Pacific Journal of Mathematics

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Volume 214 No. 1 March 2004

PACIFIC JOURNAL OF MATHEMATICS
Vol. 214, No. 1, 2004

PARTIAL REGULARITY FOR WEAK SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH SUPERCRITICAL EXPONENTS

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Let Ω be an open subset in $\mathbf{R}^n$ $(n \geq 3)$. In this paper, we study the partial regularity for stationary positive weak solutions of the equation

$$\Delta u + h_1(x)u + h_2(x)u^\alpha = 0 \quad \text{in } \Omega. \tag{1.1}$$

We prove that if $\alpha > \frac{n+2}{n-2}$, and $u \in H^1(\Omega) \cap L^{\alpha+1}(\Omega)$ is a stationary positive weak solution of (1.1), then the Hausdorff dimension of the singular set of u is less than $n - 2\frac{\alpha+1}{\alpha-1}$, which generalizes the main results in Pacard 1993 and Pacard 1994.

1. Introduction.

Let Ω be an open subset in $\mathbf{R}^n$ $(n \geq 3)$. In this paper, we prove a partial regularity result for positive weak solutions of the equation

$$\Delta u + h_1(x)u + h_2(x)u^\alpha = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $\alpha > \frac{n+2}{n-2}$, $h_i \in C^1(\Omega)$, $a_i \leq h_i(x) \leq b_i$, $0 < a_i < b_i$ and $|\nabla \log h_i(x)| \leq \beta$ $(i = 1, 2)$ for $x \in \overline{\Omega}$. As we know, there is not much known about the properties of the weak solutions of (1.1).

We say that u is a positive weak solution of (1.1) in Ω if $u(x) \geq 0$ for a.e. $x \in \Omega$ and for all $\phi \in C^\infty(\Omega)$ with compact support in Ω,

$$-\int_\Omega u \Delta \phi dx = \int_\Omega [h_1(x)u + h_2(x)u^\alpha]\phi(x)dx. \tag{1.2}$$

We say that a weak solution u is stationary, if it satisfies

$$\int_\Omega \left[\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial \phi^j}{\partial x_i} - \frac{1}{2}|\nabla u|^2 \frac{\partial \phi^i}{\partial x_i} + \frac{1}{2}u^2 \frac{\partial h_1}{\partial x_i}\phi^i + \frac{1}{2}h_1 u^2 \frac{\partial \phi^i}{\partial x_i} + \frac{1}{\alpha+1}u^{\alpha+1}\frac{\partial h_2}{\partial x_i}\phi^i + \frac{1}{\alpha+1}h_2 u^{\alpha+1}\frac{\partial \phi^i}{\partial x_i} \right] dx = 0 \tag{1.3}$$

for all regular vector field ϕ with compact support in Ω (summation over i and j is understood).

For weak solutions in $H^1(\Omega)\cap L^{\alpha+1}(\Omega)$ this identity is obtained by assuming that the functional $E(u)$ is stationary with respect to domain variations, that is,

$$\frac{d}{dt}E(u_t)|_{t=0} = 0$$

where

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2}\int_\Omega h_1 u^2 - \frac{1}{\alpha+1}\int_\Omega h_2 u^{\alpha+1} dx$$

and $u_t(x) = u(x + t\phi(x))$.

Let $u \in H^1(\Omega)\cap L^{\alpha+1}(\Omega)$ be a positive weak solution of (1.1). We denote by Σ the set of points $x \in \Omega$ such that u is not bounded in any neighborhood W of x in Ω. If u is bounded in a neighborhood of x then the classical regularity theory ensures that u is regular in the neighborhood of x. Therefore Σ is the singular set of u. Moreover, Σ is a closed subset of Ω.

If $\alpha < \frac{n}{n-2}$, a simple bootstrap argument shows that all positive weak solutions of (1.1) are regular. It is well-known that the singular set may not be empty if $\alpha \geq \frac{n}{n-2}$. Pacard [**Pa2**] constructed solutions with singular sets of Hausdorff dimension $d < n - \frac{2\alpha}{\alpha-1}$. Schoen and Yau proved in [**SY**] that the singular set of a positive weak solution of (1.1) is not always as simple as in the examples given in [**Pa2**].

In [**Pa1**] and [**Pa3**], Pacard showed that the Hausdorff dimension of the singular set of a stationary positive weak solution u of the equation $-\Delta u = u^\alpha$ in Ω is less than $n - 2\frac{\alpha+1}{\alpha-1}$.

In a recent paper [**GL**], we considered the compactness for positive solutions of Equation (1.1). Using the ideas in [**LT1**] and [**LT2**], we obtained the measure estimate of the blow up set of a sequence of positive smooth solutions $\{u_i\}$ of (1.1) with $\{\|u_i\|_{H^1(\Omega)} + \|u_i\|_{L^{\alpha+1}(\Omega)}\}$ bounded. We applied such result to a semilinear eigenvalue problem

$$-\Delta u = \lambda(u + u^\alpha) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.4}$$

when Ω is a smooth star-shaped domain and obtained that any branch of positive solutions $(\lambda(s), u(s))$ of (1.4) must converge to a (singular) positive solution u_0 of the equation

$$-\Delta u = \lambda_0(u + u^\alpha) \text{ in } \Omega \tag{1.5}$$

as $\lambda(s) + \|u(s)\|_{L^\infty(\Omega)} \to \infty$, $s \to \infty$, where $\lambda_0 = \lim_{s\to\infty}\lambda(s)$ and $0 < \lambda_0 < \infty$. The existence of such branches of positive solutions is obtained by Rabinowitz. It was proved in [**BDT**] and [**Da**] that some branches are simple curves.

In this paper, we shall prove a partial regularity theorem for a stationary positive weak solution of (1.1) with $\alpha > \frac{n+2}{n-2}$.

Theorem A. *Let* $\alpha > \frac{n+2}{n-2}$. *If* $u \in H^1(\Omega) \cap L^{\alpha+1}(\Omega)$ *is a stationary positive weak solution of* (1.1), *then the Hausdorff dimension of the singular set of* u *is less than* $n - 2\frac{\alpha+1}{\alpha-1}$.

Our result covers the main results in [**Pa1**] and [**Pa3**]. The proof is quite different, we used the duality of a weighted Hardy space and a weighted BMO, which was used in [**CLL**] to get a partial regularity result for a weak heat flow.

When $h_1 = 0$ and h_2 is a constant, it is not hard to construct solutions of (1.1) which are singular (see [**Lin**] and [**Re**]). However, when h_2 is not a constant, the problem is much harder. A singular solution was given in this case by Johnson-Pan-Yi [**JPY**]. Let $\Omega = B_R$, here $B_R \subset \mathbf{R}^n$ $(n \geq 3)$ is a ball with center at 0 and radius of $R > 0$. Consider the equation

$$\Delta u + K(|x|)u^\alpha = 0 \text{ in } B_R \tag{1.6}$$

with $K(|x|)$ satisfying the following conditions in [**JPY**]:

(K1) $K \in C^1[0,\infty)$, $K'(0) = 0$, $K(r) > 0$ for $r \geq 0$, and $\lim_{r\to\infty} K(r) = K(\infty) > 0$;

(K2) There is a $\delta > 0$ such that $\lim\limits_{r\to\infty} r^\delta(K(r) - K(\infty)) = 0$, $\lim\limits_{r\to\infty} r^{1+\delta}K'(r) = 0$;

(K3) $K'(r) \leq 0$ for $r > 0$.

It is proved in [**JPY**] (Theorem 1) that the equation

$$\Delta u + K(|x|)u^\alpha = 0 \text{ in } \mathbf{R}^n$$

has a singular solution $U_0(r)$ with $r = |x|$, which satisfies

$$\lim_{r\to 0} r^{\frac{2}{\alpha-1}} U_0(r) = \left[\frac{1}{K(0)} \cdot \frac{2}{\alpha-1}\left(n-2-\frac{2}{\alpha-2}\right)\right]^{\frac{1}{\alpha-1}},$$

$$\lim_{r\to 0} r^{\frac{2}{\alpha-1}+1} U_0'(r) = -\frac{2}{\alpha-1}\left[\frac{1}{K(0)} \cdot \frac{2}{\alpha-1}\left(n-2-\frac{2}{\alpha-1}\right)\right]^{\frac{1}{\alpha-1}}.$$

It is clear that $U_0(|x|)$ for $x \in B_R$ is a singular solution of Equation (1.6).

Throughout this paper, C will denote a universal constant depending only on α, β, n and a_i, b_i $(i = 1, 2)$, unless it is explicitly stated.

2. $H^1_w(\mathbf{R}^n)$ and $M^\sharp_{1,\nu}g(x)$.

In this section we review definitions and properties of the space $H^1_w(\mathbf{R}^n)$ and the function $M^\sharp_{1,\nu}g(x)$. See Strömberg & Torchinsky [**ST**] for more details.

Let μ be the Lebesgue measure in $\mathbf{R}^n$ and $d\mu(x) = dx$. Let ν be a weighted measure with respect to the Lebesgue measure in $\mathbf{R}^n$ with weight $w(x)$. Then

$$H^1_w(\mathbf{R}^n) = \{f \in \mathcal{S}'(\mathbf{R}^n) : \ M_1(F_\phi) \in L^1_w(\mathbf{R}^n), \quad \|f\|_{H^1_w} = \|M_1(F_\phi)\|_{L^1_w}\},$$

where

$$F_\phi(x) = \frac{1}{t^n} \int_{\mathbf{R}^n} f(y)\phi\left(\frac{y-x}{t}\right) dy,$$

ϕ is any smooth function with support in the unit ball and $M_1(F_\phi(x)) = \sup_{t>0} F_\phi(x)$.

For $g \in L^1_{\text{loc}}(\mathbf{R}^n)$, define

$$M^\sharp_{1,\nu} g(x) = \sup_{t>0} \frac{1}{\nu(B(x,t))} \int_{B(x,t)} |g(y) - (g)_{x,t}| dy,$$

where

$$(g)_{x,t} \equiv \frac{1}{B(x,t)} \int_{B(x,t)} g dy,$$

and $B(x,t) \subset \mathbf{R}^n$ is the ball centered at x with radius t. It follows from Theorem 2 in Chapter IX in [**ST**] that for $f \in \hat{\mathcal{D}}_0$, $g \in L^1_{\text{loc}}(\mathbf{R}^n)$ and $\nu \in D_d$ for some $d > 0$ (see Doubling D_d condition in Chapter I in [**ST**]), there exists $C > 0$ independent of f and g such that

$$\int_{\mathbf{R}^n} f(x)g(x)dx \le C \left(\int_{\mathbf{R}^n} M_1(F_\phi(x)) M^\sharp_{1,\nu} g(x) w(x) dx \right). \tag{2.1}$$

Since $\hat{\mathcal{D}}_0$ is dense in $H^1_w(\mathbf{R}^n)$ (see Theorem 1 of Chapter VII in [**ST**]), we conclude that (2.1) holds for $f \in H^1_w(\mathbf{R}^n)$ and $g \in L^1_{\text{loc}}(\mathbf{R}^n)$.

In this paper, we define $w(x) = |x|^{-2/(\alpha-1)}$ and $d\nu(x) = |x|^{-2/(\alpha-1)} dx$. Then ν is a doubling weighted measure with respect to the Lebesgue measure of $\mathbf{R}^n$ with weight $|x|^{-2/(\alpha-1)}$ and $\nu \in D_{n-\frac{2}{(\alpha-1)}}$. Moreover,

$$\nu(B(x,t)) = \frac{(\alpha-1)\omega_n}{n(\alpha-1)-2} t^{n-\frac{2}{\alpha-1}},$$

where ω_n is the area of the $(n-1)$-dimensional unit sphere in $\mathbf{R}^n$.

3. A monotonicity inequality and blow up.

In this section, we first recall a monotonicity inequality for stationary positive weak solutions of (1.1) established in [**GL**], using this monotonicity inequality and a blow up argument, we then obtain a decay property of the scaled energy. Assume henceforth that $u \in H^1(\Omega) \cap L^{\alpha+1}(\Omega)$ is a stationary positive solution of (1.1).

For any $x_0 \in \Omega$ and $r > 0$, define

$$E_u(x_0, r) \equiv \frac{(\alpha-1)}{2(\alpha+1)} e^{Cr} r^{-\mu} \int_{B(x_0,r)} h_2 u^{\alpha+1} dx + \frac{1}{4}\left(e^{Cr} \frac{d}{dr}\left(r^{-\mu} \int_{\partial B(x_0,r)} u^2 ds\right)\right) + \frac{1}{4}\left(e^{Cr} r^{-\mu-1}(-1+Cr) \int_{\partial B(x_0,r)} u^2 ds\right) + C\int_0^r e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi,$$

where $\mu = n - 2\frac{\alpha+1}{\alpha-1}$ and C depends only upon α, β, n and a_i, b_i $(i = 1, 2)$. It is proved in [**GL**] that $E_u(x_0, r)$ can be written to the equivalent forms:

$$E_u(x_0, r) \equiv \frac{1}{2} e^{Cr} r^{-\mu} \int_{B(x_0,r)} |\nabla u|^2 dx - \frac{1}{2} e^{Cr} r^{-\mu} \int_{B(x_0,r)} h_1 u^2 dx - \frac{1}{(\alpha+1)} e^{Cr} r^{-\mu} \int_{B(x_0,r)} h_2 u^{\alpha+1} dx + \frac{1}{(\alpha-1)} e^{Cr} r^{-\mu-1} \int_{\partial B(x_0,r)} u^2 ds + \frac{C}{4} e^{Cr} r^{-\mu} \int_{\partial B(x_0,r)} u^2 ds + C\int_0^r e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi$$

and

$$E_u(x_0, r) \equiv \left(\frac{\alpha-1}{\alpha+3}\right) e^{Cr} r^{-\mu} \left[\frac{1}{(\alpha+1)} \int_{B(x_0,r)} h_2 u^{\alpha+1} dx + \frac{1}{2}\int_{B(x_0,r)} |\nabla u|^2 dx - \frac{1}{2}\int_{B(x_0,r)} h_1 u^2 dx\right] + \frac{1}{(\alpha+3)} \frac{d}{dr}\left(e^{Cr} r^{-\mu} \int_{\partial B(x_0,r)} u^2 ds\right) + \left(\frac{C}{4} - \frac{C}{(\alpha+3)}\right) e^{Cr} r^{-\mu} \int_{\partial B(x_0,r)} u^2 ds + C\int_0^r e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi.$$

All the derivatives in the above expressions are to be understood in the sense of distributions. Lemma 3.1 and Lemma 3.2 below are proved in [**GL**].

Lemma 3.1. *If* $u \in H^1(\Omega) \cap L^{\alpha+1}(\Omega)$ *is a stationary positive weak solution of* (1.1), *then* $E_u(x_0, r)$, *defined above, is an increasing function of* r.

Lemma 3.2. $E_u(x_0, r)$ *is a continuous function of* $x_0 \in \Omega$ *and* $r > 0$.

Now we show the following lemma:

Lemma 3.3. *There exist* $0 < r_0 < 1$ *independent of* $x_0 \in \Omega$ *and some constant* $C > 0$ *depending only upon* α, n, *such that the following inequality holds:*

$$r^{-\mu} \int_{B(x_0,r)} |\nabla u|^2 dx \le C E_u(x_0, 2r) \le C E_u(x_0, r_0) \quad \text{for } r < r_0/2. \tag{3.1}$$

Proof. We consider the last one of the three equivalent formulations of $E_u(x_0, r)$ given above. By Lemma 2.3 in [**GL**] we know that there exists $0 < r_0 < 1$ such that

$$E_u(x_0, r) \ge 0 \quad \text{for all } x_0 \in \Omega, \quad 0 < r < r_0, \tag{3.2}$$

and for $r < r_0$,

(3.3)
$$\frac{1}{2}\left(\frac{\alpha-1}{\alpha+3}\right) e^{Cr} r^{-\mu} \int_{B(x_0,r)} h_1 u^2 dx$$
$$\le C \int_0^r e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi + \frac{1}{2(\alpha+1)} \left(\frac{\alpha-1}{\alpha+3}\right) e^{Cr} r^{-\mu} \int_{B(x_0,r)} h_2 u^{\alpha+1} dx.$$

We denote by $\phi(r) = r^{-\mu} \int_{B(x_0,r)} |\nabla u|^2 dx$. Since $E_u(x_0, r)$ is an increasing function of r, we integrate it from 0 to $r < r_0$ and obtain that for almost every $x_0 \in \Omega$, (note that $e^{Cr} > 1$)

$$\frac{\alpha-1}{2} \int_0^r \phi(\rho) d\rho + e^{Cr} r^{-\mu} \int_{\partial B(x_0,r)} u^2 ds \le (\alpha+3) E_u(x_0, r) r \quad \text{for } r < r_0.$$

(Here we have used $\lim_{r \to 0} r^{-\mu} \int_{\partial B(x_0,r)} u^2 ds = 0$ a.e. $x_0 \in \Omega$. This fact is proved in [**GL**].) Now we use Remark 2 in [**Pa1**] and we see that there exists some $\sigma \in [r/2, r]$ such that

$$\phi(\sigma) \le \frac{8}{r} \int_0^r \phi(\rho) d\rho \le C E_u(x_0, r),$$

for some constant $C > 0$ depending only upon α, β and n. In addition we have $\phi(r/2) \le 2^\mu \phi(\sigma)$, if $\sigma \in [r/2, r]$. This gives us the desired result for almost every x_0 and, by continuity, for every x_0.

Proposition 3.4. *Assume that there exist* $x_0 \in \Omega$ *and* $0 < r_1 < r_0$ *such that* $E_u(x_0, r_1) \le \delta$. *Then*

$$r^{-\mu} \int_{B(y,r)} |\nabla u|^2 dx \le C\delta, \tag{3.4}$$

for all $y \in B(x_0, r_1/8)$ *and* $0 < r < r_1/4$, *where* C *only depends upon* n, α, β.

Proof. Let $0 < r < r_1$. We know that for any $y \in B(x_0, r/2)$, $B(y, r/2) \subset B(x_0, r) \subset B(x_0, r_1)$. Thus,

$$\int_{B(y,r/2)} |\nabla u|^2 dx \le \int_{B(x_0,r)} |\nabla u|^2 dx.$$

Thus, (note that $e^{Cr} > 1$)

$$\begin{aligned} E_u(x_0, r) \ge\ & 2^{-\mu} \left(\frac{\alpha - 1}{2(\alpha+1)(\alpha+3)} \right) \left(\frac{r}{2} \right)^{-\mu} \int_{B(x_0,r)} h_2 u^{\alpha+1} dx \\ & + \left(\frac{\alpha-1}{\alpha+3} \right) e^{Cr} r^{-\mu} \left(\frac{1}{2} \int_{B(y,r/2)} |\nabla u|^2 dx - \widetilde{C} r^n \right) \\ & + \frac{1}{(\alpha+3)} \frac{d}{dr} \left(e^{Cr} r^{-\mu} \int_{\partial B(x_0,r)} u^2 dx \right) \\ & + C e^{Cr} r^{-\mu} \left(\frac{1}{4} - \frac{1}{(\alpha+3)} \right) \int_{\partial B(x_0,r)} u^2 ds + C \int_0^r e^{C\xi} \xi^{\frac{\alpha+1}{\alpha-1}} d\xi. \end{aligned}$$

Define $\psi(r) = \left(\frac{r}{2}\right)^{-\mu} \int_{B(y,r/2)} |\nabla u|^2 dx$. By the argument similar to that in the proof of Lemma 2.3 in [**GL**], we have, for almost every $x_0 \in \Omega$,

$$2^{\mu-1}(\alpha - 1) \int_0^r \psi(\rho) d\rho \le (\alpha + 3) E_u(x_0, r_1) r. \tag{3.5}$$

Using Remark 2 in [**Pa1**] again, we see that there exists some $\sigma \in [r/2, r]$ such that

$$\psi(\sigma) \le \frac{8}{r} \int_0^r \psi(s) ds \le C E_u(x_0, r_1), \tag{3.6}$$

for some constant $C > 0$ only depending upon α, β and n. It is clear that $\psi(r/2) \le 2^\mu \psi(\sigma)$. Since

$$\psi(r/2) = \left(\frac{r}{4} \right)^{-\mu} \int_{B(y,r/4)} |\nabla u|^2 dx, \tag{3.7}$$

we have the desired result for almost every $y \in B(x_0, r_1/8)$. By continuity, we see that it holds for every $y \in B(x_0, r_1/8)$.

Define

$$F_u(x_0, r) = r^{-\mu} \int_{B(x_0,r)} |\nabla u|^2 dx + C \int_0^r e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi,$$

where $C \int_0^r e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi$ is the function in the formulations of $E_u(x_0, r)$. Then we have the following lemma:

Lemma 3.5. *We have that*

$$r^{-\mu} \int_{B(x_0,r)} h_2 u^{\alpha+1} dx \le C F_u(x_0, r) \quad \textit{for all } x_0 \in \Omega \textit{ and } 0 < r < r_0$$

and

$$r^{-\mu}\int_{B(x_0,r)} h_1 u^2 dx \le CF_u(x_0,r) \quad \textit{for all } x_0\in\Omega \textit{ and } 0<r<r_0,$$

where C depends only upon α, n, a_i and b_i $(i=1,2)$.

Proof. We only show the first inequality, the second can be obtained by a similar argument. Since $E_u(x_0,r)\ge 0$ for all $x_0\in\Omega$ and $0<r<r_0$, it can be seen from the second of the three equivalent formulations given above that

$$r^{-\mu}\int_{B(x_0,r)} h_2 u^{\alpha+1} dx \le C\left(F_u(x_0,r)+r^{-\mu-1}\int_{\partial B(x_0,r)} u^2 ds\right),$$

for some constant C depending only upon α, n, a_i and b_i $(i=1,2)$. On the other hand, the trace embedding theorem gives

$$H^1(B(x_0,r)) \hookrightarrow W^{\frac12,2}(\partial B(x_0,r)) \hookrightarrow L^{\frac{2(n-1)}{n-2}}(\partial B(x_0,r)).$$

Therefore,

$$\|u\|_{L^{\frac{2(n-1)}{n-2}}(\partial B(x_0,r))} \le C\|u\|_{H^1(B(x_0,r))}.$$

By Hölder inequality,

$$r^{-1}\int_{\partial B(x_0,r)} u^2 ds \le C\left(\int_{\partial B(x_0,r)} u^{\frac{2(n-1)}{n-2}}\right)^{\frac{n-2}{n-1}} \le C\|u\|^2_{H^1(B(x_0,r))},$$

so we obtain

$$r^{-\mu-1}\int_{\partial B(x_0,r)} u^2 ds \le CF_u(x_0,r).$$

This implies that the first inequality in the lemma holds.

Theorem 3.6. *There exist constants $0<\epsilon_0$, $\tau<1$, $0<r_2<r_0/4$, such that*

$$E_u(x_0,r)\le\epsilon_0 \tag{3.8}$$

implies

$$F_u(x_0,\tau r)\le \frac12 F_u(x_0,r) \quad \textit{for all } x_0\in\Omega \textit{ and } 0<r<r_2. \tag{3.9}$$

Proof. It follows from Proposition 3.4 that if $E_u(x_0,r)\le\epsilon_0$, then for $\eta<r/4$,

$$\eta^{-\mu}\int_{B(x_0,\eta)} |\nabla u|^2 dx \le C\epsilon_0.$$

This implies $\lim_{\eta\to 0}\eta^{-\mu}\int_{B(x_0,\eta)}|\nabla u|^2 dx = 0$. (Otherwise we can choose ϵ_0 smaller to deduce a contradiction.)

If the result were false, there would exist balls $B(x_k, r_k) \subset \Omega$ with $r_k \to 0$ as $k \to \infty$ such that

$$F_u(x_k, r_k) \equiv \lambda_k^2 \to 0, \tag{3.10}$$

whereas

$$F_u(x_k, \tau r_k) > \frac{1}{2}\lambda_k^2, \tag{3.11}$$

for $\tau > 0$ selected as below. We rescale our variables to the unit ball $B(0,1) \subset \mathbf{R}^n$ as follows: For $z \in B(0,1)$, we set

$$v_k(z) \equiv r_k^{2/(\alpha-1)} \left(\frac{u(x_k + r_k z) - a_k}{\lambda_k} \right), \tag{3.12}$$

where

$$a_k \equiv \frac{1}{|B(x_k, r_k)|} \int_{B(x_k,r_k)} u dy = (u)_{x_k, r_k},$$

($|B(x_k, r_k)| = \mathrm{Vol}(B(x_k, r_k))$ denotes the average of u over $B(x_k, r_k)$, $k = 1, 2, \ldots$.)

Using (3.10), (3.11) and (3.12) we have

$$\sup_k \int_{B(0,1)} |v_k|^2 dz < \infty, \quad \sup_k \int_{B(0,1)} |\nabla v_k|^2 dz < \infty,$$

but

$$\frac{1}{\tau^\mu} \int_{B(0,\tau)} |\nabla v_k|^2 dz > \frac{1}{2} - e^C \tau^{2\alpha/(\alpha-1)} \geq 1/4 \ \ (k = 1, 2, \ldots), \tag{3.13}$$

if we choose $\tau < \left(\frac{1}{4} e^{-C}\right)^{\frac{\alpha-1}{2\alpha}}$. In fact, we know that

$$C \int_0^{\tau r_k} e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi \leq \frac{C e^C (\alpha - 1)}{2\alpha} (\tau r_k)^{\frac{2\alpha}{\alpha-1}}$$

and since $C \int_0^{r_k} e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi < \lambda_k^2$, it holds that

$$\frac{C(\alpha-1)}{2\alpha} r_k^{\frac{2\alpha}{\alpha-1}} < \lambda_k^2.$$

Thus, it follows from (3.11) that

$$\lambda_k^2 \left(\tau^{-\mu} \int_{B(0,\tau)} |\nabla v_k|^2 dz \right) \geq \left(\frac{1}{2} - e^C \tau^{\frac{2\alpha}{\alpha-1}} \right) \lambda_k^2.$$

The sequence $\{v_k\}_{k=1}^\infty$ is thus bounded in $H^1(B(0,1))$, so there exists a subsequence (still denoted by $\{v_k\}$) such that

$$v_k \to v \quad \text{strongly in } \ L^2(B(0,1)) \tag{3.14}$$

$$\nabla v_k \rightharpoonup \nabla v \quad \text{weakly in } \ L^2(B(0,1)). \tag{3.15}$$

Choose any function $w \in C_0^\infty(B(0,1))$. Define

$$w_k(y) \equiv w\left(\frac{y-x_k}{r_k}\right), \quad (y \in B(x_k, r_k)),$$
$$h_1(y) \equiv \widetilde{h}_1\left(\frac{y-x_k}{r_k}\right),$$
$$h_2(y) \equiv \widetilde{h}_2\left(\frac{y-x_k}{r_k}\right).$$

Since u is a weak solution of (1.1), we have

$$\int_{B(x_k,r_k)} \nabla u\, \nabla w_k dy = \int_{B(x_k,r_k)} \Big(h_1(y)u + h_2(y)u^\alpha\Big) w_k(y)dy. \tag{3.16}$$

Thus,

$$\int_{B(0,1)} \nabla v_k\, \nabla w dz = \int_{B(0,1)} \Big[r_k^2 \widetilde{h}_1(z)\Big(v_k(z) + \frac{a_k}{A_k}\Big) + \lambda_k^{\alpha-1}\widetilde{h}_2(z)\Big(v_k(z)+\frac{a_k}{A_k}\Big)^\alpha\Big] w(z)dz, \tag{3.17}$$

where $A_k = \lambda_k r_k^{-2/(\alpha-1)}$. Since

$$\begin{aligned} I_k^1 :&= r_k^2 \left|\int_{B(0,1)} \widetilde{h}_1\left(v_k + \frac{a_k}{A_k}\right) w dz\right| \\ &\le r_k^2 \left(\int_{B(0,1)} \widetilde{h}_1 \left(v_k + \frac{a_k}{A_k}\right)^2 dz\right)^{1/2} \left(\int_{B(0,1)} \widetilde{h}_1 w^2 dz\right)^{1/2} \\ &\le r_k^2 \left[\lambda_k^{-2} r_k^{-2}\left(r_k^{-\mu}\int_{B(x_k,r_k)} h_1 u^2 dx\right)\right]^{1/2} \|\widetilde{h}_1 w^2\|_{L^2(B(0,1))} \\ &\le C r_k \|\widetilde{h}_1 w^2\|_{L^2(B(0,1))} \to 0 \end{aligned}$$

(here we used Lemma 3.5) as $k \to \infty$ and

$$\begin{aligned} I_k^2 :&= \lambda_k^{\alpha-1}\left|\int_{B(0,1)} \widetilde{h}_2\left(v_k + \frac{a_k}{A_k}\right)^\alpha w dz\right| \\ &\le \lambda_k^{\alpha-1}\left(\int_{B(0,1)} \widetilde{h}_2\left(v_k+\frac{a_k}{A_k}\right)^{\alpha+1} dz\right)^{\alpha/(\alpha+1)} \\ &\quad \cdot \left(\int_{B(0,1)} \widetilde{h}_2 |w|^{\alpha+1} dz\right)^{1/(\alpha+1)} \\ &\le \lambda_k^{\alpha-1}\left(\lambda_k^{-(\alpha+1)} r_k^{-\mu}\int_{B(x_k,r_k)} h_2 u^{\alpha+1} dx\right)^{\frac{\alpha}{\alpha+1}} \|\widetilde{h}_2^{1/(\alpha+1)} w\|_{L^{\alpha+1}(B(0,1))} \end{aligned}$$

$$\le C\lambda_k^{(\alpha-1)/(\alpha+1)}\|\widetilde{h}_2^{1/(\alpha+1)}w\|_{L^{\alpha+1}(B(0,1))} \to 0$$

(here we used Lemma 3.5) as $k \to \infty$.
Letting $k \to \infty$ in (3.17), we get

$$\int_{B(0,1)} \nabla v \, \nabla w dz = 0. \tag{3.18}$$

Hence v is harmonic function, and hence smooth, and we have the bound

$$\|\nabla v\|_{L^\infty(B(0,\frac{1}{2}))} \le \frac{C}{|B(0,1)|}\int_{B(0,1)} v^2 dz < \infty, \tag{3.19}$$

where $|B(0,1)|$ =Vol$(B(0,1))$. We will show in next section that

$$\nabla v_k \to \nabla v \quad \textit{strongly} \text{ in } L^2\left(B\left(0,\frac{1}{4}\right)\right) \tag{3.20}$$

then we have,

$$\frac{1}{\tau^\mu}\int_{B(0,\tau)} |\nabla v|^2 dz \le C\tau^{n-\mu} < \frac{1}{4} \tag{3.21}$$

provided $0 < \tau < \min\left\{\left(\frac{1}{4C}\right)^{\frac{\alpha-1}{2(\alpha+1)}}, \left(\frac{e^{-C}}{4}\right)^{(\alpha+1)/(2\alpha)}, \frac{1}{4}\right\}$, which contradicts (3.13).

4. Compactness.

In this section we turn our attention to (3.20). We choose a smooth cut-off function $\zeta : \mathbf{R}^n \to \mathbf{R}$ satisfying

$$0 \le \zeta \le 1,$$
$$\zeta \equiv 1 \text{ on } B\left(0,\frac{1}{4}\right),$$
$$\zeta \equiv 0 \text{ on } \mathbf{R}^n \backslash B\left(0,\frac{5}{16}\right).$$

Lemma 4.1. *The sequence $\{\zeta v_k\}_{k=1}^\infty$ is bounded in $M^\sharp_{1,\nu}(\mathbf{R}^n,\mathbf{R})$.*

Proof. We first show that for $0 < r < r_0 < 1$,

$$E_u(x_0,r) \le CF_u(x_0,r) \text{ for all } x_0 \in \Omega. \tag{4.1}$$

In fact, it follows from the second of the three formulations of $E_u(x_0, r)$ given above that

(4.2)

$$E_u(x_0,r) \le \frac{1}{2}e^{Cr}r^{-\mu}\int_{B(x_0,r)}|\nabla u|^2dx + \frac{1}{(\alpha-1)}e^{Cr}r^{-\mu-1}\int_{\partial B(x_0,r)}u^2ds + \frac{C}{4}e^{Cr}r^{-\mu}\int_{\partial B(x_0,r)}u^2ds + C\int_0^r e^{C\xi}\xi^{(\alpha+1)/(\alpha-1)}d\xi.$$

By the trace embedding theorem and the argument similar to the one used in the proof of Lemma 3.5, we obtain

$$r^{-\mu-1}\int_{\partial B(x_0,r)}u^2ds \le CF_u(x_0,r). \tag{4.3}$$

Note that $e^{Cr} < e^C$. Our claim can be obtained from (4.2) and (4.3).

Fix any point $z_0 \in B(0, \frac{3}{4})$ and any radius $0 < r < \frac{1}{8}$, set

$$y_k = x_k + r_k z_0 \in B\left(x_k, \frac{3}{4}r_k\right).$$

By the claim above and an argument similar to the one used in the proof of Lemma 3.3, we obtain that

$$\begin{aligned}
&\frac{1}{(rr_k)^\mu}\int_{B(y_k,rr_k)}|\nabla u|^2dy \\
&\le CE_u\left(y_k, \frac{1}{4}r_k\right) \\
&\le C\left(r_k^{-\mu}\int_{B(y_k,\frac{1}{4}r_k)}|\nabla u|^2dy + C\int_0^{\frac{1}{4}r_k}e^{C\xi}\xi^{(\alpha+1)/(\alpha-1)}d\xi\right) \\
&\le CF_u(x_k,r_k) = C\lambda_k^2.
\end{aligned}$$

Rescaling this estimate we obtain,

$$r^{-\mu}\int_{B(z_0,r)}|\nabla v_k|^2dz \le C \tag{4.4}$$

for $k = 1, 2, \ldots$, all $0 < r < \frac{1}{8}$ and $z_0 \in B(0, \frac{3}{4})$. This implies that

$$\frac{1}{r^{n-\frac{2}{\alpha-1}}}\int_{B(z_0,r)}|v_k - (v_k)_{z_0,r}|dz \le C < \infty \tag{4.5}$$

for k, r and z_0 as above. This implies $v_k \in \mathcal{L}^{1,n-\frac{2}{\alpha-1}}(B(0, \frac{3}{4}))$ and $\mathcal{L}^{1,n-\frac{2}{\alpha-1}}(B(0,3/4))$ is a Campanato space (see [**Gi**]). Since $B(0, \frac{3}{4})$ is type

(A) ([**Gi**], Chapter III, Definition 1.3), Proposition 1.2 in Chapter III in [**Gi**] implies that

$$\begin{aligned}(4.6)\qquad &\sup_{z_0\in B(0,3/4),\ 0<r<1/8} \frac{1}{r^{n-\frac{2}{\alpha-1}}}\int_{B(z_0,r)}|v_k|dz\\ &\le C\sup_{z_0\in B(0,3/4),\ 0<r<1/8} \frac{1}{r^{n-\frac{2}{\alpha-1}}}\int_{B(z_0,r)}|v_k-(v_k)_{z_0,r}|dz\le C.\end{aligned}$$

Since ζ is smooth, then

$$(4.7)\qquad |(\zeta v_k)_{z_0,r}-\zeta(v_k)_{z_0,r}|\le \frac{Cr}{|B(z_0,r)|}\int_{B(z_0,r)}|v_k|dz \quad \text{on } B(z_0,r)$$

for any ball $B(z_0,r)$. Thus, if $z_0\in B(0,\frac{3}{4})$, $0<r<\frac{1}{8}$, we have,

$$\begin{aligned}(4.8)\qquad &\frac{1}{|B(z_0,r)|}\int_{B(z_0,r)}|\zeta v_k-(\zeta v_k)_{z_0,r}|dz\\ &\le \frac{1}{|B(z_0,r)|}\int_{B(z_0,r)}|v_k-(v_k)_{z_0,r}|dz+\frac{Cr}{|B(z_0,r)|}\int_{B(z_0,r)}|v_k|dz.\end{aligned}$$

On the other hand, if $z_0\in \mathbf{R}^n\backslash B(0,\frac{3}{4})$, $0<r<\frac{1}{8}$, we have

$$(4.9)\qquad \int_{B(z_0,r)}|\zeta v_k-(\zeta v_k)_{z_0,r}|dz=0.$$

It follows from (4.6), (4.8) and (4.9) that

$$(4.10)\qquad \zeta v_k\in \mathcal{L}^{1,n-\frac{2}{\alpha-1}}(\mathbf{R}^n).$$

This also implies that $\{\zeta v_k\}_{k=1}^{\infty}$ is bounded in $M^{\sharp}_{1,\nu}(\mathbf{R}^n,\mathbf{R})$ for $k=1,2,\dots$.

Proposition 4.2. *The rescaled functions $\{\nabla v_k\}_{k=1}^{\infty}$ converge strongly in $L^2(B(0,\frac{1}{4}))$.*

Proof. Subtracting (3.18) from (3.17) we obtain

$$\begin{aligned}(4.11)\qquad &\int_{B(0,1)}(\nabla v_k-\nabla v)\,\nabla w dz\\ &=r_k^2\int_{B(0,1)}\widetilde{h}_1\left(v_k+\frac{a_k}{A_k}\right)w+\lambda_k^{\alpha-1}\int_{B(0,1)}\widetilde{h}_2\left(v_k+\frac{a_k}{A_k}\right)^{\alpha}wdz\end{aligned}$$

for $w\in C_0^\infty(B(0,1))$. Hence it holds for $w\in H_0^1(B(0,1))\cap L^\infty(B(0,1))$. We now insert $w\equiv\zeta^2(v_k-v)$ into (4.11). The left-hand side of (4.11) is

$$\begin{aligned}L_k&\equiv\int_{B(0,1)}\zeta^2|\nabla v_k-\nabla v|^2dz+2\int_{B(0,1)}\zeta(v_k-v)\,(\nabla v_k-\nabla v)\,\nabla\zeta dz\\ &\ge\int_{B(0,\frac{1}{4})}|\nabla v_k-\nabla v|^2dz+o(1)\end{aligned}$$

as $k \to \infty$, in view of (3.14) and (3.15). The right-hand side of (4.11) reads

$$\begin{aligned} R_k \equiv\ & r_k^2 \int_{B(0,1)} \widetilde{h}_1 \left(v_k + \frac{a_k}{A_k} \right) \zeta^2 (v_k - v) dz \\ & + \lambda_k^{\alpha-1} \int_{B(0,1)} \widetilde{h}_2 \left(v_k + \frac{a_k}{A_k} \right)^{\alpha} \zeta^2 (v_k - v) dz \\ =\ & R_k^1 + R_k^2. \end{aligned}$$

$$\begin{aligned} R_k^1 &\le r_k^2 \left(\int_{B(0,1)} \widetilde{h}_1 \left(v_k + \frac{a_k}{A_k} \right)^2 \right)^{1/2} \left(\int_{B(0,1)} \widetilde{h}_1 \zeta^4 (v_k - v)^2 dz \right)^{1/2} \\ &= C r_k^2 \left(\lambda_k^{-2} r_k^{\frac{4}{\alpha-1} - n} \int_{B(x_k, r_k)} h_1 u^2 dx \right)^{1/2} \\ &\le C r_k^2 (r_k^{-2})^{1/2} = C r_k \to 0 \end{aligned}$$

as $k \to \infty$.

Now we show that

$$\zeta \Big(v_k + \frac{a_k}{A_k} \Big)^{\alpha} \in H_w^1(\mathbf{R}^n) \tag{4.12}$$

for $k = 1, 2, \ldots$. We first consider

$$M_1 \left(\zeta \left(v_k + \frac{a_k}{A_k} \right)^{\alpha} \right)(z) := \sup_{t>0} \frac{1}{t^n} \int_{\mathbf{R}^n} (\zeta^{1/\alpha} f_k)^{\alpha}(y) \phi \left(\frac{y-z}{t} \right) dy$$

where $f_k(y) := v_k(y) + \frac{a_k}{A_k}$, ϕ is a Schwartz function with nonvanishing integral (see [**ST**]).

If $t \ge 1 + \frac{|z|}{4}$, we have

$$\begin{aligned} & \frac{1}{t^n} \int_{\mathbf{R}^n} \left(\zeta^{1/\alpha} f_k \right)^{\alpha} (y) \phi \left(\frac{y-z}{t} \right) dy \\ & \le \frac{1}{t^n} \int_{B(0,1)} \left(\zeta^{1/\alpha} f_k \right)^{\alpha} (y) \phi_t dy \\ & \le \frac{1}{t^n} \left(\int_{B(0,1)} \left(\zeta^{1/\alpha} f_k \right)^{\alpha+1} dy \right)^{\alpha/(\alpha+1)} \left(\int_{B(0,1)} \phi_t^{\alpha+1} dy \right)^{1/(\alpha+1)} \\ & \le \frac{C}{t^n} \left[\lambda_k^{-(\alpha+1)} r_k^{-\mu} \int_{B(x_k, r_k)} u^{\alpha+1} dx \right]^{\alpha/(\alpha+1)} \\ & \le \frac{C}{t^n} \lambda_k^{\frac{\alpha(1-\alpha)}{\alpha+1}} \\ & \le \frac{C}{(4 + |z|)^n} \lambda_k^{\frac{\alpha(1-\alpha)}{\alpha+1}}. \end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\mathbf{R}^n} M_1\left(\zeta\left(v_k(z)+\frac{a_k}{A_k}\right)^\alpha\right) w(z)dz \\
&= \int_{\mathbf{R}^n} M_1\left(\zeta\left(v_k(z)+\frac{a_k}{A_k}\right)^\alpha\right) |z|^{-2/(\alpha-1)}dz \\
&\leq C\lambda_k^{\frac{\alpha(1-\alpha)}{\alpha+1}} \int_{\mathbf{R}^n} |z|^{-2/(\alpha-1)}(4+|z|)^{-n}dz \\
&\leq C\lambda_k^{\frac{\alpha(1-\alpha)}{\alpha+1}}.
\end{aligned}$$

If $t < 1+\frac{|z|}{4}$, we have, if $|y-z|<t$, then $|y-z|<1+\frac{|z|}{4}$ and $|y|>\frac{3}{4}|z|-1$. Therefore, if $|z|>8/3$, then $|y|>1$. Thus, for $0<\epsilon<1$ and $z\in B(0,3)$,

$$\begin{aligned}
&\frac{1}{t^n}\int_{\mathbf{R}^n}\left(\zeta^{1/\alpha}f_k\right)^\alpha \phi\left(\frac{y-z}{t}\right)dy \\
&\leq \frac{1}{t^n}\left(\int_{B(z,t)}\left(\zeta^{1/\alpha}f_k\right)^{\alpha+1-\epsilon}dy\right)^{\alpha/(\alpha+1-\epsilon)} \\
&\qquad \cdot\left(\int_{B(z,t)}\phi_t^{(\alpha+1-\epsilon)/(1-\epsilon)}dy\right)^{(1-\epsilon)/(\alpha+1-\epsilon)} \\
&\leq C\left(M\Big(\left(\zeta^{1/\alpha}f_k\right)^{\alpha+1-\epsilon}\Big)\right)^{\alpha/(\alpha+1-\epsilon)},
\end{aligned}$$

where $M(\cdot)$ is the Hardy-Littlewood maximal function. If $z\in\mathbf{R}^n\backslash B(0,3)$,

$$\frac{1}{t^n}\int_{\mathbf{R}^n}\left(\zeta^{1/\alpha}f_k\right)^\alpha\phi_t dy = 0.$$

Therefore,

$$\begin{aligned}
&\int_{\mathbf{R}^n} M_1\left(\zeta(z)\left(v_k(z)+\frac{a_k}{M_k}\right)^\alpha\right)|z|^{-2/(\alpha-1)}dz \\
&\leq C\int_{B(0,3)}\left[M\Big(\left(\zeta^{1/\alpha}f_k\right)^{\alpha+1-\epsilon}\Big)\right]^{\alpha/(\alpha+1-\epsilon)}|z|^{-2/(\alpha-1)}dz \\
&\leq C\left(\int_{B(0,3)}\left[M\Big(\left(\zeta^{1/\alpha}f_k\right)^{\alpha+1-\epsilon}\Big)\right]^{(\alpha+1)/(\alpha+1-\epsilon)}dz\right)^{\alpha/(\alpha+1)} \\
&\qquad\cdot\left(\int_{B(0,3)}|z|^{-2\frac{\alpha+1}{\alpha-1}}dz\right)^{1/(\alpha+1)} \\
&\leq C\left(\int_{B(0,3)}\left(\zeta^{1/\alpha}f_k\right)^{\alpha+1}dz\right)^{\alpha/(\alpha+1)}\left(\int_0^3 r^{\mu-1}dr\right)^{1/(\alpha+1)}
\end{aligned}$$

$$\leq C\lambda_k^{\frac{\alpha(1-\alpha)}{\alpha+1}},$$

where we used the facts that $\mu > 0$ if $\alpha > \frac{n+2}{n-2}$, and

$$\begin{aligned}
&\int_{B(0,3)} \left[M\Big(\big(\zeta^{1/\alpha} f_k \big)^{\alpha+1-\epsilon} \Big) \right]^{(\alpha+1)/(\alpha+1-\epsilon)} dz \\
&= \int_{\mathbf{R}^n} \left[M\Big(\big(\zeta^{1/\alpha} f_k \big)^{\alpha+1-\epsilon} \Big) \right]^{(\alpha+1)/(\alpha+1-\epsilon)} dz \\
&\leq \int_{\mathbf{R}^n} \big(\zeta^{1/\alpha} f_k \big)^{\alpha+1} dz,
\end{aligned}$$

because

$$M\Big(\big(\zeta^{1/\alpha} f_k \big)^{\alpha-1+\epsilon} \Big)(z) \equiv 0 \quad \text{for } z \in \mathbf{R}^n \backslash B(0,3).$$

It concludes that $\big(\zeta^{1/\alpha} f_k \big)^{\alpha} \in H^1_w(\mathbf{R}^n)$. Therefore, it follows from (2.1) that

$$\begin{aligned}
R_k^2 &\leq \lambda_k^{\alpha-1} \int_{\mathbf{R}^n} M_1\Big(\big(\zeta^{1/\alpha} f_k \big)^{\alpha} \Big) M^{\sharp}_{1,\nu}(\zeta(v_k - v)) |z|^{-2/(\alpha-1)} dz \\
&\leq C\lambda_k^{\alpha-1} \int_{\mathbf{R}^n} M_1\Big(\big(\zeta^{1/\alpha} f_k \big)^{\alpha} \Big) |z|^{-2/(\alpha-1)} dz \\
&\leq C\lambda_k^{\alpha-1} \lambda_k^{\frac{\alpha(1-\alpha)}{\alpha+1}} \\
&= C\lambda_k^{\frac{\alpha-1}{\alpha+1}} \to 0
\end{aligned}$$

as $k \to \infty$.

5. Proof of Theorem A.

In this section we shall prove Theorem A. We recall the definition of function space $L^{p,q}(\Omega)$:

$$L^{p,q}(\Omega) = \left\{ v \in L^p(\Omega) : \sup_{x \in \Omega, r > 0} r^{-q} \int_{B(x,r) \cap \Omega} u^p dx < +\infty \right\}.$$

This is called Morrey space (see [**Gi**]). Now we recall a theorem in [**Pa2**].

Theorem 5.1. *Let u be a positive weak solution of* (1.1), *assume that $u \in L^{\alpha, \lambda+\theta}(\Omega)$ for $\lambda = n - \frac{2\alpha}{\alpha-1}$ and some $\theta > 0$ then u is regular in Ω.*

Note that Pacard [**Pa2**] proved this theorem only for the case that $h_1 \equiv 0$ and $h_2 \equiv 1$ in Ω, but we can easily see from his proof that this theorem still holds in our case.

Set

$$V \equiv \{ x \in \Omega : \; E_u(x,r) < \epsilon_0 \;\; \text{for some } 0 < r < r_2 \},$$

where ϵ_0 and r_2 are constants in Theorem 3.6. Furthermore, using Theorem 3.6, we can show that (cf. [**Gi**]), if $x \in V$, there exists $r* > 0$ sufficiently small such that

$$F_u(y,r) \le Cr^{\gamma} \tag{5.1}$$

for some $0 < \gamma < \frac{2\alpha}{\alpha-1}$, $C > 0$, all y near x, and all sufficiently small radii $0 < r < r*$. It follows from Lemma 3.5 that

$$r^{-\mu} \int_{B(x_0,r)} u^{\alpha+1} dx \le CF_u(x_0, r) \tag{5.2}$$

for all $x_0 \in \Omega$ and $0 < r < r_0$. Note that $\gamma < \frac{2\alpha}{\alpha-1}$, by (5.1) and (5.2), we have

$$r^{-\mu} \left(\int_{B(y,r)} \left(|\nabla u|^2 + u^{\alpha+1} \right) dx \right) \le Cr^{\gamma} \tag{5.3}$$

for all y near x, and $0 < r < r*$. Now we show that

$$u \in L^{\alpha,\lambda+\theta_0}(B(x, r*/2)) \tag{5.4}$$

for some $\theta_0 > 0$. In fact, choosing $\theta_0 = \frac{\alpha\gamma}{\alpha+1}$, we have, for $0 < r < r*$,

$$r^{-(n+\theta_0)} \int_{B(x,r)} u^{\alpha} dy \le r^{-(n+\theta_0)} \left(\int_{B(x,r)} u^{\alpha+1} dy \right)^{\alpha/(\alpha+1)} r^{1/(\alpha+1)} \le C.$$

This implies (5.4) and therefore, by Theorem 5.1, u is regular at x. Hence u is regular in V.

Define $\Sigma = \Omega \backslash V$. Then

$$\Sigma \equiv \cap_{r>0} \{x \in \Omega : \ E_u(x,r) \ge \epsilon_0\}.$$

It is proved in [**GL**] that

$$\Sigma \subset \cap_{r>0} \left\{ x \in \Omega : \int_{B(x,r)} \left(u^{\alpha+1} + |\nabla u|^2 \right) dy \ge C\epsilon_0 r^{\mu} \right\}. \tag{5.5}$$

Thus, standard covering arguments imply that the Hausdorff dimension of Σ is less than $n - 2\frac{\alpha+1}{\alpha-1}$. This completes the proof of Theorem A.

Remark. The conclusion of Theorem A still holds for the stationary positive weak solutions of the equation

$$\Delta u + h_1(x)u^{\kappa} + h_2(x)u^{\alpha} = 0 \ \text{ in } \Omega$$

where $0 < \kappa < \alpha$, $\alpha > \frac{n+2}{n-2}$. It should be very interesting to know whether our partial regularity theorem holds for the equation

$$\Delta u + h(x)f(u) = 0 \ \text{ in } \Omega$$

where $f(s)$ satisfies that $f(s) > 0$ for $s > 0$ and f has the growth rate $\alpha > \frac{n+2}{n-2}$. The main difficulty is how to establish the monotonicity inequality.

Acknowledgements. Part of this work was done while the first author was visiting the Institute of Mathematics, Chinese Academy of Sciences, he would like to thank the Institute for their hospitality. The authors wish to thank Professor Ding Weiyue for bringing this problem to their attention. They also thank the referee for valuable comments. This research was partially supported by the Outstanding Young Scientists Grants in China and the National Key Basic Research Fund of China.

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Received January 30, 2001 and revised June 24, 2002.

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