Pacific Journal of Mathematics

SPECIALIZATION OF POLYNOMIAL COVERS OF PRIME DEGREE

Leonardo Zapponi

Volume 214 No. 1 March 2004

PACIFIC JOURNAL OF MATHEMATICS
Vol. 214, No. 1, 2004

SPECIALIZATION OF POLYNOMIAL COVERS OF PRIME DEGREE

Leonardo Zapponi

Let K be a complete discrete valued field of unequal characteristic $(0, p)$. The aim of this paper is to describe the semi-stable models for covers $\mathbf{P}^1_K \to \mathbf{P}^1_K$ of degree p, unramified outside $r \le p$ points and totally ramified above one of them, under the assumtion that the ramification locus has a particular reduction type (which always occurs if $r \le 4$). We are principally concerned with the minimal semi-stable models which separate the ramified fibers.

1. Introduction.

The study of the stable models for covers between algebraic curves defined over a complete discrete valuation ring is a subject which has been intensively developed during these last years. Roughly speaking, the problem can be summarized as follows: Starting from a ramified cover $\beta : C \to D$ between nonsingular projective curves defined over a complete discrete valued field (K, v) of unequal characteristic, one would like to construct a semi-stable model $\mathcal{C} \to \mathcal{D}$ of this cover over its valuation ring R. The general theory, and in particular the Semi-Stable Reduction Theorem, asserts that such a model always exists, up to a finite extension of the base field (cf. [**1**] and [**4**]). Its uniqueness is not ensured. If we require that the model separates the ramified locus $\mathcal{R}$ (that is, the ramification points specialize to pairwise distinct points) and that the curve $C - \mathcal{R}$ is hyperbolic, then there exists a minimal semi-stable model for the cover, which is unique, up to isomorphism. Moreover, this model behaves well under (finite) base change, so that it can be considered as an important birational invariant attached to the cover. In particular, there is a well-defined notion of reduction type: The number of irreducible components of the special fiber, their intersection graph and the thicknesses of the singular points (cf. Section 2).

Historically, the investigation started with the study of prime to p covers i.e., covers such that the residual characteristic p of R does not divide the order of their monodromy group G. Even if this is the simplest situation, there are no complete results. For example, there are no general criteria which can decide whether or not the cover has good reduction. The current research trend is concerned with the case where p divides some ramification indices. M. Raynaud [5] treats the case where G has a p-Sylow subgroup of order p and the cover is unramified outside three points. These results are sharpened by S. Wewers in [7]. Finally, C. Lehr [2] and M. Saïdi [6], study the semi-stable model for a p-cyclic cover in full generality. In order to have a more complete reference on the subject, see also the results of B. Green, M. Matignon and Y. Henrio.

In this paper, we concentrate on covers $\mathbf{P}^1 \to \mathbf{P}^1$ of prime degree p. The only restrictions we make are the following: There is at least one totally ramified point (wild ramification) and the branch locus has a particular reduction type (cf. Section 6). Note that these covers need not to be Galois. It turns out that the semi-stable models essentially depend only on the ramification data and on the v-adic distance between the branch points. There are four cases: If the branch locus has good reduction, we essentially recover Wewers' results. The real interesting case is where the branch points have bad reduction. We choose two branch points P and Q and consider the semi-stable model $\mathcal{C} \to \mathcal{D}$ of the cover which separates these points. There are three cases: P and Q are v-adically far from each other, P and Q have medium distance from each other (the critical case) and P and Q are v-adically close to each other. In the first case, the specializations of P and Q belong to two different irreducible components D_1 and D_2 of the special fiber of $\mathcal{D}$ and there is a unique irreducible component lying above each D_i. In the second case, P and Q specialize in the same irreducible component and there is one component above it. In the last case, P and Q specialize again in the same component but many irreducible component can lie over it. This last case is more delicate and the knowledge of the ramification data and of the p-adic distances between the branch points is not sufficient to completely determine the semi-stable model. Nevertheless, all the possible cases occurring are classified.

The strategy is to start with a natural model for the cover $\mathbf{P}^1 \to \mathbf{P}^1$ over R and then to blow it up several times until the ramification points are separated. This construction can be carried out explicitly because the covering curve is the projective line. This technique can be extended to the case where the degree of the cover is composite and even to the case where the curve C has (potentially) good reduction, but we do not do it in this paper (see [8] for more details).

2. Some notation and basic facts.

Throughout this paper, (K, v) denotes a complete valued field of unequal characteristic $(0, p)$. Its valuation ring R is a complete discrete valuation ring with maximal ideal $\mathfrak{p}$ and residue field $k = \mathcal{O}_K/\mathfrak{p}$. The valuation $v : K^* \to \mathbf{Q}$ is normalized by the condition $v(p) = 1$, so that it extends the usual p-adic valuation. In particular $v(K^*) = e^{-1}\mathbf{Z}$, where the positive integer e is the absolute ramification index of K, that is $pR = \mathfrak{p}^e$.

Let C be a projective, geometrically irreducible, nonsingular curve of genus g defined over K and consider a finite subset S of $C(K)$ of cardinality $n \geq 0$ (for $n = 0$, we simply set $S = \emptyset$). The pair (C, S) is called a *pointed curve of type* (g, n). We say that (C, S) is *hyperbolic* if the inequality $2g - 2 + n > 0$ holds. A *semi-stable model* for (C, S) over R is a proper and flat scheme $\mathcal{C}/R$ satisfying the following conditions:

(1) The generic fiber $\mathcal{C}_K = \mathcal{C} \otimes_R K$ is isomorphic to C.
(2) The special fiber $\mathcal{C}_k = \mathcal{C} \otimes_R k$ is reduced and has only ordinary double points as singularities.
(3) The elements of S specialize to pairwise distinct nonsingular points of $\mathcal{C}_k$.

If $S = \emptyset$ then this last condition must be ignored. We say that $\mathcal{C}$ is *stable* if, for any irreducible component X of $\mathcal{C}_k$, we have the relation $2g(X) - 2 + n(X) + s(X) > 0$, where $g(X)$ denotes the genus of X, $n(X)$ is the cardinality of the subset of S specializing in X and $s(X)$ is the number of singular points of X. Semi-stable models behave well under base change, i.e., if L/K is a finite extension and $\mathcal{C}/R$ is a semi-stable model of (C, S) over R then $\mathcal{C} \otimes_R R'$ is a semi-stable model for (C, S) over R', where R' is the ring of integers of L.

The Semi-Stable Reduction Theorem asserts that any pointed curve (C, S) over K admits a semi-stable model over the valuation ring R' of a finite extension L of K. Moreover, if (C, S) is hyperbolic then there exists a stable model $\mathcal{C}/R'$ which is unique up to R'-isomoprhism. The same kind of result holds for covers: More precisely, given a finite cover $\beta : C \to D$ between projective, nonsingular curves over K and a finite subset S of $D(K)$, there exist a finite extension L of K, two proper and flat schemes $\mathcal{C}, \mathcal{D}/R'$ and a finite morphism $\mathcal{C} \to \mathcal{D}$ such that the following conditions hold:

(1) $\mathcal{C}$ (resp. $\mathcal{D}$) is a semi-stable model for the pointed curve $(C, \beta^{-1}(S))$ (resp. for the pointed curve (D, S)) over R'.
(2) The generic morphism $\mathcal{C}_L \to \mathcal{D}_L$ is isomorphic to $\beta : C \to D$.
(3) The morphism $\mathcal{C}_l \to \mathcal{D}_l$ maps smooth (resp. singular) points to smooth (resp. singular) points (here, l denotes the residue field of L).

Remark 2.1. In the literature, the existence of a model satisfying the above properties is ensured only in the Galois case. Since in this paper we are working with covers which are almost never Galois, we now briefly sketch how to extend this result to the general case: Suppose that we have a cover $\beta : C \to D$. We can consider its Galois closure $\widehat{C} \to D$. Denote by G its Galois group. Then there exists a subgroup H of G such that $\widehat{C}/H = C$. Take a semi-stable model $\widehat{\mathcal{C}}$ of $\widehat{C}$ such that the elements of $\widehat{C}$ with nontrivial stabilizer (the ramified points) specialize to pairwise distinct smooth points of $\widehat{\mathcal{C}}_k$. Up to some blow-ups, we can assume that the action of G on $\widehat{C}$ extends to $\widehat{\mathcal{C}}$ and that G acts with no invertions (cf. [5]). The key result is Proposition 5 in [3] which asserts that the quotient of a semi-stable curve under the action of a finite group is again semi-stable. In this case, the curves $\mathcal{C} = \widehat{\mathcal{C}}/H$ and $\mathcal{D} = \widehat{\mathcal{C}}/G$ are semi-stable and the morphism $\mathcal{C} \to \mathcal{D}$ is the model we are looking for.

In particular, there exists a *minimal semi-stable model* satisfying the above properties, which is unique up to R'-isomorphism. Since the behaviour of its special fiber is stable under finite base change, it can be considered as an important birational invariant of the cover. In this paper, we are not concerned with rationality questions, that's why we always assume that the semi-stable model is already defined over R.

We close this section by introducing an important invariant attached to a semi-stable model $\mathcal{C}/R$: The completion of the local ring of $\mathcal{C}$ at a singular point P of its special fiber is isomorphic to the power series ring $R[\![X,Y]\!]/(XY - Z)$, with $Z \in \mathfrak{p}$. The *thickness* of P is the valuation of Z, which only depends on P. Moreover, our normalization for the valuation $v : K^* \to \mathbf{Q}$ implies that the thickness of a singular point does not change after a finite base change. Finally, if $\beta : \mathcal{C} \to \mathcal{D}$ is a semi-stable model for a cover and P is a singular point of the special fiber $\mathcal{C}_k$ of $\mathcal{C}$ of thickness ν, then $Q = \beta(P)$ is a singular point of the special fiber $\mathcal{D}_k$ of $\mathcal{D}$, of thickness $e_P\nu$, where e_P a positive integer. We refer to it as the *ramification index* at P of the cover $\mathcal{C}_k \to \mathcal{D}_k$.

The book [1] is a complete introduction to the theory of semi-stable curves and their application to the study of algebraic covers of curves. Two fundamental papers on the subject are [4] and [5].

3. Polynomial covers. Normalized models.

The main purpose of this paper is to construct semi-stable models for *polynomial covers* of prime degree p over K (i.e., covers $\beta : \mathbf{P}^1_K \to \mathbf{P}^1_K$ of degree p and totally ramified above one point) under certain assumptions on the reduction of the branch locus. We are principally concerned with the minimal model $\mathcal{X} \to \mathcal{Y}$ which separates the ramification locus. Two such covers β_1 and β_2 are *isomorphic, or equivalent* (*over* K) if there exist two elements $\sigma, \tau \in \mathrm{PGL}_2(K)$ such that $\sigma \circ \beta_1 = \beta_2 \circ \tau$. In this case, we easily see that the special fibers of the stable models of the covers β_1 and β_2 have the same behaviour. We start by constructing a "good" model of the cover over K. Let's first of all reduce to the affine case: Since $\beta : \mathbf{P}^1_K \to \mathbf{P}^1_K$ is totally ramified above one point, we obtain a finite morphism

$$\mathbf{A}^1_K = \mathrm{Spec}\,(K[X]) \xrightarrow{\beta} \mathbf{A}^1_K = \mathrm{Spec}\,(K[T])\,.$$

In terms of affine algebras, it corresponds to an injection $\beta^* : K[T] \to K[X]$, which is uniquely determined by the image $\beta(X)$ of T, which is a polynomial (that's why we are speaking about polynomial covers). From now on, by eventually enlarging K, we assume that the ramification locus of β is contained in $\mathbf{P}^1(K)$. For any $\lambda \in K$, we have a unique factorization (in $\overline{K}$)

$$\beta(X) - \lambda = c \prod_{x \in \beta^{-1}(\lambda)} (X - x)^{e_x}$$

where $c \in K$ does not depend on λ and e_x is the multiplicity of $X - x$ in the factorization of $\beta(X) - \lambda$. We clearly have

$$\sum_{x \in \beta^{-1}(\lambda)} e_x = p.$$

Furthermore, the Riemann-Hurwitz formula gives

$$\sum_{\lambda \in S} n_\lambda = (r-2)p + 1$$

where r is the cardinality of the branch locus $B = S \cup \{\infty\}$ of the cover and n_λ is the cardinality of the fiber $\beta^{-1}(\lambda)$. Up to equivalence, and after a finite extension of K, we can reduce to the following case:

(1) The finite branch locus $S = \{\lambda_1, \dots, \lambda_{r-1}\} = \beta\left(\{x \in K \mid \beta'(x) = 0\}\right)$ is contained in R and $\{0, 1\} \subset S$.
(2) The polynomial $\beta(X)$ is monic (i.e., $c = 1$) and $\beta(0) = 0$.

A polynomial $\beta(X) \in K[X]$ satisfying the above conditions is a *normalized model* for the cover. One easily checks that there exist finitely many normalized models associated to the same polynomial cover. More generally, we say that $\beta(X)$ is a *semi-normalized* model if it satisfies Condition (2) above and the following property (which is weaker than Condition (1)):

(1') The finite branch locus S is contained in R and there exists $\lambda \in R^*$ such that $\{0, \lambda\} \subset S$.

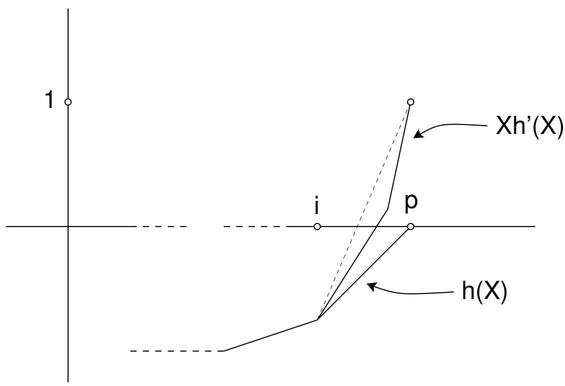
Note that there exist infinitely many semi-normalized models associated to the same cover.

4. Construction of the fundamental model.

We now start the construction of the semi-stable model associated to a polynomial cover of degree p. For any polynomial $h(X) \in R[X]$, we denote by $\overline{h}(X)$ its canonical image in $k[X]$. The importance of the (semi-) normalized models introduced in the previous paragraph is related to the following lemma:

Lemma 4.1. *Let $h(X) \in K[X]$ be a monic polynomial of degree p such that $h(0) = 0$. If the associated cover $h : \mathbf{A}^1_K \to \mathbf{A}^1_K$ is unramified outside a finite set $S \subset R$, then $h(X) \in R[X]$ and $\overline{h}(X) = X^p$.*

Proof. The case $h(X) = X^p$ is immediate. If $h(X) \neq X^p$ then its Newton polygon is not reduced to a vertical line. Suppose that the last segment has positive slope, i.e., that there exists a root x of $h(X)$ with (minimal) negative valuation. The degree of $h(X)$ is equal to p and $h(0) = 0$, so that its Newton polygon coincides with the Newton polygon associated to $Xh'(X)$, exept for the last segment (see the following picture).



In particular, we see that there is a root y of $h'(X)$ (a ramified point) such that its valuation is strictly less than the valuation of x (and thus, of any root of $h(X)$). We obtain $v(h(y)) = pv(y) < 0$, which is absurd since $h(y) \in S \subset R$. We then have $h(X) \in R[X]$ and since the (finite) branch points belong to R, we see that all the roots of $h'(X)$ belong to R. The leading coefficient of $h'(X)$ is equal to p, so that $h'(X) = pg(X)$, with $g(X) \in R[X]$. This gives $\overline{h}'(X) = 0$, which leads to $\overline{h}(X) = X^p$, since $h(X)$ is monic, of degree p and satisfies $h(0) = 0$. □

If $\beta(X) \in K[X]$ is a (semi-)normalized model associated to a polynomial cover of degree p then it satisfies the hypothesis of Lemma 4.1 so that it belongs to $R[X]$ and we have the identity $\overline{\beta}(X) = X^p$. This is our starting point for the construction of the semi-stable model associated to the cover. Let $\mathcal{D} = \mathbf{P}^1_R$ be the projective line over R, viewed as the union of the affine schemes $U_1 = \mathrm{Spec}(R[T])$ and $U_2 = \mathrm{Spec}(R[T^{-1}])$. Denote by $\mathcal{C}$ the integral closure of $\mathcal{D}$ in the function field $K(X)$ via the injection $K(T) \hookrightarrow K(X)$ which sends T to $\beta(X)$. By construction, we obtain a finite morphism

$$\mathcal{C} \xrightarrow{\beta} \mathcal{D}.$$

We have to show that $\mathcal{C}$ is a semi-stable curve. In fact, we will now prove that it is a smooth R-curve. We just have to desribe the process of normalization on the two patches U_1 and U_2. The schemes $V_i = \beta^{-1}(U_i)$ are affine (since β is finite). We have $V_1 = \mathrm{Spec}(A)$, where A is the integral closure of $R[T]$ in $K(X)$. We then find $A = R[X]$ and the map $R[T] \to R[X]$ sends T to $\beta(X)$. Indeed, $R[X]$ is integral over $R[T]$ (because $\beta(X)$ is monic) and its field of fractions if $K(X)$. Similarly, we have $V_2 = \mathrm{Spec}(B)$, where B is the integral closure of $R[T^{-1}]$ in $K(X)$. Let $\gamma(X) \in R[X]$ be the polynomial defined by the relation $\gamma(X) = X^p \beta(X^{-1})$. From a practical point of view, we have

$$\gamma(X) = \prod_{x \in \beta^{-1}(0)} (1 - xX)^{e_x}$$

where $e_x = e_x(\beta)$ is the multiplicity of $X - x$ in the factorization of $\beta(X)$. The relation $\beta(0) = 0$ implies that the degree of $\gamma(X)$ is strictly less than p. Moreover, we have $\gamma(0) = 1$ and $\overline{\gamma}(X) = 1$. With this notation, the element $T^{-1} \in K(T)$ is sent to $\beta(X)^{-1} = \frac{X^{-p}}{\gamma(X^{-1})} \in K(X)$. In particular, the ring

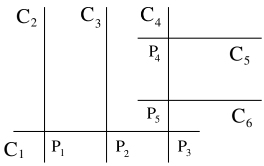
$$B' = R[T^{-1}, X^{-1}]/(X^{-p} - \gamma(X^{-1})T^{-1})$$

is a finite extension of $R[T^{-1}]$. One easily checks that B', considered as a subring of $K(X)$, coincides with $R[X^{-1}, \gamma(X^{-1})^{-1}]$. We then obtain $B' = B$ since B' is integrally closed and has $K(X)$ as field of fractions. This shows that $\mathcal{C}$ is smooth and thus isomorphic to $\mathbf{P}^1_R$.

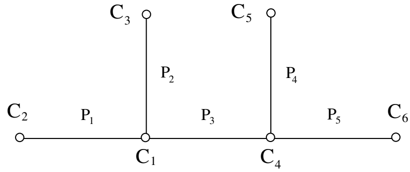
The morphism $\mathcal{C} \to \mathcal{D}$ is a semi-stable model for the cover over R (since its generic fiber is isomorphic to β over K) and we refer to it as the *fundamental model* associated to β. Lemma 4.1 implies that its specialization is purely inseparable. In particular, for a given $\lambda \in \mathcal{D}(K)$, the elements of $\beta^{-1}(\lambda)$ all have the same specialization in the special fiber $\mathcal{C}_k$ of $\mathcal{C}$, so that this morphism never separates the ramified fibers. A model $\mathcal{X} \to \mathcal{Y}$ for β which separates the ramified fibers is obtained from $\mathcal{C} \to \mathcal{D}$ after a finite number of blow-ups. In order to get a minimal one, we may need to blow-down the original irreducible component $\mathcal{C}_k$ (and $\mathcal{D}_k$ of the special fiber of $\mathcal{Y}$). This last possibility never occurs. Indeed, this would mean that all the finite branch points have the same specialization in D, which is impossible, since we are assuming $\{0, 1\} \in S$.

5. Some complements on semi-stable models of genus zero.

Before continuing, let's make some general remarks. First of all, suppose that $\mathcal{X}/R$ is any semi-stable model of a genus zero (pointed) curve and denote by $\mathcal{X}_k$ its special fiber. We classically define the *intersection graph* $\Gamma(\mathcal{X}_k)$ as the abstract graph whose vertices (resp. edges) correspond to the irreducible components of $\mathcal{X}_k$ (resp. the singular points of $\mathcal{X}_k$).

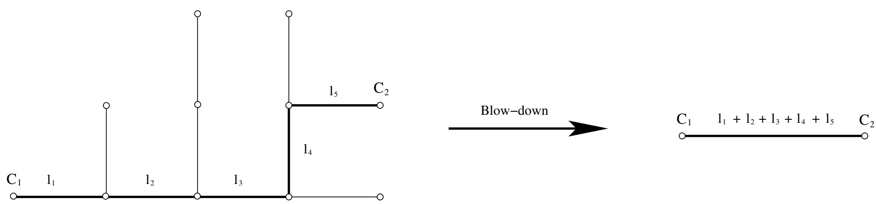


Special fiber



Associated graph

In this special case, $\Gamma(\mathcal{X}_k)$ is a tree endowed with a *metric*, obtained by associating to any edge the thickness of the corresponding singular point. If C_1 and C_2 are two irreducible components of $\mathcal{X}_k$, we can consider the *segment* $[C_1, C_2]$, which is the minimal subtree of $\Gamma(\mathcal{X}_k)$ containing the vertices corresponding to C_1 and C_2. In particular, the distance $d(C_1, C_2)$ between C_1 and C_2 is defined as the sum of the lengths of the edges belonging to $[C_1, C_2]$. From a practical point of view, consider the model $\mathcal{X}'$ of the projective line over R obtained by blowing down all the irreducible components of $\mathcal{X}_k$ different from C_1 and C_2. Then the special fiber of $\mathcal{X}'$ is the union of the curves C_1 and C_2, meeting at a unique singular point of thickness $d(C_1, C_2)$.



Suppose now that $\mathcal{X} \to \mathcal{Y}$ is any semi-stable model over R for a polynomial cover, obtained from its fundamental model $\mathcal{C} \to \mathcal{D}$ (cf. §4) after a finite number of blow-ups. Denote by C (resp. by D) the irreducible component of $\mathcal{X}_k$ (resp. of $\mathcal{Y}_k$) corresponding to the special fiber of $\mathcal{C}$ (resp. of $\mathcal{D}$). Let $C_1 \neq C$ be a tail of $\mathcal{C}_k$ (i.e., an irreducible component having only one singular point) and denote by $D_1 \neq D$ its image in $\mathcal{D}_k$. Let C_∞ be the irreducible component of $\mathcal{X}_k$ containing the specialization of the totally ramified point ∞. We suppose that the segments $[C, C_1]$ and $[C, C_\infty]$ have no common edges. Consider a point $P \in \mathcal{X}(K) = \mathbf{P}^1(K)$ specializing to a nonsingular point of C_1. Since $\mathcal{X}(K) = \mathcal{C}(K)$, the point P defines a well-defined element x of $\mathcal{C}(K) = \mathcal{C}(R)$. Moreover, the above assumption on the relative positions of C, C_0 and C_∞ implies that x belongs to $\mathcal{C}_0(R) = \mathbf{A}^1(R)$ (cf. the end of §4) and thus, it can be viewed as an element of R. Similarly the image of P in $\mathcal{Y}$ defines an element λ of $\mathcal{D}_0(R) = R$. Let π and π' be two elements of $\mathfrak{p}$ such that $v(\pi) = d(C, C_1)$ and $v(\pi') = d(D, D_1)$. We then have a unique factorization

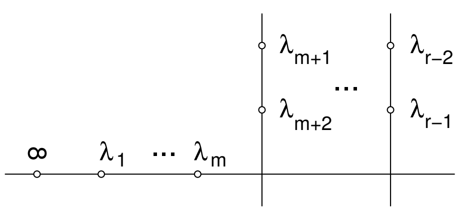
$$\beta(\pi X + x) - \lambda = \pi' \beta_0(X)\gamma_0(X) \tag{1}$$

where $\beta(X) \in R[X]$ is the normalized model from which the cover $\mathcal{C} \to \mathcal{D}$ was constructed, $\beta_0(X) \in R[X]$ is monic and $\gamma_0(X) \in R[X]$ satisfies $\overline{\gamma}_0(X) \in k^*$, i.e., $\gamma_0(X) \in R^* + \mathfrak{p}R[X]$. Moreover, the finite morphism $\mathbf{A}^1_k \to \mathbf{A}^1_k$ obtained from $C_1 \to D_1$ by removing the singular points is (isomorphic to the one) induced by the inclusion $k[T] \to k[X]$ which maps T to $\overline{\beta}_0(X)$. In particular, C_1 is the only irreducible component of $\mathcal{X}_k$ lying above D_1 if and only if $\gamma_0(X)$ has degree 0. In this case, we get the relation $d(D, D_1) = pd(C, C_1)$.

6. Simple reduction of the branch locus.

We now put some conditions on the reduction type of the branch locus $B = S \cup \{\infty\}$ of a polynomial cover over K. The curve $\mathcal{D}$ of the fundamental model associated to it (cf. §4) is a model of the projective line over R such that no point of S specialize to ∞ and at least two points have distinct specializations. These two conditions uniquely determine $\mathcal{D}$, up to R-isomorphism. The stable model $\mathcal{B}$ (over R) for the pointed curve $(\mathbf{P}^1_K, B)$ is obtained from $\mathcal{D}$ after a finite number of blow-ups.

Definition 6.1. Consider the stable (separating) model $\mathcal{B}$ associated to a pointed curve $(\mathbf{P}^1_K, B)$, where $B = S \cup \{\infty\}$. Denote by D the irreducible component of its special fiber containing the specialization of the point ∞. An element $\lambda \in S$ is *ordinary* if it specializes in D. If $\mathcal{B}$ has bad reduction, an irreducible component of $\mathcal{B}_k$ is *simple* if it only meets D and if there are exactly two elements of S specializing in it. We say that B has *simple reduction* if any tail of $\mathcal{B}_k$ (if there are any) is simple (see the following picture).



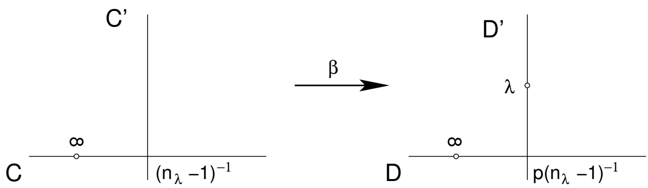
Note that if the pointed curve $(\mathbf{P}^1_K, B)$ has simple reduction then any irreducible component of its stable (separating) model different from D is automatically a tail. Moreover, if the cardinality of B is less than or equal to four then the reduction is always simple.

7. Semi-stable model separating the fiber above an ordinary branch point.

We now describe the stable model for a polynomial cover of degree p dominating its fundamental model $\mathcal{C} \to \mathcal{D}$ and separating the ramified fiber above an ordinary branch point.

Theorem 7.1. *Let $\beta : \mathbf{P}^1_K \to \mathbf{P}^1_K$ be a polynomial cover of degree p. Denote by $B = S \cup \{\infty\}$ its branch locus and suppose that $\lambda \in S$ is ordinary. Then the minimal semi-stable model $\mathcal{X} \to \mathcal{Y}$ of the cover which dominates the fundamental model $\mathcal{C} \to \mathcal{D}$ and separates the fiber above λ has the following description:*

(1) *The special fiber of $\mathcal{X}$ is the union of two projective lines C and C' meeting at a unique singular point of thickness $(n_\lambda - 1)^{-1}$ where n_λ is the cardinality of the fiber $\beta^{-1}(\lambda)$.*
(2) *The special fiber of $\mathcal{Y}$ is the union of two projective lines D and D' meeting at a unique singular point of thickness $p(n_\lambda - 1)^{-1}$.*
(3) *The morphism $C \to D$ is purely inseparable, while $C' \to D'$ is generically étale, unramified outside two points, wildly ramified above one of them and tamely ramified above the other.*



Proof. Without any loss of generality, we can assume that $\lambda = 0$ and $\beta(0) = 0$. Let ν be the rational number defined by

$$\nu = \mathrm{Min}\{v(x) \mid x \in \beta^{-1}(0)\}$$

and consider the curve $\mathcal{C}'$ obtained from $\mathcal{C}$ after a blow-up at the origin, of thickness ν. Let π be an element of $\mathfrak{p}$ such that $v(\pi) = \nu$. By construction, we have the expression

$$\beta(\pi X) = \pi^p \prod_{x \in \beta^{-1}(0)} (X - \pi^{-1}x)^{e_x} \in R[X]$$

with $\pi^{-1}x \in R$ for any $x \in \beta^{-1}(0)$. Following the notation introduced at the end of §5, we obtain $\beta_0(X) = \pi^{-p}\beta(\pi X)$ and $\gamma_0(X) = 1$. In particular, we have a model $\mathcal{C}' \to \mathcal{D}'$ of the cover β, where $\mathcal{D}'$ is the curve obtained from $\mathcal{D}$ after a blow-up at the origin, of thickness $p\nu$. We just have to prove that it separates the fiber above 0. From a practical point of view, we must show that two distinct roots of the polynomial $\beta_0(X)$ have distinct specializations. First of all, if $\mu \in S - \{0\}$ then $\beta^{-1}(\mu) \subset R^*$. Indeed, we already have $\beta^{-1}(\mu) \subset R$. If there were $x \in \beta^{-1}(\mu)$ with positive valuation, we would obtain $\overline{\mu} = \overline{\beta}(x) = \overline{x}^p = 0$, which is impossible since we are assuming that $0 \in S$ is ordinary. We have the following expression for the derivative of $\beta(X)$:

$$\beta'(X) = p \prod_{x \in \beta^{-1}(S)} (X - x)^{e_x - 1}.$$

We then obtain $\beta_0'(X) = \pi^{1-p}\beta'(\pi X)$, which leads to

$$\beta_0'(X) = p\pi^{1-n_0} \prod_{x \in \beta^{-1}(0)} (X - \pi^{-1}x)^{e_x - 1} \prod_{x \in \beta^{-1}(S - \{0\})} (\pi X - x)^{e_x - 1}$$

where n_0 is the cardinality of the fiber $\beta^{-1}(0)$. We have $\beta_0'(X) \in R[X]$, which directly implies the inequality $(n_0 - 1)\nu \leq 1$. A strict inequality would give $\overline{\beta}_0'(X) = 0$ and thus $\overline{\beta}_0(X) = X^p$ (recall that $\beta_0(X)$ is monic, of degree p and satisfies $\beta_0(0) = 0$). This is impossible since, by construction, there exists a root of $\beta_0(X)$ belonging to R^*. We then have $\nu = \frac{1}{n_0 - 1}$. Suppose now that the elements $x_1, \ldots, x_s \in \beta_0^{-1}(0)$ specialize to the same point t of C' and denote by $e_1, \ldots, e_s$ their ramification indices. We have $s < n_0$, since there are at least two distinct roots of $\overline{\beta}_0(X)$. The monomial $X - t$ appears with multiplicity $e = e_1 + \cdots + e_s < p$ in the factorization of $\overline{\beta}_0(X)$. In particular, the element t will be a root of $\overline{\beta}_0'(X)$ of multiplicity $e - 1$. But the above expression of $\beta_0'(X)$ implies that this multiplicity is equal to $e - s$, and so we have $s = 1$, i.e., the model $\mathcal{C}' \to \mathcal{D}'$ separates the fiber above 0. □

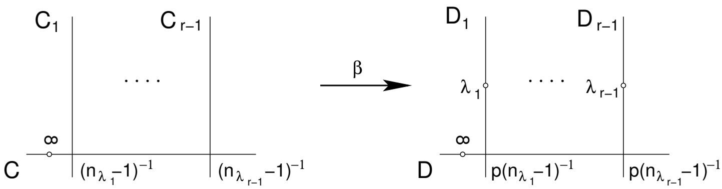
Corollary 7.2. *The notation and hypothesis being as in Theorem 7.1, there exists a smooth model of the cover* (*over* R) *which separates the fibers above* ∞ *and* λ. *In other words, the pointed curve* $(\mathbf{P}^1_K, \beta^{-1}(\{\infty, \lambda\}))$ *has good reduction.*

Proof. The desired smooth model is obtained from $\mathcal{C}' \to \mathcal{D}'$ by blowing down the irreducible components C and D. □

Corollary 7.3. *With the above assumptions, suppose that the branch locus* $B = \{\infty, \lambda_1, \ldots, \lambda_{r-1}\}$ *of the cover* β *has good reduction and that* $r \geq 3$. *Then, the minimal stable model* $\mathcal{X} \to \mathcal{Y}$ *separating the ramified fibers has the following description:*

1. $\mathcal{X}_k$ (*resp.* $\mathcal{Y}_k$) *is the union of* r *projective lines* $C, C_1, \ldots, C_{r-1}$ (*resp.* $D, D_1, \ldots, D_{r-1}$).
2. *For any* $i \in \{1, \ldots, r-1\}$, *the point* λ_i *specializes in* D_i.
3. *The irreducible component* C_i (*resp.* D_i) *only meets* C (*resp.* D). *The corresponding singular point has thickness* $(n_i - 1)^{-1}$ (*resp.* $p(n_i - 1)^{-1}$), *where* n_i *denotes the cardinality of the fiber above* λ_i.

(4) *The morphism $C \to D$ is purely inseparable and, for any $i \in \{1, \ldots, r-1\}$, the cover $C_i \to D_i$ is generically étale, unramified outside two points, wildly ramified above one of them (the singular point) and tamely ramified above the other (the specialization of λ_i).*



Proof. It suffices to repeat the construction in the proof of Theorem 7.1 for all the elements of S. □

8. Semi-stable model separating the fibers above a simple tail.

We now study the case of simple reduction of the branch locus $B = S \cup \{\infty\}$ of the cover β. Let r be the cardinality of B. We assume that $r > 3$, since for $r \leq 3$ the pointed curve $(\mathbf{P}^1, B)$ always has good reduction. If $\lambda \in S$ is an ordinary branch point, then the construction in the proof of Theorem 7.1 leads to a stable model which separates the fiber above λ. Suppose that $\lambda, \lambda' \in S$ belong to a simple tail of the minimal stable model $\mathcal{B}$ of B. In other words, viewing S as a subset of $R = \mathcal{C}_0(R)$ (cf. §4), we have $v(\lambda - \lambda') > 0$ and $v(\lambda - \lambda'') = v(\lambda' - \lambda'') = 0$ for any $\lambda'' \in S - \{\lambda, \lambda'\}$. The positive rational number $\epsilon = v(\lambda - \lambda')$ is the thickness of the singular point connecting the simple tail with the rest of $\mathcal{B}_k$. Since the reduction of the model $\mathcal{C} \to \mathcal{D}$ is purely inseparable, all the elements of $\beta^{-1}(\{\lambda, \lambda'\})$ have the same specialization.

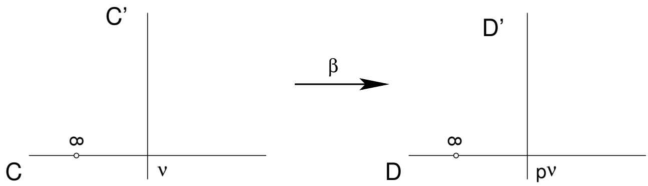
8.1. The first separating blow-up. We start the construction of the stable model separating the fibers above λ and λ' by considering the minimal model $\mathcal{C}'$ dominating $\mathcal{C}$ such that all the elements of $\beta^{-1}(\{\lambda, \lambda'\})$ specialize in the (smooth locus of the) same irreducible component and at least two of them have distinct specializations. In order to obtain a more explicit description, we can first of all reduce to the case $\lambda' = 0$ and $\beta(0) = 0$. Consider the rational number ν defined by

$$\nu = \mathrm{Min}\left\{v(x-y) \mid x, y \in \beta^{-1}(\{0, \lambda\})\right\}.$$

We have $\nu > 0$, since $\overline{\lambda} = 0$ and $\overline{\beta}(X) = X^p$. Moreover, $v(\lambda) \geq p\nu$. The curve $\mathcal{C}'$ is obtained from $\mathcal{C}$ after a blow-up at the origin, of thickness ν. Let π be an element of $\mathfrak{p}$ such that $v(\pi) = \nu$. The relation (1) at the end of §5 becomes $\beta(\pi X) = \pi^p \beta_0(X)\gamma_0(X)$, with $\gamma_0(X) = 1$ and

$$\beta_0(X) = \pi^{-p}\beta(\pi X) = \prod_{x\in\beta^{-1}(0)} (X-\pi^{-1}x)^{e_x} = \pi^{-p}\lambda + \prod_{x\in\beta^{-1}(\lambda)} (X-\pi^{-1}x)^{e_x}.$$

We then get a model $\mathcal{C}' \to \mathcal{D}'$ for β, where $\mathcal{D}'$ is the curve obtained from $\mathcal{D}$ after a blow-up at the origin, of thickness $p\nu$ (see the following picture).



The derivative of $\beta_0(X)$ can be expressed as

$$\beta_0'(X) = p\pi^{p+1-n_0-n_\lambda} \prod_{x\in\beta^{-1}(\{0,\lambda\})} (X-\pi^{-1}x)^{e_x-1} \prod_{x\in\beta^{-1}(S-\{0,\lambda\})} (\pi X - x)^{e_x-1}$$

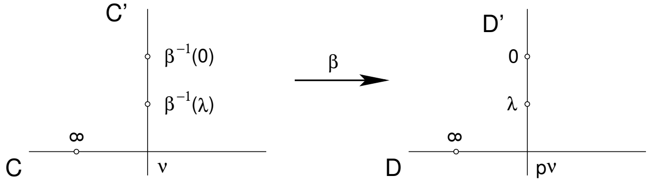
where, for any $\lambda \in S$, n_λ is the cardinality of the fiber $\beta^{-1}(\lambda)$. In particular, since $\beta_0'(X) \in R[X]$ and $\pi^{-1}x \in R$ for any $x \in \beta^{-1}(\{0,\lambda\})$, we obtain the relation

$$\nu \leq \frac{1}{n_0 + n_\lambda - p - 1}. \quad (2)$$

8.2. First case: "Far" points. Let's start by supposing that the inequality (2) is strict. This directly implies that $\overline{\beta}_0(X) = X^p$. Now, by construction, the polynomial $h(X) = \beta_0(X)(\beta_0(X) - \pi^{-p}\lambda)$ has at least two roots having distinct specializations, so that we obtain

$$v(\lambda) = p\nu.$$

Indeed, the relation $v(\lambda) > p\nu$ would give $\overline{h}(X) = X^{2p}$, which has only one root. In particular, this first blow-up separates the branch points 0 and λ but does not separate the corresponding fibers.



The above inequality now reads as

$$v(\lambda) < \frac{p}{n_0 + n_\lambda - p - 1}.$$

In order to obtain a model separating the fiber above 0, let ν_0 be the positive integer defined by

$$\nu_0 = \operatorname{Min}\left\{v(x) \mid x \in \beta_0^{-1}(0)\right\}.$$

We clearly have $\nu + \nu_0 = \operatorname{Min}\left\{v(x) \mid x \in \beta^{-1}(0)\right\}$. Consider the curve $\mathcal{C}''$ obtained from $\mathcal{C}'$ after a blow-up at the origin of the irreducible component C', of thickness ν_0. If C'' denotes the new irreducible component of $\mathcal{C}''_k$, then we have $d(C'', C) = \nu + \nu_0$. Let $\pi_0 \in \mathfrak{p}$ such that $v(\pi_0) = \nu_0$ and set $\pi_1 = \pi\pi_0$ (as in the previous paragraph, π has valuation ν). We then obtain the expression $\beta(\pi_1 X) = \pi^p \beta_0(\pi_0 X) = \pi_1^p \beta_1(X)\gamma_1(X)$, with $\gamma_1(X) = 1$ and

$$\begin{aligned}\beta_1(X) &= \pi_1^{-p}\beta(\pi_1 X) = \pi^{-p}\beta_0(\pi_0 X) \\ &= \prod_{x\in\beta^{-1}(0)} (X - \pi_1^{-1}x)^{e_x} = \prod_{x\in\beta_0^{-1}(0)} (X - \pi_0^{-1}x)^{e_x}.\end{aligned}$$

By construction, at least two roots of $\beta_1(X)$ have distinct specializations. If $\mathcal{D}''$ is the curve obtained from $\mathcal{D}'$ after a blow-up at the origin of D', of thickness $p\nu_0$, we then have a model $\mathcal{C}'' \to \mathcal{D}''$ of the cover β. Setting $u = n_0 + n_\lambda - p - 1$, $u_0 = n_0 - 1$ and $\delta = 1 - u\nu - u_0\nu_0$, we have the expression

$$\begin{aligned}\beta_1'(X) = p\pi^u\pi_0^{u_0} &\prod_{x\in\beta^{-1}(0)} (X - \pi_1^{-1}x)^{e_x-1} \prod_{x\in\beta^{-1}(\lambda)} (\pi_0 X - \pi^{-1}x)^{e_x-1} \\ &\cdot \prod_{x\in\beta^{-1}(S-\{0,\lambda\})} (\pi_1 X - x)^{e_x-1}\end{aligned}$$

which implies that $\delta = 0$ (otherwise, we would obtain $\overline{\beta}_1(X) = X^p$ and all the roots of $\beta_1(X)$ would have the same specialization). Since $v(\lambda) = p\nu$, we obtain

$$\nu_0 = \frac{p - (n_0 + n_\lambda - p - 1)v(\lambda)}{p(n_0 - 1)} < \frac{1}{n_0 - 1}.$$

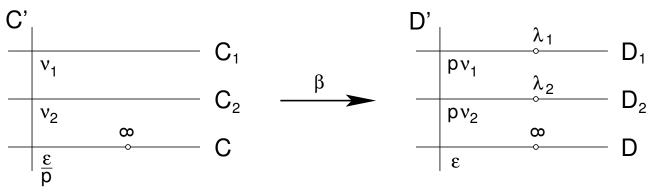
Proceeding exactly as in the end of the proof of Theorem 7.1, we finally check that this model separates the fiber above 0, i.e., that any two distinct roots of $\beta_1(X)$ have distinct specializations. The same procedure leads to a stable model separating the fiber above λ. Summarizing, we have just proved the following result:

Theorem 8.1. *Suppose that the branch locus $B = S \cup \{\infty\}$ of the polynomial cover β has bad reduction and that D' is a simple tail of the special fiber of the stable model $\mathcal{B}$ associated to the pointed curve $(\mathbf{P}^1, B)$. Denote by $\lambda_1, \lambda_2 \in S$ the two branch points specializing in D' and let $\epsilon = v(\lambda_1 - \lambda_2)$ be the thickness of the corresponding singular point of $\mathcal{B}_k$. For any $\lambda \in S$, denote by n_λ the cardinality of the fiber $\beta^{-1}(\lambda)$. If the inequality*

$$\epsilon < \frac{p}{n_{\lambda_1} + n_{\lambda_2} - p - 1}$$

holds, then the minimal semi-stable model $\mathcal{X} \to \mathcal{Y}$ of β dominating $\mathcal{C} \to \mathcal{D}$ and separating the fibers above λ_1 and λ_2 has the following description:

(1) *The curve $\mathcal{X}_k$ (resp. $\mathcal{Y}_k$) is the union of four projective lines C, C', C_1 and C_2 (resp. D, D', D_1 and D_2).*
(2) *The specializations of the points ∞, λ_1 and λ_2 belong respectively to D, D_1 and D_2.*
(3) *The irreducible component C (resp. D) only meets C' (resp. D'), at a singular point of thickness $\frac{\epsilon}{p}$ (resp. of thickness ϵ).*
(4) *For any $i \in \{1, 2\}$, the irreducible component C_i (resp. D_i) only meets C' (resp. D'), at a singular point of thickness $\nu_i = \frac{p-(n_{\lambda_1}+n_{\lambda_2}-p-1)\epsilon}{p(n_{\lambda_i}-1)}$ $\left(\text{resp. of thickness } p\nu_i = \frac{p-(n_{\lambda_1}+n_{\lambda_2}-p-1)\epsilon}{n_{\lambda_i}-1}\right)$.*
(5) *The morphisms $C \to D$ and $C' \to D'$ are purely inseparable, while the covers $C_1 \to D_1$ and $C_2 \to D_2$ are generically étale, unramified outside two points, wildly ramified above one of them and tamely ramified above the other.*



Corollary 8.2. *The notation and hypothesis being as above, if*

$$\epsilon < \frac{p}{n_{\lambda_1} + n_{\lambda_2} - p - 1}$$

then, for any $i \in \{1, 2\}$, the pointed curve $(\mathbf{P}^1, \beta^{-1}(\{\infty, \lambda_i\}))$ has good reduction.

Proof. It suffices to take the smooth R-curve obtained from $\mathcal{X}$ by blowing down the irreducible components C, C' and C_{2-i} of $\mathcal{C}_k$. □

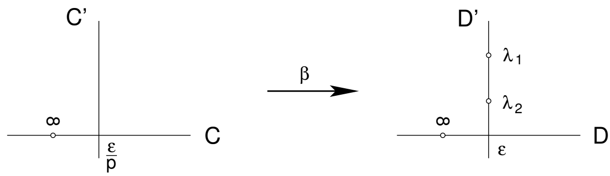
8.3. Second case: The critical distance. Keeping the above notation and hypothesis, suppose now that the inequality (2) at the end of §8.1 is in fact an equality, i.e., that $\nu = \frac{1}{n_0+n_\lambda-p-1}$, so that $\overline{\beta}'_0(X) \neq 0$. In particular, the morphism $C' \to D'$ is generically étale. We already know that $v(\lambda) \geq p\nu$. In this paragraph, we assume that this last inequality is an equality. As before, the cover $\mathcal{C}' \to \mathcal{D}'$ separates the branch points 0 and λ. Moreover, proceeding as at the end of the proof of Theorem 7.1, we can easily show that this model also separates the fibers above these two points. In particular, we have the following result:

Theorem 8.3. *The notation and hypothesis being as in Theorem 8.1, suppose that*

$$\epsilon = \frac{p}{n_{\lambda_1} + n_{\lambda_2} - p - 1}.$$

Then, the minimal semi-stable model $\mathcal{X} \to \mathcal{Y}$ of β dominating $\mathcal{C} \to \mathcal{D}$ and separating the fibers above λ_1 and λ_2 has the following description:

(1) *The curve $\mathcal{X}_k$ (resp. $\mathcal{Y}_k$) is the union of two projective lines C and C' (resp. D and D') meeting at a singular point of thickness $\frac{\epsilon}{p}$ (resp. of thickness ϵ).*
(2) *The specializations of the points λ_1 and λ_2 belong D' and ∞ specializes in D.*
(3) *The morphism $C \to D$ is purely inseparable while the cover $C' \to D'$ is generically étale, unramified outside three points, wildly ramified above one of them and tamely ramified above the others.*



Corollary 8.4. *The notation and hypothesis being as above, if*

$$\epsilon = \frac{p}{n_{\lambda_1} + n_{\lambda_2} - p - 1}$$

then the pointed curve $(\mathbf{P}^1, \beta^{-1}(\{\infty, \lambda_1, \lambda_2\}))$ has good reduction.

Proof. It suffices to take the smooth R-curve obtained from $\mathcal{X}$ by blowing down the irreducible component C of its special fiber. □

8.4. Third case: "Near" points. We now have to study the most critical case, that is, when $\nu = \frac{1}{n_0 + n_\lambda - p - 1}$ and $v(\lambda) > p\nu$. This situation occurs if and only if the model $\mathcal{C}' \to \mathcal{D}'$ introduced at the beginning of this section does not separate the elements 0 and λ of the branch locus B of the cover. As in the previous paragraph, the derivative of $\overline{\beta}_0(X)$ does not identically vanish, so that the cover $C' \to D'$, which is (isomorphic to the one) induced by the polynomial $\overline{\beta}_0(X) \in k[X]$, is generically étale, unramified outside two points, wildly ramified above one of them (the intersection with D) and tamely ramified above the other. There exist at least two roots of $\beta_0(X)$ having distinct specializations. Indeed, the contrary would give $\overline{\beta}_0(X) = X^p$, so that the cover $C' \to D'$ would be purely inseparable, which is a contradiction. The points 0 and λ are the only elements of B specializing in C' and their fibers with respect to β can be assimilated to the fibers $\beta_0^{-1}(0)$

and $\beta_0^{-1}(\lambda_0)$, where $\lambda_0 = \pi^{-p}\lambda \in \mathfrak{p}$. Set

$$\overline{\beta}_0(X) = \prod_{i=1}^{s}(X - w_i)^{d_i}$$

with $s > 1$, $d_1 + \cdots + d_s = p$ and $w_i \neq w_j$ for any $i \neq j$. We then obtain two partitions $\beta_0^{-1}(0) = S_{1,0} \cup \cdots \cup S_{s,0}$ and $\beta_0^{-1}(\lambda_0) = S_{1,\lambda_0} \cup \cdots \cup S_{s,\lambda_0}$, where we have set, for a fixed $t \in \mathfrak{p}$,

$$S_{i,t} = \{x \in \beta_0^{-1}(t) \mid \overline{x} = w_i\}.$$

For any $i \in \{1, \ldots, s\}$ we then have the identity

$$d_i = \sum_{x \in S_{i,t}} e_x$$

where e_x is the multiplicity of the root x of the polynomial $\beta_0(X) - t$. Since the cover $C' \to D'$ is generically étale and unramified outside the set $\{0, \infty\}$, we have

$$\overline{\beta}_0'(X) = c\prod_{i=1}^{s}(X - w_i)^{d_i - 1}$$

with $c \in k^*$. On the other hand, the expression of the derivative of $\beta_0(X)$ given previously leads to

$$\overline{\beta}_0'(X) = c\prod_{i=1}^{s}(X - w_i)^{2d_i - n_{i,0} - n_{i,\lambda}}$$

where $n_{i,0}$ (resp. $n_{i,\lambda}$) denotes the cardinality of $S_{i,0}$ (resp. of S_{i,λ_0}). Combining these two expressions we finally obtain the identity

$$n_{i,0} + n_{i,\lambda} = d_i + 1$$

which holds for any $i \in \{1, \ldots, s\}$. In order to completely separate the ramified fibers, we need to blow-up some projective lines at the points $w_1, \ldots, w_s$ belonging to the irreducible component C' of $\mathcal{C}'$. In other words, we have to separate the elements of $S_{i,0} \cup S_{i,\lambda_0}$ for any $i \in \{1, \ldots, s\}$. Let's start with $i = 1$: Since $0 \in S$ is a branch point and $\beta_0(0) = \beta(0) = 0$, we can assume, without any loss of generality, $w_1 = 0$. Set

$$\nu_1 = \mathrm{Min}\,\{v(x - y) \mid x, y \in S_{1,0} \cup S_{1,\lambda_0},\ x \neq y\} > 0$$

and consider the R-curve $\mathcal{C}''$, obtained from $\mathcal{C}'$ by blowing-up a projective line C_1 at w_1, of thickness ν_1. Let $\pi_2 \in \mathfrak{p}$ such that $v(\pi_2) = \nu_1$. We then have the identity

$$\beta(\pi\pi_2 X) = \pi^p\beta_0(\pi_2 X) = \pi^p\pi_2^{d_1}\beta_1(X)\gamma_1(X)$$

where

$$\beta_1(X) = \prod_{x \in S_{1,0}}(X - \pi_2^{-1}x)^{e_x}$$

and

$$\gamma_1(X) = \prod_{x\in\beta_0^{-1}(0)-S_{1,0}} (\pi_2 X - x)^{e_x}.$$

Since we are assuming that $w_1 = 0$, an element $x \in \beta_0^{-1}(0)$ belongs to $S_{1,0}$ if and only if $v(x) > 0$. In particular, $\overline{\gamma}_1(X) = c \in k^*$. We then obtain a R-morphism $\mathcal{C}'' \to \mathcal{D}''$ (which is not finite), where $\mathcal{D}''$ is the R-curve obtained from $\mathcal{D}'$ by blowing-up a projective line D_1 at the origin of D', of thickness $\epsilon_1 = d_1\nu_1$. For any $x \in S_{i,\lambda_0}$, we have $v(x) \geq \nu_1$, from which we easily deduce the inequality $v(\lambda_0) \geq \epsilon_1$, i.e., $v(\lambda) \geq \epsilon_0 + \epsilon_1$, with $\epsilon_0 = p\nu$. The specialized morphism $C_1 \to D_1$ has degree d_1 and is (isomorphic to the cover $\mathbf{P}^1_k \to \mathbf{P}^1_k$) induced by the polynomial $\overline{\beta}_1(X)$. The integer d_1 being strictly less than p, the derivative $\overline{\beta}_1'(X)$ does not identically vanish. More explicitely, using once again the expression of the derivative of $\beta_0(X)$, we obtain

$$\overline{\beta}_1'(X) = u \prod_{x\in S_{i,0}\cup S_{i,\lambda_0}} \left(X - \overline{\pi_2^{-1}x}\right)^{e_x-1}$$

with $u \in k^*$. The strict inequality $v(\lambda) > \epsilon_0 + \epsilon_1$ would imply that the cover $\mathbf{P}^1_k \to \mathbf{P}^1_k$ induced by $\overline{\beta}_1(X)$ is tame, of degree d_1 and unramified outside 0 and ∞, and thus $\overline{\beta}_1(X) = uX^{d_1}$. In particular, all the elements of $S_{1,0}$ and S_{1,λ_0}, would specialize to the same element of C_1. This is impossible, since $\mathcal{C}''$ is the minimal model dominating $\mathcal{C}'$ and separating at least two elements of $S_{1,0} \cup S_{1,\lambda_0}$. We then have the identity

$$v(\lambda) = \epsilon_0 + \epsilon_1 = \frac{p}{n_0 + n_\lambda - p - 1} + d_1\nu_1$$

which implies that the model $\mathcal{D}''$ separates the branch points 0 and λ. Then, using the above expression for the derivative of $\overline{\beta}_1(X)$ and the same arguments at the end of the proof of Theorem 7.1, one easily shows that $\mathcal{C}''$ separates the whole $S_{1,0} \cup S_{1,\lambda_0}$. Note that the above equality implies that the thickness $\epsilon_1 = d_1\nu_1$ does not depend on the point w_1 but only on the ramification data of the cover and on the valuation of λ. Iterating this construction for all the roots of $\overline{\beta}_0(X)$, we obtain a semi-stable model $\mathcal{X} \to \mathcal{Y}$ (this time the morphism is finite) which dominates $\mathcal{C} \to \mathcal{D}$ and separates the fiber above 0 and λ. Summarizing, we have proved the following result:

Theorem 8.5. *The notation and hypothesis being as in Theorem 8.1, suppose that*

$$\epsilon > \frac{p}{n_{\lambda_1} + n_{\lambda_2} - p - 1}.$$

Then, the special fiber of the minimal stable model $\mathcal{X} \to \mathcal{Y}$ of the cover β dominating $\mathcal{C} \to \mathcal{D}$ and separating the fibers above λ_1 and λ_2 has the following description:

(1) *The curve $\mathcal{Y}_k$ is the union of three projective lines D, D' and D_1.*

(2) *The curve D (resp. D_1) meets D' at a singular point of thickness* $\epsilon_0 = \frac{p}{n_{\lambda_1}+n_{\lambda_2}-p-1}$ $\left(\textit{resp. of thickness } \epsilon_1 = v(\lambda) - \frac{p}{n_{\lambda_1}+n_{\lambda_2}-p-1}\right)$.

(3) *The specializations of the points λ_1 and λ_2 belong to D_1 and ∞ specializes in D.*

(4) *There exists an integer $s \geq 2$ such that the curve $\mathcal{X}_k$ has $s+2$ irreducible components $C, C', C_1, \ldots, C_s$.*

(5) *The curve C meets C' at a singular point of thickness $\frac{\epsilon_0}{p}$.*

(6) *There exist two partitions*

$$\beta^{-1}(\lambda_1) = S_{1,1} \cup \cdots \cup S_{s,1} \quad \textit{and} \quad \beta^{-1}(\lambda_2) = S_{1,2} \cup \cdots \cup S_{s,2}$$

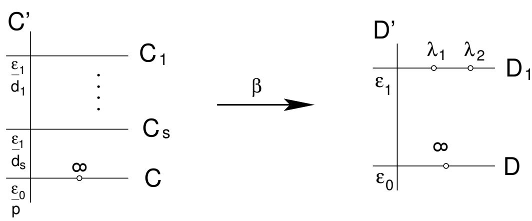
satisfying the identities

$$\sum_{x \in S_{i,1}} e_x = \sum_{x \in S_{i,2}} e_x = d_i \quad \textit{and} \quad n_{i,1} + n_{i,2} = d_i + 1$$

where e_x denotes the ramification index of β at a point $x \in \mathbf{P}^1(K)$ and, for any $i \in \{1, \ldots, s\}$ and any $j \in \{1, 2\}$, $n_{i,j}$ is the cardinality of the set $S_{i,j}$.

(7) *For any $i \in \{1, \ldots, s\}$, the curve C_i meets C at a unique singular point, of thickness $\frac{\epsilon_1}{d_i}$ and the elements of $S_{i,1} \cup S_{i,2}$ specialize to pairwise distinct points of C_i.*

(8) *The morphism $C \to D$ is purely inseparable and the cover $C' \to D'$ is generically étale, unramified outside two points, wildly ramified above one of them (the intersection with D) and tamely ramified above the other (the intersection with D_1). For any $i \in \{1, \ldots, s\}$, the cover $C_i \to D_1$ is generically étale, of degree d_i, unramified outside three points (the specializations of λ_1 and λ_2 and the intersection with C') over which the ramification is tame.*



Theorems 7.1, 8.1, 8.3 and 8.5 allow us to completely classify the reduction type of the minimal semi-stable model $\mathcal{X} \to \mathcal{Y}$ (separating the ramified fibers) for a polynomial cover β over K of degree p, under the assumption of simple reduction of its branch locus B. More precisely, the above results show that the behaviour of this model essentially depends only on the ramification data of the cover and on the thicknesses of the singular points of

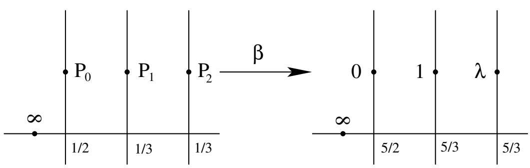
the special fiber of the stable (separating) model $\mathcal{B}$ associated to the pointed curve $(\mathbf{P}^1, B)$.

9. An example.

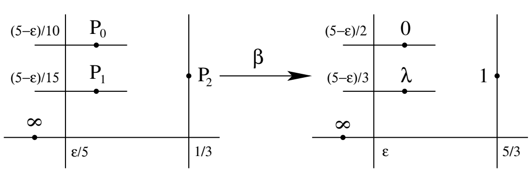
We end this paper with an explicit example. Let K be a 5-adic field, i.e., a finite extension of the field $\mathbf{Q}_5$. We want to describe the semi-stable model for a polynomial cover $\beta : \mathbf{P}^1_K \to \mathbf{P}^1_K$ of degree 5 having the following ramification data:

(1) The cover is unramified outside the set $\{\infty, 0, \lambda_1, \lambda_2\}$ with $\lambda_1, \lambda_2 \in K^*$ and $\lambda_1 \neq \lambda_2$.
(2) There is only one point above ∞.
(3) There are three points above 0, one of them, denoted by P_0 has ramification index 3, while the others are unramified.
(4) There are four points above λ_1 (resp. above λ_2), one of them, denoted by P_1 (resp. P_2) has ramification index 2 while the other three are unramified.

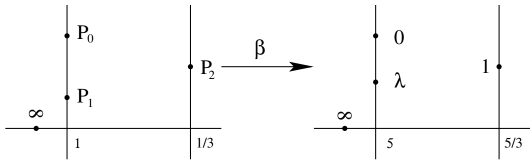
Up to equivalence (cf. §3), and since the behaviour of the ramification above λ_1 and λ_2 is the same, we can reduce to the case $\lambda_1 = 1$ and $\lambda_2 = \lambda \in R - \{0, 1\}$. The results of this paper allow us to describe the semi-stable model for β without any direct computation. More precisely, if the branch locus has good reduction, i.e., if $v(\lambda) = v(\lambda - 1) = 0$ then we can apply Theorem 7.1, which leads to the following model:



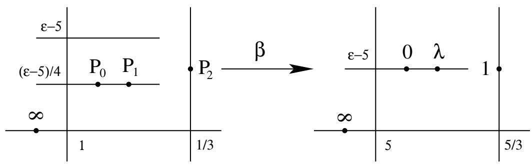
The branch locus of the cover β has bad reduction if either $v(\lambda) > 0$ or $v(\lambda - 1) > 0$. In the first case, there are three possibilities, leading to four different semi-stable models: If $\epsilon = v(\lambda) < 5$, we can apply Theorem 8.1 and obtain the following model:



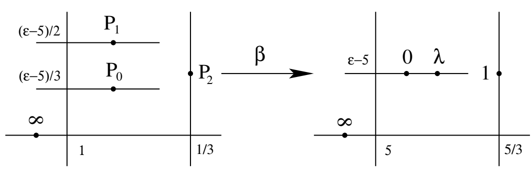
As it is shown in the following picture, and applying Theorem 8.3, the simplest case occurs for $\epsilon = 5$.



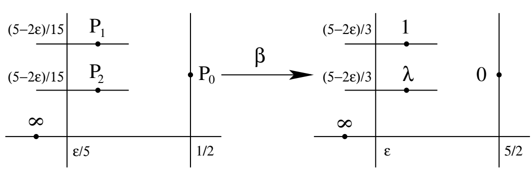
Finally, for $\epsilon > 5$ we find two possibilities, depending on the partitions of the ramification indices $(1,1,3)$ and $(1,1,1,2)$ of the fibers above 0 and λ (cf. Theorem 8.5). The first partition is given by $S_0 = \{\{1,3\},\{1\}\}$ and $S_1 = \{\{1,1,2\},\{1\}\}$, which leads to the following semi-stable model:



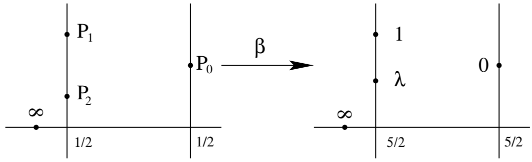
If we consider the second partition $S_0 = \{\{3\},\{1,1\}\}$ and $S_1 = \{\{1,1,1\},\{2\}\}$, we then obtain the following model:



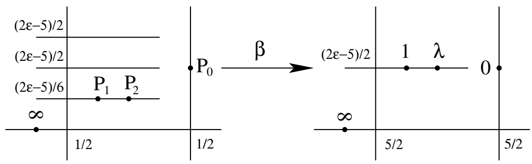
In order to completely classify the possible semi-stable models for β, we have to study the last case of bad reduction of the branch locus, i.e., when $\epsilon = v(\lambda - 1) > 0$. As before, there are three different cases, depending on ϵ, leading to four different situations. First of all, for $\epsilon < \frac{5}{2}$ we obtain the following semi-stable model (cf. Theorem 8.1):



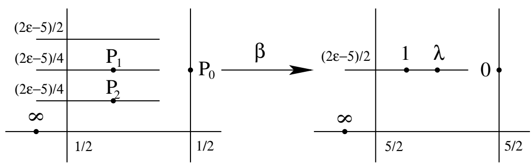
The next picture describes the case $\epsilon = \frac{5}{2}$ (cf. Theorem 8.3):



For $\epsilon > \frac{5}{2}$, according to Theorem 8.5, there are two possible partitions of the ramification indices above 1 and λ. The first is given by $S_1 = S_\lambda = \{\{1,2\},\{1\},\{1\}\}$ and leads to the following semi-stable model:



Finally, the next picture describes the semi-stable model associated to the second partition $S_1 = S_\lambda = \{\{2\},\{1,1\},\{1\}\}$:



References

[1] J.-B. Bost, F. Loeser and M. Raynaud (eds.), *Courbes Semi-stables et Groupe Fondamental en Géométrie Algébrique* (French); Proceedings of the Conference on Fundamental Group of Curves in Algebraic Geometry held in Luminy, November 30–December 4, 1998; Progress in Mathematics, **187**, Birkhäuser, Verlag, Basel, 2000, MR 1768107, Zbl 0978.14002.

[2] C. Lehr, *Reduction of p-cyclic covers of the projective line*, Manuscripta Math., **106**(**2**) (2001), 151-175, MR 1865562.

[3] M. Raynaud, *p-groupes et réduction semi-stable des courbes* (French); The Grothendieck Festschrift, Vol. III, 179-197, Progr. Math., **88**, Birkhäuser, Boston, MA, 1990, MR 1106915, Zbl 0722.14013.

[4] ———, *Revêtements de la droite affine en caractéristique $p > 0$ et conjecture d'Abhyankar*, Invent. Math., **116**(**1-3**) (1994), 425-462, MR 1253200, Zbl 0798.14013.

[5] ———, *Spécialisation des revêtements en caractéristique* $p > 0$, Ann. Sci. Ecole Norm. Sup. (4), **32**(**1**) (1999), 87-126, MR 1670532, Zbl 0999.14004.

[6] M. Saïdi, *Galois covers of degree p: Semi-stable reduction and Galois action*. Preprint, math.AG/0106249.

[7] S. Wewers, *Three point covers with bad reduction*. Preprint, math.AG/0205026.

[8] L. Zapponi, *Semi-stable model for degree p regular covers between curves with good reduction*. Preprint, GTEM n. 69.

Received November 17, 2002 and revised May 1, 2003.

Departement de Mathematiques
Ecole Polytechnique Fédérale de Lausanne
1015 Lausanne
Suisse
E-mail address: leonardo.zapponi@epfl.ch