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SEPARATION OF GLOBAL SEMIANALYTIC SUBSETS OF 2-DIMENSIONAL ANALYTIC MANIFOLDS

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In this paper we prove that two global semianalytic subsets of a real analytic manifold of dimension two are separable if and only if there is no analytic component of the Zariski closure of the boundary which intersects the interior of one of the two sets and they are separable in a neighbourhood of each singular point of the boundary.

We show also that, unlike in the algebraic case, the obstructions at infinity are not relevant.

Introduction.

This paper is mainly concerned with the problem of separation for a special class of semianalytic subsets of an analytic surface M, namely the global semianalytic sets, i.e., semianalytic subsets admitting a description of the type

$$S = \bigcup_{i=1}^{p} \{x \in M | f_i(x) = 0, g_{i1}(x) > 0, \dots, g_{ik_i}(x) > 0\}.$$

Two such semianalytic subsets A and B are said to be *separable* if there exists an analytic function $f \in \mathcal{O}(M)$ such that $f(A) > 0$ and $f(B) < 0$.

If there exists a nonzero analytic function $f \in \mathcal{O}(M)$ such that $f(A) \geq 0$ and $f(B) \leq 0$, A and B are said to be *generically separable*, equivalently A and B are generically separable if there exists a proper global analytic set $Y \subset M$ such that $A \setminus Y$ and $B \setminus Y$ are separable.

Of course A and B cannot be separated if $A \cap B \neq \emptyset$, exactly as they cannot be generically separated if $\mathring{A} \cap \mathring{B} \neq \emptyset$.

As in the algebraic case, it is easy to realize that two open sets, even if they are disjoint, are in general not separable, as for instance the open sets as in Figure 1.

The separation problem makes sense also for constructible subsets of the real spectrum of a ring and this problem has been solved by Bröcker in terms of finite spaces of orderings (see [**ABR96**]).

In the algebraic setting, in view of the Artin-Lang property, which acts as a translator between semialgebraic sets and constructible sets in the real

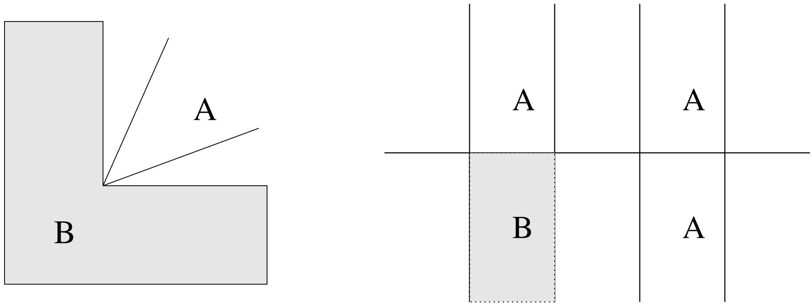

Figure 1.

spectrum of the ring of regular functions, this result makes it possible to characterize completely the geometric obstructions to separation.

Unfortunately the Artin-Lang property does not hold in general for the ring $\mathcal{O}(M)$ of global analytic functions (see [**AB90**]) and it has been proved only for the field of meromorphic functions on an analytic manifold of dimension 2 (see [**Cas94a**]).

One among the reasons is the presence in $\operatorname{Spec}_{\mathrm{r}}\mathcal{O}(M)$ of the so-called *unbounded orderings*, whose associated ultrafilter does not converge to a point.

In this paper we prove that these orderings (at least in dimension 2) have no role in such type of problems: In fact we prove that A and B can be separated in a surface M if and only if they are separated in any compact set: This is essentially because of the fact that we can use in this setting Whitney approximation theorem.

This is one of main differences between the algebraic and the analytic cases: In fact it is easy to see that the last statement does not hold for semialgebraic sets, for instance, if we consider A and B as in Figure 2.

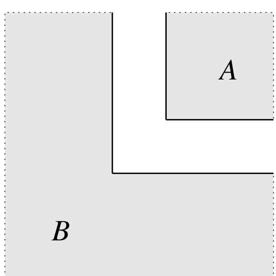

Figure 2.

We handle the separation problem in a rather direct way; we find (see Theorem 2.3 and Theorem 4.2) that the obstructions lie in the boundary of the sets A and B and that it is possible to list them in a similar way as in the algebraic case. Each one produces an obstruction for the separation of the associated constructible sets in the real spectrum which does not involve unbounded orderings.

The same methods apply to the basicness problem.

1. Generic equations for global analytic sets of codimension one.

The aim of this section is to prove the following theorem:

Theorem 1.1. *Let $M \subset \mathbb{R}^n$ be a connected real analytic manifold and let $Y \subset M$ be a global analytic set such that its irreducible components have all codimension one. Then, there is a global analytic set $Y' \subset M$ such that the ideal $\mathcal{I}(Y \cup Y') = \{f \in \mathcal{O}(M) | f_{|Y \cup Y'} = 0\}$ is principal. Moreover we can assume Y' to be smooth and $\dim Y \cap Y' < \dim Y$.*

Before proving the theorem we recall that a global analytic set Y admits coherent structures and admits complexifications, i.e., there exists a coherent ideal sheaf $\mathcal{F} \subset \mathcal{O}_M$ such that $Y = \operatorname{Supp} \mathcal{O}_M/\mathcal{F}$ and there exists a complex analytic space $\widetilde{Y}$ in a complexification $\widetilde{M}$ of M (in the sense of [**Tog67**], being M a manifold) such that $\widetilde{Y} \cap M = Y$; moreover this three properties (being global, admitting a coherent structure, being the real part of a complex analytic set) are equivalent.

One can prove that among those coherent sheaves there is a largest one, say still $\mathcal{F}$; also among those complex analytic sets there is a smallest one, say still $\widetilde{Y}$. Moreover $\mathcal{I}_{\widetilde{Y}} = \mathcal{F} \otimes_{\mathbb{R}} \mathbb{C}$, i.e., they define on Y the same structure, the so called *well reduced structure* (cf. [**ABT75**], [**Gal76**]).

Lemma 1.2. *Let $\mathcal{I}$ be the sheaf generated by the germs of the elements in $\mathcal{I}(Y) = \{f \in \mathcal{O}(M) | f_{|Y} = 0\}$. $\mathcal{I}$ is a locally principal coherent sheaf of ideals.*

Proof. The fact that $\mathcal{I}$ is coherent is known, cf. [**Fri67**], we want to prove that it defines on Y the well reduced structure. Indeed, let $\mathcal{G} \subset \mathcal{O}_M$ be a coherent ideal sheaf such that $Y = \operatorname{Supp} \mathcal{O}_M/\mathcal{G}$. By Cartan's Theorem A, the stalk of $\mathcal{G}$ in each point is generated by global sections. Each $f \in \Gamma(M, \mathcal{G})$ vanishes on Y, and then $\mathcal{G} \subset \mathcal{I}$ as wanted.

So, we can find complexifications $\widetilde{M}$, $\widetilde{Y}$ of M and Y such that $\widetilde{Y}$ has pure codimension one in $\widetilde{M}$ and $\mathcal{I}_{\widetilde{Y}} = \mathcal{I} \otimes_{\mathbb{R}} \mathbb{C}$. Hence $\mathcal{I} \otimes_{\mathbb{R}} \mathbb{C}$ is locally principal and this implies the same for $\mathcal{I}$. □

Let's recall a general result for analytic vector bundles.

Lemma 1.3. *Let $\xi = (E, \pi, M)$ be an analytic vector bundle of rank k and let $\sigma : M \to E$ be a C^∞ section. Consider $C^\infty(M, E)$ endowed with the Whitney topology. Then each neighbourhood U of σ in $C^\infty(M, E)$ contains a global analytic section.*

Proof. Arguing as in [**Tog80**] we get finitely many global analytic sections, $g^1, \dots, g^N$, such that $\sigma = \alpha_1 g^1 + \dots + \alpha_N g^N$ where $\alpha_1, \dots, \alpha_N \in C^\infty(M, \mathbb{R})$. Then, it is possible to find neighbourhoods U_i of α_i, $i = 1, \dots, N$, such that $\beta_1 g^1 + \dots + \beta_N g^N \in U$ for each $\beta_i \in U_i$. Since $\mathcal{O}(M)$ is dense in $C^\infty(M, \mathbb{R})$ (cf. [**Hir94**]) we can choose $\beta_1, \dots, \beta_N$ analytic and this proves the claim. □

Proof of Theorem 1.1. By Lemma 1.2 it follows that there exist a locally finite open covering $\{U_i\}_{i \in I}$ of M and $f_i \in \mathcal{O}(U_i)$ such that $\mathcal{I}_x = f_i \mathcal{O}_x$ for each $x \in U_i$; then, $f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)$ for each i, j such that $U_i \cap U_j \neq \emptyset$.

So, we can construct an analytic line bundle having $\{U_i\}_{i \in I}$ as its neighbourhoods of trivialization and $\{f_j/f_i\}$ as its transition functions. Observe that a global analytic section of this vector bundle is given by a collection $\{s_i\}_{i \in I}$ with $s_i \in \mathcal{O}(U_i)$ such that $s_i f_j = s_j f_i$ on $U_i \cap U_j$ for each i, j such that $U_i \cap U_j \neq \emptyset$.

Note that the functions $\{f_i\}$ give a global analytic section f such that $Y = f^{-1}(0) = \cup_i V(f_i)$. Let $\sigma = \{\sigma_i : U_i \to \mathbb{R}\}_{i \in I}$ be a C^∞ section transverse to the zero section and to the regular part of Y. Such a section exists by general theory of C^∞ vector bundle, see for instance [**Hir94**].

By Lemma 1.3 we can appoximate σ with a global analytic section $g = \{g_i\}_{i \in I}$ with the same properties. This means that $Y' = \cup_i V(g_i)$ is a smooth (global) analytic hypersurface of M and it intersects Y in codimension bigger than one. For each $x \in Y' \cap U_i$, g_{i_x} is an irreducible germ such that $\mathcal{I}(Y')_x = (g_{i_x})\mathcal{O}_x$ and f_{i_x}, g_{i_x} are coprime. We want to prove that Y' is the global analytic set we were looking for.

By taking on each open set U_i the function $h_i = f_i g_i$ we have a section of an other line bundle whose transition functions are $(f_j/f_i)^2$: Now, the exponential map and the associated usual exact sequence

$$0 \to \mathcal{O}_M \to \mathcal{O}^*_M \to \mathcal{O}^*_M/\mathcal{O}^+_M = \mathbb{Z}_2 \to 0,$$

induce an isomorphism between $H^1(M, \mathbb{Z}_2)$ and $H^1(M, \mathcal{O}^*_M)$. Under this isomorphism the image of a line bundle is the cocycle of the signs of its transition functions, so, this cocycle is trivial and hence the line bundle is trivial. This means that there exists a zero cocycle $\{\lambda_i\} \in H^0(M, \mathcal{O}^*_M)$ such that $(f_j/f_i)^2 = \lambda_j^{-1}\lambda_i$. Then the collection of functions defined on U_i by $f_i g_i \lambda_i$ defines a global analytic function h on M because $f_i g_i \lambda_i|_{U_i \cap U_j} = f_j g_j \lambda_j|_{U_i \cap U_j}$. Moreover h generates $\mathcal{I}(Y \cup Y')$. Indeed, let $F \in \mathcal{O}(M)$ be in $\mathcal{I}(Y \cup Y')$, Lemma 1.2 implies that $f_{i_x}|F_x$ for each $x \in U_i$ and by construction it follows that $g_{i_x}|F_x$ for each $x \in M$; since f_{i_x} and g_{i_x} are coprime, $F \in (h)\mathcal{O}(M)$ as wanted. □

Note that Theorem 1.1 proves that for each global analytic subset $Y \subset M$ of codimension one the ideal $\mathcal{I}(Y)$ is "generically" principal, i.e., there exists a global analytic function, h, vanishing on Y, such that $h\mathcal{O}_x = \mathcal{I}_x$ outside a global analytic set of dimension strictly lower, that is $Y \cap Y'$.

Note also that, if $H^1(M, \mathbb{Z}_2) = 0$, the ideal $\mathcal{I}(Y)$ is a principal ideal in $\mathcal{O}(M)$.

As immediate corollaries we have the following propositions:

Proposition 1.4. *Let $Y \subset M$ be a global analytic set such that its irreducible components have all codimension one and let $D \subset M$ be a discrete set of points. Then there exists an analytic hypersurface $Y' \subset M$ such that $Y' \cap D = \emptyset$ and $\mathcal{I}(Y \cup Y') \subset \mathcal{O}(M)$ is principal.*

Proof. Choose a C^∞ section σ such that $\sigma(x) \neq 0$ for each $x \in D$, by Lemma 1.3 we can approximate it by an analytic section that still has this property. □

Proposition 1.5. *If Y is an irreducible global analytic set of codimension 1 then, the ring $\mathcal{O}(M)_{\mathcal{I}(Y)}$ is a rank one discrete valuation ring of the field $\mathcal{M}(M)$.*

Proof. Let $Y' \subset M$ be as in Theorem 1.1, and let $t \in \mathcal{O}(M)$ be a generator for the ideal $\mathcal{I}(Y \cup Y')$. We want to prove that t generates the ideal $\mathcal{I}(Y)\mathcal{O}(M)_{\mathcal{I}(Y)}$ in $\mathcal{O}(M)_{\mathcal{I}(Y)}$, moreover each $f/g \in \mathcal{I}(Y)\mathcal{O}(M)_{\mathcal{I}(Y)}$ can be written as $t^n u$ with $u \notin \mathcal{I}(Y)\mathcal{O}(M)_{\mathcal{I}(Y)}$ and $n \in \mathbb{N}$.

Indeed, let $h \in \mathcal{O}(M)$ such that h vanishes on Y' and doesn't vanish on Y. We can write $fh = ts$ for some $s \in \mathcal{O}(M)$; then $\dfrac{f}{g} = t\dfrac{s}{gh}$ with $\dfrac{s}{gh} \in \mathcal{O}(M)_{\mathcal{I}(Y)}$. If s doesn't vanish on Y the claim holds with $n = 1$. Otherwise we can repeat the same arguments getting that $sh = ts'$ for some $s' \in \mathcal{O}(M)$, i.e., $t^2 | fh^2$. Let's prove that there is a maximun integer n such that $t^n | fh^n$. Indeed, choose $x \in Y$ such that $\mathcal{I}(Y_x) = (t_x)$ and $h(x) \neq 0$, since $\mathcal{O}_{M,x}$ is noetherian, there exists a maximum integer n_x such that $t_x^{n_x} | f_x$, in particular, being h_x a unit, t_x^n cannot divide $f_x h_x^n$ for $n > n_x$ and this implies our assertion.

Let $\dfrac{f}{g} \in \mathcal{M}(M)$, then $f = t^n f'$ and $g = t^m g'$ where f', g' are units in $\mathcal{O}(M)_{\mathcal{I}(Y)}$. Clearly $\dfrac{f}{g} \in \mathcal{O}(M)_{\mathcal{I}(Y)}$ if and only if $n \geq m$ and this proves our claim. □

2. Separation in a neighbourhood of the boundary.

In this section we will prove some results which make it possible to pass from separation in a neighbourhood of the boundary to global separation.

An essential tool is Łojasiewicz's inequality. It works for compact global semianalytic sets of any dimension, nevertheless this hypothesis of compactess is not needed in dimension two. The following Theorems 2.1 and 2.2 are proved in [**DC99**]:

Theorem 2.1. *Let $M \subset \mathbb{R}^N$ be a two dimensional real connected analytic manifold and let $S \subset M$ be a closed global semianalytic subset. Given $f \in \mathcal{O}(M)$ there exists $h \in \mathcal{O}(M)$ such that:*

1. $h \geq 0$, $V(h) = \overline{V(f) \cap S}^Z$,
2. $h \leq |f|$ *on* S,
3. $\dfrac{h}{f}$ *extended to zero on* $V(f) \cap S$ *is continuous on* S.

We shall often use Łojasiewicz's inequality in the following formulation, compare [**BCR87**] for the algebraic case.

Theorem 2.2. *Let S be a closed global semianalytic subset of M,* $\dim M = 2$, *and let $f, g \in \mathcal{O}(M)$, then there exists a nonnegative function $\varepsilon \in \mathcal{O}(M)$ such that:*

1. $(f + \varepsilon g)(x)$ *has the same sign as* $f(x)$*, for any* $x \in S$,
2. $V(\varepsilon) \subseteq \overline{V(f) \cap S}^Z$.

What follows is the main result of this section:

Theorem 2.3. *Let $M \subset \mathbb{R}^N$ be a real connected analytic manifold,* $\dim M = 2$, *and let $A, B \subset M$ be closed global semianalytic sets such that $A \cap B \subset X = Y \cup D$ where X is a global analytic set with Y of pure dimension one and $D = \{x_n\}_{n \in \mathbb{N}}$ discrete.*

Assume that there exist a neighbourhood U of Y and a global analytic function $f \in \mathcal{O}(M)$ such that

$$f(A \cap U \setminus Y) > 0 \qquad f(B \cap U \setminus Y) < 0.$$

Moreover assume that for each n, the semianalytic set germs $A_{x_n} \setminus \{x_n\}$ and $B_{x_n} \setminus \{x_n\}$ are separable, i.e., there exist an open neighbourhood U_n of x_n and an analytic function $f_n \in \mathcal{O}(U_n)$ such that

$$f_n(A \cap U_n \setminus \{x_n\}) > 0 \qquad f_n(B \cap U_n \setminus \{x_n\}) < 0.$$

Then there exists $F \in \mathcal{O}(M)$ that separates A from B outside X, meaning by this that

$$F(A \setminus X) > 0 \qquad F(B \setminus X) < 0.$$

The proof will be done in several steps, the aim being to pass from several functions separating A and B in a neighbourhood of some piece of X to a unique global function separating A and B in a neighbourhood of X. Then we shall pass from the neighbourhood of X to the whole M.

Lemma 2.4. *Let $D = \{x_n\}_{n\in\mathbb{N}}$ be a discrete set of points such that* $\dim(A\cap B)_{x_n} = 0$ *for each n. Assume that for each n the semianalytic set germs $A_{x_n} \setminus \{x_n\}$ and $B_{x_n} \setminus \{x_n\}$ are separable. Then, there exists a global analytic function $g \in \mathcal{O}(M)$ which separates A from B in a neighbourhood V of D.*

Proof. From the hypothesis, for each n, there exist an open neighbourhood U_n of x_n and an analytic function $f_n \in \mathcal{O}(U_n)$ such that $f_n(A\cap U_n\setminus\{x_n\}) > 0$ and $f_n(B \cap U_n \setminus \{x_n\}) < 0$.

Then, it follows that, on $A_{x_n} \cup B_{x_n}$, the germ defined by the zero set $V(f_n)$ is contained in $\{x_n\}$, which is the zero set of $\|x - x_n\|^2$. By the local Łojasiewicz inequality, being A and B closed, there exist an even integer $p_n > 0$ and a positive constant c such that

$$|f_n(x)| \geq c\|x - x_n\|^{p_n} \qquad \forall x \in A_{x_n} \cup B_{x_n}.$$

Up to take a bigger p_n we can suppose $c = 1$. Denote by $\mathfrak{m}_{x_n}$ the maximal ideal of the local ring $\mathcal{O}_{M,x_n}$. By applying Cartan's Theorem B we find a global analytic function $g \in \mathcal{O}(M)$, such that for all $x_n \in D$ we have $g - f_n \in \mathfrak{m}_{x_n}^{2p_n+2}$. Then, there exists an open neighbourhood U_n of x_n such that $g(x)$ has the same sign as $f_n(x)$ for any $x \in (A\cup B)\cap U_n$, so, g separates A from B in a neighbourhood of D, as wanted. □

Lemma 2.5. *Assume there exist a neighbourhood W of X and a global analytic function $q \in \mathcal{O}(M)$ such that*

$$q(A \cap W \setminus X) > 0 \qquad q(B \cap W \setminus X) < 0.$$

Then there exists a global analytic function $r \in \mathcal{O}(M)$ such that $q + r$ separates A from B outside X.

Proof. By hypothesis it follows that $V(q) \cap (A \cup B) \cap W \subset X$. Up to shrink it, we can assume W to be closed. Then, by Theorem 2.1, we can construct a global analytic function $t \in \mathcal{O}(M)$ such that $t \geq 0$, $V(t) = X$ and $t \leq |q|$ on $(A \cup B) \cap W$. We can suppose $t < 1$ on W.

Let $V \subset \overline{V} \subset \mathring{W}$ be an open neighbourhood of X in M. Since $A \setminus V$ and $B\setminus V$ are closed and disjoint, there exists $\varphi \in C^\infty(M)$ such that $\varphi(A\setminus V) > 0$ and $\varphi(B\setminus V) < 0$. Let $\sigma_1 : M \to \mathbb{R}$ be a C^∞-function with $\sigma_1^{-1}(0) = M\setminus\mathring{W}$, $\sigma_1^{-1}(1) = \overline{V}$ and put $\sigma_2 = 1-\sigma_1$. Then $\psi = \sigma_1 q+\sigma_2\varphi$ is C^∞ and $\psi(A\setminus X) > 0$ and $\psi(B \setminus X) < 0$.

Moreover, since $\psi = q$ on V the function $\eta : M \to \mathbb{R}$ defined by

$$\eta(x) = \begin{cases} \dfrac{\psi(x) - q(x)}{t^2(x)} & \text{if } x \notin V \\ \\ 0 & \text{if } x \in V \end{cases}$$

is C^∞ and we are to approximate it by an analytic function.

Take $\{K_n\}_n$ a sequence of compact sets in M such that $K_0 = \emptyset$, $K_n \subset \mathrm{Int}(K_{n+1})$ and $M = \cup_n K_n$.

Note that if $A \cup B \subset V$ we can take $r = 0$, so we can suppose $(A \cup B) \setminus V \neq \emptyset$.

Since $M = \cup_n K_n$, $(A \cup B) \setminus V$ intersects some K_{n_0}, and then every K_m with $m \geq n_0$. So, after replacing $\{K_n\}_n$ by $\{\widetilde{K}_n\}_n$, $\widetilde{K}_0 = \emptyset$ $\widetilde{K}_n = K_{n_0+n-1}$, we can assume $((A \cup B) \setminus V) \cap K_{m+1} \neq \emptyset$ for each m.

Set

$$s_{m+1} = \min\{|\psi(x)| \mid x \in ((A \cup B) \setminus V) \cap K_{m+1}\} \qquad \text{and}$$
$$t_{m+1} = \max\{t^2(x) \mid x \in (A \cup B) \cap K_{m+1}\}.$$

Note that s_{m+1} and t_{m+1} are well-defined and strictly positive constants.

Let $\varepsilon_{m+1} \in \mathbb{R}$ be a constant such that

$$0 < \varepsilon_{m+1} < \min\left(\frac{s_{m+1}}{t_{m+1}}, 1\right).$$

It is well-defined because $t_{m+1} > 0$.

According to Whitney's approximation theorem, there exists a global analytic function r' such that, for each $x \in K_{m+1} \setminus K_m$, $|\eta(x) - r'(x)| < \varepsilon_{m+1}$.

We want to prove that $r = r' t^2$ is the global analytic function we looked for.

We begin by showing that the analytic function $q + r$ separates A and B outside V.

For each $x \in (A \cup B) \setminus V$ we have $\psi(x) - q(x) = t^2(x)\eta(x)$ and then, if $x \in K_{m+1} \setminus K_m$, we have

$$|\psi(x) - (q + r)(x)| = t^2(x)|(\eta - r')(x)| < t_{m+1}\varepsilon_{m+1} < s_{m+1}.$$

So $q + r$ has the same sign as ψ on $(A \cup B) \setminus V$.

Consider now the sign of $q + r$ on V. By construction, being $\eta = 0$ on V, we have, for each $x \in V$, $|r'(x)| < 1$ (1 is bigger than ε_m for each m).

So if $x \in ((A \cup B) \cap V) \setminus X$ we have the following sequence of inequalities:

$$|q(x)| - |r'(x)t^2(x)| > |q(x)| - t^2(x) \geq t(x) - t^2(x) > 0.$$

Note that the first and the last inequalities are strict because $t(x) \neq 0$ and $t(x) < 1$. It follows that $q + r$ has the same sign as q on $(A \cup B) \cap V$ and this completes the proof. $\square$

Proof of Theorem 2.3. By Lemma 2.4 there is a function g that separates A and B in a neighbourhood U of D. Then, by Lemma 2.5 it is enough to glue together the function f of the statement with g, to get a function q separating A and B in a neighbourhood of X.

Let's see that, up to shrink them, we can assume that U and V are closed global semianalytic sets. Since D is a discrete set, we can assume U to be an union of disjoint balls. Let $U = \bigcup_{i \in I} B_i$ and let $f_i \in \mathcal{O}(M)$ be such that $B_i = \{f_i > 0\}$. Then the sheaf $\mathcal{I}_x = \prod_{i \in I} f_i \mathcal{O}_x$ is well-defined and coherent. We want to prove that $\mathcal{I}$ is principal, i.e., there exists a global analytic function g such that $g_x \mathcal{O}_x = \mathcal{I}_x$ for each $x \in M$. Then we get $U = \{g > 0\}$ or $U = \{g < 0\}$. In order to prove that $\mathcal{I}$ is principal we have to find a locally finite open covering of M, U_i, and generators g_i for $\mathcal{I}_{|U_i}$, such that g_i/g_j is positive on $U_i \cap U_j$, when $U_i \cap U_j \neq \emptyset$. Choose U_i and g_i in this way: Each U_i intersects at most a ball, say $B_{j(i)}$; if U_i intersects $B_{j(i)}$ we take $g_i = f_{j(i)}$, if it does not intersect any ball and it is contained in one of them, take $g_i = 1$, else choose $g_i = -1$. The same argument holds for V because V can be written as a finite union of sets that are unions of disjoint balls.

Let h be a positive equation for Y such that $\dfrac{h}{f}$, extended to 0 on $V(f) \cap (A \cup B) \cap U$, is continuous on $(A \cup B) \cap U$. Such h exists by Theorem 2.1. Similarly we can find $t \in \mathcal{O}(M)$, $t \geq 0$ and $V(t) = D$, with the same property with respect to D and V. We want to prove that the global analytic function

$$q = th\left(\frac{f}{h} + \frac{g}{t}\right)$$

has the same sign as $\dfrac{f}{h}$ near Y. ht being strictly positive outside X, it is enough to prove that $\left|\dfrac{g}{t}\right| < \left|\dfrac{f}{h}\right|$ near Y. It is true because we can clearly assume $Y \cap D = \emptyset$ so, for each $x_0 \in Y$, $\dfrac{g}{t}$ is bounded locally at x_0, while, by construction, $\lim_{x \to x_0} \left|\dfrac{f(x)}{h(x)}\right| = +\infty$. Similarly q has the same sign of g near D. □

Remark 2.6. Note that if $A \cap B = D$ is a discrete set one can remove the hypothesis on the dimension of M. Indeed, the thesis follows by Lemma 2.4 which holds without any dimension hypothesis and by Lemma 2.5 that, in this situation, can be proved using only the local Łojasiewicz inequality.

A consequence of these results is the following:

Proposition 2.7. *Let A, B be closed semianalytic subsets of M such that* $\dim A = 1$ *and* $\dim A \cap B = 0$. *Then A and B can be separated outside $A \cap B$.*

Proof. It is enough to prove that, for each $x \in A \cap B$, there exist a neighbourhood U^x of x and $f_x \in \mathcal{O}(U^x)$ such that $f_x(A \cap U^x \setminus \{x\}) > 0$ and $f_x(B \cap U^x \setminus \{x\}) < 0$ and this is true by [**Rui84**]. □

3. Generic separation versus separation.

In this section we prove that generic separation and separation are "almost" equivalent in dimension two, the proof uses essentially the same methods as in the algebraic and local analytic cases, cf. [**ABF96**], [**ABR96**, Chapter 3].

Obviously, separation implies generic separation but the converse is not true, as it is easily seen by taking for instance the sets in $\mathbb{R}^2$:

$$\begin{aligned} A &= \{(x,y)|0<x<1, y>0\} \cup \{(x,y)|0<x<1/2, y=0\} \\ B &= \{(x,y)|0<x<1, y<0\} \cup \{(x,y)|1/2<x<1, y=0\}. \end{aligned}$$

A and B are disjoint global semianalytic sets, they are obviously generically separable by the function $f=y$ but any function generically separating A and B must vanish at some points lying in $A \cup B$ and therefore they cannot be separated.

Note that any function f which generically separates two global semianalytic sets, A and B, must vanish identically on $\overline{A} \cap \overline{B}$ and therefore on $\overline{\overline{A} \cap \overline{B}}^Z$. Then a necesssary condition for A and B to be separated is that $\overline{\overline{A} \cap \overline{B}}^Z \cap (A \cup B)$ is empty.

We shall prove that this condition is also sufficient. This result follows from next theorem which shows that, given two generically separable sets, there exists a "minimal" set outside which they are separable.

Theorem 3.1. *Let A, B be global semianalytic subsets of M, $\dim M = 2$. Assume they are generically separable, then, there exists $f \in \mathcal{O}(M)$ such that*

$$f(A \setminus \overline{\overline{A} \cap \overline{B}}^Z) > 0 \qquad f(B \setminus \overline{\overline{A} \cap \overline{B}}^Z) < 0.$$

Proof. Suppose that $f \in \mathcal{O}(M)$ separates generically A and B, that is, there is a proper analytic subset $W \subset M$ such that $f(A \setminus W) > 0$ and $f(B \setminus W) < 0$. After replacing W by $\overline{\{f=0\} \cap (A \cup B)}^Z$, we may assume that $W = \overline{W \cap (A \cup B)}^Z$ and f vanishes on W.

We write $W = \overline{\overline{A} \cap \overline{B}}^Z \cup W' \cup D$ where W' and D are respectively the union of all the irreducible global components of dimension one and zero not lying in $\overline{\overline{A} \cap \overline{B}}^Z$. Suppose first $W' \neq \emptyset$. Then, $\overline{A} \cap W'$ and $\overline{B} \cap W'$ are global semianalytic sets of dimension one which intersect each other in dimension zero. By Proposition 2.7, there exists $h \in \mathcal{O}(M)$ separating them. Consider the closed set $S = (\overline{A} \cap \{h \leq 0\}) \cup (\overline{B} \cap \{h \geq 0\})$. S is global semianalytic because the closure of a global semianalytic set of M is global, ([**CA96**]). By Theorem 2.2 there exists a nonnegative analytic function $\varepsilon \in \mathcal{O}(M)$ such that $g = f + \varepsilon h$ has the same sign as f on S, and with the zero set of ε contained in $\overline{V(f) \cap S}^Z$. Thus we get that g separates $A \setminus \overline{\overline{A} \cap \overline{B}}^Z$ and

$B \setminus \overline{\overline{A} \cap \overline{B}}^Z$ up to the discrete set D. In order to remove the set D it is enough to apply two more times Theorem 2.2. We split D as $D_1 \cup D_2$ where $D_1 = D \cap A$ and $D_2 = D \cap B$. In order to remove D_1 apply Theorem 2.2 to the functions g and 1 with respect to $\overline{B}$ to obtain a function $\eta \in \mathcal{O}(M)$ such that $(g+\eta)(A \setminus \overline{\overline{A} \cap \overline{B}}^Z) > 0$ and $(g+\eta)(B \setminus (D_2 \cup \overline{\overline{A} \cap \overline{B}}^Z)) < 0$. A second application of the same theorem to $g+\eta$ and -1 removes D_2, giving a function ξ such that $g+\eta-\xi$ separates A and B outside $\overline{\overline{A} \cap \overline{B}}^Z$.

This last argument can be used also when $W' = \emptyset$. □

Corollary 3.2. *Let A, B be global semianalytic subsets of M,* $\dim M = 2$, *assume that*

$$\overline{\overline{A} \cap \overline{B}}^Z \cap (A \cup B) = \emptyset.$$

Then, A and B are separable if and only if they are generically separable. In particular if A and B are open they are separable if and only if they are generically separable.

Proof. Let's prove the last statement. One implication is trivial, for the other we have to prove that $\overline{\overline{A} \cap \overline{B}}^Z$ does not intersect $A \cup B$. Since A and B are open, an analytic component of $\overline{\overline{A} \cap \overline{B}}^Z$, say W, intersecting one of the two sets has to intersect it in dimension one. This contradicts the hypothesis that A and B are not generically separable. Indeed, let $t \in \mathcal{O}(M)$ be an uniformizer for $\mathcal{O}(M)_{\mathcal{J}(W)}$ (cf. Theorem 1.5), then any possible function f generically separating A and B can be written as $f = t^m u$, with $u \notin \mathcal{J}(W)\mathcal{O}(M)_{\mathcal{J}(W)}$. Since $W \subset \overline{\overline{A} \cap \overline{B}}^Z$, m has to be odd, since W intersects the interior of one of the two sets m has to be even. □

4. Separation and walls.

Let A, B be global semianalytic subsets in M, $\dim M = 2$, such that $\mathring{A} \cap \mathring{B} = \emptyset$.

We recall that the boundary of a global semianalytic subset $S \subset M$, $\partial S = \overline{S} \setminus \mathring{S}$, is a semianalytic set of dimension ≤ 1 contained in the zero set of the product of the functions appearing in any description of S. Therefore, ∂S is global by [**Cas94b**].

Set Y the Zariski closure of $\partial A \cup \partial B$. Note that $\partial A \cup \partial B$ being global, Y is a proper analytic subset of M.

Definition 4.1. We will call a *wall* any irreducible component of Y of dimension one. We say that a wall W is *odd* if there is a 1-dimensional subset $W' \subset W$ which is contained in $\overline{A} \cap \overline{B}$. We say that a wall W is *even* if there is a 1-dimensional subset $W' \subset W$ which is contained in $\overline{\mathring{A}}$ or $\overline{\mathring{B}}$.

Note that walls may be neither odd nor even, also they can be both odd and even, as for istance in the proof of Corollary 3.2.

For any odd (resp. even) wall W, let $t \in \mathcal{O}(M)$ be an uniformizer for $\mathcal{O}(M)_{\mathcal{I}(W)}$, then any possible function f generically separating A and B can be written as $f = t^m u$, with $u \notin \mathcal{I}(W)\mathcal{O}(M)_{\mathcal{I}(W)}$ and m odd (resp. even, possibly zero). It is clear that the parity of m doesn't depend on the choice of the generator.

In the right-hand of Figure 1 there is an example of an odd and even wall. Obviously, if some wall W is simultaneously odd and even A and B can not be generically separable, we want to show that the converse is "almost" true.

Theorem 4.2. *Let A, B be open global disjoint semianalytic subsets of M, $\dim M = 2$, set $Y = \overline{\partial A}^Z \cup \overline{\partial B}^Z$ and $Z = \overline{\overline{A} \cap \overline{B}}^Z$. Then A and B can be separated if and only if the following conditions hold:*

1. *No wall of A and B is simultaneously odd and even.*
2. *For every $x \in \operatorname{Sing} Y$ the semianalytic set germs A_x and B_x are (generically) separated.*

Denote by Y^c the union of odd walls of Y. Assuming Conditions 1 and 2 of Theorem 4.2 let's prove the following:

Lemma 4.3. *Let $X \subset M$ be an analytic set such that $\mathcal{I}(X \cup Y^c)$ is principal, $X \cap \operatorname{Sing} Y$ is empty, $X \cap Y$ is discrete. Let $g \in \mathcal{O}(M)$ be a generator for $\mathcal{I}(Y^c \cup X)$ and denote by A^g and B^g the sets*

$$A^g = \{x \in A |\, g(x) > 0\} \cup \{x \in B |\, g(x) < 0\}$$
$$B^g = \{x \in A |\, g(x) < 0\} \cup \{x \in B |\, g(x) > 0\}.$$

Set $Y^g = \overline{\partial A^g}^Z \cup \overline{\partial B^g}^Z$, $Z^g = \overline{\overline{A^g} \cap \overline{B^g}}^Z$, then the following assertions hold:

1. *$Y^g \subset X \cup Y$.*
2. *$Z^g \subset X \cup \operatorname{Sing} Y$, in particular for any $x \notin X$, $\dim Z^g_x \leq 0$.*
3. *For each $x \in \operatorname{Sing} Y$ the semianalytic sets germs $\overline{A^g}_x \setminus \{x\}$ and $\overline{B^g}_x \setminus \{x\}$ are separable.*

Proof. 1. Since $X \cup Y$ is a global analytic set, it is enough to prove that $\partial A^g \cup \partial B^g$ is contained in it.

Fix $x \in \partial A^g = \overline{A^g} \setminus A^g$. Since $A^g \subset A \cup B$, $x \in \overline{A} \cup \overline{B}$. Suppose $x \in \overline{A}$. If $x \notin A$, $x \in \partial A$, then $x \in Y$. So we can assume $x \in A$ and this implies $g(x) \leq 0$. We want to prove that $g(x) = 0$. Suppose $g(x) < 0$, being A open, we can find a neighbourhood U_x of x contained in $A \cap \{g < 0\}$. Since $A \cap B = \emptyset$, we have that $U_x \cap A^g = \emptyset$, which contradicts the hypothesis $x \in \overline{A^g}$. We argue similarly for $x \in \overline{B}$ and for $x \in \partial B^g$.

2. Note that $Z^g \subset (Y^g)^c \cup \operatorname{Sing} Y^g$, since $\operatorname{Sing} Y^g$ is contained in $X \cup \operatorname{Sing} Y$, it is enough to prove that $(Y^g)^c$ is contained in X, i.e., no wall in Y is odd with respect to A^g and B^g.

   Take a wall $W \subset Y$. If it is odd with respect to A and B then $W \subset V(g)$, hence it is even with respect to A^g and B^g. It cannot be odd, because if so the same argument shows that W is even with respect to $(A^g)^g = A$ and $(B^g)^g = B$; this is not the case by hypothesis. If W is not odd then g does not change sign through W, so W is not odd with respect to A^g and B^g.

   The second statement in 2) is clear.
3. Fix $x \in \operatorname{Sing} Y$. Since $X \cap \operatorname{Sing} Y = \emptyset$, $x \notin X$ hence $\dim(\overline{A^g} \cap \overline{B^g})_x \leq 0$. By hypothesis A_x and B_x are generically separable, say by $f_x \in \mathcal{O}_x$. Then, $f_x g$ generically separates $\overline{A^g_x}$ and $\overline{B^g_x}$. This implies that $\overline{A^g}/\{x\}$ and $\overline{B^g}/\{x\}$ are separable, (cf. [**ABR96**]).

□

Remark 4.4. Note that A and B are separable if and only if A^g and B^g are so. Note also that, if the ideal of Y^c is principal, i.e., $X = \emptyset$, A^g and B^g intersect each other only in a discrete set of points contained in $\operatorname{Sing} Y$.

Proof of Theorem 4.2. One implication is trivial, we want to prove the other. Denote by D the discrete set $\overline{A} \cap \overline{B}/Y^c$, $D \subset \operatorname{Sing} Y$. By Theorem 2.3 we know that it is enough to separate $\overline{A} \cap U/Y^c$ and $\overline{B} \cap U/Y^c$ where U is a neighbourhood of Y^c, since $\overline{A}_x/\{x\}$ and $\overline{B}_x/\{x\}$ are separeted for any $x \in D \subset \operatorname{Sing} Y$ by hypothesis. By Theorem 3.1, it is enough to separate $A \cap U$ and $B \cap U$ (we can always assume $U \cap D = \emptyset$). Let X_1 and X_2 be analytic sets as in Lemma 4.3, i.e., $\mathcal{I}(Y^c \cup X_i) = (g_i)\mathcal{O}_M$, $X_i \cap Y$ is a discrete set, $i = 1, 2$ and $X_1 \cap X_2 \cap \operatorname{Sing} Y = \emptyset$. Then we have the sets $A^i = A^{g_i}$, $B^i = B^{g_i}$, $Y^i = Y^{g_i}$ and $Z^i = Z^{g_i}$, $i = 1, 2$. We have that A and B are separable in U iff A^i and B^i are separable in U, $i = 1, 2$. As above, by Thoerem 2.3 and Theorem 3.1, it is enough to prove that $(\overline{A^i \cap U})_x/\{x\}$ and $(\overline{B^i \cap U})_x/\{x\}$ are separable for each $x \in \overline{A^i} \cap \overline{B^i}/(Y^1)^c$ and that A^i and B^i are separable in $U \cap V_i$, where V_i is a neighbourhood of X_i, $i = 1, 2$. Since the first assertion is verified by Lemma 4.3, A^i and B^i are separable in U if and only if the second one holds. More precisely, A and B are separable in U, iff A^1 and B^1 are separable in $U \cap V_1$, iff A^2 and B^2 are separable in $U \cap V_1$. But $\overline{A^2} \cap \overline{B^2} \cap U \cap V_1$ is empty, up to shrink U and V_1, so A^2 and B^2 are actually separated in $U \cap V_1$ and this in turn implies the thesis. □

As a consequence of our criterion we obtain:

Theorem 4.5. *A and B are separable if and only if they are separable in any compact set.*

This is completely different of what happens in the algebraic case where the behaviour at infinity is decisive for the separarion of semialgebraic sets, as the example in Figure 2 shows. It remains true that the important information is at the boundary but they may be hidden and appear only after compactifying and doing some blow ups, more precisely in a compact model of the variety where the semialgebraic sets are at normal crossings, see [**AAB99**]. (The field of meromorphic functions on an analytic manifold is not preserved under blowing ups, so there is no special model of M suitable to study A and B.)

Remark 4.6. We want to justify the assertion in the introduction that unbounded orderings are not useful. This is clear if the obstruction is a point. If W is a wall simultaneously odd and even, take two points $x_1, x_2 \in W$ such that W is odd in a neighbourhood of x_1 and even in a neighbourhood of x_2 and take two orderings in $\mathrm{Spec_r}\ \mathrm{Frac}\,(\mathcal{O}(M)_{\mathcal{J}(W)}/\mathcal{J}(W)_{\mathcal{J}(W)})$ centered respectively in x_1 and x_2. Since $\mathcal{O}(M)_{\mathcal{J}(W)}$ is a discrete valuation ring, cf. Proposition 1.5, arguing as in [**ABV94**], it is easy to lift these two orderings to a four element fan in $\mathrm{Spec_r}\mathcal{M}(M)$ where the corresponding constructible sets $\widetilde{A}$ and $\widetilde{B}$ are not separable. By construction the four orderings in this fan are bounded. Bearing in mind the Artin-Lang property, which holds in $\mathrm{Spec_r}\mathcal{M}(M)$, we can resume all this in the following:

Theorem 4.7. *Two global semianalytic sets A and B in a 2-dimensional analytic manifold M can be separated if and only if the associated constructible sets $\widetilde{A}, \widetilde{B} \subset \mathrm{Spec_r}\mathcal{M}(M)$ are separable in any 4-elements fan made of bounded orderings.*

5. Basicness for global semianalytic sets.

We can use the criterion for separation of the above section to prove a similar result for another kind of problems: Basicness and principality of global semianalytic sets.

Theorem 5.1. *Let $S \subset M$ be an open global semianalytic set. Then, S is basic open (resp. principal open) if and only if the following conditions hold:*

1. *If a wall is simultaneously odd and even with respect to S and $M \setminus \overline{S}$ then, it does not intersect $\mathring{\overline{S}}$.*
2. *For each $x \in \mathrm{Sing}\ \overline{\partial S}^{Z}$ the semianalytic set germ S_x is basic open.*

Resp.

3. *No wall is simultaneously odd and even with respect to S and $M \setminus \overline{S}$.*

Proof. Let's prove the statement about basicness. One implication is trivial, we want to prove the other. Since $M \setminus \overline{S}$ is an open global semianalytic set, it is a finite union of basic open sets, say $B_1 \cup \cdots \cup B_r$, such a description is possible by [**Cas94b**]. We want to prove that S can be separated from each B_i.

Fix $i \in \{1, \ldots, r\}$ and set $Y_i = \overline{\partial S}^Z \cup \overline{\partial B_i}^Z$. It is enough to prove that no wall in Y_i is simultaneously odd and even with respect to S and B_i and that, for each $x \in \operatorname{Sing} Y_i$, the semianalytic set germs S_x and $B_{i,x}$ are separable. Let $W \subset Y_i$ be an odd wall, this means that there exists an arc contained in $\overline{S} \cap \overline{B_i}$. If W is even, we can find another arc contained in $\overline{\mathring{S}}$ or in $\overline{\mathring{B}_i}$. The first assertion contradicts Hypothesis 1, the second the basicness of B_i. This proves that no odd wall can be even.

Since two open basic semianalytic set germs in dimension 2 can always be separated, cf. [**ABR96**], it is enough to prove that $B_{i,x}$ and S_x are basic for each $x \in \operatorname{Sing} Y_i$. This is true by Hypothesis 2 (note that S_x is principal if $x \notin \operatorname{Sing} \overline{\partial S}^Z$).

Hence, we can find $f_i \in \mathcal{O}(M)$ such that $f_i(S) > 0$ and $f_i(B_i) < 0$. Then, $S = \{f_1 > 0, \ldots, f_r > 0\}$. Indeed, if $x \notin S$, $x \in \overline{B_i}$ for some $i \in \{1, \ldots, r\}$ and this implies that $f_i(x) \leq 0$ for some $i \in \{1, \ldots, r\}$.

As it concerne the statement about principality, with the same argument we can separate S from $M \setminus \overline{S}$ proving that S is principal. $\square$

Arguing as in Remark 4.6, we see that unbounded orderings are again useless for the basicness and principality properties, more precisely:

Theorem 5.2. *An open global semianalytic set S is basic open (principal) if and only if $\#\widetilde{S} \cap F \neq 3$ ($\#\widetilde{S} \cap F \neq 1, 3$) for any 4-elements fan F made of bounded orderings.*

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References

[AAB99] F. Acquistapace, C. Andradas and F. Broglia, *Separation of semialgebraic sets*, J. Amer. Math. Soc., **12**(**3**) (1999), 703-728, MR 99m:14109, Zbl 0917.14032.

[ABF96] F. Acquistapace, F. Broglia and E. Fortuna, *A separation theorem in dimension* 3, Nagoya Math. J., **143** (1996), 171-193, MR 97k:14056, Zbl 0864.14035.

[ABT75] F. Acquistapace, F. Broglia and A. Tognoli, *Sulla normalizzazione degli spazi analitici reali*, Boll. Un. Mat. Ital. (4), **12**(**1**-**2**) (1975), 26-36, MR 53 #853, Zbl 0318.32001.

[ABV94] F. Acquistapace, F. Broglia and M.P. Vélez, *An algorithmic criterion for basicness in dimension* 2, Manuscripta Math., **85**(**1**) (1994), 45-66, MR 95k:14074, Zbl 0854.14024.

[AB90] C. Andradas and E. Becker, *A note on the real spectrum of analytic functions on an analytic manifold of dimension one*, in 'Real analytic and algebraic geometry' (Trento, 1988), 1-21, Lecture Notes in Math., **1420**, Springer, Berlin, 1990, MR 92c:14055, Zbl 0779.32004.

[ABR96] C. Andradas, L. Bröcker and Jesús M. Ruiz, *Constructible Sets in Real Geometry*, Ergebnisse Math. Grenz., **33**, Springer-Verlag, Berlin, 1996, MR 98e:14056, Zbl 0873.14044.

[BCR87] J. Bochnak, M. Coste and M.-F. Roy, *Géométrie Algébrique Réelle*, Ergebnisse Math. Grenz., **12**, Springer-Verlag, Berlin, 1987, MR 90b:14030, Zbl 0633.14016.

[Cas94a] A. Castilla, *Artin-Lang property for analytic manifolds of dimension two*, Math. Z., **217**(**1**) (1994), 5-14, MR 95g:32011, Zbl 0832.32005.

[Cas94b] ______, *Proprietad de Artin-Lang para Variedades Analíticas de Dimensión Dos*, Tesis Doctoral, Universitad Complutense de Madrid, 1994.

[CA96] A. Castilla and C. Andradas, *Connected components of global semianalytic subsets of* 2*-dimensional analytic manifolds*, J. Reine Angew. Math., **475** (1996), 137-148, MR 97h:32006, Zbl 0862.32003.

[DC99] A. Díaz-Cano, *Índice de Estabilidad y Descripción de Conjuntos Semianalíticos*, Ph.D. Thesis, Universidad Complutense de Madrid, 1999.

[Fri67] J. Frisch, *Points de platitude d'un morphisme d'espaces analytiques complexes*, Invent. Math., **4** (1967), 118-138, MR 36 #5388, Zbl 0167.06803.

[Gal76] M. Galbiati, *Stratifications et ensemble de non-cohérence d'un espace analytique réel*, Invent. Math., **34**(**2**) (1976), 113-128, MR 54 #5493, Zbl 0314.32006.

[Hir94] M.W. Hirsch, *Differential Topology*, Grad. Texts in Math., **33**, Springer-Verlag, New York, 1994, Corrected reprint of the 1976 original, MR 96c:57001, Zbl 0356.57001.

[Rui84] J.M. Ruiz, *A note on a separation problem*, Arch. Math. (Basel), **43**(**5**) (1984), 422-426, MR 86f:32007, Zbl 0547.14017.

[Tog67] A. Tognoli, *Proprietà globali degli spazi analitici reali*, Ann. Mat. Pura Appl. (4), **75** (1967), 143-218, MR 38 #4714, Zbl 0158.33001.

[Tog80] ______, *Une remarque sur les fibrés vectoriels analytiques et de Nash*, C.R. Acad. Sci. Paris Sér. A-B, **290**(**7**) (1980), A321-A323, MR 81f:32010, Zbl 0465.32006.

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