ISOMORPHISM THEOREM ON LOW DIMENSIONAL LIE ALGEBRAS

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Let \( \mathfrak{g} \) (resp. \( \mathfrak{g}' \)) be a Lie algebra of dimension \( d \leq 3 \) (resp. of finite dimension) over a field \( k \) of characteristic \( \neq 2 \). We prove that \( \mathfrak{g} \) is isomorphic to \( \mathfrak{g}' \) as Lie algebras over \( k \) if and only if the enveloping algebra \( U(\mathfrak{g}) \) of \( \mathfrak{g} \) is isomorphic to \( U(\mathfrak{g}') \) as \( k \)-algebras.

1. Introduction.

In this article, we study the isomorphism theorem on Lie algebras of dimension \( \leq 3 \). Our goal is the following theorem:

**Theorem 1.1.** Let \( \mathfrak{g} \) (resp. \( \mathfrak{g}' \)) be a Lie algebra of dimension \( d \leq 3 \) (resp. of finite dimension) over a field \( k \) (of characteristic not equal to 2). Then \( \mathfrak{g} \) is isomorphic to \( \mathfrak{g}' \) if and only if the universal enveloping algebra \( U(\mathfrak{g}) \) of \( \mathfrak{g} \) is isomorphic to the one \( U(\mathfrak{g}') \) of \( \mathfrak{g}' \).

For a Lie algebra of dimension 1 or 2, the theorem is clear by classification of low dimensional Lie algebras [3, I.4]. Malcolmson [4] proved the isomorphism theorem for 3-dimensional simple Lie algebras by using their Killing forms. We describe the simplicity of a 3-dimensional Lie algebra in terms of its enveloping algebra. To complete the isomorphism theorem on 3-dimensional Lie algebras, we prove the theorem for non-simple Lie algebras of dimension 3.

**Notation.** We denote by \( \sigma = \sigma_{\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g}) \) a canonical map from a Lie algebra to its enveloping algebra \( U(\mathfrak{g}) \).

2. Preliminaries on enveloping algebras.

We prove some preliminary properties on the enveloping algebra \( U(\mathfrak{g}) \).

**Proposition 2.1.** The two-sided ideal \( I_{\text{com}} \) generated by \( \{ [a, b] := ab - ba \in U(\mathfrak{g}); a, b \in U(\mathfrak{g}) \} \) is equal to the one \( I_{[\mathfrak{g}, \mathfrak{g}]} \) generated by \( \sigma([\mathfrak{g}, \mathfrak{g}]) \).

**Proof.** We have only to verify \( I_{\text{com}} \subseteq I_{[\mathfrak{g}, \mathfrak{g}]} \). Since \( \sigma(g_1)\sigma(g_2) \cdots \sigma(g_s) \) (\( g_i \in \mathfrak{g} \)) generate \( U(\mathfrak{g}) \) as a \( k \)-vector space, it is enough to show that

\[
[\sigma(g_1) \cdots \sigma(g_s), \sigma(h_1) \cdots \sigma(h_r)] \in I_{[\mathfrak{g}, \mathfrak{g}]}.
\]
for \(g_i, h_j \in \mathfrak{g}\). It follows from the formula
\[
[g, hh'] = [g, h]h' + h[g, h'] \quad \text{for} \quad g, h, h' \in U(\mathfrak{g}).
\]

\[\square\]

**Proposition 2.2.** In the notation of Proposition 2.1, we have a canonical isomorphism \(U(\mathfrak{g})/I_{\text{com}} = U(\mathfrak{g})/I_{[\mathfrak{g}, \mathfrak{g}]} \to U(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])\) as \(k\)-algebras.

**Proof.** See [2, 2.2.14, p. 72]. By the functoriality of \(U(\mathfrak{g})\) with respect to \(\mathfrak{g}\), we have a canonical \(k\)-algebra homomorphism \(\varphi: U(\mathfrak{g}) \to U(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])\). Since \(\sigma(\mathfrak{g})\) generates \(U(\mathfrak{g})\) as \(k\)-algebra, the homomorphism \(\varphi\) is surjective. On the other hand, every (Lie algebra) homomorphism from \(\mathfrak{g}\) to the Lie algebra associated to a commutative ring factors through \(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]\). Since \(U(\mathfrak{g})/I_{\text{com}}\) is commutative, we have a \(k\)-algebra homomorphism \(\psi: U(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) \to U(\mathfrak{g})/I_{\text{com}}\). Hence we can prove the kernel of \(\varphi\) is equal to \(I_{\text{com}}\) by the fact that the composite \(\psi \varphi\) is the canonical projection \(U(\mathfrak{g}) \to U(\mathfrak{g})/I_{\text{com}}\).

\[\square\]

**Proposition 2.3.** Let \(\text{GK-dim}_k U(\mathfrak{g})\) be the Gelfand-Kirillov dimension of \(U(\mathfrak{g})\). Then we have \(\text{GK-dim}_k U(\mathfrak{g}) = \dim_k \mathfrak{g}\).

**Proof.** See [5, 8.1.15 (iii)].

**Proposition 2.4.** Let \(\mathfrak{h}\) be an ideal of \(\mathfrak{g}\) which is abelian. Let \(I_{\mathfrak{h}}\) be the right ideal of \(U(\mathfrak{g})\) generated by \(\sigma(\mathfrak{h})\), which is a two-sided ideal (cf. [2, 2.2.14]). Then, for any two-sided maximal ideal \(m\) with \(U(\mathfrak{g})/m \cong k\) which contains \(I_{\mathfrak{h}}\), we have a Lie algebra isomorphism \(\mathfrak{h} \to I_{\mathfrak{h}}/I_{\mathfrak{h}} m\) via \(\sigma\). Here the Lie algebra structure of \(I_{\mathfrak{h}}/I_{\mathfrak{h}} m\) is defined by that of \(U(\mathfrak{g})\).

**Proof.** First, we prove the proposition in the case \(m = \langle \sigma(\mathfrak{g}) \rangle\). Let \(g_1, \ldots, g_d\) be a basis of \(\mathfrak{g}\) such that \(g_1, \ldots, g_l\) is a basis of \(\mathfrak{h}\). By Poincaré-Birkhoff-Witt theorem, we have
\[
U(\mathfrak{g}) = k \oplus \bigoplus_{s \geq 1} k\sigma(g_{i_1}) \cdots \sigma(g_{i_s}) \quad \text{for} \quad 1 \leq i_1 \leq \cdots \leq i_s \leq d.
\]
Here \(\sigma(g_{i_1}) \cdots \sigma(g_{i_s}) \quad (1 \leq i_1 \leq \cdots \leq i_s \leq d)\) form a \(k\)-basis of \(U(\mathfrak{g})\). Since \(\mathfrak{h}\) is abelian, we have similar decompositions:
\[
I_{\mathfrak{h}} = \bigoplus_{s \geq 1} k\sigma(g_{i_1}) \cdots \sigma(g_{i_s}) \quad \text{for} \quad 1 \leq i_1 \leq \cdots \leq i_s \leq d;
\]
\[
I_{\mathfrak{h}} m = \bigoplus_{s \geq 2} k\sigma(g_{i_1}) \cdots \sigma(g_{i_s}) \quad \text{for} \quad 1 \leq i_1 \leq \cdots \leq i_s \leq d.
\]
as \(k\)-vector spaces. Hence we have an isomorphism

\[
\mathfrak{h} \xrightarrow{\cong} I_\mathfrak{h}/I_\mathfrak{h}m = \bigoplus_{s=1}^{1 \leq i_1 \leq l} k\sigma(g_{i_1})
\]

via \(\sigma\). Using \(I_\mathfrak{h} \subset m\) and \(I_\mathfrak{h}^2 \subset I_\mathfrak{h}m\), one can verify that the Lie algebra structure of \(I_\mathfrak{h}/I_\mathfrak{h}m\) is well-defined and abelian.

Next we show the proposition in the general case. Let \(\alpha: U(\mathfrak{g}) \to k\) be a surjective \(k\)-algebra homomorphism with kernel \(m\). Using \(\alpha\), we have an automorphism \(i: U(\mathfrak{g}) \to U(\mathfrak{g})\) with \(i\sigma(g) = \sigma(g) - \alpha(\sigma(g)) \cdot 1\) for all \(g \in \mathfrak{g}\). Since \(m\) contains \(I_\mathfrak{h}\), the restriction of \(i\) to \(\sigma(\mathfrak{h})\) is the identity of \(\sigma(\mathfrak{h})\). One can easily verify \(i(\langle \sigma(\mathfrak{g}) \rangle) = m\). Hence we have an isomorphism \(\mathfrak{h} \to I_\mathfrak{h}/I_\mathfrak{h}m\) using the isomorphism (1) and a commutative diagram

\[
\begin{array}{ccc}
\mathfrak{h} & \longrightarrow & I_\mathfrak{h}/I_\mathfrak{h}\langle \sigma(\mathfrak{g}) \rangle \\
\uparrow & \cong & \downarrow \ i \\
\mathfrak{h} & \longrightarrow & I_\mathfrak{h}/I_\mathfrak{h}m.
\end{array}
\]

Corollary 2.5. In the notation of Proposition 2.1, we regard the ideal \(I := I_{[\mathfrak{g},\mathfrak{g}]}\) as ideal of the underlying Lie algebra \(U(\mathfrak{g})\). Assume that \([\mathfrak{g},\mathfrak{g}]\) is abelian. Then, for any maximal ideal \(m\) with \(U(\mathfrak{g})/m \cong k\), the composite \([\mathfrak{g},\mathfrak{g}] \xrightarrow{\cong} I \xrightarrow{pr} I/Im\) is an isomorphism of Lie algebras.

Remark 2.6. The composition \([\mathfrak{g},\mathfrak{g}] \xrightarrow{\cong} I \xrightarrow{pr} I/Im\) is surjective for any Lie algebra, but not necessarily injective if \([\mathfrak{g},\mathfrak{g}]\) is not abelian. For example, consider a simple Lie algebra.

Proposition 2.7. Let \(\mathfrak{g}_0\) be an ideal of \(\mathfrak{g}\). Suppose that there exists a subalgebra \(\mathfrak{g}_1\) of \(\mathfrak{g}\) such that \(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1\) as \(k\)-vector spaces. Then \(\mathfrak{g}\) is isomorphic to the semidirect product \(\mathfrak{g}_0 \rtimes \mathfrak{g}_1\).

Proof. It is straightforward to show that the \(k\)-linear map \(\mathfrak{g}_0 \rtimes \mathfrak{g}_1 \to \mathfrak{g}\) defined by \((g_0, g_1) \mapsto g_0 + g_1\) is a Lie algebra isomorphism. \(\Box\)

Proposition 2.8. Let \(\mathfrak{g}_i\) and \(\mathfrak{g}_i' (i = 0, 1)\) be Lie algebras and \(\mathfrak{g}_1 \xrightarrow{d} \text{Der}_k \mathfrak{g}_0\) (resp. \(\mathfrak{g}_1' \xrightarrow{d'} \text{Der}_k \mathfrak{g}_0'\)) a derivation of \(\mathfrak{g}_0\) (resp. \(\mathfrak{g}_0'\)). Assume that there exist Lie algebra isomorphisms \(\varphi_0: \mathfrak{g}_0 \to \mathfrak{g}_0'\) and \(\varphi_1: \mathfrak{g}_1 \to \mathfrak{g}_1'\) with a commutative diagram

\[
\begin{array}{ccc}
\mathfrak{g}_1 & \xrightarrow{d} & \text{Der}_k \mathfrak{g}_0 \\
\varphi_1 \downarrow & & \varphi_0^* \downarrow \\
\mathfrak{g}_1' & \xrightarrow{d'} & \text{Der}_k \mathfrak{g}_0'.
\end{array}
\]
Here $\varphi^*_0$ is the induced homomorphism by $\varphi_0$. Then the semidirect product $g_0 \rtimes g_1$ is isomorphic to $g'_0 \rtimes g'_1$ by $(g_0, g_1) \mapsto (\varphi_0(g_0), \varphi_1(g_1))$.

Proof. Straightforward. See [1, Chapitre 1 §7].

3. Proof of Theorem 1.1.

We have only to show that, if $U(g)$ is isomorphic to $U(g')$, the Lie algebra $g$ is isomorphic to $g'$.

Assume that $U(g)$ is isomorphic to $U(g')$. We remark that $\dim_k g = \dim_k g'$ and $\dim_k [g, g] = \dim_k [g', g']$ by Propositions 2.2 and 2.3.

In the case of $\dim_k g = 1, 2$, the theorem follows from a result of Malcolmson [3, I.4].

We now assume $\dim_k g = \dim_k g' = 3$. We carry out the proof in each case of $\dim_k [g, g] = 0, 1, 2, 3$.

If $\dim_k [g, g] = 0$, i.e., $g$ is abelian, then the theorem is clear.

Suppose $\dim_k [g, g] = 3$. Then one can verify that $g$ is simple (cf. [3, I.4]). Hence the theorem follows from a result of Malcolmson [4, Corollary 1].

Finally, we treat the case $\dim_k [g, g] = 1, 2$. Let $\psi : U(g) \to U(g')$ be an isomorphism. We denote by $m$ (resp. $m'$) the (two-sided) maximal ideal generated by $\sigma(g)$ (resp. $\sigma(g')$). Let $I := I[g, g]$ and $I' := I[g', g']$ be as in Proposition 2.1. Note that $[g, g]$ is abelian in this case (cf. [3, I.4]). By Proposition 2.2 and Corollary 2.5, we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] & \overset{\sigma}\longrightarrow & U(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) \\
\downarrow \quad & & \downarrow \rho \\
\text{Der}_k([\mathfrak{g}, \mathfrak{g}]) & \overset{\cong}{\longrightarrow} & \text{Der}_k(I/I\mathfrak{m}) \\
\end{array}
\begin{array}{ccc}
U(\mathfrak{g}'/[\mathfrak{g}', \mathfrak{g}']) & \overset{\sigma_{g'}}\longleftarrow & \mathfrak{g}'/[\mathfrak{g}', \mathfrak{g}'] \\
\downarrow \rho' & & \downarrow \\
\text{Der}_k(I'/I'\mathfrak{m}') & \overset{\cong}{\longleftarrow} & \text{Der}_k([\mathfrak{g}', \mathfrak{g}']).
\end{array}
\]

Here $\rho, \rho'$ are Lie homomorphisms defined by inner derivation as usual, and $\psi$ (resp. $\psi^*$) is the isomorphism induced by $\psi$.

Suppose $\dim_k [g, g] = 1$. Then there are just two isomorphism classes of 3-dimensional Lie algebras [3, I.4]: One is nilpotent; the other is not nilpotent. In this case, a Lie algebra $g$ is nilpotent if and only if its center contains $[g, g]$, i.e., the above $\rho$ is trivial for a maximal ideal $m$ with $U/m \cong k$. The theorem follows from the above diagram.

Next, we suppose $\dim_k [g, g] = 2$. Take elements $z \in g \setminus [g, g]$ and $z' \in \mathfrak{g}' \setminus [\mathfrak{g}', \mathfrak{g}']$. We denote by $g_1$ (resp. $g'_1$) the subalgebra of $g$ (resp. $g'$) generated by $z$ (resp. $z'$). Since

$$\bar{\psi}([g, g]) = a\sigma_{g'}(z' \mod [\mathfrak{g}', \mathfrak{g}']) + b$$

for some $a \in k^*, b \in k$,
we have the following commutative diagram of Lie algebras:

\[
\begin{array}{ccc}
\mathfrak{g}_1 & \xrightarrow{\psi_1} & \mathfrak{g}_1' \\
\downarrow & & \downarrow \\
\text{Der}_k([\mathfrak{g}, \mathfrak{g}]) & \xrightarrow{\psi^*} & \text{Der}_k([\mathfrak{g}', \mathfrak{g}'])
\end{array}
\]

where $\psi_1$ maps $z$ to $az'$, and $\psi^*$ is the composite of the lower horizontal maps in the above diagram. The theorem follows from Propositions 2.7 and 2.8.

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References


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