Pacific Journal of Mathematics

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Volume 214 No. 1 March 2004

PACIFIC JOURNAL OF MATHEMATICS
Vol. 214, No. 1, 2004

PROPERLY EMBEDDED MINIMAL DISKS BOUNDED BY NONCOMPACT POLYGONAL LINES

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In this paper we give a uniqueness and existence result for minimal disks with some noncompact, U-shaped boundaries in a slab of $\mathbb{R}^3$.

1. Introduction and preliminaries.

Minimal surfaces containing straight lines have special properties that distinguish them from the rest of minimal surfaces. In this article, we emphasize Schwarz's reflection principle. Examples of this type were well studied during the last two centuries.

Recently, in [**11**], F.J. López and F. Wei obtained an existence and uniqueness theorem for properly immersed minimal disks whose boundaries consist of two disjoint straight lines and a segment which meets the lines orthogonally.

Following this, López and the second author of this paper have constructed a deformation of López-Wei disks which consists of properly embedded minimal disks bounded by straight lines and contained in a wedge of a slab (see [**9**] and [**10**]). Essentially, the deformation modifies the angle formed by the two halfplanes containing the connected components of the boundary. The surfaces that appear in this deformation for angle zero correspond to some Jenkins-Serrin graphs (see [**6**]). The López-Martín examples have nice geometric properties such as the convex hull property. These examples are a solution to Plateau's problem for a polygonal noncompact boundary consisting of a double **U** shaped contour (see Figure 1). These surfaces can be used as a new type of barrier for the maximum principle application ([**8**] and [**9**]). Examples of this kind are also closely related to minimal surfaces with helicoidal ends ([**15**]).

In this paper, we obtain all the solutions to the aforementioned Plateau problem with noncompact polygonal boundary, which are contained in the slab, but not lie necessarily in the convex hull of their boundary (see Figure 2). To be more precise, we deal with the study of properly embedded minimal surfaces whose boundary $\Gamma_{\theta\, d}$ consists of the following configuration of straight lines:

Fix $\theta \in [0, \pi]$ and $d \geq 0$, and consider two half-lines r_1^+ and r_1^- in $\mathbb{R}^3$, meeting at an angle of θ. If $\theta = 0$ this means that the straight lines are parallel. Let q_1^+ and q_1^- be two points in r_1^+ and r_1^-, respectively, such that they are symmetric with respect the inner bisector of these half-lines. We choose q_1^+ and q_1^- in such a way that either $q_1^+ = q_1^-$ or the half-lines ℓ_1^+ and ℓ_1^- on r_1^+ and r_1^- starting at q_1^+ and q_1^-, respectively, do not intersect. Write $d = \mathrm{dist}(q_1^+, q_1^-)$.

Let Π_1 be the plane determined by ℓ_1^+ and ℓ_1^-, Π_2 a plane parallel and distinct to Π_1 and let S denote the slab determinated by Π_1 and Π_2. Let ℓ_2^+ and ℓ_2^- be the orthogonal projections to Π_2 of ℓ_1^+ and ℓ_1^-, respectively. Denote q_2^+ (resp. q_2^-) as the orthogonal projection to Π_2 of q_1^+ (resp. q_1^-), and label ℓ_0^+ (resp. ℓ_0^-) as the segment $[q_1^+, q_2^+]$ (resp. $[q_1^-, q_2^-]$). Finally, we define

$$\Gamma_{\theta\, d}^+ = \bigcup_{i=0}^{2} \left(\ell_i^+\right), \quad \Gamma_{\theta\, d}^- = \bigcup_{i=0}^{2} \left(\ell_i^-\right), \quad \Gamma_{\theta\, d} = \Gamma_{\theta\, d}^+ \cup \Gamma_{\theta\, d}^-.$$

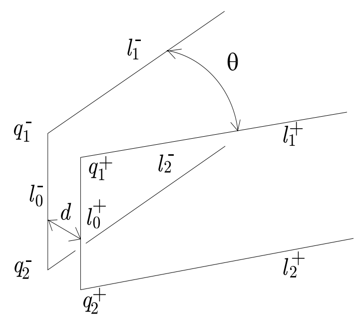

Figure 1. The curve $\Gamma_{\theta\, d}$.

We consider the following generalized Plateau problem:

Problem 1. Determine a properly immersed minimal surface $X : M \to \mathbb{R}^3$ satisfying:

(1) M is homeomorphic to the closed unit disk $\overline{\mathbb{D}}$ minus two boundary points E_1 and E_2, that we call the ends of M.
(2) $X(\partial(M)) = \Gamma_{\theta\, d}$.
(3) If $d > 0$, X is an *embedding*.
(4) In the limit case $\ell_0^+ = \ell_0^-$ (i.e., $d = 0$), the maps $X|_{M-\gamma^+}$ and $X|_{M-\gamma^-}$ are injective, where γ^+ and γ^- are the two connected components of $\partial(M)$.
(5) $X(M)$ lies in a slab that contains S.

Observe that if (5) is satisfied then it is easy to prove (see Lemma 2.1 in [**12**]) that $X(M)$ lies in the slab S. Then Condition (5) is equivalent to

(5) $X(M)$ lies in S.

We prove the following:

Main Theorem. *If $0 < \theta < \pi$ there exist d_θ and d'_θ with $0 < d'_\theta < d_\theta$ such that:*

(i) *If $d > d_\theta$ there are no solutions of Problem* 1.
(ii) *If $d = d_\theta$, Problem* 1 *has a unique solution.*
(iii) *If $d \in]d'_\theta, d_\theta[$ or $d = 0$, Problem* 1 *has two solutions.*
(iv) *If $d = d'_\theta$, Problem* 1 *has three solutions.*
(v) *If $d \in]0, d'_\theta[$, Problem* 1 *has four solutions.*

If $\theta = \pi$ there exist d_π with $0 < d_\pi$ such that:

(i) *If $d > d_\pi$ there are no solutions of Problem* 1.
(ii) *If $d = 0$ or $d = d_\pi$, Problem* 1 *has a unique solution.*
(iii) *If $d \in]0, d_\pi[$, Problem* 1 *has two solutions.*

If $\theta = 0$ there exist d'_0 with $0 < d'_0 < \mathrm{dist}(\Pi_1, \Pi_2)$ such that:

(i) *If $d \geq \mathrm{dist}(\Pi_1, \Pi_2)$ there are no solutions of Problem* 1.
(ii) *If $d = 0$ or $d \in]d'_0, \mathrm{dist}(\Pi_1, \Pi_2)[$, Problem* 1 *has a unique solution.*
(iii) *If $d = d'_0$, Problem* 1 *has two solutions.*
(iv) *If $d \in]0, d'_0[$, Problem* 1 *has three solutions.*

López and Wei proved in [**11**] that there exists a unique solution of Problem 1 when $\theta = \pi$ and $d = 0$. Therefore, we always omit this case in our discussions.

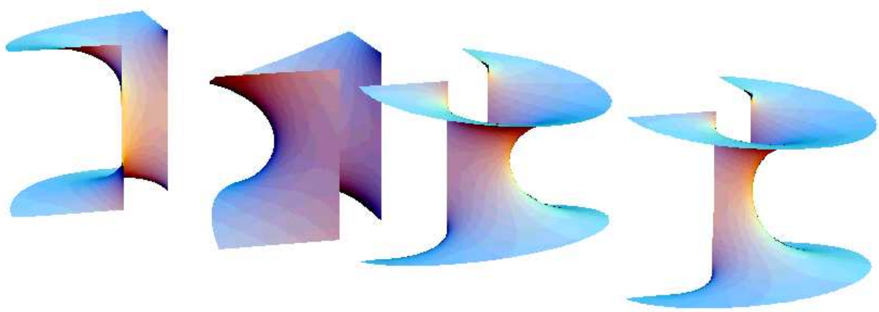

Figure 2. The four solutions in case $\theta = \frac{\pi}{2}$, $d = \frac{\mathrm{dist}(\Pi_1, \Pi_2)}{2}$. The first and second one on the left corresponds to López-Martín examples.

The aim of this paper is to prove the uniqueness and existence of the solutions stated in the Main Theorem. The paper is set out as follows:

In Section 2, we obtain the uniqueness result stated in the above theorem. For the sake of clarity we divide the proof in several subsections. In the first one, we shall see that if M is a solution of our Plateau problem then M is conformally equivalent to a twice punctured closed disk with piecewise analytic boundary and its meromorphic data extend to the closed disk.

Among the results obtained in this subsection, we emphasize the following proposition:

Proposition 1. *Assume $X : M \longrightarrow \mathbb{R}^3$ is a solution of Problem 1 for $0 \leq \theta < \pi$ and denote by $\mathcal{E}(\Gamma_{\theta\, d})$ the convex hull of $\Gamma_{\theta\, d}$. Then $X(M)$ lies either in $\mathcal{E}(\Gamma_{\theta\, d})$ or in $(S \setminus \mathcal{E}(\Gamma_{\theta\, d})) \cup \Gamma_{\theta\, d}$. If $\theta = \pi$, then $X(M)$ lies in one of the half-slabs determinated by the strip $\mathcal{E}(\Gamma_{\pi\, d})$.*

Roughly speaking, the above proposition asserts that if $0 \leq \theta < \pi$, then the solutions of our problem lie either in the *interior* of the convex hull of the boundary or in the *exterior* of it. Subsection 2.2 is devoted to proving that M inherits the horizontal symmetry of its boundary and also the vertical symmetry in case $d = 0$. Finally, in Subsection 2.3, taking into account the preceding steps, we determine a model of the complex structure and Weierstrass representation of any solution of Plateau's problem above. As a consequence, we obtain that, in the general case, a solution of our Plateau problem also inherits the vertical symmetry of its boundary.

The existence part of the Main Theorem can be found in Section 3. We prove that the Weierstrass data obtained in Section 2 really correspond to solutions of our problem.

As we mentioned before, López-Martín examples can be used as barriers in order to prove nonexistence results for minimal surfaces with planar boundaries in a wedge of a slab. Furthermore, they extended the family of minimal surfaces satisfying the convex hull property. To state these results we need some notation. Define $L = \{(0,0,t) \mid -\frac{1}{2} < t < \frac{1}{2}\}$ and $W = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid -\frac{1}{2} \leq x_3 \leq \frac{1}{2}\}$. For $\theta \in [0,\pi]$, we also write $W_\theta = L \cup \{(x_1,x_2,x_3) \in W \setminus L \mid \operatorname{Arg}((x_1,x_2)) \in [0,\theta]\}$ and $\Sigma_\theta = L \cup \{(x_1,x_2,x_3) \in W_\pi \setminus L \mid \operatorname{Arg}((x_1,x_2)) = \theta\}$. Using this notation López and the second author have proved the following:

Theorem 1 ([**10**]). *Let M be a connected properly immersed minimal surface in a wedge $W_{2\pi-\epsilon}$ for some $0 < \epsilon < 2\pi$. Then one has:*

(i) *If $\partial(M) \subset \Sigma_0$, then M is a planar region in Σ_0.*
(ii) *If $\partial(M) \subset W_\theta$, for $\theta \in]0,\pi[$, then M lies in the convex hull of its boundary.*

Finally, we would like to mention that the Main Theorem and Proposition 1 are announced in the proceedings of the conference *Differential Geometry Valencia* 2001 ([**5**]).

2. Conformal structure and Weierstrass representation.

As we mentioned before, this section is devoted to study the underlying complex structure and Weierstrass data of the solutions of our problem.

Throughout this paper (x_1,x_2,x_3) denotes a set of Cartesian coordinates such that:

- ℓ_0^+ and ℓ_0^- have the direction of x_3-axis,
- $\{x_3 = \frac{1}{2}\}$ and $\{x_3 = -\frac{1}{2}\}$ are the equations of planes Π_1 and Π_2, respectively,
- the origin is the middle point between $\frac{q_1^+ + q_2^+}{2}$ and $\frac{q_1^- + q_2^-}{2}$,
- the x_2-axis is the inner bisector of the orthogonal projection of ℓ_i^+ and ℓ_i^- to the plane $x_3 = 0$, $i = 1, 2$, and
- $\Gamma_{\theta\, d}^+ \subset \{x_1 \geq 0\}$.

Along this section, $X : M \longrightarrow \mathbb{R}^3$ denote a proper conformal minimal immersion satisfying Conditions (1)-(5) of Problem 1. For the sake of simplicity, we use Γ, Γ^+ and Γ^- instead of $\Gamma_{\theta\, d}$, $\Gamma_{\theta\, d}^+$ and $\Gamma_{\theta\, d}^-$. Taking into account that $X(M) \subset S$ and the maximum principle, we deduce that $\Pi_i \cap X(M) \subset \Pi_i \cap \Gamma$, for $i = 1, 2$.

As we announced, we shall divide the study of conformal structure in several subsections.

2.1. Conformal type of M. The conformal type of M can be easily determined using a global result on conformal structure of properly immersed minimal surfaces by P. Collin, R. Kusner, W.H. Meeks and H. Rosenberg (see [**4**]). From Theorem 3.1 of [**4**] we obtain that M is parabolic and hence, taking into account the topological type of M, M is conformally equivalent to the closed unit disk $\overline{\mathbb{D}}$ minus two boundary points E_1 and E_2, where the biholomorphism extends piecewise analytically to the boundary.

Next, we prove that the Gauss map and Weierstrass data extend continuously to the ends. To obtain this, we need some additional results.

Let $U(E_i)$, $i = 1, 2$, be two open disjoint neighbourhoods of the ends of M and let C_a denote the catenoid given by the equation $x_1^2 + x_2^2 = a^2 \cosh^2\left(\frac{x_3}{a}\right)$, where $a \in \mathbb{R}^+$. Define $\sigma_a = X^{-1}(X(M) \cap C_a)$, for $a > 0$. With this notation we shall prove the following:

Lemma 1. *There exists $a_0 > 0$ such that for $a \geq a_0$, $\sigma_a = \sigma_a^1 \cup \sigma_a^2$, where σ_a^1 and σ_a^2 are two disjoint simple compact analytic curves such that $\sigma_a^i \subset U(E_i)$, for $i = 1, 2$.*

Proof. Clearly, since $X(M) \cap C_a$ is compact and X is proper, we have that σ_a is compact for $a > 0$. Furthermore, σ_a is a set of properly immersed analytic lines, because it is the intersection of distinct minimal surfaces. Denote by $\mathrm{Int}(C_a)$ and $\mathrm{Ext}(C_a)$, the interior and exterior connected component of $\mathbb{R}^3 \setminus C_a$, respectively.

Note that we can consider a_1 sufficiently large to insure the following: $\ell_0^+ \cup \ell_0^- \subset \mathrm{Int}(C_a)$ and $S \cap \mathrm{Int}(C_a) \subset S \cap \mathrm{Int}(C_{a'})$ if $a_1 \leq a \leq a'$. As C_a and $X(M)$ are transverse along $\bigcup_{i=1}^2 \ell_i^+ \cup \ell_i^-$, for $a \geq a_1$, we can assert that only one curve lying in σ_a approaches to each one of the four points in $\sigma_a \cap \partial(M)$.

Moreover, since σ_{a_1} is compact, we can find $U'(E_i)$, connected neighbourhoods of E_i such that $U'(E_i) \subset U(E_i)$ and $X(U'(E_i)) \subset \mathrm{Ext}(C_{a_1})$. Consider now $a_0 > a_1$ such that $\sigma_{a_0} \subset U'(E_1) \cup U'(E_2)$.

Therefore, if $a \geq a_0$ we deduce that $\sigma_a = \sigma_a^1 \cup \sigma_a^2$, with $\sigma_a^i \subset U'(E_i)$, $i = 1, 2$ and $\sigma_a^1 \cap \sigma_a^2 = \emptyset$. Now, we must prove that σ_a^i are simple curves for $i = 1, 2$. Suppose that there exists a disk Ω in $U'(E_i)$ bounded by an arc of σ_a^i. In this case, either $X(\Omega) \subset \mathrm{Ext}(C_a)$ or $X(\Omega) \subset \mathrm{Int}(C_a)$. In the first case, we have that $X(\Omega) \subset \mathrm{Ext}(C_a) \cap \mathrm{Int}(C_{a_2})$, for some $a \leq a_2$. Hence, using the family of catenoids $\{C_t\}_{a \leq t \leq a_2}$ and the maximum principle, we obtain that $X(\Omega)$ is contained in the catenoid C_a, which is contrary to our assumptions. Moreover, since $a_0 \leq a$ we can assert that $X(\Omega) \subset \mathrm{Ext}(C_{a_1})$. Consequently, if $X(\Omega) \subset \mathrm{Int}(C_a)$ we may consider the family of catenoids $\{C_t\}_{a_1 \leq t \leq a}$. The maximum principle gives again a contradiction. □

Label $\gamma_i^+ = X^{-1}(\Gamma_i^+)$ and $\gamma_i^- = X^{-1}(\Gamma_i^-)$, for $i = 1, 2$. Consider also $\gamma_0^+ = X^{-1}(\ell_0^+ \cup \ell_0^-) \cap \gamma^+$ and $\gamma_0^- = X^{-1}(\ell_0^+ \cup \ell_0^-) \cap \gamma^-$.

Concerning the boundary behaviour we have, up to relabellings, three possibilities:

Case 1. $X(\gamma^+) = \Gamma^+$, $X(\gamma^-) = \Gamma^-$ and $\gamma_i^+ \cup \gamma_i^-$ diverges to E_i, for $i = 1, 2$ (see Figure 3.(1)).

Case 2. $X(\gamma^+) = \Gamma^+$, $X(\gamma^-) = \Gamma^-$, $\gamma_1^+ \cup \gamma_2^-$ diverges to E_1 and $\gamma_2^+ \cup \gamma_1^-$ diverges to E_2 (see Figure 3.(2)).

Case 3. When $d = 0$ we have also the case $X(\gamma^+) = \ell_1^+ \cup \ell_0 \cup \ell_2^-$, $X(\gamma^-) = \ell_1^- \cup \ell_0 \cup \ell_2^+$ and $\gamma_i^+ \cup \gamma_i^-$ diverges to E_i, for $i = 1, 2$, where $\ell_0 = \ell_0^+ = \ell_0^-$ (see Figure 3(3)).

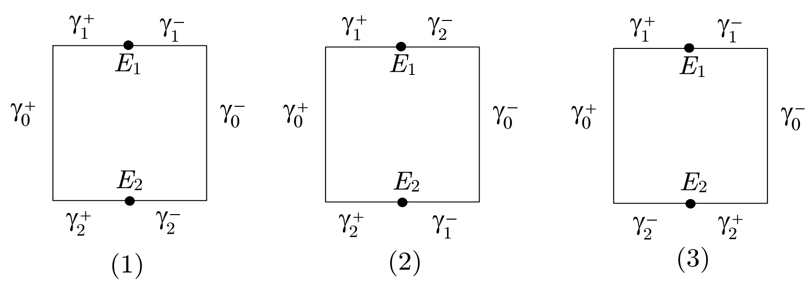

Figure 3.

Now, we shall prove that if $d = 0$ then Case 2 and Case 3 do not occur.

Lemma 2. *Assume $d = 0$. Then the boundary of $X : M \longrightarrow \mathbb{R}^3$ is as in Case 1.*

Proof. Note that if $d = 0$ and the boundary is either as in Case 2 or as in Case 3, $X(M)$ contains a Möbius strip. Let us define t_ρ as the translation of vector $(0, 0, \rho)$ and consider N the topological surface of $\mathbb{R}^3$ given by

$N = \bigcup_{n\in\mathbb{Z}} t_n(X(M))$. Since $X(M) \subset S$ we have that N is a connected embedded topological surface in $\mathbb{R}^3$. A well-known topological result asserts that then N must be orientable, but this is absurd because N contains Möbius strips. □

Denote $\vec{a}_2 = (0,1,0)$. Observe that $\vec{a}_2$ is the unitary vector in the direction of the inner bisector of ℓ_2^+ and ℓ_2^- pointing to $\mathcal{E}(\Gamma)$. At this point we can prove a restatement of Proposition 1 above:

Proposition 1. *If $0 \le \theta < \pi$, then $X(M)$ lies either in $\mathcal{E}(\Gamma)$ or in $(S \setminus \mathcal{E}(\Gamma)) \cup \Gamma$. If $\theta = \pi$, then $X(M)$ lies in one of the half-slabs determinated by the strip $\mathcal{E}(\Gamma)$.*

Proof. Assume $0 \le \theta < \pi$. In accordance to Lemma 2 we have that the boundary behaviour is either as in Case 1 or as in Case 2. Consider $\beta = X^{-1}(X(M) \cap \{x_2 = 0\})$. Since β is a nodal set of an harmonic function we have that β is a set of properly immersed analytic lines. Using the maximum principle we obtain that there are no compact connected regions of M bounded by curves in β. Furthermore, as we are assuming $0 \le \theta < \pi$, the theorem of the order of contact (see [**13**, §437]) gives us that there are no curves in β approaching to either γ_i^+ or γ_i^-, for $i = 1,2$. Now, we consider the following half-strips:

$$B^+ = \{(x_1, 0, x_3) \mid x_1 \ge \tfrac{d}{2}, \tfrac{-1}{2} \le x_3 \le \tfrac{1}{2}\},$$
$$B^- = \{(x_1, 0, x_3) \mid x_1 \le \tfrac{-d}{2}, \tfrac{-1}{2} \le x_3 \le \tfrac{1}{2}\}.$$

First, we shall prove that if there exists a curve in β starting at either γ_0^+ or γ_0^- and diverging to one end, then there are no curves starting at the same vertical segment and diverging to the same end. Assume that β' and β'' are two curves starting at γ_0^+ and diverging to E_1. The other cases can be treated in the same way. Clearly, if $d > 0$ we have $X(\beta') \cup X(\beta'') \subset B^+$. Suppose that $d = 0$ and $X(\beta')$ and $X(\beta'')$ are contained in different half-strips. Then we can consider, taking a piece of γ_0^+ if necessary, a piecewise analytic curve $\hat{\beta}$ that diverge to E_1 and contains β' and β''. It is not difficult to see that there is an angle between the curves in $\hat{\beta}$, Θ, that goes by X to an angle greater or equal than 2Θ. Since $X : M \longrightarrow \mathbb{R}^3$ is conformal this is a contradiction. Then we conclude that in both cases $X(\beta')$ and $X(\beta'')$ are contained in the same half-strip. Therefore, we can find a connected component, Ω, of $M \setminus \beta$ such that $X(\Omega)$ is contained in one of the half-slabs determinated by $\{x_2 = 0\}$ and $X(\partial(\Omega))$ is in a half-strip. Consequently, applying Statement (i) in Theorem 1 we obtain that $X(\Omega)$ is a planar domain of $\{x_2 = 0\}$ which contradicts our assumptions.

Moreover, we shall prove that there are no compact curves in $\beta \setminus (\gamma_0^+ \cup \gamma_0^-)$ starting at γ_0^+ and ending at γ_0^-. Assume there exists τ such a curve. As there are no compact regions of $M \setminus \beta$, we infer that τ is the unique curve that starts at γ_0^+ and ends at γ_0^-. Taking into account the above paragraph and the fact that Γ^+ and Γ^- are in the same half-slab of S determinated by $\{x_2 = 0\}$ we conclude that must exist a pair of curves, τ_1 and τ_2 starting at γ_0^+ and γ_0^-, respectively and diverging to either E_1 or E_2. We assert that both curves must diverge to the same end. Indeed, if τ_1 and τ_2 diverge each one to one different end, then there exists a curve τ_3 diverging to both ends. But this curve τ_3 intersects τ transversally in a odd number of points while $X(\tau_3)$ intersects $X(\tau)$ transversally in a even number of points.

Without loss of generality, we can assume that τ_1 and τ_2 diverge to E_1. Now, we may consider, taking pieces of γ_0^+ and γ_0^- if necessary, a piecewise analytic curve τ' from E_1 to E_1 that encloses a disk Ω of $M \setminus \beta$. If $X(\tau_1)$ and $X(\tau_2)$ are contained in the same half-strip, the domain Ω verifies the conditions of statement (i) in Theorem 1 and we obtain a contradiction. Assume that $X(\tau_1)$ and $X(\tau_2)$ are contained in different half-strips. Note that then there is an angle between the curves in τ', Θ, that goes by X to an angle greater or equal than 2Θ. Using again that $X : M \longrightarrow \mathbb{R}^3$ is conformal we get a contradiction.

Consequently, $\beta \setminus (\gamma_0^+ \cup \gamma_0^-)$ consists of curves starting at $\gamma_0^+ \cup \gamma_0^-$ and diverging to one end and divergent curves. Next we prove that there are no curves diverging to only one end. As before we have that one of these curves would be contained either in B^+ or in B^-. Otherwise, in each of these cases it is possible to find a connected component, Ω, of $M \setminus \beta$ such that $X(\Omega)$ is contained in a half-slab of S and $X(\partial(\Omega))$ is contained in a half-strip of $\{x_2 = 0\}$. Consequently, applying Statement (i) in Theorem 1 we obtain that $X(\Omega)$ is a planar domain of $\{x_2 = 0\}$ which contradicts our assumptions. Furthermore, using again that Γ^+ and Γ^- are in one of the half-slabs determinated by $\{x_2 = 0\}$ we deduce that if there exists a curve that starts at $\gamma_0^\pm$ and diverge to one end, then there exists a curve that starts at $\gamma_0^\pm$ and diverge to the other end. All these facts allows us to assert that in $\beta \setminus (\gamma_0^+ \cup \gamma_0^-)$ either there are no curves starting at $\gamma_0^+ \cup \gamma_0^-$, or there are a pair starting at γ_0^+ or γ_0^- and diverging to different ends or there are four curves, a pair starting at γ_0^+ and diverging to different ends and another pair starting at γ_0^- and diverging to different ends. Moreover, we may find curves in $\beta \setminus (\gamma_0^+ \cup \gamma_0^-)$ diverging to the two ends. Note that then the number of curves diverging to E_1 is the same as the number of curves diverging to E_2. It is not hard to see, using Statement (i) in Theorem 1, that two consecutive curves diverging to the same end have to be in different half-strips, it is to say, if one is in B^+ the other one is in B^- and that all divergent curves are disjoint. Assume that there are more than two curves in β diverging to E_1 and consider the compact curves $\sigma^i = \sigma_a^i$, for $i = 1, 2$ and $a \geq a_0$ given in Lemma 1.

Now, we analyze each of the possibilities for the boundary separately.

Case 1. Denote $p_1^+ = \gamma_1^+ \cap \sigma^1$ and $p_1^- = \gamma_1^- \cap \sigma^1$. Then, denoting $\mathfrak{p}_3 : \mathbb{R}^3 \longrightarrow \{x_3 = 0\}$ as the orthogonal projection over the plane $\{x_3 = 0\}$ we deduce that $\mathfrak{p}_3(X(\sigma^1))$ is a curve in $\{x_3 = 0\}$ such that $|\arg(\mathfrak{p}_3(X(p_1^-))) - \arg(\mathfrak{p}_3(X(p_1^+)))| > 2\pi$. Since $X_3(p_1^+) = X_3(p_1^-) = \frac{1}{2}$ we infer that $X(\sigma^1)$ has self-intersections, which is contrary to our assumptions. As a consequence, there is at most two curves in β diverging to E_1 and the same for E_2.

Case 2. Denote $p_1^+ = \gamma_1^+ \cap \sigma^1$, $p_2^- = \gamma_2^- \cap \sigma^1$, $p_2^+ = \gamma_2^+ \cap \sigma^2$ and $p_1^- = \gamma_1^- \cap \sigma^2$ and suppose that σ^1 has been parametriced so that it starts at p_1^+ and ends at p_2^- and σ^2 has been parametriced so that it starts at p_1^- and ends at p_2^+. Using the same notation as above we can see that $\mathfrak{p}_3(X(\sigma^i))$ are curves in $\{x_3 = 0\}$ satisfying $|\arg(\mathfrak{p}_3(X(p_2^-))) - \arg(\mathfrak{p}_3(X(p_1^+)))| > 2\pi$ and $|\arg(\mathfrak{p}_3(X(p_2^+))) - \arg(\mathfrak{p}_3(X(p_1^-)))| > 2\pi$. Moreover, $\mathfrak{p}_3(X(\sigma^1))$ and $\mathfrak{p}_3(X(\sigma^2))$ rotates around $(0,0,0)$ in reverse sense, it is to say, if $\mathfrak{p}_3(X(\sigma^1))$ rotates clockwise then $\mathfrak{p}_3(X(\sigma^2))$ rotates counterclockwise, and vice versa. Since $X_3(p_i^+) = X_3(p_i^-)$ for $i = 1, 2$ we infer that $X(\sigma^1)$ and $X(\sigma^2)$ intersect each other. This contradicts our assumptions and therefore there is at most two curves diverging to each end in β.

The same argument used in both cases proves that if $\beta \setminus (\gamma_0^+ \cup \gamma_0^-)$ consists of two curves, τ_1 and τ_2, diverging to the two ends such that $X(\tau_1) \subset B^+$ and $X(\tau_2) \subset B^-$, then the boundaries of the three connected components of $M \setminus \beta$ are $\gamma^+ \cup \tau_1$, $\tau_1 \cup \tau_2$ and $\gamma^- \cup \tau_2$.

Taking into account this and the fact that $\Gamma^+ \cup \Gamma^-$ is in one of the half-slabs determined by $\{x_2 = 0\}$ we have that either $\beta \setminus (\gamma_0^+ \cup \gamma_0^-)$ is empty or it consists of:

1. Two curves diverging to the two ends (see Figure 4(1)),
2. a curve starting at γ_0^+ and diverging to E_1, a curve starting at γ_0^+ and diverging to E_2 and a curve diverging to the two ends (see Figure 4(2)),
3. a curve starting at γ_0^- and diverging to E_1, a curve starting at γ_0^- and diverging to E_2 and a curve diverging to the two ends (see Figure 4(3)),
4. a curve starting at γ_0^+ and diverging to E_1, a curve starting at γ_0^+ and diverging to E_2, a curve starting at γ_0^- and diverging to E_1 and a curve starting at γ_0^- and diverging to E_2 (see Figure 4(4)).

Clearly, if $\beta \setminus (\gamma_0^+ \cup \gamma_0^-)$ is empty we obtain that $X(M)$ is contained in the half-slab $\{x_2 \geq 0\}$. Therefore $X(M)$ satisfies the conditions of Statement (ii) in Theorem 1 and so $X(M) \subset \mathcal{E}(\Gamma)$.

Assume that we have one of the other possibilities. Then we shall prove that $X(M) \subset (S \setminus \mathcal{E}(\Gamma)) \cup \Gamma$. Note that it is sufficient to study the connected components of $M \setminus \beta$ whose image is contained in the half-slab $\{x_2 \geq 0\}$. Note that these connected components are those whose contains any of the curves γ_i^+ or γ_i^-, for $i = 1, 2$. At this point, it can be easily check that each

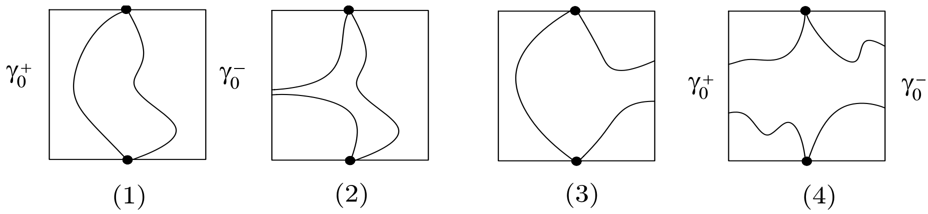

Figure 4.

of these connected components have the boundary contained in one of the following wedges:

$$(S\setminus \overset{\circ}{\mathcal{E}}(\Gamma)) \cap \{(x_1, x_2, x_3) \mid x_1 \geq 0, x_2 \geq 0, \tfrac{-1}{2} \leq x_3 \leq \tfrac{1}{2}\},$$
$$(S\setminus \overset{\circ}{\mathcal{E}}(\Gamma)) \cap \{(x_1, x_2, x_3) \mid x_1 \leq 0, x_2 \geq 0, \tfrac{-1}{2} \leq x_3 \leq \tfrac{1}{2}\}.$$

Using again Assertion (ii) of Theorem 1, we conclude that the image of these connected components is contained entirely in the correspondent wedge. Summarizing, we have proved that $X(M) \subset (S\setminus \overset{\circ}{\mathcal{E}}(\Gamma))$. Now the Proposition is an easy consequence of the maximum principle.

Next, we analyze the case $\theta = \pi$, $d > 0$. Let us define $\Delta = X^{-1}(X(M) \cap \{x_1 = \frac{d}{2}\})$. It is well-known that Δ is a nodal set of an harmonic function and so it is a set of properly immersed analytic lines. Using the maximum principle we obtain that there are no compact connected regions of M bounded by curves in Δ. Then, $\Delta \setminus \gamma_0^+$ consists of a set of divergent curves. Since $\Gamma^+ \subset \{x_1 \geq \frac{d}{2}\}$ and $\Gamma^- \subset \{x_1 \leq \frac{d}{2}\}$, we infer that if there exist a curve in Δ starting at γ_0^+ and diverging to one end, then another curve starting at γ_0^+ and diverging to the other end must exist. Reasoning as in case $0 \leq \theta < \pi$, $d = 0$ we can see that the image of such a pair of curves is contained in one of the following half-strips:

$$C^+ = \{(\tfrac{d}{2}, x_2, x_3) \mid x_2 \geq 0, \tfrac{-1}{2} \leq x_3 \leq \tfrac{1}{2}\},$$
$$C^- = \{(\tfrac{d}{2}, x_2, x_3) \mid x_2 \leq 0, \tfrac{-1}{2} \leq x_3 \leq \tfrac{1}{2}\},$$

and if two curves in Δ diverge to the same end there must be one of them with the image contained in C^+ and the other one with the image in C^-. Therefore, adapting to this situation the argument presented above for the two different possibilities of the boundary, it is not hard to see that there is at most a curve diverging to each end. And then $\Delta \setminus \gamma_0^+$ consists of either a curve diverging to the two ends or a pair of curves starting at γ_0^+ and diverging to different ends. Note that in both cases $X(\Delta)$ is contained either in C^+ or C^-. In order to conclude the proposition it is sufficient to apply Statement (ii) in Theorem 1 to each of connected components of $M \setminus \Delta$. □

Remark 1. Assume that T is a plane in $\mathbb{R}^3$ and that the divergent curves in $X^{-1}(X(M) \cap T)$ verifies that two consecutive divergent curves are in different half-strips of T. Then reasoning as in the proof of Proposition 1 we can see that there are at most two curves diverging to each end.

Corollary 1. *The boundary of the immersion X is as in Case* 1.

Proof. Assuming that $0 \leq \theta < \pi$ and taking into account Proposition 1 we have that either $X(M) \subset \mathcal{E}(M)$ or $X(M) \subset (S \setminus \mathcal{E}(\Gamma)) \cup \Gamma$. Suppose that the boundary behaviour is as in Case 2 and consider the compact curves $\sigma^i = \sigma^i_a$, $i = 1, 2$ given in Lemma 1 for some $a \geq a_0$. Clearly, $X(\sigma^1)$ starts at ℓ_1^+ and ends at ℓ_2^- and $X(\sigma^2)$ is a curve starting at ℓ_1^- and ending at ℓ_2^+. Since both curves lie either in $C_a \cap \mathcal{E}(\Gamma)$ or in $C_a \cap ((S \setminus \mathcal{E}(\Gamma)) \cup \Gamma)$, they intersect, which contradicts our assumptions. □

Taking the above corollary into account and the fact that X_3 is a bounded harmonic function one has the following:

Corollary 2. *The function $X_3 : M \longrightarrow \mathbb{R}^3$ extends continuously to the ends.*

Let us consider $\delta_t = X^{-1}(X(M) \cap \{x_3 = t\})$ for $t \geq -\frac{1}{2}$. Concerning δ_t we can prove:

Corollary 3. *The set δ_t is compact and consists of a simple arc, for all $t \in]-\frac{1}{2}, \frac{1}{2}[$. Moreover, the Gauss map g of X omits the points* 0 *and* ∞.

Proof. Clearly, from Corollary 2, we deduce that δ_t is compact. Since δ_t is the nodal set of a harmonic function we have that δ_t is a one-dimensional proper real analytic subvariety of M. Then, taking into account the maximum principle we deduce that there are no regions in M bounded by curves in $X^{-1}(\delta_t)$. Therefore, δ_t is a regular simple curve in M starting at ℓ_0^+ and ending at ℓ_0^-. Moreover, the theorem of the order of contact (see [**13**, §437]) gives that there are no points in M with vertical normal vector. □

In the case $0 \leq \theta < \pi$ the uniqueness of solutions $X : M \longrightarrow \mathbb{R}^3$ of Problem 1 satisfying $X(M) \subset \mathcal{E}(M)$ were completely studied by F.J. López and F. Martín in [**9**]. Henceforth, in the remainder of the section we assume that $X(M) \subset (S \setminus \mathcal{E}(\Gamma)) \cup \Gamma$. Furthermore, we always assume that $X(M) \subset \{x_2 \leq 0\}$ in the case $\theta = \pi$. With this assumptions, we can prove:

Lemma 3. *We have the following possibilities for the set $\tau_0 = X^{-1}(X(M) \cap \{x_1 = 0\})$:*

i) *If $d > 0$, τ_0 consists of a curve diverging to both ends, E_1 and E_2.*

ii) *If* $d = 0$, $\tau_0 \setminus (\gamma_0^+ \cup \gamma_0^-)$ *either consists of a curve diverging to both ends or* $\tau_0 \setminus (\gamma_0^+ \cup \gamma_0^-) = \tau_0^1 \cup \tau_0^2$, *where* τ_0^i *are curves starting at* γ_0^+ *or* γ_0^- *and diverging to* E_i, *for* $i = 1, 2$.

Proof. Since τ_0 is the nodal set of a harmonic function we have that τ_0 is a set of properly immersed analytic curves in M. Observe that $\tau_0 \neq \emptyset$. If not, applying Statement (ii) in Theorem 1 we obtain that $X(M)$ are two planar domains. Moreover, by the maximum principle, there are no compact connected regions in $M \setminus \tau_0$ bounded by curves in τ_0. Clearly, taking into account that $X(M) \cap \{x_1 = 0\} \subset \{(x_1, x_2, x_3) \mid x_1 = 0, x_2 \leq 0, -\frac{1}{2} \leq x_3 \leq \frac{1}{2}\}$ and Statement (i) in Theorem 1 we obtain i).

Assume now that $d = 0$. In this case we have $\gamma_0^+ \cup \gamma_0^- \subset \tau_0$. Since $\Gamma^+ \subset \{x_1 \geq 0\}$ and $\Gamma^- \subset \{x_1 \leq 0\}$, we infer that if there exists a curve in τ_0 starting at $\gamma_0^\pm$ and diverging to one end, then there exist another curve which starts at $\gamma_0^\pm$ and diverges to the other end. Then reasoning as in the above paragraph we obtain ii). □

Proposition 2. *Counting multiplicities* $\sharp[N^{-1}(-\vec{a}_2, \vec{a}_2)] \leq 5$ *and*

$$\sharp[N^{-1}(-\vec{a}_2, \vec{a}_2) \cap (M \setminus \partial(M))] \leq 3,$$

where $N : M \longrightarrow \mathbb{S}^2$ *is the Gauss map of* X. *Furthermore, if* $0 < \theta < \pi$ *we have*

$$\sharp[N^{-1}(-\vec{a}, \vec{a})] \leq 5, \ \forall \vec{a} \in \mathbb{S}^2 \cap \{x_3 = 0\}.$$

Proof. We prove the first assertion in Proposition 2. The second assertion can be proved using similar arguments. Let us consider $\beta_t = X^{-1}(X(M) \cap \{x_2 = t\})$, for $t \in \mathbb{R}$. Note that β_t is a set of properly immersed analytic lines, because it is the nodal set of a harmonic function. Hence, using the maximum principle, we infer that there are no compact domains in M bounded by curves in β_t, for all $t \in \mathbb{R}$. Therefore, any two curves in β_t do not intersect in more than one point. If not, we can find a compact domain of $M \setminus \beta_t$ bounded by curves in β_t.

We start with the case $t < 0$. Observe that in this case β_t is a nonempty set of divergent curves, converging to a unique end or to the two ends. If $\beta_t = \emptyset$ for some $t < 0$ we deduce that $X(M) \subset \{x_2 \geq 0\}$ and applying Statement (ii) in Theorem 1 we obtain $X(M) \subset \mathcal{E}(\Gamma)$ which contradicts our assumption.

Let $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ be a pair of arcs in β_t diverging to E_i such that $\widetilde{\alpha}_1 \cap \widetilde{\alpha}_2 \neq \emptyset$. Therefore, there exists U_i, a neighbourhood of E_i, verifying that:

- $U_i \setminus (U_i \cap \tau_0)$ has two connected components, U_i^+ and U_i^-, where τ_0 was defined in Lemma 3.
- $\widetilde{\alpha}_1 \cap U_i \subset U_i^+$ and $\widetilde{\alpha}_2 \cap U_i \subset U_i^-$.

If not, one can find a neighbourhood of E_i, U_i, verifying the first condition and such that either $(\widetilde{\alpha}_1 \cup \widetilde{\alpha}_2) \cap U_i \subset U_i^+$ or $(\widetilde{\alpha}_1 \cup \widetilde{\alpha}_2) \cap U_i \subset U_i^-$. Hence, we deduce that $X((\widetilde{\alpha}_1 \cup \widetilde{\alpha}_2) \cap U_i)$, $i = 1, 2$ are contained in a half strip of $\{x_2 = t\}$ and so there is a connected component in $M \setminus \beta_t$, Ω, whose boundary is in a half-strip of $\{x_2 = t\}$. Therefore, we can apply Statement (i) in Theorem 1 to conclude that $X(\Omega)$ is a planar domain in $\{x_2 = t\}$, which is a contradiction.

Let us prove that if α_1 and α_2 are two curves in β_t diverging to E_i, then they are disjoint. Indeed, if α_1 and α_2 intersect then we have four arcs $\{\widetilde{\alpha}_i\}_{i=1}^4$ in β_t diverging to E_i and $\widetilde{\alpha}_j \cap \widetilde{\alpha}_l \neq \emptyset$. But this contradicts the above result.

Assume now that α_1 is a curve diverging to E_i and that α_2 is a curve diverging to the two ends. If α_1 and α_2 intersect each other then we have three arcs $\{\widetilde{\alpha}_i\}_{i=1}^3$ in β_t diverging to E_i and $\widetilde{\alpha}_j \cap \widetilde{\alpha}_l \neq \emptyset$. But, again this contradicts the above result.

As a consequence, only curves diverging to the two ends can intersect. Now, we shall prove that there are at most two of these curves whose intersection is not empty. Assume there exist α_i for $i = 1, 2, 3$ curves in β_t diverging to the two ends such that $\alpha_1 \cap \alpha_i \neq \emptyset$, for $i = 2, 3$. Then, by considering apropriate arcs in α_i, for $i = 1, 2, 3$ and using the assertion proved above about arcs diverging to one end, we can find a connected component of $M \setminus \beta_t$ that satisfies the conditions of Statement (i) in Theorem 1 and then $X(\Omega)$ must be a planar domain in $\{x_2 = t\}$, which contradicts our assumptions. Moreover, it is clear that if α_1 and α_2 are two curves in β_t diverging to the two ends whose intersection is not empty then $\alpha_1 \cap \alpha_2$ is a unique point.

Lastly, if we have two curves in β_t diverging to the two ends whose intersection is not empty, then there are no more intersections in $\beta_{t'}$ for any $t' < 0$, $t' \neq t$. If not, using again the above assertion, we deduce that the pair of divergent curves in β_t intersect the pair of divergent curves in $\beta_{t'}$. Since β_t and $\beta_{t'}$ are contained in parallel planes, this is a contradiction.

Now, we tackle the case $t > 0$. Observe that this case only has sense if $0 \leq \theta < \pi$ and that $\beta_t \cap \partial(M) = p_1^+ \cup p_2^+ \cup p_1^- \cup p_2^-$ where $p_i^+ \in \gamma_i^+$ and $p_i^- \in \gamma_i^-$, for $i = 1, 2$. Since $X(M) \subset (S \setminus \mathcal{E}(M)) \cup \Gamma$ we deduce that connected curves in β_t are contained in a half-strip of $\{x_2 = t\}$. Therefore, there are no curves in β_t diverging to one end. Indeed, we have a connected component of $M \setminus \beta_t$ satisfying the conditions in Statement (i) in Theorem 1 and so we get a contradiction. Hence, it is clear that curves in β_t diverging to two ends are disjoint and moreover a divergent curve starting at $\partial(M)$ and a curve diverging to the two ends can not intersect each other. Then, $\alpha_1 \cap \alpha_2 \neq \emptyset$ only in two situations:

i) When α_1 is a curve starting at γ_1^+ and diverging to E_2 and α_2 is a curve starting at γ_2^+ and diverging to E_1,

ii) when α_1 is a curve starting at γ_1^- and diverging to E_2 and α_2 is a curve starting at γ_2^- and diverging to E_1.

Moreover, we observe that if there exist a pair of curves as in i) for $t_0 > 0$, this pair is unique. It is to say, $\{x_2 = t\} \cap \{x_1 \geq 0\} \cap X(N^{-1}(\{-\vec{a}_2, \vec{a}_2\})) = \emptyset$ for $t > 0$, $t \neq t_0$. And the same occurs for a pair of curves as in ii). Therefore, we have at most two points in $\delta_t \cap N^{-1}(\{-\vec{a}_2, \vec{a}_2\})$ for $t > 0$.

We recall that the set $\beta_0 = \beta$ was studied in the proof of Proposition 1. Since we are assuming $X(M) \subset (S \setminus \mathcal{E}(\Gamma)) \cup \Gamma$ the possibilities for β_0 are those described in 1, 2, 3 and 4. We also point out that Case 2 is not compatible with Case i) analyzed in the case $t > 0$, Case 3 is not compatible with Case ii) analyzed in the case $t > 0$ and so Case 4 is not compatible with either i) or ii). Therefore, there exist at most five points of ordinary contact in M and only three of them can lie in $M \setminus \partial(M)$. ☐

Lemma 4. *For any $p \in \gamma_1^+ \cup \gamma_1^- \cup \gamma_2^+ \cup \gamma_2^-$, counting multiplicities, one has $\sharp[g^{-1}(g(p)) \cap \partial(M)] < 6$.*

Proof. Assume $X(p) \in \ell_1^+$. The proofs of the other possibilities are similar.

Label Σ as the tangent plane to $X(M)$ at $X(p)$ and let us consider $\Lambda = X^{-1}(\Sigma \cap X(M))$. Since Λ is the nodal set of a harmonic function, then Λ is a set of properly immersed analytic curves. Using the interior maximum principle we also deduce that there are no compact simply connected region of M bounded by curves in Λ.

First, we study the case $0 \leq \theta < \pi$. In this case $\ell_1^+ \subset \Sigma$ and since $X(M) \subset (S \setminus \mathcal{E}(\Gamma)) \cup \Gamma$, it is straightforward to prove that $\Lambda \cap X^{-1}(\Gamma - \ell_1^+) = \emptyset$. Hence, taking into account the reasoning at the beginning of the proof and that M is simply connected, we have that if λ is a curve in Λ starting at γ_1^+, then λ diverges to an end and when two curves in Λ start at γ_1^+, they do not intersect.

If $\sharp[g^{-1}(g(p)) \cap \partial(M)] \geq 6$ then, using once again the theorem of the order of contact (see [**13**, §437]), there are at least 6 curves in Λ starting at γ_1^+. Observe that then there exist at least three curves diverging to the same end. Consider this set of diverging curves. If there is a pair of consecutive curves in this set contained in the same half-strip of Σ then the connected component between them, that we call Ω, satisfies the conditions of Statement (i) in Theorem 1 and then $X(\Omega)$ must be a planar domain in Σ, which contradicts our assumptions. On the contrary, if each pair of consecutive curves are in different half-strips, we can use Remark 1 and obtain a new contradiction.

Finally, we analyze the case $\theta = \pi$. Observe that in this case $\Lambda \cap \partial(M) = \gamma_1^+ \cup \gamma_1^-$. Suppose $\sharp[g^{-1}(g(p)) \cap \partial(M)] \geq 6$. We note that in this case compact curves starting at γ_1^+ and ending at γ_1^- can appear in Λ. Otherwise, we can only have one of these curves, because if there exist two or more curves of this type in Λ we would get a compact domain, Ω, in $M \setminus \Lambda$ satisfying

$X(\Omega) \subset \Sigma$, and this is a contradiction. Either there exists a compact curve in Λ or Λ consists only of divergent curves starting at $\gamma_1^+ \cup \gamma_1^-$, it is not hard to see, using Statement (i) in Theorem 1, that there are at least a set of three curves (counting the curve γ_1^+) diverging to the same end in Λ such that two consecutive curves in this set are in different half-strips of Σ. Then we can conclude as in the former case. $\square$

Using the above lemmas we can now prove:

Proposition 3. *The map g extends continuously to the ends. In particular, the total curvature of M is finite. Furthermore, the limit tangent plane to M at E_i is π_i, $i = 1, 2$.*

Proof. We shall prove that the map g extends to E_1. The same argument can be used for E_2. Taking into account Lemma 4 it is not difficult to prove that the following limits exist:

$$\lim_{\substack{p \to E_1 \\ p \in \gamma^+}} g(p)\ , \qquad \lim_{\substack{p \to E_1 \\ p \in \gamma^-}} g(p)\ , \quad i = 1, 2. \tag{1}$$

For the proof of this fact see Claim 3.15 in [**9**].

Since M is conformally equivalent to a sector $S_{\theta_1} = \{r\mathrm{e}^{\mathrm{i}\Theta} \mid r > 0, \Theta \in [0, \theta_1]\}$, a truncated sector $S_{\theta_1}(R) = S_{\theta_1} \setminus \overline{D(0, R)}$ can be seen as a neighbourhood of E_1 in M. Furthermore, we can assume that R is sufficiently large so that $X(r\mathrm{e}^{\mathrm{i}\theta_1}) \in \ell_1^+$ for $r > R$. According to Schwarz Principle we can consider the reflection respect to ℓ_1^+ of $X(S_{\theta_1}(R))$. Taking into account Proposition 2 we deduce that the Gauss map N on the truncated sector $S_{2\theta_1}(R) = \{r\mathrm{e}^{\mathrm{i}\Theta} \mid r > 0, \Theta \in [0, 2\theta_1]\} \setminus \overline{D(0, R)}$, assumes the values $\vec{a}_2$ and $-\vec{a}_2$ a finite number of times. Then it is possible to choose $R' > R$ sufficiently large so that g restricted to $S_{2\theta_1}(R')$ omits the values -1 and 1.

At this point, we need the following technical result:

> *Let $\alpha < \gamma < \beta$, $S(R) = \{r\mathrm{e}^{\mathrm{i}\Theta} \mid r > 0, \Theta \in]\alpha, \beta[\} \setminus \overline{D(0, R)}$, for $R > 0$ and let f be holomorphic in $S(R)$ and for some complex c satisfy $\lim_{r \to +\infty} f(re^{i\gamma}) = c$. Suppose that there are two distinct complex number absent from the range of f. Then $\lim_{r \to +\infty} f(re^{i\Theta}) = c$ for every $\Theta \in]\alpha, \beta[$.*

We refer to reference book [**1**, pp. 441-445].

This theorem and (1) imply that g extends continuously to E_1. Finally, since $M \subset S$ we have that the limit tangent plane at E_i coincides with π_i, $i = 1, 2$. $\square$

Consider again the compact curves σ_a^i given in Lemma 1 for $i = 1, 2$ and $a \geq a_0$ sufficiently large. Let us denote by U_a^i the connected component of $M \setminus (\sigma_a^1 \cup \sigma_a^2)$ that contains the end E_i, $i = 1, 2$. With this notation we can prove the following useful result:

Corollary 4. *For a sufficiently large, $X(U_a^i)$ is a graph over the plane π_i, $i = 1, 2$.*

Proof. Take $a \geq a_0$ sufficiently large so that $g(U_a^i)$ does not intersect the equator $\{z \in \overline{\mathbb{C}} \mid |z| = 1\}$.

Taking into account that $X(M) \subset (S \setminus \mathcal{E}(\Gamma)) \cup \Gamma$ and the definition of U_a^i, it is not hard to see that $\mathfrak{p}_3|_{X(U_a^i)}$ is a local diffeomorphism onto Ω_a^i, where Ω_a^i is the exterior unbounded domain in the plane $\{x_3 = 0\}$ determined by the curve $\mathfrak{p}_3(\ell_i^+) \cup \mathfrak{p}_3(\ell_i^-) \cup \mathfrak{p}_3(\sigma_a^i)$. As X is proper, the same occurs for the map $\mathfrak{p}_3 \circ X|_{U_a^i}$. So, $\mathfrak{p}_3 \circ X|_{U_a^i}$ is a covering map, and taking into account that Ω_a^i is simply connected we deduce that $\mathfrak{p}_3 \circ X|_{U_a^i}$ is one-to-one. This concludes the proof. $\square$

2.2. The symmetries of the surface. The method for proving that $\{x_3 = 0\}$ is a plane of symmetry of $X(M)$ is based on the well-known Alexandrov's reflection method and consists of a generalization of Schoen's ideas (see [**14**]) to our particular case of noncompact boundary. For a precise presentation of our result the following notation is required. Recall that $\delta_t = X^{-1}(X(M) \cap \{x_3 = t\})$. We also denote for $t \geq -1/2$:

$$M_+(t) = \{(x_1, x_2, x_3) \in X(M) \ / \ x_3 \geq t\},$$
$$M_-(t) = \{(x_1, x_2, x_3) \in X(M) \ / \ x_3 \leq t\}.$$

A thorough reading of the paragraph 3.2.2 of [**9**] will convince the readers that, sharpening some arguments, the proof of Theorem 3.24 still works in the case $X(M) \subset (S \setminus \mathcal{E}(\Gamma)) \cup \Gamma$. Then, we have:

Proposition 4. *$X(M)$ is symmetric with respect the plane $\{x_3 = 0\}$. Furthermore, $M_+(0) \setminus (\ell_0^+ \cup \ell_0^-)$ and $M_-(0) \setminus (\ell_0^+ \cup \ell_0^-)$ are graphs over $\{x_3 = 0\}$.*

Now, we recover two consequences of the above proposition that we need in what follows.

Corollary 5. *There are only two branch points $R_0^+ \in \gamma^+$ and $R_0^- \in \gamma^-$ of g along $\gamma_0^+ \cup \gamma_0^-$, g has multiplicity two at these points and $R_0^+, R_0^- \in \delta_0$. Furthermore, the set $\mathcal{G} = \{p \in M \ / \ |g(p)| = 1\}$ consists of $\gamma_0^+ \cup \gamma_0^- \cup \delta_0$.*

Corollary 6. *The limit normal vectors at the ends are opposite.*

In the remainder of the paper and without loss of generality, we assume that

$$g(E_1) = 0, \quad g(E_2) = \infty. \tag{2}$$

Next we prove that if $d = 0$ then $\{x_1 = 0\}$ is a plane of symmetry of $X(M)$. As in the horizontal symmetry case, the proof is inspired on Alexandrov's reflection method. However, the argument exhibited here is slightly different from classical Alexandrov's technique which uses a family of parallel planes.

In this case we use the pencil of vertical planes that contains the vertical segment $\ell_0 = \ell_0^+ = \ell_0^-$.

For the sake of simplicity, in this paragraph we consider a new set of Cartesian coordinates obtained from the old one by a rotation of $-\frac{\pi}{2}$ around the x_3-axis. Observe that in the new coordinate the x_1-axis is the inner bisector of the orthogonal projection of ℓ_i^+ and ℓ_i^- to the plane $\{x_3 = 0\}$, $i = 1, 2$, and $\Gamma^+ \subset \{x_2 \geq 0\}$. Moreover, we need to introduce some notation. For $\xi \in \left[0, \pi - \frac{\theta}{2}\right]$, $t \in]-\infty, 0]$ and a set $A \subset \mathbb{R}^3$ we define:

$$
\begin{aligned}
&H_\xi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \operatorname{Arg}(x_1 + \mathrm{i}x_2) = \xi\}\ ,\ H_{\xi,t} = H_\xi + (t, 0, 0),\\
&P_\xi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \operatorname{Arg}(x_1 + \mathrm{i}x_2) = \xi - \tfrac{\pi}{2}\}\ , P_{\xi,t} = P_\xi + (t, 0, 0),\\
&H_\xi^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \operatorname{Arg}(x_1 + \mathrm{i}x_2) \in \left[\xi, \pi - \tfrac{\theta}{2}\right]\},\\
&H_\xi^- = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \operatorname{Arg}(x_1 + \mathrm{i}x_2) \in \left[-\pi + \tfrac{\theta}{2}, \xi\right]\},\\
&\delta(\xi) = X(M) \cap H_\xi\ ,\ A_+(\xi) = A \cap H_\xi^+\ ,\ A_-(\xi) = A \cap H_\xi^-,\\
&\delta(\xi, t) = X(M) \cap H_{\xi,t},\ A_+(\xi, t) = A \cap (H_\xi^+ + (t, 0, 0)),\\
&A_-(\xi, t) = A \cap (H_\xi^- + (t, 0, 0)),
\end{aligned}
$$

where $\operatorname{Arg} : \mathbb{C} \setminus]-\infty, 0] \longrightarrow \mathbb{R}$ denotes the principal argument. Note that $H_\xi \perp P_\xi$. In addition we label $\mathfrak{s}_\xi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ and $\mathfrak{s}_{\xi,t} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ as the orthogonal symmetries with respect to the planes containing H_ξ and $H_{\xi,t}$, respectively. In the same way, we label $\mathfrak{p}_\xi : \mathbb{R}^3 \longrightarrow H_\xi$ as the orthogonal projection. With these definitions we denote $A_+^*(\xi) = \mathfrak{s}_\xi(A_+(\xi))$ and $A_+^*(\xi, t) = \mathfrak{s}_{\xi,t}(A_+(\xi, t))$. In particular we denote $M_+(\xi) = X(M)_+(\xi)$, $M_+(\xi, t) = X(M)_+(\xi, t)$, $M_-(\xi) = X(M)_-(\xi)$, $M_-(\xi, t) = X(M)_-(\xi, t)$ and $\Delta_\xi = M_+^*(\xi) \cap M_-(\xi)$. If $\xi \in \left[\frac{\pi}{2}, \pi - \frac{\theta}{2}\right]$ we also consider $\Delta_{\xi,t} = M_+^*(\xi, t) \cap M_-(\xi, t)$.

Since the following argument is valid for all $\xi \in \left[\frac{\pi}{2}, \pi - \frac{\theta}{2}\right]$ and $t \in]-\infty, 0]$ we omit the parameters ξ and t in the description of the different sets.

With the above notations, it is not difficult to see that

$$
\Delta \cap (\Gamma_+^* \cup \Gamma_-) = \delta \cap \Gamma. \tag{3}
$$

From Proposition 3 and Corollary 4 we can also consider a_1 sufficiently large so that $a_1 \geq a_0$, $X(U^i) = X(U_{a_1}^i)$ is a graph over the plane $\{x_3 = 0\}$, $i = 1, 2$ and $x_3|_{X(U^1)} > 0$ and $x_3|_{X(U^2)} < 0$, where a_0 is as in Lemma 1 and $U_{a_1}^i$ is defined in Subsection 2.1.

Now, we can prove the following assertion:

Claim 1. If $X^{-1}(\Delta \setminus \delta) \subset U^1 \cup U^2$, then $\Delta = \delta$.

The proof of this claim is similar to the proof of Claim 3.19 in [**9**]. We refer the reader to [**9**] for details.

Now we define the set

$$\mathcal{I} = \Big\{\xi_1 \in \left[0, \pi - \tfrac{\theta}{2}\right] \mid \left[M_+^*(\xi) \cap \{x_3 \geq 0\}\right] \succeq \left[M_-(\xi) \cap \{x_3 \geq 0\}\right], \\ \pi - \tfrac{\theta}{2} \leq \xi \leq \xi_1 \Big\}.$$

Our objective is to prove that $\mathcal{I} = \left[0, \pi - \frac{\theta}{2}\right]$. We divide the proof of this fact into several points:

Claim 2. $\left]\frac{\pi}{2}, \pi - \frac{\theta}{2}\right] \subset \mathcal{I}$.

If $A, B \subset \mathbb{R}^3$, we say that $A \geq_\xi B$ provided for every $x \in \mathbb{R}^3$ for which $\mathfrak{p}_\xi^{-1}(\{x\}) \cap A \neq \emptyset$ and $\mathfrak{p}_\xi^{-1}(\{x\}) \cap B \neq \emptyset$, we have that the orthogonal coordinate to $H_{\xi,t}$ of any point in $\mathfrak{p}_\xi^{-1}(\{x\}) \cap A$ is equal to or greater than the respective orthogonal coordinate of any point in $\mathfrak{p}_\xi^{-1}(\{x\}) \cap B$.

Given $\xi \in \left]\frac{\pi}{2}, \pi - \frac{\theta}{2}\right]$, we define the set

$$\mathcal{I}_\xi = \Big\{t \in]-\infty, 0] \mid M_+(\xi, t) \quad \text{is a graph over} \quad H_{\xi,t} \\ \text{and} \quad M_+^*(\xi, t) \leq_\xi M_-(\xi, t)\Big\}.$$

Our purpose is to show that $\mathcal{I}_\xi =]-\infty, 0]$, for all $\xi \in \left]\frac{\pi}{2}, \pi - \frac{\theta}{2}\right]$. Note that this fact implies $\left]\frac{\pi}{2}, \pi - \frac{\theta}{2}\right] \subset \mathcal{I}$.

First, we are going to see that $\mathcal{I}_\xi \neq \emptyset$. To do this, let $t' < 0$ such that $X^{-1}(M_+(\xi, t)) \subset U^1 \cup U^2$, where U^i are defined as above, $\forall t \leq t'$ (observe that $X^{-1}(\Delta_{\xi,t}) \subset U^1 \cup U^2$). Hence from Claim 1, $\Delta_{\xi,t} = \delta_{\xi,t}$, $t \leq t'$.

Then, it is clear that $X^{-1}(M_+(\xi, t))$ consists of two simply connected components, one of them in U^1 and the other one in U^2, $\forall t \geq t'$, and thus $M_+(\xi, t)$ is the union of two disjoint graphs $G_+^1(\xi, t)$ and $G_+^2(\xi, t)$ over the same simply connected domain $G_+(\xi, t)$ in the plane $\{x_3 = 0\}$.

From the definition of U^1 and U^2, we have that $\mathfrak{p}_\xi(G_+^1(t')) \cap \mathfrak{p}_\xi(G_+^2(t')) = \emptyset$.

Let us see that $M_+(\xi, t)$ is a graph over the halfplane $H_{\xi,t}$, $t \leq t'$.

First, observe that $\mathfrak{p}_\xi$ is injective on $\delta_{\xi,t}$, $t \leq t'$. Indeed, note that $\delta_{\xi,t} \subset H_{\xi,t}$ is a graph over a connected piece of a straight line, and so the $\mathfrak{p}_\xi$ is injective.

Moreover, a similar argument gives that the set $P_{\xi,s} \cap M_+(\xi, t')$ is a connected curve, for sufficiently large s. Furthermore, the function x_3 is monotone over $P_{\xi,s} \cap M_+(\xi, t')$. Otherwise, there would exist some points in $M_+(\xi, t')$ whose normal vector lie in $\{x_1 = 0\} \cap \mathbb{S}^2$. Thus, we could take $t \leq t'$ in such a way that δ_t contains a point with normal vector in $P_\xi \cap \mathbb{S}^2$. Hence, from the theorem of the order of contact (see [**13**, §437]) and taking into account that $X(U^i)$ are graphs over the plane $x_3 = 0$, $i = 1, 2$, we deduce that $\Delta_{\xi,t} - \delta_{\xi,t} \neq \emptyset$, which is contrary to Claim 1.

This proves that $M_+(\xi, t')$ is a graph over the plane H_ξ, and so the same holds for $M_+(\xi, t)$, $t \le t'$.

Taking into account that $\Delta_{\xi,t} = \delta_{\xi,t}$, for $t \le t'$ and $X(M) \in (S \backslash \mathcal{E}(M)) \cup \Gamma$, we deduce that $M_+^*(\xi, t) \le_\xi M_-(\xi, t)$, $t \le t'$. Thus, $]-\infty, t'] \subset \mathcal{I}_\xi$.

It is obvious that $\mathcal{I}_\xi$ is closed in $]-\infty, 0]$.

Next, we shall see that $0 = \text{Maximum}(\mathcal{I}_\xi)$. We proceed by contradiction. Assume $t_0 = \text{Maximum}(\mathcal{I}_\xi) < 0$. Let $K = X(M \setminus (U^1 \cup U^2))$. Since $t_0 \in \mathcal{I}_\xi$, $K_+(\xi, t_0)$ is a graph over the plane H_ξ. Using the interior maximum principle and the maximum principle at the boundary, it is not hard to see that there exists $\varepsilon > 0$ such that $K_+(\xi, t)$ is a graph over H_ξ and $\Delta_{\xi,t} \cap K = \delta_{\xi,t} \cap K$ for $t \in]t_0, t_0 + \varepsilon]$(for details, see Claim 3.21 in [**9**]). Hence, using Claim 1, we deduce that $\Delta_{\xi,t} = \delta_{\xi,t}$. However, the maximality of t_0 leads us to $\Delta_{\xi,t} \setminus \delta_{\xi,t} \neq \emptyset$, which is absurd. This finishes the proof of the claim.

Claim 3. The set $\mathcal{I}$ is closed in $\left[0, \pi - \frac{\theta}{2}\right]$ and its minimum is 0.

Obviously, $\mathcal{I}$ is closed. To prove $\text{Minimum}(\mathcal{I}) = 0$ we proceed by contradiction. Let us assume that $\text{Minimum}(\mathcal{I}) = \xi_0 > 0$. As in the preceding claim, we consider $K = X(M \setminus (U^1 \cap U^2))$. Taking into account that $\xi_0 \in \mathcal{I}$ we have

$$\left[M_+^*(\xi_0) \cap \{x_3 \ge 0\}\right] \succeq \left[M_-(\xi_0) \cap \{x_3 \ge 0\}\right]. \tag{4}$$

Hence, since $K_+(\xi_0)$ is compact, there exists $\varepsilon_0 > 0$ sufficiently small so that $\left[K_+^*(\xi_0 - \varepsilon) \cap \{x_3 \ge 0\}\right] \succeq \left[K_-(\xi_0 - \varepsilon) \cap \{x_3 \ge 0\}\right]$, for all $0 \le \varepsilon \le \varepsilon_0$. Otherwise, we could find sequences $\{\xi_n\} \nearrow \xi_0$, with $\xi_n \in [0, \xi_0[$, and $\{x_n\}$, $\{y_n\}$ in K, fulfilling the following conditions:

i) $x_n \in K_+(\xi_n)$, $y_n \in K_-(\xi_n)$ and $\mathfrak{s}_{\xi_n}(x_n) = y_n$, $\forall n \in \mathbb{N}$.

ii) $\{x_n\} \to x \in K_+(\xi_0)$, $\{y_n\} \to y \in K_-(\xi_0)$.

From i) and ii) we deduce that $\mathfrak{s}_{\xi_0}(x) = y$. On the other hand, (3) implies that any point lying in $\Delta_{\xi_0} \setminus \delta_{\xi_0}$ is an interior point of contact between $M_+^*(\xi_0)$ and $M_-(\xi_0)$. Assume $\Delta_{\xi_0} \setminus \delta_{\xi_0} \neq \emptyset$. Then, making use of the interior maximum principle we deduce $M_+^*(\xi_0) = M_-(\xi_0)$, which is absurd because, since $\xi_0 > 0$, Γ is not symmetric with respect to the plane H_{ξ_0}.

Therefore $x = y \in K \cap \delta_{\xi_0}$. Hence, taking into account i) and ii), we have that $N(X^{-1}(x)) = \mathfrak{s}_{\xi_0}(N(X^{-1}(x)))$ and so $N(X^{-1}(x))$ is parallel to H_{ξ_0}. Therefore, by the theorem of the order of contact (see [**13**, §437]) we have that $x \in K \cap (\delta_{\xi_0} \setminus (\delta_{\xi_0} \cap \ell_0))$ and $x \notin \{q_1^+ = q_1^-, q_2^+ = q_2^-\}$. From this fact and taking into account (4), the maximum principle at the boundary can be applied to a neighbourhood of the point x. We get $M_+^*(\xi_0) = M_-(\xi_0)$, which is as above a contradiction.

By the preceding reasoning, we have $\Delta_\xi \setminus \delta_\xi \subset X(U^1 \cup U^2)$ for $\xi \ge \xi_0 - \varepsilon_0$. From Claim 1 we conclude that $\Delta_\xi = \delta_\xi$ for $\xi \ge \xi_0 - \varepsilon_0$. Clearly this implies $\left[M_+^*(\xi) \cap \{x_3 \ge 0\}\right] \succeq \left[M_-(\xi) \cap \{x_3 \ge 0\}\right]$ for $\xi \ge \xi_0 - \varepsilon_0$ and so $\xi_0 - \varepsilon_0 \in \mathcal{I}$, which contradicts that ξ_0 is a minimum.

Summarizing we have $\left[M_+^*(0) \cap \{x_3 \geq 0\}\right] \succeq [M_-(0) \cap \{x_3 \geq 0\}]$. We can repeat the above argument starting from Γ^- instead of Γ^+ and obtain $\left[M_-^*(0) \cap \{x_3 \geq 0\}\right] \succeq [M_+(0) \cap \{x_3 \geq 0\}]$ and so $X(M) \cap \{x_3 \geq 0\}$ is symmetric with respect to the plane $\{x_2 = 0\}$. Finally, by the horizontal symmetry mentioned in 2.2 we have that $X(M)$ is symmetric with respect to the plane $\{x_2 = 0\}$.

2.3. Determination of conformal structure and Weierstrass data of $\boldsymbol{M}$. This subsection is devoted to determining the Weierstrass data associated to the minimal immersion $X : M \longrightarrow \mathbb{R}^3$. We define $n = \frac{2\pi}{3\pi - \theta}$. Observe that $n \in [\frac{2}{3}, 1]$. As M is simply-connected, the map $(-\mathrm{i}g)^{\frac{n}{2}}$ has a well-defined branch on M. Let f be the branch of $(-\mathrm{i}g)^{\frac{n}{2}}$ such that $\mathrm{Arg}(f(p)) = 0$, whenever $-\mathrm{i}g(p) \in \mathbb{R}^+$.

As before, $Q_j^+ = \gamma_0^+ \cap \gamma_j^+$, $Q_j^- = \gamma_0^- \cap \gamma_j^-$, $j = 1, 2$. Observe that $f(Q_j^+) = \mathrm{i}$ and $f(Q_j^-) = -\mathrm{i}$, $j = 1, 2$. Moreover, since $X(M) \subset (S \setminus \mathcal{E}(\Gamma)) \cup \Gamma$ it is easy to see that $f(p) \neq f(Q_j^\pm)$ for all $p \in \gamma_j^\pm$, $j = 1, 2$. Then, taking into account that $g(E_1) = 0$ and $g(E_2) = \infty$, one has:

$$f(\gamma_1^-) = s_1^- = \{\lambda \mathrm{i} \mid \lambda \in [-1, 0[\}, \quad f(\gamma_2^-) = s_2^- = \{\lambda \mathrm{i} \mid \lambda \in]-\infty, -1]\},$$
$$f(\gamma_1^+) = s_1^+ = \{\lambda \mathrm{i} \mid \lambda \in]0, 1]\}, \quad f(\gamma_2^+) = s_2^+ = \{\lambda \mathrm{i} \mid \lambda \in [1, \infty[\}.$$

Using Corollary 5, we deduce that $f|_{[Q_j^\pm, R_0^\pm]}$ are injective for $j = 1, 2$. Hence, if we write $f(R_0^+) = \mathrm{e}^{\mathrm{i}\frac{t_0}{2}}$ and $f(R_0^-) = \mathrm{e}^{\mathrm{i}\frac{t_1}{2}}$, it is not hard to check that

(5)
$$f(\gamma_0^-) = s_0^- = \left\{\mathrm{e}^{\mathrm{i}\frac{t}{2}} \mid t \in [-\pi, t_1]\right\}, \quad f(\gamma_0^+) = s_0^+ = \left\{\mathrm{e}^{\mathrm{i}\frac{t}{2}} \mid t \in [t_0, \pi]\right\}.$$

Using again that $X(M) \subset (S \setminus \mathcal{E}(\Gamma)) \cup \Gamma$, the fact that there are at most two points on $\gamma_0^\pm$ where the Gauss map achieves the values $\pm\vec{a}_2$ (see the proof of Proposition 1) and Corollary 5, one deduces that $-\pi < t_1 < 0 < t_0 < \pi$.

Let Λ denote the connected component of $\mathbb{C} \setminus (f(\partial(M)) \cup \{0\})$ containing the point $\{1\}$ (see Figure 5). We have the following result:

Lemma 5. *The map $f : M \to \mathbb{C}$ fulfills that:*
(i) *$f(M \setminus \partial(M)) = \Lambda$, and*
(ii) *$f|_{M \setminus \partial(M)} : M \setminus \partial M \to \Lambda$ is a biholomorphism.*

Proof. In order to prove (i) we note that f is holomorphic and nonconstant, and so $f(M \setminus \partial(M))$ is an open subset of $\mathbb{C}^*$ (note that no points in M have vertical normal vector). On the other hand, taking into account that $\overline{M} = M \cup \{E_1, E_2\}$ is compact, $f(\overline{M})$ is a closed subset of $\overline{\mathbb{C}}$. Therefore the set $W = f(M \setminus \partial(M)) \cap (\mathbb{C} \setminus (f(\partial(M)) \cup \{0\})) = f(\overline{M}) \cap (\mathbb{C} \setminus (f(\partial(M)) \cup \{0\}))$ is a closed subset of $\mathbb{C} \setminus (f(\partial(M)) \cup \{0\})$. Then, either $W = \mathbb{C} \setminus (f(\partial(M)) \cup \{0\})$ or W is a connected component of $\mathbb{C} \setminus (f(\partial(M)) \cup \{0\})$.

To see that the first possibility does not occur we proceed by contradiction. Assume $W = \mathbb{C} \setminus (f(\partial(M)) \cup \{0\})$. Observe that $f(p) \in \mathbb{S}^1$ if and only if $g(p) \in \mathbb{S}^1$. Hence, taking into account Corollary 5 and (5) we infer that $\{e^{i\frac{t}{2}}, t \in [t_1, t_0] \cup [\frac{\pi}{2}, \frac{3\pi}{2}]\} \subset f(\delta_0)$. Moreover, since there are no ramification points in $\delta_0 \setminus \{R_0^+, R_0^-\}$, we deduce that $\mathrm{Arg}(f)$ is a monotone function in δ_0 and $f(\delta_0) = \mathbb{S}^1$. Since $\{1, e^{\frac{1}{2}n\pi i}, e^{n\pi i}, e^{\frac{3}{2}n\pi i}\} \subset \mathbb{S}^1$, we have that there are at least four points in $\delta_0 \cap N^{-1}(\{\vec{a}_2, -\vec{a}_2\})$. But it is contrary to Proposition 2. Since $f(\gamma_0^+)$ and $f(\gamma_0^-)$ lie in the boundary of W, we easily obtain that $W = \Lambda$ and so $f(M \setminus \partial(M)) = \Lambda$.

The same argument presented above gives us that $f(\delta_0) = \{e^{i\frac{t}{2}} \mid t \in [t_1, t_0]\}$ and $f|_{\delta_0} : \delta_0 \longrightarrow \{e^{i\frac{t}{2}} \mid t \in [t_1, t_0]\}$ is a one-to-one function.

To finish the proof, we define $\gamma = \partial(\overline{M})$. Since $\overline{M}$ is conformally a closed disk with piecewise analytic boundary, then γ is a piecewise analytic curve homeomorphic to $\mathbb{S}^1$. Note that since $g|_M$ has no poles and g extends continuously to $\overline{M}$, we can assert the same for f. Then, we know that for any $w \in \Lambda = f(M \setminus \partial(M))$, $\sharp\left(f^{-1}(w)\right) = \frac{1}{2\pi i}\int_\gamma \frac{df}{f-w} \in \mathbb{Z}$. Thus, if we define $h : \Lambda \longrightarrow \mathbb{Z}$ by $h(w) = \sharp(f^{-1}(w))$, the function h is continuous on Λ, and so it is constant. From the above arguments we have that $h(w) = 1$, $\forall w \in f(\delta_0 \setminus \{R_0^+, R_0^-\})$ and this concludes the proof. □

Let $\mathcal{N}_{t_0}^{t_1}$ be the following four-punctured torus:

$$\mathcal{N}_{t_0}^{t_1} = \{(u, v) \in \mathbb{C}^* \times \mathbb{C} \mid v^2 = (u - e^{i\frac{t_0}{2}})(u + e^{-i\frac{t_0}{2}})(u - e^{i\frac{t_1}{2}})(u + e^{-i\frac{t_1}{2}})\}.$$

Consider $\mathcal{U}_{t_0}^{t_1} \subset \mathcal{N}_{t_0}^{t_1}$ as the connected component of $u^{-1}(\mathbb{C} \setminus f(\partial(M)))$ containing the point $P_0 = \left(1, 2\sqrt{-\sin(\frac{t_0}{2})\sin(\frac{t_1}{2})}\right)$. Define $\mathcal{N}' = \overline{\mathcal{U}_{t_0}^{t_1}}$. At this point we prove the following proposition:

Proposition 5. *M is biholomorphic to $\mathcal{N}'$. Furthermore, the Weierstrass data are given on $\mathcal{N}'$ by*

$$g(u) = i u^{\frac{2}{n}}, \quad \Phi_3 = \lambda \frac{du}{v},$$

where $\lambda \in \mathbb{R}^+$ and we choose the branch of $u^{\frac{2}{n}}$ satisfying $1^{\frac{2}{n}} = 1$.

Proof. Consider $f : M \setminus \partial(M) \longrightarrow \Lambda$, the biholomorphism defined in Lemma 5. Observe that the u-projection is a biholomorphism from $\mathcal{N}' \setminus \partial(\mathcal{N}')$ onto Λ and so $F = f^{-1} \circ u : \mathcal{N}' \setminus \partial(\mathcal{N}') \longrightarrow M \setminus \partial(M)$ is a biholomorphism. Since $\mathcal{N}' \setminus \partial(\mathcal{N}')$ and $M \setminus \partial(M)$ are conformally equivalent to Jordan regions in $\mathbb{C}$, a well-known result of complex analysis asserts that F can be extended to a biholomorphism from $\mathcal{N}'$ onto M. For the sake of simplicity, in what follows we identify M with $\mathcal{N}'$.

Now, we consider the holomorphic function $\omega = \frac{\Phi_3 v}{du}$ on $\mathcal{N}'$. Let $p_0 \in s_0^+ \cup s_0^-$ and $\{p_0^1, p_0^2\} = u^{-1}(p_0)$. Since the surface is symmetric with respect to the plane $\{x_3 = 0\}$ (see 2.2), we have that $\Phi_3(p_0^1) = -\Phi_3(p_0^2)$. But this is also the behaviour of v. Consequently, $\widetilde{\omega} = \omega \circ u^{-1}$ is a holomorphic function in Λ that can be extended to $s_0^+ \cup s_0^-$.

As Φ_3 and v extend to the punctures in a natural way, $\widetilde{\omega}$ can be seen as a holomorphic function on $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0\}$. Furthermore, $\widetilde{\omega}$ is real on $\{z \in \mathbb{C} \mid \mathrm{Re}(z) = 0\}$ and so, using the Schwarz Principle, $\widetilde{\omega}$ extends to the whole $\mathbb{C}$. Then, $\widetilde{\omega}$ is a holomorphic function on $\mathbb{C}$ without zeroes or poles and therefore $\widetilde{\omega} = \lambda \in \mathbb{C}^*$. Finally, using again that $\widetilde{\omega}$ is real on $\{z \in \mathbb{C} \mid \mathrm{Re}(z) = 0\}$ we deduce $\lambda \in \mathbb{R}^*$. Observe that, up to a rigid motion, we can assume that $\lambda \in \mathbb{R}^+$. $\square$

As we announced, we now prove that $X(M)$ also inherits the vertical symmetry of its boundary when $d > 0$.

Proposition 6. *In the above setting, $t_1 = -t_0$.*

Proof. First of all, observe that the result about the vertical symmetry proved in Paragraph 2.2 implies $t_1 = -t_0$ if $d = 0$. We shall see that this fact suffices to prove the general case.

Since $X : M \longrightarrow \mathbb{R}^3$ is a solution of Problem 1 we have that $X_1(R_0^-) = -\frac{d}{2}$ and $X_1(R_0^+) = \frac{d}{2}$. Thus

$$d = X_1(R_0^+) - X_1(R_0^-) = \mathrm{Re} \int_{\widehat{\delta}} \Phi_1,$$

where $\widehat{\delta}$ is the lift to M of the curve $e^{i\frac{t}{2}}$, $t_1 \leq t \leq t_0$, in the u-plane. Taking into account the expressions for g and Φ_3 given in Proposition 5, it is not difficult to obtain that

$$d = \frac{\lambda}{4} f_1(t_0, t_1) = \frac{\lambda}{4} \int_{t_1}^{t_0} \frac{\cos(\frac{t}{n})}{v(t, t_0, t_1)} dt,$$

where $v(t, t_0, t_1) = \sqrt{\left(\sin(\frac{t_0}{2}) - \sin(\frac{t}{2})\right)\left(\sin(\frac{t}{2}) - \sin(\frac{t_1}{2})\right)}$. Furthermore, we have that $X_2(R_0^+) = X_2(R_0^-)$. Thus

$$X_2(R_0^+) - X_2(R_0^-) = \mathrm{Re} \int_{\widehat{\delta}} \Phi_2 = 0.$$

A direct computation using again the expressions of g and Φ_3 given in Proposition 5 gives

$$\mathrm{Re} \int_{\widehat{\delta}} \Phi_2 = \frac{\lambda}{4} f_2(t_0, t_1) = \frac{\lambda}{4} \int_{t_1}^{t_0} \frac{\sin(\frac{t}{n})}{v(t, t_0, t_1)}.$$

From the definitions of the functions f_1 and f_2 we have

$$f_1(t_0, t_1) = f_1(-t_1, -t_0), \quad f_2(t_0, t_1) = -f_2(-t_1, -t_0), \tag{6}$$

for all $(t_0, t_1) \in [0, \pi[\times] - \pi, 0]$. Observe that if $X : M \longrightarrow \mathbb{R}^3$ is the solution of Problem 1 given in Proposition 5, then (t_0, t_1) must satisfy $f_1(t_0, t_1) \geq$

0 and $f_2(t_0, t_1) = 0$. From the properties of f_1 and f_2 given in (6), it suffices to study the zeros of the functions f_1 and f_2 in the triangle given by $T = \{(t_0, t_1) \in [0, \pi[\times] - \pi, 0] \mid t_0 \leq -t_1\}$.

Let us denote by $L_1 = \{0\} \times [-\pi, 0]$, $L_2 = \{(t_0, t_1) \in [0, \pi] \times [-\pi, 0] \mid t_0 = -t_1\}$ and $L_3 = [0, \pi] \times \{-\pi\}$ the sides of the triangle T. We also define the sets $C_1 = \{(t_0, t_1) \in T \mid f_1(t_0, t_1) = 0\}$, $C_2 = \{(t_0, t_1) \in T \mid f_2(t_0, t_1) = 0\}$. It is clear from (6) that $L_2 \subset C_2$. Furthermore, by the vertical symmetry proved in Paragraph 2.2 we deduce that C_1 can only intersect C_2 in points of L_2. For the sake of clarity, we divide the rest of the proof in several steps.

Step 1. The objective of this step is to show that C_i is a set of analytic curves in T, for $i = 1, 2$.

Consider the meromorphic 1-form given by $\Phi = \frac{u^{\frac{2}{n}}}{v} du$. Note that $-f_2$ and f_1 are the real and imaginary part of $4 \int_{\widehat{\delta}} \Phi$. By deriving, one has

$$\mathcal{L}(\Phi) = \frac{\partial^2 \Phi}{\partial t_0^2} + \frac{\partial^2 \Phi}{\partial t_1^2} + a_1^1 \frac{\partial \Phi}{\partial t_0} + a_1^2 \frac{\partial \Phi}{\partial t_1} + a_0^0 \Phi = d(F), \tag{7}$$

where

$$a_1^1 = \frac{\cos(t_0) + \sin(\frac{t_0}{2}) \sin(\frac{t_1}{2})}{2 \cos(\frac{t_0}{2})(\sin(\frac{t_0}{2}) - \sin(\frac{t_1}{2}))},$$
$$a_1^2 = -\frac{\cos(t_1) + \sin(\frac{t_0}{2}) \sin(\frac{t_1}{2})}{2 \cos(\frac{t_1}{2})(\sin(\frac{t_0}{2}) - \sin(\frac{t_1}{2}))},$$
$$a_0^0 = \frac{4 - n^2}{4n^2}$$

and F is the following meromorphic function:

$$F = \frac{u^{\frac{2}{n}+1}(au^4 + bu^3 + cu^2 + du + e)}{4nv^3},$$

where $a = 2+n$, $b = -\mathrm{i}(4+n)(\sin(\frac{t_0}{2})+\sin(\frac{t_1}{2}))$, $c = -4(1+2\sin(\frac{t_0}{2})\sin(\frac{t_1}{2}))$, $d = -\mathrm{i}(-4 + n)(\sin(\frac{t_0}{2}) + \sin(\frac{t_1}{2}))$ and $e = 2 - n$.

Integrating by parts in (7), we have that f_1 and f_2 are zeroes of the second order elliptic operator given by $\mathcal{L}$. As C_1 and C_2 are the nodal sets of f_1 and f_2, respectively, we can assert (see [**2**]) that C_i is a set of regular curves and the critical points on the nodal lines are isolated. Furthermore, when the nodal lines meet, they form an equiangular system. Moreover, by the Maximum Principle for elliptic operators, C_i cannot contain closed curves.

Step 2. The purpose of the present step is to study the behaviour of the curves in C_1 at the boundary of the triangle T. We shall see that $C_1 \cap L_1 = (0, \widetilde{t}_1)$, $C_1 \cap L_2 = (\widetilde{t}_0, -\widetilde{t}_0)$ and that there are no curves in C_1 approaching L_3.

First, we shall prove that $\frac{\partial f_1}{\partial t_1}(t_0, t_1) > 0$, for all $(t_0, t_1) \in T$. It is not difficult to see that

$$\frac{\partial}{\partial t_1}\left(\frac{\cos(\frac{t}{n})}{v(t,t_0,t_1)}\right) = \frac{\cos(\frac{t}{n})\cos(\frac{t_1}{2}) - \cos(\frac{t}{2})\cos(\frac{t_1}{n})}{4v(t,t_0,t_1)(\sin(\frac{t}{2}) - \sin(\frac{t_1}{2}))}$$
$$- \frac{\cos(\frac{t_1}{n})}{\sin(\frac{t_0}{2}) - \sin(\frac{t_1}{2})} d\left(\frac{\sin(\frac{t_0}{2}) - \sin(\frac{t}{2})}{v(t,t_0,t_1)}\right).$$

Thus integrating by parts in the above equality we obtain

$$\frac{\partial f_1}{\partial t_1}(t_0, t_1) = \int_{t_1}^{t_0} \frac{\cos(\frac{t}{n})\cos(\frac{t_1}{2}) - \cos(\frac{t}{2})\cos(\frac{t_1}{n})}{4v(t,t_0,t_1)(\sin(\frac{t}{2}) - \sin(\frac{t_1}{2}))}. \tag{8}$$

In order to prove $\frac{\partial f_1}{\partial t_1}(t_0, t_1) > 0$ we shall see that the function

$$h(t,n) = \cos\left(\tfrac{t}{n}\right)\cos\left(\tfrac{t_1}{2}\right) - \cos\left(\tfrac{t}{2}\right)\cos\left(\tfrac{t_1}{n}\right) \geq 0,$$

for $t \in [t_1, t_0]$. Since $h(-t,n) = h(t,n)$ and $t_0 \leq -t_1$, it suffices to prove that $h(t,n) \geq 0$ for $t \in [t_1, 0]$. Moreover, taking into account $h(t_1, n) = 0$, it is enough to see that $\frac{\partial h}{\partial t}(t,n) = -\frac{1}{n}\sin(\frac{t}{n})\cos(\frac{t_1}{2}) + \frac{1}{2}\sin(\frac{t}{2})\cos(\frac{t_1}{n}) \geq 0$ for $t \in [t_1, 0]$.

Assume first $t_1 \in [-\frac{n\pi}{2}, 0]$. As $n \in [\frac{2}{3}, 1]$ we have $0 \leq -\sin(\frac{t}{2}) \leq -\sin(\frac{t}{n})$ and $0 \leq \cos(\frac{t_1}{n}) \leq \cos(\frac{t_1}{2})$. Hence $\frac{\partial h}{\partial t}(t,n) \geq 0$.

If $t_1 \in [-n\pi, -\frac{n\pi}{2}]$, the study of the signs in the expression of $\frac{\partial h}{\partial t}$ gives directly that this partial is nonnegative.

Finally, we consider the case $t_1 \in [-\pi, -n\pi]$. As in the former case, studying the signs in the expression of $\frac{\partial h}{\partial t}$ we obtain that $\frac{\partial h}{\partial t}(t,n) \geq 0$ for $t \in [-n\pi, 0]$. Otherwise, it is not difficult to see that $\frac{\partial^2 h}{\partial n \partial t}(t,n) \geq 0$ for $t \in [t_1, -n\pi]$. As $\frac{\partial h}{\partial t}(t, \frac{2}{3}) = 2\cos(\frac{t_1}{2})\sin(\frac{t}{2})\left(-2\cos(\frac{t}{2})^2 + (\cos(\frac{t_1}{2})^2 - \cos(\frac{t}{2})^2)\right)$ ≥ 0 for $t \in [t_1, -n\pi]$, we conclude that $\frac{\partial h}{\partial t}(t,n) \geq 0$.

Our next objective is to prove that $f_1(0, -n\,\pi) < 0$. Indeed, making the change of variable $s = t + n\pi/2$ one has:

$$f_1(0,-n\pi) = \int_{\frac{-n\,\pi}{2}}^{\frac{n\,\pi}{2}} \frac{\sin(\frac{s}{n})}{\sqrt{-\sin(\frac{-n\,\pi+2\,s}{4})\,\left(\sin(\frac{n\,\pi}{2}) + \sin(\frac{-n\,\pi+2\,s}{4})\right)}} ds =$$

$$\int_0^{\frac{n\pi}{2}} \sin\left(\frac{s}{n}\right) \frac{\sqrt{\left(\sin(\frac{n\,\pi}{2}) - \sin(\frac{n\,\pi+2\,s}{4})\right)\,\sin(\frac{n\,\pi+2\,s}{4})} - \sqrt{\left(\sin(\frac{n\,\pi}{2}) - \sin(\frac{n\,\pi-2\,s}{4})\right)\,\sin(\frac{n\,\pi-2\,s}{4})}}{\sqrt{\left(\sin(\frac{n\,\pi}{2}) - \sin(\frac{n\,\pi-2\,s}{4})\right)\,\sin(\frac{n\,\pi-2\,s}{4})}\sqrt{\left(\sin(\frac{n\,\pi}{2}) - \sin(\frac{n\,\pi+2\,s}{4})\right)\,\sin(\frac{n\,\pi+2\,s}{4})}} ds.$$

An easy computation gives us that the numerator in the last integral is always nonpositive, and then $f_1(0,-n\pi) < 0$. From the definition of f_1 we also have $f_1(0, -\frac{n\pi}{2}) > 0$. Then, taking into account that $\frac{\partial f_1}{\partial t_1}(0, t_1) > 0$, we have that there exists a unique $\widetilde{t}_1 \in]-n\pi, -\frac{n\pi}{2}[$ such that $f_1(0, \widetilde{t}_1) = 0$.

We now consider the function $\widetilde{f_1}(t_0) = f_1(t_0, -t_0)$, that is the function f_1 restricted to the side L_2. Taking into account (6) one has

$$\frac{\partial \widetilde{f_1}}{\partial t_0}(t_0) = \frac{\partial f_1}{\partial t_0}(t_0, -t_0) - \frac{\partial f_1}{\partial t_1}(t_0, -t_0) = -2\frac{\partial f_1}{\partial t_1}(t_0, -t_0) < 0. \tag{9}$$

According to the definition of f_1 we have $\widetilde{f_1}(\frac{n\pi}{2}) = f_1(\frac{n\pi}{2}, -\frac{n\pi}{2}) > 0$. Our next purpose is to see that $\widetilde{f_1}(n\pi) = f_1(n\pi, -n\pi) < 0$. Note that

$$\widetilde{f_1}(n\pi) = 2\sqrt{2}\int_{-n\pi}^{0} \frac{\cos(\frac{t}{n})}{\sqrt{\cos(t) - \cos(n\pi)}}.$$

A direct computation gives

$$\begin{aligned}&\widetilde{f_1}(n\pi)\\ &= 2\sqrt{2}\int_0^{\frac{n\pi}{2}} \frac{\sin(\frac{s}{n})(\sqrt{\cos(s+\frac{n\pi}{2}) - \cos(n\pi)} - \sqrt{\cos(s-\frac{n\pi}{2}) - \cos(n\pi)})}{\sqrt{\cos(s+\frac{n\pi}{2}) - \cos(n\pi)}\sqrt{\cos(s-\frac{n\pi}{2}) - \cos(n\pi)}}.\end{aligned}$$

It is not hard to see that the numerator in the above integrand is nonpositive, in particular $\widetilde{f_1}(n\pi) < 0$.

Therefore there exists $\widetilde{t_0} \in]\frac{n\pi}{2}, n\pi[$ such that $f_1(\widetilde{t_0}, -\widetilde{t_0}) = 0$, f_1 is positive in $\{(t_0, -t_0) \in T \mid 0 < t_0 < \widetilde{t_0}\}$ and negative in $\{(t_0, -t_0) \in T \mid \widetilde{t_0} < t_0 < \pi\}$.

Now, we prove that $\lim_{t_1 \to -\pi} f_1(t_0, t_1) = -\infty$. In order to do this we consider a new set of parameters

$$s(t_0, t_1) = \sin(\tfrac{t_0}{2}), \quad r(t_0, t_1) = \sin(\tfrac{t_1}{2}). \tag{10}$$

Note that $r(t_0, -\pi) = -1$. Our next objective is to see that $\lim_{r \to -1} f_1(s, r) = -\infty$. In order to do this, we derive again the 1-form Φ defined in Step 1 and we obtain the following equality:

$$\frac{\partial^4 \Phi}{\partial r^4} + b_3\frac{\partial^3 \Phi}{\partial r^3} + b_2\frac{\partial^2 \Phi}{\partial r^2} + b_1\frac{\partial \Phi}{\partial r} + b_0\Phi = d(\varphi), \tag{11}$$

where

$$\begin{aligned}
b_3 &= \frac{2(5r^3 + 2s - 8r^2 s + r(-2 + 3s^2))}{(-1 + r^2)(r - s)^2},\\
b_2 &= \frac{-16(r - s)^2 + n^2(-8 + 99r^2 - 116rs + 25s^2)}{4n^2(-1 + r^2)(r - s)^2},\\
b_1 &= \frac{(-48 + 57n^2)r + (48 - 39n^2)s}{4n^2(-1 + r^2)(r - s)^2},\\
b_0 &= \frac{3(n^2 - 4)}{4n^2(-1 + r^2)(r - s)^2}
\end{aligned}$$

and φ is the following meromorphic function:

$$\varphi(u) = \frac{u^{\frac{2}{n}+1}(a'u^4 + b'u^3 + c'u^2 + d'u + e')(u^2 - 1 - 2siu)^{\frac{1}{2}}}{4n^2(-1+r^2)(r-s)^2(u^2-1-2riu)^{\frac{7}{2}}},$$

with $a' = -3n(2+n)$, $b' = -3in((-4+3n)r - (4+5n)s)$, $c' = 12n(1+2rs)$, $d' = -3in((4+3n)r + (4-5n)s)$ and $e' = 3n(n-2)$. Integrating by parts in (11), we have that f_1 is a solution of the fourth order ordinary differential equation given by

$$\frac{\partial^4 f_1}{\partial r^4} + b_3 \frac{\partial^3 f_1}{\partial r^3} + b_2 \frac{\partial^2 f_1}{\partial r^2} + b_1 \frac{\partial f_1}{\partial r} + b_0 f_1 = 0. \tag{12}$$

Observe that this equation presents a regular singular point in $r = -1$ and then we can use the Frobenius method to compute the limit of f_1 when r tends to -1 (see §4.8 in [**3**]). Taking into account the coefficients of Equation (12) and the aforementioned method, we deduce that

$$f_1(s,r) = c_1 \log(1+r)\phi_1(s,r) + c_2\, \phi_2(s,r) + c_3(r+1)\phi_3(s,r) + c_4(r+1)^2\phi_4(s,r),$$

where $c_i \in \mathbb{R}$, $\phi_1(s,-1) \neq 0$ and ϕ_i are analytic at the points $(s,-1)$, $i \in \{1,2,3,4\}$. A direct computation using (8) and (10) proves that $\lim_{r\to -1} \frac{\partial f_1}{\partial r} = +\infty$. Thus $c_1 \neq 0$ and $\lim_{r\to -1} f_1(s,r) = -\infty$.

Step 3. With regard to C_2, we shall check that $C_2 \cap \{(0,t_1) \in T \mid \widetilde{t}_1 \leq t_1 < 0\} = \emptyset$ and that if $(\widehat{t}_0, -\widehat{t}_0)$ is a critical point of C_2 in L_2, then $\widehat{t}_0 > \widetilde{t}_0$.

Clearly, from the definition of f_2 one has $f_2(0,t_1) < 0$ for $t_1 \in [-n\pi, 0[$. Thus we obtain the first assertion in the present step.

In order to prove the second one we need an apropriate expression for $\frac{\partial f_2}{\partial t_0}$. Observe that

$$\frac{\partial \Phi}{\partial t_0} = u^{2/n}\tau - \frac{2a}{n} u^{2/n-1}\psi\, du + a\, d\left(u^{2/n}\psi\right),$$

where

$$\tau = \frac{-\mathrm{i}\left(-1 + \mathrm{e}^{\mathrm{i}\frac{t_0}{2}}\, u\right)}{4\left(1 + \mathrm{e}^{\mathrm{i}\frac{t_0}{2}}\, u\right)} \frac{du}{v},$$

$$\psi = \frac{\left(u + \mathrm{e}^{-\mathrm{i}\frac{t_1}{2}}\right)\left(u - \mathrm{e}^{\mathrm{i}\frac{t_1}{2}}\right)}{v},$$

$$a = \frac{1}{4\left(\sin(\frac{t_1}{2}) - \sin(\frac{t_0}{2})\right)}.$$

Integrating by parts once again in the above equality and computing the real part, we get:

$$\frac{\partial f_2}{\partial t_0}(t_0,t_1)=\int_{t_1}^{t_0}\left(\frac{\cos(\frac{t}{n})\sin(\frac{t}{2})}{nv(t,t_0,t_1)\left(\sin(\frac{t_0}{2})-\sin(\frac{t_1}{2})\right)}+\frac{\sin(\frac{t}{n})\tan(\frac{t+t_0}{4})}{4v(t,t_0,t_1)}\right)dt$$
$$-\frac{\sin(\frac{t_1}{2})}{n\left(\sin(\frac{t_0}{2})-\sin(\frac{t_1}{2})\right)}f_1(t_0,t_1).$$

In particular, if $t_1=-t_0$ one has that the integral of the first summand vanishes, and so

$$\frac{\partial f_2}{\partial t_0}(t_0,-t_0)=\int_{-t_0}^{t_0}\frac{\sin(\frac{t}{n})\tan(\frac{t+t_0}{4})}{4v(t,t_0,t_1)}dt+\frac{1}{2n}f_1(t_0,-t_0).$$

Taking into account that $\widetilde{t_0}<n\pi$, it is not difficult to see that

$$\int_{-t_0}^{t_0}\frac{\sin(\frac{t}{n})\tan(\frac{t+t_0}{4})}{4v(t,t_0,t_1)}dt>0,$$

for $0<t_0\leq\widetilde{t_0}$, and so $\frac{\partial f_2}{\partial t_0}(t_0,-t_0)$ is positive in the points where the function f_1 is nonnegative. This concludes the assertion.

By Steps 1, 2 and 3 we deduce that there are no points (t_0,t_1) in T with $f_1(t_0,t_1)\geq 0$ and $f_2(t_0,t_1)=0$ apart from the points $\{(t_0,-t_0)\mid 0\leq t_0\leq\widetilde{t_0}\}$. □

Corollary 7. *M is invariant under the antiholomorphic involution $S_v(u)=\overline{u}$ which corresponds to the reflection in the plane $\{x_1=0\}$.*

3. The existence results.

In the former section we have seen that if $X:M\longrightarrow\mathbb{R}^3$ is a solution of Problem 1 satisfying $X(M)\subset((S\setminus\mathcal{E}(\Gamma_{\theta\,d}))\cup\Gamma_{\theta\,d})$ then M is biholomorphic to $\mathcal{N}'=\overline{\mathcal{U}_{t_0}^{-t_0}}$ and their Weierstrass data are given by

$$g=\mathrm{i}u^{\frac{2}{n}},\quad\Phi_3=\lambda\frac{du}{v},\quad v=\sqrt{u^4-2u^2\cos(t_0)+1},\quad n\in\left[\tfrac{2}{3},1\right].$$

At this point, we observe that, up to an easy conformal transformation, the above Weierstrass data for $n\in[1,2[$ correspond to a López-Martín example, it is to say, solutions of Problem 1 verifying $X(M)\subset\mathcal{E}(\Gamma_{\theta\,d})$. For the existence of this examples we refer the reader to [**10**].

Moreover, if $n=2$ then the surface is a Jenkins-Serrin graph. Extension by Schwarz reflection of these surfaces gives embedded doubly periodic examples with two orthogonal planes of symmetry between adjacent saddle towers. These examples were studied by H. Karcher in [**7**].

Therefore, to complete the existence part of the Main Theorem it suffices to prove that indeed for $n\in\left[\frac{2}{3},1\right]$ the above conformal representation leads

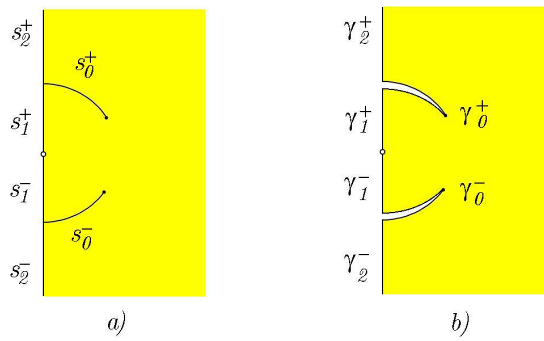

Figure 5. a) The domain $u(M)$. b) The surface M.

to a solution of Problem 1 which lies in the *exterior* of the convex hull of its boundary and this is the purpose of the present section.

In what follows we denote $M = \mathcal{N}'$ and $\gamma_i^+ = u^{-1}(s_i^+)$, $\gamma_i^- = u^{-1}(s_i^-)$, $i = 0,1,2$, where s_i^+ and s_i^- are those defined in Section 2 corresponding to $t_1 = -t_0$ for $i = 0,1,2$ (see Figure 5). Moreover, we label $\gamma^+ = \cup_{i=0}^2 \gamma_i^+$ and $\gamma^- = \cup_{i=0}^2 \gamma_i^-$. We use the notation $\ell_i^\pm$ introduced in Section 1 for the half-lines in the polygonal $\Gamma_{\theta\, d}$. Furthermore, recall that the set of Cartesian coordinates was introduced at the beginning of Section 2.

Now, we consider the curve γ_0^+ which consists of two copies, δ_1^+ and δ_2^+, of s_0^+. We can assume that $\delta_1^+(t)$ and $\delta_2^+(t)$ are the two lifts to M of the curve $\mathrm{e}^{\mathrm{i}\frac{t}{2}}$, $t \in [t_0, \pi]$, in the u-plane, satisfying $\delta_1^+(\pi) \in \gamma_1^+$ and $\delta_2^+(\pi) \in \gamma_2^+$, respectively. Define $\widetilde{h} :]0,\pi[\longrightarrow \mathbb{R}$ as $\widetilde{h}(t_0) = 2\mathrm{Re}\left(\int_{\delta_2^+} \tau\right)$, where $\tau = \frac{du}{v}$. A direct computation gives $\widetilde{h}(t_0) = \frac{\sqrt{2}}{2} \int_{t_0}^{\pi} \frac{dt}{\sqrt{\cos(t_0) - \cos(t)}} > 0$. As we are assuming that the immersion $X : M \longrightarrow \mathbb{R}^3$ is normalized so that the distance between the planes π_1 and π_2 is 1, we have $\lambda = \frac{1}{\widetilde{h}(t_0)}$.

As usual, we define $(\Phi_1, \Phi_2, \Phi_3) = \frac{1}{2}\left(-\mathrm{i}(u^{-\frac{2}{n}} + u^{\frac{2}{n}}), u^{-\frac{2}{n}} - u^{\frac{2}{n}}, 1\right)\Phi_3$. Since M is homeomorphic to a closed disk minus two boundary points, we have that $X : M \longrightarrow \mathbb{R}^3$ given by $X(p) = \int_{p_0}^{p} (\Phi_1, \Phi_2, \Phi_3)$ is a well-defined conformal minimal immersion verifying Condition (1) in the statement of Problem 1. Let us see that (M, g, Φ_3) fulfill also the other conditions.

Denote S_h, S_v the antiholomorphic transformations on M given by $S_h((u,v)) = (1/\overline{u}, \overline{v}/\overline{u}^2)$, $S_v((u,v)) = (\overline{u}, \overline{v})$. Observe that the point $P_0 = (1, 2\sin(\frac{t_0}{2}))$ is invariant under S_h and S_v. Moreover, we have

$$g \circ S_h = 1/\overline{g}\,, \quad g \circ S_v = -\overline{g}\,, \quad S_h^*(\phi_3) = -\overline{\phi_3}\,, \quad S_v^*(\phi_3) = \overline{\phi_3}\,. \tag{13}$$

Hence elementary arguments imply that S_h (resp. S_v) induces on $X(M)$ a symmetry with respect to the plane $\{x_3 = 0\}$ (resp. $\{x_1 = 0\}$).

First, notice that

$$S_v(\gamma_i^+) = \gamma_i^-,\ i = 0,1,2 \tag{14}$$

$$S_h(\gamma_1^+) = \gamma_2^+,\ S_h(\gamma_1^-) = \gamma_2^-,\ S_h(\gamma_0^+) = \gamma_0^+,\ S_h(\gamma_0^-) = \gamma_0^-. \tag{15}$$

It is straightforward to check that the conditions on the boundary are satisfied. Actually, we have:

Lemma 6. *The maps $X|_{\gamma^+}$, $X|_{\gamma^-}$ are injective, $X(\gamma_i^+) = \ell_i^+$ and $X(\gamma_i^-) = \ell_i^-$, for $i = 0,1,2$, it is to say, $X(\partial(M)) = \Gamma_{\theta\, d}$ for $\theta = (3n-2)\pi/n$.*

Taking into account Lemma 6, the expression of the oriented distance $d :]0,\pi[\longrightarrow \mathbb{R}$ between ℓ_0^+ and ℓ_0^- is $d(t_0) = \operatorname{Re}\left(\int_{\widehat{\delta}} \Phi_1\right)$, where now $\widehat{\delta}$ is the lift of the oriented curve $\mathrm{e}^{\mathrm{i}\frac{t}{2}}$, $t \in [-t_0, t_0]$, in the u-plane. Since $(S_h)_*(\widehat{\delta}) = \widehat{\delta}$, $(S_v)_*(\widehat{\delta}) = -\widehat{\delta}$ and taking into account (13) we deduce that $\int_{\widehat{\delta}} \Phi_1 = \overline{\int_{\widehat{\delta}} \Phi_1} = \frac{\lambda}{4}\widetilde{f}_1(t_0)$, where, as in Section 2, $\Phi = \frac{u^{\frac{2}{n}}}{v}du$ and $\widetilde{f}_1(t_0) = f_1(t_0, -t_0)$. Hence we have

$$d(t_0) = \frac{1}{4}\frac{\widetilde{f}_1(t_0)}{\widetilde{h}(t_0)}. \tag{16}$$

A thoughtful study of the function d will be very useful in order to prove the rest of the conditions on the immersion $X : M \longrightarrow \mathbb{R}^3$. In this context we shall see:

Lemma 7. *The function $d :]0,\pi[\rightarrow \mathbb{R}$ satisfies:*

1. *It vanishes at only one point $\widetilde{t}_0 \in]\frac{n\pi}{2}, n\pi[$. Furthermore, d is positive in $]0, \widetilde{t}_0[$ and negative in $]\widetilde{t}_0, \pi[$.*
2. $\lim_{t_0 \to 0} d(t_0) = 0$. *In particular, d is bounded in $]0, \widetilde{t}_0[$.*
3. *It has only a critical point $t_0' \in]0, \widetilde{t}_0[$ which is a maximum. In particular, $\sharp\left[d^{-1}(\{x\})\right] = 2,\ \forall x \in]0, d(\widetilde{t}_0)[$.*

Proof. We had seen (see Step 2 in Proposition 6) that there exists a unique $\widetilde{t}_0 \in]\frac{n\pi}{2}, n\pi[$ such that $\widetilde{f}_1(\widetilde{t}_0) = 0$, $\widetilde{f}_1(t_0) > 0$ in $]0, \widetilde{t}_0[$ and $\widetilde{f}_1(t_0) < 0$ in $]\widetilde{t}_0, \pi[$. Note that this proves the first assertion.

In order to prove the second statement, observe that

$$\lim_{t_0 \to 0} \widetilde{f}_1(t_0) = \lim_{t_0 \to 0} \sqrt{2}\int_{-1}^{1} \frac{t_0 \cos(\frac{t_0 s}{n})}{\sqrt{\cos(t_0 s) - \cos(t_0)}} ds = 2\int_{-1}^{1} \frac{ds}{\sqrt{1-s^2}} = 2\pi. \tag{17}$$

Moreover, it is clear that $0 \le \sqrt{\cos(t_0) - \cos(t)} \le \sqrt{1-\cos(t)}$, $t \in [t_0, \pi]$, then $\widetilde{h}(t_0) \ge \frac{\sqrt{2}}{2}\int_{t_0}^{\pi} \frac{dt}{\sqrt{1-\cos(t)}} = -\log\left[\tan\left(\frac{t_0}{4}\right)\right]$, and so

$$\lim_{t_0 \to 0} \widetilde{h}(t_0) \ge \lim_{t_0 \to 0} \left(-\log\left[\tan\left(\frac{t_0}{4}\right)\right]\right) = +\infty. \tag{18}$$

Both (17) and (18) give Assertion 2. Concerning Assertion 3, we shall prove that the functions $\widetilde{f}_1$, $\widetilde{h}$ satisfy the following differential equations:

$$\widetilde{f_1}''(t_0) + \cot(t_0)\widetilde{f_1}'(t_0) + \tfrac{4-n^2}{4n^2}\widetilde{f}_1(t_0) = 0, \tag{19}$$

$$\widetilde{h}''(t_0) + \cot(t_0)\widetilde{h}'(t_0) - \tfrac{1}{4}\widetilde{h}(t_0) = 0. \tag{20}$$

The above ordinary differential equations can be obtained from the following equalities integrating by parts:

$$\frac{\partial^2\Phi}{\partial t_0^2} + \cot(t_0)\frac{\partial\Phi}{\partial t_0} + \frac{4-n^2}{4n^2}\Phi = d(\widetilde{G}), \quad \frac{\partial^2\tau}{\partial t_0^2} + \cot(t_0)\frac{\partial\tau}{\partial t_0} - \frac{1}{4}\tau = d(\widetilde{H}),$$

where

$$\widetilde{G}(u) = \frac{u^{\frac{2+n}{n}}\left(n\left(-1+u^4\right) + 2\left(1+u^4\right) - 4\,u^2\cos(t_0)\right)}{4\,n\,(1+u^4-2\,u^2\cos(t_0))^{\frac{3}{2}}}$$

and

$$\widetilde{H}(u) = \frac{u\left(-1+u^4\right)}{4\,(1+u^4-2\,u^2\cos(t_0))^{\frac{3}{2}}}.$$

Let t_0' a critical point of d in $]0,\widetilde{t_0}[$. This implies that $d'(t_0') = 0$ and so $(\widetilde{f_1}'\widetilde{h} - \widetilde{f}_1\widetilde{h}')(t_0') = 0$. Therefore we have the following expression for the second derivative of d at the point t_0':

$$d''(t_0') = \frac{1}{4}\frac{\widetilde{f_1}''\widetilde{h} - \widetilde{f}_1\widetilde{h}''}{\widetilde{h}^2}(t_0').$$

Hence, using (19) and (20), we obtain that $d''(t_0') = -\cot(t_0)d'(t_0') - \frac{1}{n^2}d(t_0') < 0$. Consequently, there exists only one critical point of d, t_0', in $]0,\widetilde{t_0}[$ and it is a *maximum*. Obviously, $d(t_0') = \text{Maximum}\{d(t_0) \mid t_0 \in]0,\widetilde{t_0}[\}$.

Hence, it is clear that $\sharp\left[d^{-1}(\{x\})\right] \geq 2$, $\forall x \in]0, d(t_0')[$. If $\sharp\left[d^{-1}(\{x\})\right] > 2$, for some $x \in]0, d(t_0')[$, then it implies the existence of a local minimum of d in $]0,\widetilde{t_0}[$, which is absurd. This concludes the proof. $\square$

Remark 2. For each $n \in [\frac{2}{3}, 1]$, we denote either by d_n or by d'_θ the maximum of the distance function $d(t_0)$, $t_0 \in]0,\widetilde{t_0}]$. Observe that the function $d(n,t_0)$ is a differentiable function on $[\frac{2}{3},1] \times]0,\pi[$. Let us check that $\frac{\partial d}{\partial n} > 0$.

Taking into account that $\widetilde{t_0} < n\pi$ and the definition of the function $\widetilde{f}_1$, we obtain

$$\frac{\partial d}{\partial n} = \frac{1}{4\widetilde{h}}\frac{\partial\widetilde{f}_1}{\partial n} = \frac{\sqrt{2}}{4n^2\widetilde{h}}\int_{-t_0}^{t_0}\frac{t\sin(\frac{t}{n})}{\sqrt{\cos(t)-\cos(t_0)}} > 0.$$

Then $n \to d_n$ is a continuous increasing function in $[\frac{2}{3},1]$, equivalently the function $\theta \to d'_\theta$ is increasing in $[0,\pi]$. Therefore, $d'_{\theta_0} < d'_{\theta_1}$ for $0 \leq \theta_0 < \theta_1 \leq \pi$.

At this point, we recall that the distance function for the examples which lie in the convex hull of their boundary coincides with our function d for $n \in [1,2]$ (see [**10**]). As $\frac{\partial d}{\partial n} > 0$ we have that $d_{n_0} < d_{n_1}$ for $\frac{2}{3} \leq n_0 < n_1 \leq 2$. Hence, we infer that $d'_\theta < d_\theta$, for $\theta \in [0,\pi]$, where d_θ is the maximum of the distance function for the López-Martín examples (see Remark 4 in [**10**]).

From the asymptotic expansion of the Weierstrass data we have that X can be expressed locally around E_1 as

$$
\begin{aligned}
(21) \qquad X(u) &= (X_1(u), X_2(u), X_3(u)) \\
&= \left(-\mathrm{i}\frac{n\lambda}{2(2-n)}\overline{u}^{1-\frac{2}{n}}\left(1+O_1(u)+\mathrm{i}O_2(u)\right), O_3(u)\right),
\end{aligned}
$$

where $O_i(u)/|u|$ is a bounded function in a neighbourhood of E_1, $i = 1,2,3$. Using this fact and Lemma 2.1 in [**12**] it is not difficult to prove the following lemma:

Lemma 8. *The minimal immersion $X : M \to \mathbb{R}^3$ is proper and $X(M)$ is contained in the slab S.*

Let us consider $M_1 = \{(u,v) \in M \mid |u| \leq 1\}$ and $M_2 = \{(u,v) \in M \mid |u| \geq 1\}$. We recall that we had denoted by $\widehat{\delta}$ and δ_i^+ the lifts to M of the curves of $u(M)$ given by $\mathrm{e}^{\mathrm{i}\frac{t}{2}}$, for $t \in [-t_0, t_0]$ and $t \in [t_0, \pi]$, respectively satisfying $\delta_i^+(\pi) \in \gamma_i^+$, $i = 1,2$. Clearly, the surfaces M_1 and M_2 are topologically a closed disk minus one boundary point. Moreover, $M = M_1 \cup M_2$, $M_1 \cap M_2 = \widehat{\delta}$ and $\partial(M_1) = \widehat{\delta} \cup \gamma_1^+ \cup \gamma_1^- \cup \delta_1^+ \cup \delta_1^-$, $\partial(M_2) = \widehat{\delta} \cup \gamma_2^+ \cup \gamma_2^- \cup \delta_2^+ \cup \delta_2^-$, where $\delta_i^- = S_v(\delta_i^+)$ for $i = 1,2$.

Our next objective is to prove the following assertion:

Claim 4. $X|_{\widehat{\delta}}$ is injective and $X(\widehat{\delta}) \subset \{x_2 \leq 0, x_3 = 0\}$.

Proof. To see this we observe that $X_2(\hat{\delta}(t)) = X_2(\hat{\delta}(t)) - X_2(\hat{\delta}(-t_0)) = -\sqrt{2}\int_{-t_0}^{t} \frac{\sin(s/n)}{\sqrt{\cos(s)-\cos(t_0)}}ds$. Since $0 < t_0 \leq n\pi$ we have that $X_2 \circ \widehat{\delta}$ is a nonpositive decreasing function for $t \in [-t_0, 0]$. This fact and the vertical symmetry S_v prove the assertion. □

Let us denote $M^+ = \{(u,v) \in M \mid \mathrm{Im}(u) \geq 0\}$ and $M^- = \{(u,v) \in M \mid \mathrm{Im}(u) \leq 0\}$, and define ρ as the lift to M of the divergent curve $]0,+\infty[$ in the u-plane. We parametrize ρ as follows: $\rho(t) = u^{-1}(t)$, $t \in]0,+\infty[$.

Obviously, the surfaces M^+ and M^- are topologically a closed disk minus two boundary points. Furthermore, $M = M^+ \cup M^-$, $M^+ \cap M^- = \rho$ and $\partial(M^+) = \rho \cup \gamma_1^+ \cup \gamma_2^+ \cup \gamma_0^+$, $\partial(M^-) = \rho \cup \gamma_1^- \cup \gamma_2^- \cup \gamma_0^-$. Next we prove:

Claim 5. $X(\rho(t)) \subset \{(x_1,x_2,x_3) \mid x_1 = 0, x_2 \leq -\varepsilon, -\frac{1}{2} < x_3 < \frac{1}{2}\}$, where $\varepsilon = -X_2(P_0) > 0$.

Proof. Taking the symmetry S_h into account, it suffices to prove the assertion for $X(\rho(]0,1]))$. From the Weierstrass data we deduce that

$$X_2(\rho(t)) = -\varepsilon - \int_t^1 \frac{s^{-\frac{2}{n}} - s^{\frac{2}{n}}}{\sqrt{s^4 - 2\cos(t_0)s^2 + 1}} ds.$$

Using Claim 4 we have $\varepsilon = -X_2(P_0) > 0$. So $X_2 \circ \rho$ is decreasing and nonpositive in $]0,1]$. $\square$

Moreover, taking (21) and the symmetry S_h into account, we obtain that

$$\lim_{t \to 0} X_2(\rho(t)) = \lim_{t \to +\infty} X_2(\rho(t)) = -\infty.$$

At this point we can prove the following lemma:

Lemma 9. *The minimal immersion $X : M \longrightarrow \mathbb{R}^3$ verifies:*

1. $X(M) \subset ((S \setminus \mathcal{E}(\Gamma_{\theta\, d})) \cup \Gamma_{\theta\, d})$.
2. *The surfaces $X(M_1 \setminus (\delta_1^+ \cup \delta_1^-))$ and $X(M_2 \setminus (\delta_2^+ \cup \delta_2^-))$ are graphs on the plane $\{x_3 = 0\}$.*
3. *$d > 0$ implies that X is an embedding. If $d = 0$, then $X|_{M \setminus \gamma^+}$ and $X|_{M \setminus \gamma^-}$ are injective.*

Proof. Denote by $\mathfrak{p}_3$ the orthogonal projection on the plane $\{x_3 = 0\}$. Using (21) once again and the symmetries, it is not hard to see that $X(M^+)$ and $X(M^-)$ are contained in a wedge of the slab S. Then we can apply Statement (ii) in Theorem 1 to conclude that $X(M^+)$ and $X(M^-)$ lies in the convex hull of their boundary.

In case $d = 0$, taking into account Lemma 6, Claim 5 and the interior maximum principle we have the proof of Assertion 1. Moreover, in this case the interior maximum principle also gives us that $X(M^+) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0, -\varepsilon < x_2 < 0\} = \emptyset$, and so, taking into account the symmetry S_v, we deduce that

$$\mathfrak{p}_3(X(M)) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0, -\varepsilon < x_2 < 0\} = \emptyset. \tag{22}$$

In case $d > 0$ the above reasoning implies

$$X(M) \subset S \setminus \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ \mid \ x_1 = 0, \ x_2 > 0\}. \tag{23}$$

Next, we prove Assertion 1 for $d > 0$ and Assertion 2. In what follows we denote $\alpha = \mathfrak{p}_3(\partial(X(M_1)))$. We also introduce the following notation:

(A) If $d = 0$, W_0 will denote the bounded connected component of $\mathbb{R}^2 \setminus \alpha$, whereas W_1 will denote the unbounded connected component which is disjoint from $\mathfrak{p}_3(\mathcal{E}(\Gamma_{\theta\, d}))$.

(B) If $d > 0$, observe that α is a connected simple curve. In this case we denote W_0 as the connected component of $\mathbb{R}^2 \setminus \alpha$ containing the point $(0,0)$ and W_1 the other one.

In both cases $\mathfrak{p}_3(M_1) \subset W_0 \cup W_1 \cup \alpha$. Since $\mathfrak{p}_3(X(\partial(M_1))) = \alpha$ and $\mathfrak{p}_3 \circ X$ is proper on M_1 (see Lemma 8), then $\mathfrak{p}_3 \circ X(M_1 \setminus \partial(M_1)) \cap W_0$ is closed in W_0. Furthermore, $g(M_1 \setminus \partial(M_1)) \cap \mathbb{S}^1 = \emptyset$ and so $(\mathfrak{p}_3 \circ X)|_{M_1 \setminus \partial(M_1)}$ is a local diffeomorphism. In particular, $\mathfrak{p}_3 \circ X(M_1 \setminus \partial(M_1)) \cap W_0$ is an open set of W_0. Hence we deduce that either $\mathfrak{p}_3 \circ X(M_1 \setminus \partial(M_1)) \cap W_0 = \emptyset$ or $\mathfrak{p}_3 \circ X(M_1 \setminus \partial(M_1)) \cap W_0 = W_0$. According to (22) and (23) we have that $\mathfrak{p}_3 \circ X(M_1 \setminus \partial(M_1)) \cap W_0 = \emptyset$. A similar argument yields $\mathfrak{p}_3(X(M_1)) \cap W_1 = W_1$, i.e., $\mathfrak{p}_3(X(M_1)) = W_1 \cup \alpha$. Hence, using the symmetry S_h and the interior maximum principle we conclude the proof of Assertion 1.

From the above reasoning we have that $\mathfrak{p}_3 \circ X : M_1 \setminus (\delta_1^+ \cup \delta_1^-) \longrightarrow W_1 \cup \alpha$ is a proper local diffeomorphism and so $\mathfrak{p}_3 \circ X$ is a covering map. Since $(\mathfrak{p}_3 \circ X)|_{\gamma_1^+}$ is one-to-one we obtain that $X(M_1 \setminus (\delta_1^+ \cup \delta_1^-))$ is a graph on the plane $\{x_3 = 0\}$.

Using that $\partial(X(M_1)) \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \le x_3 \le \frac{1}{2}\}$ and Lemma 2.1 in [**12**] we infer that

$$X(M_1) \subset \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \,\middle|\, 0 \le x_3 \le \frac{1}{2} \right\}. \tag{24}$$

Then, taking into account the symmetry S_h, we obtain Assertion 2.

Finally, Assertion 2 and (24) give us Assertion 3. □

The Main Theorem is a consequence of the following results: Propositions 1, 5, and 6, Lemmas 7, 8, and 9, Remark 2, Theorem 4 in [**10**] and Theorem 3.32 in [**9**].

Acknowledgements. We would like to thank Professor F.J. López for helpful conversations.

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Received November 6, 2002. This research was partially supported by MCYT-FEDER Grant number BFM2001-3489.

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