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ISOMORPHISM THEOREM ON LOW DIMENSIONAL LIE ALGEBRAS

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Let $\mathfrak{g}$ (resp. $\mathfrak{g}'$) be a Lie algebra of dimension $d \leq 3$ (resp. of finite dimension) over a field k of characteristic $\neq 2$. We prove that $\mathfrak{g}$ is isomorphic to $\mathfrak{g}'$ as Lie algebras over k if and only if the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is isomorphic to $U(\mathfrak{g}')$ as k-algebras.

1. Introduction.

In this article, we study the isomorphism theorem on Lie algebras of dimension ≤ 3. Our goal is the following theorem:

Theorem 1.1. *Let* $\mathfrak{g}$ (*resp.* $\mathfrak{g}'$) *be a Lie algebra of dimension* $d \leq 3$ (*resp. of finite dimension*) *over a field* k (*of characteristic not equal to* 2). *Then* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{g}'$ *if and only if the universal enveloping algebra* $U(\mathfrak{g})$ *of* $\mathfrak{g}$ *is isomorphic to the one* $U(\mathfrak{g}')$ *of* $\mathfrak{g}'$.

For a Lie algebra of dimension 1 or 2, the theorem is clear by classification of low dimensional Lie algebras [**3**, I.4]. Malcolmson [**4**] proved the isomorphism theorem for 3-dimensional simple Lie algebras by using their Killing forms. We describe the simplicity of a 3-dimensional Lie algebra in terms of its enveloping algebra. To complete the isomorphism theorem on 3-dimensional Lie algebras, we prove the theorem for non-simple Lie algebras of dimension 3.

Notation. We denote by $\sigma = \sigma_{\mathfrak{g}} \colon \mathfrak{g} \to U(\mathfrak{g})$ a canonical map from a Lie algebra to its enveloping algebra $U(\mathfrak{g})$.

2. Preliminaries on enveloping algebras.

We prove some preliminary properties on the enveloping algebra $U(\mathfrak{g})$.

Proposition 2.1. *The two-sided ideal* I_{com} *generated by* $\{[a,b] := ab - ba \in U(\mathfrak{g}); a, b \in U(\mathfrak{g})\}$ *is equal to the one* $I_{[\mathfrak{g},\mathfrak{g}]}$ *generated by* $\sigma([\mathfrak{g},\mathfrak{g}])$.

Proof. We have only to verify $I_{\mathrm{com}} \subset I_{[\mathfrak{g},\mathfrak{g}]}$. Since $\sigma(g_1)\sigma(g_2)\cdots\sigma(g_s)$ ($g_i \in \mathfrak{g}$) generate $U(\mathfrak{g})$ as a k-vector space, it is enough to show that

$$[\sigma(g_1)\cdots\sigma(g_s), \sigma(h_1)\cdots\sigma(h_r)] \in I_{[\mathfrak{g},\mathfrak{g}]}$$

for $g_i, h_j \in \mathfrak{g}$. It follows from the formula

$$[g, hh'] = [g, h]h' + h[g, h'] \text{ for } g, h, h' \in U(\mathfrak{g}).$$

□

Proposition 2.2. *In the notation of Proposition* 2.1*, we have a canonical isomorphism* $U(\mathfrak{g})/I_{\mathrm{com}} = U(\mathfrak{g})/I_{[\mathfrak{g},\mathfrak{g}]} \to U(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$ *as k-algebras.*

Proof. See [**2**, 2.2.14, p. 72]. By the functoriality of $U(\mathfrak{g})$ with respect to $\mathfrak{g}$, we have a canonical k-algebra homomorphism $\varphi\colon U(\mathfrak{g}) \to U(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$. Since $\sigma(\mathfrak{g})$ generates $U(\mathfrak{g})$ as k-algebra, the homomorphism φ is surjective. On the other hand, every (Lie algebra) homomorphism from $\mathfrak{g}$ to the Lie algebra associated to a commutative ring factors through $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. Since $U(\mathfrak{g})/I_{\mathrm{com}}$ is commutative, we have a k-algebra homomorphism $\psi\colon U(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]) \to U(\mathfrak{g})/I_{\mathrm{com}}$. Hence we can prove the kernel of φ is equal to I_{com} by the fact that the composite $\psi\varphi$ is the canonical projection $U(\mathfrak{g}) \to U(\mathfrak{g})/I_{\mathrm{com}}$. □

Proposition 2.3. *Let* $\text{GK-dim}_k U(\mathfrak{g})$ *be the Gelfand-Kirillov dimension of* $U(\mathfrak{g})$*. Then we have* $\text{GK-dim}_k U(\mathfrak{g}) = \dim_k \mathfrak{g}$.

Proof. See [**5**, 8.1.15 (iii)]. □

Proposition 2.4. *Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$ which is abelian. Let $I_{\mathfrak{h}}$ be the right ideal of $U(\mathfrak{g})$ generated by $\sigma(\mathfrak{h})$, which is a two-sided ideal* (*cf.* [**2**, 2.2.14]). *Then, for any two-sided maximal ideal $\mathfrak{m}$ with $U(\mathfrak{g})/\mathfrak{m} \cong k$ which contains $I_{\mathfrak{h}}$, we have a Lie algebra isomorphism $\mathfrak{h} \to I_{\mathfrak{h}}/I_{\mathfrak{h}}\mathfrak{m}$ via σ. Here the Lie algebra structure of $I_{\mathfrak{h}}/I_{\mathfrak{h}}\mathfrak{m}$ is defined by that of $U(\mathfrak{g})$.*

Proof. First, we prove the proposition in the case $\mathfrak{m} = \langle \sigma(\mathfrak{g}) \rangle$. Let $g_1, \ldots, g_d$ be a basis of $\mathfrak{g}$ such that $g_1, \ldots, g_l$ is a basis of $\mathfrak{h}$. By Poincaré-Birkhoff-Witt theorem, we have

$$U(\mathfrak{g}) = k \oplus \bigoplus_{\substack{s \geq 1 \\ 1 \leq i_1 \leq \cdots \leq i_s \leq d}} k\sigma(g_{i_1}) \cdots \sigma(g_{i_s}).$$

Here $\sigma(g_{i_1}) \cdots \sigma(g_{i_s})$ $(1 \leq i_1 \leq \cdots \leq i_s \leq d)$ form a k-basis of U. Since $\mathfrak{h}$ is abelian, we have similar decompositions:

$$I_{\mathfrak{h}} = \bigoplus_{\substack{s \geq 1 \\ 1 \leq i_1 \leq \cdots \leq i_s \leq d \\ i_1 \leq l}} k\sigma(g_{i_1}) \cdots \sigma(g_{i_s});$$

$$I_{\mathfrak{h}}\mathfrak{m} = \bigoplus_{\substack{s \geq 2 \\ 1 \leq i_1 \leq \cdots \leq i_s \leq d \\ i_1 \leq l}} k\sigma(g_{i_1}) \cdots \sigma(g_{i_s})$$

as k-vector spaces. Hence we have an isomorphism

$$\mathfrak{h} \xrightarrow{\cong} I_{\mathfrak{h}}/I_{\mathfrak{h}}\mathfrak{m} = \bigoplus_{\substack{s=1\\ 1\le i_1\le l}} k\sigma(g_{i_1}) \tag{1}$$

via σ. Using $I_{\mathfrak{h}} \subset \mathfrak{m}$ and $I_{\mathfrak{h}}^2 \subset I_{\mathfrak{h}}\mathfrak{m}$, one can verify that the Lie algebra structure of $I_{\mathfrak{h}}/I_{\mathfrak{h}}\mathfrak{m}$ is well-defined and abelian.

Next we show the proposition in the general case. Let $\alpha\colon U(\mathfrak{g}) \to k$ be a surjective k-algebra homomorphism with kernel $\mathfrak{m}$. Using α, we have an automorphism $i\colon U(\mathfrak{g}) \to U(\mathfrak{g})$ with $i\sigma(g) = \sigma(g) - \alpha(\sigma(g)) \cdot 1$ for all $g \in \mathfrak{g}$. Since $\mathfrak{m}$ contains $I_{\mathfrak{h}}$, the restriction of i to $\sigma(\mathfrak{h})$ is the identity of $\sigma(\mathfrak{h})$. One can easily verify $i(\langle\sigma(\mathfrak{g})\rangle) = \mathfrak{m}$. Hence we have an isomorphism $\mathfrak{h} \to I_{\mathfrak{h}}/I_{\mathfrak{h}}\mathfrak{m}$ using the isomorphism (1) and a commutative diagram

$$\begin{tikzcd}
\mathfrak{h} \arrow[r] \arrow[d, equal] & I_{\mathfrak{h}}/I_{\mathfrak{h}}\langle\sigma(\mathfrak{g})\rangle \arrow[d, "i", "\cong"'] \\
\mathfrak{h} \arrow[r] & I_{\mathfrak{h}}/I_{\mathfrak{h}}\mathfrak{m}.
\end{tikzcd}$$

☐

Corollary 2.5. *In the notation of Proposition* 2.1, *we regard the ideal* $I := I_{[\mathfrak{g},\mathfrak{g}]}$ *as ideal of the underlying Lie algebra* $U(\mathfrak{g})$. *Assume that* $[\mathfrak{g},\mathfrak{g}]$ *is abelian. Then, for any maximal ideal* $\mathfrak{m}$ *with* $U(\mathfrak{g})/\mathfrak{m} \cong k$*, the composite* $[\mathfrak{g},\mathfrak{g}] \xrightarrow{\sigma} I \xrightarrow{\mathrm{pr}} I/I\mathfrak{m}$ *is an isomorphism of Lie algebras.*

Remark 2.6. The composition $[\mathfrak{g},\mathfrak{g}] \xrightarrow{\sigma} I \xrightarrow{\mathrm{pr}} I/I\mathfrak{m}$ is surjective for any Lie algebra, but not necessarily injective if $[\mathfrak{g},\mathfrak{g}]$ is not abelian. For example, consider a simple Lie algebra.

Proposition 2.7. *Let* $\mathfrak{g}_0$ *be an ideal of* $\mathfrak{g}$. *Suppose that there exists a subalgebra* $\mathfrak{g}_1$ *of* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *as* k*-vector spaces. Then* $\mathfrak{g}$ *is isomorphic to the semidirect product* $\mathfrak{g}_0 \rtimes \mathfrak{g}_1$.

Proof. It is straightforward to show that the k-linear map $\mathfrak{g}_0 \rtimes \mathfrak{g}_1 \to \mathfrak{g}$ defined by $(g_0, g_1) \mapsto g_0 + g_1$ is a Lie algebra isomorphism. ☐

Proposition 2.8. *Let* $\mathfrak{g}_i$ *and* $\mathfrak{g}'_i$ $(i = 0, 1)$ *be Lie algebras and* $\mathfrak{g}_1 \xrightarrow{d} \mathrm{Der}_k\,\mathfrak{g}_0$ (*resp.* $\mathfrak{g}'_1 \xrightarrow{d'} \mathrm{Der}_k\,\mathfrak{g}'_0$) *a derivation of* $\mathfrak{g}_0$ (*resp.* $\mathfrak{g}'_0$). *Assume that there exist Lie algebra isomorphisms* $\varphi_0\colon \mathfrak{g}_0 \to \mathfrak{g}'_0$ *and* $\varphi_1\colon \mathfrak{g}_1 \to \mathfrak{g}'_1$ *with a commutative diagram*

$$\begin{tikzcd}
\mathfrak{g}_1 \arrow[r, "d"] \arrow[d, "\varphi_1"'] & \mathrm{Der}_k\,\mathfrak{g}_0 \arrow[d, "\varphi_0^*"] \\
\mathfrak{g}'_1 \arrow[r, "d'"'] & \mathrm{Der}_k\,\mathfrak{g}'_0.
\end{tikzcd}$$

Here φ_0^* *is the induced homomorphism by* φ_0. *Then the semidirect product* $\mathfrak{g}_0 \rtimes \mathfrak{g}_1$ *is isomorphic to* $\mathfrak{g}'_0 \rtimes \mathfrak{g}'_1$ *by* $(g_0, g_1) \mapsto (\varphi_0(g_0), \varphi_1(g_1))$.

Proof. Straightforward. See [**1**, Chapitre 1 §7]. ☐

3. Proof of Theorem 1.1.

We have only to show that, if $U(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g}')$, the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{g}'$.

Assume that $U(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g}')$. We remark that $\dim_k \mathfrak{g} = \dim_k \mathfrak{g}'$ and $\dim_k[\mathfrak{g}, \mathfrak{g}] = \dim_k[\mathfrak{g}', \mathfrak{g}']$ by Propositions 2.2 and 2.3.

In the case of $\dim_k \mathfrak{g} = 1, 2$, the theorem follows from the classification of Lie algebras (e.g., [**3**, I.4]).

We now assume $\dim_k \mathfrak{g} = \dim_k \mathfrak{g}' = 3$. We carry out the proof in each case of $\dim_k[\mathfrak{g}, \mathfrak{g}] = 0, 1, 2, 3$.

If $\dim_k[\mathfrak{g}, \mathfrak{g}] = 0$, i.e., $\mathfrak{g}$ is abelian, then the theorem is clear.

Suppose $\dim_k[\mathfrak{g}, \mathfrak{g}] = 3$. Then one can verify that $\mathfrak{g}$ is simple (cf. [**3**, I.4]). Hence the theorem follows from a result of Malcolmson [**4**, Corollary 1].

Finally, we treat the case $\dim_k[\mathfrak{g}, \mathfrak{g}] = 1, 2$. Let $\psi\colon U(\mathfrak{g}) \to U(\mathfrak{g}')$ be an isomorphism. We denote by $\mathfrak{m}$ (resp. $\mathfrak{m}'$) the (two-sided) maximal ideal generated by $\sigma(\mathfrak{g})$ (resp. the maximal ideal $\psi(\mathfrak{m})$). Let $I := I_{[\mathfrak{g},\mathfrak{g}]}$ and $I' := I_{[\mathfrak{g}',\mathfrak{g}']}$ be as in Proposition 2.1. Note that $[\mathfrak{g}, \mathfrak{g}]$ is abelian in this case (cf. [**3**, I.4]). By Proposition 2.2 and Corollary 2.5, we have the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \arrow[r, "\sigma_{\mathfrak{g}}"] \arrow[d] & U(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]) \arrow[r, "\bar{\psi}"] \arrow[d, "\rho"] & U(\mathfrak{g}'/[\mathfrak{g}',\mathfrak{g}']) \arrow[d, "\rho'"] & \mathfrak{g}'/[\mathfrak{g}',\mathfrak{g}'] \arrow[l, "\sigma_{\mathfrak{g}'}"'] \arrow[d] \\
\mathrm{Der}_k([\mathfrak{g},\mathfrak{g}]) \arrow[r, "\cong"'] & \mathrm{Der}_k(I/I\mathfrak{m}) \arrow[r, "\psi^*"'] & \mathrm{Der}_k(I'/I'\mathfrak{m}') & \mathrm{Der}_k([\mathfrak{g}',\mathfrak{g}']). \arrow[l, "\cong"]
\end{tikzcd}$$

Here ρ, ρ' are Lie homomorphisms defined by inner derivation as usual, and $\bar{\psi}$ (resp. ψ^*) is the isomorphism induced by ψ.

Suppose $\dim_k[\mathfrak{g}, \mathfrak{g}] = 1$. Then there are just two isomorphism classes of 3-dimensional Lie algebras [**3**, I.4]: One is nilpotent; the other is not nilpotent. In this case, a Lie algebra $\mathfrak{g}$ is nilpotent if and only if its center contains $[\mathfrak{g}, \mathfrak{g}]$, i.e., the above ρ is trivial for a maximal ideal $\mathfrak{m}$ with $U/\mathfrak{m} \cong k$. The theorem follows from the above diagram.

Next, we suppose $\dim_k[\mathfrak{g}, \mathfrak{g}] = 2$. Take elements $z \in \mathfrak{g} \setminus [\mathfrak{g}, \mathfrak{g}]$ and $z' \in \mathfrak{g}' \setminus [\mathfrak{g}', \mathfrak{g}']$. We denote by $\mathfrak{g}_1$ (resp. $\mathfrak{g}'_1$) the subalgebra of $\mathfrak{g}$ (resp. $\mathfrak{g}'$) generated by z (resp. z'). Since

$$\bar{\psi}(\sigma_{\mathfrak{g}}(z \bmod [\mathfrak{g}, \mathfrak{g}])) = a\sigma_{\mathfrak{g}'}(z' \bmod [\mathfrak{g}', \mathfrak{g}']) + b \text{ for some } a \in k^*, b \in k,$$

we have the following commutative diagram of Lie algebras:

$$\begin{tikzcd}
\mathfrak{g}_1 \arrow[r, "\psi_1"] \arrow[d] & \mathfrak{g}_1' \arrow[d] \\
\mathrm{Der}_k([\mathfrak{g},\mathfrak{g}]) \arrow[r, "\psi^*"'] & \mathrm{Der}_k([\mathfrak{g}',\mathfrak{g}']),
\end{tikzcd}$$

where ψ_1 maps z to az', and ψ^* is the composite of the lower horizontal maps in the above diagram. The theorem follows from Propositions 2.7 and 2.8.

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