ specialize to pairwise distinct nonsingular points of $\mathcal{C}_k$.

If $S = \emptyset$ then this last condition must be ignored. We say that $\mathcal{C}$ is stable if, for any irreducible component $X$ of $\mathcal{C}_k$, we have the relation $2g(X) - 2 + n(X) + s(X) > 0$, where $g(X)$ denotes the genus of $X$, $n(X)$ is the cardinality of the subset of $S$ specializing in $X$ and $s(X)$ is the number of singular points of $X$. Semi-stable models behave well under base change, i.e., if $L/K$ is a finite extension and $\mathcal{C}/R$ is a semi-stable model of $(C, S)$ over $R$ then $\mathcal{C} \otimes_R R'$ is a semi-stable model for $(C, S)$ over $R'$, where $R'$ is the ring of integers of $L$.

The Semi-Stable Reduction Theorem asserts that any pointed curve $(C, S)$ over $K$ admits a semi-stable model over the valuation ring $R'$ of a finite extension $L$ of $K$. Moreover, if $(C, S)$ is hyperbolic then there exists a stable model $\mathcal{C}/R'$ which is unique up to $R'$-isomorphism. The same kind of result holds for covers: More precisely, given a finite cover $\beta : C \to D$ between projective, nonsingular curves over $K$ and a finite subset $S$ of $D(K)$, there exist a finite extension $L$ of $K$, two proper and flat schemes $\mathcal{C}, \mathcal{D}/R'$ and a finite morphism $\mathcal{C} \to \mathcal{D}$ such that the following conditions hold:

(1) $\mathcal{C}$ (resp. $\mathcal{D}$) is a semi-stable model for the pointed curve $(C, \beta^{-1}(S))$ (resp. for the pointed curve $(D, S)$) over $R'$.
(2) The generic morphism $\mathcal{C}_L \to \mathcal{D}_L$ is isomorphic to $\beta : C \to D$.
(3) The morphism $\mathcal{C}_l \to \mathcal{D}_l$ maps smooth (resp. singular) points to smooth (resp. singular) points (here, $l$ denotes the residue field of $L$).

Remark 2.1. In the literature, the existence of a model satisfying the above properties is ensured only in the Galois case. Since in this paper we are working with covers which are almost never Galois, we now briefly sketch how to extend this result to the general case: Suppose that we have a cover $\beta : C \to D$. We can consider its Galois closure $\widehat{C} \to D$. Denote by $G$ its Galois group. Then there exists a subgroup $H$ of $G$ such that $\widehat{C}/H = C$. Take a semi-stable model $\mathcal{C}$ of $\widehat{C}$ such that the elements of $\mathcal{C}$ with nontrivial stabilizer (the ramified points) specialize to pairwise distinct smooth points
of $\hat{C}_k$. Up to some blow-ups, we can assume that the action of $G$ on $\hat{C}$ extends to $\hat{C}$ and that $G$ acts with no inversions (cf. [5]). The key result is Proposition 5 in [3] which asserts that the quotient of a semi-stable curve under the action of a finite group is again semi-stable. In this case, the curves $C = \hat{C}/H$ and $D = \hat{C}/G$ are semi-stable and the morphism $C \rightarrow D$ is the model we are looking for.

In particular, there exists a minimal semi-stable model satisfying the above properties, which is unique up to $R'$-isomorphism. Since the behaviour of its special fiber is stable under finite base change, it can be considered as an important birational invariant of the cover. In this paper, we are not concerned with rationality questions, that’s why we always assume that the semi-stable model is already defined over $R$.

We close this section by introducing an important invariant attached to a semi-stable model $C/R$: The completion of the local ring of $C$ at a singular point $P$ of its special fiber is isomorphic to the power series ring $R[X,Y]/(XY - Z)$, with $Z \in p$. The thickness of $P$ is the valuation of $Z$, which only depends on $P$. Moreover, our normalization for the valuation $v : K^* \rightarrow \mathbb{Q}$ implies that the thickness of a singular point does not change after a finite base change. Finally, if $\beta : C \rightarrow D$ is a semi-stable model for a cover and $P$ is a singular point of the special fiber $C_k$ of $C$ of thickness $\nu$, then $Q = \beta(P)$ is a singular point of the special fiber $D_k$ of $D$, of thickness $e_P \nu$, where $e_P$ a positive integer. We refer to it as the ramification index at $P$ of the cover $C_k \rightarrow D_k$.

The book [1] is a complete introduction to the theory of semi-stable curves and their application to the study of algebraic covers of curves. Two fundamental papers on the subject are [4] and [5].


The main purpose of this paper is to construct semi-stable models for polynomial covers of prime degree $p$ over $K$ (i.e., covers $\beta : \mathbb{P}^1_K \rightarrow \mathbb{P}^1_K$ of degree $p$ and totally ramified above one point) under certain assumptions on the reduction of the branch locus. We are principally concerned with the minimal model $X \rightarrow Y$ which separates the ramification locus. Two such covers $\beta_1$ and $\beta_2$ are isomorphic, or equivalent (over $K$) if there exist two elements $\sigma, \tau \in \text{PGL}_2(K)$ such that $\sigma \circ \beta_1 = \beta_2 \circ \tau$. In this case, we easily see that the special fibers of the stable models of the covers $\beta_1$ and $\beta_2$ have the same behaviour. We start by constructing a “good” model of the cover over $K$. Let’s first of all reduce to the affine case: Since $\beta : \mathbb{P}^1_K \rightarrow \mathbb{P}^1_K$ is totally ramified above one point, we obtain a finite morphism

$$\mathbb{A}^1_K = \text{Spec}(K[X]) \xrightarrow{\beta} \mathbb{A}^1_K = \text{Spec}(K[T])$$.
In terms of affine algebras, it corresponds to an injection $\beta^* : K[T] \to K[X]$, which is uniquely determined by the image $\beta(X)$ of $T$, which is a polynomial (that’s why we are speaking about polynomial covers). From now on, by eventually enlarging $K$, we assume that the ramification locus of $\beta$ is contained in $\mathbb{P}^1(K)$. For any $\lambda \in K$, we have a unique factorization (in $\overline{K}$)

$$\beta(X) - \lambda = c \prod_{x \in \beta^{-1}(\lambda)} (X - x)^{e_x}$$

where $c \in K$ does not depend on $\lambda$ and $e_x$ is the multiplicity of $X - x$ in the factorization of $\beta(X) - \lambda$. We clearly have

$$\sum_{x \in \beta^{-1}(\lambda)} e_x = p.$$ 

Furthermore, the Riemann-Hurwitz formula gives

$$\sum_{\lambda \in S} n_\lambda = (r - 2)p + 1$$

where $r$ is the cardinality of the branch locus $B = S \cup \{\infty\}$ of the cover and $n_\lambda$ is the cardinality of the fiber $\beta^{-1}(\lambda)$. Up to equivalence, and after a finite extension of $K$, we can reduce to the following case:

1. The finite branch locus $S = \{\lambda_1, \ldots, \lambda_{r-1}\} = \beta(\{x \in K \mid \beta'(x) = 0\})$ is contained in $R$ and $\{0, 1\} \subset S$.

2. The polynomial $\beta(X)$ is monic (i.e., $c = 1$) and $\beta(0) = 0$.

A polynomial $\beta(X) \in K[X]$ satisfying the above conditions is a normalized model for the cover. One easily checks that there exist finitely many normalized models associated to the same polynomial cover. More generally, we say that $\beta(X)$ is a semi-normalized model if it satisfies Condition (2) above and the following property (which is weaker than Condition (1)):

1’ The finite branch locus $S$ is contained in $R$ and there exists $\lambda \in R^*$ such that $\{0, \lambda\} \subset S$.

Note that there exist infinitely many semi-normalized models associated to the same cover.

4. Construction of the fundamental model.

We now start the construction of the semi-stable model associated to a polynomial cover of degree $p$. For any polynomial $h(X) \in R[X]$, we denote by $\overline{h}(X)$ its canonical image in $\overline{k}[X]$. The importance of the (semi-) normalized models introduced in the previous paragraph is related to the following lemma:
Lemma 4.1. Let \( h(X) \in K[X] \) be a monic polynomial of degree \( p \) such that \( h(0) = 0 \). If the associated cover \( h : A^1_K \to A^1_K \) is unramified outside a finite set \( S \subset R \), then \( h(X) \in R[X] \) and \( \overline{h}(X) = X^p \).

Proof. The case \( h(X) = X^p \) is immediate. If \( h(X) \neq X^p \) then its Newton polygon is not reduced to a vertical line. Suppose that the last segment has positive slope, i.e., that there exists a root \( x \) of \( h(X) \) with (minimal) negative valuation. The degree of \( h(X) \) is equal to \( p \) and \( h(0) = 0 \), so that its Newton polygon coincides with the Newton polygon associated to \( Xh'(X) \), except for the last segment (see the following picture).

![Newton polygon diagram]

In particular, we see that there is a root \( y \) of \( h'(X) \) (a ramified point) such that its valuation is strictly less than the valuation of \( x \) (and thus, of any root of \( h(X) \)). We obtain \( v(h(y)) = pv(y) < 0 \), which is absurd since \( h(y) \in S \subset R \). We then have \( h(X) \in R[X] \) and since the (finite) branch points belong to \( R \), we see that all the roots of \( h'(X) \) belong to \( R \). The leading coefficient of \( h'(X) \) is equal to \( p \), so that \( h'(X) = pg(X) \), with \( g(X) \in R[X] \). This gives \( \overline{h'}(X) = 0 \), which leads to \( \overline{h}(X) = X^p \), since \( h(X) \) is monic, of degree \( p \) and satisfies \( h(0) = 0 \). \( \square \)

If \( \beta(X) \in K[X] \) is a (semi-)normalized model associated to a polynomial cover of degree \( p \) then it satisfies the hypothesis of Lemma 4.1 so that it belongs to \( R[X] \) and we have the identity \( \overline{\beta}(X) = X^p \). This is our starting point for the construction of the semi-stable model associated to the cover. Let \( D = \mathbb{P}^1_R \) be the projective line over \( R \), viewed as the union of the affine schemes \( U_1 = \text{Spec}(R[T]) \) and \( U_2 = \text{Spec}(R[T^{-1}]) \). Denote by \( \mathcal{C} \) the integral closure of \( D \) in the function field \( K(X) \) via the injection \( K(T) \hookrightarrow K(X) \) which sends \( T \) to \( \beta(X) \). By construction, we obtain a finite morphism

\[
\mathcal{C} \xrightarrow{\beta} D.
\]
We have to show that $C$ is a semi-stable curve. In fact, we will now prove that it is a smooth $R$-curve. We just have to describe the process of normalization on the two patches $U_1$ and $U_2$. The schemes $V_i = \beta^{-1}(U_i)$ are affine (since $\beta$ is finite). We have $V_1 = \text{Spec}(A)$, where $A$ is the integral closure of $R[T]$ in $K(X)$. We then find $A = R[X]$ and the map $R[T] \to R[X]$ sends $T$ to $\beta(X)$. Indeed, $R[X]$ is integral over $R[T]$ (because $\beta(X)$ is monic) and its field of fractions if $K(X)$. Similarly, we have $V_2 = \text{Spec}(B)$, where $B$ is the integral closure of $R[T^{-1}]$ in $K(X)$. Let $\gamma(X) \in R[X]$ be the polynomial defined by the relation $\gamma(X) = X^p \beta(X^{-1})$. From a practical point of view, we have

$$\gamma(X) = \prod_{x \in \beta^{-1}(0)} (1 - xX)^{e_x}$$

where $e_x = e_x(\beta)$ is the multiplicity of $X - x$ in the factorization of $\beta(X)$. The relation $\beta(0) = 0$ implies that the degree of $\gamma(X)$ is strictly less than $p$. Moreover, we have $\gamma(0) = 1$ and $\tau_\gamma(0) = 1$. With this notation, the element $T^{-1} \in K(T)$ is sent to $\beta(X)^{-1} = \frac{X^{-p}}{\gamma(X^{-1})} \in K(X)$. In particular, the ring

$$B' = R[T^{-1}, X^{-1}]/(X^{-p} - \gamma(X^{-1})T^{-1})$$

is a finite extension of $R[T^{-1}]$. One easily checks that $B'$, considered as a subring of $K(X)$, coincides with $R[X^{-1}, \gamma(X^{-1})^{-1}]$. We then obtain $B' = B$ since $B'$ is integrally closed and has $K(X)$ as field of fractions. This shows that $C$ is smooth and thus isomorphic to $\mathbb{P}^1_R$.

The morphism $C \to D$ is a semi-stable model for the cover over $R$ (since its generic fiber is isomorphic to $\beta$ over $K$) and we refer to it as the fundamental model associated to $\beta$. Lemma 4.1 implies that its specialization is purely inseparable. In particular, for a given $\lambda \in D(K)$, the elements of $\beta^{-1}(\lambda)$ all have the same specialization in the special fiber $C_k$ of $C$, so that this morphism never separates the ramified fibers. A model $\mathcal{X} \to \mathcal{Y}$ for $\beta$ which separates the ramified fibers is obtained from $C \to D$ after a finite number of blow-ups. In order to get a minimal one, we may need to blow-down the original irreducible component $C_k$ (and $D_k$ of the special fiber of $\mathcal{Y}$). This last possibility never occurs. Indeed, this would mean that all the finite branch points have the same specialization in $D$, which is impossible, since we are assuming $\{0,1\} \subseteq S$.

### 5. Some complements on semi-stable models of genus zero.

Before continuing, let’s make some general remarks. First of all, suppose that $\mathcal{X}/R$ is any semi-stable model of a genus zero (pointed) curve and denote by $X_k$ its special fiber. We classically define the intersection graph $\Gamma(X_k)$ as the abstract graph whose vertices (resp. edges) correspond to the irreducible components of $X_k$ (resp. the singular points of $X_k$).
In this special case, $\Gamma(X_k)$ is a tree endowed with a metric, obtained by associating to any edge the thickness of the corresponding singular point. If $C_1$ and $C_2$ are two irreducible components of $X_k$, we can consider the segment $[C_1, C_2]$, which is the minimal subtree of $\Gamma(X_k)$ containing the vertices corresponding to $C_1$ and $C_2$. In particular, the distance $d(C_1, C_2)$ between $C_1$ and $C_2$ is defined as the sum of the lengths of the edges belonging to $[C_1, C_2]$. From a practical point of view, consider the model $X'$ of the projective line over $\mathbb{R}$ obtained by blowing down all the irreducible components of $X_k$ different from $C_1$ and $C_2$. Then the special fiber of $X'$ is the union of the curves $C_1$ and $C_2$, meeting at a unique singular point of thickness $d(C_1, C_2)$.

Suppose now that $X \rightarrow Y$ is any semi-stable model over $\mathbb{R}$ for a polynomial cover, obtained from its fundamental model $C \rightarrow D$ (cf. §4) after a finite number of blow-ups. Denote by $C$ (resp. by $D$) the irreducible component of $X_k$ (resp. of $Y_k$) corresponding to the special fiber of $C$ (resp. of $D$). Let $C_1 \neq C$ be a tail of $C_k$ (i.e., an irreducible component having only one singular point) and denote by $D_1 \neq D$ its image in $D_k$. Let $C_\infty$ be the irreducible component of $X_k$ containing the specialization of the totally ramified point $\infty$. We suppose that the segments $[C, C_1]$ and $[C, C_\infty]$ have no common edges. Consider a point $P \in X(K) = \mathbb{P}^1(K)$ specializing to a nonsingular point of $C_1$. Since $X(K) = \mathcal{C}(K)$, the point $P$ defines a well-defined element $x$ of $\mathcal{C}(K) = \mathcal{C}(\mathbb{R})$. Moreover, the above assumption on the relative positions of $C, C_0$ and $C_\infty$ implies that $x$ belongs to $C_0(R) = \mathbb{A}^1(R)$ (cf. the end of §4) and thus, it can be viewed as an element of $R$. Similarly, the image of $P$ in $Y$ defines an element $\lambda$ of $\mathcal{D}_0(R) = R$. Let $\pi$ and $\pi'$ be two elements of $\mathfrak{p}$ such that $v(\pi) = d(C, C_1)$ and $v(\pi') = d(D, D_1)$. We then
have a unique factorization

\[ \beta(\pi X + x) - \lambda = \pi' \beta_0(X) \gamma_0(X) \]

where \( \beta(X) \in R[X] \) is the normalized model from which the cover \( C \rightarrow D \) was constructed, \( \beta_0(X) \in R[X] \) is monic and \( \gamma_0(X) \in R[X] \) satisfies \( \gamma_0(X) \in k^* \), i.e., \( \gamma_0(X) \in R^* + pR[X] \). Moreover, the finite morphism \( A_k^1 \rightarrow A_k^1 \) obtained from \( C_1 \rightarrow D_1 \) by removing the singular points is (isomorphic to the one) induced by the inclusion \( k[T] \rightarrow k[X] \) which maps \( T \) to \( \beta_0(X) \).

In particular, \( C_1 \) is the only irreducible component of \( A_k \) lying above \( D_1 \) if and only if \( \gamma_0(X) \) has degree 0. In this case, we get the relation \( d(D, D_1) = pd(C, C_1) \).


We now put some conditions on the reduction type of the branch locus \( B = S \cup \{ \infty \} \) of a polynomial cover over \( K \). The curve \( D \) of the fundamental model associated to it (cf. §4) is a model of the projective line over \( R \) such that no point of \( S \) specialize to \( \infty \) and at least two points have distinct specializations. These two conditions uniquely determine \( D \), up to \( R \)-isomorphism. The stable model \( B \) (over \( R \)) for the pointed curve \( (P_1^1, B) \) is obtained from \( D \) after a finite number of blow-ups.

**Definition 6.1.** Consider the stable (separating) model \( B \) associated to a pointed curve \( (P_1^1, B) \), where \( B = S \cup \{ \infty \} \). Denote by \( D \) the irreducible component of its special fiber containing the specialization of the point \( \infty \).

An element \( \lambda \in S \) is **ordinary** if it specializes in \( D \). If \( B \) has bad reduction, an irreducible component of \( B_k \) is **simple** if it only meets \( D \) and if there are exactly two elements of \( S \) specializing in it. We say that \( B \) has **simple reduction** if any tail of \( B_k \) (if there are any) is simple (see the following picture).

\[
\begin{array}{ccccccc}
\infty & \lambda_1 & \cdots & \lambda_m & \lambda_{m+1} & \cdots & \lambda_{r-2} \\
& \lambda_{m+2} & \cdots & \lambda_{r-1} & & & \\
\end{array}
\]

Note that if the pointed curve \( (P_1^1, B) \) has simple reduction then any irreducible component of its stable (separating) model different from \( D \) is automatically a tail. Moreover, if the cardinality of \( B \) is less than or equal to four then the reduction is always simple.
7. Semi-stable model separating the fiber above an ordinary branch point.

We now describe the stable model for a polynomial cover of degree $p$ dominating its fundamental model $C \to D$ and separating the ramified fiber above an ordinary branch point.

**Theorem 7.1.** Let $\beta : \mathbb{P}^1_K \to \mathbb{P}^1_K$ be a polynomial cover of degree $p$. Denote by $B = S \cup \{\infty\}$ its branch locus and suppose that $\lambda \in S$ is ordinary. Then the minimal semi-stable model $X \to Y$ of the cover which domi-
nates the fundamental model $C \to D$ and separates the fiber above $\lambda$ has the following description:

1. The special fiber of $X$ is the union of two projective lines $C$ and $C'$ meeting at a unique singular point of thickness $(n_\lambda - 1)^{-1}$ where $n_\lambda$ is the cardinality of the fiber $\beta^{-1}(\lambda)$.

2. The special fiber of $Y$ is the union of two projective lines $D$ and $D'$ meeting at a unique singular point of thickness $p(n_\lambda - 1)^{-1}$.

3. The morphism $C \to D$ is purely inseparable, while $C' \to D'$ is generically étale, unramified outside two points, wildly ramified above one of them and tamely ramified above the other.

**Proof.** Without any loss of generality, we can assume that $\lambda = 0$ and $\beta(0) = 0$. Let $\nu$ be the rational number defined by

$$\nu = \text{Min}\{v(x) \mid x \in \beta^{-1}(0)\}$$

and consider the curve $C'$ obtained from $C$ after a blow-up at the origin, of thickness $\nu$. Let $\pi$ be an element of $\mathfrak{p}$ such that $v(\pi) = \nu$. By construction, we have the expression

$$\beta(\pi X) = \pi^p \prod_{x \in \beta^{-1}(0)} (X - \pi^{-1}x)^{e_x} \in R[X]$$

with $\pi^{-1}x \in R$ for any $x \in \beta^{-1}(0)$. Following the notation introduced at the end of §5, we obtain $\beta_0(X) = \pi^{-\nu}\beta(\pi X)$ and $\gamma_0(X) = 1$. In particular, we have a model $C' \to D'$ of the cover $\beta$, where $D'$ is the curve obtained from $D$ after a blow-up at the origin, of thickness $p\nu$. We just have to prove that it separates the fiber above 0. From a practical point of view, we must show that two distinct roots of the polynomial $\beta_0(X)$ have distinct specializations. First of all, if $\mu \in S - \{0\}$ then $\beta^{-1}(\mu) \subset R^\times$. Indeed, we
already have $\beta^{-1}(\mu) \subset R$. If there were $x \in \beta^{-1}(\mu)$ with positive valuation, we would obtain $\mu = \beta(x) = \pi^p = 0$, which is impossible since we are assuming that $0 \in S$ is ordinary. We have the following expression for the derivative of $\beta(X)$:

$$\beta'(X) = p \prod_{x \in \beta^{-1}(S)} (X - x)^{e_x - 1}.$$  

We then obtain $\beta'_0(X) = \pi^{1-p}\beta'(\pi X)$, which leads to

$$\beta'_0(X) = p\pi^{1-n_0} \prod_{x \in \beta^{-1}(0)} (X - \pi^{-1}x)^{e_x - 1} \prod_{x \in \beta^{-1}(S \setminus \{0\})} (\pi X - x)^{e_x - 1}$$

where $n_0$ is the cardinality of the fiber $\beta^{-1}(0)$. We have $\beta'_0(X) \in R[X]$, which directly implies the inequality $(n_0 - 1)\nu \leq 1$. A strict inequality would give $\beta'_0(X) = 0$ and thus $\beta'_0(X) = \pi^p$ (recall that $\beta_0(X)$ is monic, of degree $p$ and satisfies $\beta_0(0) = 0$). This is impossible since, by construction, there exists a root of $\beta_0(X)$ belonging to $R^*$. We then have $\nu = \frac{1}{n_0 - 1}$.

Suppose now that the elements $x_1, \ldots, x_s \in \beta_0^{-1}(0)$ specialize to the same point $t$ of $C'$ and denote by $e_1, \ldots, e_s$ their ramification indices. We have $s < n_0$, since there are at least two distinct roots of $\beta_0(X)$. The monomial $X - t$ appears with multiplicity $e = e_1 + \cdots + e_s < p$ in the factorization of $\beta_0(X)$. In particular, the element $t$ will be a root of $\beta'_0(X)$ of multiplicity $e - 1$. But the above expression of $\beta'_0(X)$ implies that this multiplicity is equal to $e - s$, and so we have $s = 1$, i.e., the model $C' \to D'$ separates the fiber above $0$.

**Corollary 7.2.** The notation and hypothesis being as in Theorem 7.1, there exists a smooth model of the cover (over $R$) which separates the fibers above $\infty$ and $\lambda$. In other words, the pointed curve $(\mathbb{P}^1_K, \beta^{-1}((\infty, \lambda)))$ has good reduction.

**Proof.** The desired smooth model is obtained from $C' \to D'$ by blowing down the irreducible components $C$ and $D$. \qed

**Corollary 7.3.** With the above assumptions, suppose that the branch locus $B = \{\infty, \lambda_1, \ldots, \lambda_{r-1}\}$ of the cover $\beta$ has good reduction and that $r \geq 3$. Then, the minimal stable model $X \to \mathcal{Y}$ separating the ramified fibers has the following description:

1. $\mathcal{X}_k$ (resp. $\mathcal{Y}_k$) is the union of $r$ projective lines $C, C_1, \ldots, C_{r-1}$ (resp. $D, D_1, \ldots, D_{r-1}$).
2. For any $i \in \{1, \ldots, r-1\}$, the point $\lambda_i$ specializes in $D_i$.
3. The corresponding irreducible component $C_i$ (resp. $D_i$) only meets $C$ (resp. $D$). The corresponding singular point has thickness $(n_i-1)^{-1}$ (resp. $p(n_i-1)^{-1}$), where $n_i$ denotes the cardinality of the fiber above $\lambda_i$. 

The morphism \( C \to D \) is purely inseparable and, for any \( i \in \{1, \ldots, r-1\} \), the cover \( C_i \to D_i \) is generically étale, unramified outside two points, wildly ramified above one of them (the singular point) and tamely ramified above the other (the specialization of \( \lambda_i \)).

\[ \begin{array}{ccc}
C_1 & \cdots & C_{r-1} \\
\downarrow \sigma_k \downarrow & \cdots & \downarrow \sigma_k \\
C & \cdots & D_{r-1} \\
\downarrow \infty & \cdots & \downarrow \infty \\
(\sigma_k^{-1})^{-1} & \cdots & (\sigma_k^{-1})^{-1} \\
(\pi^{-1})^{-1} & \cdots & (\pi^{-1})^{-1} \\
\end{array} \]

\[ \beta \]

Proof. It suffices to repeat the construction in the proof of Theorem 7.1 for all the elements of \( S \). \qed

8. Semi-stable model separating the fibers above a simple tail.

We now study the case of simple reduction of the branch locus \( B = S \cup \{\infty\} \) of the cover \( \beta \). Let \( r \) be the cardinality of \( B \). We assume that \( r > 3 \), since for \( r \leq 3 \) the pointed curve \( (\mathbb{P}^1, B) \) always has good reduction. If \( \lambda \in S \) is an ordinary branch point, then the construction in the proof of Theorem 7.1 leads to a stable model which separates the fiber above \( \lambda \).

Suppose that \( \lambda, \lambda' \in S \) belong to a simple tail of the minimal stable model \( B \) of \( B \). In other words, viewing \( S \) as a subset of \( R = \mathbb{C}^0(R) \) (cf. §4), we have \( v(\lambda - \lambda') > 0 \) and \( v(\lambda - \lambda'') = v(\lambda' - \lambda'') = 0 \) for any \( \lambda'' \in S - \{\lambda, \lambda'\} \).

The positive rational number \( \epsilon = v(\lambda - \lambda') \) is the thickness of the singular point connecting the simple tail with the rest of \( B_k \). Since the reduction of the model \( C \to D \) is purely inseparable, all the elements of \( \beta^{-1}(\{\lambda, \lambda'\}) \) have the same specialization.

8.1. The first separating blow-up. We start the construction of the stable model separating the fibers above \( \lambda \) and \( \lambda' \) by considering the minimal model \( C' \) dominating \( C \) such that all the elements of \( \beta^{-1}(\{\lambda, \lambda'\}) \) specialize in the (smooth locus of the) same irreducible component and at least two of them have distinct specializations. In order to obtain a more explicit description, we can first of all reduce to the case \( \lambda' = 0 \) and \( \beta(0) = 0 \). Consider the rational number \( \nu \) defined by

\[ \nu = \text{Min} \left\{ v(x - y) \mid x, y \in \beta^{-1}(\{0, \lambda\}) \right\}. \]

We have \( \nu > 0 \), since \( \lambda = 0 \) and \( \beta(X) = X^p \). Moreover, \( v(\lambda) \geq p\nu \). The curve \( C' \) is obtained from \( C \) after a blow-up at the origin, of thickness \( \nu \). Let \( \pi \) be an element of \( \mathfrak{p} \) such that \( v(\pi) = \nu \). The relation (1) at the end of §5...
becomes $\beta(\pi X) = \pi^p \beta_0(X) \gamma_0(X)$, with $\gamma_0(X) = 1$ and

$$
\beta_0(X) = \pi^{-p} \beta(\pi X) = \prod_{x \in \beta^{-1}(0)} (X - \pi^{-1} x)^{e_x} = \pi^{-p} \lambda + \prod_{x \in \beta^{-1}(\lambda)} (X - \pi^{-1} x)^{e_x}.
$$

We then get a model $C' \to D'$ for $\beta$, where $D'$ is the curve obtained from $D$ after a blow-up at the origin, of thickness $p\nu$ (see the following picture).

The derivative of $\beta_0(X)$ can be expressed as

$$
\beta_0'(X) = P \pi^{p+1-n_0-n_\lambda} \prod_{x \in \beta^{-1}(\{0, \lambda\})} (X - \pi^{-1} x)^{e_x-1} \prod_{x \in \beta^{-1}(\{0, \lambda\})} (\pi X - x)^{e_x-1}
$$

where, for any $\lambda \in S$, $n_\lambda$ is the cardinality of the fiber $\beta^{-1}(\lambda)$. In particular, since $\beta_0'(X) \in R[X]$ and $\pi^{-1} x \in R$ for any $x \in \beta^{-1}(\{0, \lambda\})$, we obtain the relation

$$
\nu \leq \frac{1}{n_0 + n_\lambda - p - 1}.
$$

### 8.2. First case: “Far” points.

Let’s start by supposing that the inequality (2) is strict. This directly implies that $\overline{\beta}_0(X) = X^p$. Now, by construction, the polynomial $h(X) = \beta_0(X)(\beta_0(X) - \pi^{-p} \lambda)$ has at least two roots having distinct specializations, so that we obtain

$$
v(\lambda) = p\nu.
$$

Indeed, the relation $v(\lambda) > p\nu$ would give $h(X) = X^{2p}$, which has only one root. In particular, this first blow-up separates the branch points 0 and $\lambda$ but does not separate the corresponding fibers.

The above inequality now reads as

$$
v(\lambda) < \frac{p}{n_0 + n_\lambda - p - 1}.
$$
In order to obtain a model separating the fiber above 0, let \( \nu_0 \) be the positive integer defined by

\[
\nu_0 = \text{Min} \{ v(x) \mid x \in \beta_0^{-1}(0) \}.
\]

We clearly have \( \nu + \nu_0 = \text{Min} \{ v(x) \mid x \in \beta^{-1}(0) \} \). Consider the curve \( C'' \) obtained from \( C' \) after a blow-up at the origin of the irreducible component \( C' \), of thickness \( \nu_0 \). If \( C'' \) denotes the new irreducible component of \( C'_k \), then we have \( d(C'', C) = \nu + \nu_0 \). Let \( \pi_0 \in \mathfrak{p} \) such that \( v(\pi_0) = \nu_0 \) and set \( \pi_1 = \pi \pi_0 \) (as in the previous paragraph, \( \pi \) has valuation \( \nu \)). We then obtain the expression

\[
\beta_1(X) = \pi_{p,v}^p \beta(\pi_1 X) = \pi_{p,v} \beta(\pi_0 X)
\]

\[
= \prod_{x \in \beta^{-1}(\mathfrak{p})} (X - \pi_{p,v}^{-1}x)^{e_x} = \prod_{x \in \beta_0^{-1}(\mathfrak{p})} (X - \pi_0^{-1}x)^{e_x}.
\]

By construction, at least two roots of \( \beta_1(X) \) have distinct specializations. If \( D'' \) is the curve obtained from \( D' \) after a blow-up at the origin of \( D' \), of thickness \( p \nu_0 \), we then have a model \( C'' \rightarrow D'' \) of the cover \( \beta \). Setting \( u = n_0 + n_\lambda - p - 1 \), \( u_0 = n_0 - 1 \) and \( \delta = 1 - uw - u_0 \nu_0 \), we have the expression

\[
\beta_1(X) = \pi_{p,v}^n \pi_0^{u_0} \prod_{x \in \beta^{-1}(0)} (X - \pi_{p,v}^{-1}x)^{e_x} \prod_{x \in \beta^{-1}(\lambda)} (\pi_0 X - \pi_{p,v}^{-1}x)^{e_x}
\]

\[
\cdot \prod_{x \in \beta^{-1}(S \setminus \{0, \lambda\})} (\pi_1 X - x)^{e_x}.
\]

which implies that \( \delta = 0 \) (otherwise, we would obtain \( \beta_1(X) = X^p \) and all the roots of \( \beta_1(X) \) would have the same specialization). Since \( v(\lambda) = \nu_0 \), we obtain

\[
\nu_0 = \frac{p - (n_0 + n_\lambda - p - 1)v(\lambda)}{p(n_0 - 1)} < \frac{1}{n_0 - 1}.
\]

Proceeding exactly as in the end of the proof of Theorem 7.1, we finally check that this model separates the fiber above 0, i.e., that any two distinct roots of \( \beta_1(X) \) have distinct specializations. The same procedure leads to a stable model separating the fiber above \( \lambda \). Summarizing, we have just proved the following result:

**Theorem 8.1.** Suppose that the branch locus \( B = S \cup \{\infty\} \) of the polynomial cover \( \beta \) has bad reduction and that \( D' \) is a simple tail of the special fiber of the stable model \( B \) associated to the pointed curve \((\mathbb{P}^1, B)\). Denote by \( \lambda_1, \lambda_2 \in S \) the two branch points specializing in \( D' \) and let \( \epsilon = v(\lambda_1 - \lambda_2) \) be the thickness of the corresponding singular point of \( B_k \). For any \( \lambda \in S \), denote by \( n_\lambda \) the cardinality of the fiber \( \beta^{-1}(\lambda) \). If the inequality

\[
\epsilon < \frac{p}{n_{\lambda_1} + n_{\lambda_2} - p - 1}
\]
holds, then the minimal semi-stable model $X \rightarrow \mathcal{Y}$ of $\beta$ dominating $C \rightarrow D$ and separating the fibers above $\lambda_1$ and $\lambda_2$ has the following description:

1. The curve $X_k$ (resp. $Y_k$) is the union of four projective lines $C, C', C_1$ and $C_2$ (resp. $D, D', D_1$ and $D_2$).
2. The specializations of the points $\infty, \lambda_1$ and $\lambda_2$ belong respectively to $D, D_1$ and $D_2$.
3. The irreducible component $C$ (resp. $D$) only meets $C'$ (resp. $D'$), at a singular point of thickness $\frac{\epsilon}{p}$ (resp. of thickness $\epsilon$).
4. For any $i \in \{1, 2\}$, the irreducible component $C_i$ (resp. $D_i$) only meets $C'$ (resp. $D'$), at a singular point of thickness $\nu_i = \frac{p-(n\lambda_1+n\lambda_2-p-1)\epsilon}{p(n\lambda_i-1)}$ (resp. of thickness $p\nu_i = \frac{p-(n\lambda_1+n\lambda_2-p-1)\epsilon}{n\lambda_i-1}$).
5. The morphisms $C \rightarrow D$ and $C' \rightarrow D'$ are purely inseparable, while the covers $C_1 \rightarrow D_1$ and $C_2 \rightarrow D_2$ are generically étale, unramified outside two points, wildly ramified above one of them and tamely ramified above the other.

![Diagram](image)

**Corollary 8.2.** The notation and hypothesis being as above, if

$$\epsilon < \frac{p}{n\lambda_1 + n\lambda_2 - p - 1}$$

then, for any $i \in \{1, 2\}$, the pointed curve $(\mathbb{P}^1, \beta^{-1}([\infty, \lambda_i]))$ has good reduction.

**Proof.** It suffices to take the smooth $R$-curve obtained from $X$ by blowing down the irreducible components $C, C'$ and $C_{2-i}$ of $C_k$. \qed

**8.3. Second case: The critical distance.** Keeping the above notation and hypothesis, suppose now that the inequality (2) at the end of §8.1 is in fact an equality, i.e., that $\nu = \frac{1}{n\lambda_1 + n\lambda_2 - p - 1}$, so that $\beta_0(X) \neq 0$. In particular, the morphism $C' \rightarrow D'$ is generically étale. We already know that $\nu(\lambda) \geq p\nu$. In this paragraph, we assume that this last inequality is an equality. As before, the cover $C' \rightarrow D'$ separates the branch points 0 and $\lambda$. Moreover, proceeding as at the end of the proof of Theorem 7.1, we can easily show that this model also separates the fibers above these two points. In particular, we have the following result:
**Theorem 8.3.** The notation and hypothesis being as in Theorem 8.1, suppose that
\[ \epsilon = \frac{p}{n \lambda_1 + n \lambda_2 - p - 1}. \]
Then, the minimal semi-stable model \( X \to Y \) of \( \beta \) dominating \( C \to D \) and separating the fibers above \( \lambda_1 \) and \( \lambda_2 \) has the following description:

1. The curve \( X_k \) (resp. \( Y_k \)) is the union of two projective lines \( C \) and \( C' \) (resp. \( D \) and \( D' \)) meeting at a singular point of thickness \( \frac{\epsilon}{p} \) (resp. of thickness \( \epsilon \)).

2. The specializations of the points \( \lambda_1 \) and \( \lambda_2 \) belong \( D' \) and \( \infty \) specializes in \( D \).

3. The morphism \( C \to D \) is purely inseparable while the cover \( C' \to D' \) is generically étale, unramified outside three points, wildly ramified above one of them and tamely ramified above the others.

**Corollary 8.4.** The notation and hypothesis being as above, if
\[ \epsilon = \frac{p}{n \lambda_1 + n \lambda_2 - p - 1} \]
then the pointed curve \((\mathbb{P}^1, \beta^{-1}(\{\infty, \lambda_1, \lambda_2\}))\) has good reduction.

**Proof.** It suffices to take the smooth \( R \)-curve obtained from \( X \) by blowing down the irreducible component \( C \) of its special fiber. \( \square \)

**8.4. Third case: “Near” points.** We now have to study the most critical case, that is, when \( \nu = \frac{1}{n \lambda_1 + n \lambda_2 - p - 1} \) and \( \nu(\lambda) > p \nu \). This situation occurs if and only if the model \( C' \to D' \) introduced at the beginning of this section does not separate the elements 0 and \( \lambda \) of the branch locus \( B \) of the cover. As in the previous paragraph, the derivative of \( \beta_0(X) \) does not identically vanish, so that the cover \( C' \to D' \), which is (isomorphic to the one) induced by the polynomial \( \beta_0(X) \in k[X] \), is generically étale, unramified outside two points, wildly ramified above one of them (the intersection with \( D \)) and tamely ramified above the other. There exist at least two roots of \( \beta_0(X) \) having distinct specializations. Indeed, the contrary would give \( \beta_0(X) = X^p \), so that the cover \( C' \to D' \) would be purely inseparable, which is a contradiction. The points 0 and \( \lambda \) are the only elements of \( B \) specializing in \( C' \) and their fibers with respect to \( \beta \) can be assimilated to the fibers \( \beta_0^{-1}(0) \).
and $\beta_0^{-1}(\lambda_0)$, where $\lambda_0 = \pi^{-p}\lambda \in \mathfrak{p}$. Set

$$\bar{\beta}_0(X) = \prod_{i=1}^{s} (X - w_i)^{d_i}$$

with $s > 1$, $d_1 + \cdots + d_s = p$ and $w_i \neq w_j$ for any $i \neq j$. We then obtain two partitions $\beta_0^{-1}(0) = S_{1,0} \cup \cdots \cup S_{s,0}$ and $\beta_0^{-1}(\lambda_0) = S_{1,\lambda_0} \cup \cdots \cup S_{s,\lambda_0}$, where we have set, for a fixed $t \in \mathfrak{p}$,

$$S_{i,t} = \{ x \in \beta_0^{-1}(t) \mid \pi = w_i \}.$$ 

For any $i \in \{1, \ldots, s\}$ we then have the identity

$$d_i = \sum_{x \in S_{i,t}} e_x$$

where $e_x$ is the multiplicity of the root $x$ of the polynomial $\beta_0(X) - t$. Since the cover $C' \to D'$ is generically étale and unramified outside the set $\{0, \infty\}$, we have

$$\bar{\beta}_0^p(X) = c \prod_{i=1}^{s} (X - w_i)^{d_i - 1}$$

with $c \in k^*$. On the other hand, the expression of the derivative of $\beta_0(X)$ given previously leads to

$$\bar{\beta}_0'(X) = c \prod_{i=1}^{s} (X - w_i)^{2d_i - n_{i,0} - n_{i,\lambda}}$$

where $n_{i,0}$ (resp. $n_{i,\lambda}$) denotes the cardinality of $S_{i,0}$ (resp. of $S_{i,\lambda_0}$). Combining these two expressions we finally obtain the identity

$$n_{i,0} + n_{i,\lambda} = d_i + 1$$

which holds for any $i \in \{1, \ldots, s\}$. In order to completely separate the ramified fibers, we need to blow-up some projective lines at the points $w_1, \ldots, w_s$ belonging to the irreducible component $C'$ of $C'$. In other words, we have to separate the elements of $S_{i,0} \cup S_{i,\lambda_0}$ for any $i \in \{1, \ldots, s\}$. Let’s start with $i = 1$: Since $0 \in S$ is a branch point and $\beta_0(0) = \beta(0) = 0$, we can assume, without any loss of generality, $w_1 = 0$. Set

$$\nu_1 = \operatorname{Min} \{ v(x - y) \mid x, y \in S_{1,0} \cup S_{1,\lambda_0}, x \neq y \} > 0$$

and consider the $R$-curve $C''$, obtained from $C'$ by blowing-up a projective line $C_1$ at $w_1$, of thickness $\nu_1$. Let $\pi_2 \in \mathfrak{p}$ such that $v(\pi_2) = \nu_1$. We then have the identity

$$\beta(\pi_2X) = \pi^p \beta_0(\pi_2X) = \pi^p \pi_2^{d_1} \beta_1(X) \gamma_1(X)$$

where

$$\beta_1(X) = \prod_{x \in S_{1,0}} (X - \pi_2^{-1}x)^{e_x}$$
and
\[ \gamma_1(X) = \prod_{x \in \beta_0^{-1}(0)-S_{1,0}} (\pi_2 X - x)^{\epsilon_x}. \]

Since we are assuming that \( w_1 = 0 \), an element \( x \in \beta_0^{-1}(0) \) belongs to \( S_{1,0} \) if and only if \( v(x) > 0 \). In particular, \( \pi_1(X) = c \in k^* \). We then obtain a \( R \)-morphism \( C'' \to D'' \) (which is not finite), where \( D'' \) is the \( R \)-curve obtained from \( D' \) by blowing-up a projective line \( D_1 \) at the origin of \( D' \), of thickness \( \epsilon_1 = d_1 \nu_1 \). For any \( x \in S_{i,\lambda_0} \), we have \( v(x) \geq \nu_1 \), from which we easily deduce the inequality \( v(\lambda_0) \geq \epsilon_1 \), i.e., \( v(\lambda) \geq \epsilon_0 + \epsilon_1 \), with \( \epsilon_0 = p \nu \).

The specialized morphism \( C_1 \to D_1 \) has degree \( d_1 \) and is (isomorphic to the cover \( P^1_k \to P^1_k \)) induced by the polynomial \( \beta_1(X) \). The integer \( d_1 \) being strictly less than \( p \), the derivative \( \beta_1(X) \) does not identically vanish. More explicitly, using once again the expression of the derivative of \( \beta_0(X) \), we obtain
\[ \beta_1'(X) = u \prod_{x \in S_{1,0} \cup S_{1,\lambda_0}} \left( X - \pi_2^{-1}x \right)^{\epsilon_x-1} \]
with \( u \in k^* \). The strict inequality \( v(\lambda) > \epsilon_0 + \epsilon_1 \) would imply that the cover \( P^1_k \to P^1_k \) induced by \( \beta_1(X) \) is tame, of degree \( d_1 \) and unramified outside 0 and \( \infty \), and thus \( \beta_1(X) = uX^{d_1} \). In particular, all the elements of \( S_{1,0} \) and \( S_{1,\lambda_0} \), would specialize to the same element of \( C_1 \). This is impossible, since \( C'' \) is the minimal model dominating \( C' \) and separating at least two elements of \( S_{1,0} \cup S_{1,\lambda_0} \). We then have the identity
\[ v(\lambda) = \epsilon_0 + \epsilon_1 = \frac{p}{n_0 + n_\lambda - p - 1} + d_1 \nu_1 \]
which implies that the model \( D'' \) separates the branch points 0 and \( \lambda \). Then, using the above expression for the derivative of \( \beta_1(X) \) and the same arguments at the end of the proof of Theorem 7.1, one easily shows that \( C'' \) separates the whole \( S_{1,0} \cup S_{1,\lambda_0} \). Note that the above equality implies that the thickness \( \epsilon_1 = d_1 \nu_1 \) does not depend on the point \( w_1 \) but only on the ramification data of the cover and on the valuation of \( \lambda \). Iterating this construction for all the roots of \( \beta_0(X) \), we obtain a semi-stable model \( X \to Y \) (this time the morphism is finite) which dominates \( C \to D \) and separates the fiber above 0 and \( \lambda \). Summarizing, we have proved the following result:

**Theorem 8.5.** The notation and hypothesis being as in Theorem 8.1, suppose that
\[ \epsilon > \frac{p}{n_\lambda_1 + n_\lambda_2 - p - 1}. \]
Then, the special fiber of the minimal stable model \( X \to Y \) of the cover \( \beta \) dominating \( C \to D \) and separating the fibers above \( \lambda_1 \) and \( \lambda_2 \) has the following description:

1. The curve \( Y_k \) is the union of three projective lines \( D, D' \) and \( D_1 \).
(2) The curve $D$ (resp. $D_1$) meets $D'$ at a singular point of thickness $\epsilon_0 = \frac{p}{n_{\lambda_1} + n_{\lambda_2} - p - 1}$ (resp. of thickness $\epsilon_1 = v(\lambda) - \frac{p}{n_{\lambda_1} + n_{\lambda_2} - p - 1}$).
(3) The specializations of the points $\lambda_1$ and $\lambda_2$ belong to $D_1$ and $\infty$ specializes in $D$.
(4) There exists an integer $s \geq 2$ such that the curve $X_k$ has $s+2$ irreducible components $C, C', C_1, \ldots, C_s$.
(5) The curve $C$ meets $C'$ at a singular point of thickness $\frac{\alpha}{p}$.
(6) There exist two partitions

$$
\beta^{-1}(\lambda_1) = S_{1,1} \cup \cdots \cup S_{s,1} \quad \text{and} \quad \beta^{-1}(\lambda_2) = S_{1,2} \cup \cdots \cup S_{s,2}
$$

satisfying the identities

$$
\sum_{x \in S_{i,1}} e_x = \sum_{x \in S_{i,2}} e_x = d_i \quad \text{and} \quad n_{i,1} + n_{i,2} = d_i + 1
$$

where $e_x$ denotes the ramification index of $\beta$ at a point $x \in \mathbf{P}^1(K)$ and, for any $i \in \{1, \ldots, s\}$ and any $j \in \{1, 2\}$, $n_{i,j}$ is the cardinality of the set $S_{i,j}$.
(7) For any $i \in \{1, \ldots, s\}$, the curve $C_i$ meets $C$ at a unique singular point, of thickness $\frac{\alpha}{d_i}$ and the elements of $S_{i,1} \cup S_{i,2}$ specialize to pairwise distinct points of $C_i$.
(8) The morphism $C \to D$ is purely inseparable and the cover $C' \to D'$ is generically étale, unramified outside two points, wildly ramified above one of them (the intersection with $D$) and tamely ramified above the other (the intersection with $D_1$). For any $i \in \{1, \ldots, s\}$, the cover $C_i \to D_1$ is generically étale, of degree $d_i$, unramified outside three points (the specializations of $\lambda_1$ and $\lambda_2$ and the intersection with $C'$) over which the ramification is tame.

Theorems 7.1, 8.1, 8.3 and 8.5 allow us to completely classify the reduction type of the minimal semi-stable model $X \to \mathcal{Y}$ (separating the ramified fibers) for a polynomial cover $\beta$ over $K$ of degree $p$, under the assumption of simple reduction of its branch locus $B$. More precisely, the above results show that the behaviour of this model essentially depends only on the ramification data of the cover and on the thicknesses of the singular points of
the special fiber of the stable (separating) model $B$ associated to the pointed curve $(\mathbb{P}^1, B)$.

9. An example.

We end this paper with an explicit example. Let $K$ be a $5$-adic field, i.e., a finite extension of the field $\mathbb{Q}_5$. We want to describe the semi-stable model for a polynomial cover $\beta : \mathbb{P}^1_K \to \mathbb{P}^1_K$ of degree $5$ having the following ramification data:

(1) The cover is unramified outside the set $\{\infty, 0, \lambda_1, \lambda_2\}$ with $\lambda_1, \lambda_2 \in K^*$ and $\lambda_1 \neq \lambda_2$.
(2) There is only one point above $\infty$.
(3) There are three points above 0, one of them, denoted by $P_0$ has ramification index 3, while the others are unramified.
(4) There are four points above $\lambda_1$ (resp. above $\lambda_2$), one of them, denoted by $P_1$ (resp. $P_2$) has ramification index 2 while the other three are unramified.

Up to equivalence (cf. §3), and since the behaviour of the ramification above $\lambda_1$ and $\lambda_2$ is the same, we can reduce to the case $\lambda_1 = 1$ and $\lambda_2 = \lambda \in R - \{0, 1\}$. The results of this paper allow us to describe the semi-stable model for $\beta$ without any direct computation. More precisely, if the branch locus has good reduction, i.e., if $v(\lambda) = v(\lambda - 1) = 0$ then we can apply Theorem 7.1, which leads to the following model:

The branch locus of the cover $\beta$ has bad reduction if either $v(\lambda) > 0$ or $v(\lambda - 1) > 0$. In the first case, there are three possibilities, leading to four different semi-stable models: If $\epsilon = v(\lambda) < 5$, we can apply Theorem 8.1 and obtain the following model:

As it is shown in the following picture, and applying Theorem 8.3, the simplest case occurs for $\epsilon = 5$. 
Finally, for $\epsilon > 5$ we find two possibilities, depending on the partitions of the ramification indices $(1, 1, 3)$ and $(1, 1, 1, 2)$ of the fibers above 0 and $\lambda$ (cf. Theorem 8.5). The first partition is given by $S_0 = \{\{1, 3\}, \{1\}\}$ and $S_1 = \{\{1, 1, 2\}, \{1\}\}$, which leads to the following semi-stable model:

![Diagram 1](image1)

If we consider the second partition $S_0 = \{\{3\}, \{1, 1\}\}$ and $S_1 = \{\{1, 1, 1\}, \{2\}\}$, we then obtain the following model:

![Diagram 2](image2)

In order to completely classify the possible semi-stable models for $\beta$, we have to study the last case of bad reduction of the branch locus, i.e., when $\epsilon = v(\lambda - 1) > 0$. As before, there are three different cases, depending on $\epsilon$, leading to four different situations. First of all, for $\epsilon < \frac{5}{2}$ we obtain the following semi-stable model (cf. Theorem 8.1):

![Diagram 3](image3)

The next picture describes the case $\epsilon = \frac{5}{2}$ (cf. Theorem 8.3):
For $\epsilon > \frac{5}{2}$, according to Theorem 8.5, there are two possible partitions of the ramification indices above 1 and $\lambda$. The first is given by $S_1 = S_\lambda = \{\{1, 2\}, \{1\}, \{1\}\}$ and leads to the following semi-stable model:

\[
\begin{array}{c|c|c|c}
\infty & P_1 & P_2 & \beta \\
1/2 & 1/2 & 5/2 & 5/2 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\infty & P_0 & \lambda & 0 \\
\end{array}
\]

Finally, the next picture describes the semi-stable model associated to the second partition $S_1 = S_\lambda = \{\{2\}, \{1, 1\}, \{1\}\}$:

\[
\begin{array}{c|c|c|c}
\infty & P_1 & P_2 & \beta \\
1/2 & 1/2 & 5/2 & 5/2 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\infty & P_0 & \lambda & 0 \\
\end{array}
\]

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Received November 17, 2002 and revised May 1, 2003.

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ON THE EXTREMAL FUNCTIONS OF
SBOLEV–POINCARÉ INEQUALITY

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Dedicated to Elliott H. Lieb on the occasion of his 70th birthday

We prove the existence of extremal functions of Sobolev-Poincaré inequality on \( S^n \) for \( p \in (1, (1 + \sqrt{1 + 8n})/4) \). For general \( n \)-dimensional compact Riemannian manifolds embedded in \( \mathbb{R}^{n+1} \), such an existence result is proved for \( p \in (n/(n-1), (1 + \sqrt{1 + 8n})/4) \).

1. Introduction.

Let \((M^n, g)\) be a \( n \)-dimensional compact Riemannian manifold without boundary. The standard Sobolev-Poincaré inequality can be stated as the following: For any \( p \in (1, n) \), there is a constant \( A(p, M^n, g) > 0 \) such that

\[
\left( \int_{M^n} |u - u_a|^p \right)^{p/p_*} \leq A(p, M^n, g) \int_{M^n} |\nabla_g u|^p, \quad \forall u \in W^{1,p}(M^n)
\]

where \( u_a = \frac{1}{\text{vol}(M^n)} \int_{M^n} u \), \( p_* = np/(n - p) \) is the Sobolev conjugate of \( p \). This inequality can be proved by combining Sobolev inequality with Poincaré inequality; see, for example, Hebey’s book \[8\]. In this paper we are interested in the estimates of the best constant and the existence of extremal functions to the above inequality. Analytically these are naturally motivated questions. On the other hand they may have interesting geometric implications, in particular, in the study of Poincaré’s isoperimetric inequality. We shall discuss this geometric issue in another paper \[9\].

Generally speaking, the existence of extremal functions is not a trivial issue, given the fact that \( p_* \) is the critical Sobolev exponent for the Sobolev embedding theorem. In this paper, for a model manifold -the unit sphere \( S^n \) with the standard metric \( g_0 = \sum_{i=1}^{n+1} dx_i^2 \), we obtain a fairly satisfying result. Define the Sobolev-Poincaré quotient by

\[
I_p(u) := \frac{\int_{S^n} |\nabla u|^p}{(\int_{S^n} |u - u_A|^{p_*)^{p/p_*}})
\]

where \( u_A = \frac{1}{\text{vol}(S^n)} \int_{S^n} u \). We are going to prove:
Theorem 1.1. If $1 < p < (1 + \sqrt{1 + 8n})/4$, then

$$P_1(S^n, p) := \inf \{ I_p(u) : u \in C^1(S^n) \setminus \{0\} \}$$

is achieved.

The main idea in the proof is to show that there is a minimizing sequence which strongly converges (in $L^{p^*}$ sense) to a nonzero function. Let

$$k(n, p) = \pi^{-1/2} n^{-1/p} \left( \frac{p - 1}{n - p} \right)^{1 - 1/p} \left\{ \frac{\Gamma(1 + n/2) \Gamma(n)}{\Gamma(n/p) \Gamma(1 + n - n/p)} \right\}^{1/n}$$

and $k(n, 1) = \lim_{p \to 1^+} k(n, p) = \pi^{-1/2} n^{-1} \left\{ \Gamma(1 + n/2) \right\}^{1/n}$. If one can prove that

$$P_1(S^n, p) < \frac{1}{k^p(n, p)}, \quad (1.3)$$

the convergence of a minimizing sequence will follow from some standard arguments. However, for general $p$ such a strict inequality may not be true. For example, from Bernstein inequality on $S^2$ one can prove (see more details in [9]) that

$$\inf_{u \in BV(S^2) \setminus \{0\}} \frac{\int_{S^2} |\nabla u|}{\int_{S^2} |u - A|^n} = \frac{1}{k(2, 1)} = 2\sqrt{\pi}.$$

Nevertheless, by choosing a suitable (but standard) test function we can show the following:

Theorem 1.2. Let $(M^n, g)$ be a $n$-dimensional compact Riemannian manifold without boundary embedded in $R^{n+1}$. If $p \in (n/(n - 1), (1 + \sqrt{1 + 8n})/4)$ for $n \geq 4$, then

$$P_1(M^n, p) := \inf_{u \in C^1(M^n) \setminus \{0\}} \frac{\int_{M^n} |\nabla g u|^p}{\int_{M^n} |u - u_0|^{p/p^*}} < \frac{1}{k^p(n, p)}, \quad (1.4)$$

and the infimum is achieved at some $u_0 \in C^1(M^n)$.

The case of $p \in (n/(n - 1), (1 + \sqrt{1 + 8n})/4)$ in Theorem 1.1 is obviously covered by Theorem 1.2. It is unclear whether the strict inequality (1.4) is still true for $p \geq (1 + \sqrt{1 + 8n})/4$ or not. On the other hand, one may check that if $n = 3$ inequality (1.4) holds for $p = 1$, thus it is true for $p \in [1, 1 + \delta_0]$ for some positive number $\delta_0$. Unfortunately, we have no information about this $\delta_0$. We may guess that the strict inequality (1.4) holds for all $p \in [1, n]$ and $n \geq 2$ except the case of $p = 1$ and $n = 2$. Even though we do not know whether (1.4) holds for general $p$ or not, however, if we return to the special manifold $S^n$ and change the constraint on $u$ slightly, we obtain the
following result for all $p \in (1, n)$. For convenience, throughout the paper we denote

$$H_a(S^n) := \left\{ u \in C^1(S^n) : \int_{S^n} |u|^{p_* - 2} u = 0 \right\}.$$  

**Theorem 1.3.** If $1 < p < n$, then

$$P_{II}(S^n, p) := \inf_{u \in H_a(S^n) \setminus \{0\}} \frac{\int_{S^n} |\nabla u|^p}{(\int_{S^n} |u|^{p_*})^{p/p_*}},$$

is achieved at some $u_0 \in H_a(S^n)$.

It shall be pointed out that in the case of $n = 2$ and $p = 1$ we have $p_* - 2 = 0$. Therefore the constraints on $u$ in Theorem 1.3 is the same as that in Theorem 1.1.

Based on the symmetrization result on $S^n$ (see, for example, Baernstein [2]), we can assume that there is a minimizing sequence depending only on one variable. Amazingly, the case of $p < 2n/(n + 1)$ in Theorem 1.1 can be handled in the same spirit as that in the proof of Theorem 1.3 (note that this upper bound of $p$ matches the lower bound of $p$ in Theorem 1.2 perfectly).

We make a final remark in this introduction. When the manifold is a sphere with the standard metric, if one can show that there is an antisymmetric minimizing sequence $u_k$ (that is, $u(x', x_n) = -u(x', -x_n)$) to $\inf I_p(u)$, one easily obtains (1.3) and Theorem 1.1 follows. Unfortunately, such an expectation may not be realized in practice, given the fact that there are the extremal functions for $\inf I_1(u)$ on $S^2$ which are not antisymmetric. We refer to [4] for some related issues on the symmetric properties of the extremal functions.

We organize the paper as follows: In Section 2, we prove Theorem 1.3; in Section 3, we prove Theorem 1.1 and Theorem 1.2.

**2. Proof of Theorem 1.3.**

For a small positive parameter $0 < \epsilon \ll 1$, we define $q_\epsilon = p_* - \epsilon$,

$$H_{a, \epsilon}(S^n) := \left\{ u \in C^1(S^n) : \int_{S^n} |u|^{q_\epsilon - 2} u = 0 \right\},$$

and

$$S_\epsilon := \inf_{u \in H_{a, \epsilon}(S^n) \setminus \{0\}} J_\epsilon(u) := \inf_{u \in H_{a, \epsilon}(S^n) \setminus \{0\}} \frac{\int_{S^n} |\nabla u|^p}{(\int_{S^n} |u|^{q_\epsilon})^{p/q_\epsilon}}.$$

Let $(\alpha_1, \ldots, \alpha_{n-1}, \theta)$ be the spherical coordinates of $S^n$, where $0 \leq \alpha_i < 2\pi$, $-\pi/2 \leq \theta < \pi/2$. Standard variational method shows that $\inf J_\epsilon(u)$ is attained in $H_{a, \epsilon}(S^n)$. Further, the symmetrization argument (see, e.g., Baernstein [2]) yields that the extremal function $u_\epsilon(x)$ only depends on $\theta$ and is a
monotonically non-decreasing function of $\theta$. We can normalize $u_\varepsilon$ such that
\[
\int_{S^n} |u_\varepsilon|^q = 1.
\]
Thus, $u_\varepsilon$ satisfies the following equation:
\[
\begin{cases}

\nabla(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) + S_\varepsilon |u_\varepsilon|^{q-2}u_\varepsilon = 0 \quad \text{in } S^n \\
\frac{du_\varepsilon(\theta)}{d\theta} \geq 0.
\end{cases}
\]

If $\|u_\varepsilon\|_{L^\infty} \leq C$, then from the elliptic estimates (see, for example, [5]) we know that $\|u_\varepsilon\|_{C^{1,\alpha}} \leq C$ for some $\alpha \in (0,1)$, and conclude that up to a subsequence of $\varepsilon$, $u_\varepsilon \to u_0$ in $C^{1,\alpha}$ as $\varepsilon \to 0$, where $u_0$ is the minimizer of $J_0$. Hence Theorem 1.3 is proved. So we shall focus on ruling out the case:

Up to a subsequence of $\varepsilon$,
\[
\|u_\varepsilon\|_{L^\infty} \to \infty \quad \text{as } \varepsilon \to 0.
\]

We denote $\theta_\varepsilon$ as the zero point of $u_\varepsilon$, and assume that $\theta_\varepsilon \to \theta_0$ (up to a subsequence of $\varepsilon$).

Further, without loss of generality we can assume that
\[
\begin{cases}
u_\varepsilon(\theta) < 0 \quad \text{for } -\frac{\pi}{2} \leq \theta < \theta_\varepsilon \\
u_\varepsilon(\theta) > 0 \quad \text{for } \theta_\varepsilon < \theta \leq \frac{\pi}{2},
\end{cases}
\]
and $u_\varepsilon(\pi/2) = \max_{\theta} u_\varepsilon(\theta) = \|u_\varepsilon\|_{L^\infty}$.

**Proposition 2.1.** Given $\delta > 0$, for any $\theta < \pi/2 - \delta$,
\[
u_\varepsilon(\theta) \to 0 \quad \text{uniformly as } \varepsilon \to 0.
\]

**Proof.** The proposition can be proved via the following standard two steps:

First of all we claim that
\[
\lim_{\varepsilon \to 0} S_\varepsilon \leq \frac{1}{k^p(n,p)}.
\]

We relegate the proof of this inequality to the end of this paper.

Then we have the following concentration phenomena: For any fixed $\delta > 0$,
\[
\liminf_{\varepsilon \to 0} \int_{S^n \cap \{\theta > \pi/2 - \delta\}} u_\varepsilon^q = 1.
\]

As a consequence, one can obtain Proposition 2.1. We refer readers to, for example, [7] for more details.

Now, for any $\tau \in (-\pi/2, \pi/2)$, we define $\mu_\varepsilon = \max_{\theta \in [-\pi/2, \pi]} |u_\varepsilon(\theta)|$. From Proposition 2.1 we know that $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$. Let $v_\varepsilon(\theta) = -u_\varepsilon(\theta)/\mu_\varepsilon$. 

\[
\begin{align*}
\int_{S^n} |u_\varepsilon|^q &= 1 \\
abla(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) + S_\varepsilon |u_\varepsilon|^{q-2}u_\varepsilon &= 0 \\
du_\varepsilon(\theta) &= 0.
\end{align*}
\]
Then \( v_\epsilon \) satisfies:

\[
\begin{align*}
\nabla(|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon) + S_\epsilon \mu_\epsilon^{q-2} v_\epsilon^{q-1} = 0 & \quad \text{in } S^n \cap \{-\pi/2 < \theta < \tau\} \\
0 \leq |v_\epsilon| \leq 1 & \quad \text{for } -\pi/2 < \theta < \tau.
\end{align*}
\]

(2.11)

It follows from the standard elliptic theory that \( v_\epsilon \to v_0 \) in any compact set of \( \{x = (\alpha_1, \ldots, \alpha_{n-1}, \theta) \in S^n \mid -\pi/2 < \theta < \tau\} \), where \( v_0 \) satisfies:

\[
\begin{align*}
\nabla(|\nabla v_0|^{p-2}\nabla v_0) = 0 & \quad \text{for } -\pi/2 < \theta < \tau \\
\max|v_0(\theta)| = 1, \quad 0 \leq |v_0| \leq 1 & \quad \text{for } -\pi/2 < \theta < \tau.
\end{align*}
\]

Since \( v_0(\theta) \) is monotonically non-decreasing in \( \theta \), in both cases (\( \tau > \theta_0 \) or \( \tau < \theta_0 \)) we obtain a contradiction due to the maximum principle!

### 3. Proof of Theorems 1.1 and 1.2.

Let us first establish Theorem 1.1 in the case of \( p < 2n/(n + 1) \). We follow the main stream in the proof of Theorem 1.3. For a small positive parameter \( 0 < \epsilon \ll 1 \), we define \( q_\epsilon = p_\epsilon - \epsilon \), and

\[
Z_\epsilon := \inf_{u \in H_{a,p_\epsilon}(S^n) \setminus \{0\}} I_\epsilon(u) := \inf_{u \in H_{a,p_\epsilon}(S^n) \setminus \{0\}} \frac{\int_{S^n} |\nabla u|^p}{\left(\int_{S^n} |u|^{q_\epsilon}\right)^{p/q_\epsilon}},
\]

where

\[
H_{a,p_\epsilon}(S^n) = \left\{ u \in C^1(S^n) : \int_{S^n} u = 0 \right\}.
\]

Standard variational method and the symmetrization argument show that \( \inf I_\epsilon(u) \) is attained by \( u_\epsilon(x) \) which depends only on \( \theta \) and is a monotonically non-decreasing function of \( \theta \). We normalize \( u_\epsilon \) such that

\[
\int_{S^n} |u_\epsilon|^{|\epsilon|} = 1.
\]

(3.13)

If \( \|u_\epsilon\|_{L^\infty} < \infty \), we are done. Otherwise, we assume, up to some subsequence of \( \epsilon \), that

\[
\|u_\epsilon\|_{L^\infty} \to \infty \quad \text{as } \epsilon \to 0.
\]

Let \( \theta_\epsilon \) be the zero point of \( u_\epsilon \) and denote \( c_\epsilon = \int_{S^n} |u_\epsilon|^{q_\epsilon-2}u_\epsilon \). Up to a further subsequence of \( \epsilon \), we can assume that \( \theta_\epsilon \to \theta_0 \). It follows easily from Hölder inequality and (3.13) that

\[
|c_\epsilon| \leq C.
\]

Without loss of generality, we can assume that

\[
\begin{align*}
\begin{cases}
\begin{align*}
u_\epsilon(\theta) &< 0 & \text{for } -\pi/2 \leq \theta < \theta_\epsilon \\
u_\epsilon(\theta) &> 0 & \text{for } \theta_\epsilon < \theta \leq \pi/2,
\end{align*}
\end{cases}
\end{align*}
\]

(3.14)
and $u_\epsilon(\pi/2) = \max u_\epsilon(\theta) = \|u_\epsilon\|_{L^\infty}$. The Euler-Lagrange equation of $u_\epsilon$ is given by:

$$\begin{cases}
\nabla(|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon) + Z_\epsilon|u_\epsilon|^{q_\epsilon-2}u_\epsilon - Z_\epsilon c_\epsilon = 0 & \text{in } S^n \\
du_\epsilon(\theta)/d\theta \geq 0,
\end{cases} \tag{3.15}$$

(3.15) implies $s = 1/(q_\epsilon - 1)$. Using the fact that $p < 2n/(n + 1)$, one can check that for small $\epsilon$, $s + 1 - p > 0$. Thus it follows from the standard elliptic theory that $v_\epsilon \to v_0$ in $C^{1,\alpha}(K)$ for any compact set $K$ of $\{x = (\alpha_1, \ldots, \alpha_{n-1}, \theta) \in S^n : -\pi/2 < \theta < \theta_0\}$, where $v_0$ satisfies:

$$\begin{cases}
\nabla(|\nabla v_0|^{p-2}\nabla v_0) = 0 & \text{for } -\pi/2 < \theta < \theta_0 \\
v_0(-\pi/2) = \max_{-\pi/2 \leq \theta \leq \theta_0} |v_0(\theta)| = 1, \\
0 \leq v_0 \leq 1 & \text{for } -\pi/2 < \theta < \theta_0.
\end{cases}$$

We thus derive a contradiction due to the maximum principle.
If $\theta_0 = -\frac{\pi}{2}$, then for sufficiently small $\epsilon$, $\theta_\epsilon < 0$, thus $u_\epsilon(\theta) \geq 0$ for $\theta \geq 0$. Therefore $\int_{S^n \cap \{-\pi/2 \leq \theta \leq \pi/4\}} u_\epsilon \leq 0$. It follows that

$$|u_\epsilon(-\pi/2)| \cdot \left(\theta_\epsilon + \frac{\pi}{2}\right) \geq C \int_{S^n \cap \{\theta_\epsilon \leq \pi/4\}} u_\epsilon$$

$$\geq C \int_{S^n \cap \{0 \leq \theta \leq \pi/4\}} u_\epsilon \geq C u_\epsilon(0).$$

This yields that $|u_\epsilon(-\pi/2)| \geq C u_\epsilon(0)$ for sufficiently small $\epsilon$. We then consider $v_\epsilon = -u_\epsilon/\mu_\epsilon$ in the lower hemisphere and obtain $v_\epsilon \to v_0$ in $C^{1,\alpha}(K)$ for any compact set $K$ of the lower hemisphere, where $v_0$ satisfies:

$$\begin{cases}
\nabla(|\nabla v_0|^{p-2}\nabla v_0) = 0 & \text{for } -\pi/2 < \theta < 0 \\
v_0(-\pi/2) = \max_{-\pi/2 \leq \theta \leq 0} v_0(\theta) = 1, \ 0 \leq |v_0| \leq 1 & \text{for } -\pi/2 < \theta < 0.
\end{cases}$$

Again this contradicts the maximum principle! We hereby complete the proof of Theorem 1.1 in the case of $p < 2n(n+1)$.

To prove Theorem 1.1 in the case of $p \geq 2n(n+1)$, we only need to prove Theorem 1.2.

**Proof of Theorem 1.2.** We first establish Theorem 1.2 under the assumption that (1.4) holds.

We quote the following lemma from Aubin’s book [1]:

**Lemma 3.1.** Given $\delta > 0$, there is a constant $C(\delta)$ such that

$$\left(\int_{M^n} |u|^{p_\ast} dv_g\right)^{p/p_\ast} \leq (k^p(n,p) + \delta) \int_{M^n} |\nabla_g u|^p dv_g + C(\delta) \int_{M^n} |u|^p dv_g$$

holds for all $u \in W^{1,p}(M^n)$.

Let $\{u^{(m)}\}$ be a minimizing sequence with $\int_{M^n} u^{(m)} dv_g = 0$ and

$$\|u^{(m)}\|_{P_\ast, M^n} = \left(\int_{M^n} |u^{(m)}|^{p_\ast} dv_g\right)^{1/p_\ast} = 1.$$

Clearly, $\|u^{(m)}\|_{W^{1,p}(M^n)} \leq C$. After passing to a subsequence, $u^{(m)}$ converges weakly to some $\tilde{u} \in W^{1,p}(M^n)$, and $u^{(m)}$ converges strongly to $\tilde{u}$ in $L^q$ for any $q < p_\ast$. Thus $\int_{M^n} \tilde{u} dv_g = 0$. Due to Brezis-Lieb’s lemma [3], it is not difficult to see that

$$\int_{M^n} (|u^{(m)}|^{p_\ast} - |u^{(m)} - \tilde{u}|^{p_\ast}) dv_g = \int_{M^n} |\tilde{u}|^{p_\ast} dv_g + o(1),$$

and consequently

$$\int_{M^n} (|u^{(m)}|^{p_\ast} - |u^{(m)} - \tilde{u}|^{p_\ast}) dv_g \leq 1 + o(1), \quad \int_{M^n} |\tilde{u}|^{p_\ast} dv_g \leq 1,$$

where $o(1)$ denotes various quantity tending to zero as $m$ tends to $\infty$. 
Choose \( \delta \) such that \( \frac{1}{k^p(n,p) + \delta} - P_1(M^n, p) \geq \delta/2 \). By the Sobolev embedding theorem and Lemma 3.1, we have:

\[
P_1(M^n, p) = \int_{M^n} |\nabla g u^{(m)}|^p + o(1)
\]

\[
= \int_{M^n} |\nabla g (u^{(m)} - \tilde{u})|^p + \int_{M^n} |\nabla g \tilde{u}|^p + o(1)
\]

\[
= \int_{M^n} |\nabla (u^{(m)} - \tilde{u})|^p + \frac{C(\delta)}{k^p(n,p) + \delta} \int_{M^n} |u^{(m)} - \tilde{u}|^p + \int_{M^n} |\nabla g \tilde{u}|^p + o(1)
\]

\[
\geq \frac{1}{k^p(n,p) + \delta} \left( \int_{M^n} |u^{(m)} - \tilde{u}|^{p/p^*} \right)^{p/p^*} + P_1(M^n, p) \left( \int_{M^n} \tilde{u}^{p^*} \right)^{p/p^*} + o(1)
\]

\[
\geq \frac{1}{k^p(n,p) + \delta} \int_{M^n} |u^{(m)} - \tilde{u}|^{p^*} + P_1(M^n, p) \int_{M^n} \tilde{u}^{p^*} + o(1)
\]

\[
= \left( \frac{1}{k^p(n,p) + \delta} - P_1(M^n, p) \right) \int_{M^n} |u^{(m)} - \tilde{u}|^{p^*} + P_1(M^n, p) + o(1).
\]

This yields that \( \int_{M^n} |u^{(m)} - \tilde{u}|^{p^*} = o(1) \). It follows easily that \( \tilde{u} \) is a minimizer. Theorem 1.2 is proved.

We are left to verify (1.4).

**Proof of (1.4).**

We follow closely the computations in Druet [6]. Since \( M^n \) is embedded in \( R^{n+1} \), there is a point \( x_0 \in M \) such that the scalar curvature at \( x_0 \) (denoted as \( R_g(x_0) \)) is positive. We choose a normal geodesic coordinates system near \( x_0 \). Let \( r \) be the distance from point \( x \) to \( x_0 \), we define, for small positive constants \( \epsilon \) and \( \delta \),

\[
u_{r}(x) = u_{r}(r) = \left( \frac{1}{\epsilon + r^{p/(p-1)}} \right)^{\frac{2}{p-1}} \cdot \varphi(r),
\]

where \( \varphi(r) \) is a nonnegative cut-off function satisfying:

\[
\varphi(r) = \begin{cases} 
1 & \text{for } r \leq \frac{\delta}{2}, \\
\leq 1 & \text{for } \frac{\delta}{2} \leq r \leq \delta, \\
0 & \text{for } r \geq \delta.
\end{cases}
\]

Notice that in this system,

\[
dv_g = \left[ 1 - \frac{1}{6} R_{ij}(x_0) x^i x^j + o_r(1)(r^2) \right] dx^i dx^j,
\]

where and throughout this section we denote \( o_r(1) \) as the term tending to 0 as \( \gamma \to 0 \).

We divide our computation into four steps.
Step 1. Estimate of \( \int_{M^n} u_p^* dv_g \).

We have, from (3.18), that

\[
\int_{M^n} u_p^* dv_g \geq \int_{B_{\delta/2}(x_0)} \left( \frac{1}{\epsilon + r^{p/(p-1)}} \right)^{(n-1) \cdot \frac{n}{p-1}} dv_g
\]

\[
= \omega_{n-1} \int_0^{\delta/2} (\epsilon + r^{p/(p-1)})^{-n} r^{n-1} dr
\]

\[
- \frac{\omega_{n-1} R_g(x_0)}{6n} (1 + o_\delta(1)) \int_0^{\delta/2} (\epsilon + r^{p/(p-1)})^{-n} r^{n+1} dr
\]

\[
= \omega_{n-1} \epsilon^{-\frac{n}{p}} \int_0^{\delta/2} (1 + \beta^{p/(p-1)})^{-n} \beta^{n-1} d\beta,
\]

where \( \omega_{n-1} \) is the surface area of \( S^{n-1} \). For \( p < \frac{n+2}{2} \) we know that \( n + 2 - \frac{pn}{p-1} < 0 \). Thus

\[
\int_0^{\delta/2} (1 + \beta^{p/(p-1)})^{-n} \beta^{n+1} d\beta \leq C.
\]

It follows that (we also use \( n > 2(p-1) \))

\[
\omega_{n-1} \epsilon^{-\frac{n}{p}} \int_0^{\delta/2} (1 + \beta^{p/(p-1)})^{-n} \beta^{n-1} d\beta
\]

\[
\geq D_1 \cdot \epsilon^{-\frac{n}{p}} - C \delta^{-\frac{n}{p-1}} \epsilon^{n/p} \cdot \epsilon^{-\frac{n}{p}}
\]

\[
= D_1 \cdot \epsilon^{-\frac{n}{p}} - o_\epsilon(1) \delta^{-\frac{n}{p-1}} \epsilon^{n+2(p-1)/p},
\]

where

\[
D_1 = \omega_{n-1} \int_0^{\infty} \beta^{n-1} (1 + \beta^{p/(p-1)})^{-n} d\beta;
\]

and

\[
\frac{\omega_{n-1} R_g(x_0) \epsilon^{-\frac{n}{p}}}{6n} (1 + o_\delta(1)) \int_0^{\delta/2} (1 + \beta^{p/(p-1)})^{-n} \beta^{n+1} \cdot \epsilon^{2(p-1)/p} d\beta
\]

\[
\leq D_2 (1 + o_\delta(1)) \epsilon^{-\frac{n}{p} + 2(p-1)/p}
\]

where

\[
D_2 = \frac{\omega_{n-1} R_g(x_0) \epsilon^{-\frac{n}{p}}}{6n} \int_0^{\infty} \beta^{n+1} (1 + \beta^{p/(p-1)})^{-n} d\beta.
\]
Therefore, for $p < \frac{n+2}{2}$,

\begin{equation}
\int_{M^n} u_{\varepsilon}^p \, dv_g \geq D_1 \varepsilon^{-n/p} - (D_2 + o_{\varepsilon, \delta}(1)) \varepsilon^{-n/p + 2(p-1)/p},
\end{equation}

where $o_{\varepsilon, \delta}(1)$ means that for any $\gamma > 0$ there is a $\delta_0$ such that for $\delta < \delta_0$, $\liminf_{\varepsilon \to 0} o_{\varepsilon, \delta}(1) < \gamma$. Notice that $(1 + \sqrt{1 + 8n})/4 < (n + 2)/2$ for $n \geq 2$.

**Step 2.** Estimate of $\int_M |\nabla u_\varepsilon|^p \, dv_g$.

If we denote

$$
\tau(r) = (\varepsilon + r^{p/(p-1)})^{1-n/p},
$$

then

$$
\nabla u_\varepsilon = \nabla \tau(r) \varphi(r) + \tau(r) \nabla \varphi(r),
$$

and

$$
|\nabla u_\varepsilon|^p_g \leq |\varphi(r) \nabla \tau(r)|^p_g + \nu |\nabla \varphi(r) \tau(r)|^p_g + \mu |\nabla \varphi(r) \tau(r)| |\nabla \tau(r)|^{p-1}
$$

for some positive constants $\nu$ and $\mu$.

Since $\nabla \varphi(r) = 0$ for $0 \leq r \leq \delta/2$, we know that

$$
\int_M (\nu |\nabla \varphi(r) \tau(r)|^p + \mu |\nabla \varphi(r) \tau(r)| |\nabla \tau(r)|^{p-1}) \, dv_g \leq C(\delta)
$$

for some constant $C(\delta)$ independent of $\varepsilon$. On the other hand, noting that $u_\varepsilon$ is radially symmetric and $g^{rr} = 1$, we know that

$$
|\varphi(r) \nabla \tau(r)|^p_g = \left( \frac{n-p}{p-1} \right)^p \cdot r^{p/(p-1)} \cdot (\varepsilon + r^{p/(p-1)})^{-n}.
$$

Therefore

$$
\int_{M^n} |\nabla u_\varepsilon|^p \, dv_g 
\leq C(\delta) + \int_M |\nabla \tau(r)|^p \, dv_g 
\leq C(\delta) + \omega_{n-1} \int_0^\delta \left( \frac{n-p}{p-1} \right)^p \cdot r^{p/(p-1)}(\varepsilon + r^{p/(p-1)})^{-n} \cdot r^{n-1} \, dr 
- \frac{\omega_{n-1} R_g(x_0)}{6n} (1 + o_\delta(1)) \int_0^\delta \left( \frac{n-p}{p-1} \right)^p \cdot r^{p/(p-1)}(\varepsilon + r^{p/(p-1)})^{-n} \cdot r^{n+1} \, dr.
$$

One can easily check:

$$
\omega_{n-1} \left( \frac{n-p}{p-1} \right)^p \int_0^\delta (\varepsilon + r^{p/(p-1)})^{-n} \cdot r^{p/(p-1)+n-1} \, dr 
= \omega_{n-1} \left( \frac{n-p}{p-1} \right)^p \epsilon^{1-n} \cdot \int_0^{\delta \epsilon} \frac{1}{r^{p-1}} \cdot (1 + \beta^{p/(p-1)})^{-n} \beta^{p/(p-1)+n-1} \, d\beta 
\leq \left( E_1 + C \delta \epsilon^{n+1} \cdot \epsilon^{\frac{n-p}{p}} \right) \epsilon^{1-n},
$$
where
\[ E_1 = \omega_{n-1} \left( \frac{n-p}{p-1} \right)^p \int_0^\infty (1 + \beta^{p/(p-1)})^{-n/2} \beta^{p/(p-1)+n-1} d\beta. \]

Note that \( p/(p-1) + n - 1 > -1 \), and \( p/(p-1) + n - p/(p-1) = (p-n)/(p-1) < 0 \), thus \( E_1 < \infty \). Also for \( p < (n+2)/3 \),
\[
\frac{\omega_{n-1} R_g(x_0)}{6n} \left( \frac{n-p}{p-1} \right)^p \int_0^\delta (\epsilon + \gamma p/(p-1))^{-n} \gamma^{p/(p-1)+n+1} d\gamma
\]
\[ = \frac{\omega_{n-1} R_g(x_0)}{6n} \left( \frac{n-p}{p-1} \right)^p \epsilon^{3p-2-n/p} \int_0^{\delta \epsilon^{1-p/p}} (1 + \beta^{p/(p-1)})^{-n \beta^{p/(p-1)+n+1}} d\beta \]
\[ \leq \left( E_2 + C(\beta^{p-1}/p) \cdot \epsilon^{1-\frac{n}{p} + \frac{2(p-1)}{p}} \right), \]
where
\[ E_2 = \frac{\omega_{n-1} R_g(x_0)}{6n} \left( \frac{n-p}{p-1} \right)^p \int_0^\infty (1 + \beta^{p/(p-1)})^{-n \beta^{p/(p-1)+n+1}} d\beta < \infty. \]

For \( p < (n+2)/3 \), we have \((n-p)/p > 2(p-1)/p\). Thus,
\[
(3.21) \quad \int_{M^n} |\nabla u_\epsilon|^p dv_g \leq C(\delta) + E_1 \epsilon^{1-\frac{n}{p}} - E_2 (1 + o_\epsilon(1)) \epsilon^{1-\frac{n}{p} + \frac{2(p-1)}{p}}. \]

We also note that \((1 + \sqrt{1+8n})/4 < (n+2)/3 \) for \( n \geq 2 \).

**Step 3.** Estimates of \( \int_{M^n} u_\epsilon \).
\[
(3.22) \quad \int_{M^n} u_\epsilon \leq \omega_{n-1} \int_0^{\delta} (\epsilon + \gamma^{p/(p-1)})^{1-n/p} \gamma^{n-1} d\gamma
\]
\[ \leq C_1 \epsilon^{1-\frac{n}{p} + \frac{n(p-1)}{p}} \cdot \int_0^{\delta \epsilon^{1-p/p}} (1 + \beta^{p/(p-1)})^{1-\frac{n}{p} \beta^{n-1}} d\beta \]
\[ \leq \begin{cases} C(\delta) \epsilon^{\frac{np+p-2n}{p}} & \text{if } np + p - 2n < 0 \\ C(\delta) \ln(\delta \epsilon^{1-p/p} + C) & \text{if } np + p - 2n = 0 \\ C(\delta) & \text{if } np + p - 2n > 0. \end{cases} \]

**Step 4.** Define \( w_\epsilon = u_\epsilon - \int_{M^n} u_\epsilon \). Then \( \int_{M^n} |\nabla w_\epsilon|^p = \int_{M^n} |\nabla u_\epsilon|^p \). From Minkowski inequality, we know
\[
\left| \left( \int_{M^n} |w_\epsilon|^p \right)^{1/p} - \left( \int_{M^n} |u_\epsilon|^p \right)^{1/p} \right| \leq \text{vol}^{1/p}(M^n) \cdot \int_{M^n} |u_\epsilon|. \]
From (3.20) we have

\[(3.23) \quad \left( \int_{M^n} |u_\epsilon|^{p^*_*} \right)^{1/p^*_*} \geq (D_1 \epsilon^{-n/p} - (D_2 + o_(\epsilon,\delta)(1)) \epsilon^{-n/p + 2(p-1)/p})^{1/p^*_*} = D_1^{1/p^*_*} \epsilon^{-\frac{n}{pp^*_*}} - \left( \frac{D_1^{1/p^*_*} D_2}{D_1 p^*_*} + o_{\epsilon,\delta}(1) \right) \epsilon^{-\frac{n}{pp^*_*} + \frac{2(p-1)}{p}}.\]

If \( p \geq 2n/(n+1) \) (i.e., \( np + p - 2n \geq 0 \)) and

\[(3.24) \quad -\frac{n}{pp^*_*} + \frac{2(p-1)}{p} < 0,\]

we obtain from (3.22) that

\[(3.25) \quad \left( \int_{M^n} |w_\epsilon|^{p^*_*} \right)^{1/p^*_*} \geq D_1^{1/p^*_*} \epsilon^{-\frac{n}{pp^*_*}} - \left( \frac{D_1^{1/p^*_*} D_2}{D_1 p^*_*} + o_{\epsilon,\delta}(1) \right) \epsilon^{-\frac{n}{pp^*_*} + \frac{2(p-1)}{p}}.\]

This is the place where we need the condition \( p < (1 + \sqrt{1 + 8n})/4 \) to guarantee that (3.24) holds.

On the other hand, if the dimension \( n \geq 4 \), for any \( p \in (n/(n-1), 2n/(n+1)) \), one can check:

\[-\frac{n}{pp^*_*} + \frac{2(p-1)}{p} < \frac{np + p - 2n}{p} < 0.\]

From (3.22) and (3.23) we know that (3.25) still holds. From (3.25) and (3.21) one can derive that for some small enough \( \delta \) and \( \epsilon \),

\[\frac{\int_{M^n} |
abla w_\epsilon|^p}{(\int_{M^n} |w_\epsilon|^{p^*_*})^{p/p^*_*}} \leq C(\delta) + E_1 \epsilon^{-\frac{n}{p}} - E_2(1 + o_{\epsilon,\delta}(1)) \epsilon^{-\frac{n}{p} + \frac{2(p-1)}{p}} \leq D_1^{p/p^*_*} \epsilon^{-\frac{n}{p^*_*}} - \left( \frac{pD_1^{p/p^*_*} D_2}{p^*_* D_1} + o(1) \right) \epsilon^{-\frac{n}{p^*_*} + \frac{2(p-1)}{p}} \leq \frac{1}{k^p(n,p)}.\]

More details about the derivation of the second inequality in the above can be found in Druet’s paper [6]. We hereby complete the Proof of (1.4).
Now we return to the proof of (2.9). We use the same test function. Similar to Step 3, we have
\[
\begin{align*}
\int_{M^n} |u_\epsilon|^{p_\ast-2}u_\epsilon & \\
\leq \omega n^{-1} \int_0^\delta \left( \epsilon + r^{p/(p-1)} \right)^{-n(p-n)+p/p} r^{n-1} dr \\
\leq C_1\epsilon^{-\frac{np-n+p}{p} + \frac{(p-1)}{p}} \cdot \int_0^\delta \left( 1 + \beta^{p/(p-1)} \right)^{-n(p-n)+p/p} \beta^{n-1} d\beta \\
\leq C\epsilon^{-1}.
\end{align*}
\]
We choose \( C_\epsilon \) such that for \( v_\epsilon = u_\epsilon - C_\epsilon \),
\[
\int_{M^n} |v_\epsilon|^{p_\ast-2}v_\epsilon dv_g = 0.
\]
Thus
\[
\begin{align*}
\int_{\{x \in M^n: u_\epsilon \geq C_\epsilon\}} |u_\epsilon - C_\epsilon|^{p_\ast-2}(u_\epsilon - C_\epsilon) dv_g & \\
= -\int_{\{x \in M^n: u_\epsilon < C_\epsilon\}} |u_\epsilon - C_\epsilon|^{p_\ast-2}(u_\epsilon - C_\epsilon) dv_g.
\end{align*}
\]
This implies that \( C_\epsilon \geq 0 \). It follows that
\[
(3.26) \quad \int_{M^n} |u_\epsilon - C_\epsilon|^{p_\ast-1} dv_g = 2 \int_{\{x \in M^n: u_\epsilon \geq C_\epsilon\}} |u_\epsilon - C_\epsilon|^{p_\ast-1} dv_g \\
\leq 2 \int_{M^n} |u_\epsilon|^{p_\ast-1} dv_g \leq 2C\epsilon^{-1}.
\]
But for any fixed \( p_\ast \), there are two positive constants \( a \) and \( b \) depending only on \( p_\ast \) such that
\[
(x - 1)^{p_\ast-1} \geq ax^{p_\ast-1} - b, \quad \forall x \geq 1.
\]
Applying the above in (3.26) we obtain that \( C_\epsilon \leq C\epsilon^{1/(p_\ast-1)} \). Using Minkowski inequality, we have
\[
\left( \int_{M^n} |u_\epsilon|^{p_\ast} \right)^{1/p_\ast} - C \cdot C_\epsilon \leq \left( \int_{M^n} |v_\epsilon|^{p_\ast} \right)^{1/p_\ast} \leq \left( \int_{M^n} |u_\epsilon|^{p_\ast} \right)^{1/p_\ast} + C \cdot C_\epsilon.
\]
For \( p \in (1, n) \), we have \(-1/(p_\ast-1) > -n/(pp_\ast)\). Similar to the computation in (3.20) we know that
\[
\left( \int_{M^n} |u_\epsilon|^{p_\ast} \right)^{1/p_\ast} = \left( 1 + a_\epsilon(1) \right)D_1^{1/p_\ast} \epsilon^{-n/(pp_\ast)}, \quad \forall p \in (1, n).
\]
Thus $C_\epsilon = o_\epsilon(1)(\int_{M^n} |u_\epsilon|^{p_\epsilon})^{1/p_\epsilon}$, and

$$\left(\int_{M^n} |v_\epsilon|^{p_\epsilon}\right)^{1/p_\epsilon} = (1 + o_\epsilon(1)) \left(\int_{M^n} |u_\epsilon|^{p_\epsilon}\right)^{1/p_\epsilon}.$$ 

It follows that

$$\lim_{\epsilon \to 0} S_\epsilon \leq S_0 \leq \lim_{\epsilon \to 0} \frac{\int_{M^n} |\nabla g v_\epsilon|^p}{\left(\int_{M^n} |v_\epsilon|^{p_\epsilon}\right)^{p/p_\epsilon}} = \lim_{\epsilon \to 0} \frac{\int_{M^n} |\nabla g u_\epsilon|^p}{\left(\int_{M^n} |u_\epsilon|^{p_\epsilon}\right)^{p/p_\epsilon}} \leq \frac{1}{k^p(n, p)}.$$

Acknowledgments. This paper should be a joint work with Elliott H. Lieb. Most ideas came out from some stimulative discussions with him; the author respects his decision with admiration that he will not put his name on this paper, and would like to dedicate this paper to him. The author thanks A. Baernstein and E. Hebey for some useful discussions. This work has been done while the author was visiting Princeton University, and was partially supported by the American Mathematical Society Centennial fellowship. He also would like to thank A. Chang, P. Yang and Mathematics Department of Princeton University for their hospitality during his visit.

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Received February 6, 2003 and revised June 2, 2003.

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