PARTIAL REGULARITY FOR WEAK SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH SUPERCritical EXPONENTS

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Let \( \Omega \) be an open subset in \( \mathbb{R}^n \) \((n \geq 3)\). In this paper, we study the partial regularity for stationary positive weak solutions of the equation

\[
\Delta u + h_1(x)u + h_2(x)u^\alpha = 0 \quad \text{in} \; \Omega.
\]

We prove that if \( \alpha > \frac{n+2}{n-2} \), and \( u \in H^1(\Omega) \cap L^{\alpha+1}(\Omega) \) is a stationary positive weak solution of (1.1), then the Hausdorff dimension of the singular set of \( u \) is less than \( n - \frac{2\alpha+1}{\alpha-1} \), which generalizes the main results in Pacard 1993 and Pacard 1994.

1. Introduction.

Let \( \Omega \) be an open subset in \( \mathbb{R}^n \) \((n \geq 3)\). In this paper, we prove a partial regularity result for positive weak solutions of the equation

\[
\Delta u + h_1(x)u + h_2(x)u^\alpha = 0 \quad \text{in} \; \Omega,
\]

where \( \alpha > \frac{n+2}{n-2} \), \( h_i \in C^1(\Omega) \), \( a_i \leq h_i(x) \leq b_i \), \( 0 < a_i < b_i \) and \( |\nabla \log h_i(x)| \leq \beta \) \((i = 1, 2)\) for \( x \in \Omega \). As we know, there is not much known about the properties of the weak solutions of (1.1).

We say that \( u \) is a positive weak solution of (1.1) in \( \Omega \) if \( u(x) \geq 0 \) for a.e. \( x \in \Omega \) and for all \( \phi \in C^\infty(\Omega) \) with compact support in \( \Omega \),

\[
- \int_\Omega u \Delta \phi dx = \int_\Omega [h_1(x)u + h_2(x)u^\alpha] \phi(x) dx.
\]

We say that a weak solution \( u \) is stationary, if it satisfies

\[
\int_\Omega \left[ \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial \phi^i}{\partial x_j} - \frac{1}{2} |\nabla u|^2 \frac{\partial \phi^i}{\partial x_i} + \frac{1}{2} u^2 \frac{\partial h_1}{\partial x_i} \phi^i + \frac{1}{2} \frac{\partial h_1}{\partial x_i} u^2 \frac{\partial \phi^i}{\partial x_i} + \frac{1}{\alpha+1} u^{\alpha+1} \frac{\partial h_2}{\partial x_i} \phi^i + \frac{1}{\alpha+1} h_2 u^{\alpha+1} \frac{\partial \phi^i}{\partial x_i} \right] dx = 0
\]

for all regular vector field \( \phi \) with compact support in \( \Omega \) (summation over \( i \) and \( j \) is understood).

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For weak solutions in $H^1(\Omega) \cap L^{\alpha+1}(\Omega)$ this identity is obtained by assuming that the functional $E(u)$ is stationary with respect to domain variations, that is,

$$
\frac{d}{dt}E(u_t)|_{t=0} = 0
$$

where

$$
E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{1}{2} \int_\Omega h_1 u^2 - \frac{1}{\alpha+1} \int_\Omega h_2 u^{\alpha+1} dx
$$

and $u_t(x) = u(x + t \phi(x))$.

Let $u \in H^1(\Omega) \cap L^{\alpha+1}(\Omega)$ be a positive weak solution of (1.1). We denote by $\Sigma$ the set of points $x \in \Omega$ such that $u$ is not bounded in any neighborhood $W$ of $x$ in $\Omega$. If $u$ is bounded in a neighborhood of $x$ then the classical regularity theory ensures that $u$ is regular in the neighborhood of $x$. Therefore $\Sigma$ is the singular set of $u$. Moreover, $\Sigma$ is a closed subset of $\Omega$.

If $\alpha < \frac{n}{n-2}$, a simple bootstrap argument shows that all positive weak solutions of (1.1) are regular. It is well-known that the singular set may not be empty if $\alpha \geq \frac{n}{n-2}$. Pacard [Pa2] constructed solutions with singular sets of Hausdorff dimension $d < n - \frac{2\alpha+1}{\alpha-1}$. Schoen and Yau proved in [SY] that the singular set of a positive weak solution of (1.1) is not always as simple as in the examples given in [Pa2].

In [Pa1] and [Pa3], Pacard showed that the Hausdorff dimension of the singular set of a stationary positive weak solution $u$ of the equation $-\Delta u = u^\alpha$ in $\Omega$ is less than $n - \frac{2\alpha+1}{\alpha-1}$.

In a recent paper [GL], we considered the compactness for positive solutions of Equation (1.1). Using the ideas in [LT1] and [LT2], we obtained the measure estimate of the blow up set of a sequence of positive smooth solutions $\{u_i\}$ of (1.1) with $\{\|u_i\|_{H^1(\Omega)} + \|u_i\|_{L^{\alpha+1}(\Omega)}\}$ bounded. We applied such result to a semilinear eigenvalue problem

(1.4) \quad $-\Delta u = \lambda(u + u^\alpha)$ in $\Omega$, \quad $u = 0$ on $\partial \Omega$

when $\Omega$ is a smooth star-shaped domain and obtained that any branch of positive solutions $(\lambda(s), u(s))$ of (1.4) must converge to a (singular) positive solution $u_0$ of the equation

(1.5) \quad $-\Delta u = \lambda_0(u + u^\alpha)$ in $\Omega$

as $\lambda(s) + \|u(s)\|_{L^\infty(\Omega)} \to \infty$, $s \to \infty$, where $\lambda_0 = \lim_{s \to \infty} \lambda(s)$ and $0 < \lambda_0 < \infty$. The existence of such branches of positive solutions is obtained by Rabinowitz. It was proved in [BDT] and [Da] that some branches are simple curves.

In this paper, we shall prove a partial regularity theorem for a stationary positive weak solution of (1.1) with $\alpha > \frac{n+2}{n-2}$. 

Theorem A. Let $\alpha > \frac{n+2}{n-2}$. If $u \in H^1(\Omega) \cap L^{\alpha+1}(\Omega)$ is a stationary positive weak solution of (1.1), then the Hausdorff dimension of the singular set of $u$ is less than $n - 2\frac{\alpha+1}{\alpha-1}$.

Our result covers the main results in [Pa1] and [Pa3]. The proof is quite different, we used the duality of a weighted Hardy space and a weighted BMO, which was used in [CLL] to get a partial regularity result for a weak heat flow.

When $h_1 = 0$ and $h_2$ is a constant, it is not hard to construct solutions of (1.1) which are singular (see [Lin] and [Re]). However, when $h_2$ is not a constant, the problem is much harder. A singular solution was given in this case by Johnson-Pan-Yi [JPY]. Let $\Omega = B_R$, here $B_R \subset \mathbb{R}^n$ ($n \geq 3$) is a ball with center at 0 and radius of $R > 0$. Consider the equation

$$\Delta u + K(|x|)u^\alpha = 0 \text{ in } B_R$$

with $K(|x|)$ satisfying the following conditions in [JPY]:

(K1) $K \in C^1[0, \infty)$, $K'(0) = 0$, $K(r) > 0$ for $r \geq 0$, and $\lim_{r \to \infty} K(r) = K(\infty) > 0$;

(K2) There is a $\delta > 0$ such that $\lim_{r \to \infty} r^\delta(K(r) - K(\infty)) = 0$, $\lim_{r \to \infty} r^{1+\delta}K'(r) = 0$;

(K3) $K'(r) \leq 0$ for $r > 0$.

It is proved in [JPY] (Theorem 1) that the equation

$$\Delta u + K(|x|)u^\alpha = 0 \text{ in } \mathbb{R}^n$$

has a singular solution $U_0(r)$ with $r = |x|$, which satisfies

$$\lim_{r \to 0} r^{\frac{2}{\alpha-1}}U_0(r) = \left[ \frac{1}{K(0)} \cdot \frac{2}{\alpha-1} \left( n - 2 - \frac{2}{\alpha-2} \right) \right]^{\frac{1}{\alpha-1}},$$

$$\lim_{r \to 0} r^{\frac{2}{\alpha-1}+1}U_0'(r) = -\frac{2}{\alpha-1} \left[ \frac{1}{K(0)} \cdot \frac{2}{\alpha-1} \left( n - 2 - \frac{2}{\alpha-2} \right) \right]^{\frac{1}{\alpha-1}}.$$

It is clear that $U_0(|x|)$ for $x \in B_R$ is a singular solution of Equation (1.6).

Throughout this paper, $C$ will denote a universal constant depending only on $\alpha$, $\beta$, $n$ and $a_i, b_i$ ($i = 1, 2$), unless it is explicitly stated.

2. $H^1_w(\mathbb{R}^n)$ and $M_{1,\nu}^w g(x)$.

In this section we review definitions and properties of the space $H^1_w(\mathbb{R}^n)$ and the function $M_{1,\nu}^w g(x)$. See Strömberg & Torchinsky [ST] for more details.

Let $\mu$ be the Lebesgue measure in $\mathbb{R}^n$ and $d\mu(x) = dx$. Let $\nu$ be a weighted measure with respect to the Lebesgue measure in $\mathbb{R}^n$ with weight $w(x)$. Then

$$H^1_w(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : M_1(F_\phi) \in L^1_w(\mathbb{R}^n), \| f \|_{H^1_w} = \| M_1(F_\phi) \|_{L^1_w} \}. $$
where
\[
F_\phi(x) = \frac{1}{t^n} \int_{\mathbb{R}^n} f(y) \phi \left( \frac{y - x}{t} \right) dy,
\]
\(\phi\) is any smooth function with support in the unit ball and \(M_1(F_\phi(x)) = \sup_{t > 0} F_\phi(x)\).

For \(g \in L^1_{\text{loc}}(\mathbb{R}^n)\), define
\[
M^\sharp_{1,\nu} g(x) = \sup_{t > 0} \frac{1}{\nu(B(x,t))} \int_{B(x,t)} |g(y) - (g)_{x,t}| dy,
\]
where
\[(g)_{x,t} \equiv \frac{1}{B(x,t)} \int_{B(x,t)} g dy,
\]
and \(B(x,t) \subset \mathbb{R}^n\) is the ball centered at \(x\) with radius \(t\). It follows from Theorem 2 in Chapter IX in [ST] that for \(f \in \hat{D}_0\), \(g \in L^1_{\text{loc}}(\mathbb{R}^n)\) and \(\nu \in D_d\) for some \(d > 0\) (see Doubling \(D_d\) condition in Chapter I in [ST]), there exists \(C > 0\) independent of \(f\) and \(g\) such that
\[
\int_{\mathbb{R}^n} f(x) g(x) dx \leq C \left( \int_{\mathbb{R}^n} M_1(F_\phi(x)) M^\sharp_{1,\nu} g w(x) dx \right).
\]

Since \(\hat{D}_0\) is dense in \(H^1_{\text{w}}(\mathbb{R}^n)\) (see Theorem 1 of Chapter VII in [ST]), we conclude that (2.1) holds for \(f \in H^1_{\text{w}}(\mathbb{R}^n)\) and \(g \in L^1_{\text{loc}}(\mathbb{R}^n)\).

In this paper, we define \(w(x) = |x|^{-2/(\alpha - 1)}\) and \(d\nu(x) = |x|^{-2/(\alpha - 1)} dx\). Then \(\nu\) is a doubling weighted measure with respect to the Lebesgue measure of \(\mathbb{R}^n\) with weight \(|x|^{-2/(\alpha - 1)}\) and \(\nu \in D_n^{-2/(\alpha - 1)}\). Moreover,

\[
\nu(B(x,t)) = \frac{(\alpha - 1) \omega_n}{n(\alpha - 1) - 2} t^{n-\frac{2}{\alpha - 1}},
\]
where \(\omega_n\) is the area of the \((n - 1)\)-dimensional unit sphere in \(\mathbb{R}^n\).

3. A monotonicity inequality and blow up.

In this section, we first recall a monotonicity inequality for stationary positive weak solutions of (1.1) established in [GL], using this monotonicity inequality and a blow up argument, we then obtain a decay property of the scaled energy. Assume henceforth that \(u \in H^1(\Omega) \cap L^{\alpha + 1}(\Omega)\) is a stationary positive solution of (1.1).
For any \( x_0 \in \Omega \) and \( r > 0 \), define
\[
E_u(x_0, r) \equiv \frac{(\alpha - 1)}{2(\alpha + 1)} e^{Cr r^{-\mu}} \int_{B(x_0, r)} h_2 u^{\alpha + 1} dx
+ \frac{1}{4} \left( e^{Cr} \frac{d}{dr} \left( r^{-\mu} \int_{\partial B(x_0, r)} u^2 ds \right) \right)
+ \frac{1}{4} \left( e^{Cr r^{-\mu} - 1} (-1 + C r) \int_{\partial B(x_0, r)} u^2 ds \right)
+ C \int_0^r e^{C \xi \xi^{(\alpha + 1)/(\alpha - 1)}} d\xi,
\]
where \( \mu = n - \frac{2a_i + 1}{n - 1} \) and \( C \) depends only upon \( \alpha, \beta, n \) and \( a_i, b_i (i = 1, 2) \).

It is proved in [GL] that \( E_u(x_0, r) \) can be written to the equivalent forms:
\[
E_u(x_0, r) \equiv \frac{1}{2} e^{Cr r^{-\mu}} \int_{B(x_0, r)} |\nabla u|^2 dx - \frac{1}{2} e^{Cr r^{-\mu}} \int_{B(x_0, r)} h_1 u^2 dx
- \frac{1}{(\alpha + 1)} e^{Cr r^{-\mu}} \int_{B(x_0, r)} h_2 u^{\alpha + 1} dx
+ \frac{1}{(\alpha - 1)} e^{Cr r^{-\mu} - 1} \int_{\partial B(x_0, r)} u^2 ds
+ \frac{C}{4} e^{Cr r^{-\mu}} \int_{\partial B(x_0, r)} u^2 ds + C \int_0^r e^{C \xi \xi^{(\alpha + 1)/(\alpha - 1)}} d\xi,
\]
and
\[
E_u(x_0, r) \equiv \frac{(\alpha - 1)}{\alpha + 3} e^{Cr r^{-\mu}} \left[ \frac{1}{(\alpha + 1)} \int_{B(x_0, r)} h_2 u^{\alpha + 1} dx \right]
+ \frac{1}{2} \int_{B(x_0, r)} |\nabla u|^2 dx - \frac{1}{2} \int_{B(x_0, r)} h_1 u^2 dx)
+ \frac{1}{(\alpha + 3)} \frac{d}{dr} \left( e^{Cr r^{-\mu}} \int_{\partial B(x_0, r)} u^2 ds \right)
+ \left( \frac{C}{4} - \frac{C}{(\alpha + 3)} \right) e^{Cr r^{-\mu}} \int_{\partial B(x_0, r)} u^2 ds
+ C \int_0^r e^{C \xi \xi^{(\alpha + 1)/(\alpha - 1)}} d\xi.
\]

All the derivatives in the above expressions are to be understood in the sense of distributions.

Lemma 3.1 and Lemma 3.2 below are proved in [GL].

**Lemma 3.1.** If \( u \in H^1(\Omega) \cap L^{\alpha + 1}(\Omega) \) is a stationary positive weak solution of (1.1), then \( E_u(x_0, r) \), defined above, is an increasing function of \( r \).

**Lemma 3.2.** \( E_u(x_0, r) \) is a continuous function of \( x_0 \in \Omega \) and \( r > 0 \).
Now we show the following lemma:

**Lemma 3.3.** There exist $0 < r_0 < 1$ independent of $x_0 \in \Omega$ and some constant $C > 0$ depending only upon $\alpha$, $n$, such that the following inequality holds:

\begin{equation}
(3.1) \quad r^{-\mu} \int_{B(x_0, r)} |\nabla u|^2 dx \leq CE_u(x_0, 2r) \leq CE_u(x_0, r_0) \quad \text{for } r < r_0/2.
\end{equation}

**Proof.** We consider the last one of the three equivalent formulations of $E_u(x_0, r)$ given above. By Lemma 2.3 in [GL] we know that there exists $0 < r_0 < 1$ such that

\begin{equation}
(3.2) \quad E_u(x_0, r) \geq 0 \quad \text{for all } x_0 \in \Omega, \quad 0 < r < r_0,
\end{equation}

and for $r < r_0$,

\begin{equation}
(3.3) \quad \frac{1}{2} \left( \frac{\alpha - 1}{\alpha + 3} \right) e^{Cr r^{-\mu}} \int_{B(x_0, r)} h_1 u^2 dx \leq \frac{1}{2} \left( \frac{\alpha - 1}{\alpha + 3} \right) e^{Cr r^{-\mu}} \int_{B(x_0, r)} u^{2(\alpha + 1)} dx.
\end{equation}

We denote by $\phi(r) = r^{-\mu} \int_{B(x_0, r)} |\nabla u|^2 dx$. Since $E_u(x_0, r)$ is an increasing function of $r$, we integrate it from 0 to $r < r_0$ and obtain that for almost every $x_0 \in \Omega$, (note that $e^{Cr} > 1$)

\begin{equation}
\alpha - 1 \int_0^r \phi(\rho) d\rho + e^{Cr r^{-\mu}} \int_{\partial B(x_0, r)} u^2 ds \leq (\alpha + 3) E_u(x_0, r) r \quad \text{for } r < r_0.
\end{equation}

(Here we have used $\lim_{r \to 0} r^{-\mu} \int_{\partial B(x_0, r)} u^2 ds = 0$ a.e. $x_0 \in \Omega$. This fact is proved in [GL].) Now we use Remark 2 in [Pa1] and we see that there exists some $\sigma \in [r/2, r]$ such that

\begin{equation}
\phi(\sigma) \leq \frac{8}{\pi} \int_0^r \phi(\rho) d\rho \leq CE_u(x_0, r),
\end{equation}

for some constant $C > 0$ depending only upon $\alpha$, $\beta$ and $n$. In addition we have $\phi(r/2) \leq 2^\mu \phi(\sigma)$, if $\sigma \in [r/2, r]$. This gives us the desired result for almost every $x_0$ and, by continuity, for every $x_0$.

**Proposition 3.4.** Assume that there exist $x_0 \in \Omega$ and $0 < r_1 < r_0$ such that $E_u(x_0, r_1) \leq \delta$. Then

\begin{equation}
(3.4) \quad r^{-\mu} \int_{B(y, r)} |\nabla u|^2 dx \leq C\delta,
\end{equation}

for all $y \in B(x_0, r_1/8)$ and $0 < r < r_1/4$, where $C$ only depends upon $n$, $\alpha$, $\beta$. 
Proof. Let \(0 < r < r_1\). We know that for any \(y \in B(x_0, r/2), B(y, r/2) \subset B(x_0, r) \subset B(x_0, r_1)\). Thus,

\[
\int_{B(y, r/2)} |\nabla u|^2 dx \leq \int_{B(x_0, r)} |\nabla u|^2 dx.
\]

Thus, (note that \(e^{Cr} > 1\))

\[
E_u(x_0, r) \geq 2^{-\mu} \left( \frac{\alpha - 1}{2(\alpha + 1)(\alpha + 3)} \right) \left( \frac{r}{2} \right)^{-\mu} \int_{B(x_0, r)} h_2 u_{\alpha+1} dx
\]

\[
+ \left( \frac{\alpha - 1}{\alpha + 3} \right) e^{Cr} r^{-\mu} \left( \frac{1}{2} \int_{B(y, r/2)} |\nabla u|^2 dx - \tilde{C} r^n \right)
\]

\[
+ \frac{1}{(\alpha + 3)} \frac{d}{dr} \left( e^{Cr} r^{-\mu} \int_{\partial B(x_0, r)} u^2 d\sigma \right)
\]

\[
+ Ce^{Cr} r^{-\mu} \left( \frac{1}{4} - \frac{1}{(\alpha + 3)} \right) \int_{\partial B(x_0, r)} u^2 ds + C \int_0^r e^{C\xi} \xi^{\frac{\alpha + 1}{\alpha - 1}} d\xi.
\]

Define \(\psi(r) = \left( \frac{r}{2} \right)^{-\mu} \int_{B(y, r/2)} |\nabla u|^2 dx\). By the argument similar to that in the proof of Lemma 2.3 in [GL], we have, for almost every \(x_0 \in \Omega\),

\[
2^{\mu - 1}(\alpha - 1) \int_0^r \psi(\rho) d\rho \leq (\alpha + 3) E_u(x_0, r_1)r.
\]

Using Remark 2 in [Pa1] again, we see that there exists some \(\sigma \in [r/2, r]\) such that

\[
\psi(\sigma) \leq \frac{8}{r} \int_0^r \psi(s) ds \leq CE_u(x_0, r_1),
\]

for some constant \(C > 0\) only depending upon \(\alpha, \beta\) and \(n\). It is clear that \(\psi(r/2) \leq 2^\mu \psi(\sigma)\). Since

\[
\psi(r/2) = \left( \frac{r}{4} \right)^{-\mu} \int_{B(y, r/4)} |\nabla u|^2 dx,
\]

we have the desired result for almost every \(y \in B(x_0, r_1/8)\). By continuity, we see that it holds for every \(y \in B(x_0, r_1/8)\).

Define

\[
F_u(x_0, r) = r^{-\mu} \int_{B(x_0, r)} |\nabla u|^2 dx + C \int_0^r e^{C\xi} \xi^{(\alpha + 1)/(\alpha - 1)} d\xi,
\]

where \(C \int_0^r e^{C\xi} \xi^{(\alpha + 1)/(\alpha - 1)} d\xi\) is the function in the formulations of \(E_u(x_0, r)\). Then we have the following lemma:

**Lemma 3.5.** We have that

\[
r^{-\mu} \int_{B(x_0, r)} h_2 u_{\alpha+1} dx \leq CF_u(x_0, r) \quad \text{for all } x_0 \in \Omega \text{ and } 0 < r < r_0
\]
and
\[ r^{-\mu} \int_{B(x_0,r)} h_1 u^2 dx \leq C F_u(x_0,r) \quad \text{for all } x_0 \in \Omega \text{ and } 0 < r < r_0, \]
where \( C \) depends only upon \( \alpha, n, a_i \) and \( b_i \) \((i = 1, 2)\).

**Proof.** We only show the first inequality, the second can be obtained by a similar argument. Since \( E_u(x_0,r) \geq 0 \) for all \( x_0 \in \Omega \) and \( 0 < r < r_0 \), it can be seen from the second of the three equivalent formulations given above that
\[ r^{-\mu} \int_{B(x_0,r)} h_2 u^{\alpha+1} dx \leq C \left( F_u(x_0,r) + r^{-\mu-1} \int_{\partial B(x_0,r)} u^2 ds \right), \]
for some constant \( C \) depending only upon \( \alpha, n, a_i \) and \( b_i \) \((i = 1, 2)\). On the other hand, the trace embedding theorem gives
\[ H^1(B(x_0,r)) \hookrightarrow W^{1,2}(\partial B(x_0,r)) \hookrightarrow L^{2(n-1)}(\partial B(x_0,r)). \]
Therefore,
\[ \| u \|_{L^{2(n-1)}(\partial B(x_0,r))} \leq C \| u \|_{H^1(B(x_0,r))}. \]
By Hölder inequality,
\[ r^{-1} \int_{\partial B(x_0,r)} u^2 ds \leq C \left( \int_{\partial B(x_0,r)} u^{\frac{2(n-1)}{n-2}} \right)^{\frac{n-2}{n-1}} \leq C \| u \|^2_{H^1(B(x_0,r))}, \]
so we obtain
\[ r^{-\mu-1} \int_{\partial B(x_0,r)} u^2 ds \leq CF_u(x_0,r). \]
This implies that the first inequality in the lemma holds.

**Theorem 3.6.** There exist constants \( 0 < \epsilon_0, \tau < 1, 0 < r_2 < r_0/4 \), such that
(3.8) \( E_u(x_0,r) \leq \epsilon_0 \)
implies
(3.9) \( F_u(x_0,\tau r) \leq \frac{1}{2} F_u(x_0,r) \) for all \( x_0 \in \Omega \) and \( 0 < r < r_2 \).

**Proof.** It follows from Proposition 3.4 that if \( E_u(x_0,r) \leq \epsilon_0 \), then for \( \eta < r/4 \),
\[ \eta^{-\mu} \int_{B(x_0,\eta)} |\nabla u|^2 dx \leq C \epsilon_0. \]
This implies \( \lim_{\eta \to 0} \eta^{-\mu} \int_{B(x_0,\eta)} |\nabla u|^2 dx = 0 \). (Otherwise we can choose \( \epsilon_0 \) smaller to deduce a contradiction.)
If the result were false, there would exist balls $B(x_k, r_k) \subset \Omega$ with $r_k \to 0$ as $k \to \infty$ such that

\[(3.10) \quad F_u(x_k, r_k) \equiv \lambda_k^2 \to 0,\]

whereas

\[(3.11) \quad F_u(x_k, \tau r_k) > \frac{1}{2} \lambda_k^2,\]

for $\tau > 0$ selected as below. We rescale our variables to the unit ball $B(0, 1) \subset \mathbb{R}^n$ as follows: For $z \in B(0, 1)$, we set

\[(3.12) \quad v_k(z) \equiv \frac{r_k^{2/(\alpha-1)}}{\lambda_k} \left( \frac{u(x_k + r_k z) - a_k}{\lambda_k} \right),\]

where

\[a_k \equiv \frac{1}{|B(x_k, r_k)|} \int_{B(x_k, r_k)} u dy = (u)_{x_k, r_k},\]

$|B(x_k, r_k)| = \text{Vol}(B(x_k, r_k))$ denotes the average of $u$ over $B(x_k, r_k)$, $k = 1, 2, \ldots$)

Using (3.10), (3.11) and (3.12) we have

\[\sup_k \int_{B(0,1)} |v_k|^2 dz < \infty, \quad \sup_k \int_{B(0,1)} |\nabla v_k|^2 dz < \infty,\]

but

\[(3.13) \quad \frac{1}{\tau^\mu} \int_{B(0,\tau)} |\nabla v_k|^2 dz > \frac{1}{2} - e^{C \frac{\alpha^1}{\alpha}} \geq 1/4 \quad (k = 1, 2, \ldots),\]

if we choose $\tau < \left( \frac{1}{4} e^{-C} \right)^{\frac{\alpha^1}{\alpha}}$. In fact, we know that

\[C \int_0^{\tau r_k} e^{C \xi (\alpha+1)/(\alpha-1)} d\xi \leq \frac{CeC(\alpha - 1)}{2\alpha} (\tau r_k)^{\frac{2\alpha}{\alpha - 1}}\]

and since $C \int_0^{\tau r_k} e^{C \xi (\alpha+1)/(\alpha-1)} d\xi < \lambda_k^2$, it holds that

\[\frac{C(\alpha - 1)}{2\alpha} r_k^{\frac{2\alpha}{\alpha - 1}} < \lambda_k^2.\]

Thus, it follows from (3.11) that

\[\lambda_k^2 \left( \frac{1}{\tau^\mu} \int_{B(0,\tau)} |\nabla v_k|^2 dz \right) \geq \left( \frac{1}{2} - e^{C \tau^{\frac{2\alpha}{\alpha - 1}}} \right) \lambda_k^2.\]

The sequence $\{v_k\}_{k=1}^\infty$ is thus bounded in $H^1(B(0,1))$, so there exists a subsequence (still denoted by $\{v_k\}$) such that

\[(3.14) \quad v_k \to v \quad \text{strongly in } L^2(B(0,1))\]

\[(3.15) \quad \nabla v_k \rightharpoonup \nabla v \quad \text{weakly in } L^2(B(0,1)).\]
Choose any function $w \in C_0^\infty(B(0,1))$. Define
\[ w_k(y) \equiv w \left( \frac{y - x_k}{r_k} \right), \quad (y \in B(x_k, r_k)), \]
\[ h_1(y) \equiv \bar{h}_1 \left( \frac{y - x_k}{r_k} \right), \]
\[ h_2(y) \equiv \bar{h}_2 \left( \frac{y - x_k}{r_k} \right). \]
Since $u$ is a weak solution of (1.1), we have
\[
\int_{B(x_k, r_k)} \nabla u \nabla w_k dy = \int_{B(x_k, r_k)} \left( h_1(y)u + h_2(y)u^\alpha \right) w_k(y) dy.
\]
Thus,
\[
\int_{B(0,1)} \nabla v_k \nabla w dy = \int_{B(0,1)} \left[ r_k^2 \bar{h}_1(z) \left( v_k(z) + \frac{a_k}{A_k} \right) + \lambda_k^{-1} \bar{h}_2(z) \left( v_k(z) + \frac{a_k}{A_k} \right)^{\alpha} \right] w(z) dz,
\]
where $A_k = \lambda_k r_k^{-2/(\alpha-1)}$. Since
\[
I_k^1 := r_k^2 \int_{B(0,1)} \bar{h}_1 \left( v_k + \frac{a_k}{A_k} \right) w dz 
\leq r_k^2 \left( \int_{B(0,1)} \bar{h}_1 \left( v_k + \frac{a_k}{A_k} \right)^2 dz \right)^{1/2} \left( \int_{B(0,1)} \bar{h}_1 w^2 dz \right)^{1/2}
\leq r_k^2 \left[ \lambda_k^{-2} r_k^{-2} \left( r_k^{-\mu} \int_{B(x_k, r_k)} h_1 u^2 dx \right) \right]^{1/2} \bar{h}_1 w^2 \| L^2(B(0,1))
\leq Cr_k \| \bar{h}_1 w^2 \| L^2(B(0,1)) \rightarrow 0
\]
(here we used Lemma 3.5) as $k \to \infty$ and
\[
I_k^2 := \lambda_k^{\alpha-1} \left| \int_{B(0,1)} \bar{h}_2 \left( v_k + \frac{a_k}{A_k} \right)^{\alpha} w dz \right|
\leq \lambda_k^{\alpha-1} \left( \int_{B(0,1)} \bar{h}_2 \left( v_k + \frac{a_k}{A_k} \right)^{\alpha+1} dz \right)^{\alpha/(\alpha+1)}
\cdot \left( \int_{B(0,1)} \bar{h}_2 |w|^{\alpha+1} dz \right)^{1/(\alpha+1)}
\leq \lambda_k^{\alpha-1} \left( \lambda_k^{-(\alpha+1)} r_k^{-\mu} \int_{B(x_k, r_k)} h_2 u^{\alpha+1} dx \right)^{\alpha/(\alpha+1)} \| h_2 \| L^{\alpha+1}(B(0,1))
\[ \leq C \Delta_k^{(\alpha-1)/(\alpha+1)} \| \tilde{h}_2^{1/(\alpha+1)} w \|_{L^{\alpha+1}(B(0,1))} \to 0 \]

(here we used Lemma 3.5) as \( k \to \infty \).

Letting \( k \to \infty \) in (3.17), we get

\[ \int_{B(0,1)} \nabla v \nabla w dz = 0. \]  

(3.18)

Hence \( v \) is harmonic function, and hence smooth, and we have the bound

\[ \| \nabla v \|_{L^\infty(B(0,\frac{1}{4}))} \leq \frac{C}{|B(0,1)|} \int_{B(0,1)} v^2 dz < \infty, \]  

(3.19)

where \( |B(0,1)| = \text{Vol}(B(0,1)) \). We will show in next section that

\[ \nabla v_k \to \nabla v \quad \text{strongly in} \quad L^2(B\left(0,\frac{1}{4}\right)) \]  

(3.20)

then we have,

\[ \frac{1}{\tau^\mu} \int_{B(0,\tau)} |\nabla v|^2 dz \leq C \tau^{n-\mu} < \frac{1}{4} \]  

(3.21)

provided \( 0 < \tau < \min\left\{ \left(\frac{1}{4C}\right)^{\frac{\alpha-1}{\alpha+1}}, \left(\frac{\epsilon - C}{4}\right)^{(\alpha+1)/(2\alpha)}, \frac{1}{4} \right\} \), which contradicts (3.13).

4. Compactness.

In this section we turn our attention to (3.20). We choose a smooth cut-off function \( \zeta : \mathbb{R}^n \to \mathbb{R} \) satisfying

\[ 0 \leq \zeta \leq 1, \]

\[ \zeta \equiv 1 \quad \text{on} \quad B\left(0,\frac{1}{4}\right), \]

\[ \zeta \equiv 0 \quad \text{on} \quad \mathbb{R}^n \setminus B\left(0,\frac{5}{16}\right). \]

Lemma 4.1. The sequence \( \{\zeta v_k\}_{k=1}^\infty \) is bounded in \( M^2_{1,\nu}(\mathbb{R}^n, \mathbb{R}) \).

Proof. We first show that for \( 0 < r < r_0 < 1 \),

\[ E_u(x_0,r) \leq CF_u(x_0,r) \quad \text{for all} \quad x_0 \in \Omega. \]  

(4.1)
In fact, it follows from the second of the three formulations of $E_u(x_0, r)$ given above that

\[(4.2)\]

\[
E_u(x_0, r) \leq \frac{1}{2} e^{Cr} r^{-\mu} \int_{B(x_0, r)} |\nabla u|^2 dx + \frac{1}{(\alpha - 1)} e^{Cr} r^{-\mu - 1} \int_{\partial B(x_0, r)} u^2 ds \\
+ C \frac{1}{4} e^{Cr} r^{-\mu} \int_{\partial B(x_0, r)} u^2 ds + C \int_0^r e^{C \xi \xi^{(\alpha+1)/(\alpha-1)}} d\xi.
\]

By the trace embedding theorem and the argument similar to the one used in the proof of Lemma 3.5, we obtain

\[(4.3)\]

\[
r^{-\mu - 1} \int_{\partial B(x_0, r)} u^2 ds \leq CF_u(x_0, r).
\]

Note that $e^{Cr} < e^C$. Our claim can be obtained from (4.2) and (4.3).

Fix any point $z_0 \in B(0, \frac{3}{4})$ and any radius $0 < r < \frac{1}{8}$, set

\[y_k = x_k + r_k z_0 \in B(x_k, \frac{3}{4} r_k).\]

By the claim above and an argument similar to the one used in the proof of Lemma 3.3, we obtain that

\[
\frac{1}{(rr_k)^\mu} \int_{B(y_k, rr_k)} |\nabla u|^2 dy \\
\leq CE_u \left( \frac{r_k}{4} \right) \\
\leq C \left( r_k^{\mu} \int_{B(y_k, \frac{1}{4} r_k)} |\nabla u|^2 dy + C \int_0^{\frac{1}{4} r_k} e^{C \xi \xi^{(\alpha+1)/(\alpha-1)}} d\xi \right) \\
\leq CF_u(x_k, r_k) = C\lambda_k^2.
\]

Rescaling this estimate we obtain,

\[(4.4)\]

\[
r^{-\mu} \int_{B(z_0, r)} |\nabla v_k|^2 dz \leq C
\]

for $k = 1, 2, \ldots$, all $0 < r < \frac{1}{8}$ and $z_0 \in B(0, \frac{3}{4})$. This implies that

\[(4.5)\]

\[
\frac{1}{r^{n - \frac{2}{\alpha - 1}}} \int_{B(z_0, r)} |v_k - (v_k)_{z_0, r}| dz \leq C < \infty
\]

for $k, r$ and $z_0$ as above. This implies $v_k \in L^{1, n - \frac{2}{\alpha - 1}}(B(0, \frac{3}{4}))$ and $L^{1, n - \frac{2}{\alpha - 1}}(B(0, 3/4))$ is a Campanato space (see [Gi]). Since $B(0, \frac{3}{4})$ is type
(A) ([Gi], Chapter III, Definition 1.3), Proposition 1.2 in Chapter III in [Gi] implies that

\[ (4.6) \quad \sup_{z_0 \in B(0,3/4), 0 < r < 1/8} \frac{1}{r^{n-\frac{2}{\alpha}-1}} \int_{B(z_0,r)} |v_k|dz \leq C \sup_{z_0 \in B(0,3/4), 0 < r < 1/8} \frac{1}{r^{n-\frac{2}{\alpha}-1}} \int_{B(z_0,r)} |v_k - (v_k)_{z_0,r}|dz \leq C. \]

Since \( \zeta \) is smooth, then

\[ (4.7) \quad |(\zeta)_{z_0,r} - (\zeta)_{z_0,r}| \leq C \frac{r}{|B(z_0,r)|} \int_{B(z_0,r)} |v_k|dz \quad \text{on } B(z_0,r) \]

for any ball \( B(z_0,r) \). Thus, if \( z_0 \in B(0, \frac{3}{4}) \), \( 0 < r < \frac{1}{8} \), we have,

\[ (4.8) \quad \frac{1}{|B(z_0,r)|} \int_{B(z_0,r)} |\zeta v_k - (\zeta v_k)_{z_0,r}|dz \leq \frac{1}{|B(z_0,r)|} \int_{B(z_0,r)} |v_k - (v_k)_{z_0,r}|dz + \frac{Cr}{|B(z_0,r)|} \int_{B(z_0,r)} |v_k|dz. \]

On the other hand, if \( z_0 \in \mathbb{R}^n \setminus B(0, \frac{3}{4}) \), \( 0 < r < \frac{1}{8} \), we have

\[ (4.9) \quad \int_{B(z_0,r)} |\zeta v_k - (\zeta v_k)_{z_0,r}|dz = 0. \]

It follows from (4.6), (4.8) and (4.9) that

\[ (4.10) \quad \zeta v_k \in \mathcal{L}^{1,n-\frac{2}{\alpha}}(\mathbb{R}^n). \]

This also implies that \( \{\zeta v_k\}_{k=1}^\infty \) is bounded in \( M_{1,\nu}^2(\mathbb{R}^n, \mathbb{R}) \) for \( k = 1, 2, \ldots \).

**Proposition 4.2.** The rescaled functions \( \{\nabla v_k\}_{k=1}^\infty \) converge strongly in \( L^2(B(0, \frac{1}{4})) \).

**Proof.** Subtracting (3.18) from (3.17) we obtain

\[ (4.11) \quad \int_{B(0,1)} (\nabla v_k - \nabla v) \nabla wdz \]

\[ = r_k^2 \int_{B(0,1)} \widetilde{h}_1 \left( v_k + \frac{a_k}{A_k} \right) w + \lambda_k^{\alpha-1} \int_{B(0,1)} \widetilde{h}_2 \left( v_k + \frac{a_k}{A_k} \right)^\alpha wdz \]

for \( w \in C_0^\infty(B(0,1)) \). Hence it holds for \( w \in H_0^1(B(0,1)) \cap L^\infty(B(0,1)) \). We now insert \( w \equiv \zeta^2(v_k - v) \) into (4.11). The left-hand side of (4.11) is

\[ L_k \equiv \int_{B(0,1)} \zeta^2 |\nabla v_k - \nabla v|^2dz + 2 \int_{B(0,1)} \zeta(v_k - v) (\nabla v_k - \nabla v) \nabla \zeta dz \]

\[ \geq \int_{B(0,\frac{1}{4})} |\nabla v_k - \nabla v|^2dz + o(1) \]
as \( k \to \infty \), in view of (3.14) and (3.15). The right-hand side of (4.11) reads

\[
R_k = r_k^2 \int_{B(0,1)} \tilde{h}_1 \left( v_k + \frac{a_k}{A_k} \right) \zeta^2(v_k - v) \, dz \\
+ \lambda_k^{\alpha-1} \int_{B(0,1)} \tilde{h}_2 \left( v_k + \frac{a_k}{A_k} \right) \zeta^2(v_k - v) \, dz \\
= R_1^k + R_2^k.
\]

\[
R_1^k \leq r_k^2 \left( \int_{B(0,1)} \tilde{h}_1 \left( v_k + \frac{a_k}{A_k} \right)^2 \right)^{1/2} \left( \int_{B(0,1)} \tilde{h}_1 \zeta^4(v_k - v)^2 \, dz \right)^{1/2} \\
= C r_k^2 \left( \frac{1}{\lambda_k^{2-n}} \int_{B(x_k,r_k)} h_1 u^2 \, dx \right)^{1/2} \\
\leq C r_k^2 (r_k^{-2})^{1/2} = Cr_k \to 0
\]
as \( k \to \infty \).

Now we show that

\[
(4.12) \quad \zeta \left( v_k + \frac{a_k}{A_k} \right)^\alpha \in H^1_{w}(\mathbb{R}^n)
\]

for \( k = 1, 2, \ldots \). We first consider

\[
M_1 \left( \zeta \left( v_k + \frac{a_k}{A_k} \right)^\alpha \right) (z) := \sup_{t > 0} \frac{1}{t^n} \int_{\mathbb{R}^n} (\zeta^{1/\alpha} f_k)^\alpha(y) \phi \left( \frac{y - z}{t} \right) \, dy
\]

where \( f_k(y) := v_k(y) + \frac{a_k}{A_k} \), \( \phi \) is a Schwartz function with nonvanishing integral (see [ST]).

If \( t \geq 1 + \frac{|z|}{4} \), we have

\[
\frac{1}{t^n} \int_{\mathbb{R}^n} (\zeta^{1/\alpha} f_k)^\alpha(y) \phi \left( \frac{y - z}{t} \right) \, dy \\
\leq \frac{1}{t^n} \int_{B(0,1)} (\zeta^{1/\alpha} f_k)^\alpha(y) \phi_t \, dy \\
\leq \frac{1}{t^n} \left( \int_{B(0,1)} (\zeta^{1/\alpha} f_k)^{\alpha+1} \, dy \right)^{\alpha/(\alpha+1)} \left( \int_{B(0,1)} \phi_t^{\alpha+1} \, dy \right)^{1/(\alpha+1)} \\
\leq C \frac{1}{t^n} \left[ \frac{1}{\lambda_k^{(\alpha+1)}} r_k^{-\mu} \int_{B(x_k,r_k)} u^{\alpha+1} \, dx \right]^{\alpha/(\alpha+1)} \\
\leq C \frac{\alpha(1-\alpha)}{t^n} \frac{\alpha(1-\alpha)}{\alpha+1} \\
\leq \frac{C}{(4 + |z|)^n} \lambda_k^{\alpha(1-\alpha)/\alpha+1}.
\]
Therefore,
\[
\int_{\mathbb{R}^n} M_1 \left( \zeta \left( v_k(z) + \frac{a_k}{A_k} \right)^\alpha \right) w(z) dz \\
= \int_{\mathbb{R}^n} M_1 \left( \zeta \left( v_k(z) + \frac{a_k}{A_k} \right)^\alpha \right) |z|^{-2/(\alpha-1)} dz \\
\leq C \lambda_k^{\frac{\alpha(1-\alpha)}{\alpha+1}} \int_{\mathbb{R}^n} |z|^{-2/(\alpha-1)} (4 + |z|)^{-n} dz \\
\leq C \lambda_k^{\frac{\alpha(1-\alpha)}{\alpha+1}}.
\]

If \( t < 1 + \frac{|z|}{t} \), we have, if \(|y - z| < t\), then \(|y - z| < 1 + \frac{|z|}{t} \) and \(|y| > \frac{3}{4}|z| - 1\). Therefore, if \(|z| > 8/3\), then \(|y| > 1\). Thus, for \(0 < \epsilon < 1\) and \(z \in B(0, 3)\),
\[
\frac{1}{t^n} \int_{\mathbb{R}^n} \left( \zeta^{1/\alpha} f_k \right)^\alpha \phi \left( \frac{y - z}{t} \right) dy \\
\leq \frac{1}{t^n} \left( \int_{B(z, t)} \left( \zeta^{1/\alpha} f_k \right)^{\alpha+1-\epsilon} \phi_t \right)^{\alpha/(\alpha+1-\epsilon)} \\
\cdot \left( \int_{B(z, t)} \phi_t^{(\alpha+1-\epsilon)/\alpha-\epsilon} dy \right)^{(1-\epsilon)/(\alpha+1-\epsilon)} \\
\leq C \left( \frac{M}{(\zeta^{1/\alpha} f_k)^{\alpha+1-\epsilon}} \right)^{\alpha/(\alpha+1-\epsilon)},
\]
where \( M(\cdot) \) is the Hardy-Littlewood maximal function. If \(z \in \mathbb{R}^n \setminus B(0, 3)\),
\[
\frac{1}{t^n} \int_{\mathbb{R}^n} \left( \zeta^{1/\alpha} f_k \right)^\alpha \phi_t dy = 0.
\]
Therefore,
\[
\int_{\mathbb{R}^n} M_1 \left( \zeta \left( v_k(z) + \frac{a_k}{M_k} \right)^\alpha \right) |z|^{-2/(\alpha-1)} dz \\
\leq C \int_{B(0,3)} \left[ M \left( \left( \zeta^{1/\alpha} f_k \right)^{\alpha+1-\epsilon} \right)^{\alpha/(\alpha+1-\epsilon)} |z|^{-2/(\alpha-1)} dz \\
\leq C \left( \int_{B(0,3)} \left[ M \left( \left( \zeta^{1/\alpha} f_k \right)^{\alpha+1-\epsilon} \right)^{\alpha+1}/(\alpha+1-\epsilon) \right] dz \right)^{\alpha/(\alpha+1)} \\
\cdot \left( \int_{B(0,3)} |z|^{-\frac{2+\alpha}{\alpha-1}} dz \right)^{1/(\alpha+1)} \\
\leq C \left( \int_{B(0,3)} \left( \zeta^{1/\alpha} f_k \right)^{\alpha+1} dz \right)^{\alpha/(\alpha+1)} \left( \int_0^3 r^{\alpha-1} dr \right)^{1/(\alpha+1)}
\]
where we used the facts that \( \mu > 0 \) if \( \alpha > \frac{n}{n-2} \), and
\[
\int_{B(0,3)} \left[ M\left(\left(\zeta^{1/\alpha} f_k\right)^{\alpha+1}\right) \right]^{(\alpha+1)/(\alpha+1-\epsilon)} \, dz \\
= \int_{\mathbb{R}^n} \left[ M\left(\left(\zeta^{1/\alpha} f_k\right)^{\alpha+1}\right) \right]^{(\alpha+1)/(\alpha+1-\epsilon)} \, dz \\
\leq \int_{\mathbb{R}^n} \left(\zeta^{1/\alpha} f_k\right)^{\alpha+1} \, dz,
\]

because
\[
M\left(\left(\zeta^{1/\alpha} f_k\right)^{\alpha-1+\epsilon}\right)(z) \equiv 0 \quad \text{for} \quad z \in \mathbb{R}^n \setminus B(0,3).
\]

It concludes that \( \left(\zeta^{1/\alpha} f_k\right)^\alpha \in H^1_{w}(\mathbb{R}^n) \). Therefore, it follows from (2.1) that
\[
R_k^2 \leq \lambda_k^{\alpha-1} \int_{\mathbb{R}^n} M_1\left(\left(\zeta^{1/\alpha} f_k\right)^\alpha\right) M_{1,\nu}^2(\zeta(v_k - v)) |z|^{-2/(\alpha-1)} \, dz \\
\leq C \lambda_k^{\alpha-1} \int_{\mathbb{R}^n} M_1\left(\left(\zeta^{1/\alpha} f_k\right)^\alpha\right) |z|^{-2/(\alpha-1)} \, dz \\
\leq C \lambda_k^{\alpha-1} \lambda_k^{\frac{\alpha}{\alpha+1}} \\
= C \lambda_k^{\frac{\alpha}{\alpha+1}} \to 0
\]
as \( k \to \infty \).

5. Proof of Theorem A.

In this section we shall prove Theorem A. We recall the definition of function space \( L^{p,q}(\Omega) \):
\[
L^{p,q}(\Omega) = \left\{ v \in L^p(\Omega) : \sup_{x \in \Omega, r > 0} r^{-q} \int_{B(x,r) \cap \Omega} u^p \, dx < +\infty \right\}.
\]
This is called Morrey space (see [Gi]). Now we recall a theorem in [Pa2].

**Theorem 5.1.** Let \( u \) be a positive weak solution of (1.1), assume that \( u \in L^{\alpha,\lambda+\theta}(\Omega) \) for \( \lambda = n - \frac{2\alpha}{\alpha-1} \) and some \( \theta > 0 \) then \( u \) is regular in \( \Omega \).

Note that Pacard [Pa2] proved this theorem only for the case that \( h_1 \equiv 0 \) and \( h_2 \equiv 1 \) in \( \Omega \), but we can easily see from his proof that this theorem still holds in our case.

Set
\[
V \equiv \{ x \in \Omega : E_u(x,r) < \epsilon_0 \quad \text{for some} \quad 0 < r < r_2 \},
\]
where $\epsilon_0$ and $r_2$ are constants in Theorem 3.6. Furthermore, using Theorem 3.6, we can show that (cf. [Gi]), if $x \in V$, there exists $r^* > 0$ sufficiently small such that
\[(5.1) \quad F_u(y, r) \leq Cr^\gamma\]
for some $0 < \gamma < \frac{2\alpha}{\alpha - 1}$, $C > 0$, all $y$ near $x$, and all sufficiently small radii $0 < r < r^*$. It follows from Lemma 3.5 that
\[(5.2) \quad r^{-\mu} \int_{B(x_0, r)} u^{\alpha+1} dx \leq CF_u(x_0, r)\]
for all $x_0 \in \Omega$ and $0 < r < r_0$. Note that $\gamma < \frac{2\alpha}{\alpha - 1}$, by (5.1) and (5.2), we have
\[(5.3) \quad r^{-\mu} \left( \int_{B(y, r)} (|\nabla u|^2 + u^{\alpha+1}) dx \right) \leq Cr^\gamma\]
for all $y$ near $x$, and $0 < r < r^*$. Now we show that $u \in L^{\alpha,\lambda+\theta_0}(B(x, r^*/2))$ for some $\theta_0 > 0$. In fact, choosing $\theta_0 = \frac{\alpha\gamma}{\alpha+1}$, we have, for $0 < r < r^*$,
\[
r^{-\mu(\alpha+1) - \theta_0} \int_{B(x, r)} u^{\alpha} dy \leq r^{-\mu(\alpha+1)} \left( \int_{B(x, r)} u^{\alpha+1} dy \right)^{\alpha/(\alpha+1)} r^{1/(\alpha+1)} \leq C.
\]
This implies (5.4) and therefore, by Theorem 5.1, $u$ is regular at $x$. Hence $u$ is regular in $V$.

Define $\Sigma = \Omega \setminus V$. Then
\[
\Sigma \equiv \cap_{r>0} \{ x \in \Omega : E_u(x, r) \geq \epsilon_0 \}.
\]
It is proved in [GL] that
\[(5.5) \quad \Sigma \subset \cap_{r>0} \left\{ x \in \Omega : \int_{B(x, r)} \left( u^{\alpha+1} + |\nabla u|^2 \right) dy \geq C\epsilon_0 r^\mu \right\}.
\]
Thus, standard covering arguments imply that the Hausdorff dimension of $\Sigma$ is less than $n - 2\frac{n+1}{\alpha - 1}$. This completes the proof of Theorem A.

**Remark.** The conclusion of Theorem A still holds for the stationary positive weak solutions of the equation
\[
\Delta u + h_1(x) u^\kappa + h_2(x) u^\alpha = 0 \quad \text{in } \Omega
\]
where $0 < \kappa < \alpha$, $\alpha > \frac{n+2}{n-2}$. It should be very interesting to know whether our partial regularity theorem holds for the equation
\[
\Delta u + h(x) f(u) = 0 \quad \text{in } \Omega
\]
where $f(s)$ satisfies that $f(s) > 0$ for $s > 0$ and $f$ has the growth rate $\alpha > \frac{n+2}{n-2}$. The main difficulty is how to establish the monotonicity inequality.
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