THE MAXIMUM PRINCIPLE FOR SYSTEMS OF PARABOLIC EQUATIONS SUBJECT TO AN AVOIDANCE SET

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Hamilton’s maximum principle for systems states that given a reaction-diffusion equation (semi-linear heat-type equation) for sections of a vector bundle over a manifold, if the solution is initially in a subset invariant under parallel translation and convex in the fibers and if the ODE associated to the PDE preserves the subset, then the solution remains in the subset for positive time. We generalize this result to the case where the subsets are time-dependent and where there is an avoidance set from which the solution is disjoint. In applications the existence of an avoidance set can sometimes be used to prove the preservation of a subset of the vector bundle by the PDE.

1. Introduction.

For scalar parabolic equations, maximum principles are well-known [PW] and have been applied in numerous settings in partial differential equations and geometric analysis. In the case of systems of parabolic equations, maximum principles are not as well-known and appear to be much less frequent. Notable exceptions are given by Richard Hamilton ([H1] and [H2]) and Joel Smoller [S] (see Chapter 14). Hamilton’s maximum principle holds for solutions of reaction-diffusion equations (PDE) which are time-dependent sections of a vector bundle over a Riemannian manifold; in particular, it holds for the reaction-diffusion equations satisfied by the curvature operator under Ricci flow. When the convex subsets of the fibers are independent of time, Hamilton proved such a maximum principle in [H2], which roughly says that if the convex subsets are preserved by the system of ordinary differential equations (ODE) associated to the PDE then the convex subsets are preserved by the PDE. A special case of this result, which applied to symmetric 2-tensors, was proved earlier by Hamilton in [H1] and applied to obtain crucial curvature pinching estimates in his proof that a compact 3-manifold with positive Ricci curvature converges to a constant curvature metric under the volume preserving Ricci flow. The general formulation in

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1 We would like to thank Yung-Sze Choi for informing us of this reference.
greatly simplified the computations in [H1] and facilitated the more complicated convex analysis of the ODE associated to the evolution of the Riemann curvature operator in dimension four in [H5].

The main purpose of this paper is to prove two extensions of Hamilton’s maximum principle for systems which should be useful for the study of the Ricci flow and some other geometric evolution equations such as the mean curvature flow. We present an extension where the convex sets are allowed to depend on time, we call the extension the maximum principle with time-dependent convex sets (see Theorem 3). We also present a souped-up version where both the convex sets are allowed to depend on time and the convex set may not be preserved by the ODE on a subset of the boundary but the solution to the PDE avoids that part of the boundary. We call such subsets of the convex sets, which contain this part of their boundary, avoidance sets, and call this extension the maximum principle subject to an avoidance set (see Theorem 4).

A special case of this maximum principle with time-dependent convex sets has already been proved by Hamilton in Theorem 3.3 in Section 2.3 of [H5]. In particular, Hamilton adjoins to the solution \( \sigma \) of the PDE the function \( r = \frac{1}{T^* - t} \), where \( T^* \) is the singularity time, which trivially satisfies the equation \( \frac{\partial r}{\partial t} = \Delta r + r^2 \) and applies the maximum principle to the pair \((\sigma, r)\). However, in general this device of adjoining the function \( r = \frac{1}{T^* - t} \) has the drawback that the sets in space and time to be preserved may not be convex even though the space slices are. In such a case Hamilton’s proof would not directly apply. In the proof of the somewhat more general form of the maximum principle given in this paper, we modify Hamilton’s original proof of the maximum principle in [H2]. The difficulty in this approach is reconciling the time-dependence of the sets over which one takes the maximum of certain functions with the framework of Hamilton’s “ODE to PDE” formulation. More precisely, when the convex sets depend on time, the lemma used by Hamilton (see Lemma 9) on taking the time derivative of the function \( \sup_{s \in S(t)} g(s, t) \) must have a correction term, since now the set \( S(t) \) depends on time, which is difficult to control in the later applications of the lemma. This is overcome by considering the space-time track of the time-dependent convex sets and finding suitable splitting of certain quantities which arise in the study of both the ODE and the PDE (see the proof of Proposition 10 and the proof of Theorem 3). Formulating the proof this way enables us to generalize our result to the case where the PDE is subject to an avoidance set without much difficulty. A special case of this maximum principle has already been applied in [H5] and [H6] to obtain refined and subtle pointwise curvature estimates. In addition, although it is not necessary, our maximum principle may be used in the proofs of results in [H3] and [H4].
Our maximum principle subject to avoidance sets is a more general formulation of a form of the maximum principle implicitly used in the proof of certain estimates in Section 2.3 of [H5] which are used to detect necks. Some of these estimates, have analogues in dimension three (see § 24 of [H4]). In regards to this, as suggested by Mao-Pei Tsui, our souped-up version can also be used to give an alternate proof of Theorem 24.6 of [H4] (cf. [CK]). However, in the study of 4-manifolds with positive isotropic curvature [H5], a souped-up version is necessary and is implicitly used by Hamilton.

2. Main results.
Let $M^n$ be a closed oriented $n$-dimensional manifold with a smooth family of Riemannian metrics $g(t)$, $t \in [0, T]$. Let $V \to M$ be a real vector bundle with a time-independent bundle metric $h$ and $\Gamma(V)$ be the vector space of $C^\infty$ sections of $V$. Let

\[ \nabla(t) : \Gamma(V) \to \Gamma(V \otimes TM^*), t \in [0, T] \]

be a smooth family of time-dependent connections compatible with $h$, that is,

\[ X[h(\sigma, \tau)] = h(\nabla(t)X\sigma, \tau) + h(\sigma, \nabla(t)X\tau) \]

for all $X \in TM$, $\sigma, \tau \in \Gamma(V)$ and $t \in [0, T]$. The time-dependent Laplacian $\Delta(t)$ acting on a section $\sigma \in \Gamma(V)$ is defined by

\[ \Delta(t)\sigma = \text{trace}_{g(t)}(\tilde{\nabla}(t)(\nabla(t)\sigma)), \]

where

\[ \tilde{\nabla}(t) : \Gamma(V \otimes TM^*) \to \Gamma(V \otimes TM^* \otimes TM^*) \]

is defined using the connection $\nabla(t)$ on $V$ and the Levi-Civita connection $D(t)$ on $TM^*$ associated with metric $g(t)$. That is,

\[ \tilde{\nabla}(t)X(\sigma \otimes \alpha) = (\nabla(t)X\sigma) \otimes \alpha + \sigma \otimes (D(t)X\alpha) \]

for all $X \in TM$, $\sigma \in \Gamma(V)$, $\alpha \in \Gamma(TM^*)$.

Let $F : V \times [0, T] \to V$ be a fiber preserving map; i.e., $F(\sigma, t)$ is a time-dependent vector field defined on the bundle $V$ and tangent to the fibers. Then we can form a system of reaction-diffusion equations (PDE)

\[ \frac{\partial}{\partial t} \sigma(x, t) = \Delta(t)\sigma(x, t) + F(\sigma(x, t), t), \]

where $\sigma(\cdot, t), t \in [0, T]$ are sections of $V$. In each fiber $V_x$ the system of ordinary differential equations (ODE) associated to the PDE (1) obtained by dropping the Laplacian term is

\[ \frac{d}{dt} \sigma_x(t) = F(\sigma_x(t), t), \]

where $\sigma_x(t) \in V_x$. 


Let $\mathcal{K}$ be closed subset of $V$. Denote $\mathcal{K}_x = \mathcal{K} \cap V_x$. For any initial time $t_0 \in [0, T)$ we say that the solution $\sigma(x, t) : t \in [t_0, T]$ of the PDE (1) starts in $\mathcal{K}$ if $\sigma(x, t_0) \in \mathcal{K}_x$ for all $x \in M$. We say that the solution $\sigma(x, t)$ remains in $\mathcal{K}$ for all later times if $\sigma(x, t) \in \mathcal{K}_x$ for all $x \in M$ and all $t \in (t_0, T]$. For any $x \in M$ and for any initial time $t_0 \in [0, T)$ we say that the solution $\sigma_x(t) : t \in [t_0, T]$ of the ODE (2) starts in $\mathcal{K}_x$ if $\sigma_x(t_0) \in \mathcal{K}_x$. We say that the solution $\sigma_x(t)$ remains in $\mathcal{K}_x$ for all later times if $\sigma_x(t) \in \mathcal{K}_x$ for all $t \in (t_0, T]$.

One important question is: When will an arbitrary solution of the PDE (1) which starts in $\mathcal{K}$ at an arbitrary initial time $t_0 \in [0, T)$ remain in $\mathcal{K}$ for all later times? To answer this question, we need to impose two conditions on $\mathcal{K}$:

I. $\mathcal{K}$ is invariant under parallel translation defined by the connection $\nabla(t)$ for each $t \in [0, T]$.

II. In each fiber $V_x$ set $\mathcal{K}_x$ is closed and convex.

The following theorem is the maximum principle of Hamilton (Theorem 4.3 in [H2]):

**Theorem 2.** Let $\mathcal{K} \subset V$ be a closed subset satisfying Conditions I and II. Assume that $F(\sigma, t)$ is continuous in $t$ and is Lipschitz in $\sigma$. Suppose that for any $x \in M$ and any initial time $t_0 \in [0, T)$, and any solution $\sigma_x(t)$ of the ODE (2) which starts in $\mathcal{K}_x$ at $t_0$, the solution $\sigma_x(t)$ will remain in $\mathcal{K}_x$ for all later times. Then for any initial time $t_0 \in [0, T)$ the solution $\sigma(x, t)$ of the PDE (1) will remain in $\mathcal{K}$ for all later times if $\sigma(x, t)$ starts in $\mathcal{K}$ at time $t_0$.

In applications to the Ricci flow the vector bundle $V$ is a tensor bundle and the subsets $\mathcal{K}_x \subset V_x$, which are invariant under the action of $O(n)$, are identified under the isomorphism between two fibers $V_x$ and $V_y$ induced by any choice of orthonormal frames in $TM$ at the two points $x$ and $y$. The ODEs in $V_x$ and $V_y$ are also $O(n)$-invariant and identical under this identification. When this is the case the requirement on the solutions of the ODE (2) in Theorem 2 will hold for every fiber if it holds for one fiber.

Next we formulate the maximum principle where the convex sets are time-dependent. Let $U$ be an open subset of $V$ and $\mathcal{K}(t) \subset U$ be a closed subset for each $t \in [0, T]$. We impose two conditions on $\mathcal{K}(t)$ for each $t$:

III. $\mathcal{K}(t)$ is invariant under parallel translation defined by the connection $\nabla(t)$ for each $t \in [0, T]$.

IV. In each fiber $V_x$ set $\mathcal{K}_x(t) = \mathcal{K}(t) \cap V_x$ is nonempty, closed and convex for each $t \in [0, T]$.

We define the space-time track

$$T \doteq \{(v, t) \in V \times [0, T] : v \in \mathcal{K}(t), t \in [0, T]\},$$
and define

$$\mathcal{T}_x \doteq \mathcal{T} \cap (V_x \times [0,T]).$$

Let $F : U \times [0,T] \to V$ be a fiber preserving map, i.e., $F(x,\sigma,t)$ is a time-dependent vector field defined on $U$ and tangent to the fibers. Let $u(x,t) : V_x \times T_x M^* \to V_x$ be a smooth family of bundle maps of diagonal form, i.e.,

$$u(x,t)(\sigma, dx^i) = u^i(x,t) \cdot \sigma$$

where $u^i(x,t)$ are smooth functions (in applications to the Ricci flow, $u \equiv 0$; that is, there is no gradient term). Then we can form a system of reaction-diffusion equations (PDE)

$$\frac{\partial}{\partial t} \sigma(x,t) = \Delta(t) \sigma(x,t) + u(x,t)(\nabla(t) \sigma(x,t)) + F(x,\sigma(x,t),t).$$

In each fiber $V_x$ the associated system of ordinary differential equations (ODE) is

$$\frac{d}{dt} \sigma_x(t) = F(x,\sigma_x(t),t).$$

Hamilton’s maximum principle is an answer to the following question: For any $t_0 \in [0,T)$ when will the solution $\sigma(x,t)$, $t \in [t_0,T]$ of the PDE (3) which starts in $\mathcal{K}(t_0)$, remain in $\mathcal{K}(t)$ for all later times, i.e., $\sigma(x,t) \in \mathcal{K}_x(t)$ for all $x \in M$ and $t \in [t_0,T]$?

In this paper we extend Hamilton’s techniques established in [H2] to the case where the convex sets are time-dependent (see Theorem 3 below) and use our extension to also prove a maximum principle for the PDE (3) subject to an avoidance set (see Theorem 4 below). Our first main result is the following:

**Theorem 3.** Let $\mathcal{K}(t) \subset V, t \in [0,T]$ be closed subsets which satisfy Conditions III and IV above, and such that the space-time track $\mathcal{T}$ is closed. Assume that $u(x,t) : V_x \times T_x M^* \to V_x$ is a smooth family of bundle maps of diagonal form and assume that $F(x,\sigma,t)$ is continuous in $x$, $t$ and is Lipschitz in $\sigma$. Suppose that, for any $x \in M$ and any initial time $t_0 \in [0,T)$, any solution $\sigma_x(t)$ of the ODE (4) which starts in $\mathcal{K}_x(t_0)$ will remain in $\mathcal{K}_x(t)$ for all later times, i.e., $\sigma_x(t) \in \mathcal{K}_x(t)$ for all $t \in [t_0,T]$. Then for any initial time $t_0 \in [0,T)$ the solution $\sigma(x,t) : t \in [t_0,T]$ of the PDE (3) will remain in $\mathcal{K}(t)$ for all later times if $\sigma(x,t)$ starts in $\mathcal{K}(t_0)$ at time $t_0$.

Special cases of this result have been proved by Hamilton and applied to the study of the Ricci flow (see for example Section 2.2 and Section 2.3 in [H5]).

There is also a souped-up version of the maximum principle for systems of reaction-diffusion equations. The idea is that for applications sometimes we are in the situation where the reason that a convex set $\mathcal{K}(t)$ is not preserved
by the ODE (4) is that the solution wants to escape from a certain part of the convex set (which we will call the avoidance set $A(t) \subset K(t)$). In this case, if we assume that any solution $\sigma(x, t) : t \in [t_0, T]$ to the PDE (3) starts in $K(t_0) \setminus A(t_0)$ and assume that the solution $\sigma(x, t)$ does not enter subsets $A(t)$ for all $t \geq t_0$ (i.e., $\sigma(x, t) \notin A_x(t) = A(t) \cap V_x$ for all $x \in M$ and all $t \geq t_0$), then $\sigma(x, t)$ remains in $K(t)$ for $t \geq t_0$. A typical example where this happens is when the solution to the Ricci flow is assumed not to have any necklike points (see Theorem 3.3 and 3.4 in Section 2.3 of [H5]).

We define the avoidance space-time track

$$\mathcal{A}T \doteq \{(v, t) \in V \times [0, T] : v \in A(t), \ t \in [0, T]\},$$

and define

$$\mathcal{A}T_x \doteq (\mathcal{A}T) \cap (V_x \times [0, T]).$$

Our second main result is the following:

**Theorem 4.** Let $K(t) \subset V, t \in [0, T]$ be closed subsets which satisfy Conditions III and IV above, and such that the space-time and the avoidance space-time tracks $T$ and $\mathcal{A}T$ are closed. Assume that $u(x, t) : V_x \times T_x M^* \to V_x$ is a smooth family of bundle maps of the diagonal form and that $F(x, \sigma, t)$ is continuous in $x, t$ and is Lipschitz in $\sigma$. Suppose that for any $x \in M, t_0 \in [0, T)$ and any solution $\sigma_x(t)$ of the ODE (4) with initial condition $\sigma_x(t_0) \in K_x(t_0) \setminus A_x(t_0)$, either $\sigma_x(t) \in K_x(t)$ for all $t \geq t_0$, or there is time $t_1$ such that $\sigma_x(t) \in K_x(t) \setminus A_x(t)$ for all $t_0 \leq t < t_1$ and $\sigma_x(t_1) \in A_x(t_1)$. Then for any $t_0 \in [0, T)$ and any solution $\sigma(x, t) : t \in [t_0, T]$ of the PDE (3) satisfying initial condition $\sigma(x, t_0) \in K_x(t_0) \setminus A_x(t_0)$ for all $x \in M$ and satisfying $\sigma(x, t) \notin A_x(t)$ for all $x \in M$ and all $t \geq t_0$, the solution $\sigma(x, t)$ will remain in $K(t)$ for later times.

3. Hamilton’s proof of Theorem 2.

In order to present the proof of our main result, Theorem 3, more clearly and to exhibit the differences between the proofs of Theorem 3 and Hamilton’s Theorem 2, we will first review his proof of Theorem 2 in this section. This will enable us to omit the common parts of the two proofs when we prove Theorem 3.

**Theorem 5.** Let $K \subset V$ be a closed subset satisfying Conditions I and II. Assume that $F(x, \sigma, t)$ is continuous in $x, t$ and is Lipschitz in $\sigma$. Suppose that for any $x \in M$ and any initial time $t_0 \in [0, T)$, and any solution $\sigma_x(t)$ of the ODE (4) which starts in $K_x$ at $t_0$, the solution $\sigma_x(t)$ will remain in $K_x$ for all later times. Then for any initial time $t_0 \in [0, T)$ the solution $\sigma(x, t)$ of the PDE (3) will remain in $K$ for all later times if $\sigma(x, t)$ starts in $K$ at time $t_0$. 
Remark 6. The above result is slightly more general than Theorem 2 in that it allows for a gradient term in the equation (along the lines of the maximum principle for symmetric 2-tensors in [H1]). This does not affect Hamilton’s proof in [H2].

Before proving Theorem 5, we need to recall three lemmas, which are essentially in [H2].

Let \( f : [a, b] \rightarrow \mathbb{R} \) be a function. Then we define \( \frac{d^+ f(t)}{dt} \) at \( t \in [a, b) \) to be the lim sup of forward difference quotients:

\[
\frac{d^+ f(t)}{dt} = \limsup_{s \to 0^+} \frac{f(t + s) - f(t)}{s}.
\]

Lemma 7. Suppose function \( f : [a, b] \rightarrow \mathbb{R} \) is left lower semi-continuous and right-continuous with \( f(a) \leq 0 \). Assume either:

(i) \( \frac{d^+ f(t)}{dt} \leq 0 \) when \( f(t) \geq 0 \) on \( (a, b) \), or

(ii) for some constant \( C < +\infty \), \( \frac{d^+ f(t)}{dt} \leq C \cdot f(t) \) when \( f(t) \geq 0 \) on \( (a, b) \).

Then \( f(t) \leq 0 \) on \( [a, b] \).

Proof. By checking the proof of Lemma 3.1 in [H2], one can prove:

Sublemma: Suppose function \( f(t) : [a, b] \rightarrow \mathbb{R} \) is left lower semi-continuous and right-continuous with \( f(a) \leq 0 \). Assume \( \frac{d^+ f(t)}{dt} \leq 0 \) when \( f(t) \geq 0 \) on \( [a, b] \). Then \( f(t) \leq 0 \) on \( [a, b] \).

Hypothesis (i) in Lemma 7 is a little weaker than the hypothesis in the sublemma since we do not require the inequality to hold at the left endpoint \( a \). By the right continuity at \( t = a \), given any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( f(t) \leq \varepsilon \) on \( [a, a + \delta] \). Now \( f(a + \delta) - \varepsilon \leq 0 \) and \( \frac{d^+ f(t)}{dt} \leq C \cdot f(t) \) when \( f(t) \geq 0 \) on \( [a, a + \delta] \). Hence we may apply the sublemma to \( f(t) - \varepsilon \) to obtain \( f(t) - \varepsilon \leq 0 \) on \( [a, a + \delta, b] \). We conclude that for any \( \varepsilon > 0 \), we have \( f(t) \leq \varepsilon \) for all \( t \in [a, b] \). This proves the lemma under Hypothesis (i).

To prove the lemma under Hypothesis (ii), we set \( g(t) = e^{-C t} \cdot f(t) \). Then \( \frac{d^+ g(t)}{dt} \leq 0 \) when \( g(t) \geq 0 \) on \( (a, b) \). Applying the lemma under Hypothesis (i) to \( g(t) \), we get \( g(t) \leq 0 \) on \( [a, b] \). This implies \( f(t) \leq 0 \) on \( [a, b] \). \( \Box \)

The second lemma below gives a useful characterization of when systems of ordinary differential equations preserve closed convex sets in Euclidean space. Let \( J \subset \mathbb{R}^n \) be a closed convex subset and \( \partial J \) be the boundary of \( J \) in \( \mathbb{R}^n \). For any \( v \in \partial J \) we define the tangent cone \( C_v J \) of \( J \) at \( v \) to be the smallest convex cone in \( \mathbb{R}^n \) with vertex at \( v \) which contains \( J \).

Lemma 8. Let \( U \subset \mathbb{R}^n \) be an open subset and \( J \subset U \) be a closed convex subset. Consider the ODE

\[
\frac{d \tau}{dt} = F(\tau, t),
\]
where $F : U \times [0, T] \to \mathbb{R}^n$ is continuous in $t$ and is Lipschitz in $\tau$. Then the following two statements are equivalent:

(i) For any initial time $t_0 \in [0, T)$, any solution of the ODE (6) which starts in $\mathcal{J}$ at $t_0$ will remain in $\mathcal{J}$ for all later times;

(ii) $v + F(v, t) \in C_v \mathcal{J}$ for all $v \in \partial \mathcal{J}$ and $t \in [0, T)$.

Proof. This is Lemma 4.1 in [H2]. The fact that $F(\tau, t)$ depends on time $t$ does not pose any difficulties for the original proof. □

The third lemma gives a general principle on how to take the derivative of a $\sup$-function which plays an important role in proving Theorem 5. Note that $\mathcal{S}$ in the lemma below is independent of time $t$.

Lemma 9. Let $\mathcal{S}$ be a sequentially compact topological space and let $g : \mathcal{S} \times [a, b] \to \mathbb{R}$ be a function. If $g$ is continuous in $s$ and $t$ and $\frac{\partial g}{\partial t}$ is continuous in $s$ and $t$, then the function $f : [a, b] \to \mathbb{R}$ defined by

$$f(t) = \sup_{s \in \mathcal{S}} g(s, t),$$

is Lipschitz and

$$\frac{d^+ f(t)}{dt} \leq \sup \left\{ \frac{\partial g}{\partial t}(s, t) : s \in \mathcal{S} \text{ satisfies } g(s, t) = f(t) \right\}.$$

Proof. See Lemma 3.5 in [H2]. □

The rest of this section will be devoted to proving Theorem 5. As remarked in the proof of Lemma 4.1 on p. 160 of [H2] we may assume that $\mathcal{K}$ is compact. For if there were a counterexample $\sigma_0(x, t)$ for $t \in [t_0, T]$, then $\sigma_0(x, t)$ will be contained in $V(r)$ for some $r$ large enough, where $V(r)$ is the tubular neighborhood of the zero section in $V$ whose intersection with each fiber $V_x$ is a ball of radius $r$ around origin measured by metric $h$. Let $\eta$ be a cut-off function on $V$ which equals 1 on $V(r)$ and equals to zero on $V \setminus V(2r)$. Then we can modify the PDE (3) as

$$\frac{\partial}{\partial t} \sigma(x, t) = \Delta(t) \sigma(x, t) + u(x, t)(\nabla(t) \sigma(x, t)) + \eta(\sigma(x, t)) \cdot F(x, \sigma(x, t), t).$$

Note that the paths of the counterexample solution $\sigma_0(x, t)$ do not change inside $V(r)$, hence $\sigma_0(x, t)$ still is a solution of (7). If we intersect $\mathcal{K}$ with $V(2r)$, we get a counterexample of Theorem 5 for (7) with the closed compact convex set $V(2r) \cap \mathcal{K} \neq \emptyset$ replacing $\mathcal{K}$, since using Lemma 8 it is easy to check that the ODE

$$\frac{d}{dt} \sigma_x(t) = \eta(\sigma_x(t)) \cdot F(x, \sigma_x(t), t)$$

and $V(2r) \cap \mathcal{K}$ satisfy the assumption of Theorem 5.
Now we assume that \( \mathcal{K} \) is compact. We define the distance between \( \sigma \in V_x \) and \( v \in V_x \) using the metric \( h \) and denote it by \(|\sigma - v|\). We will prove Theorem 5 by contradiction. Suppose we have a solution \( \sigma(x,t) \) of the PDE (3) which starts with \( \sigma(x,t_0) \in \mathcal{K}_x \) for all \( x \in M \) and which goes out of \( \mathcal{K} \) at some time \( t_2 \). Since \( \mathcal{K} \) is closed, we can find a time \( t_1 \geq t_0 \) such that \( \sigma(x,t_1) \in \mathcal{K}_x \) for all \( x \in M \), and for any \( t \in (t_1,t_2) \) there is \( x \) such that \( \sigma(x,t) \notin \mathcal{K}_x \). Below we will focus on the time interval \([t_1,t_2]\).

Define the function

\[
f(t) = \sup_{x \in M} d(\sigma(x,t), \mathcal{K}_x) = \sup_{x \in M} \inf_{v \in \mathcal{K}_x} |\sigma(x,t) - v| \text{ for } t \in [t_1,t_2].
\]

We have \( f(t_1) = 0 \) and \( f(t) > 0 \) for \( t \in (t_1,t_2] \) by assumption. It is easy to check using Condition I that \( f(t) \) is a continuous function of \( t \). Below we will prove that there is a constant \( C < \infty \) such that \( \frac{d^+ f(t)}{dt} \leq C \cdot f(t) \) for \( t \in (t_1,t_2) \). Once this is proved, then \( f(t) \leq 0 \) for \( t \in [t_1,t_2] \) by Lemma 7(ii). Hence \( \sigma(x,t) \in \mathcal{K}_x \) for all \( x \in M \) and all \( t \in (t_1,t_2) \), we get the required contradiction.

For any \( v \in \partial \mathcal{K}_x \), let \( S_v \subset V_x \) be the set of outward normal directions \( n \) of the supporting hyperplanes of \( \mathcal{K}_x \) at \( v \); we require that \( n \) be unit with respect to the metric \( h \). Then, since \( \mathcal{K} \) is nonempty and for each \( t \in (t_1,t_2) \), \( \sigma(x,t) \) is not in \( \mathcal{K}_x \) for some \( x \in M \), it is well-known that

\[
f(t) = \sup_{x \in M} \sup_{v \in \partial \mathcal{K}_x} n \cdot (\sigma(x,t) - v),
\]

where \( \cdot \) is the inner product in \( V_x \) defined by the metric \( h \). Define the set

\[
S = \{(x,v,n) : x \in M, v \in \partial \mathcal{K}_x, n \in S_v \}
\]

and the function

\[
g((x,v,n),t) = n \cdot (\sigma(x,t) - v),
\]

then

\[
f(t) = \sup_{(x,v,n) \in S} g((x,v,n),t).
\]

Note that \( S \) is a compact subset of \( V \otimes V \) independent of time \( t \), we can apply Lemma 9 and get for any \( t \in (t_1,t_2) \)

\[
\frac{d^+ f(t)}{dt} \leq \sup \frac{\partial}{\partial t} [n \cdot (\sigma(x,t) - v)],
\]

where the sup is over all \( (x,v,n) \in S \) such that \( n \cdot (\sigma(x,t) - v) = f(t) \); in particular we have \(|\sigma(x,t) - v| = f(t)\) for these \( (x,v,n) \). We compute at
these \((x,v,n)\)

\[
\frac{\partial}{\partial t} [n \cdot (\sigma(x,t) - v)] = n \cdot \left( \frac{\partial}{\partial t} \sigma(x,t) \right) = n \cdot [\Delta(t)\sigma(x,t)] + n \cdot [u(x,t)(\nabla t)\sigma(x,t)] + n \cdot F(x,\sigma(x,t),t).
\]

By the assumption of Theorem 5 and Lemma 8 we have \(v + F(x,v,t) \in C_vK_x\). Hence \(n \cdot F(x,v,t) \leq 0\) for any \(n \in S_v\) and any \(t \in (t_1,t_2)\). We have

\[
n \cdot F(x,\sigma(x,t),t) \\
\leq n \cdot F(x,\sigma(x,t),t) - n \cdot F(x,v,t) \\
\leq n \cdot [F(x,\sigma(x,t),t) - F(x,v,t)] \\
\leq |F(x,\sigma(x,t),t) - F(x,v,t)| \\
\leq C \cdot |\sigma(x,t) - v| \\
= C \cdot f(t),
\]

where \(C\) is some constant from the assumption that \(F(x,\sigma,t)\) is Lipschitz in \(\sigma\).

We claim that

\[
n \cdot [u(x,t)(\nabla t)\sigma(x,t))] = 0, \\
n \cdot [\Delta(t)\sigma(x,t)] \leq 0,
\]

which will be proved in a moment. This shows

\[
\frac{d^+ f(t)}{dt} \leq C \cdot f(t) \text{ on } (t_1,t_2).
\]

We are left to prove the claim. We will prove \(n \cdot [u(x,t)(\nabla t)\sigma(x,t))] = 0\) and \(n \cdot [\Delta(t)\sigma(x,t)] \leq 0\) together. Recall that \((x,v,n)\) satisfies \(n \cdot (\sigma(x,t) - v) = f(t)\). If we extend a vector in the bundle \(V\) from a point \(x\) by parallel translation along geodesics emanating radially out of \(x\), we get a smooth section of the bundle on some small neighborhood of \(x\) such that all the symmetrized covariant derivatives of the section at \(x\) are zero. Let \(y\) be an arbitrary point in some small neighborhood \(U_x\) of \(x\). We extend \(v \in \partial K_x\) and \(n \in V_x\) in this manner using the connection \(\nabla(t)\) to get \(v_y\) and \(n_y\). Since the connection \(\nabla(t)\) is compatible with the metric \(h\) we continue to have \(|n_y| = 1\), and since \(K\) is invariant under parallel translation we have \(v_y \in \partial K_y\) and \(n_y \in S_{v_y}\) for \(K_y\) at \(v_y\). Therefore

\[
n_y \cdot (\sigma(y,t) - v_y) \leq f(t),
\]
for all \( y \in U_x \). It follows that function \( n_y \cdot (\sigma(y,t) - v_y) \) of \( y \in U_x \) has a local maximum at \( y = x \). So

\[
\frac{\partial}{\partial y}[n_y \cdot (\sigma(y,t) - v_y)] = 0 \text{ at } y = x,
\]

\[
\Delta(t)[n_y \cdot (\sigma(y,t) - v_y)] \leq 0 \text{ at } y = x.
\]

Let \( \nabla_{t,i} \) be the covariant derivative in direction \( \frac{\partial}{\partial y} \) defined by the connection \( \nabla(t) \). Since \( v_y \) and \( n_y \) have their symmetrized covariant derivatives equal to zero at \( y = x \), so \( \nabla_{t,i} n_y = \nabla_{t,i} v_y = 0 \) and \( \Delta(t)n_y = \Delta(t)v_y = 0 \) at \( y = x \). Hence

\[
n \cdot [\nabla_{t,i} \sigma(x,t)] = 0, \quad n \cdot [\Delta(t)\sigma(x,t)] \leq 0.
\]

Then

\[
n \cdot [u(x,t)(\nabla(t)\sigma(x,t))] = n \cdot \left[ \sum_i u^i(x,t) \cdot \nabla_{t,i} \sigma(x,t) \right]
\]

\[
= \sum_i u^i(x,t) \cdot (n \cdot [\nabla_{t,i} \sigma(x,t)]) = 0.
\]

The claim is proved and so is Theorem 5.

4. Proof of Theorem 3.

Throughout this section we will use the same index notation \( i \) to denote a sequence or its subsequence or the subsequence of its subsequence. Our arguments below will involve taking subsequences from time to time. The convention will simplify our notations. Before proving Theorem 3, we first formulate a useful characterization of when systems of ordinary differential equations preserve time-dependent closed convex sets in Euclidean space, i.e., a more general version of Lemma 8.

Let \( J(t) \subset \mathbb{R}^n, 0 \leq t \leq T \) be a family of nonempty closed convex subsets. Define the space-time track

\[
\mathcal{L} = \{(v,t) \in \mathbb{R}^n \times \mathbb{R} : v \in J(t), 0 \leq t \leq T\}.
\]

For each \( (v,t) \in \mathcal{L} \) we define a time-like tangent cone in the forward direction of \( \mathcal{L} \) at \( (v,t) \) and denote it by \( C_{(v,t)}\mathcal{L} \). \( C_{(v,t)}\mathcal{L} \) consists of all \( (W,1) \in \mathbb{R}^n \times \mathbb{R} \) satisfying the following condition: For any sequence \( s_i \to 0^+ \) (i.e., \( s_i \) approaches to zero from positive side), there is a subsequence of \( s_i \) and vectors \( W_i \to W \) such that points \( (v + s_iW_i) \in J(t + s_i) \). Note that the definition is stronger than the conventional definition where one sequence of \( s_i \) is enough. When \( J(t) = J \) is independent of time \( t \), then \( C_{(v,t)}\mathcal{L} = \{C_vJ - v\} \times \{1\} \).
Proposition 10. Let $U \subset \mathbb{R}^n$ be an open subset and $\mathcal{J}(t) \subset U, 0 \leq t \leq T$ be a family of nonempty closed convex subsets such that the space-time track $\mathcal{L}$ is closed. Consider the ODE
\[
\frac{d\tau}{dt} = F(\tau, t),
\]
where $F : U \times [0, T] \to \mathbb{R}^n$ is continuous in $t$ and is Lipschitz in $\tau$. Then the following two statements are equivalent:

(i) For any initial time $t_0 \in [0, T)$, any solution of the ODE (9) which starts in $\mathcal{J}(t_0)$ at time $t_0$ will remain in $\mathcal{J}(t)$ for all later times;

(ii) $(F(v, t), 1) \in C_{(v,t)}\mathcal{L}$ for all $(v, t) \in \partial \mathcal{L}$, where $\partial \mathcal{L}$ is the boundary of $\mathcal{L} \subset \mathbb{R}^{n+1}$.

Proof. (i) $\Rightarrow$ (ii). For any $(v_0, t_0) \in \partial \mathcal{L}$, we consider the solution of (9) with initial condition $\tau(t_0) = v_0$. (i) implies that $\tau(t_0 + s) \in \mathcal{J}(t_0 + s)$ for any $s \in [0, T - t_0]$. Hence

\[
\lim_{s \to 0^+} \frac{(\tau(t_0 + s), t_0 + s) - (\tau(t_0), t_0)}{s} = (F(v_0, t_0), 1) \in C_{(v_0, t_0)}\mathcal{L}.
\]

(ii) $\Rightarrow$ (i). We prove it by contradiction. We will not assume $\mathcal{L}$ to be compact. Suppose we have a solution $\tau(t)$ starting with $\tau(t_0) \in \mathcal{J}(t_0)$ and going out of $\mathcal{L}$ at some time $t_2$, i.e., $\tau(t_2) \notin \mathcal{J}(t_2)$. Since $\mathcal{L}$ is closed, we can find a time $t_1$ such that $\tau(t_1) \in \mathcal{J}(t_1)$ and $\tau(t) \notin \mathcal{J}(t)$ for all $t \in (t_1, t_2)$. Below we will focus on the time interval $[t_1, t_2]$.

Let $\partial \mathcal{J}(t)$ be the boundary of $\mathcal{J}(t) \subset \mathbb{R}^n$. Define the function
\[
l(t) = d(\tau(t), \mathcal{J}(t)) \text{ for } t \in [t_1, t_2]
\]
where $d$ is the Euclidean distance on $\mathbb{R}^n$. It is clear that $l(t_1) = 0$ and $l(t) > 0$ for $t \in (t_1, t_2]$. Because $\mathcal{L}$ is not assumed to be a domain with smooth boundary, the function $l(t)$ is not necessarily continuous.

Lemma 11. Let $\mathcal{J}(t) \subset U, 0 \leq t \leq T$ be a family of nonempty closed convex subsets. If the space-time track $\mathcal{L}$ is closed and satisfies (ii) in Proposition 10, then $l(t)$ is left lower semi-continuous and is right continuous on $[t_1, t_2]$.

Proof of the lemma. To see that $l(t)$ is lower semi-continuous, for any $t \in [t_1, t_2]$ and any $s_i \to 0$ with $t + s_i \in (t_1, t_2]$, we choose $v_i \in \partial \mathcal{J}(t + s_i)$ such that
\[
l(t + s_i) = d(\tau(t + s_i), v_i).
\]
Then either a subsequence $v_i$ will converge to some $v_\infty \in \mathcal{J}(t)$ since $\mathcal{L}$ is closed, or $v_i$ will diverge to $\infty$. In the case of convergence, we have $l(t + s_i) \to d(\tau(t), v_\infty) \geq l(t)$. The lower semi-continuity is true. In the case of divergence, then $l(t + s_i) \to +\infty$. Since $\partial \mathcal{J}(t)$ is nonempty, $l(t)$ is finite. Hence the lower semi-continuity of $l(t)$ is also true.
To prove the right-continuity of \( l(t) \), it suffices to prove the upper right-continuity. We will use (ii) in Proposition 10 which actually puts some restriction on the space-time track \( \mathcal{L} \). It follows from \((\tau(t), t) \notin \mathcal{L} \) for \( t > t_1 \) that \((\tau(t_1), t_1) \in \partial \mathcal{L} \). We denote \( \tau(t_1) \) by \( v_{t_1} \). For any \( t \in (t_1, t_2) \) it follows from \( \tau(t) \notin \mathcal{J}(t) \) that there is \( v_t \in \partial \mathcal{J}(t) \) such that \( l(t) = d(\tau(t), v_t) \). Hence for any \( t \in [t_1, t_2] \) we can find \( v_t \in \mathcal{J}(t) \) such that \( l(t) = d(\tau(t), v_t) \) and \((v_t, t) \in \partial \mathcal{L} \). By (ii) \((F(v_t, t), 1) \in C_{(v_t, t)} \mathcal{L} \). If we fix a \( t \in [t_1, t_2] \), then for any sequence \( s_i \to 0^+ \) we can find a subsequence \( s_i \) such that \((v_t + s_i W_t) \in \mathcal{J}(t + s_i) \) and \( W_t \to F(v_t, t) \). So

\[
l(t + s_i) \leq d(\tau(t + s_i), v_t + s_i W_t).
\]

Letting \( i \to \infty \), we get \( \limsup_{i \to \infty} l(t + s_i) \leq d(\tau(t), v_t) = l(t) \). Hence \( \limsup_{i \to \infty} l(t + s_i) = l(t) \) by the lower semi-continuity of \( l(\cdot) \). The lemma is proved.

Now we go back to the proof of (ii) \( \Rightarrow \) (i) in Proposition 10. Below we will prove that there is some constant \( C < \infty \) such that \( \frac{d^+ l(t)}{dt} \leq C \cdot l(t) \) for all \( t \in (t_1, t_2) \). Once this is proved, then \( l(t) \leq 0 \) for all \( t \in [t_1, t_2] \) by Lemma 11 and Lemma 7. Hence \( \tau(t) \in \mathcal{J}(t) \) for \( t \in [t_1, t_2] \), which is the required contradiction.

Now our proof of the maximum principle with time-dependent convex sets diverges from Hamilton’s proof of his maximum principle. This is a necessity in our approach. The key difference is that we will not use the general principle (Lemma 9). We will calculate \( \frac{d^+ l(t)}{dt} \) directly from the definition. Also our proof will not need the cutoff argument which appeared after Lemma 9. For any \( t \in (t_1, t_2) \) there is a sequence \( s_i \to 0^+ \) such that

\[
\frac{d^+ l(t)}{dt} = \lim_{i \to \infty} \frac{l(t + s_i) - l(t)}{s_i}.
\]

For any \( v \in \partial \mathcal{J}(t) \), as in previous section we define \( S_v \subset \mathbb{R}^n \) to be the set of outward normal directions \( n \) of the supporting hyperplanes of \( \mathcal{J}(t) \) at \( v \); we require that \( n \) be unit with respect to the Euclidean metric. Define

\[
g(v, n, t) = n \cdot [\tau(t) - v].
\]

Since \( \tau(t) \notin \mathcal{J}(t) \) for \( t \in (t_1, t_2) \), we have

\[
l(t) = \sup_{v \in \partial \mathcal{J}(t)} \sup_{n \in S_v} g(v, n, t)
\]

and so we can find a sequence of points \( v_i \in \partial \mathcal{J}(t + s_i) \) and \( n_i \in S_{v_i} \) such that \( g(v_i, n_i, t + s_i) = \sup_{v \in \partial \mathcal{J}(t)} g(v, n, t) \). We can also find \( v_\infty \in \partial \mathcal{J}(t) \) and \( n_\infty \in S_{v_\infty} \) such that \( g(v_\infty, n_\infty, t) = l(t) = |\tau(t) - v_\infty| \). It is not obvious that such \( v_\infty \) exists when \( t = t_1 \); this is one of the reason why we use Lemma 7. The proof below does not need a subsequence of \( v_i \) to converge to \( v_\infty \) or a
subsequence of \( n_i \) to converge to \( n_\infty \).

\[
\frac{d^+l(t)}{dt} = \lim_{i \to \infty} \frac{g(v_i, n_i, t + s_i) - g(v_\infty, n_\infty, t)}{s_i}
= \lim_{i \to \infty} \frac{n_i \cdot [\tau(t + s_i) - v_i] - n_\infty \cdot [\tau(t) - v_\infty]}{s_i}
= \lim_{i \to \infty} \frac{n_i \cdot [\tau(t + s_i) - \tau(t)] + n_i \cdot \tau(t) - n_i \cdot v_i - n_\infty \cdot [\tau(t) - v_\infty]}{s_i}.
\]

Since \((F(v_\infty, t), 1) \in C(v_\infty, t)\mathcal{L}\), we can find a subsequence \( s_i \) and vectors \( F_i \to F(v_\infty, t) \) as \( i \to \infty \) such that \((v_\infty + s_i F_i) \in \mathcal{J}(t + s_i)\). Note that \( v_\infty \in \partial \mathcal{J}(t + s_i) \) and \( n_i \) is the outward normal direction of the supporting hyperplane at \( v_\infty \). We have

\[
 n_i \cdot [v_\infty + s_i F_i - v_i] \leq 0.
\]

Hence

\[
\frac{d^+l(t)}{dt} = \lim_{i \to \infty} \left\{ n_i \cdot \left[ \frac{\tau(t + s_i) - \tau(t)}{s_i} - F_i \right] + \frac{n_i \cdot [v_\infty + s_i F_i - v_i]}{s_i} \right\}
\leq \lim_{i \to \infty} \left\{ n_i \cdot \left[ \frac{\tau(t + s_i) - \tau(t)}{s_i} - F_i \right] + \frac{(n_i - n_\infty) \cdot [\tau(t) - v_\infty]}{s_i} \right\}
\leq \lim_{i \to \infty} n_i \cdot \left[ \frac{\tau(t + s_i) - \tau(t)}{s_i} - F_i \right]
\leq \lim_{i \to \infty} \left| \frac{\tau(t + s_i) - \tau(t)}{s_i} - F_i \right|
= |F(\tau(t), t) - F(v_\infty, t)|
\leq C \cdot |\tau(t) - v_\infty|
= C \cdot l(t).
\]

We have used \((n_i - n_\infty) \cdot [\tau(t) - v_\infty] \leq 0\) to get the second inequality above. This is because \( n_i \cdot [\tau(t) - v_\infty] \leq |\tau(t) - v_\infty| \) and \( |\tau(t) - v_\infty| = n_\infty \cdot [\tau(t) - v_\infty] \). We have used \(|n_i| = 1\) to get the third inequality above. Proposition 10 is now proved.

The rest of this section is devoted to the proof of Theorem 3. We will prove it by contradiction. Suppose we have a solution \( \sigma(x, t) \) of the PDE (3) on \([t_0, T]\) which starts with \( \sigma(x, t_0) \in \mathcal{K}_x(t_0) \) for all \( x \in M \) and which goes out of space-time track \( \mathcal{J} \) at some time \( t_2 \). Since \( \mathcal{J} \) is closed, there is a time \( t_1 \geq t_0 \) such that \( \sigma(x, t_1) \in \mathcal{K}_x(t_1) \) for all \( x \in M \) and for any \( t_1 < t < t_2 \) there is \( x \) such that \( \sigma(x, t) \notin \mathcal{K}_x(t) \). Below we will focus on the time interval \([t_1, t_2]\).
Define the function
\begin{equation}
(10) \quad f(t) = \sup_{x \in M} d(\sigma(x,t), K_x(t)) \text{ for } t \in [t_1, t_2]
\end{equation}
where \( d \) is distance on \( V_x \) defined by the metric \( h \). It is clear from our choice that \( f(t_1) = 0 \), \( f(t) > 0 \) for \( t > t_1 \). Note that \( f(t) \) is not necessarily continuous.

Next we prove a lemma which will enable us later to apply Lemma 7 to \( f(t) \) defined by (10). Let \( \hat{\sigma}(x,t) \) be any continuous section of bundle \( V \) which satisfies that \( \hat{\sigma}(x,t_1) \in K_x(t_1) \) for all \( x \in M \) and where for each \( t \in (t_1, t_2) \) there is \( x \) such that \( \hat{\sigma}(x,t) \) is not in \( K_x(t) \). We define the function \( \hat{g}: M \times [t_1, t_2] \to \mathbb{R} \) by
\[
\hat{g}(x,t) = d(\hat{\sigma}(x,t), K_x(t)),
\]
and define the function
\[
\hat{f}(t) = \sup_{x \in M} \hat{g}(x,t) \text{ for } t \in [t_1, t_2].
\]
By assumption \( \hat{f}(t_1) = 0 \), and \( \hat{f}(t) > 0 \) for \( t \in (t_1, t_2) \). For any \( t \in [t_1, t_2] \) and any sequence \( s_i \to 0^+ \), there is a subsequence \( s_i \) and a sequence \( x_i \in M \) such that \( \hat{g}(x_i, t + s_i) = \sup_{x \in M} \hat{g}(x, t + s_i) \) and \( x_i \to x_\infty \).

**Lemma 12.** For the space-time track \( T \) satisfying the assumption of Theorem 3, \( \hat{f}(t) \) is left lower semi-continuous and is right-continuous on \( [t_1, t_2] \), and for \( t \in [t_1, t_2] \) the above chosen \( x_\infty \) satisfies
\[
\hat{g}(x_\infty, t) = \hat{f}(t).
\]

**Proof of the lemma.** First we show that \( \hat{f}(t) \) is lower semi-continuous. \( \hat{f}(t) \) is obviously lower semi-continuous at \( t = t_1 \). At any \( t = t_a \in (t_1, t_2) \), we have \( \hat{f}(t_a) > 0 \). We fix \( x_a \) such that \( \hat{f}(t_a) = \hat{g}(x_a, t_a) \). Then since \( T \) is closed, there is an \( \varepsilon > 0 \) such that \( \hat{\sigma}(x_a, t) \notin K_{x_a}(t) \) for \( t \in (t_a - \varepsilon, t_a + \varepsilon) \).

We can apply Lemma 11 to \( \hat{g}(x_a, t) \) in the fiber \( V_{x_a} \) to conclude that \( \hat{g}(x_a, \cdot) \) is lower semi-continuous at \( t = t_a \). Hence for any \( s_i \to 0 \)
\[
\lim_{i \to +\infty} \inf \hat{f}(t_a + s_i) \geq \lim_{i \to +\infty} \inf \hat{g}(x_a, t_a + s_i) \geq \hat{g}(x_a, t_a) = \hat{f}(t_a).
\]
Hence \( \hat{f}(t) \) is lower semi-continuous at time \( t = t_a \), and hence on \([t_1, t_2] \).

To prove the right-continuity of \( \hat{f}(t) \), it suffices to prove the upper right-continuity. For any \( t_a \in [t_1, t_2] \) and any sequence \( s_i \to 0^+ \) we will show that there is a subsequence \( s_i \) such that \( \lim_{i \to +\infty} \hat{f}(t_a + s_i) \leq \hat{f}(t_a) \). By passing to a subsequence if necessarily we may assume that \( \lim_{i \to +\infty} \hat{f}(t_a + s_i) \) exists. Choose \( x_i \in M \) satisfying \( \hat{f}(t_a + s_i) = \hat{g}(x_i, t_a + s_i) \); without loss of generality, we may assume that \( x_i \to x_\infty \) by taking a subsequence if necessary. Let \( v_\infty \in K_{x_\infty}(t_a) \) such that \( \hat{g}(x_\infty, t_a) = d(\hat{\sigma}(x_\infty, t_a), v_\infty) \). It follows from \( \hat{f}(t_a + s_i) > 0 \), the invariance of \( T_x \) under parallel translation and
the closedness of $T$, that $(v_\infty, t_a) \in \partial T_{x_\infty}$. By the assumption of Theorem 3
and Proposition 10, $C_{(v_\infty, t_a)} T_{x_\infty}$ is nonempty. Then there is a subsequence
$(v_\infty + s_i W_i) \in K_{x_\infty} (t_a + s_i)$ with $W_i \rightarrow W$ for some $W \in V_{x_\infty}$. Hence
\begin{equation}
\lim_{i \rightarrow \infty} d(\hat{\sigma}(x_\infty, t_a + s_i), v_\infty + s_i W_i) \geq d(\hat{\sigma}(x_\infty, t_a + s_i), K_{x_\infty} (t_a + s_i)).
\end{equation}
Since $\hat{\sigma}(x, t_a)$ is continuous in $x$ and $K_{x} (t_a + s_i)$ is invariant under parallel
translation $V(t_a + s_i)$ for any $x \in M$, $d(\hat{\sigma}(x_\infty, t_a + s_i), K_{x_\infty} (t_a + s_i))$
can be chosen arbitrarily close to $d(\hat{\sigma}(x_i, t_a + s_i), K_{x_i} (t_a + s_i))$ when $i$ is large, so the right side of (11) approaches $\lim_{i \rightarrow +\infty} f(t_a + s_i)$. The left side of (11) approaches
\[d(\hat{\sigma}(x_\infty, t_a), v_\infty) = d(\hat{\sigma}(x_\infty, t_a), K_{x_\infty} (t_a)) \leq \hat{f}(t_a).
\]
Now we have proved $\lim_{i \rightarrow +\infty} \hat{f}(t_a + s_i) \leq \hat{f}(t_a)$ and hence the right continuity of $\hat{f}(t)$.

By taking the limit of (11) we have
\[\hat{g}(x_\infty, t_a) = d(\hat{\sigma}(x_\infty, t_a), K_{x_\infty} (t_a)) \geq \lim_{i \rightarrow +\infty} \hat{f}(t_a + s_i).
\]
Since $\hat{f}(t)$ is right-continuous and $\hat{g}(x_\infty, t_a) \leq \hat{f}(t_a)$, we conclude that
\[\hat{g}(x_\infty, t_a) = \hat{f}(t_a) \text{ for any } t_a \in [t_1, t_2].\]
The lemma is proved.

Now we go back to the proof of Theorem 3. Let $f(t)$ be the function defined in (10), we will prove that there is a constant $C < +\infty$ such that
\[\frac{d^+ f(t_a)}{dt} \leq C \cdot f(t) \text{ for } t \in (1, t_2).\]
Once this is proved, from Lemma 12 and Lemma 7 we conclude that $f(t) = 0$ for $t \in [t_1, t_2]$, and hence $\sigma(x, t) \in K_x(t)$ for all $x \in M$ and $t \in [t_1, t_2]$. We get the required contradiction.

For any $t_a \in (t_1, t_2)$ there exists a sequence $s_i \rightarrow 0^+$ such that
\[\frac{d^+ f(t_a)}{dt} = \lim_{i \rightarrow -\infty} \frac{f(t_a + s_i) - f(t_a)}{s_i}.
\]
We define the function
\[g(x, v, n, t) = n \cdot [\sigma(x, t) - v], \text{ for } x \in M, n \in V_x, v \in V_x, \text{ and } t \in [t_1, t_2].
\]
For any $v \in \partial K_x(t)$, we define $S_v \subset V_x$ to be the set of the outward unit normal directions $n$ of the supporting hyperplanes of $K_x(t)$ in $V_v$ at $v$. Then, for any $t > t_1$ since $K_x(t)$ is not empty and $\sigma(x, t)$ is not in the interior of $K_x(t)$ for some $x \in M$, it is well-known that
\[f(t) = \sup_{x \in M} \sup_{v \in \partial K_x(t)} \sup_{n \in S_v} g(x, v, n, t).
\]
Note that the set over which we take the supremum in the definition of $f(t)$ depends on time. This is why we compute $\frac{d^+ f(t_a)}{dt}$ directly rather than using Lemma 9.

We can find a sequence of points $x_i \in M$, $v_i \in \partial K_{x_i}(t_a + s_i)$, and $n_i \in S_{v_i}$, such that $g(x_i, v_i, n_i, t_a + s_i) = f(t_a + s_i)$, by Lemma 12 we may assume $x_i \rightarrow
We claim that there is a sequence of vectors \( n_i \) such that for any \( \varepsilon > 0 \) there is an \( i_0 \) such that for any \( i \geq i_0 \) we have \( v^*_i + s_i F_i \in K_{x_i(t_a)}(t_a) \) and \( |F_i - F(x, v^*_i, t_a)| \leq \varepsilon \). The claim can be proved by studying a family indexed by \( i \) of ODE (4) in \( V_{x_i} \) with initial time \( t_a \) and initial value \( \sigma_{x_i}(t_a) = v^*_i \). We write the solution \( \sigma_{x_i}(t_a + s_i) = v^*_i + s_i F_i \). It follows from the assumption of Theorem 3 that \( \sigma_{x_i}(t_a + s_i) \in K_{x_i(t_a + s_i)} \). Since \( F(x, \sigma, t) \) is Lipschitz in \( \sigma \), the inequality \( |F_i - F(x, v^*_i, t_a)| \leq \varepsilon \) follows from the fact that solutions of
ordinary differential equations depend continuously on their parameters, in this case the parameters are \( x_i \in \mathcal{M} \) and \( v_{i^*}^* \in \partial \mathcal{K}_{x_i}(t_a) \) varying in compact domain.

Since \( d(\sigma(x_i, t_a), \mathcal{K}_{x_i}(t_a)) = d(\sigma(x_i, t_a), v_{i^*}^*) \leq f(t_a) < \infty \), we can rule out the divergence of \( v_{i^*}^* \) to \( \infty \). We may assume that a subsequence \( v_{i^*}^* \) converges to \( v_\infty^* \in V_{\infty}^* \), and get \( d(\sigma(x_\infty, t_a), \mathcal{K}_{x_\infty}(t_a)) = d(\sigma(x_\infty, t_a), v_\infty^*) \). By the closedness of the space-time track \( T \) we have \( v_\infty^* \in \mathcal{K}_{x_\infty}(t_a) \).

Hence

\[
d(\sigma(x_\infty, t_a), v_{i^*}^*) = d(\sigma(x_\infty, t_a), \mathcal{K}_{x_\infty}(t_a)) = d(\sigma(x_\infty, t_a), v_\infty^*)
\]

and \( \mathcal{K}_{x_\infty}(t_a) \) is convex, we conclude that \( v_\infty^* = v_\infty \). Our choice of \( F_i \) ensures that \( \lim_{i \to \infty} F_i = F(\mathcal{M}_\infty, v_{i^*}^*, t_a) = F(\mathcal{M}_\infty, v_\infty^*, t_a) \). Recall that \( v_i \in \partial \mathcal{K}_{x_i}(t_a + s_i) \) and \( n_i \) is the outward normal direction of the supporting hyperplane at \( v_i \). We have in each fiber \( V_{x_i} \) and at time \( t_a + s_i \)

\[
n_i \cdot [v_i^* + s_i F_i - v_i] \leq 0.
\]

Hence

\[
\frac{d^+ f(t_a)}{dt} = \lim_{i \to \infty} \left\{ n_i \cdot \left[ \frac{\sigma(x_i, t_a + s_i) - \sigma(x_i, t_a)}{s_i} - F_i \right] + n_i \cdot \frac{[v_i^* + s_i F_i - v_i]}{s_i} + n_i \cdot \frac{[\sigma(x_i, t_a) - v_i^*] - n_\infty \cdot [\sigma(x_\infty, t_a) - v_\infty]}{s_i} \right\}
\]

\[
\leq \lim_{i \to \infty} \left\{ n_i \cdot \left[ \frac{\sigma(x_i, t_a + s_i) - \sigma(x_i, t_a)}{s_i} - F_i \right] + n_i \cdot \frac{[\sigma(x_i, t_a) - v_i^*] - n_\infty \cdot [\sigma(x_\infty, t_a) - v_\infty]}{s_i} \right\}
\]

\[
\leq \lim_{i \to \infty} \left\{ n_i \cdot \left[ \frac{\sigma(x_i, t_a + s_i) - \sigma(x_i, t_a)}{s_i} - F_i \right] \right\},
\]

where to get the last inequality above we have used

\[
n_i \cdot [\sigma(x_i, t_a) - v_i^*] \leq n_\infty \cdot [\sigma(x_\infty, t_a) - v_\infty].
\]

This is because

\[
n_i \cdot [\sigma(x_i, t_a) - v_i^*] \leq |\sigma(x_i, t_a) - v_i^*| = d(\sigma(x_i, t_a), \mathcal{K}_{x_i}(t_a)),
\]

and at time \( t_a \)

\[
d(\sigma(x_i, t_a), \mathcal{K}_{x_i}(t_a)) \leq f(t_a) = d(\sigma(x_\infty, t_a), \mathcal{K}_{x_\infty}(t_a)) = n_\infty \cdot [\sigma(x_\infty, t_a) - v_\infty].
\]
by our choice of \( x_\infty, v_\infty, \) and \( n_\infty. \)

\[
\frac{d^+ f(t_a)}{dt} \leq \left[ \lim_{i \to \infty} n_i \right] \cdot \left[ \lim_{i \to \infty} \frac{\sigma(x_i, t_a + s_i) - \sigma(x_i, t_a)}{s_i} \right] - \lim_{i \to \infty} F_i
\]

\[
= n_\infty \cdot \left[ \frac{\partial}{\partial t} \sigma(x_\infty, t_a) - F(x_\infty, v_\infty, t_a) \right]
\]

\[
= n_\infty \cdot \left[ \Delta(t_a)\sigma(x_\infty, t_a) + u(x_\infty, t_a)(\nabla(t_a)\sigma(x_\infty, t_a)) + F(x_\infty, \sigma(x_\infty, t_a), t_a) - F(x_\infty, v_\infty, t_a) \right]
\]

By the same argument as in Section 3 we conclude that

\[
n_\infty \cdot [\Delta(t_a)\sigma(x_\infty, t_a)] \leq 0,
\]

\[
n_\infty \cdot [u(x_\infty, t_a)(\nabla(t_a)\sigma(x_\infty, t_a))] = 0.
\]

So

\[
\frac{d^+ f(t_a)}{dt} \leq n_\infty \cdot [F(x_\infty, \sigma(x_\infty, t_a), t_a) - F(x_\infty, v_\infty, t_a)]
\]

\[
\leq |F(x_\infty, \sigma(x_\infty, t_a), t_a) - F(x_\infty, v_\infty, t_a)|
\]

\[
\leq C \cdot |\sigma(x_\infty, t_a) - v_\infty|
\]

\[
= C \cdot f(t_a).
\]

Theorem 3 is proved.

5. Proof of Theorem 4.

First we prove a version of Proposition 10 subject to an avoidance set.

**Proposition 13.** Let \( U \subset \mathbb{R}^n \) be an open subset, \( J(t) \subset U, t \in [0, T] \) be a family of nonempty closed convex subsets and \( B(t) \subset J(t) \) be avoidance sets such that the space-time track \( \mathcal{L} \) and the avoidance space-time track \( \mathcal{B} \mathcal{L} = \{(v, t) \in \mathbb{R}^n \times \mathbb{R} : v \in B(t), t \in [0, T]\} \) are closed. Consider the ODE

\[
(15) \quad \frac{d\tau}{dt} = F(\tau, t),
\]

where \( F : U \times [0, T] \to \mathbb{R}^n \) is continuous in \( t \) and is Lipschitz in \( \tau \). Then the following two statements are equivalent:

(i) For any \( t_0 \in [0, T] \) and any solution \( \tau(t), t \in [t_0, T] \) of the ODE (15) with initial condition \( \tau(t_0) \in J(t_0) \setminus B(t_0) \), either \( \tau(t) \in J(t) \) for all \( t \geq t_0 \), or there is a time \( t_1 > t_0 \) such that \( \tau(t) \in J(t) \setminus B(t) \) for all \( t \in [t_0, t_1) \) and \( \tau(t_1) \in B(t_1) \).

(ii) \( (F(v, t), 1) \in C(v, t) \mathcal{L} \) for all \( (v, t) \in (\partial \mathcal{L}) \setminus (\mathcal{B} \mathcal{L}) \).
Proof. This proposition can be proved as Proposition 10 except for the following issue which arises in proving (ii) \( \implies \) (i): In the proof of Proposition 10 we have used the property \((F(v, t), 1) \in C(v, t) \mathcal{L}\) for all \((v, t) \in \partial \mathcal{L}\), however here this property holds only for \((v, t) \in (\partial \mathcal{L}) \setminus (\mathcal{B} \mathcal{L})\). We need to ensure that \((v, t)\) can be chosen in \((\partial \mathcal{L}) \setminus (\mathcal{B} \mathcal{L})\) when we use this property in the proof of Proposition 10.

We adopt the notations used in the proof of Proposition 10 and resolve the issue. Since \(\mathcal{B} \mathcal{L}\) is closed and the solution \(\tau(t), t \in [t_1, t_2]\) in the proof of Lemma 11 does not enter in \(\mathcal{B} \mathcal{L}\), there is a constant \(\varepsilon > 0\) such that

\[
\inf_{t \in [t_1, t_2]} d(\tau(t), B(t)) \geq 3\varepsilon.
\]

Since \((v_1, t_1) = (\tau(t_1), t_1) \in (\partial \mathcal{L}) \setminus (\mathcal{B} \mathcal{L}), l(t)\) is right-continuous at \(t_1\) by the proof of Lemma 11. Hence there is \(t_3 \in (t_1, t_2)\) such that \(f(t) \leq \varepsilon\) for all \(t \in (t_1, t_3)\). For any \(t \in (t_1, t_3)\)

\[
d(v_t, B(t)) \geq d(\tau(t), B(t)) - d(v_t, \tau(t)) \geq 2\varepsilon,
\]

hence \((v_t, t) \in (\partial \mathcal{L}) \setminus (\mathcal{B} \mathcal{L})\) for all \(t \in (t_1, t_3)\) and again \(l(t)\) can be shown to be left lower semi-continuous and right-continuous on \([t_1, t_3]\).

For any \(t \in (t_1, t_3)\), choose the points \((v_{\infty}, t)\) in \(\partial \mathcal{L}\) as in the proof of Proposition 10. These points are at least \(2\varepsilon\) away from \(\mathcal{B} \mathcal{L}\), so by Statement (ii) we still have the property \((F(v_{\infty}, t), 1) \in C(v_{\infty}, t) \mathcal{L}\), which was use in the proof of Proposition 10. We may now repeat the rest of the proof of Proposition 10 to conclude that there is a constant \(C < +\infty\) such that

\[
\frac{d^+ f(t)}{dt} \leq C \cdot f(t)
\]

for all \(t \in (t_1, t_3)\). By Lemma 7 we get \(l(t) = 0\) on \([t_1, t_3]\), which is the required contradiction.

The intuition behind the proof of Theorem 4 is as follows: Outside the avoidance set (where the solution is assumed not to enter) the reaction term of the PDE (i.e., corresponding to the associated ODE) wants to push the solution back into the convex set. The diffusion part wants to keep the solution in the convex set, possibly trying (but not succeeding) to push it into the avoidance part.

Proof of Theorem 4. We will prove it by contradiction. As in the proof of Theorem 3, suppose we have a solution \(\sigma(x, t)\) of PDE (3) on \([t_0, T]\) which starts with \(\sigma(x, t_0) \in K_x(t_0) \setminus A_x(t_0)\) for all \(x \in M\) and which goes out of the space-time track \(T\) at some time \(t_2\). Since \(T\) is closed, there is a time \(t_1 \geq t_0\) such that \(\sigma(x, t_1) \in K_x(t_1)\) for all \(x\) and for any \(t \in (t_1, t_2)\) there is \(x\) such that \(\sigma(x, t) \notin K_x(t)\). Below we will focus on the time interval \([t_1, t_2]\).

We define function

\[
f(t) = \sup_{x \in M} d(\sigma(x, t), K_x(t))\text{ for } t \in [t_1, t_2]
\]
where \( d \) is the distance on \( V \) defined by the metric \( h \). It is clear that 
\[ f(t_1) = 0 \] and \( f(t) > 0 \) for \( t > t_1 \).

Since the avoidance space-time track \( \mathcal{AT} \) is closed and \( \sigma(x,t) \notin \mathcal{AT} \) for all \( x \in M \) and \( t \in [t_1, t_2] \), there is an \( \varepsilon > 0 \) such that 
\[ \inf_{x \in M, t \in [t_1, t_2]} d(\sigma(x,t), A_x(t)) \geq 3\varepsilon. \]

By Proposition 13 we have \( (F(x,v,t), 1) \in C_{(v,t)}(T_x) \) for all \( (v,t) \in (\partial T_x) \setminus (\mathcal{AT}_x) \), however we have used the property \( (F(x,v,t), 1) \in C_{(v,t)}(T_x) \) for all \( (v,t) \in \partial T_x \) in the proof of Lemma 12, we need to modify the proof of Lemma 12 to show that \( f(t) \) is left lower semi-continuous and right-continuous. We adopt the notations used in the proof of Lemma 12 and replace \( \tilde{\sigma}(x,t) \) by \( \sigma(x,t) \). When \( t_a = t_1 \), \( (v_\infty, t_1) = (\sigma(x,t_1), t_1) \in (\partial T_{x_\infty}) \setminus (\mathcal{AT}_{x_\infty}) \), \( f(t) \) is right-continuous at \( t_1 \) by the same proof. Hence there is \( t_3 \in (t_1, t_2) \) such that \( f(t) \leq \varepsilon \) for all \( t \in (t_1, t_3) \). For any \( t_a \in (t_1, t_3) \)
\[ d(v_\infty, A(t_a)) \geq d(\sigma(x_\infty, t_a), A(t_a)) - d(v_\infty, \sigma(x_\infty, t_a)) \geq 2\varepsilon, \]
so \( (v_\infty, t_a) \in (\partial T_{x_\infty}) \setminus (\mathcal{AT}_{x_\infty}) \) for all \( t_a \in (t_1, t_3) \) and \( f(t) \) is left lower semi-continuous and right-continuous on \([t_1, t_3] \).

We will prove that there is a constant \( C < +\infty \) such that \( \frac{d^+ f(t)}{dt} \leq C \cdot f(t) \) for all \( t \in (t_1, t_3) \), then by Lemma 7 we get \( f(t) = 0 \) for all \( t \in [t_1, t_3] \), which is the required contradiction.

For any \( t \in (t_1, t_3) \) since \( f(t) = \sup_{x \in M} d(\sigma(x,t), \mathcal{K}_x(t)) < \varepsilon \), all the points in \( T \) we choose in the proof of Theorem 3 are at least \( 2\varepsilon \) away from \( \mathcal{AT} \), so we can repeat the proof of Theorem 3 to conclude that \( \frac{d^+ f(t)}{dt} \leq C \cdot f(t) \) for all \( t \in (t_1, t_3) \). Hence Theorem 4 is proved. \( \square \)

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References


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COMPLETE EMBEDDINGS OF THE COHEN ALGEBRA INTO THREE FAMILIES OF C.C.C., NON-MEASURABLE BOOLEAN ALGEBRAS

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The Cohen algebra embeds as a complete subalgebra into three classic families of complete, atomless, c.c.c., non-measurable Boolean algebras; namely, the families of Argyros algebras and Galvin-Hajnal algebras, and the atomless part of each Gaifman algebra. It immediately follows that the weak \((\omega, \omega)\)-distributive law fails everywhere in each of these Boolean algebras.

1. Introduction.

Von Neumann conjectured that the countable chain condition and the weak \((\omega, \omega)\)-distributive law characterize measurable algebras among Boolean \(\sigma\)-algebras \([\text{Mau}]\). Consistent counter-examples have been obtained by Maharam \([\text{Mah}]\), Jensen \([\text{J}]\), Glówczyński \([\text{Gl}]\), and Veličković \([\text{V}]\). However, whether von Neumann’s proposed characterization of measurable algebras fails within ZFC remains an open problem.

In searching for possible counter-examples to von Neumann’s proposed characterization of measurable algebras within ZFC, we investigated three families of complete, c.c.c., non-measurable Boolean algebras, namely, the Argyros, Galvin-Hajnal, and Gaifman algebras, to find out whether these Boolean algebras sustain any weak form of distributivity. By the Cohen algebra we mean the completion of any countable, atomless Boolean algebra. By the \(\kappa\)-Cohen algebra we mean the completion of the free Boolean algebra on \(\kappa\)-many generators. We found the following:

**Theorem 1.1.**

1) It is possible to construct both atomless Gaifman algebras and Gaifman algebras with atoms.

2) The \(\text{cf}(2^\omega)\)-Cohen algebra embeds as a complete subalgebra into each Galvin-Hajnal algebra.

3) The Cohen algebra embeds as a complete subalgebra into each Argyros algebra and related variants, and into the atomless part of each Gaifman algebra.

We show 1) in §4, 2) in §2, and 3) in §§3 and 5.
Throughout this paper, let $B$ denote a Boolean algebra and $P$ denote a partial ordering. Let $B^+$ denote $B \setminus \{0\}$. Galvin and Hajnal, and Argyros constructed families of partial orderings and Gaifman constructed a family of Boolean algebras to establish strict implications between various chain conditions, including the c.c.c., the $\sigma$-bounded c.c., and $\text{CUP}(B^+)$. Recall the following definitions:

**Definition 1.2 ([Ko]).** $P$ satisfies the countable chain condition (c.c.c.) if for each pairwise incompatible subset $X \subseteq P$, $|X| \leq \omega$.

**Definition 1.3 ([F]).** $P$ satisfies the $\sigma$-bounded chain condition ($\sigma$-bounded c.c.) if there exist subsets $X_n \subseteq P$, $n < \omega$, such that $P = \bigcup_{n<\omega} X_n$, where $\forall n < \omega$ each pairwise incompatible subset of $X_n$ has cardinality $\leq n + 1$.

**Definition 1.4 ([GP]).** Let $S \subseteq P$ be a nonempty set. For $s_0, \ldots, s_n \in S$, not necessarily distinct, let

\[(1.1) \quad \alpha^*(s_0, \ldots, s_n) = \frac{1}{n+1} \max\{|I| : I \subseteq n + 1, \exists p \in P \forall i \in I s_i \geq p\}.\]

The intersection number of $S$ is

\[(1.2) \quad \alpha(S) = \inf \{\alpha^*(s_0, \ldots, s_n) : n \in \omega, s_0, \ldots, s_n \in S\}.\]

We say that $\text{CUP}(P)$ holds if $P$ is a countable union of subsets of $P$, each of which has positive intersection number; i.e., there exist subsets $X_n \subseteq P$, $n < \omega$, such that $P = \bigcup_{n<\omega} X_n$ and $\forall n < \omega$, $\alpha(X_n) > 0$.

The above three notions are defined for a Boolean algebra $B$ by replacing $P$ with $B^+$ in Definitions 1.2-1.4. Note that for a partial ordering $P$, the above three notions are preserved under completions. That is, $P$ satisfies the c.c.c. ($\sigma$-bounded c.c.) iff $r.o. (P)$ satisfies the c.c.c. ($\sigma$-bounded c.c.), respectively, and $\text{CUP}(P)$ holds iff $\text{CUP}(r.o.(P)^+)$ holds.

It is easy to see that $\text{CUP}(P)$ implies the $\sigma$-bounded c.c., which in turn implies the c.c.c. Gaifman showed that $\text{CUP}(P)$ is strictly stronger than the $\sigma$-bounded c.c. [Ga]. Galvin and Hajnal showed that the $\sigma$-bounded c.c. is strictly stronger than the c.c.c. [CN].

**Definition 1.5 ([Ko]).** $B$ satisfies the weak $(\omega, \omega)$-distributive law (weak $(\omega, \omega)$-d.l.) if for each family $(b_{ij})_{i<\omega, j<\omega}$ of elements of $B$,

\[(1.3) \quad \bigwedge_{i<\omega} \bigvee_{j<\omega} b_{ij} = \bigvee_{f: \omega \rightarrow \omega} \bigwedge_{i<\omega} \bigvee_{j \leq f(i)} b_{ij},\]

provided that $\bigvee_{j<\omega} b_{ij}$ for each $i < \omega$, $\bigwedge_{i<\omega} \bigvee_{j<\omega} b_{ij}$, and $\bigwedge_{i<\omega} \bigvee_{j \leq f(i)} b_{ij}$ for each $f : \omega \rightarrow \omega$ exist in $B$. We say that the weak $(\omega, \omega)$-d.l. fails everywhere in $B$ if there exist $(b_{ij})_{i<\omega, j<\omega} \subseteq B$ such that $\bigwedge_{i<\omega} \bigvee_{j<\omega} b_{ij} = 1$ and $\bigvee_{f: \omega \rightarrow \omega} \bigwedge_{i<\omega} \bigvee_{j \leq f(i)} b_{ij} = 0$. 
Definition 1.6 ([Ko]). A Boolean \( \sigma \)-algebra \( B \) is measurable if there exists a function \( \mu : B \to [0, \infty) \) which is strictly positive (\( \mu(b) = 0 \iff b = 0 \)) and \( \sigma \)-additive (\( \mu(\bigvee_{i<\omega} b_i) = \Sigma_{i<\omega} \mu(b_i) \) for all pairwise disjoint \( \{ b_i : i < \omega \} \subseteq B \)).

Theorem 1.7 ([Ko]). Every measurable Boolean algebra \( B \) satisfies the weak \((\omega, \omega)\)-d.l. and the c.c.c.

Theorem 1.8 (Kelley, [Ke]). CUP\((B^+)\) holds iff \( B \) carries a strictly positive, finitely-additive measure \( \mu : B \to [0, \infty) \).

The following Theorem 1.9 of Kelley completely characterizes measurable algebras among Boolean \( \sigma \)-algebras in ZFC by strengthening the c.c.c. to CUP\((B^+)\).

Theorem 1.9 (Kelley, [Ke]). If \( B \) is a \( \sigma \)-algebra, then \( B \) is measurable iff \( B \) satisfies the weak \((\omega, \omega)\)-d.l. and CUP\((B^+)\) holds.

Theorem 1.10 (Folklore). The weak \((\omega, \omega)\)-d.l. fails everywhere in the Cohen algebra.

Each of our complete embeddings of the Cohen algebra will involve the following notions and lemmas:

Definition 1.11 ([Ko]). A subalgebra \( A \) of a Boolean algebra \( B \) is a regular subalgebra of \( B \) if for each \( M \subseteq A \) such that \( \bigvee^A M \) exists in \( A \), \( \bigvee^B M \) exists in \( B \) and \( \bigvee^A M = \bigvee^B M \).

Lemma 1.12. Let \( B \) be a Boolean algebra, \( P \) a dense subset of \( B^+ \), and \( A \) a subalgebra of \( B \). \( A \) is a regular subalgebra of \( B \) iff \( \forall p \in P \exists a_p \in A^+ \) such that whenever \( a \in A \) and \( a_p \land a \neq 0 \), then \( p \land a \neq 0 \).

Definition 1.13 ([Ko]). A subalgebra \( A \) of a Boolean algebra \( B \) is a complete subalgebra of \( B \) if for each subset \( M \subseteq A \) such that \( \bigvee^B M \) exists, \( \bigvee^A M \) exists and \( \bigvee^B M = \bigvee^A M \).

Definition 1.14 ([Ko]). A monomorphism \( f : A \to B \) is complete if for each \( M \subseteq A \) for which \( \bigvee^A M \) exists, \( f(\bigvee^A M) = \bigvee^B \{ f(b) : b \in M \} \).

The following lemma is a natural consequence of the Sikorski Extension Theorem:

Lemma 1.15 ([Ko]). If \( B \) is a complete Boolean algebra and \( A \) is a regular subalgebra of \( B \), then there is a complete monomorphism from r.o.\((A^+)\) into \( B \).

Throughout this paper, let \( P_{GH}, P_A, \) and \( B_G \) denote members of the families of Galvin-Hajnal partial orderings, Argyros partial orderings, and Gaifman algebras, respectively. Galvin and Hajnal, Argyros, and Gaifman...
showed that in $r.o.(P_{GH})$, $r.o.(P_A)$, and $B_G$, respectively, $\text{CUP}(B^+)$ fails; thus, by Kelley's Theorem 1.9, these three Boolean algebras are not measurable. However, this is not the only reason measurability fails in these algebras. By Theorem 1.10, completely embedding the Cohen algebra into $r.o.(P_{GH})$, $r.o.(P_A)$, and the atomless part of $r.o.(B_G^+)$ shows that no weak form of distributivity holds in these Boolean algebras.

**Remark 1.16.** We thank the referee for pointing out the following: By a result of Shelah, in c.c.c. Suslin forcings, adding a Cohen real is equivalent to the weak $(\omega, \omega)$-d.l. failing everywhere $[S]$. Hence, to prove 3) of Theorem 1.1, it would suffice to show that the weak $(\omega, \omega)$-d.l. fails everywhere in the Argyros and Gaifman algebras. However, Shelah’s result does not apply to the Galvin-Hajnal algebra, since $P_{GH}$ is not Suslin.

For $\kappa = \text{cf}(2^\omega)$ in the case of Galvin-Hajnal algebras, and for $\kappa = \omega$ in the case of Argyros algebras and the atomless part of Gaifman algebras, the method we employ for completely embedding the $\kappa$-Cohen algebra is the following: Choose $\kappa$ many independent elements $\{c_i : i < \kappa\} \subseteq B$ in such a way that the subalgebra $C_\kappa$ generated by $\{c_i : i < \kappa\}$ satisfies the conditions of Lemma 1.12. Then $C_\kappa$ is isomorphic to the free Boolean algebra on $\kappa$-many generators and is a regular subalgebra of $B$. Since $B$ is complete, Lemma 1.15 implies that the completion of $C_\kappa$, the $\kappa$-Cohen algebra, embeds into $B$ as a complete subalgebra.

### 2. A complete embedding of the Cohen algebra into the Galvin-Hajnal algebra.

Galvin and Hajnal constructed a separative, atomless partial ordering $P_{GH}$ which satisfies the c.c.c. but not the $\sigma$-bounded c.c. [CN]. To do this, they used the following family of sets:

Recall that for a set $S$, $[S]^2 = \{\{\beta, \gamma\} : \beta, \gamma \in S \text{ and } \beta \neq \gamma\}$, the collection of all two-element subsets of $S$. In this section, we fix a well-ordering on $2^\omega$.

**Lemma 2.1 (Galvin-Hajnal, [CN]).** There is a family of sets $\{S_\alpha : \alpha < 2^\omega\}$ with the following four properties:

(S1) $\forall \alpha < 2^\omega \quad S_\alpha \subseteq \alpha$;
(S2) $\forall \alpha < 2^\omega \quad [S_\alpha]^2 \subseteq \bigcup_{\gamma < 2^\omega} \{\{\beta, \gamma\} : \beta \in S_\gamma\}$;
(S3) $\forall \alpha < 2^\omega \quad \text{o.t.}(S_\alpha) \leq \omega$;
(S4) If $S \subseteq 2^\omega$, $[S]^2 \subseteq \bigcup_{\gamma < 2^\omega} \{\{\beta, \gamma\} : \beta \in S_\gamma\}$, and $\text{o.t.}(S) \leq \omega$,

then $\exists \alpha < 2^\omega$ such that $S = S_\alpha$.

The following properties of the collection of sets $\{S_\alpha : \alpha < 2^\omega\}$ will be used extensively:
Lemma 2.2. Suppose $\eta, \zeta$ are ordinals with $\eta, \zeta < 2^\omega$ and $|S_\eta| = \omega$. Then there exist $\zeta < \alpha < \beta < 2^\omega$ such that $S_\alpha \cup S_\beta = S_\eta$.

Proof. Let $X = \{S_\alpha : \alpha \leq \zeta, S_\alpha \subseteq S_\eta\} \cup \{S_\eta \setminus S_\alpha : \alpha \leq \zeta\}$. $|X| < 2^\omega$, so choose some $S \subseteq S_\eta$ such that $S \not\in X$ and $S_\eta \setminus S \not\in X$. $[S]^2, [S\setminus S]^2 \subseteq [S_\eta]^2 \subseteq \bigcup_{\alpha < 2^\omega} \{\{\beta, \alpha\} : \beta \in S_\alpha\}$, by (S2). (S4) implies $\exists \alpha, \beta < 2^\omega$ such that $S_\alpha = S$ and $S_\beta = S_\eta \setminus S$. $\alpha, \beta > \zeta$, since $S \not\in X$. \hfill \Box

Lemma 2.3. Given $\alpha_0 < 2^\omega$, there is a sequence $\alpha_0 < \alpha_1 < \alpha_2 < \cdots < \lambda < 2^\omega$ of order type $\omega + 1$ such that for each $i < \omega$,

\begin{equation}
S_{\alpha_{i+1}} = \{\alpha_0, \ldots, \alpha_i\};
S_\lambda = \{\alpha_i : 0 < i < \omega\}.
\end{equation}

Proof. Let $\alpha_0 < 2^\omega$. $[\{\alpha_0\}]^2 = \emptyset$, so $\exists \alpha_1 < 2^\omega$ for which $S_{\alpha_1} = \{\alpha_0\}$, by (S4). (S1) implies $\alpha_1 > \alpha_0$. Given $\alpha_0 < \cdots < \alpha_n$ where for each $0 < j \leq n$ $S_{\alpha_j} = \{\alpha_0, \ldots, \alpha_{j-1}\}$, the set $[\{\alpha_0, \ldots, \alpha_n\}]^2 \subseteq \bigcup_{\alpha < 2^\omega} \{\{\beta, \alpha\} : \beta \in S_\alpha\}$. By (S1) and (S4), there is some $\alpha_{n+1} > \alpha_n$ such that $S_{\alpha_{n+1}} = \{\alpha_0, \ldots, \alpha_n\}$.

By our choice of the $\alpha_n$, $[\{\alpha_1, \alpha_2, \ldots\}]^2 \subseteq \bigcup_{\alpha < 2^\omega} \{\{\beta, \alpha\} : \beta \in S_\alpha\}$. By (S1) and (S4), there is a $\lambda < 2^\omega$ such that $S_\lambda = \{\alpha_i : 0 < i < \omega\}$ and $\lambda > \alpha_j$ for all $j < \omega$. \hfill \Box

Galvin and Hajnal constructed the following partial ordering $P_{GH}$: $\forall \alpha < 2^\omega$; let
\begin{equation}
V_\alpha = \{f : 2^\omega \rightarrow 2 : \forall \beta \in S_\alpha (f(\beta) = 0), \text{ and } f(\alpha) = 1\}.
\end{equation}
That is, $V_\alpha$ is the collection of all functions from $2^\omega$ into 2 which send each element of $S_\alpha$ to 0 and send $\alpha$ to 1. Let
\begin{equation}
P_{GH} = \left\{ \bigcap_{\alpha \in F} V_\alpha : F \in [2^\omega]^{<\omega} \text{ and } \bigcap_{\alpha \in F} V_\alpha \neq \emptyset \right\}.
\end{equation}

$P_{GH}$ is the collection of nonempty intersections of finitely many of the $V_\alpha$'s, partially ordered by inclusion.

Theorem 2.4 (Galvin-Hajnal, [CN]). $(P_{GH}, \subseteq)$ is a separative, atomless partial ordering which satisfies the c.c.c., but not the $\sigma$-bounded c.c.

Let $e : P_{GH} \rightarrow r.o.(P_{GH})$ denote the canonical embedding of $P_{GH}$ into its completion $r.o.(P_{GH})$. We shall call $r.o.(P_{GH})$ the Galvin-Hajnal algebra. Since the $\sigma$-bounded c.c. fails in $P_{GH}$, $\text{CUP}(r.o.(P_{GH})^+) \text{ fails}$. By Kelley’s Theorem 1.9, $r.o.(P_{GH})$ is not measurable.

Theorem 2.5. The $\text{cf}(2^\omega)$-Cohen algebra embeds as a complete subalgebra into $r.o.(P_{GH})$. 

Proof. Let $\kappa$ denote $\text{cf}(2^{\omega})$. We construct a subalgebra $C_\kappa$ of $\text{r.o.}(P_{GH})$ and show that $C_\kappa$ is isomorphic to the free Boolean algebra on $\kappa$-many generators, and is a regular subalgebra of $\text{r.o.}(P_{GH})$.

**Construction of $C_\kappa$:** By Lemma 2.3, there exist $\kappa$-many sequences, each of order type $\omega + 1$, of the form $\alpha(i,0) < \alpha(i,1) < \alpha(i,2) < \cdots < \alpha(i,j) < \alpha(i,j+1) < \cdots < \lambda(i)$, $i < \kappa$, such that the following hold: $\forall i < \kappa$, $\forall j < j' < \omega$, $\alpha(i,j) < \alpha(i,j') < \lambda(i)$; $\forall i < i' < \kappa$, $\lambda(i) < \alpha(i',0)$; and $\forall i < \kappa$, $\forall 0 < j < \omega$,

\begin{equation}
S_{\alpha(i,j)} = \{\alpha(i,k) : k < j\};
S_{\lambda(i)} = \{\alpha(i,j) : 0 < j < \omega\}.
\end{equation}

Note that the sets $\{\alpha(i,j) : j < \omega\} \cup \{\lambda(i)\}$, $i < \kappa$, are pairwise disjoint. For each $i < \kappa$, let

\begin{equation}
c_i = \bigvee_{0 < j < \omega} e(V_{\alpha(i,j)})
\end{equation}

in $\text{r.o.}(P_{GH})$. Let

\begin{equation}
C_\kappa = \{c_i : i < \kappa\},
\end{equation}

the subalgebra of $\text{r.o.}(P_{GH})$ generated by $\{c_i : i < \kappa\}$. The elements $V_{\lambda(i)}$, $i < \kappa$, will be used in Proposition 2.11 to ensure that the generators of $C_\kappa$ are independent. The following simple facts will be useful:

**Fact 2.6.** $\forall p, q \in P_{GH}$, $e(p) \land e(q) = e(p \cap q)$.

**Fact 2.7.** For each finite $F \subseteq 2^{\omega}$, $\bigcap_{a \in F} V_a \neq \emptyset$ iff $(\bigcup_{a \in F} S_a) \cap F = \emptyset$.

**Fact 2.8.** If $\{p_i : i < \omega\} \subseteq P_{GH}$ is infinite, then $\bigcap_{i < \omega} e(p_i) = \emptyset$.

If $\{p_i : i < \omega\}$ is infinite and $q \in \bigcap_{i < \omega} e(p_i)$, then every $f \in q$ must take infinitely many elements of $2^{\omega}$ to 1. There are no such $q \in P_{GH}$.

**Fact 2.9.** Given $i < \kappa$, if $F$ is a finite subset of $2^{\omega}$ and $\bigcup_{a \in F} S_a \supseteq S_{\lambda(i)}$, then $e(\bigcap_{a \in F} V_a) \leq -c_i$. In particular, $\forall i < \kappa$, $e(V_{\lambda(i)}) \leq -c_i$.

Suppose $\bigcup_{a \in F} S_a \supseteq S_{\lambda(i)}$. Then $\forall p \in e(\bigcap_{a \in F} V_a)$, $\forall f \in p$, $\forall 0 < j < \omega$, $f(\alpha(i,j)) = 0$. However, $\forall q \in c_i$, $\exists f \in q$ and $\exists 0 < j < \omega$ such that $f(\alpha(i,j)) = 1$. So $e(\bigcap_{a \in F} V_a) \cap c_i = \emptyset$.

**Fact 2.10.** $\forall i < \kappa$, $c_i = \bigcup_{0 < j < \omega} e(V_{\alpha(i,j)})$.

Proof. Clearly, $c_i \supseteq \bigcup_{0 < j < \omega} e(V_{\alpha(i,j)})$. Suppose $p \notin \bigcup_{0 < j < \omega} e(V_{\alpha(i,j)})$. Let $F$ be the finite subset of $2^{\omega}$ such that $p = \bigcap_{a \in F} V_a$, $\forall 0 < j < \omega$, $p \nsubseteq V_{\alpha(i,j)}$, so $S_{\lambda(i)} \cap F = \emptyset$. By Lemma 2.2, $\exists \gamma < \beta < 2^{\omega}$ with $\gamma > \text{sup}(F \cup \{\lambda(i)\})$ such that $S_{\gamma} \cup S_{\beta} = S_{\lambda(i)}$. By Fact 2.7, $V_{\gamma} \cap V_{\beta} \cap p \neq \emptyset$. By Fact 2.9, $e(V_{\gamma} \cap V_{\beta} \cap p) \subseteq -c_i$, so $e(p) \nsubseteq c_i$. Since $c_i$ is an open subset of $P_{GH}$, $e(p) \nsubseteq c_i$ implies $p \notin c_i$. Thus, $c_i \subseteq \bigcup_{0 < j < \omega} e(V_{\alpha(i,j)})$. \qed
Next we show that the generators of $C_\kappa$ are independent.

**Proposition 2.11.** For finite sets $I, J \subseteq \kappa$, $\bigwedge_{i \in I} c_i \land \bigwedge_{j \in J} -c_j = 0$ iff $I \cap J \neq \emptyset$.

**Proof.** Let $I, J \subseteq \kappa$ be disjoint, finite sets, and let $r = \bigcap_{i \in I} V_{\alpha(i,1)} \cap \bigcap_{j \in J} V_{\lambda(j)}$. Fact 2.7 implies $r \neq \emptyset$. By Fact 2.9, $\bigwedge_{i \in I} c_i \land \bigwedge_{j \in J} -c_j \geq \bigwedge_{i \in I} e(V_{\alpha(i,1)}) \land \bigwedge_{j \in J} e(V_{\lambda(j)}) = e(r) > 0$. \hfill $\Box$

Since $C_\kappa$ is generated by $\text{cf}(2^\omega)$-many independent elements, $C_\kappa$ is isomorphic to the free Boolean algebra on $\text{cf}(2^\omega)$-many generators.

The next proposition will aid us in constructing elements $c_p$ satisfying the conditions of Lemma 1.12.

**Proposition 2.12.** Given $p \in \mathcal{P}_{GH}$, there are at most finitely many $i < \kappa$ for which either $e(p) \land c_i = 0$ or $e(p) \land -c_i = 0$.

**Proof.** Suppose $p \in \mathcal{P}_{GH}$ and let $J \subseteq \kappa$ be defined by $j \in J \leftrightarrow e(p) \land c_j = 0$.

For $j \in J$, 
\[(2.7) \quad 0 = e(p) \land c_j = e(p) \cap \left( \bigcup_{0 < k < \omega} e(V_{\alpha(j,k)}) \right) = \bigcup_{0 < k < \omega} (e(p) \cap e(V_{\alpha(j,k)})�),\]

the second equality following from Fact 2.10. Thus, $\forall j \in J, \forall 0 < k < \omega$, $p \cap e(V_{\alpha(j,k)}) = \emptyset$.

Let $F$ be the finite subset of $2^\omega$ such that $p = \bigcap_{\alpha \in F} V_\alpha$. The following can be shown by an easy induction argument using Fact 2.7: If $e(p) \land c_j = 0$, then either $\alpha(j, 0) \in F$ or $S_{\lambda(j)} \subseteq \bigcup_{\alpha \in F} S_\alpha$.

$\alpha(j, 0) \in F$ for at most finitely many $j < \kappa$, since $F$ is finite. By (S3), $\omega \cdot \omega < \omega \cdot \omega$, so $S_{\lambda(j)} \subseteq \bigcup_{\alpha \in F} S_\alpha$ for at most finitely many $j < \kappa$.

Hence, $e(p) \land c_j = 0$ for at most finitely many $j < \kappa$. Thus, $|J| < \omega$.

Let $I \subseteq \kappa$ be defined by $i \in I \leftrightarrow e(p) \land -c_i = 0$. Then
\[(2.8) \quad e(p) \subseteq \bigcap_{i \in I} c_i = \bigcap_{i \in I} \bigcup_{0 < k < \omega} e(V_{\alpha(i,k)}) = \bigcup_{\gamma : I \rightarrow \omega \setminus \{0\}} \bigcap_{i \in I} e(V_{\alpha(i,\gamma(i))}),\]

where the first equality follows from Fact 2.10. If $I$ is infinite, then Fact 2.8 implies that $\forall g : I \rightarrow \omega \setminus \{0\}, \bigcap_{i \in I} e(V_{\alpha(i,\gamma(i))}) = \emptyset$, contradicting $p \neq \emptyset$. Thus, $|I| < \omega$. \hfill $\Box$

Now we use Proposition 2.12 to show that $C_\kappa$ satisfies the conditions of Lemma 1.12.

**Proposition 2.13.** $\forall p \in \mathcal{P}_{GH} \exists c_p \in C_\kappa^+$ such that $\forall c \in C_\kappa$, if $c_p \land c \neq 0$, then $e(p) \land c \neq 0$. 
Proof. Fix \( p \in \mathcal{P}_{GH} \) and let \( F \) be the finite subset of \( 2^\omega \) such that \( p = \bigcap_{\alpha \in F} V_\alpha \). Let \( I, J \subseteq \kappa \) be the finite disjoint sets of Proposition 2.12. Let \( c_p = (\bigwedge_{i \in I} e_i) \land (\bigwedge_{j \in J} -e_j) \). \( c_p \in \mathcal{C}_\kappa^+ \), by Proposition 2.11. \( (c_p \text{ is actually the minimal cover for } e(p) \text{ in } \mathcal{C}) \)

Suppose \( c = (\bigwedge_{k \in K} c_k) \land (\bigwedge_{l \in L} -e_l) \in \mathcal{C}_\kappa \) and \( c_p \land c \neq \emptyset \). Then \( I \cap L = \emptyset \) implies \( J \cap K = \emptyset \). \( J \cap K = \emptyset \) implies \( \forall k \in K, \exists \emptyset < m_k < \omega \) such that \( p \cap V_{\alpha(k,m_k)} \neq \emptyset \). By Fact 2.7, \( e(p) \not\subseteq c \).

(2.10) \[
F \bigcap \left( \bigcup_{l \in L} S_{\lambda(l)} \right) = \emptyset.
\]

Using Lemma 2.2, \( \forall l \in L \) choose \( \beta_l, \alpha_l > \sup(F \cup \{ \lambda(n) : n \in K \cup L \}) \) such that \( S_{\alpha_l} \cup S_{\beta_l} = S_{\lambda(l)} \). Let

(2.11) \[
r = \bigcap_{k \in K} V_{\alpha(k,m_k)} \bigcap_{l \in L} (V_{\alpha_l} \cap V_{\beta_l}).
\]

\( r \neq \emptyset \), by Fact 2.7. \( e(r) \leq c \), by Fact 2.9. Furthermore, \( r \cap p \neq \emptyset \), by Fact 2.7, (2.9), (2.10), and the fact that \( \forall l \in L, \alpha_l, \beta_l \) are larger than \( \sup(F \cup \{ \lambda(n) : n \in K \cup L \}) \). Hence, \( 0 < e(r \cap p) \leq c \), so \( e(p) \land c \neq \emptyset \). \( \square \)

It follows from Lemma 1.12 and Proposition 2.13 that \( \mathcal{C}_\kappa \) is a regular subalgebra of r.o.(\( \mathcal{P}_{GH} \)). Thus, by Lemma 1.15, r.o.(\( \mathcal{C}_\kappa^+ \)), the \( \kappa \)-Cohen algebra, embeds into r.o.\( \mathcal{P}_{GH} \) as a complete subalgebra. That is, \( \mathcal{P}_{GH} \) adds \( \text{cf}(2^\omega) \)-many side-by-side Cohen reals.

3. A complete embedding of the Cohen algebra into the Argyros algebra.

Argyros constructed a separative, atomless partial ordering \( \mathcal{P}_A \) in which the \( \sigma \)-bounded c.c. holds but CUP(\( \mathcal{P}_A \)) fails, and, assuming CH, property \( K_3 \) also fails [A]. He constructed \( \mathcal{P}_A \) using three basic types of elements. In this section, let \( 2^\omega \) denote the set of functions from \( \omega \) to \( 2 \). For \( X, Y \in [\omega]^{<\omega} \), let

(3.1) \[
B_X = \{ f \in 2^\omega : \forall x \in X \ f(x) = 1 \},
\]

(3.2) \[
\overline{B}_Y = \{ f \in 2^\omega : \forall y \in Y \ f(y) = 0 \}.
\]

For the third type of element, Argyros constructed a tree \( T \subseteq [\omega]^2 \) as follows: Let \( \{ S_{nm} : n < \omega, \ 1 \leq m \leq 3^n \} \) be a family of sets such that \( \forall n, m < \omega, \).
\[ S_{nm} \subseteq [\omega]^3 \text{ and } S_{nm} \cap S_{nm'} = \emptyset \text{ whenever } (n, m) \neq (n', m'). \] For each \( n < \omega \), let \( \text{Lev}(n) = \bigcup_{1 \leq m \leq 3^n} [S_{nm}]^2 \). For each \( n < \omega \), index the elements of \( \text{Lev}(n) \) as \( s_nj \), \( 1 \leq j \leq 3^{n+1} \). The partial ordering on \( T \) is defined at level \( n + 1 \) as follows: For \( s = s_nj \in \text{Lev}(n) \) and \( t \in \text{Lev}(n + 1) \), \( s < t \iff t \in [S_{n+1,j}]^2 \). Let \( T = \bigcup_{n \in \omega} \text{Lev}(n) \). \((T, \prec)\) is the Argyros tree.

For \( s = \{k, l\} \in T \), let

\[ K_s = (B_{\{k\}} \cap \overline{B}_{\{l\}}) \cup (B_{\{l\}} \cap \overline{B}_{\{k\}}). \]

\( K_s \) is the set of all functions in \( 2^\omega \) which are nonconstant on \( s \). Let \( Br \) be the set of all branches (finite and infinite) of \( T \). For \( \sigma \in Br \), let

\[ A_\sigma = \bigcap_{s \in \sigma} K_s. \]

\( A_\sigma \) is the set of all functions in \( 2^\omega \) which are nonconstant on every node \( s \in \sigma \). Let

\[ \mathbf{P}_A = \left\{ B_X \cap \overline{B}_Y \cap \bigcap_{\sigma \in \Sigma} A_\sigma : X, Y \subseteq [\omega]^{<\omega} \text{ and } \Sigma \subseteq [Br]^{<\omega} \right\} \setminus \{\emptyset\}, \]

the collection of all nonempty intersections of finitely many elements of the three forms \( B_X, \overline{B}_Y, \) and \( A_\sigma \), partially ordered by inclusion. We call \( \langle \mathbf{P}_A, \subseteq \rangle \) the Argyros partial ordering.

**Theorem 3.1** (Argyros, [A]). \( \langle \mathbf{P}_A, \subseteq \rangle \) is a separative, atomless partial ordering which satisfies the \( \sigma \)-bounded c.c. but not CUP(\( \mathbf{P}_A \)), and, assuming CH, does not satisfy property \( K_3 \).

Let \( e : \mathbf{P}_A \to \text{r.o.}(\mathbf{P}_A) \) be the canonical embedding of \( \mathbf{P}_A \) into its completion. We shall call \( \text{r.o.}(\mathbf{P}_A) \) the Argyros algebra. By Argyros’ Theorem 3.1, \( \text{r.o.}(\mathbf{P}_A) \) is a complete, atomless Boolean algebra which satisfies the \( \sigma \)-bounded c.c. but not CUP(\( \text{r.o.}(\mathbf{P}_A)^+ \)), and hence, by Kelley’s Theorem 1.9, is not measurable.

**Definition 3.2.** We will say that \( s, t \in T \) are siblings if \( s \neq t \) and \( \exists v \in T \) such that \( s \) and \( t \) are both immediate successors of \( v \). \( s, t, u \in T \) are triplets if they are pairwise siblings.

**Note:** If \( s \) and \( t \) are siblings, then there exist unique \( m, n < \omega \) such that \( s, t \subseteq [S_{mn}]^2 \), and \( K_s \cap K_t \neq \emptyset \). If \( s, t, u \) are triplets, then \( s = t \triangle u \) (the set-theoretic difference of \( t \) and \( u \) in \( \omega \)), and \( K_s \cap K_t \cap K_u = \emptyset \).

**Remark 3.3.** The elements of \( \mathbf{P}_A \) are not uniquely represented by the form \( B_X \cap \overline{B}_Y \cap \bigcap_{\sigma \in \Sigma} A_\sigma \). For instance, if \( s = \{k, l\} \in \sigma \in Br \), then \( B_{\{k\}} \cap A_\sigma = B_{\{k\}} \cap \overline{B}_{\{l\}} \cap A_\sigma = \overline{B}_{\{l\}} \cap A_\sigma \). We shall hold to the following convention: Given \( S \subseteq T \) and \( X, Y \subseteq [\omega]^{<\omega} \), the representation \( B_X \cap \overline{B}_Y \cap \bigcap_{s \in S} K_s \) of a subset of \( 2^\omega \) is said to be in the normal form if and only if \( X \cap Y = \emptyset \),
sets: $\Gamma = \{\}$, and $S$ contains no triplets. If $B_X \cap \overline{B_Y} \cap \bigcap_{s \in S} K_s$ is the normal form representation of some $B_U \cap \overline{B_V} \cap \bigcap_{\sigma \in \Sigma} A_\sigma \in P_A$, then $X \supseteq U$, $Y \supseteq V$, $X \cap Y = \emptyset$, and $S \subseteq \bigcup\{\sigma : \sigma \in \Sigma\}$. It is not hard to see that for each element $p \in P_A$ there is a unique normal form representation of $p$. The normal form is not necessary for the proceeding proofs, but rather serves to simplify notation.

**Lemma 3.4.** Let $p = B_X \cap \overline{B_Y} \cap \bigcap_{s \in S} K_s$ be a subset of $2^\omega$, (not necessarily in the normal form, and not necessarily in $P_A$). Then $p \neq \emptyset$ iff the following four conditions hold:

(L1) $p$ $X \cap Y = \emptyset$;
(L2) $\forall s \in S$, $s \not\supseteq X$ and $s \not\supseteq Y$;
(L3) $S$ does not contain any triplets;
(L4) If $s, t \in S$ are siblings, then either $(s \triangle t) \cap X = \emptyset$ or $(s \triangle t) \cap Y = \emptyset$.

In particular, if $p = B_X \cap \overline{B_Y} \cap \bigcap_{s \in S} K_s$ is in the normal form, then $p \neq \emptyset$.

**Proof.** The forward direction is trivial. Assume (L1)–(L4) hold. We show there is a partial function $f_p$ with domain $X \cup Y \cup \bigcup S$ such that every total extension of $f_p$ is in $B_X \cap \overline{B_Y} \cap \bigcap_{s \in S} K_s$.

By (L1) $p$, $B_X \cap \overline{B_Y} \neq \emptyset$. By (L3) $p$, we can divide $S$ into two disjoint sets: $\Gamma = \{s \in S : s$ has no sibling in $S\}$ and $\Theta = \{s \in S : s$ has exactly one sibling in $S\}$. $\forall s \in \Gamma$, by (L2) $s \not\supseteq X$ and $s \not\supseteq Y$. Hence, there is a partial function $f_s$ such that $\text{dom}(f_s) = X \cup Y \cup s$, $f_s \upharpoonright X \equiv 1$, $f_s \upharpoonright Y \equiv 0$, and $f_s \upharpoonright s$ is nonconstant. Every extension of $f_s$ to a total function is in $B_X \cap \overline{B_Y} \cap K_s$.

For each pair of siblings $t,u \in \Theta$, there is a partial function $f_{t,u}$ such that $\text{dom}(f_{t,u}) = X \cup Y \cup t \cup u$, $f_{t,u} \upharpoonright X \equiv 1$, $f_{t,u} \upharpoonright Y \equiv 0$, and every extension of $f_{t,u}$ to a total function is in $B_X \cap \overline{B_Y} \cap K_t \cap K_u$. To define such an $f_{t,u}$ on $t \cup u$ while preserving $f_{t,u} \upharpoonright X \equiv 1$ and $f_{t,u} \upharpoonright Y \equiv 0$, we consider 3 cases. If $(t \triangle u) \cap X \neq \emptyset$, then $(t \cap u) \cap X = \emptyset$ by (L2) $p$, and $(t \triangle u) \cap Y = \emptyset$ by (L4) $p$; so let $f_{t,u} \upharpoonright (t \triangle u) \equiv 1$ and $f_{t,u} \upharpoonright (t \cap u) \equiv 0$. If $(t \cap u) \cap Y \neq \emptyset$, then by (L2) $p$ $(t \triangle u) \cap Y = \emptyset$; so let $f_{t,u} \upharpoonright (t \triangle u) \equiv 1$ and $f_{t,u} \upharpoonright (t \cap u) \equiv 0$. If $(t \triangle u) \cap X$ and $(t \cap u) \cap Y = \emptyset$, let $f_{t,u} \upharpoonright (t \triangle u) \equiv 0$ and $f_{t,u} \upharpoonright (t \cap u) \equiv 1$.

Since two elements of $S$ have nonempty intersection only if they are siblings, the partial function $f_p = \bigcup\{f_s : s \in \Gamma\} \cup \bigcup\{f_{t,u} : t,u$ are siblings in $\Theta\}$ is well-defined. Every total extension of $f_p$ is in $B_X \cap \overline{B_Y} \cap \bigcap_{s \in S} K_s$. □

**Theorem 3.5.** The Cohen algebra embeds into $\text{r.o.}(P_A)$ as a complete subalgebra.

**Proof.** We construct a countable, atomless, regular subalgebra $\mathbf{C}$ of $\text{r.o.}(P_A)$.
**Construction of $C$.** Choose an infinite branch of $T$ and call it $\beta_0$. $\forall n < \omega$, let $t_n$ be the unique element in $\beta_0 \cap \text{Lev}(n)$, and choose one $s_{n+1} \in T \setminus \beta_0$ such that $s_{n+1}$ and $t_{n+1}$ are siblings. $\forall 0 < n < \omega$, let $\beta_n$ be an infinite branch in $T$ which contains $s_n$. For $0 < m < n$, $\beta_m \cap \beta_n = \{t_0, \ldots, t_{m-1}\}$. Define the following sets:

\[(3.5) \quad T_C = \left\{ t \in T : t \in \bigcup_{n<\omega} \beta_n \text{ or } t \text{ is a sibling of some } s \in \bigcup_{n<\omega} \beta_n \right\}, \]

\[(3.6) \quad N_C = \bigcup T_C = \{k < \omega : \exists l < \omega \text{ such that } \{k, l\} \in T_C\}. \]

$T_C \subseteq T$ and $N_C \subseteq \omega$. Let $Br_C$ denote the set of all branches (finite and infinite) of $T_C$. $Br_C$ is countable, since $T_C$ has only countably many infinite branches. Let

\[(3.7) \quad C = \left\{ B_X \cap B_Y \cap \bigcap_{\sigma \in \Sigma} A_\sigma : X, Y \in [N_C]^{<\omega} \text{ and } \Sigma \in [Br_C]^{<\omega} \right\}. \]

Let

\[(3.8) \quad C = \langle \{e(p) : p \in C\} \rangle, \]

the subalgebra of $r.o.(P_A)$ generated by the set $\{e(p) : p \in C\}$. Note that $|C| = \omega$, since $N_C$ and $Br_C$ are countable.

**Remark 3.6.** The idea behind the choice of $T_C$ and $N_C$ is as follows: $T_C$ was chosen so that for any finite set of branches $\Sigma \subseteq Br_C$: $\bigcap_{\sigma \in \Sigma} A_\sigma \neq \emptyset$. However, the subalgebra generated by $\{e(\bigcap_{\sigma \in \Sigma} A_\sigma) : \Sigma \in [Br_C]^{<\omega}\}$ is not a regular subalgebra of $r.o.(P_A)$. To enlarge it to a regular subalgebra, we chose $N_C$ so that we can tell exactly how elements of the form $\bigcap_{\sigma \in \Sigma} A_\sigma$, $\Sigma \in [Br_C]^{<\omega}$, interact with members of $P_A$. This allows us to find for each $p \in P_A$ a minimal cover for $e(p)$ in $C$, and thus ensure that $C$ is a regular subalgebra of $r.o.(P_A)$.

The following are two simple facts which we shall use without mention in subsequent proofs:

**Fact 3.7.** $\forall x \in \omega$, $-e(B_{\{x\}}) = e(\overline{B_{\{x\}}})$.

**Fact 3.8.** $\forall p, q \in P_A$, $e(p) \land e(q) = e(p \cap q)$.

**Proposition 3.9.** $\{e(p) : p \in C\}$ is dense in $C^+$.

**Proof.** Every element of $C$ is a finite disjunction of elements of the form

\[(3.9) \quad e\left(B_X \cap B_Y \cap \bigcap_{\sigma \in \Sigma} A_\sigma\right) \land \bigwedge_{\gamma \in \Gamma} -e(A_\gamma), \]
where \( X, Y \in [N_C]^{<\omega} \) and \( \Sigma, \Gamma \in [B_{rC}]^{<\omega} \). Let \( c \in C^{+} \) be of the form (3.9), and let \( p \in P_A \) be such that \( e(p) \leq c \). For each \( \gamma \in \Gamma \), \( e(p) \land e(A_\gamma) = 0 \implies p \subseteq \bigcup_{s \in \gamma}(B_s \cup B_s) \). Fix \( f \in p \). For each \( \gamma \in \Gamma \) choose an \( s_\gamma \in \Gamma \) for which \( f \in (B_{s_\gamma} \cup \overline{B}_{s_\gamma}) \). Define \( \Gamma' = \{ \gamma \in \Gamma : f \in B_{s_\gamma} \} \) and \( \Gamma'' = \{ \gamma \in \Gamma : f \in \overline{B}_{s_\gamma} \} \). Let

\[
q = B_{X \cup (\bigcup_{s_\gamma \in \Gamma'} \bigcup \overline{B}_Y \cup (\bigcup_{s_\gamma \in \Gamma''}) \bigcap \bigcap_{\sigma \in \Sigma} A_\sigma.
\]

\( f \in q \), so \( q \neq \emptyset \). By its construction, \( q \subseteq C \). Furthermore, \( \forall \gamma \in \Gamma' \), \( e(B_{s_\gamma}) \leq -e(A_\gamma) \); \( \forall \gamma \in \Gamma'' \), \( e(\overline{B}_{s_\gamma}) \leq -e(A_\gamma) \); and \( q \subseteq B_X \cap B_Y \cap \bigcap_{\sigma \in \Sigma} A_\sigma \). Hence, \( e(q) \leq c \).

**Proposition 3.10.** \( C \) is atomless.

**Proof.** It suffices to show that \( \{ e(p) : p \in C \} \) is atomless. Suppose \( p \in C \) and \( B_X \cap B_Y \cap \bigcap_{s \in S} K_s \) is the normal form of \( p \). Choose some \( z \in N_C \setminus (X \cup Y) \) such that \( z \) is neither in any member of \( S \) nor in any sibling of any member of \( S \). Then \( B_X \cup \{ z \} \cap B_Y \cap \bigcap_{s \in S} K_s \) is the normal form of \( p \cap B_{\{ z \}} \); so by Lemma 3.4, \( p \cap B_{\{ z \}} \neq \emptyset \). Hence, \( p \cap B_{\{ z \}} \in C \).

To see that \( e(p \cap B_{\{ z \}}) < e(p) \), note that \( -e(B_{\{ z \}}) \land e(p) = e(B_X \cap B_Y \cup \{ z \} \cap \bigcap_{s \in S} K_s) \). This is strictly greater than \( 0 \), by Lemma 3.4, since \( B_X \cap B_Y \cup \{ z \} \cap \bigcap_{s \in S} K_s \) is in the normal form.

We now show that \( C \) satisfies the conditions of Lemma 1.12.

**Proposition 3.11.** \( \forall p \in P_A, \exists c_p \in C^{+} \) such that \( \forall c \in C \), if \( c_p \land c \neq 0, \) then \( e(p) \land e(r) > 0 \).

**Proof.** Let \( p = B_X \cap B_Y \cap \bigcap_{s \in S} K_s \in P_A \). \( p \neq \emptyset \), so by Lemma 3.4, (L1 p)-(L4 p) hold. Let \( X' = X \cap N_C, Y' = Y \cap N_C, S' = S \cap T_C \), and \( q = B_{X'} \cap B_{Y'} \cap \bigcap_{s' \in S'} K_{s'} \). \( q \geq p \), and \( q \in C \) since \( S' \) is a finite union of branches in \( T_C \). Let \( c_p = e(q) \). \( c_p \in C^{+} \) and \( c_p \geq e(p) > 0 \). \( (c_p \) is actually the minimal cover for \( e(p) \) in \( C \).) It suffices to show that \( \forall r \in C \), if \( c_p \land e(r) > 0 \), then \( e(p) \land e(r) > 0 \).

Suppose \( r = B_U \cap B_V \cap \bigcap_{w \in W} K_w \in C \) and \( c_p \land e(r) > 0 \). Then

\[
q \land r = B_{(X' \cup U)} \cap B_{(Y' \cup V)} \cap \bigcap_{t \in S' \cup W} K_t \neq \emptyset.
\]

Thus, Lemma 3.4 implies

\[
\begin{align*}
& (L1 \ q \land r) \ (X' \cup U) \cap (Y' \cup V) = \emptyset; \\
& (L2 \ q \land r) \ \forall s \in S' \cup W, \ s \not\subseteq X' \cup U \text{ and } s \not\subseteq Y' \cup V; \\
& (L3 \ q \land r) \ S' \cup W \text{ has no triplets}; \\
& (L4 \ q \land r) \ s, t \text{ are siblings in } S' \cup W \implies ((s \Delta t) \cap (X' \cup U) = \emptyset \text{ or } (s \Delta t) \cap (Y' \cup V) = \emptyset).
\end{align*}
\]
It suffices to show that \( p \cap r \neq \emptyset \). First, note that
\[
(3.12) \quad p \cap r = B_{((X \setminus X') \cup Y') \cup U} \cap \bigcap_{s \in (S \setminus S') \cup S'' \cup W} K_s,
\]
where \((X \setminus X') \cap N_C = (Y \setminus Y') \cap N_C = \emptyset; X' \cup U, Y' \cup V \subseteq N_C; (S \setminus S') \cap T_C = \emptyset; \) \( S' \cup W \subseteq T_C \).

**Claim.** (L1 \( p \cap r \))- (L4 \( p \cap r \)) hold.

(L1 \( p \cap r \)) and (L2 \( p \cap r \)) follow naturally from (L1 \( p \)), (L1 \( q \cap r \)), (L2 \( q \cap r \)) and (L2 \( q \cap r \)). \( S' \cup W \) and \( S \setminus S' \) each contain no triplets, by (L3 \( q \cap r \)) and (L3 \( p \)), respectively. \( S \setminus S' \subseteq T \setminus T_C \) and \( S' \cup W \subseteq T_C \) imply \( S \setminus S' \) and \( S' \cup W \) have no common siblings. Hence, \( S \cup W \) contains no triplets, so (L3 \( p \cap r \)) holds.

Suppose \( s, t \) are siblings in \( S' \cup W \). Then either \( s, t \in S \setminus S' \) or else \( s, t \in S' \cup W \). Suppose \( s, t \in S \setminus S' \). Then \((s \cup t) \cap N_C = \emptyset, \) so \((s \Delta t) \cap (X' \cup Y' \cup U \cup V) = \emptyset. \) Further, (L4 \( p \)) implies either \((s \Delta t) \cap X = \emptyset \) or else \((s \Delta t) \cap Y = \emptyset. \) On the other hand, if \( s, t \in S' \cup W, \) then \( s \cup t \in N_C, \) so \((s \cup t) \cap ((X \setminus X') \cup (Y \setminus Y')) = \emptyset. \) By (L4 \( q \cap r \)), either \((s \Delta t) \cap (X' \cup U) = \emptyset \) or else \((s \Delta t) \cap (Y' \cup V) = \emptyset. \) Thus, in both cases, either \((s \Delta t) \cap (X \cup U) = \emptyset \) or else \((s \Delta t) \cap (Y \cup V) = \emptyset. \) Hence, (L4 \( p \cap r \)) holds.

By Lemma 3.4, \( p \cap r \neq \emptyset \). Thus, \( e(p) \wedge e(r) > 0. \)

By Proposition 3.11 and Lemmas 1.12 and 1.15, \( r.o.(C^+ \cap \omega) \) is a complete subalgebra of \( r.o.(P_A) \). This completes our construction of a complete embedding of the Cohen algebra into the Argyros algebra.

**Remark 3.12.** We are very grateful to the referee for suggesting that we investigate Argyros’ other variants of this example in which stronger chain conditions hold, to see whether the Cohen algebra completely embeds in these. We have found that it does.

**Theorem 3.13** (Argyros, [CN], p. 156.). For each \( 2 \leq m < \omega, \) there is a family of atomless, separative partial orderings \( \mathbb{P}_m \) such that each \( \mathbb{P}_m \) satisfies the \( \sigma \)-bounded c.c. and property \( K_m, \) \( \text{CUP}(\mathbb{P}_m) \) fails, and, assuming \( CH, \) property \( K_{m+1} \) fails.

Our preceding construction can be easily modified to completely embed the Cohen algebra into each \( r.o.(\mathbb{P}_m). \)

**Theorem 3.14.** For each \( 2 \leq m < \omega, \) the Cohen algebra embeds as a complete subalgebra into \( r.o.(\mathbb{P}_m). \)

The modification is as follows: Let \( m \) be given. Argyros constructed a family of trees \( T \) such that each \( T \) is an \((m + 1)\)-branching tree of height \( \omega \) with the following properties: For each node \( \sigma \in T, \) there is a finite subset, \( \text{dom}(\sigma) \subseteq \omega, \) such that \( \sigma \) is a set of functions from \( \text{dom}(\sigma) \) to 2. All
siblings have the same domain, and any two nodes which are not siblings have disjoint domains. Any \( m \) siblings have nonempty intersection, but the intersection of \( (m + 1) \)-many siblings is empty. (See [CN] p. 156 for the precise definition of \( T_m \).) For \( \sigma \in T \), let \( A_\sigma = \{ f \in 2^\omega : f \upharpoonright \operatorname{dom}(\sigma) \in \sigma \} \). 

\[
\mathbf{P}_m = \{ B_X \cap \overline{B}_Y \cap \bigcap_{\sigma \in \Sigma} A_\sigma : X, Y \in [\omega]^{<\omega} \text{ and } \Sigma \text{ is a finite set of branches in } T \} \setminus \{ \emptyset \},
\]

partially ordered by inclusion. 

To construct a countable, atomless, regular subalgebra of \( \mathfrak{r}(\mathbf{P}_m) \), take, as before, \( \omega \)-many infinite branches \( \beta_i \) in \( T \) such that for each node \( \sigma \in \bigcup_{i<\omega} \beta_i \), \( \sigma \) has at most one sibling in \( \bigcup_{i<\omega} \beta_i \). Let \( T_C \) be the subtree of \( T \) consisting of \( \bigcup_{i<\omega} \beta_i \) and all siblings of nodes in \( \bigcup_{i<\omega} \beta_i \). Let \( N_C = \bigcup \{ \operatorname{dom}(\sigma) : \sigma \in T_C \} \). Let \( \mathbf{C} \) be the subalgebra of \( \mathfrak{r}(\mathbf{P}_m) \) generated by the elements of the form \( B_X \cap \overline{B}_Y \cap \bigcap_{\sigma \in \Sigma} A_\sigma \) such that \( X, Y \) are finite subsets of \( N_C \) and \( \Sigma \) is a finite set of branches in \( T_C \). The proof that this subalgebra is atomless and regular proceeds as before.


Gaifman constructed a family of Boolean algebras \( \mathbf{B}_G \) as follows [Ga]: Let \( \operatorname{Clop}(2^{(0,1)}) \) denote the clopen subsets of \( 2^{(0,1)} \). For \( X, Y \in [(0,1)]^{<\omega} \), let

\[
(4.1) \quad B_X = \left\{ f \in 2^{(0,1)} : \forall x \in X, \ f(x) = 1 \right\},
\]

\[
\overline{B}_Y = \left\{ f \in 2^{(0,1)} : \forall y \in Y, \ f(y) = 0 \right\}.
\]

Let \( \{ T_i : 2 \leq i < \omega \} \) be an enumeration of the open subintervals of \( (0,1) \) with rational endpoints. For each \( 2 \leq i < \omega \), choose \( i^2 \)-many disjoint, open subintervals of \( T_i \) and label them \( T_{i1}, T_{i2}, \ldots, T_{ii^2} \). We let \( T_0 = (0,1) \setminus \bigcup_{1 \leq j \leq i^2} T_{ij} \), so that \( \{ T_{ij} : 0 \leq j \leq i^2 \} \) is a partition of \( (0,1) \). Let \( I \) be the set of those elements of \( \operatorname{Clop}(2^{(0,1)}) \) of the form \( B_X \cap \overline{B}_Y \) such that for some \( 2 \leq i < \omega \), \( X \) intersects at least \( i \)-many of the open intervals \( T_{i1}, \ldots, T_{ii^2} \). Let \( I \) be the ideal generated by \( I \) in \( \operatorname{Clop}(2^{(0,1)}) \). The \emph{Gaifman algebra} is the quotient algebra

\[
(4.2) \quad \mathbf{B}_G = \operatorname{Clop}(2^{(0,1)}) / I.
\]

For \( c \in \operatorname{Clop}(2^{(0,1)}) \) we shall denote the equivalence class of \( c \) in \( \mathbf{B}_G \) by \( [c] \).

Notice that the set \( \{ [B_X \cap \overline{B}_Y] : X, Y \in [(0,1)]^{<\omega} \text{ and } B_X \cap \overline{B}_Y \not\in I \} \) is dense in \( \mathbf{B}_G^+ \). We will use this fact implicitly in this and the next section.

**Theorem 4.1** (Gaifman, [Ga]). \( \mathbf{B}_G \) satisfies the \( \sigma \)-bounded c.c., but does not satisfy \( \operatorname{CUP}((\mathbf{B}_G^+)) \).

By Kelley’s Theorem 1.9, \( \mathbf{B}_G \) is not measurable.

Depending on how the \( T_i \)'s and \( T_{ij} \)'s are chosen, \( \mathbf{B}_G \) may or may not have atoms. To show this, we will use Lemmas 4.2 and 4.4 and Fact 4.3.
Lemma 4.2 (Gaifman, [Ga]). For each \( B_X \cap \overline{B}_Y \in \text{Clop}(2^{(0,1)}) \) with \( X \cap Y = \emptyset \), \( B_X \cap \overline{B}_Y \in \mathcal{I} \) iff \( B_X \in \mathcal{I} \).

The next fact follows easily from Lemma 4.2.

Fact 4.3. Suppose \([B_U \cap \overline{B}_V] \leq [B_X \cap \overline{B}_Y] \) in \( \mathcal{B}_G^+ \). Then \( X \subseteq U \) and \((V \cup Y) \cap U = \emptyset \). If in addition \( U \setminus X \neq \emptyset \), then \([B_U \cap \overline{B}_V] < [B_X \cap \overline{B}_Y] \).

The following lemma gives necessary and sufficient conditions for an element of \( \mathcal{B}_G^+ \) to be an atom:

Lemma 4.4. \( b \in \mathcal{B}_G^+ \) is an atom \( \iff \) for some \( X,Y \in [(0,1)]^{<\omega} \), \( b = [B_X \cap \overline{B}_Y] > [0] \) and for each \( x \in (0,1) \setminus (X \cup Y) \), \( B_X \cap B_{\{x\}} \in \mathcal{I} \).

Proof. Suppose \( b \) is an atom. Then there must exist \( X,Y \in [(0,1)]^{<\omega} \) such that \( b = [B_X \cap \overline{B}_Y] \). If \( \exists x \in (0,1) \setminus (X \cup Y) \) such that \( B_X \cap B_{\{x\}} \notin \mathcal{I} \), then by Lemma 4.2 and Fact 4.3, \([0] < [B_X \cap B_{\{x\}} \cap \overline{B}_Y] < [B_X \cap \overline{B}_Y] \).

Contradiction.

Conversely, suppose \( b = [B_X \cap \overline{B}_Y] > [0] \) and \( \forall x \in (0,1) \setminus (X \cup Y), \quad B_X \cap B_{\{x\}} \in \mathcal{I} \). Suppose also that \([B_U \cap \overline{B}_V] \) is such that \([B_U \cap \overline{B}_V] \wedge [B_X \cap \overline{B}_Y] > [0] \). Then \( X \cap V = \emptyset \), so \( \forall v \in V \setminus Y, \quad B_X \cap B_{\{v\}} \in \mathcal{I} \). Thus, 
\([\overline{B}_V \setminus Y] \supseteq [B_X] \). Furthermore, \( B_U \cap B_X \notin \mathcal{I} \), so \( U \) must be contained in \( X \).

Therefore, \([B_U \cap \overline{B}_V] \geq [B_X \cap \overline{B}_Y] \). Hence, \( b \) is an atom.

Depending on the intervals \( T_i, T_{ij} \) used in the construction, a Gaifman algebra may have atoms. The following is an atom in many Gaifman algebras:

Example 4.5 (Some Gaifman algebras have atoms). Let \( T_2 = (0,1) \), \( T_3 = (0, \frac{31}{32}) \), and \( T_4 = (0, \frac{59}{64}) \), and choose the following \( T_{ij} \)'s in these \( T_i \)'s:

\[
\{T_{2,1}, \ldots, T_{2,4}\} = \left\{ \left(0, \frac{1}{4}\right), \left( \frac{3}{16}, \frac{5}{16}\right), \left( \frac{7}{16}, \frac{15}{16}\right), \left( \frac{11}{16}, 1\right) \right\}
\]

\[
\{T_{3,1}, \ldots, T_{3,9}\} = \left\{ \left(0, \frac{1}{16}\right), \left( \frac{1}{16}, \frac{3}{16}\right), \left( \frac{3}{16}, \frac{7}{16}\right), \left( \frac{5}{16}, \frac{7}{16}\right), \left( \frac{5}{16}, \frac{9}{16}\right), \left( \frac{9}{16}, \frac{11}{16}\right), \left( \frac{11}{16}, \frac{13}{16}\right), \left( \frac{13}{16}, \frac{29}{32}\right), \left( \frac{29}{32}, \frac{31}{32}\right) \right\}
\]

\[
\{T_{4,1}, \ldots, T_{4,16}\} = \left\{ \left(0, \frac{1}{32}\right), \left( \frac{1}{32}, \frac{3}{32}\right), \left( \frac{5}{32}, \frac{3}{32}\right), \left( \frac{5}{32}, \frac{7}{32}\right), \left( \frac{7}{32}, \frac{9}{32}\right), \left( \frac{9}{32}, \frac{11}{32}\right), \left( \frac{11}{32}, \frac{13}{32}\right), \left( \frac{13}{32}, \frac{15}{32}\right), \left( \frac{15}{32}, \frac{17}{32}\right), \left( \frac{17}{32}, \frac{19}{32}\right), \left( \frac{19}{32}, \frac{21}{32}\right), \left( \frac{21}{32}, \frac{23}{32}\right), \left( \frac{23}{32}, \frac{27}{32}\right), \left( \frac{27}{32}, \frac{37}{32}\right), \left( \frac{37}{32}, \frac{57}{64}\right) \right\}
\]

Let
\[
(4.3) \quad X = \left\{ \frac{1}{32}, \frac{3}{32}, \frac{5}{32}, \frac{10}{32}, \frac{14}{32}, \frac{18}{32} \right\}
\]
Remark 4.8. Hence, $T_i$ is a Gaifman algebra. For instance, if $T_5, T_6, T_7 \subseteq (\frac{31}{32}, 31)$, then no matter how the $T_i$ for $i \geq 8$ and $T_{ij}$ for $i \geq 5$ are chosen, $[B_X] \neq [0]$. Furthermore, for any $x \in (0, 1) \setminus X$, $B_{X \cup \{x\}} \in \mathcal{I}$, since $x$ must lie in at least one of the $T_{2,j}$’s, $T_{3,j}$’s, or $T_{4,j}$’s ($j \geq 1$) which $X$ does not intersect. Thus, by Lemma 4.4, $[B_X]$ is an atom.

\[ \text{Remark 4.6.} \] Every Gaifman algebra has at most countably many atoms, since the c.c.c. holds.

Every Gaifman algebra has a large atomless part. Let

\[ E = \bigcup_{2 \leq i < \omega} (T_{0i}\setminus \text{int}(T_{0i})). \tag{4.4} \]

$E$ is the set of all endpoints of the intervals $T_{ij}$, $2 \leq i < \omega$, $1 \leq j \leq i^2$.

**Lemma 4.7.** If $z \in (0, 1) \setminus E$, then there are no atoms in $B_G$ below $[B_{\{z\}}]$.

**Proof.** Suppose $z \in (0, 1) \setminus E$ and $[0] < [B_{U} \cap B_V] \leq [B_{\{z\}}]$. Then $z \in U$, by Fact 4.3. For each $i \geq 2$ let $j(i) \leq i^2$ be such that $z \in \text{int}(T_{i,j(i)})$, and let $N = |U| + 2$. Then $\text{int}(\bigcap_{2 \leq i \leq N} T_{i,j(i)}) \neq \emptyset$, so choose a $z' \in \text{int}(\bigcap_{2 \leq i \leq N} T_{i,j(i)} \setminus (U \cup V))$. \forall 2 \leq i \leq N, $z, z' \in T_{i,j(i)}$, so $U \cup \{z'\}$ has nonempty intersection with at most $(i - 1)$-many $T_{ij}$’s. Thus, $[0] < [B_{U \cup \{z'\}} \cap B_V]$. Fact 4.3 implies $[B_{U \cup \{z'\}} \cap B_V] < [B_{U} \cap B_V]$, since $z' \notin U$. Hence, $[B_{U} \cap B_V]$ is not an atom.

**Remark 4.8.**

1. By Lemma 4.7, $[B_X \cap B_Y]$ is an atom only if $X \subseteq E$. Moreover, if $[B_X \cap B_Y]$ is an atom, then $X \subseteq \bigcup_{2 \leq i \leq |X| + 1} (T_{0i}\setminus \text{int}(T_{0i}))$.

2. Atomless Gaifman algebras do exist. For example, if the $T_{ij}$’s are nested so that ($i < k$, $0 \leq j \leq i^2$, $1 \leq l \leq k^2$, and $T_{ij} \cap T_{kl} \neq \emptyset$) $\Rightarrow T_{kl} \subseteq T_{ij}$, then the resulting Gaifman algebra is atomless.

**5. The Cohen algebra completely embeds into the atomless part of each Gaifman algebra.**

In this section, we work in the atomless part of Gaifman algebras. Let $B_G$ be a Gaifman algebra. We identify $B_G^+$ with its image under the canonical dense embedding of $B_G^+$ into $r.o.(B_G^+)$ and work in $r.o.(B_G^+)$. Let

\[ a = [1] \setminus \left( \bigvee \{ b \in r.o.\left(B_G^+\right) : b \text{ is an atom} \} \right) \tag{5.1} \]

in $r.o.(B_G^+)$, and define

\[ A_G = r.o.\left(B_G^+\right) \upharpoonright a. \tag{5.2} \]
\(A_G\) is the atomless part of \(\text{r.o.}(B_G^+)\). Since \(\text{r.o.}(B_G^+)\) satisfies the \(\sigma\)-bounded c.c., \(A_G\) also satisfies the \(\sigma\)-bounded c.c. Since \(\text{CUP}(\text{r.o.}(B_G^+))\) fails and \(\text{r.o.}(B_G^+)\) has at most countably many atoms, \(\text{CUP}(A_G^+)\) must fail. Thus, by Kelley’s Theorem 1.9, \(A_G\) is non-measurable.

To avoid confusing notation between elements of \(\text{r.o.}(B_G^+)\) and \(A_G\), we will hold to the convention that \([B_X \cap B_Y] \leq a\) always refers to an element of \(\text{r.o.}(B_G^+)\). We will often use without mention the fact that \(\left\{ [B_X \cap B_Y] : [0] < [B_X \cap B_Y] \leq a \right\}\) is a dense subset of \(A_G^+\).

**Lemma 5.1.** If \(F \subseteq (0,1)\) is finite and \([B_X \cap B_Y] \in A_G^+\), then \(\exists z \in (0,1) \setminus (F \cup X \cup Y)\) such that \([0] < [B_X \cup \{z\} \cap B_Y]\).

**Proof.**

**Claim.** \(\forall [B_X \cap B_Y] \in A_G^+, \exists z \in (0,1) \setminus (X \cup Y)\) for which \([0] < [B_X \cup \{z\} \cap B_Y]\). Suppose \([B_X \cap B_Y] \in A_G^+\). Since \(A_G\) is atomless, \(\exists [B_U \cap B_V] \in A_G^+\) such that \([B_U \cap B_V] < [B_X \cap B_Y]\). By Fact 4.3, \(U \supseteq X\), so \([B_U \cap B_V] = [B_U \cap B_{U \cup Y}]\). Since \(A_G\) is atomless, Lemma 4.4 implies \(\exists z \in (0,1) \setminus (U \cup V \cup Y)\) such that \([B_U \cup \{z\} \cap B_{U \cup Y}] \not\in I\). Lemma 4.2 implies \([B_U \cup \{z\} \cap B_{U \cup Y}] > [0]\). Hence, the Claim holds.

Let \(n = |F|\). By the Claim, we can inductively choose a sequence of distinct elements \(z_0, \ldots, z_n \in (0,1) \setminus (X \cup Y)\) such that \([0] < [B_{X \cup \{z_0, \ldots, z_n\} \cap B_Y]\). Thus, there is some \(0 \leq i \leq n\) such that \(z_i \not\in F \cup X \cup Y\) and \([0] < [B_{X \cup \{z_i\} \cap B_Y}]\).

**Theorem 5.2.** The Cohen algebra embeds as a complete subalgebra into \(A_G\).

**Proof.** We construct a countable, atomless, regular subalgebra \(C\) of \(A_G\). Recall that \(E\) denotes the set of all endpoints of the intervals \(T_{ij}\), \(2 \leq i < \omega\), \(1 \leq j \leq i^2\) (see (4.4)). Our construction uses two types of sets: \(F_i\)’s which keep track of elements of \(E\), and \(X_i\)’s which keep track of elements of \((0,1)\) \(\setminus E\). We start by constructing the \(F_i\)’s recursively.

**Construction of \(C\).** Let \(E_2 = T_{20} \setminus \text{int}(T_{20})\), the endpoints of the open intervals \(T_{21}, T_{22}, T_{23}, T_{24}\). Let

\[
F_2 = E_2.
\]

Let \(E_3 = T_{30} \setminus (\text{int}(T_{30}) \cup F_2)\). Recall that \(a\) is the complement of the supremum of the atoms in \(\text{r.o.}(B_G^+)\) (see (5.1)). \(\forall F \subseteq F_2\) for which \([B_F] \leq a\), choose one \(x_F \in (0,1) \setminus F_2\) such that \([B_{F \cup \{x_F\}}] \geq a\). This is possible by Lemma 5.1 and the fact that \(\left\{ [B_X \cap B_Y] : [0] < [B_X \cap B_Y] \leq a \right\}\) is dense in \(A_G^+\). Let

\[
F_3 = E_3 \cup \{x_F : F \subseteq F_2, [B_F] \geq [0]\}.
\]
Given $F_2, \ldots, F_n$, let $E_{n+1} = T_{n+1,0} \setminus (\text{int}(T_{n+1,0}) \cup \bigcup_{2 \leq i \leq n} F_i)$. Again, $\forall F \subseteq \bigcup_{2 \leq i \leq n} F_i$ for which $[B_F] \land a > [0]$, choose one $x_F \in (0,1) \setminus (\bigcup_{2 \leq i \leq n} F_i)$ such that $[B_{F \cup \{x_F\}}] \land a > [0]$. Let

$$\text{(5.5)} \quad F_{n+1} = E_{n+1} \cup \left\{ x_F : F \subseteq \bigcup_{2 \leq i \leq n} F_i, [B_F] \land a > [0] \right\}.$$ 

The sets $F_i$ are finite and have the following properties:

(F1) $E \subseteq \bigcup_{2 \leq i < \omega} F_i$;

(F2) $\forall 2 \leq i < j < \omega, F_i \cap F_j = \emptyset$;

(F3) $\forall 2 \leq j < \omega, \forall F \subseteq \bigcup_{2 \leq k < j} F_k$ such that $[B_F] \land a > [0]$, there is an $x_F \in F_j$ for which $[B_{F \cup \{x_F\}}] \land a > [0]$.

Taking the $F_i$'s into consideration, we construct finite sets $X_i$ recursively. For $i \geq 2$, let

$$\text{(5.6)} \quad J_i = \left\{ s = (s(2), \ldots, s(i+1)) \in \prod_{k=2}^{i+1} (k^2 + 1) : \text{int} \left( \bigcap_{k=2}^{i+1} T_{k,s(k)} \right) \neq \emptyset \right\}.$$ 

$J_i \neq \emptyset$, since $\forall 2 \leq k \leq i^2 + 1, \{T_{k,j} : j \leq k^2\}$ is a partition of $(0,1)$ into finitely many open intervals and a finite union of closed intervals. For each $s \in J_i$, choose one $x_s \in \text{int}(\bigcap_{2 \leq k \leq i+1} T_{k,s(k)}) \setminus (\bigcup_{2 \leq k < \omega} F_k \cup \bigcup_{2 \leq k < i} X_k)$. Let

$$\text{(5.7)} \quad X_i = \{ x_s : s \in J_i \}.$$ 

The sets $X_i, F_i$ have the following properties:

(XF1) $\bigcup_{2 \leq i < \omega} X_i \cap (\bigcup_{2 \leq j < \omega} F_j) = \emptyset$;

(XF2) $\forall 2 \leq i < j < \omega, (X_i \cup F_i) \cap (X_j \cup F_j) = \emptyset$.

For each $2 \leq i < \omega$, define (in r.o.($B_{G^+}$))

$$\text{(5.8)} \quad c_i = \left( \bigvee_{x \in X_i} [B_{\{x\}}] \lor \bigvee_{f \in F_i} [B_{\{f\}}] \right) \land a.$$ 

Note that by Lemma 4.7, for each $x \in \bigcup_{2 \leq i < \omega} X_i$, $[B_{\{x\}}] \in A_G$, since $x \notin E$. Hence, $c_i = \bigvee_{x \in X_i} [B_{\{x\}}] \lor (\bigvee_{f \in F_i} [B_{\{f\}}] \land a)$. Let

$$\text{(5.9)} \quad C = \{ c_i : 2 \leq i < \omega \}.$$ 

the subalgebra of $A_G$ generated by $\{ c_i : 2 \leq i < \omega \}$. By our notational convention, the complement of $c_i$ in $A_G$ will be written as $-c_i \land a$, where $-c_i$ denotes the complement of $c_i$ in r.o.($B_{G^+}$).

**Proposition 5.3.** The generators of $C$ are independent.
Proof. Suppose $K, L$ are finite, disjoint subsets of $\{2, 3, 4, \ldots\}$. For all $c_d \in C^+$ such that whenever $c \in C$ and $c \wedge c_d \neq [0]$, then $c \wedge d \neq [0]$.

Proof. It suffices to show the proposition for all $d \in A_G^+$ of the form $[B X \cap B Y]$. Let $d = [B X \cap B Y] \in A_G^+$. By (XF2), we can fix an $N \geq |X \cup Y| + 1$ such that $(X \cup Y) \cap (\bigcup_{N < i < c_d} (X_i \cup F_i)) = \emptyset$. Define $I = \{2 \leq i \leq N : X \cap (X_i \cup F_i) \neq \emptyset\}$ and $J = \{2 \leq i \leq N : X \cap (X_i \cup F_i) = \emptyset\}$. Let

$$c_d = \bigwedge_{i \in I} c_i \wedge \bigwedge_{j \in J} (-c_j \wedge a).$$

$c_d > [0]$ by Proposition 5.3. (Unlike in our constructions of complete embeddings of the Cohen algebra into the families of Galvin-Hajnal and Argyros algebras, $c_d$ is not necessarily a minimal cover for $d$ in $C$.)

Suppose $c = \bigwedge_{k \in K} c_k \wedge \bigwedge_{l \in L} (-c_l \wedge a) \in C^+$, where $K, L$ are finite subsets of $\{2, 3, \ldots\}$, and $c_d \wedge c > [0]$. Then by Proposition 5.3, $(I \cup K) \cap (J \cup L) = \emptyset$. Let $K' = K \cap \{2, \ldots, N\}$ and $K'' = K \setminus K'$. $K \cap J = \emptyset$ implies $K' \subseteq I$. Since $d \leq a$,

$$c \wedge d = \bigwedge_{k \in K'} \left( \bigvee_{u \in X_k \cup F_k} [B_{(u)}] \right) \wedge \left[ B X \cap B Y \cup \bigcup_{l \in L} (X_l \cup F_l) \right].$$

This follows from (5.12).
Hence, \( \langle r \rangle \) and \( \langle f \rangle \). To our attention that A. Kamburelis has done some similar work on these \( \omega, \omega \)-algebras in an unpublished paper. Specifically, Kamburelis used forcing methods (in contrast to our constructive, purely Boolean-algebraic methods) to show that the weak \((\omega, \omega)\)-d.l. fails in the Galvin-Hajnal, Agryros, and Gaifman algebras.

In this paper, Kamburelis mentions K. Skandalis’ remark that his proof of the failure everywhere of the weak \((\omega, \omega)\)-d.l. in the the Argyros algebra can be easily modified to produce a Cohen real. By the result of Shelah [S] mentioned in Remark 1.16, we now know that Kamburelis’ proof of the failure everywhere of the weak \((\omega, \omega)\)-d.l. actually implies the Argyros algebra.
adds a Cohen real. In addition, Kamburelis showed that the Gaifman algebra adds a Cohen real, although to do this, he assumed that the Gaifman algebra contains no atoms, which, as we showed, is not always the case.

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References


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INTERSECTION OF CONJUGACY CLASSES WITH
BRUHAT CELLS IN CHEVALLEY GROUPS

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Let $G = \tilde{G}(K)$ where $\tilde{G}$ is a simple and simply-connected algebraic group that is defined and quasi-split over a field $K$. We investigate properties of intersections of Bruhat cells $B\dot{w}B$ of $G$ with conjugacy classes $C$ of $G$, in particular, we consider the question, when is $B\dot{w}B \cap C \neq \emptyset$.

1. Introduction.

Let $(G, B, N, S)$ be a Tits system. Some aspects of intersections of conjugacy classes of $G$ with Bruhat cells $B\dot{w}B$ have been investigated by several authors (see e.g., [St1], [K], [V] and [VS]). Here $w \in W = N/(B \cap N)$ and $\dot{w} \in N$ is a preimage of $w$ with respect to the natural surjection $N \rightarrow W$. In particular, it is desirable to learn how a conjugacy class $C$ of $G$ is related to those conjugacy classes $C_w$ of $W$ for which $B\dot{w}B \cap C \neq \emptyset$, where $w \in C_w$.

Here we deal with the case where $G$ is a Chevalley group, i.e., $G$ is the group of points $\tilde{G}(K)$ of a simple algebraic group $\tilde{G}$ that is defined and quasi-split over a field $K$, thus $G$ is a proper or a twisted Chevalley group (see [St2]). Therefore, one can define a Tits system $(G, B, N, S)$, where $S = \{w_{\alpha_i} \mid \alpha_i \in \Pi\}$ for a simple root system $\Pi$ corresponding to $G$ ([St2] and [C1]).

A crucial step to investigate intersections $B\dot{w}B \cap C$ was done by R. Steinberg [St1] who constructed the cross-section of regular conjugacy classes in $B\dot{w}_SB$, where $w_S$ is a Coxeter element of $W$ with respect to the fixed set of generators $S$ of $W$, i.e., $w_S$ is a product of elements in $S$ in any order, where each $s \in S$ occurs exactly once. The next natural step is to consider intersections of regular classes with cells of the form $B\dot{w}_S\dot{w}^{-1}B$. Here we prove the following:

**Theorem 1.1.** Let $\tilde{G}$ be a simple and simply-connected algebraic group that is defined and quasi-split over a field $K$ and let $G = \tilde{G}(K)$. Further, let $C \subset G$ be a conjugacy class of $G$ such that

\[(*) \quad B\dot{w}_SB \cap C \neq \emptyset,\]

where $w_S$ is a Coxeter element of $W$ with respect to $S$. Then $C$ intersects all cells of the form $B\dot{w}_S\dot{w}^{-1}B$, where $w \in W$. 

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Note that the condition $B\dot{w}SB \cap C \neq \emptyset$ implies that every element of $C$ is regular in $\tilde{G}$, except the case when $\tilde{G}$ is not split and has the type $A_{2l}$ ([St1, Remark 8.8]). Condition (⋆) holds, for instance, for regular conjugacy classes of $G$ in the following cases (as shown in Section 4):

(a) $G = SL_n(K)$;
(b) $K = \overline{K}$ (where $\overline{K}$ is the algebraic closure of $K$).

In the cases (c) to (f) below, the field $K$ is supposed to be perfect:

(c) $\tilde{G}$ is split over $K$ and $C = \tilde{C} \cap G$ for a conjugacy class $\tilde{C}$ of $\tilde{G}$;
(d) $\dim K \leq 1$ and $C$ is a semisimple class (here $\dim K$ is the homological dimension of $K$);
(e) $\tilde{G}$ is split over $K$, $C \cap B \neq \emptyset$, and $C$ is a semisimple class;
(f) $C$ is a unipotent class, $\text{char } K$ is not a bad prime for $\tilde{G}$, and if $\tilde{G}$ is not split, then $\tilde{G}$ is not of type $A_{2l}$.

Theorem 1.1 implies:

**Corollary 1.2.** Let $\tilde{G}$ be a simple and simply-connected algebraic group that is defined and quasi-split over a field $K$ and let $G = \tilde{G}(K)$. Further, let $C \subset G$ be a regular conjugacy class of $G$. If one of Conditions (a) to (f) holds, then $C$ intersects all Bruhat cells of the form $B\dot{w}\dot{w}^{-1}B$.

**Remark.** The statement of the Corollary in Case (a) follows from the existence of a normal rational form. Case (b) follows from a much more general fact: Every regular conjugacy class of a simple algebraic group (i.e., $G = G(\overline{K})$) intersects all Bruhat cells (see Appendix). Also, in Case (f), if $K$ is a finite field, then a theorem of Kawanaka [K] shows that any regular unipotent conjugacy class intersects all Bruhat cells.

Now let $X \subset S$, $W_X = \langle X \rangle$. By $w_X$ we denote a product (in any order) of elements of $X$, where each $x \in X$ occurs exactly once, i.e., $w_X$ is a Coxeter element of $W_X$ with respect to $X$. It is natural to consider intersections $B\dot{w}\dot{w}^{-1}X \cap C$ next. In [GS] it has been shown that $B\dot{w}\dot{w}^{-1}X \cap C \neq \emptyset$ for some $X \subset S$ if $C$ is a semisimple class and $K$ is a finite field. Here we prove:

**Theorem 1.3.** Let $\tilde{G}$ be a simple and simply-connected algebraic group that is defined and quasi-split over a perfect field $K$ such that $\dim K \leq 1$, and let $G = \tilde{G}(K)$. Further, let $C \subset G$ be a noncentral semisimple conjugacy class of $G$. Then $C$ intersects all Bruhat cells of the form $B\dot{w}\dot{w}^{-1}X$ for some $X \subset S$, $X \neq \emptyset$.

**Remark.** This theorem generalizes Proposition 6 from [GS].

We thank the referee for drawing our attention to a result of Geck and Pfeiffer (see Proposition 3.3) which allows us to extend our results to all Chevalley groups.
2. S-Coxeter elements in Coxeter groups.

Let $W$ be a finite group of orthogonal transformations of a Euclidean space $V$ generated by reflections. Then $W$ is a Coxeter group. Let $S = \{s_1, \ldots, s_r\}$ be a Coxeter system of generators of $W$, i.e., $s_i^2 = 1$ for every $i = 1, \ldots, r$ and $(s_i s_j)^{m_{ij}} = 1$ is the system of basic relations for the group $W$ (see [Bou, IV, 1]). Then every element of the form $s_{\pi(1)} s_{\pi(2)} \ldots s_{\pi(r)}$, where $\pi \in S_r$, is called a Coxeter element of $W$. All Coxeter elements of $W$ constructed for all possible Coxeter systems of generators are conjugate in $W$ (see [Bou, V, 6, Proposition 1]), and if $V^W = \{0\}$, each Coxeter element acts on $V \setminus \{0\}$ without fixed points ([Bou, V, 6, 2]).

Definition 2.1. Let $X \subset S$ and let $W_X$ be the subgroup of $W$ generated by $X$. Every element of $W$ that is conjugate to a Coxeter element in $W_X$ will be called a generalized Coxeter element of $W$.

Definition 2.2. For a fixed system $S$ of generators the elements of the form $s_{\pi(1)} s_{\pi(2)} \ldots s_{\pi(r)}$, where $|S| = r$, will be called $S$-Coxeter elements. If $X \subset S$, then $X$-Coxeter elements in $W_X$ will be called generalized $S$-Coxeter elements of $W$.

Let $l_S(w)$ be the $S$-length of $w$, i.e., the length of $w$ with respect to $S$. Obviously, a Coxeter element $w \in W$ is $S$-Coxeter if and only if $l_S(w) = r$.

Below, we shall work with a fixed system $S$ and we shall write $l(w)$ instead of $l_S(w)$. We shall use the well-known fact that $l_X(w) = l_S(w)$ for any $w \in W_X$.

Example 2.3. Let $W = S_4$ and $S = \{(12), (23), (34)\}$. Then we have six Coxeter elements (4-cycles) in $W$. Among them there are four $S$-Coxeter elements:

$$(12)(23)(34), \ (34)(23)(12), \ (23)(12)(34), \ (12)(34)(23),$$

and two elements that are not $S$-Coxeter elements:


Lemma 2.4. Let $w_1, w_2$ be two $S$-Coxeter elements of $W$. Then there exists a sequence $\sigma_1, \sigma_2, \ldots, \sigma_n \in S$ (possibly $\sigma_i = \sigma_j$ for $i \neq j$) such that

$$w_2 = \sigma_n \sigma_{n-1} \ldots \sigma_1 w_1 \sigma_1 \sigma_2 \ldots \sigma_n$$

and $l(\sigma_i \sigma_{i-1} \ldots \sigma_1 w_1 \sigma_1 \sigma_2 \ldots \sigma_{i-1} \sigma_i) = r$ for every $i = 1, \ldots, n$.

Proof. See [C2, Section 10.3].

3. A condition for the intersection of a conjugacy class with Bruhat cells and Gauss cells.

We are going to use the concepts of $S$-ascent and $S$-descent and derive some of their properties. The notion of descent was introduced and considered in
[GP] (without the name “descent”) as a binary relation between elements of conjugacy classes of Coxeter groups. The notion of ascent is dual to that of descent.

**Definition 3.1.** Let \( w_1, w_2 \in W \). We say that there exists an \( S \)-ascent (resp. \( S \)-descent) from \( w_1 \) to \( w_2 \) if there is a sequence \( \sigma_1, \ldots, \sigma_n \in S \) such that

\[
w_2 = \sigma_n \sigma_{n-1} \cdots \sigma_1 w_1 \sigma_1 \sigma_2 \cdots \sigma_n
\]

and

\[
l(\sigma_1 \sigma_{i-1} \cdots \sigma_1 w_1 \sigma_1 \sigma_2 \cdots \sigma_i) \\
\geq (\text{resp. } \leq) \ l(\sigma_{i-1} \cdots \sigma_1 w_1 \sigma_1 \sigma_2 \cdots \sigma_{i-1})
\]

for every \( i = 1, \ldots, n \).

**Remark.** As before, we fix a set \( S \) of generators for \( W \). In [GP] an \( S \)-descent from an element \( w \in W \) to an element \( w' \in W \) is denoted by \( w \rightarrow w' \). It is logical to denote an \( S \)-ascent from \( w' \in W \) to \( w \in W \) by \( w' \leftarrow w \).

**Definition 3.2.** Let \( C \subset W \) be a conjugacy class. We define

\[
l(C) = \min \{ l(w) \mid w \in C \}.
\]

The following proposition is due to M. Geck and G. Pfeifer ([GP, Theorem 3.2.9.(a)]):

**Proposition 3.3.** Let \( C \subset W = W(R) \) be a conjugacy class. Then for every \( w \in C \) there exists an \( S \)-descent to an element \( w' \in C \) such that

\[
l(w') = l(C).
\]

Let \( G \) be a Chevalley group (proper or twisted) corresponding to a root system \( R \) in the sense of [St2]. We fix a simple root system \( \Pi = \{ \alpha_1, \ldots, \alpha_r \} \) and a corresponding Borel subgroup \( B = HU \). Let \( W = W(R) \) be the Weyl group of \( G \) and \( S = \{ w_{\alpha_1}, \ldots, w_{\alpha_r} \} \) the corresponding Coxeter system of generators. By \( X_\alpha \) we denote below a root subgroup of \( G \) (see [St2]).

The meaning of Definition 3.1 becomes clear from the following:

**Proposition 3.4.** Let \( g \in B \dot{w} B \) (resp. \( g \in B \dot{w}' B \)) and let \( w' \in W \) be an element that is conjugate to \( w \). If there exists an \( S \)-ascent (resp. \( S \)-descent) from \( w \) to \( w' \), then there exists an element \( g' \in B \dot{w}' B \) (resp. \( B \dot{w}' B \)) that is conjugate to \( g \).

**Proof.** We shall use the following lemma:

**Lemma 3.5.** Let \( w \in W \). Suppose

\[
w(\alpha_i) < 0, \ \text{and} \ w^{-1}(\alpha_i) < 0
\]

for some \( \alpha_i \in \Pi \). Then either \( w = w_{\alpha_i} w' w_{\alpha_i} \), where \( l(w') = l(w) - 2 \), or \( w = w_{\alpha_i} w' = w' w_{\alpha_i} \), where \( l(w') = l(w) - 1 \).
Proof. The assumption $w^{-1}(\alpha_i) < 0$ implies
\[ w = w_{\alpha_i} w_1, \]
where $l(w_1) = l(w) - 1$ ([C2, Section 2.2]). Suppose $w_1(\alpha_i) = \beta > 0$. Since $w(\alpha_i) = w_{\alpha_i}(\beta) < 0$, we have $\beta = \alpha_i$ and we have the second possibility. Now let $w_1(\alpha_i) < 0$. Then $w_1 = w' w_{\alpha_i}$, where $l(w') = l(w_1) - 1$ and we have the first possibility. \qed

First, let $g \in B\dot{w}B$, then $g = b_1 \dot{w} b_2$. We may assume $b_1 = 1$ and $b_2 = u \in U$. Also, it is sufficient to prove the assertion for an S-ascent of one step, i.e., $w' = w_{\alpha} w w_{\alpha}$ for some $\alpha \in \Pi$. We can write $u = u_{\alpha} v$, where $u_{\alpha}$ is a root subgroup element corresponding to $\alpha$ and where $v \in U$ is an element that has no $\alpha$-factors in any decomposition into positive root subgroup elements.

If $u_{\alpha} = 1$, then $w' = w_{\alpha} \dot{w} w_{\alpha}^{-1} \in U$ and
\[ g' = w_{\alpha} \dot{w} w_{\alpha}^{-1} = (w_{\alpha} \dot{w} w_{\alpha}^{-1})(\dot{w} w_{\alpha}^{-1}) = u' \in B\dot{w}B. \]

Let $u_{\alpha} \neq 1$. Suppose $\beta = w(\alpha) > 0$. We may assume $\beta \neq \alpha$ (otherwise $w' = w_{\alpha} w w_{\alpha}^{-1} = w$). We have $g = \dot{w} u_{\alpha} \dot{w} = u_{\beta} \dot{w} v$. Now we can consider the element $u_{\beta}^{-1} g u_{\beta}$ instead of $g$ which satisfies the previous condition $u_{\alpha} = 1$.

Suppose $\beta = w(\alpha) < 0$ and $\gamma = w^{-1}(\alpha) > 0$. We have $g = \dot{w} u_{\alpha} v = \dot{w} u_{\alpha} v u_{\alpha}^{-1} w_{\alpha}$. Note that $v' = u_{\alpha} v u_{\alpha}^{-1}$ has no factors corresponding to $\alpha$. Consider now the element $\tilde{g} = u_{\alpha} g u_{\alpha}^{-1}$ instead of $g$. We have $\tilde{g} = u_{\alpha} \dot{w} v = \dot{w}^{-1} u_{\alpha} \dot{w} v = u_{\alpha} \dot{w} v$, an element which also satisfies the condition $u_{\alpha} = 1$.

Now let $\beta = w(\alpha) < 0$, $\gamma = w^{-1}(\alpha) < 0$. Then, by Lemma 3.5, either $w_{\alpha} w w_{\alpha} = w$ and, therefore, there is nothing to prove, or $l(w_{\alpha} w w_{\alpha}) < l(w)$ which contradicts our assumption.

Second, let $g \in B^{-} \dot{w} B$. We may assume $g = v v_{\alpha} \dot{w} u_{\alpha} u$, where $v \in U^{-}$, $v_{\alpha} \in X_{-\alpha}$, $u_{\alpha} \in X_{\alpha}$, $u \in U$ and the elements $v, u$ have no factors from the group $X_{\pm \alpha}$. Note, $\tilde{w}_{\alpha} v v_{\alpha}^{-1} \in U^{-}$, $\tilde{w}_{\alpha} w \tilde{w} w_{\alpha}^{-1} \in U$ (because $\alpha$ is a simple root). Thus, if $v_{\alpha} = u_{\alpha} = 1$, then $\tilde{w}_{\alpha} g \tilde{w}^{-1} \in B^{-} \dot{w} B$. Now put $\beta = w(\alpha), \gamma = w^{-1}(\alpha)$. If $\beta < 0$, $\gamma < 0$, we have $g = v v_{\alpha} \dot{w} u_{\alpha} \dot{w}^{-1} w = v v_{\alpha} v_{\beta} \dot{w} u = v v_{\alpha} v_{\beta} \tilde{w} w = v v_{\alpha} v_{\beta} v_{\alpha}^{-1} v_{\beta} \tilde{w} w = v (v_{\alpha} v_{\beta} v_{\alpha}^{-1}) \tilde{w} w u$, where $v_{\beta} = \tilde{w} w u \in X_{\beta}$, $u_{\beta} = \tilde{w}^{-1} v_{\beta} \tilde{w} \in X_{-\beta}$. We may assume $\beta, \gamma \neq -\alpha$ (otherwise we have $w_{\alpha} w w_{\alpha} = w$). Thus the elements $v (v_{\alpha} v_{\beta} v_{\alpha}^{-1}), u_{\gamma} u$ have no factors from $X_{\pm \alpha}$ and we are in the preceding case.

Let $\beta > 0, \gamma < 0$. Then $g = v v_{\alpha} u_{\beta} \dot{w} u = v v_{\alpha} v_{\beta} \tilde{w} u$, where the element $v' \in U^{-}$ has no factor from $X_{-\alpha}$. Put $u_{\alpha} = \tilde{w}_{\alpha} v_{\alpha} \tilde{w}_{\alpha}^{-1}$. Then $\tilde{w}_{\alpha} \tilde{w}_{\alpha}^{-1} = u_{\alpha} v_{\beta} \tilde{w} u$ for some $v'' \in U^{-}, u' \in U$. Thus $u_{\alpha}^{-1} \tilde{w}_{\alpha} g \tilde{w}_{\alpha}^{-1} u_{\alpha} \in B^{-} \dot{w} B$.

The case $\beta < 0, \gamma > 0$ is similar to the preceding case.

Let $\beta > 0, \gamma > 0$. Again, as above, we may assume $\beta, \gamma \neq \alpha$. Thus by Lemma 3.5, we have $l(w') = l(w) + 2$ which contradicts our assumption. \qed
Example 3.6. Let \( G = SL_3(K) \) and let
\[
\dot{w} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \dot{w}' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]
Let \( g \) be a semisimple element of \( G \) that has no eigenvalues in \( K \). Then \( g \) is a regular element and therefore its conjugacy class \( C_g \) intersects the big Bruhat cell \( B \dot{w} B \) (see [EGH, Lemma 4]). But \( C_g \cap B \dot{w}' B = \emptyset \) because every element of the form \( b_1 \dot{w}' b_2 \) is conjugate to an element of the form
\[
\dot{w}' b = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & a_{22} & a_{23} \\ -a_{11} & -a_{12} & -a_{13} \\ 0 & 0 & a_{33} \end{pmatrix}
\]
which has an eigenvalue \( a_{33} \in K \). Note that here \( S = \{w_{12}, w_{23}\} \) (where \( w_{ij} \) is the matrix in which the \( i \)th and \( j \)th elements of the standard basis are interchanged) and \( l(w) = 3, l(w') = l(w_{12}) = 1 \).

Example 3.7. Let \( G = SL_4(K) \) and let
\[
\dot{w} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \dot{w}' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
\]
Here \( S = \{w_{12}, w_{23}, w_{34}\} \) and \( \dot{w} = \dot{w}_{12} \dot{w}_{34} \) is a generalized S-Coxeter element. Thus every noncentral conjugacy class of \( G \) intersects \( B^- \dot{w} B \) ([EG]).

In particular, one can find a transvection \( g \in B^- \dot{w} B \). But there are no transvections in \( B^- \dot{w}' B \) because \( B^- \dot{w}' B = \dot{w}' B \dot{w}'^{-1} \dot{w}' B = \dot{w}' B \), and every matrix \( x \in \dot{w}' B \) satisfies the condition \( \text{rank } (x - 1) \geq 2 \).

The examples above show that if there is no \( S \)-ascent (resp. \( S \)-descent) from \( w \in W \) to its conjugate \( w' \in W \), the condition \( C \cap B \dot{w} B \neq \emptyset \) (resp. \( C \cap B^- \dot{w}' B \neq \emptyset \)) for a conjugacy class \( C \in G \) does not necessarily imply \( C \cap B \dot{w} B \neq \emptyset \) (resp. \( C \cap B^- \dot{w}' B \neq \emptyset \)).

Proposition 3.8. Let \( g \in B \dot{w} B \) (resp. \( g \in B^- \dot{w} B \)). Suppose \( l(w_{\alpha} w_{\alpha}) = l(w) - 2 \) (resp. \( l(w_{\alpha} w_{\alpha}) = l(w) + 2 \)). Then the conjugacy class \( C_g \) of \( g \) intersects either \( B \dot{w}_{\alpha} \dot{w}_{\alpha}^{-1} B \) (resp. \( B^- \dot{w}_{\alpha} \dot{w}_{\alpha}^{-1} B \)) or \( B \dot{w}_{\alpha} \dot{w}_{\alpha} B \) and \( B \dot{w}_{\alpha} \dot{w}_{\alpha} B \) (resp. \( B^- \dot{w}_{\alpha} \dot{w}_{\alpha} B \) and \( B^- \dot{w}_{\alpha} \dot{w}_{\alpha} B \)).

Proof. Let \( g \in B \dot{w} B \). We may assume, as in the proof of Proposition 3.4, that \( g = \dot{w} a u \) and \( w = w_{\alpha} w_{\alpha} w_{\alpha} \), where \( l(w_{\alpha}) = l(w) - 2 \). Moreover, \( \beta = w_{1}(\alpha) > 0, \gamma = w_{1}^{-1}(\alpha) > 0, \) and \( w_{1}(\alpha) w_{1}^{-1}(\alpha) \neq \alpha \). If \( u_{\alpha} = 1 \), then \( \dot{w}_{\alpha} \dot{w}_{\alpha}^{-1} \in B \dot{w}_{\alpha} B \). Suppose \( u_{\alpha} \neq 1 \). Put \( u_{\alpha} = \dot{w}_{\alpha} a \dot{w}_{\alpha}^{-1} \). There exists \( u'_{\alpha} \in X_{\alpha} \) (here \( X_{\alpha} \) is the corresponding root subgroup) such that \( u'_{\alpha} u_{\alpha} = \dot{w}_{\alpha} a u_{\alpha} \) for some \( u''_{\alpha} \in X_{\alpha} \). Further, \( g_{1} = \dot{w}_{\alpha} g_{\alpha}^{-1} = \dot{w}_{1} u_{\alpha} u'_{\alpha} \) for some \( u'_{\alpha} \in U \). Put \( u_{\beta} = \dot{w}_{1} u'_{\alpha}^{-1} \) (recall \( \beta = w_{1}(\alpha) > 0 \)). Then
\[ g_2 = u_2 g_1 u_2^{-1} = \hat{w}_1 u_1' \hat{w}_1^{-1} \hat{w}_1 u_{-\alpha} u' u_\beta^{-1} = \hat{w}_1 w'_\alpha u' u_\beta^{-1} \in B \hat{w}_1 \hat{w}_\alpha B. \] Since \[ l(w_\alpha) = l(w_\alpha w_1), \] we also can find an element in \( C_g \cap B \hat{w}_\alpha \hat{w}_1 B \) (by Proposition 3.4).

Now let \( g \in B^{-\hat{w}} B. \) As in the proof of Proposition 3.4 we may assume \( g = \nu \alpha w_{\alpha} u, \alpha \neq w(\alpha) > 0, \alpha \neq w^{-1}(\alpha) > 0. \) If \( u_\alpha = u_\alpha = 1, \) then \( \hat{w}_\alpha g \hat{w}_\alpha^{-1} \in B^{-\hat{w}_\alpha \hat{w}_\alpha^{-1}} B. \) Let \( v_\alpha = 1, u_\alpha \neq 1. \) Then
\[ g_1 = \hat{w}_\alpha \hat{g}_\alpha^{-1} = (\hat{w}_\alpha \nu \nu^{-1} w'_\alpha u (\hat{w}_\alpha \nu \nu^{-1} w'_\alpha u (\hat{w}_\alpha \nu \nu^{-1} w'_\alpha u (\hat{w}_\alpha \nu \nu^{-1} w'_\alpha u (\hat{w}_\alpha \nu \nu^{-1} w'_\alpha u (\hat{w}_\alpha \nu \nu^{-1} w'_\alpha u (\hat{w}_\alpha \nu \nu^{-1} w'_\alpha u (\hat{w}_\alpha \nu \nu^{-1} w'_\alpha u (\hat{w}_\alpha \nu \nu^{-1} w'_\alpha u (\hat{w}_\alpha \nu \nu^{-1} w'_\alpha u (\hat{w}_\alpha \nu \nu^{-1} w'_\alpha u (\hat{w}_\alpha \nu \nu^{-1} w'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'' \alpha u'
Lemma 4.1. Let $\tilde{C}$ be a conjugacy class of $\tilde{G}$ such that $C = \tilde{C} \cap G \neq \emptyset$. Further, let $g \in C$. If $H^1(K, C_{\tilde{G}}(g)) = 1$ then $C$ is a conjugacy class of $G$ (here $C_{\tilde{G}}(g)$ is the centralizer of $g$ in $\tilde{G}$).

Proof. The argument here is the same as in ([C2, Proposition 3.7.3]). Indeed, if $g' \in C$, then there exists an element $\gamma \in \tilde{G}$ such that $g' = \gamma g \gamma^{-1}$. Thus, for every element $\sigma \in \text{Gal}(\overline{K}/K)$ of the Galois group we have

$$\sigma(\gamma) g \sigma(\gamma^{-1}) = \gamma g \gamma^{-1}$$

and therefore $x_\sigma = \gamma^{-1} \sigma(\gamma) \in C_{\tilde{G}}(g)$. Since $x_\sigma$ is a 1-cocycle, we have $x_\sigma = y\sigma(y^{-1})$ for some $y \in C_{\tilde{G}}$ and therefore $\sigma(\gamma y) = \gamma y$ for every $\sigma \in \text{Gal}(\overline{K}/K)$. Thus, $\gamma y \in G$ and $g' = \gamma yy^{-1}\gamma^{-1}$.

Lemma 4.2. Let $\tilde{C}$ be a semisimple conjugacy class of $\tilde{G}$ and let $C = \tilde{C} \cap G \neq \emptyset$. If $\dim K \leq 1$, then $C$ is a conjugacy class of $G$.

Proof. Since $\tilde{G}$ is simply-connected, $C_{\tilde{G}}(s)$ is a connected reductive group for $s \in \tilde{C} \cap T$ ([C2, Theorem 3.5.6]) and therefore $H^1(K, C_{\tilde{G}}(s)) = 1$ ([St1, 11.2]). Now the assertion follows from Lemma 4.1.

Lemma 4.3. Let $C$ be the same as in the preceding lemma. Suppose that $G$ is split and $\tilde{C}$ is a regular semisimple class such that $\tilde{C} \cap T \neq \emptyset$. Then $C$ is a conjugacy class of $G$.

Proof. If $s \in \tilde{C} \cap T$, then $C_{\tilde{G}}(s) = \tilde{T}$ is a $K$-split torus and therefore $H^1(K, C_{\tilde{G}}(s)) = 1$ ([Sp, 12.3.5.(3)]). Now the assertion follows from Lemma 4.1.

Lemma 4.4. Let $u_1, u_2 \in G$ be two regular unipotent elements of $\tilde{G}$. Assume that $\text{char } K$ is not a bad prime for $\tilde{G}$. Then there exist elements $t \in T$ and $\gamma \in G$ such that $u_1 = t\gamma u_2\gamma^{-1}t^{-1}$.

Proof. Let $\tilde{G} = \tilde{G}/Z(\tilde{G}), \tilde{T} = \tilde{T}/Z(\tilde{G})$. Then $\tilde{G}$ is defined and quasi-split over $K$ and $Z(\tilde{G}) = 1$. Further, let $u \in \tilde{G}$ be a regular unipotent element and let $\overline{u}$ be its image in $\tilde{G}$. The char $K$ is not a bad prime for $\tilde{G}$, thus $V = C_{\tilde{G}}(u)$ is a connected unipotent subgroup of $\tilde{G}$ ([C2, Proposition 5.1.6]) which is defined and split over $K$ ([Sp, 14.3.8]) and therefore $H^1(K, V) = 1$ ([Sp, 12.3.5.(3)]). Hence any two regular unipotent elements of $\tilde{G}(K)$ are conjugate (Lemma 4.1). If $\tilde{G}_1(K) \leq \tilde{G}(K)$ is a subgroup generated by unipotent elements of $\tilde{G}$, then it is a normal subgroup and $\tilde{G}(K) = \tilde{G}_1(K) \tilde{T}(K)$ (this follows from the Bruhat decomposition). Now let $\overline{u}_1, \overline{u}_2 \in \tilde{G}(K)$ be images of regular unipotent elements $u_1, u_2 \in G$. Then there exist elements $\overline{\gamma} \in \tilde{G}_1(K), \overline{t} \in \tilde{T}(K)$ such that $\overline{u}_1 = \overline{t}\overline{\gamma}\overline{u}_2\overline{\gamma}^{-1}\overline{t}^{-1}$. If $\gamma \in \tilde{G}(K) = G, t \in \tilde{T}$ are preimages of $\overline{\gamma}, \overline{t}$, then $u_1 \equiv t\gamma u_2\gamma^{-1}t^{-1}(\mod Z(\tilde{G}))$. Since $u_1, u_2$ are both unipotent elements, we have $u_1 = t\gamma u_2\gamma^{-1}t^{-1}$. □
Now we check Condition (*) for (a) to (f):

(a) If \( G = SL_n(K) \), Condition (*) is an immediate consequence of the
representation of elements of \( GL_n(K) \) in rational canonical form.

(b) Consider the case where \( K \) is an algebraically closed field. According

to Steinberg’s theorem ([St1, 1.4]), the set

\[
\mathfrak{R} = \tilde{w}_{\gamma_1}X_{\gamma_1}\tilde{w}_{\gamma_2}X_{\gamma_2} \ldots \tilde{w}_{\gamma_s}X_{\gamma_s}
\]

is a cross-section of all regular conjugacy classes of the group \( \tilde{G} \), where

\( \tilde{w}_{\gamma_1}, \ldots, \tilde{w}_{\gamma_s} \) is any fixed system of preimages of the basic reflections \( w_{\gamma_1}, \ldots, w_{\gamma_s} \) in any fixed order (here \( X_{\gamma_i} \) is the corresponding root subgroup). Moreover, we can rewrite \( \mathfrak{R} \) in the form

\[
\mathfrak{R} = \tilde{w}_{\gamma_1} \tilde{w}_{\gamma_2} \ldots \tilde{w}_{\gamma_s} X_{\theta_1}X_{\theta_2} \ldots X_{\theta_s},
\]

where \( \theta_i = w_{r_i} \ldots w_{r_{i+2}}w_{r_{i+1}}(\gamma_i) > 0 \). Since \( K \) is an algebraically closed field, \( \{\alpha_1, \ldots, \alpha_s\} = \{\gamma_1, \ldots, \gamma_s\} \) and any element in the intersection \( C \cap \tilde{N}(K) \) lies in the S-Coxeter cell \( B\tilde{w}_{\alpha_1}\tilde{w}_{\alpha_2} \ldots \tilde{w}_{\alpha_r}B \). This proves (*).

(c) If \( \tilde{G} \) is split over \( K \), the closed subset \( \mathfrak{R} \) (defined above) of \( \tilde{G} \) is defined over \( K \) and \( \mathfrak{R} \cap \tilde{C} \subset \tilde{G} \) ([St1, Section 9]).

(d) There exists a closed subset \( \mathfrak{R}' \) of \( \mathfrak{R} \) which is defined over \( K \) and such

that every regular semisimple conjugacy class \( \tilde{C} \) of \( \tilde{G} \) intersects \( \mathfrak{R}' \) in just

one point (and this point belongs to \( G \) if \( \tilde{C} \cap G \neq \emptyset \)) ([St1, 9.11]). Since

\( \mathfrak{R} \subset B\tilde{w}_SB \) for some S-Coxeter element \( w_S \), the assertion follows from

Lemma 4.2.

(e) We may use the same argument as in (d), and Lemma 4.3.

(f) If \( \tilde{G} \) is split or \( \tilde{G} \) is not of type \( A_2 \), the cross-section of regular classes \( \mathfrak{R} \)

is defined over \( K \) and for the conjugacy class of regular unipotent elements

\( C \) we have \( u = \tilde{C} \cap \mathfrak{R} \in B\tilde{w}_SB \), where \( \tilde{w}_S \in N \) for some S-Coxeter element

\( w_S \) in \( W \) ([St1, Section 9]). Now let \( u' \in \tilde{C} \cap \tilde{G} \). By Lemma 4.4 we have

\[
t\gamma u' \gamma^{-1} t^{-1} = u = u_1\tilde{w}_Sb_1 \text{ for some } t \in \tilde{T}, \gamma \in \tilde{G} \text{ and } u_1 \in U, b_1 \in B.
\]

Hence \( u'' = \gamma u' \gamma^{-1} = (t^{-1}ut)(t^{-1}\tilde{w}_st)(t^{-1}b_1t) \). Thus \( u'' \in B\tilde{w}_SB \). But \( u'' \in G \)

and, therefore, \( u'' \in B\tilde{w}B \) for some \( \tilde{w} \in N \). Since \( B\tilde{w}B \subset B\tilde{w}B \), we have

\( w = w_S \). This implies that the conjugacy class \( C \) of \( u' \) in \( G \) has a nontrivial

intersection with \( B\tilde{w}_SB \), where \( \tilde{w}_S \in N \).

Proof of Theorem 1.3.

Below, \( \tilde{\Gamma} \) is a connected reductive algebraic group defined over a perfect
field \( K \) such that \( \dim K \leq 1 \).

**Lemma 4.5.** Let \( \tilde{P} = \tilde{L}R_u(\tilde{P}) \) be a parabolic subgroup of \( \tilde{\Gamma} \) defined over \( K \).
Let \( \tilde{L} \) be a fixed Levi factor (defined over \( K \)) and let \( R_u(\tilde{P}) \) be the unipotent
radical of \( \tilde{P} \). Further, let \( s \in \tilde{P}(K), s = lu \), where \( l \in \tilde{L} \) and \( u \in R_u(\tilde{P}) \).
If \( s \in \tilde{\Gamma}(K) \), then \( l \in \tilde{L}(K) \) and \( u \in R_u(\tilde{P})(K) \). If, in addition, \( s \) is a semisimple element, then \( s \) is conjugate to \( l \) in \( \tilde{P} \).

**Proof.** The first assertion follows from the uniqueness of the decomposition \( lu \).

Further, if \( s \) is semisimple, it is contained in a maximal torus in \( \tilde{P} \) which is contained in a Levi subgroup \( L' \). ([Sp, 8.4.4]). Since all Levi subgroups are conjugate in \( \tilde{P} \) ([Sp, 16.1.1]) by elements of \( \tilde{P} \), one can find an element \( p = l_1u_1 \in \tilde{P} \) where \( l_1 \in \tilde{L} \), \( u_1 \in R_u(\tilde{P}) \) such that \( ps^{-1} \in \tilde{L} \). Then \( l_1^{-1}ps^{-1}l_1 = u_1su_1^{-1} = l(l^{-1}u_1l)u_1^{-1} \in \tilde{L} \). Hence \( (l^{-1}u_1l)u_1^{-1} = 1 \) (because \( (l^{-1}u_1l)u_1^{-1} \in R_u(\tilde{P}) \)) and therefore \( l_1^{-1}ps^{-1}l_1 = l \). \( \Box \)

**Lemma 4.6.** Let \( s \in \tilde{\Gamma}(K) \) be a semisimple element of \( \tilde{\Gamma} \) such that \( C_{\tilde{\Gamma}}(s)^0 \) is not a torus. Then there exists a parabolic subgroup \( \tilde{P} \) of \( \tilde{\Gamma} \) defined over \( K \) such that \( s \in \tilde{P} \).

**Proof.** The group \( C_{\tilde{\Gamma}}(s)^0 \) is defined over \( K \) ([Sp, 12.1.4]). Further, the condition \( \dim K \leq 1 \) implies that there exists a Borel subgroup \( \tilde{B}_s \) of \( C_{\tilde{\Gamma}}(s)^0 \) which is also defined over \( K \) ([St1, 10.2]). Since \( C_{\tilde{\Gamma}}(s)^0 \) is not a torus, the unipotent radical \( R_u(\tilde{B}_s) \) is not trivial. The group \( \tilde{U}_1 = R_u(\tilde{B}_s) \) is also defined over \( K \) ([Sp, 14.4.5(v)]). Further, let

\[
\begin{align*}
\tilde{N}_1 &= N_{\tilde{G}}(\tilde{U}_1), \\
\tilde{U}_2 &= \tilde{U}_1 \cdot R_u(\tilde{N}_1), \\
\tilde{N}_2 &= N_{\tilde{G}}(\tilde{U}_2), \ldots, \\
\tilde{U}_i &= \tilde{U}_{i-1} \cdot R_u(\tilde{N}_{i-1}), \\
\tilde{N}_i &= N_{\tilde{G}}(\tilde{U}_i), \ldots.
\end{align*}

\]

Then all members of (1) are closed subgroups of \( \tilde{\Gamma} \) and \( \tilde{U}_k = \tilde{U}_{k+1}, \tilde{N}_k = \tilde{N}_{k+1} \) for some positive integer \( k \) ([Hu, 30.3]). Further, all groups in (1) are defined over \( K \); indeed, the field \( K \) is perfect and all groups are defined as normalizers of \( K \)-defined groups, their unipotent radicals, and the images of \( K \)-defined groups with respect to maps \( \tilde{U}_{i-1} \times R_u(\tilde{N}_{i-1}) \longrightarrow \tilde{U}_{i-1} \cdot R_u(\tilde{N}_{i-1}) \), induced by multiplication in \( \tilde{G} \). Since \( \tilde{U}_1 \) is connected, the last member \( \tilde{N}_k \) of this sequence is a parabolic subgroup of \( \tilde{\Gamma} \) ([Hu, 30.3]). From the definitions we have \( s \in \tilde{N}_1 \leq \tilde{N}_k \). \( \Box \)

Now we can prove Theorem 1.3. Let \( s \in G \) be a noncentral semisimple element. We may assume that \( s \) is not a regular element of \( \tilde{G} \) (otherwise the statement follows from Theorem 1.1 and Property (d)). By Lemma 4.6 we have \( s \in \tilde{P} \) for some parabolic subgroup defined over \( K \). Since \( g\tilde{P}g^{-1} = \tilde{P}_l \) for some standard parabolic subgroup \( \tilde{P}_l \) and \( g \in G \) ([Sp, 15.4.6]), we may assume \( s \in \tilde{P}_l \), where \( I \subset \tilde{\Pi} \) is a Gal \((\overline{K}/K)\)-invariant subset (note that the group Gal \((\overline{K}/K)\) acts on \( \tilde{\Pi} \) by permutation and the orbits of this action
correspond to \( \Pi \); see [St1, Section 9]). Let \( \tilde{L}_I = \tilde{T}\tilde{G}_I \), where \( \tilde{G}_I = \langle X_\alpha \mid \alpha \in \langle I \rangle \rangle \). Then \( \tilde{L}_I \) is a \( K \)-defined Levi factor of \( \tilde{P}_I \).

By Lemmas 4.5 and 4.2 we may assume \( s \in \tilde{L}_I \). (Indeed, by Lemma 4.5 we have an element \( l \in \tilde{L}_I(K) \) which is conjugate to \( s \) in \( \tilde{P}_I \). By Lemma 4.2 the elements \( s, l \) are conjugate by an element of the group \( G \). Hence we may take the element \( l \in C \) instead of \( s \).

Again by Lemma 4.6 we may assume that \( C_{\tilde{L}_I}(s)^0 = \tilde{T}' \), where \( \tilde{T}' \) is a maximal torus of \( \tilde{L}_I \) defined over \( K \) (otherwise, we can take a smaller set \( I \) using the same procedure as above). Note that the derived subgroup \( \tilde{L}_I \) is equal to \( \tilde{G}_I \) and therefore is a simply-connected semisimple group (because \( \tilde{G} \) is simply-connected). Hence \( C_{\tilde{L}_I}(s)^0 = C_{\tilde{L}_I}(s) \) ([C2, Theorem 3.5.6]) and thus

\[
(2) \quad C_{\tilde{L}_I}(s) = \tilde{T}'.
\]

Further, if \( I = \emptyset \) we have \( \tilde{P}_I = \tilde{B} \) and \( \tilde{T}' = \tilde{T} \). Hence \( s \in \tilde{T}(K) = T \). Since \( s \) is a noncentral element of \( G \), there exists a root \( \alpha \in \Pi \) such that \( s \) is not in the center of the group \( T\tilde{G}_\alpha(K) \) (here, \( \tilde{G}_\alpha = \langle X_\beta \mid \beta \in \langle I_\alpha \rangle \rangle \) where \( I_\alpha \subset \tilde{\Pi} \) is the Gal \( (K/K) \)-orbit of \( \alpha \)). Since the Borel subgroup \( B_\alpha \) of \( T\tilde{G}_\alpha(K) \) (with respect to \( T \)) is not a normal subgroup, one can find an element \( \gamma \in T\tilde{G}_\alpha(K) \) such that \( \gamma s \gamma^{-1} = \tilde{w}_\alpha b \), where \( w_\alpha \in W \) is the corresponding reflection and \( b \in B_\alpha \). Hence \( C \cap B\tilde{w}_\alpha B \neq \emptyset \). Further, let \( \omega \in W \). Then \( \omega w_\alpha \omega^{-1} = w_\beta \), where \( \beta = \omega(\alpha) \). Let \( \tilde{\omega}, \tilde{w}_\beta \) be preimages of \( \omega, w_\beta \) in the group \( N \). Then \( \tilde{\omega}T\tilde{G}_\alpha(K)\tilde{\omega}^{-1} = T\tilde{G}_\beta(K) \). The element \( \tilde{s}' = \tilde{\omega} s \tilde{\omega}^{-1} \) is not a central element in \( T\tilde{G}_\beta(K) \). Now, as above, we have \( \gamma' s' \gamma'^{-1} \in B\tilde{w}_\beta B \) for some \( \gamma' \in T\tilde{G}_\beta(K) \). Thus, if \( I = \emptyset \), the assertion of the theorem holds for \( X = \{ \alpha \} \).

Now we may assume that \( I \neq \emptyset \) and Condition (2) holds.

We have \( s = tg \), \( t \in \tilde{T} \cap C_{\tilde{L}_I} (\tilde{G}_I) \), and \( g \in \tilde{G}_I \) ([Hu, 27.5]). Note that the elements \( t \) and \( g \) do not necessarily belong to \( G \) but \( t, g \in \tilde{L}_I(K') \) for some extension \( K'/K \). The element \( s \in G \) is Gal \( (K/K) \)-invariant and \( t \in Z(\tilde{L}_I) \). Hence \( g = h_1 g_1 \), where \( h_1 \in \tilde{T}(K') \), \( g_1 \in \tilde{G}_I(K) \) (this follows from the Bruhat decomposition of \( g \)). Further, (2) implies that \( g \) is a regular element of \( \tilde{G}_I \). If \( \mathfrak{N}' \) is a cross-section (defined over \( K \)) of regular semisimple conjugacy classes of \( \tilde{G}_I \) ([St1, Section 9]) then \( h_1 \mathfrak{N}' \) is also a cross-section (defined over \( K' \)) of regular semisimple conjugacy classes of \( \tilde{G}_I \). Hence the conjugacy class \( C_g \) of \( g \) in \( \tilde{G}_I \) intersects \( h_1 \mathfrak{N}' \) in just one point. Thus the conjugacy class \( C_s = tC_g \) of \( s \) in \( \tilde{L}_I \) intersects \( th_1 \mathfrak{N}' \) also in one point \( x \) (recall, \( t \in Z(\tilde{L}_I) \)). Since the conjugacy class \( C_s \) is defined over \( K \) and the closed subset \( th_1 \mathfrak{N}' \) is also defined over \( K \) (because \( th_1 = sg_1^{-1} \in \tilde{L}_I(K) \)),
the point $x$ is Gal($\overline{K}/K$)-invariant and therefore it belongs to $L/I(K)$. Since $s, x \in L/I(K) \leq G$ are conjugate in $L/I$ (and therefore in $\tilde{G}$), we have $x = \sigma s \sigma^{-1}$ for some $\sigma \in G$ (Lemma 4.2). Further,

$$\text{(3)} \quad \tilde{t}_1 \mathfrak{M}' \subset \left( \prod_{\alpha \in X} \tilde{w}_\alpha \right) \tilde{U},$$

where $X \subset \Pi$ is the set of Gal($\overline{K}/K$)-orbits of $I \subset \tilde{\Pi}$ and $w_\alpha$ in (3) is the product of basic reflections $w_\gamma$, where $\gamma$ runs through the orbit corresponding to $\alpha$ or $w_\alpha = w_{\gamma_1 + \gamma_2}$ if such orbit consists of two roots $\gamma_1, \gamma_2$ such that $\gamma_1 + \gamma_2$ is a root (see [St1, Section 9]). From (3) we obtain

$$\text{(4)} \quad x = \sigma s \sigma^{-1} \in \tilde{B} \prod_{\alpha \in X} \tilde{w}_\alpha \tilde{B}.$$

Since $x \in G$, we have

$$\text{(5)} \quad x = \sigma s \sigma^{-1} \in B \tilde{w} B$$

for some $w \in W$. But

$$\text{(6)} \quad B \tilde{w} B \subset \tilde{B} \tilde{w} \tilde{B}. $$

From (4), (5), (6) we get

$$\text{(7)} \quad w = \prod_{\alpha \in X} w_\alpha,$$

i.e., $w$ is a generalized $S$-Coxeter element of $W$. Now (5) and (7) imply that the conjugacy class of $s$ in $G$ intersects $B \tilde{w} B$ for some generalized $S$-Coxeter element $w$ of $W$.

Suppose that $w' = \omega w \omega^{-1}$ is also an $S$-Coxeter element of $W$ for some $\omega \in W$. Then $w' = \prod_{\alpha \in Y} w_\alpha$ for some $Y \subset \Pi, |Y| = |X|$. Let $X' = \{\omega(\alpha) \mid \alpha \in X\}$. Then

$$w' = \prod_{\alpha \in Y} w_\alpha = \prod_{\beta \in X'} w_\beta.$$

The element $w'$ is a Coxeter element of the root systems generated by $Y$ and $X'$. It acts without fixed points on the vector space (over $\mathbb{R}$) generated by $Y$ and on the vector space generated by $X'$. Moreover, $l(w') = |Y| = |X'|$. Hence the vector spaces (over $\mathbb{R}$) generated by $Y$ and $X'$ coincide (it is the $\langle w' \rangle$-complement to the vector space of $w'$-invariant vectors). Since $X$ is a simple root system for the root system $\langle X \rangle$, the set $X'$ is a simple root system for $\langle X' \rangle$. On the other hand, the set $Y$ is a simple root system for the root system $\langle Y \rangle$. Now $X' \subset \omega(\Pi), Y \subset \Pi$ and the linear spaces generated by $X'$ and $Y$ coincide. Moreover, the root subsystems $\langle X' \rangle, \langle Y \rangle$ have the same Coxeter element $w'$. Hence $\langle X' \rangle = \langle Y \rangle$. Now let $I'$ be a subset of $\Pi$
that is Gal($\overline{K}/K$)-invariant and such that the set of Gal($\overline{K}/K$)-orbits of $I'$ coincides with $Y$. Since $\omega((X)) = \langle X' \rangle = \langle Y \rangle$, we have

\[ \widetilde{G}_{I'} = \langle X_\beta \mid \beta \in \langle I' \rangle \rangle \cong \tilde{\omega} \tilde{G}_{I} \tilde{\omega}^{-1}. \]

From (8) we get

\[ \tilde{L}_{I'} = \tilde{T} \tilde{G}_{I'} = \tilde{\omega} \tilde{L}_I \tilde{\omega}^{-1}. \]

Since $\omega \in W$, we can choose the preimage $\tilde{\omega} \in G$. From (9)

\[ s' = \tilde{\omega} s \tilde{\omega}^{-1} \in \tilde{L}_I \cap G. \]

Now we have a semisimple regular element $s' \in \tilde{L}_I(K)$. The same arguments as above show that there exists an element $\tau \in G$ such that $s'' = \tau s' \tau^{-1} \in B \tilde{w}' B$, where

\[ w'' = \prod_{\beta \in Y} w_\beta \]

(the order of the roots $\beta$ in this product can be different from the order of the roots $\alpha$ in the product corresponding to $w'$). By Lemma 2.4 there exists an $S$-ascent from $w''$ to $w'$ (both elements are $Y$-Coxeter elements for the Weyl group of the system $\langle Y \rangle$). Proposition 3.4 implies

\[ \delta s'' \delta^{-1} \in B \tilde{w}' B \]

for some $\delta \in G$.

The inclusions (5) and (10) show that the conjugacy class $C$ of $s$ in $G$ intersects all Bruhat cells $B \tilde{w}'' B$, where $w''$ runs through all generalized $S$-Coxeter elements that are conjugate to $w$. Now let $\tilde{w} \in W$ be an element from the conjugacy class of $w$. Proposition 3.3 implies that there exists an $S$-ascent from some generalized $S$-Coxeter element $w'''$ to $\tilde{w}$. Now the assertion of the theorem follows from Proposition 3.4.

Theorem 1.3 has been proved.

Remarks to Theorem 1.3.

1. Intersection with a parabolic subgroup. In the proof of Theorem 1.3 we showed that

\[ (***) \quad C \cap P_X \neq \emptyset \]

for every noncentral semisimple conjugacy class $C$ that is not regular, where $X \subseteq \Pi$ and $P_X = BW_X B$ is the corresponding parabolic subgroup (if $K$ is a perfect field and $\dim K \leq 1$). More generally, Equation (**) holds for every noncentral conjugacy class $C$ that is not a regular semisimple class (if $K$ is a perfect field and $\dim K \leq 1$). Indeed, we consider the Jordan decomposition $g = su$ of an element $g \in C$. Applying the same construction as in Lemma 4.6, we get a parabolic subgroup $P$ which is defined over $K$ and contains $s, u$. Then by an appropriate conjugation we can embed $g$ in
a standard parabolic subgroup. (Note, if \( K \) is a finite field, then Condition (**) is a consequence of the properties of the Steinberg representation \([C2, Proposition 6.4.5]\).)

2. The condition: \( \dim K \leq 1 \). The example below shows that if this condition does not hold, the conclusion of Theorem 1.3 may be false.

Let \( n = 4k \) and let \( V \) be a linear space over the real number field \( \mathbb{R} \) such that \( \dim V = 4k \). Further, let \( \{e_1, \ldots, e_{4k}\} \) be a basis of \( V \) and let \( V^+ = \langle e_1, \ldots, e_{2k} \rangle, V^- = \langle e_{2k+1}, \ldots, e_{4k} \rangle \). Further, let \( (x_1, \ldots, x_{4k}) \) be the coordinates of an element in \( V \) with respect to the basis \( \{e_i\} \) and let \( \Phi = x_1^2 + \cdots + x_{2k}^2 - x_{2k+1}^2 - \cdots - x_{4k}^2 \). Let \( \Omega = \Omega(V, \Phi) = [SO(V, \Phi), SO(V, \Phi)] \). Then \( \Omega \) is a Chevalley group in the sense of \([St2]\), corresponding to the root system \( D_{2k} \). Let \( g \in GL(V) \) be the linear operator such that \( g|_{V^+} = -1, g|_{V^-} = 1 \). One can easily check that \( g \in \Omega \) and \( gug^{-1} \neq u^{\pm 1} \) for every nontrivial unipotent element \( u \in \Omega \) (the latter follows from the fact that \( v \pm g(v) \) is not an isotropic vector if \( v \neq 0 \) is isotropic). Hence the element \( g \) cannot normalize any nontrivial unipotent subgroup of \( \Omega \) and therefore \( g \) cannot belong to any proper parabolic subgroup of \( \Omega \). This implies that a preimage \( \hat{g} \) of \( g \) in \( G = \text{Spin}_{4k}(\mathbb{R}) \) (with respect to the natural homomorphism \( G \longrightarrow \Omega \)) also cannot belong to a proper parabolic subgroup of \( G \). Hence \( C \cap B_wX = \emptyset \) for every \( X \subset \Pi \), where \( C \) is the conjugacy class of \( \hat{g} \) in \( G \), \( B \) is a Borel subgroup of \( G \), and \( \Pi \) is a simple root system corresponding to \( \hat{G} = \text{Spin}_{4k} \) (note, \( BW_XB = P_X \) is a standard parabolic subgroup).

3. The ordered set of \( \mathcal{X}_C \). Recall, for any set \( X \subset \Pi \) we define \( w_X = \prod_{\alpha \in X} w_\alpha \), where the product can be taken in any fixed order. For the set

\[
\mathcal{X}_C = \{X \subset \Pi \mid C \cap Bw_XB \neq \emptyset\}
\]

one can consider the natural order with respect to inclusion.

Let \( G = SL_n(\mathbb{C}) \) and \( C \) a noncentral semisimple conjugacy class. Let \( \lambda(C) = (\lambda_1, \ldots, \lambda_r) \) be the partition of \( n \), i.e., \( \lambda_1 \geq \cdots \geq \lambda_r \), where \( \lambda_1 + \cdots + \lambda_r = n \), which corresponds to the multiplicities of eigenvalues of elements of \( C \) (i.e., \( \lambda_1 \) is the biggest multiplicity, then \( \lambda_2 \), etc.) and let \( \lambda^*(C) \) be the dual partition (i.e., the rows and columns of \( \lambda \) are interchanged). Further, to every partition \( \mu = (\mu_1, \ldots, \mu_s) \) of \( n \) we assign a subset \( X(\mu) \subset \Pi = \{\alpha_1, \ldots, \alpha_{n-1}\} \), namely,

\[
X(\mu) \stackrel{\text{def}}{=} \Pi \setminus \{\alpha_{\mu_1}, \alpha_{\mu_1+\mu_2}, \ldots, \alpha_{\mu_1+\cdots+\mu_{s-1}}\}.
\]

It is easy to see that \( X(\lambda^*(C)) \) is a maximal element of \( \mathcal{X}_C \). Moreover, every maximal element \( Y \in \mathcal{X}_C \) is \( W \)-conjugate to \( X(\lambda^*(C)) \). Thus we have just one conjugacy class \( \{ww_Xw^{-1}\} \) in \( W \) for each maximal \( X \in \mathcal{X}_C \).

For other types of groups we can have several conjugacy classes in \( W \) of elements of the form \( w_X \), where \( X \in \mathcal{X}_C \) is a maximal element. Say, consider
the root system $R = B_2 = \langle \alpha_1, \alpha_2 \rangle$, where $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2$ (in the notation of [Bou]), and let $G$ be the corresponding simple and simply connected group over $\mathbb{C}$. Let $g = h_{\varepsilon_1}(t)h_{\varepsilon_2}(t^{-1}) \in G$ be a semisimple element, where $h_{\varepsilon_1}(t), h_{\varepsilon_2}(t^{-1})$ are the corresponding root semisimple elements (in the notation of [St2]) and $t \neq \pm 1$. Let $C$ be the conjugacy class of $g$. Then $\{\alpha_1\}$ and $\{\alpha_2\}$ both are maximal elements of $X_C$. Thus here we have two different conjugacy classes in $W$ of elements $w_X$ for maximal $X$ in $X_C$.

5. Appendix.

The following result as well as the line of proof was pointed out to the second author by T.A. Springer in the discussion of relevant questions:

**Proposition 5.1.** Let $\tilde{G}$ be a simple algebraic group defined over an algebraically closed field $K$ and let $G = \tilde{G}(K)$. Further, let $C$ be the conjugacy class of a regular element of $G$. Then $C \cap B \tilde{w}B \neq \emptyset$ for every $w \in W$.

**Proof.** For $b \in B$ we put $\mathcal{O}_B(b) = \{xbx^{-1} \mid x \in B\}$.

**Lemma 5.2.** There exists a nonempty finite set $\{b_1, \ldots, b_n\} \subset C \cap B$ such that

$$C \cap B = \bigcup_{1 \leq i \leq n} \mathcal{O}_B(b_i).$$

**Proof.** Let $x = s_1u_1, y = s_2u_2 \in B$ be two regular elements, where $s_1, s_2 \in T$ and $u_1, u_2 \in U$. We show

$$\mathcal{O}_B(x) = \mathcal{O}_B(y) \text{ if and only if } s_1 = s_2.$$ (11)

Indeed, “only if” is obvious. Now let

$$b_1 = su_1, b_2 = su_2, \quad s \in T, \quad u_1, u_2 \in U.$$ (12)

Since we can consider the Jordan decompositions of $x, y$ as elements of $B$, we may assume that (12) gives the Jordan decompositions of $b_1$ and $b_2$. Put $\Gamma = [C_G(s), C_G(s)], B_\Gamma = B \cap \Gamma$. Then (12) implies $u_1, u_2 \in B_\Gamma$. Moreover, the elements $u_1, u_2$ are regular unipotent elements of $\Gamma$ ([St1, 3.7]) and therefore the elements $u_1, u_2$ are conjugate in $B_\Gamma$ (see [C2, the proof of Proposition 5.1.3]). Hence we have (11).

Now let $b = su \in B, \quad s \in T, \quad u \in U, \quad g \in G, \quad gb^{-1} \in B$. Further, let $g \in B\tilde{w}B$. Then $gb^{-1} = w(s)u'$ for some $u' \in U$. Together with (11), this implies our assertion.

**Lemma 5.3.** Let $b \in C \cap B$ be a fixed element and let $w \in W$. Then every irreducible component $\mathcal{C}_w$ of $C \cap \overline{B\tilde{w}B}$ such that $\mathcal{O}_B(b) \subset \mathcal{C}_w$ satisfies the following condition:

$$\dim \mathcal{C}_w = \dim C + \dim \overline{B\tilde{w}B} - \dim G.$$
Proof. Since $b$ is a regular element, $\dim C_B(b) = \text{rank } G$ ([St1, 3.11]). If $C_1$ is an irreducible component of $\overline{C} \cap B$ containing $\mathcal{O}_B(b)$, then Lemma 5.2 implies $C_1 = \overline{\mathcal{O}_B(b)}$ and, therefore,

(13) \[ \dim C_1 = \dim B - \text{rank } G = \dim \overline{C} + \dim B - \dim G. \]

Let $\mathcal{O}_B(b) \subset C_w$ for some irreducible component $C_w$ of $\overline{C} \cap B\hat{w}B$. Suppose

(14) \[ \dim C_w > \dim \overline{C} + \dim B\hat{w}B - \dim G. \]

Since $B$ is a closed subset of $B\hat{w}B$ ([Sp, 8.15]) and $C_1$ is an irreducible component of $C_w \cap B$, we have

(15) \[ \dim C_1 \geq \dim C_w + \dim B - \dim B\hat{w}B. \]

Now (14) and (15) contradict (13). Thus we have our statement.

Now we return to the proof of Proposition 5.1.

Take $C_w$ as in Lemma 5.3. Assume $C \cap B\hat{w}B = \emptyset$. Then

(16) \[ C_w \subset \bigcup_{w' < w} B\hat{w}'B = \bigcup_{w' < w} B\hat{w'}B \]

([Sp, 8.15]). From (16) we have $C_w \subset B\hat{w'}B$ for some $w' < w$ and we may consider $C_w$ as an irreducible component of $\overline{C} \cap B\hat{w'}B$ that contains $\mathcal{O}_B(b)$. Then, by Lemma 5.3, we have

(17) \[ \dim C_w = \dim \overline{C} + \dim B\hat{w'}B - \dim G. \]

But (17) contradicts Lemma 5.3 because $\dim B\hat{w'}B < \dim B\hat{w}B$. This proves Proposition 5.1. \[ \square \]

References


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PROPERTIES OF THE RESIDUAL CIRCLE ACTION
ON A HYPERTORIC VARIETY

MEGUMI HARADA AND NICHOLAS PROUDFOOT

We consider an orbifold $X$ obtained by a Kähler reduction of $\mathbb{C}^n$, and we define its “hyperkähler analogue” $M$ as a hyperkähler reduction of $T^*\mathbb{C}^n \cong \mathbb{H}^n$ by the same group. In the case where the group is abelian and $X$ is a toric variety, $M$ is a toric hyperkähler orbifold, as defined in Bielawski and Dancer, 2000, and further studied by Konno and by Hausel and Sturmfels. The variety $M$ carries a natural action of $S^1$, induced by the scalar action of $S^1$ on the fibers of $T^*\mathbb{C}^n$. In this paper we study this action, computing its fixed points and its equivariant cohomology. As an application, we use the associated $\mathbb{Z}_2$ action on the real locus of $M$ to compute a deformation of the Orlik-Solomon algebra of a smooth, real hyperplane arrangement $\mathcal{H}$, depending nontrivially on the affine structure of the arrangement. This deformation is given by the $\mathbb{Z}_2$-equivariant cohomology of the complement of the complexification of $\mathcal{H}$, where $\mathbb{Z}_2$ acts by complex conjugation.

In order to construct a toric variety as a Kähler quotient of $\mathbb{C}^n$ by a torus, one begins with the combinatorial data of an arrangement $\mathcal{H}$ of $n$ cooriented, rational, affine hyperplanes in $\mathbb{R}^d$. The normal vectors to these hyperplanes determine a subtorus $T^k \subseteq T^n$ ($k = n - d$), and the affine structure determines a value $\alpha \in (t^k)^*$ at which to reduce, so that we may define $X = \mathbb{C}^n \big/_{\alpha} T^k$. Using the same combinatorial data, one can also construct a hypertoric variety, which is defined as the hyperkähler quotient $M = \mathbb{H}^n \big/_{(\alpha,0)} T^k$ of $\mathbb{H}^n \cong T^*\mathbb{C}^n$ by the induced action of the same subtorus $T^k \subseteq T^n$ [BD]. It is well-known that the toric variety $X$ does not retain all of the information of $\mathcal{H}$; indeed, it depends only on the polyhedron $\Delta$ obtained by intersecting the half-spaces associated to each of the cooriented hyperplanes. Thus it is always possible to add an extra hyperplane to $\mathcal{H}$ without changing $X$. In contrast, the hypertoric variety $M$ remembers the number of hyperplanes in $\mathcal{H}$, but its equivariant diffeomorphism type depends neither on the coorientations nor on the affine structure of $\mathcal{H}$ (see Theorem 4.1, and Lemmas 2.1 and 2.2).

1In [BD, K1, K2] and [HS] $M$ is called a “toric hyperkähler” variety, but as it is a complex variety that is not toric in the standard sense, we prefer the term “hypertoric.”
The purpose of this paper is to study the Hamiltonian $S^1$ action on $M$ descending from the scalar action of $S^1$ on the fibers of $T^*\mathbb{C}^n$. This action is sensitive to both the coorientations and the affine structure of $\mathcal{H}$, even on the level of equivariant cohomology (Section 4). One can recover the toric variety $X$ as the minimum of the $S^1$ moment map, hence the geometric structure of $M$ along with its circle action carries strictly more information than either $X$ or $M$ alone. In Section 3 we give an explicit description of this action when restricted to the core $C$, a deformation retract of $M$ which is a union of projective subvarieties. In Section 4 we compute the $S^1$ and $T^d \times S^1$-equivariant cohomologies of $M$, using the full combinatorial data of $\mathcal{H}$.

In Section 5, we examine the real locus $M_\mathbb{R} \subseteq M$, i.e., the fixed point set of an involution of $M$ that is anti-holomorphic with respect to the first complex structure. By studying the topology of $M_\mathbb{R}$, we interpret the results of Section 4 in terms of the Orlik-Solomon algebra $\mathcal{O}_= S^* (\mathcal{M}(\mathcal{H}))$, where $\mathcal{M}(\mathcal{H})$ is the complement of the complexification of $\mathcal{H}$. We show how to interpret Theorem 4.4 as a computation of $H^*_{\mathbb{Z}_2} (\mathcal{M}(\mathcal{H}); \mathbb{Z}_2)$, a deformation of the Orlik-Solomon algebra of a smooth, real arrangement that depends nontrivially on the affine structure.$^2$

1. Hyperkähler reductions.

A hyperkähler manifold is a smooth manifold, necessarily of real dimension $4n$, which admits three complex structures $J_1, J_2, J_3$ satisfying the usual quaternionic relations, in a manner compatible with a metric. Just as in the Kähler case, we can define three different symplectic forms on $N$ as follows:

$$\omega_1(v, w) = g(J_1 v, w), \quad \omega_2(v, w) = g(J_2 v, w), \quad \omega_3(v, w) = g(J_3 v, w).$$

Note that the complex-valued two-form $\omega_2 + i\omega_3$ is nondegenerate and covariant constant, hence closed and holomorphic with respect to the complex structure $J_1$. Any hyperkähler manifold can therefore be considered as a holomorphic symplectic manifold with complex structure $J_1$, real symplectic form $\omega_\mathbb{R} := \omega_1$, and holomorphic symplectic form $\omega_\mathbb{C} := \omega_2 + i\omega_3$. This is the point of view that we will adopt in this paper.

We will refer to an action of $G$ on a hyperkähler manifold $N$ as hyperHamiltonian if it is Hamiltonian with respect to $\omega_\mathbb{R}$ and holomorphic Hamiltonian with respect to $\omega_\mathbb{C}$, with $G$-equivariant moment map

$$\mu_{HK} := \mu_\mathbb{R} \oplus \mu_\mathbb{C} : N \to g^* \oplus g_\mathbb{C}^*.$$

**Theorem 1.1** ([HKLR]). Let $(N^{4n}, g)$ be a hyperkähler manifold with real symplectic form $\omega_\mathbb{R}$ and holomorphic symplectic form $\omega_\mathbb{C}$. Suppose that $N$

$^2$A more general computation of $H^*_{\mathbb{Z}_2} (\mathcal{M}(\mathcal{H}); \mathbb{Z}_2)$, in which $\mathcal{H}$ is not assumed to be simple, rational, or smooth, will appear in [Pr].
is equipped with a hyperhamiltonian action of a compact Lie group $G$, with moment map $\mu_{HK} = \mu_R \oplus \mu_C$. Suppose $\xi = \xi_R \oplus \xi_C$ is a central regular value of $\mu_{HK}$. Then there is a unique hyperkähler structure on the hyperkähler quotient $M = N\sslash G := \mu_{HK}^{-1}(\xi)/G$, with associated symplectic and holomorphic symplectic forms $\omega^\xi_R$ and $\omega^\xi_C$, such that $\omega^\xi_R$ and $\omega^\xi_C$ pull back to the restrictions of $\omega_R$ and $\omega_C$ to $\mu_{HK}^{-1}(\xi)$.

If $\xi \in \mathfrak{g}^* \oplus \mathfrak{g}_C^*$ is fixed by the coadjoint action of $G$, the inverse image $\mu_C^{-1}(\xi)$ is preserved by $G$, and is a (singular) Kähler subvariety with respect to $\omega_R$. Then by [HL] (see also [Na, 3.2] and [Sj, 2.5]), we have

$$N\sslash G = \mu_C^{-1}(\xi)/\xi_G = \mu_C^{-1}(\xi)^{ss}/G_C,$$

where

$$\mu_C^{-1}(\xi)^{ss} = \{ x \in \mu_C^{-1}(\xi) \mid GX \cap \mu_R^{-1}(\xi_R) \neq \emptyset \}. $$

We now specialize to the case where $G$ is a compact Lie group acting linearly on $\mathbb{C}^n$ with moment map $\mu : \mathbb{C}^n \to \mathfrak{g}^*$, taking $0 \in \mathbb{C}^n$ to $0 \in \mathfrak{g}^*$. This action induces an action of $G$ on the holomorphic cotangent bundle $T^*\mathbb{C}^n \cong \mathbb{C}^n \times (\mathbb{C}^n)^*$.

If we choose a bilinear inner product on $\mathbb{C}^n$, we can coordinatize this representation as $\{(z, w) \mid z, w \in \mathbb{C}^n\}$ with $g(z, w) = (gz, g^{-1}w)$. Choose an identification of $\mathbb{H}^n$ with $T^*\mathbb{C}^n$ such that the complex structure $J_1$ on $\mathbb{H}^n$ given by right multiplication by $i$ corresponds to the natural complex structure on $T^*\mathbb{C}^n$. Then $T^*\mathbb{C}^n$ inherits a hyperkähler, and therefore also a holomorphic symplectic, structure, with $\omega_R$ given by adding the standard symplectic structures on $\mathbb{C}^n$ and $(\mathbb{C}^n)^* \cong \mathbb{C}^n$, and $\omega_C = d\eta$, where $\eta$ is the canonical holomorphic 1-form on $T^*\mathbb{C}^n$.

Note that $G$ acts $\mathbb{H}$-linearly on $T^*\mathbb{C}^n \cong \mathbb{H}^n$ (where $n \times n$ matrices act on the left on $\mathbb{H}^n$, and scalar multiplication by $\mathbb{H}$ is on the right), and does so hyperhamiltonianly with moment map $\mu_{HK} = \mu_C \oplus \mu_R$, where

$$\mu_R(z, w) = \mu(z) - \mu(w) \quad \text{and} \quad \mu_C(z, w)(v) = w(\hat{v})$$

for $w \in T_z^*\mathbb{C}^n$, $v \in \mathfrak{g}_C$, and $\hat{v}$ the element of $T_v\mathbb{C}^n$ induced by $v$. Consider a central regular value $\alpha \in \mathfrak{g}^*$ for $\mu$, and suppose that $(\alpha, 0) \in \mathfrak{g}^* \oplus \mathfrak{g}_C^*$ is a central regular value for $\mu_{HK}$. We refer to the hyperkähler reduction $M = \mathbb{H}^n \sslash (\alpha, 0)G$ as the hyperkähler analogue of the corresponding Kähler reduction $X = \mathbb{C}^n \sslash_\alpha G$. The following proposition is proven for the case where $G$ is a torus in [BD, 7.1]:

**Proposition 1.2.** The cotangent bundle $T^*X$ is isomorphic to an open subset of $M$.

**Proof.** Let $Y = \{(z, w) \in \mu_C^{-1}(0)^{ss} \mid z \in (\mathbb{C}^n)^{ss}\}$, where we ask $z$ to be semistable with respect to $\alpha$ for the action of $G_C$ on $\mathbb{C}^n$, so that $X \cong (\mathbb{C}^n)^{ss}/G_C$. Let $[z]$ denote the element of $X$ represented by $z$. The tangent
space $T_{[z]}X$ is equal to the quotient of $T_z\mathbb{C}^n$ by the tangent space to the $G_\mathbb{C}$ orbit through $z$, hence

$$T^*_{[z]}X \cong \{ w \in T^*_{[z]}\mathbb{C}^n \mid w(\tilde{v}_z) = 0 \text{ for all } v \in \mathfrak{g}_\mathbb{C} \}$$

$$= \{ w \in (\mathbb{C}^n)^* \mid \mu_\mathbb{C}(z, w) = 0 \}.$$

Then

$$T^*X \cong \{(z, w) \mid z \in (\mathbb{C}^n)^{ss} \text{ and } \mu_\mathbb{C}(z, w) = 0 \} / G_\mathbb{C} = Y / G_\mathbb{C}$$

is an open subset of $M$. □

Consider the action of $\mathbb{S}^1$ on $\mathbb{H}^n \cong T^*\mathbb{C}^n$ given by “rotating the fibers” of the cotangent bundle, given explicitly by $\tau(z, w) = (z, \tau w)$. This action is hamiltonian with respect to the real symplectic structure $\omega_\mathbb{R}$ with moment map $\Phi(z, w) = \frac{1}{2} |w|^2$. Because it commutes with the action of $G$, the action descends to a hamiltonian action on $M$, where we will still denote the moment map by $\Phi$. Since $\mathbb{S}^1$ acts trivially on $z$, and by scalars on $w$, it does not preserve the complex symplectic form $\omega_\mathbb{C}(z, w) = dw \wedge dz$, and does not act $\mathbb{H}$-linearly.

**Proposition 1.3.** If the original moment map $\mu : \mathbb{C}^n \to \mathfrak{g}^*$ is proper, then so is $\Phi : M \to \mathbb{R}$.

**Proof.** We would like to show that $\Phi^{-1}[0, R]$ is compact for any $R$. Since

$$\Phi^{-1}[0, R] = \{(z, w) \mid \mu_\mathbb{R}(z, w) = \alpha, \mu_\mathbb{C}(z, w) = 0, \Phi(z, w) \leq R \}/G$$

and $G$ is compact, it is sufficient to show that the set $\{(z, w) \mid \mu_\mathbb{R}(z, w) = \alpha, \Phi(z, w) \leq R \}$ is compact. Since $\mu_\mathbb{R}(z, w) = \mu(z) - \mu(w)$, this set is a closed subset of

$$\mu^{-1}\left\{ \alpha + \mu(w) \mid \frac{1}{2} |w|^2 \leq R \right\} \times \left\{ w \mid \frac{1}{2} |w|^2 \leq R \right\},$$

which is compact by the properness of $\mu$. □

In the case where $G$ is abelian and $X$ is a nonempty toric variety, properness of $\mu$ (and therefore of $\Phi$) is equivalent to compactness of $X$.

### 2. Hypertoric varieties.

In this section we restrict our attention to hypertoric varieties, which are the hyperkähler analogues of toric varieties in the sense of Section 1. We begin with the full $n$-dimensional torus $T^n$ acting on $\mathbb{C}^n$, and the induced action on $\mathbb{H}^n \cong T^*\mathbb{C}^n$ given by $t(z, w) = (tz, t^{-1}w)$. Let $\{a_i\}_{1 \leq i \leq n}$ be nonzero primitive integer vectors in $t^d \cong \mathbb{R}^d$ defining a map $\beta : t^n \to t^d$ by $\varepsilon_i \mapsto a_i$, where $\{\varepsilon_i\}$ is the standard basis for $t^n \cong \mathbb{R}^n$, dual to $\{u_i\}$. This map fits into an exact sequence

$$0 \to t^k \to t^n \to \beta : t^d \to 0,$$
where $t^k := \ker(\beta)$. Exponentiating, we get the exact sequence

$$0 \longrightarrow T^k \xrightarrow{\iota} T^n \xrightarrow{\beta} T^d \longrightarrow 0,$$

whereas by dualizing, we get

$$0 \longrightarrow (t^d)^* \xrightarrow{\beta^*} (t^n)^* \xrightarrow{\iota^*} (t^k)^* \longrightarrow 0,$$

where we abuse notation by using $\iota$ and $\beta$ to denote maps on the level of groups as well as on the level of algebras. Note that $T^k$ is connected if and only if the vectors $\{a_1, \ldots, a_n\}$ span $t^d$ over the integers.

Consider the restriction of the action of $T^n$ on $H^n$ to the subgroup $T^k$. This action is hyperhamiltonian with hyperkahler moment map

$$\mu_R(z, w) = \iota^* \left( \frac{1}{2} \sum_{i=1}^n (|z_i|^2 - |w_i|^2)u_i \right)$$

and

$$\mu_C(z, w) = \iota^* \left( \sum_{i=1}^n (z_iw_i)u_i \right),$$

where $\{u_i\}$ is the standard basis in $(t^n)^* \cong \mathbb{R}^n$. In contrast with the Kähler situation, the hyperkahler moment map is surjective onto $(t^n)^* \oplus (t^k)^*$.

We denote by $M$ the hyperkahler reduction of $H^n$ by the subtorus $T^k$ at $(\alpha, 0) \in (t^k)^* \oplus (t^k)^*$, which is the hyperkahler analogue of the Kähler toric variety $X = \mathbb{C}^n // T^k$. Choose a lift $\bar{\alpha} \in (t^n)^*$ of $\alpha$ along $\iota^*$. Then $M$ has a natural residual action of $T^d$ with hyperkahler moment map $\mu_{HK} = \mu_R \oplus \mu_C$. Note that the choice of subtorus $T^k \subseteq T^n$ is equivalent to choosing a central arrangement of cooriented hyperplanes in $(t^d)^*$, where the $i$th hyperplane is the annihilator of $a_i \in t^d$. (The coorientation comes from the fact that we know for which $x$ we have $\langle x, a_i \rangle > 0$.) The choice of $\bar{\alpha}$ corresponds to an affinization $\mathcal{H}$ of this arrangement, where the $i$th hyperplane is

$$H_i = \{ x \in (t^d)^* \mid \langle x, a_i \rangle = \langle -\bar{\alpha}, \varepsilon_i \rangle \}.$$

Changing $\bar{\alpha}$ by an element $c \in (t^d)^*$ has the effect of translating $\mathcal{H}$ by $c$, and adding $c$ to the residual moment map $\mu_R$. In order to record the information about coorientations, we define the half-spaces

$$F_i = \{ x \in (t^d)^* \mid \langle x, a_i \rangle \geq \langle -\bar{\alpha}, \varepsilon_i \rangle \} \quad \text{and} \quad G_i = \{ x \in (t^d)^* \mid \langle x, a_i \rangle \leq \langle -\bar{\alpha}, \varepsilon_i \rangle \},$$

which intersect in the hyperplane $H_i$. Our convention will be to draw pictures, as in Figure 1, in which we specify the coorientations of the hyperplanes by shading the polyhedron $\Delta = \bigcap_{i=1}^n F_i$ (which works as long as $\Delta \neq \emptyset$). Note that the Kähler variety $X$ is precisely the Kähler toric variety determined by $\Delta$. 
Figure 1. A hypertoric variety of real dimension 8 obtained by reducing $\mathbb{H}^4$ by $T^2$.

The variety $M$ is an orbifold if and only if $H$ is simple, i.e., if and only if every subset of $m$ hyperplanes intersect in codimension $m$ [BD, 3.2]. Furthermore, $M$ is smooth if and only if whenever some subset of $d$ hyperplanes $\{H_i\}$ has nonempty intersection, the corresponding vectors $\{a_i\}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^d \subseteq \mathbb{R}^d$. In this case we will refer to the arrangement itself as smooth. We will always assume that $\mathcal{H}$ is simple, and at times we will also assume that it is smooth.

The hyperplanes $\{H_i\}$ divide $(\mathbb{R}^d)^\ast \cong \mathbb{R}^d$ into a finite family of closed, convex polyhedra

$\Delta_A = (\cap_{i \in A} F_i) \cap (\cap_{i \notin A} G_i)$,

indexed by subsets $A \subseteq \{1, \ldots, n\}$. Consider the subset

$I = \{A \subseteq \{1, \ldots, n\} \mid \Delta_A \text{ bounded}\}$

of the power set of $\{1, \ldots, n\}$. For each $A \subseteq \{1, \ldots, n\}$, let

$M_A = \mu^{-1}_{\mathbb{R}}(\Delta_A) \cap \mu^{-1}_{\mathbb{C}}(0)$.

The Kähler submanifold $(M_A, \omega_{\mathbb{R}}|_{M_A})$ of $(M, \omega_{\mathbb{R}})$ is $d$-dimensional and invariant under the action of $T^d$, and is therefore $T^d$-equivariantly isomorphic to the Kähler toric variety determined by $\Delta_A$ [BD, 6.5]. We define the core $C$ and extended core $D$ of a hypertoric variety by setting

$C = \cup_{A \in I} M_A$ and $D = \cup_{A} M_A = \mu^{-1}_C(0) = \{(z, w) \mid z_iw_i = 0 \text{ for all } i\}$,

where $[z, w]$ denotes the $T^k$-equivalence class in $M$ of the element $(z, w) \in \mathbb{P}^{-1}_{HR}(\alpha, 0)$. Bielawski and Dancer [BD] show that $C$ and $D$ are each $T^d$-equivariant deformation retracts of $M$. See Corollary 3.6 for a Morse theoretic proof.

We take a minute to discuss the differences between the combinatorial data determining a toric variety $X = \mathbb{C}^n \sslash_{\alpha} T^k$ and its hypertoric analogue $M = \mathbb{H}^n \sslash_{(\alpha, 0)} T^k$. Each is determined by $\mathcal{H}$, a simple, cooriented, affine
arrangement of \( n \) hyperplanes in \((t^d)^*\), defined up to simultaneous translation. The toric variety \( X \) is in fact determined by less information than this; it depends only on the polyhedron \( \Delta = \cap_{i=1}^n F_i \). Thus if the last hyperplane \( H_n \) has the property that \( \cap_{i=1}^{n-1} F_i \subseteq F_n \), then this hyperplane is superfluous to \( X \). This is not the case for \( M \), which means that it is slightly misleading to call \( M \) the hyperkähler analogue of \( X \); more precisely, it is the hyperkähler analogue of a given presentation of \( X \) as a Kähler reduction of \( \mathbb{C}^n \). On the other hand, the \( T^d \)-equivariant diffeomorphism type of \( M \) also does not depend on all of the information of \( H \), as evidenced by the two following results:

**Lemma 2.1.** The hypertoric varieties \( M_\alpha = H^n / / / (\alpha, 0) T^k \) and \( M_{\alpha'} = H^n / / / (\alpha', 0) T^k \) are \( T^d \)-equivariantly diffeomorphic, and their cohomology rings can be naturally identified.

**Lemma 2.2.** The hypertoric variety \( M \) does not depend on the coorientations of the hyperplanes \( \{ H_i \} \).

This means that, unlike that of \( X \), the \( T^d \)-equivariant diffeomorphism type of \( M \) depends only on the unoriented central arrangement underlying \( \mathcal{H} \). A weaker version of Lemma 2.1, involving the (nonequivariant) homeomorphism type of \( M \), appears in [BD].

**Proof of 2.1.** The set of nonregular values for \( \overline{\mu}_{HK} \) has codimension 3 inside of \((t^d)^* \oplus (t_c^d)^*\). This tells us that the set of regular values is simply connected, and we can choose a path connecting any two regular values \((\alpha, 0)\) and \((\alpha', 0)\), unique up to homotopy.

Since the moment map \( \overline{\mu}_{HK} \) is not proper, we must take some care in showing that two fibers are diffeomorphic. To this end, we note that the norm-square function \( \psi(z, w) = \|z\|^2 + \|w\|^2 \) is \( T^n \)-invariant and proper on \( \mathbb{H}^n \). Let \( \mathbb{H}^n_{\text{reg}} \) denote the open submanifold of \( \mathbb{H}^n \) consisting of the preimages of the regular values of \( \overline{\mu}_{HK} \). By a direct computation, it is easy to see that the kernels of \( d\psi \) and \( d(\overline{\mu}_{HK}) \) intersect transversely at any point \( p \in \mathbb{H}^n_{\text{reg}} \). Using the standard \( T^n \)-invariant metric on \( \mathbb{H}^n \), we define an Ehresmann connection on \( \mathbb{H}^n_{\text{reg}} \) with respect to \( \overline{\mu}_{HK} \) such that the horizontal subspaces are contained in the kernel of \( d\psi \).

This connection allows us to lift a path connecting the two regular values to a horizontal vector field on its preimage in \( \mathbb{H}^n_{\text{reg}} \). Since the horizontal subspaces are tangent to the kernel of \( d\psi \), the flow preserves level sets of \( \psi \). Note that the function

\[
\overline{\mu}_{HK} \oplus \psi : \mathbb{H}^n \to (t^d)^* \oplus (t_c^d)^* \oplus \mathbb{R}
\]

is proper. By a theorem of Ehresmann [BJ, 8.12], the properness of this map implies that the flow of this vector field exists for all time, and identifies the inverse image of \((\alpha, 0)\) with that of \((\alpha', 0)\). Since the metric, \( \psi \), and
\( \mathcal{P}_{HK} \) are all \( T^n \)-invariant, the Ehresmann connection is also \( T^n \)-invariant, therefore the diffeomorphism identifying the fibers is \( T^n \)-equivariant, making the reduced spaces are \( T^d \)-equivariantly diffeomorphic.

\( \square \)

**Proof of 2.2.** It suffices to consider the case when we change the orientation of a single hyperplane within the arrangement. Changing the coorientation of a hyperplane \( H_l \) is equivalent to defining a new map \( \beta' : t^n \rightarrow t^d \), with \( \beta'(\varepsilon_i) = a_i \) for \( i \neq l \), and \( -a_i \) for \( i = l \). This map exponentiates to a map \( \beta' : T^n \rightarrow T^d \), and we want to show that the hyperkähler variety obtained by reducing \( \mathbb{H}^n \) by the torus \( \ker(\beta') \) is isomorphic to \( M \), which is obtained by reducing \( \mathbb{H}^n \) by the torus \( T^k = \ker(\beta) \). To see this, note that \( \ker(\beta') \) and \( \ker(\beta) \) are conjugate inside of \( M(n, \mathbb{H}) \) by the element \((1, \ldots, 1, j, 1, \ldots, 1) \in M(1, \mathbb{H})^n \subseteq M(n, \mathbb{H}) \), where the \( j \) appears in the \( l \)th slot. \( \square \)

**Example 2.3.** The three cooriented arrangements of Figure 2 all specify the same hyperkähler variety \( M \) up to equivariant diffeomorphism. The first has \( X \cong \mathbb{F}_1 \) (the first Hirzebruch surface) and the second and the third have \( X \cong \mathbb{C}P^2 \). Note that if we flipped the coorientation of \( H_3 \) in Figure 2(a) or 2(c), then we would get a noncompact \( X \cong \mathbb{C}^2 \), the blow-up of \( \mathbb{C}^2 \) at a point. If we flipped the coorientation of \( H_3 \) in Figure 2(b), then \( X \) would be empty. We make no assumptions about \( X \) in this section.

**Figure 2.** Three arrangements related by flipping coorientations and translating hyperplanes.

The purpose of this paper is to study not just the topology of \( M \), but the topology of \( M \) along with the natural hamiltonian \( S^1 \) action defined in Section 1. In order to define this \( S^1 \) action, it is necessary that we reduce at a regular value of the form \((\alpha, 0) \in (t^d)^* \oplus (t^d_{\mathbb{C}})^* \), and although the set of regular values of \( \mathcal{P}_{HK} \) is simply connected, the set of regular values of the form \((\alpha, 0) \) is not even connected. Furthermore, left multiplication by the diagonal matrix \((1, \ldots, 1, j, 1, \ldots, 1) \in U(n, \mathbb{H}) \) is not an \( S^1 \)-equivariant automorphism of \( \mathbb{H}^n \), therefore the geometric structure of \( M \) along with
a circle action may depend nontrivially both on the affine structure and the coorientations of the arrangement \( \mathcal{H} \). Indeed it must, because we can recover \( X \) from \( M \) by taking the minimum \( \Phi^{-1}(0) \) of the \( S^1 \) moment map \( \Phi : M \to \mathbb{R} \). In this sense, the structure of a hypertoric variety \( M \) along with a circle action is the universal geometric object from which both \( M \) and \( X \) can be recovered.

3. Gradient flow on the core.

Although \( S^1 \) does not act on \( M \) as a subtorus of \( T^d \), we show below that when restricted to any single component \( M_A \) of the extended core, \( S^1 \) does act as a subtorus of \( T^d \), with the subtorus depending combinatorially on \( A \). This will allow us to give a combinatorial analysis of the gradient flow of \( \Phi \) on the extended core.

**Lemma 3.1.** Let \( x \) be an element of \( (t^d)^* \), and consider a point \([z,w] \in \mu^{-1}_R(x) \cap D\). We have \( x \in F_i \) if and only if \( w_i = 0 \), and \( x \in G_i \) if and only if \( z_i = 0 \).

**Proof.** The fact that \([z,w] \in D\) tells us that \( z_i w_i = 0 \). Then

\[
x \in F_i \iff \langle \tilde{\alpha}, \varepsilon_i \rangle \leq \langle \mu_R[z,w], a_i \rangle = \langle \mu_R(z,w) - \tilde{\alpha}, \varepsilon_i \rangle \\
\iff \frac{1}{2} |z_i|^2 - \frac{1}{2} |w_i|^2 = \langle \mu_R(z,w), \varepsilon_i \rangle \geq 0.
\]

Since \( z_i w_i = 0 \), this is equivalent to the condition \( w_i = 0 \). The second half of the lemma follows similarly. \( \square \)

On the suborbifold \( M_A \subseteq D \subseteq M \) we have \( z_i = 0 \) for all \( i \in A \) and \( w_i = 0 \) for all \( i \notin A \), therefore for \( \tau \in S^1 \) and \([z,w] \in M_A\),

\[
\tau[z,w] = [z,\tau w] = [\tau_1 z_1, \ldots, \tau_n z_n, \tau_1^{-1} w_1, \ldots, \tau_n^{-1} w_n],
\]

where \( \tau_i = \begin{cases} \tau^{-1} & \text{if } i \in A, \\ 1 & \text{if } i \notin A. \end{cases} \)

In other words, the \( S^1 \) action on \( M_A \) is given by the one dimensional subtorus \((\tau_1, \ldots, \tau_n)\) of the original torus \( T^n \), hence the moment map \( \Phi|M_A \) is given (up to an additive constant) by

\[
\Phi[z,w] = \left\langle \mu_R[z,w], \sum_{i \in A} a_i \right\rangle.
\]

This formula allows us to compute the fixed point sets of the \( S^1 \) action. Since \( S^1 \) acts freely on \((t^d_C)^* \setminus \{0\}\) and \( \mu_C : M \to (t^d_C)^* \) is \( S^1 \)-equivariant, we must have \( M^{S^1} \subseteq \mu_C^{-1}(0) = D \). For any subset \( B \subseteq \{1, \ldots, n\} \), let \( M^B_A \) be the toric subvariety of \( M_A \) defined by the conditions \( z_i = w_i = 0 \)
for all $i \in B$. Geometrically, $M^B_A$ is defined by the (possibly empty) face $\cap_{i \in B} H_i \cap \Delta_A$ of the polyhedron $\Delta_A$.

**Proposition 3.2.** The fixed point set of the action of $S^1$ on $M_A$ is the union of those toric subvarieties $M^B_A$ such that $\sum_{i \in A} a_i \in t^*_d := \text{Span}_{j \in B} a_j$.

**Proof.** The moment map $\Phi|_{M^B_A}$ will be constant if and only if $\sum_{i \in A} a_i$ is perpendicular to $\ker ((t^d)^* \to (t^d_B)^*)$, i.e., if $\sum_{i \in A} a_i$ lies in the kernel of the projection $t^d \to t^d/t^d_B$. $\square$

**Corollary 3.3.** Every vertex $v \in (t^d)^*$ of the polyhedral complex defined by $\mathcal{H}$ is the image of an $S^1$-fixed point in $M$. Every component of $M^{S^1}$ has dimension less than or equal to $d$, and the only component of dimension $d$ is $M_0 = X = \Phi^{-1}(0)$.

For any point $p \in M^{S^1}$, the stable orbifold $S(p)$ at $p$ is defined to be the set of $x \in M$ such that $x$ approaches $p$ when flowing along the vector field $-\text{grad}(\Phi)$, and the unstable orbifold $U(p)$ at $p$ is defined to be the stable orbifold with respect to the function $-\Phi$. For any suborbifold $Y \subseteq M^{S^1}$, the unstable orbifold $U(Y)$ at $Y$ is defined to be the union of $U(y)$ for all $y \in Y$. In general, for $y \in Y$, we have the identity $\dim_R U(Y) + \dim_S S(y) = 4d$.

Let $Y \subseteq M^{S^1}$ be a component of the fixed point set of $M$. Let $v \in (t^d)^*$ be a vertex in the polyhedron $\mu_{\mathbb{R}}(Y)$, and let $y$ be the unique preimage of $v$ in $Y$.

**Proposition 3.4.** The unstable orbifold $U(Y)$ is a complex suborbifold of complex dimension at most $d$, contained in the core $C \subseteq M$. If $\mathcal{H}$ is smooth at $y$, then $\dim_C U(Y) = d$, and the closure of $U(Y)$ is an irreducible component of $C$.

**Proof.** For simplicity, we will assume that $v = \cap_{j=1}^d H_j$. For all $l \in \{1, \ldots, d\}$, let $b_l \in t^*_d$ be the smallest integer vector such that $\langle a_j, b_l \rangle = 0$ for $j \neq l$ and $\langle a_l, b_l \rangle > 0$. Geometrically, $b_l$ is the primitive integer vector on the line $\cap_{j \neq l} H_j$ pointing in the direction of $\Delta$. Note that $M$ is smooth at the $T^d$-fixed point above $v$ if and only if $\langle a_l, b_l \rangle = 1$ for all $l \in \{1, \ldots, d\}$. Let $R_l \subseteq (t^d)^*$ be the ray emanating from $v$ in the direction of $b_l$, and ending before it hits another vertex. Let $Q_l$ be the analogous ray in the opposite direction.

Let $\Delta_A$ be a region (not necessarily bounded) of the polyhedral complex defined by $\mathcal{H}$ adjacent to $R_l$. The preimage $\mu_{\mathbb{R}}^{-1}(R_l) \cap D$ of $R_l$ in $D$ is a complex line, and it is contained in the unstable orbifold at $U(Y)$ if and only if $\langle b_l, \sum_{i \in A} a_i \rangle \geq 0$. If $\langle b_l, \sum_{i \in A} a_i \rangle < 0$, it is contained in the stable orbifold $S(y)$. The preimage $\mu_{\mathbb{R}}^{-1}(Q_l) \cap D$ of $Q_l$ in $D$ is also a complex line, contained in the unstable orbifold $U(Y)$ if and only if $\langle -b_l, a_l + \sum_{i \in A} a_i \rangle \geq 0$, and otherwise in $S(y)$. Since $\langle b_l, \sum_{i \in A} a_i \rangle + \langle -b_l, a_l + \sum_{i \in A} a_i \rangle = -\langle a_l, b_l \rangle < 0$,
at most one of these two directions can be unstable. In the smooth case, \( \langle a_l, b_l \rangle = 1 \) for all \( l \), and exactly one of the two directions is unstable.

Consider the polytope \( \Delta_v \) incident to \( v \) and characterized by the property that its edges at the vertex \( v \) are exactly the unstable directions. The toric variety \( X_{\Delta_v} \subseteq D \) is contained in the closure of \( U(Y) \), and a dimension count tells us that this containment is an equality. In the smooth case, \( \Delta_v \) is \( d \)-dimensional, and \( X_{\Delta_v} \) is a component of the core. \( \square \)

Note that, even in the smooth case, it is not necessarily the case that the \( R_l \) direction is stable and the \( Q_l \) direction is unstable. See, for example, the vertex \( v = H_1 \cap H_2 \) in Figure 2(c).

**Corollary 3.5.** There is a natural injection from the set of bounded regions \( \{ \Delta_A \mid A \in I \} \) to the set of connected components of \( M^{S^1} \). If \( \mathcal{H} \) is smooth, this map is a bijection.

**Proof.** To each \( A \in I \), we associate the fixed subvariety \( M_A^B \) corresponding to the face of \( \Delta_A \) on which the linear functional \( \sum_{i \in A} a_i \) is minimized, so that \( M_A \) is the closure of \( U(M_A^B) \). If \( \mathcal{H} \) is smooth, then every connected component of the fixed point set will have a component of the core as its closed unstable orbifold. \( \square \)

**Corollary 3.6.** The core of \( M \) is equal to the union of the unstable orbifolds of the connected components of \( M^{S^1} \), hence \( C \) is a \( T^d \times S^1 \)-equivariant deformation retract of \( M \).

**Example 3.7.** In Figure 3, representing a reduction of \( \mathbb{H}^5 \) by \( T^3 \), we choose a metric on \( (t^2)^* \) in order to draw the linear functional \( \sum_{i \in A} a_i \) as a vector in each region \( \Delta_A \). We see that \( M^{S^1} \) has three components, one of them \( X \cong F_1 \), one of them a projective line, with another \( F_1 \) as its unstable manifold, and one of them a point, with a \( \mathbb{C}P^2 \) as its unstable manifold.

**Example 3.8.** The hypertoric variety represented by Figure 4 has a fixed point set with four connected components (three points and a \( \mathbb{C}P^2 \)), but only three components in its core. This phenomenon can be blamed on the orbifold point represented by the intersection of \( H_3 \) and \( H_4 \), which has a one-dimensional unstable orbifold.

4. Equivariant cohomology.

In this section we extend Konno’s computations of the ordinary and \( T^d \)-equivariant cohomologies of \( M \) to the \( S^1 \)-equivariant setting. We follow Konno’s approach of restricting to the smooth case to simplify arguments involving line bundles on \( M \). Hausel and Sturmfels, however, prove theorems analogous to 4.1 and 4.3 with rational coefficients in the orbifold case, and Theorems 4.4 and 4.5 extend to this setting as well (see Remark 4.12).
The gradient flow of $\Phi : M \to \mathbb{R}$.

A singular example.

**Theorem 4.1** ([K2]). The $T^d$-equivariant cohomology ring of a smooth hypertoric variety $M$ is given by

$$H^*_T(M) = \mathbb{Z}[u_1, \ldots, u_n]/\left(\prod_{i \in S} u_i \mid \bigcap_{i \in S} H_i = \emptyset\right).$$

**Remark 4.2.** This is precisely the Stanley-Reisner ring of the unoriented matroid determined by the arrangement $\mathcal{H}$ [HS].

Just as the cohomology of a toric variety is obtained from the equivariant cohomology by introducing linear relations that generate $\ker \eta^* = (\ker \beta)^\perp$, the same is true for hypertoric varieties:

**Theorem 4.3** ([K1]). The ordinary cohomology ring of a smooth hypertoric variety $M$ is given by

$$H^*(M) = H^*_T(M)/\langle \sum_i a_i u_i \in \ker \eta^* \rangle.$$

The rest of this section will be devoted to the proof of the following two theorems:
Theorem 4.4. Let $M$ be the hypertoric variety corresponding to a smooth, cooriented arrangement $\mathcal{H}$. Given any minimal set $S \subseteq \{1, \ldots, n\}$ such that $\bigcap_{i \in S} H_i = \emptyset$, let $S = S_1 \sqcup S_2$ be the unique splitting of $S$ such that $(\bigcap_{i \in S_1} G_i) \cap (\bigcap_{j \in S_2} F_j) = \emptyset$ (see (1)). Then the $T^d \times S^1$-equivariant cohomology of $M$ is given by

$$H^*_T(M) \cong \mathbb{Z}[u_1, \ldots, u_n, x]/\left( \prod_{i \in S_1} u_i \times \prod_{j \in S_2} (x - u_j) \bigg| \bigcap_{i \in S} H_i = \emptyset \right).$$

Theorem 4.5. In the notation of Theorem 4.4, the $S^1$-equivariant cohomology ring of $M$ is given by

$$H^*_S(M) \cong H^*_{T^d \times S^1}(M)/\langle \Sigma u_i \in \ker \iota^* \rangle.$$

Remark 4.6. Konno observes that the quotient map from the abstract polynomial ring $\mathbb{Z}[u_1, \ldots, u_n] \to H^*_T(M)$ is precisely the $T^d$-equivariant Kirwan map

$$\kappa_T : H^*_T(M) \to H^*_T(M)$$

which is induced by the inclusion $\mu^{-1}(\alpha, 0) \hookrightarrow T^* \mathbb{C}^n$. Likewise, the map from $\mathbb{Z}[u_1, \ldots, u_n]/\ker \iota^*$ to $H^*(M)$ is the ordinary Kirwan map

$$\kappa : H^*_T(M) \to H^*(M).$$

The analogous maps for Kähler reductions are known to always be surjective [Ki, 5.4], but the hyperkähler case remains open. Thus Theorems 4.1 and 4.3 can be interpreted as saying that the Kirwan maps for hypertoric varieties are surjective, and computing the kernel. Likewise, Theorems 4.4 and 4.5 assert that the $S^1$-equivariant Kirwan maps

$$\kappa_{T^d \times S^1} : H^*_{T^d \times S^1}(T^* \mathbb{C}^n) \to H^*_T(M)$$

and

$$\kappa_{S^1} : H^*_{T^k \times S^1}(T^* \mathbb{C}^n) \to H^*_S(M)$$

are surjective, and provide computations of their kernels.

In order to apply Konno’s results, we will make use of the principle of equivariant formality, proven for compact manifolds in [Ki], which we adapt to our situation in Proposition 4.7. For the sake of simplicity, we will restrict our attention to the case where $X$ is compact and nonempty. This condition will be necessary for the application of Proposition 4.7 and the proof of Theorem 5.1, both of which require a proper Morse function, which we get from Proposition 1.3. We note, however, that both Proposition 4.7 and Theorem 5.1 can be extended to the case of a general hypertoric variety by a Mayer-Vietoris argument, using the fact that the core $C \subseteq M$ is a compact $T^d \times S^1$-equivariant deformation retract. We present the slightly less general Morse theoretic proofs only because we find them more pleasant.
Proposition 4.7. Let $M$ be a symplectic orbifold, possibly noncompact but of finite topological type. Suppose that $M$ admits a Hamiltonian action of a torus $T \times S^1$, and that the $S^1$-component $\Phi : M \to \mathbb{R}$ of the moment map is proper and bounded below. Then $H^*_T \times S^1 (M)$ is a free module over $H^*_S (pt)$.

Proof. Because $\Phi$ is a moment map, it is a Morse-Bott function such that all of the critical suborbifold and their normal bundles carry almost complex structures. Thus we get a Morse-Bott stratification of $M$ into even-dimensional $T$-invariant suborbifolds. This tells us, as in [Ki, 5.8], that the spectral sequence associated to the fibration $M \hookrightarrow EG \times_G M \to BG$ collapses, and we get the desired result. □

Consider the following commuting square of maps, where $\phi$ and $\psi$ are each given by setting $x$ to zero:

$$
\begin{array}{c}
H^*_T \times S^1 (T^* \mathbb{C}^n) \xrightarrow{\kappa_{T^d \times S^1}} H^*_T \times S^1 (M) \\
\phi \downarrow \quad \downarrow \psi \\
H^*_T (T^* \mathbb{C}^n) \xrightarrow{\kappa_{T^d}} H^*_T (M)
\end{array}
$$

Proposition 4.7 has the following consequence:

Corollary 4.8. Let $\mathcal{I} \subseteq \ker \kappa_{T^d \times S^1}$ be an ideal with $\phi (\mathcal{I}) = \ker \kappa_{T^d}$. Then $\mathcal{I} = \ker \kappa_{T^d \times S^1}$.

Proof. Suppose that $a \in \ker \kappa_{T^d \times S^1} \setminus \mathcal{I}$ is a homogeneous class of minimal degree, and choose $b \in \mathcal{I}$ such that $\phi (a - b) = 0$. Then $a - b = cx$ for some $c \in H^*_T (T^* \mathbb{C}^n)$. By Proposition 4.7, $cx \in \ker \kappa_{T^d \times S^1} \Rightarrow c \in \ker \kappa_{T^d \times S^1}$, hence $c \in \ker \kappa_{T^d \times S^1} \setminus \mathcal{I}$ is a class of lower degree than $a$. □

Lemma 4.9. The equivariant Kirwan map $\kappa_{T^d \times S^1}$ is surjective.

Proof. Suppose that $\gamma \in H^*_T (M)$ is a homogeneous class of minimal degree that is not in the image of $\kappa_{T^d \times S^1}$. By Theorem 4.1 $\kappa_{T^d}$ is surjective, hence we may choose a class $\eta \in \phi^{-1} \kappa_{T^d}^{-1} \psi (\gamma)$. Then $\kappa_{T^d \times S^1} (\eta) - \gamma = x \delta$ for some $\delta \in H^*_T \times S^1 (M)$, and therefore $\delta$ is a class of lower degree that is not in the image of $\kappa_{T^d \times S^1}$. □

Proof of 4.4. For any element $h \in H^2_{T^n \times S^1} (\mathbb{H}^n ; \mathbb{Z})$, let $L_h = \mathbb{H}^n \times C_h$ be the $T^n \times S^1$-equivariant line bundle on $\mathbb{H}^n$ with equivariant Euler class $h$. This gives $L_h$, as well as its dual $L_h^*$, the structure of a $T^n \times S^1$-equivariant bundle. Let

$$
L_h = \tilde{L}_h |_{\bar{C}^{-1} (0)_{ss}} / T^2_C
$$

be the quotient $T^d \times S^1$-equivariant line bundle on $M$. Let $\{ u_i \}$ be the standard basis of $(T^2_C)^*$. Identifying $H^2_{T^n \times S^1} (\mathbb{H}^n ; \mathbb{Z})$ with $(T^2_C)^* \oplus \mathbb{Z} x$, we
will use $\tilde{L}_i$ to denote the bundle $L_{u_i,0}$, and $\tilde{K}$ to denote the bundle $L_{0,0}$, with quotients $L_i$ and $K$. Since the $T^d \times S^1$-equivariant Euler class $e(L_i)$ is the image of $u_i \oplus 0$ under the hyperkähler Kirwan map $H^*_{T^d \times S^1}(\mathbb{H}^n) \to H^*_{T^d \times S^1}(M)$, we will abuse notation and denote it by $u_i$. Similarly, we will denote $e(K)$ by $x$. Corollary 4.9 tells us that $H^*_{T^d \times S^1}(M)$ is generated by $u_1, \ldots, u_n, x$.

Consider the $T^n \times S^1$-equivariant section $\tilde{s}_i$ of $L_i$ given by the function $\tilde{s}_i(z, w) = z_i$. This descends to a $T^d \times S^1$-equivariant section $s_i$ of $L_i$ with zero-set

$$Z_i := \{ [z, w] \in M \mid z_i = 0 \}.$$ 

Similarly, the function $\tilde{t}_i(z, w) = w_i$ defines a $T^d \times S^1$-equivariant section of $L_i^* \otimes K$ with zero set

$$W_i := \{ [z, w] \in M \mid w_i = 0 \}.$$ 

Thus the divisor $Z_i$ represents the cohomology class $u_i$, and $W_i$ represents $x - u_i$. Note, by the proof of Lemma 3.1, that $\mu_R(Z_i) = G_i$ and $\mu_R(W_i) = F_i$ for all $1 \leq i \leq n$.

Let $S = S_1 \cup S_2$ be a subset of $\{1, \ldots, n\}$ such that

$$\left( \cap_{i \in S_1} G_i \right) \cap \left( \cap_{j \in S_2} F_j \right) = \emptyset,$$

and hence

$$\left( \cap_{i \in S_1} Z_i \right) \cap \left( \cap_{j \in S_2} W_j \right) \subseteq \mu_R^{-1}\left( \left( \cap_{i \in S_1} G_i \right) \cap \left( \cap_{j \in S_2} F_j \right) \right) = \emptyset.$$ 

Now consider the vector bundle $E_S = (\oplus_{i \in S_1} L_i) \oplus \left( \oplus_{j \in S_2} L_j^* \otimes K \right)$ with equivariant Euler class

$$e(E_S) = \prod_{i \in S_1} u_i \times \prod_{j \in S_2} (x - u_j).$$

The section $(\oplus_{i \in S_1} s_i) \oplus (\oplus_{i \in S_2} t_i)$ is a nonvanishing equivariant global section of $E_S$, hence for any such $S$, $e(E_S)$ is trivial in $H^*_{T^d \times S^1}(M)$.

The fact that $u_1, \ldots, u_n, x$ generate $H^*_{T^d \times S^1}(M)$ is proven in Lemma 4.9, and the fact that we have found all of the relations follows from Theorem 4.1 and Corollary 4.8. \qed

**Proof of 4.5.** The proof of this theorem is identical to the proof of Theorem 4.4, making use of Theorem 4.3 rather than Theorem 4.1. \qed

How sensitive are the invariants $H^*_{T^d \times S^1}(M)$ and $H^*_S(M)$? We can recover $H^*_{T^d}(M)$ and $H^*_S(M)$ by setting $x$ to zero, hence they are at least as fine as the ordinary or $T^d$-equivariant cohomology rings. The ring $H^*_{T^d \times S^1}(M)$ does not depend on coorientations, for if $M'$ is related to $M$ by flipping the coorientation of the $l$th hyperplane $H_k$, then the map taking $u_i$ to $u_i$ for $i \neq l$
and $u_l$ to $x - u_l$ is an isomorphism between $H^*_{T^d \times S^1}(M)$ and $H^*_{T^d \times S^1}(M')$. It is, however, dependent on the affine structure of the arrangement $H$.

**Example 4.10.** We compute the equivariant cohomology ring $H^*_{T^d \times S^1}(M)$ for the hypertoric varieties $M_a$, $M_b$, and $M_c$ defined by the arrangements in Figure 2(a), (b), and (c), respectively.

$$
H^*_{T^d \times S^1}(M_a) = \mathbb{Z}[u_1, \ldots, u_4, x]/\langle u_2 u_3, u_1(x - u_2)u_4, u_1 u_3 u_4 \rangle,
$$
$$
H^*_{T^d \times S^1}(M_b) = \mathbb{Z}[u_1, \ldots, u_4, x]/\langle (x - u_2)u_3, u_1 u_2 u_4, u_1 u_3 u_4 \rangle,
$$
$$
H^*_{T^d \times S^1}(M_c) = \mathbb{Z}[u_1, \ldots, u_4, x]/\langle u_2 u_3, (x - u_1)u_2(x - u_4), u_1 u_3 u_4 \rangle.
$$

As we have already observed, $H^*_{T^d \times S^1}(M_a)$ and $H^*_{T^d \times S^1}(M_b)$ are isomorphic by interchanging $u_2$ with $x - u_2$. One can check that the annihilator of $u_2$ in $H^*_{T^d \times S^1}(M_a)$ is the principal ideal generated by $u_3$, while the ring $H^*_{T^d \times S^1}(M_c)$ has no degree 2 element whose annihilator is generated by a single element of degree 2. Hence $H^*_{T^d \times S^1}(M_c)$ is not isomorphic to the other two rings.

The ring $H^*_{S^1}(M)$, on the other hand, is sensitive to coorientations as well as the affine structure of $H$.

**Example 4.11.** We now compute the ring $H^*_{S^1}(M)$ for $M_a$, $M_b$, and $M_c$ of Figure 2. Theorem 4.5 tells us that we need only to quotient the ring $H^*_{T^d \times S^1}(M)$ by $\ker(\iota^*)$. For $M_a$, the kernel of $\iota^*_a$ is generated by $u_1 + u_2 - u_3$ and $u_1 - u_4$, hence we have

$$
H^*_{S^1}(M_a) = \mathbb{Z}[u_2, u_3, x]/\langle u_2 u_3, (u_3 - u_2)^2(x - u_2), (u_3 - u_2)^2 u_3 \rangle
$$
$$
\cong \mathbb{Z}[u_2, u_3, x]/\langle u_2 u_3, (u_3 - u_2)^2(x - u_2), u_3^3 \rangle.
$$

Since the hyperplanes of 2(c) have the same coorientations as those of 2(a), we have $\ker \iota^*_b = \ker \iota^*_a$, hence

$$
H^*_{S^1}(M_c) = \mathbb{Z}[u_2, u_3, x]/\langle u_2 u_3, (x - u_3 + u_2)^2 u_2, (u_3 - u_2)^2 u_3 \rangle
$$
$$
\cong \mathbb{Z}[u_2, u_3, x]/\langle u_2 u_3, (x - u_3 + u_2)^2 u_2, u_3^3 \rangle.
$$

Finally, since Figure 2(b) is obtained from 2(a) by flipping the coorientation of $H_2$, we find that $\ker(\iota^*_b)$ is generated by $u_1 - u_2 - u_3$ and $u_1 - u_4$, therefore

$$
H^*_{S^1}(M_b) = \mathbb{Z}[u_2, u_3, x]/\langle (x - u_2)u_3, (u_2 + u_3)^2 u_2, (u_2 + u_3)^2 u_3 \rangle.
$$

As in Example 4.10, $H^*_{S^1}(M_a)$ and $H^*_{S^1}(M_c)$ can be distinguished by the fact that the annihilator of $u_2 \in H^*_{S^1}(M_a)$ is generated by a single element of degree 2, and no element of $H^*_{S^1}(M_c)$ has this property. On the other hand, $H^*_{S^1}(M_b)$ is distinguished from $H^*_{S^1}(M_a)$ and $H^*_{S^1}(M_c)$ by the fact that neither $x - u_2$ nor $u_3$ cubes to zero.
Remark 4.12. Theorems 4.4 and 4.5 can be interpreted in light of the recent work of Hausel and Sturmfels [HS] on Lawrence toric varieties. The Lawrence toric variety \( N \) associated to the arrangement \( \mathcal{H} \) is the Kähler reduction \( T^*\mathbb{C}^n//T^k \), so that \( M \) sits inside of \( N \) as the complete intersection cut out by the equation \( \mu_{\mathbb{C}}^*(z,w) = 0 \). The residual torus acting on \( N \) has dimension \( d + n \), and includes the \((d + 1)\)-dimensional torus \( T^d \times S^1 \) on \( M \), and the inclusion of \( M \) into \( N \) induces an isomorphism on \( T^d \times S^1 \)-equivariant cohomology. One can use geometric arguments similar to those that were applied to prove Theorem 4.4, or the purely combinatorial approach of [HS], to show that

\[
H^*_T \mathbb{Z}(N) = \mathbb{Q}[u_1, \ldots, u_n, v_1, \ldots, v_n] / \left< \prod_{i \in S_1} u_i \times \prod_{j \in S_2} v_j \mid \bigcap_{i \in S} H_i = \emptyset \right>.
\]

From here we can recover \( H^*_T \mathbb{Z}(M) = H^*_T \mathbb{Z}(N) \) by setting \( u_i + v_i = u_j + v_j \) for all \( i, j \leq n \). Note that Hausel and Sturmfels’ work applies to the general orbifold case.

5. A deformation of the Orlik-Solomon algebra of \( \mathcal{H} \).

Let \( M_\mathbb{R} \subseteq M \) be the real locus \( \{ [z,w] \in M \mid z,w \text{ real} \} \) of \( M \) with respect to the complex structure \( J_1 \). The full group \( T^d \times S^1 \) does not act on \( M_\mathbb{R} \), but the subgroup \( T^d \mathbb{Z}_2 \) does act, where \( T^d \mathbb{Z}_2 := \mathbb{Z}_2^d \subseteq T^d \) is the fixed point set of the involution of \( T^d \) given by complex conjugation.\(^3\) In this section we will study the geometry of the real locus, focusing in particular on the properties of the residual \( \mathbb{Z}_2 \) action.

A proof of a more general statement of the following theorem is forthcoming in [HH]:

**Theorem 5.1.** Let \( G = T^d \times S^1 \) or \( T^d \), and \( G_\mathbb{R} = T^d \mathbb{R} \times \mathbb{Z}_2 \) or \( T^d \mathbb{R} \). Then we have \( H^*_G(M;\mathbb{Z}_2) \cong H^*_G(M_{\mathbb{R}};\mathbb{Z}_2) \), i.e., the rings are isomorphic by an isomorphism that halves the grading. Furthermore, this isomorphism identifies the class \( u_i \in H^*_G(M;\mathbb{Z}_2) \) represented by the divisor \( Z_i \), with the class in \( H^*_G(M_{\mathbb{R}};\mathbb{Z}_2) \) represented by the divisor \( Z_i \cap M_{\mathbb{R}} \), and likewise takes \( x - u_i \) (if \( G = T^d \times S^1 \)) or \( -u_i \) (if \( G = T^d \)) to the class represented by \( W_i \cap M_{\mathbb{R}} \).

**Sketch of proof.** Consider the injection \( H^*_G(M;\mathbb{Z}_2) \hookrightarrow H^*_G(M^G;\mathbb{Z}_2) \) given by the inclusion of the fixed point set into \( M \). The essential idea is to show that a class in \( H^*_G(M^G;\mathbb{Z}_2) \) extends over \( M \) if and only if it extends to the set of points on which \( G \) acts with a stabilizer of codimension at most 1, and

\(^3\)It is interesting to note that the real locus with respect to the complex structure \( J_1 \) is in fact a complex submanifold with respect to the one of the other complex structures on \( M \). The action of \( T^d \mathbb{R} \) is holomorphic because \( T^d \mathbb{R} \) is a subgroup of \( T^d \), which preserves all of the complex structures on \( M \). The action of \( \mathbb{Z}_2 \), on the other hand, is anti-holomorphic, i.e., it can be thought of as complex conjugation.
then to show that a similar statement in \( G_\mathbb{R} \)-equivariant cohomology also holds for the real locus \( M_\mathbb{R} \) with its \( G_\mathbb{R} \) action. One then uses a canonical isomorphism \( H^2_G(pt, \mathbb{Z}) \cong H^1_G(pt, \mathbb{Z}_2) \) to give the result.

The key to the proof is a noncompact \( G_\mathbb{R} \) version of the proposition in \([TW]\) stating that the \( G_\mathbb{R} \)-equivariant Euler class of the negative normal bundle of a critical point \( p \) is not a zero divisor, which can be shown explicitly using a local normal form for the actions of \( G \) and \( G_\mathbb{R} \). The proposition then follows from standard \( G_\mathbb{R} \) versions of the Thom isomorphism theorem with coefficients in \( \mathbb{Z}_2 \). Since a component of the moment map is proper, bounded below, and has finitely many fixed points, one can then check that the inductive argument, given in Section 3 of \([TW]\) to complete the proof of \([TW, \text{Thm 1}]\) also holds in this case. \( \square \)

Let us consider the restriction of the hyperkähler moment map \( \mu_{\text{HK}} = \mu_\mathbb{R} \oplus \mu_\mathbb{C} \) to \( M_\mathbb{R} \). Since \( z \) and \( w \) are real for every \( [z, w] \in M_\mathbb{R} \), the map \( \mu_\mathbb{C} \) takes values in \( t^d_\mathbb{R} \subseteq t^d \), which we will identify with \( i\mathbb{R}^n \), so that \( f = \mu_\mathbb{R}|_{M_\mathbb{R}} \oplus \mu_\mathbb{C}|_{M_\mathbb{R}} \) takes values in \( \mathbb{R}^n \oplus i\mathbb{R}^n \cong \mathbb{C}^n \). Note that \( f \) is \( \mathbb{Z}_2 \)-equivariant, with \( \mathbb{Z}_2 \) acting on \( \mathbb{C}^n \) by complex conjugation.

**Lemma 5.2.** The map \( f : M_\mathbb{R} \to \mathbb{C}^n \) is surjective, and the fibers are the orbits of \( T^d_\mathbb{R} \). The stabilizer of a point \( x \in M_\mathbb{R} \) has order \( 2^r \), where \( r \) is the number of hyperplanes in the complexified arrangement \( \mathcal{H}_\mathbb{C} \) containing the point \( f(x) \).

**Proof.** For any point \( p = a + bi \in \mathbb{C}^n \), choose a point \( [z, w] \in M \) such that \( \mu_\mathbb{R}[z, w] = a \) and \( \mu_\mathbb{C}[z, w] = b \). We can hit \([z, w]\) with an element of \( T^d = T^n/T^k \) to make \( z \) real, and the fact that \( \mu_\mathbb{C}[z, w] \in \mathbb{R}^n \) forces \( w \) to be real as well, hence we may assume that \( [z, w] \in M_\mathbb{R} \). Then \( f^{-1}(p) = \mu_{\text{HK}}^{-1}(a, b) \cap M_\mathbb{R} = T^d[z, w] \cap M_\mathbb{R} = T^d_k[z, w] \). The second statement follows easily from \([BD, 3.1]\). \( \square \)

Let \( Y \subseteq M_\mathbb{R} \) be the locus of points on which \( T^d_\mathbb{R} \) acts freely, i.e., the preimage under \( f \) of the space \( \mathcal{M}(\mathcal{H}) := \mathbb{C}^n \setminus \cup_{i=1}^n H^i_{\mathbb{C}} \). The inclusion map \( Y \to M_\mathbb{R} \) induces maps backward on cohomology, which we will denote

\[
\phi : H^*_{T^d_\mathbb{R}}(M_\mathbb{R}; \mathbb{Z}_2) \to H^*_{T^d}(Y; \mathbb{Z}_2) \cong H^*(\mathcal{M}(\mathcal{H}); \mathbb{Z}_2)
\]

and

\[
\phi_2 : H^*_{T^d_\mathbb{R} \times S^2}(M_\mathbb{R}; \mathbb{Z}_2) \to H^*_{T^d \times \mathbb{Z}_2}(Y; \mathbb{Z}_2) \cong H^*_{\mathbb{Z}_2}(\mathcal{M}(\mathcal{H}); \mathbb{Z}_2).
\]

The ring \( H^*(\mathcal{M}(\mathcal{H}); \mathbb{Z}) \) has been studied extensively, and is called the Orlik-Solomon algebra \([OT]\), which we will denote by \( \mathcal{OS} \). A remarkable fact about the Orlik-Solomon algebra is that it depends only on the combinatorial structure of \( \mathcal{H} \); the following is a presentation in terms of anticommuting generators \( e_1, \ldots, e_n \) \([OS]\):

\[
\mathcal{OS} \cong H^*(\mathcal{M}(\mathcal{H}); \mathbb{Z}) \cong \mathbb{Z}[e_1, \ldots, e_n]/ \langle \Pi_i \in S e_i | \cap_{i \in S} H_i = \emptyset \rangle.
\]
Rather than working with anticommuting generators, we can work over the ground field \( \mathbb{Z}_2 \), in which case commutativity and anticommutativity are the same. Because \( OS \) is torsion-free [OT, 3.74], we have

\[
OS \otimes \mathbb{Z}_2 \cong H^*(\mathcal{M}(\mathcal{H}); \mathbb{Z}_2)
\]

\[
\cong \mathbb{Z}_2[e_1, \ldots, e_n]/ \langle \Pi_{i \in S} e_i \mid \cap_{i \in S} H_i = \emptyset \rangle + \langle e_i^2 \mid i \leq n \rangle,
\]

where \( \deg(e_i) = 1 \).

**Claim 5.3.** The map \( \phi : H^*_{\mathbb{T}_{\mathbb{R}}}(M_{\mathbb{R}}; \mathbb{Z}_2) \to OS \otimes \mathbb{Z}_2 \) takes \( u_i \) to \( e_i \), hence \( \ker \phi \) is generated by the set \( \{u_i^2 \mid i \leq n\} \).

**Proof.** Recall from Section 2 that the hyperplane \( H_i \subseteq (t^d)^* \) is defined by the equation \( \langle x, a_i \rangle = \langle -\alpha, \epsilon_i \rangle \). Let \( \eta_i : \mathbb{C}^n \to \mathbb{C} \) be the affine map taking \( x \) to \( \langle x, a_i \rangle + \langle \alpha, \epsilon_i \rangle \), so that \( H_i^{\circ} \) is cut out of \( \mathbb{C}^n \) by \( \eta_i \). Then \( \eta_i \) restricts to a map \( \mathcal{M}(\mathcal{H}) \to \mathbb{C}^* \), and Orlik and Terao identify \( e_i \) with the cohomology class represented by the pull-back of the submanifold \( \mathbb{R}_- \) (the negative reals) along \( \eta_i \) [OT, 5.90]. Theorem 5.1 tells us that the cohomology class \( u_i \) is represented by the divisor \( Z_i \cap M_{\mathbb{R}} \). By Lemma 3.1, \( f(Z_i \cap M_{\mathbb{R}}) = G_i \cap \mathbb{R}^n \), hence \( \phi(u_i) \) is the class represented by the submanifold \( G_i \cap \mathbb{R}^n \cap \mathcal{M}(\mathcal{H}) = \eta^{-1}_i(\mathbb{R}_-) \).

**Remark 5.4.** The fact that the classes \( u_i^2 \) lie in the kernel of \( \phi \) can be seen by noting that \( u_i \) is represented both by the divisor

\[
Z_i = \{[z, w] \in M_{\mathbb{R}} \mid z_i = 0\},
\]

and, since \( u_i = -u_i \) over \( \mathbb{Z}_2 \), also by the divisor

\[
W_i = \{[z, w] \in M_{\mathbb{R}} \mid w_i = 0\}.
\]

The condition \( x \in Z_i \cap W_i \) says exactly that \( \mu_{\mathbb{R}}(x) \in H_i \), therefore the intersection \( Z_i \cap W_i \cap \hat{Y} \) is empty.

In some sense we have cheated here; we have concluded that we can recover a presentation of \( OS \otimes \mathbb{Z}_2 \) from a presentation of \( H^*_{\mathbb{T}_{\mathbb{R}}}(M) \), but we used the fact that we already have a presentation of \( OS \otimes \mathbb{Z}_2 \). In the \( \mathbb{Z}_2 \)-equivariant picture, however, our trivial observation turns magically into new information, giving us a presentation of the equivariant cohomology ring \( H^*_{\mathbb{Z}_2}(\mathcal{M}(\mathcal{H}); \mathbb{Z}_2) \).

**Theorem 5.5.** The map \( \phi_2 \) is surjective, with kernel generated by \( \{u_i(x - u_i) \mid i \leq n\} \). Hence

\[
H^*_{\mathbb{Z}_2}(\mathcal{M}(\mathcal{H}); \mathbb{Z}_2) \cong H^*_{\mathbb{T}_{\mathbb{R}} \times \mathbb{Z}_2}(M_{\mathbb{R}}; \mathbb{Z}_2)/ \ker \phi_2
\]

\[
\cong \mathbb{Z}_2[u_1, \ldots, u_n, x]/ \langle \Pi_{i \in S_1} u_i \times \Pi_{j \in S_2} (x - u_j) \mid \cap_{i \in S} H_i = \emptyset \rangle + \langle u_i(x - u_i) \mid i \leq n \rangle.
\]
Corollary 4.9. By Theorem 5.1 and Proposition 4.7, follows from surjectivity of $\phi$. Hence we are done.

The fact that $H^*_2(c)$ is not a complete invariant of smooth, rational, affine arrangements up to combinatorial equivalence.

Example 5.6. Consider the arrangements $\mathcal{H}_a$ and $\mathcal{H}_c$ in Figure 2(a) and 2(c). By Theorem 5.5 and Example 4.10 we have

$$H^*_2(\mathcal{M}(\mathcal{H}_a); \mathbb{Z}_2) \cong \mathbb{Z}_2[ u_1, \ldots, u_4, x] / \left< u_1(x - u_1), u_2(x - u_2), u_3(x - u_3), u_4(x - u_4) \right>$$

and

$$H^*_2(\mathcal{M}(\mathcal{H}_c); \mathbb{Z}_2) \cong \mathbb{Z}_2[ u_1, \ldots, u_4, x] / \left< u_1(x - u_1), u_2(x - u_2), u_3(x - u_3), u_4(x - u_4) \right>.$$ 

The map $f : H^*_2(\mathcal{M}(\mathcal{H}_a); \mathbb{Z}_2) \to H^*_2(\mathcal{M}(\mathcal{H}_b); \mathbb{Z}_2)$ given by

$$f(u_1) = u_1 + u_2, \quad f(u_2) = u_2 + u_3 + x, \quad f(u_3) = u_3, \quad f(u_4) = u_2 + u_4, \quad \text{and} \quad f(x) = x$$

is an isomorphism of graded $\mathbb{Z}_2[x]$-algebras, hence the ring $H^*_2(\mathcal{M}(\mathcal{H}); \mathbb{Z}_2)$ is not a complete invariant of smooth, rational, affine arrangements up to combinatorial equivalence.\footnote{We thank Graham Denham for finding this isomorphism.}

Example 5.7. Now consider the arrangements $\mathcal{H}'_a$ and $\mathcal{H}'_c$ obtained from $\mathcal{H}_a$ and $\mathcal{H}_c$ by adding a vertical line on the far left, as shown below.

Again by Theorem 5.5, we have

$$H^*_2(\mathcal{M}(\mathcal{H}'_a); \mathbb{Z}_2) \cong \mathbb{Z}_2[ u, x] / \left< u_1(x - u_1), u_2(x - u_2), u_3(x - u_3), u_4(x - u_4) \right>$$

and

$$H^*_2(\mathcal{M}(\mathcal{H}'_c); \mathbb{Z}_2) \cong \mathbb{Z}_2[ u, x] / \left< u_1(x - u_1), u_2(x - u_2), u_3(x - u_3), u_4(x - u_4) \right>.$$

We have used Macaulay 2 to check that the annihilator of the element $u_2 \in H^*_2(\mathcal{M}(\mathcal{H}'_a); \mathbb{Z}_2)$ is generated by two linear elements (namely $u_3$ and $x - u_2$) and nothing else, while there is no element of $H^*_2(\mathcal{M}(\mathcal{H}'_c); \mathbb{Z}_2)$ with this property. Hence the two rings are not isomorphic.
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References


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UNITARY REPRESENTATIONS OF CLASSICAL LIE GROUPS OF EQUAL RANK WITH NONZERO DIRAC COHOMOLOGY

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In this paper, we consider unitary representations of classical groups of equal rank (rank $G = \text{rank } K$) except type $CI$ with regular lambda-lowest $K$-type and get the necessary and sufficient condition such that those unitary representations considered have nonzero Dirac cohomology.

1. Introduction.

In the past twenty years, people are interested in unitary representations with nonzero cohomologies, that is, $(g, K)$-cohomology and Dirac cohomology. The former was studied by Vogan and Zuckerman in [10]. Since every representation with nonzero $(g, K)$-cohomology has nonzero Dirac cohomology, maybe it is this fact that motivates people to pay more attention to Dirac cohomology.

In 1997, Vogan explained a conjecture on Dirac cohomology at MIT Lie groups seminar. The conjecture can be stated as follows: Let $G$ be a connected semisimple Lie group with Lie algebra $g_0$ and let $K$ be the maximal compact subgroup of $G$ corresponding to the Cartan involution $\theta$. Suppose $X$ is an irreducible unitarizable $(g, K)$-module and $(\gamma, S)$ is a space of spinors for $p_0$. Here $g_0 = k_0 + p_0$ is the Cartan decomposition of $g_0$. Let $x_1, \ldots, x_n$ be an orthonormal basis of $p_0$, then the Dirac operator $D = \sum \pi(x_i) \otimes \gamma(x_i)$ acts on $X \otimes S$. Vogan’s conjecture says that if $D$ has nonzero Dirac cohomology, which by definition is just $\text{Ker } D$, then the infinitesimal character of $X$ can be expressed in terms of the highest weight of a $K$-type of $X$.

The conjecture was proved by Huang and Pandžić [2]. Furthermore, they get that an irreducible unitarizable $(g, K)$-module $X$ has nonzero Dirac cohomology, say $\gamma \subseteq \text{Ker } D$, if and only if the infinitesimal character $\Lambda$ of $X$ is given by $\gamma + \rho_c$. To be precise, $\gamma$ has highest weight $\omega(\mu - \rho_n)$, where $\mu$ is a $K$-type of $X$, $\omega \in W(K)$ such that $\omega(\mu - \rho_n)$ is dominant and $\Lambda = \omega(\mu - \rho_n) + \rho_c$. One could ask: For what kinds of $K$-types does the expression $\|\omega(\mu - \rho_n) + \rho_c\|$ reach the minimum? For what cases is $\mu$ a lambda-lowest $K$-type of $X$ when $\omega(\mu - \rho_n) \subseteq \text{Ker } D$?
In this paper, we will answer the above problems partially. We study the representations of classical group $G$ of equal rank except type $CI$, with regular lambda-lowest $K$-type. First we recall the definition of $\theta$-stable data.

**Definition 1.1** (Vogan [8], Definition 6.5.1). A set of $\theta$-stable data for $G$ is a quadruple $(q, H, \delta, \nu)$, such that:

- a) $q = t + u$ is a $\theta$-stable parabolic subalgebra of $g$. Let $L$ be the normalizer of $q$ in $G$.
- b) $L$ is quasisplit, and $H = TA \subseteq L$ is a maximally split $\theta$-stable Cartan subgroup of $L$.
- c) $\delta \in \hat{T}$ is fine with respect to $L$.
- d) $\nu \in \hat{A}$.
- e) Write $\lambda^L \in t^*$ for the differential of $\delta$, and $\lambda = \lambda^L + \rho(\Delta(u, t)) \in t^* \subseteq h^*$.

Then $\lambda$ is strictly dominant for $\Delta(u, h)$.

There is a surjective map from the set of equivalence classes of irreducible $(g, K)$-module to $K$ conjugacy classes of set of $\theta$-stable data for $G$ ([8], Corollary 6.5.13). And following Vogan's method ([8], Chapter 5), one can construct $\theta$-stable data from any given irreducible $(g, K)$-module $X$.

Now we can state our main theorem.

**Theorem 1.2.** Let $X$ be an irreducible $(g, K)$-module with regular lambda-lowest $K$-type $\mu$. Then $X$ is unitary and has nonzero Dirac cohomology if and only if the parameter $\nu$ in the $\theta$-stable data $(q, H, \delta, \nu)$ corresponding to $X$ is just $\frac{1}{2} \sum_{\beta_i \in \Gamma_1} \beta_i$ under $G$-conjugation. Here, $\Gamma_1$ is a set of roots defined by the lambda-lowest $K$-type $\mu$ during the construction of $\theta$-stable data (see Section 3.1 for details).

The paper is organized as follows: We first collected some notations and results on Dirac operator and Dirac cohomology in Section 2. Then we followed Vogan’s method to construct $\theta$-stable data $(q, H, \delta, \nu)$ for corresponding $(g, K)$-module $X$. Actually, we found that the quasisplit subgroup $L$ is simple enough under our assumption. Locally $L$ is a product of copies of $SL(2, \mathbb{R})$ and Euclidean space. In Section 4, we find out that if a lambda-lowest $K$-type $\mu$ of $X$ is regular, then $\mu - \rho_n$ is dominant (Proposition 4.2) and $\|\mu - \rho_n + \rho_c\| \leq \|\omega(\mu' - \rho'_n) + \rho_c\|$. Then $X$ has nonzero Dirac cohomology only if $\|\Lambda\| = \|\mu - \rho_n + \rho_c\|$. Fortunately, in this case, $\Lambda$ is dominant. Then Vogan’s result, Theorem 1.3 [9], implies that $X$ is unitary, hence $X$ has nonzero Dirac cohomology by Huang and Pandžić’s result (Proposition 2.4) since $\|\Lambda\| = \|\mu - \rho_n + \rho_c\|$. Thus we get the main theorem.

**2. Preliminary.**

Let $G$ be a real semisimple group with Lie algebra $g_0$ and let $K$ be the maximal compact subgroup of $G$ corresponding to Cartan involution $\theta$. Let
$g_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition of $g_0$. Fix a maximally compact Cartan subalgebra $h_0^c$ of $g_0$ with decomposition $h_0^c = t_0^c + a_0^c$. Denote by $g$, $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{h}^c$, $t^c$ and $a^c$ the complexifications of $g_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$, $h_0^c$, $t_0^c$ and $a_0^c$, respectively. Let $\Delta(g, \mathfrak{h}^c)$ be the root system of $g$ with respect to $h_0^c$. Fix a system of positive roots, $\Delta^+(\mathfrak{k}, t^c)$, for $\Delta(\mathfrak{k}, t^c)$ and choose a compatible system of positive roots, $\Delta^+(g, \mathfrak{h}^c)$, for $\Delta(g, \mathfrak{h}^c)$ with the set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_l\}$. Let $G_0$ be the identity component of $G$.

**Definition 2.1 ([11]).** Let $(\pi, X)$ be a $(g, K)$-module, set $S = S(p_0)$, a space of spinors of $p_0$. Let $x_1, \ldots, x_n$ be an orthonormal basis of $p_0$, then the **Dirac operator**

$$D : X \otimes S \to X \otimes S$$

is defined by

$$D = \sum \pi(x_i) \otimes \gamma(x_i),$$

which is a $K$-module homomorphism (sometimes $\tilde{K}$-module homomorphism, where $\tilde{K}$ is a two-fold spin cover of $K$).

The **Dirac cohomology** of $X$ is defined by

$$\text{Ker} D / (\text{Ker} D \cap \text{Im} D).$$

When $X$ is unitary, then Dirac operator is self-dual, then we can see that the Dirac cohomology of $X$ is just $\text{Ker} D$.

The following result of Pathasarathy is well-known. It can be found in many papers.

**Proposition 2.2** (Pathasarathy’s Dirac Inequality). Let $X$ be an irreducible unitary $(g, K)$-module with infinitesimal character $\Lambda$. Fix a representation of $K$ occurring in $X$ of highest weight $\mu \in (t^c)^*$, and a positive root system $\Delta^+(g, t^c)$ for $t^c$ in $g$. Here $t^c$ is Cartan subalgebra of $\mathfrak{t}$. Write

$$\rho_c = \rho(\Delta^+(\mathfrak{t}, t)), \quad \rho_n = \rho(\Delta^+(\mathfrak{p}, t)).$$

Fix an element $\omega \in W_K$ such that $\omega(\mu - \rho_n)$ is dominant for $\Delta^+(\mathfrak{t}, t)$. Then

$$(\omega(\mu - \rho_n) + \rho_c, \omega(\mu - \rho_n) + \rho_c) \geq (\Lambda, \Lambda).$$

The equality holds if and only if

$$\Lambda = \omega(\mu - \rho_n) + \rho_c.$$

The last assertion was obtained by Huang and Pandžić [2].

We also have another similar inequality.

**Proposition 2.3.** Let $V$ be an irreducible unitary $(g, K)$-module with Hermitian form $\langle \cdot, \cdot \rangle$ and infinitesimal character $\Lambda$. Assume $\mu \in \hat{K}$ occurs in $V$. Then

$$||\Lambda||^2 \leq ||\mu + \rho_c||^2 - ||\rho_c||^2 + ||\rho||^2.$$
Proof. Let \( \{x_i\} \) be an orthonormal basis of \( p \) with respect to the Killing form. For \( v \in V_\mu \) we have
\[
\langle x_i v, x_i v \rangle \geq 0 \Rightarrow \langle x_i^2 v, v \rangle \leq 0 \Rightarrow \langle (c - c_k) v, v \rangle \leq 0.
\]
Then the assertion follows easily. \qed

In 1997, Vogan explained a conjecture on Dirac cohomology, which was proved by Huang and Pandžić [2]. We summarize their results as follows:

**Proposition 2.4 ([2]).** Let \( X \) be an irreducible unitarizable \((\mathfrak{g}, K)\)-module with infinitesimal character \( \Lambda \). Assume \( X \otimes S \) contains a \( \tilde{K} \)-type \( \gamma \), i.e., \((X \otimes S)(\gamma) \neq 0\). Then the Dirac cohomology of \( X \), \( \text{Ker} D \), contains \((X \otimes S)(\gamma)\) if and only if \( \Lambda = \gamma + \rho_c \). Here \( \gamma \) must be of the form \( \omega(\mu - \rho_n) \) for some \( \rho_n \) and \( K \)-type \( \mu \) contained in \( X \).

### 3. Construction of \( \theta \)-stable data.

In this section, we will make the following assumption:

**Assumption 3.1.** \( G \) is a classical group with rank \( G = \text{rank} K \), i.e., \( \theta \) is an inner automorphism of \( \mathfrak{g}_0 \). Consequently \( \mathfrak{h}_0^c = t_0^c \).

We will follow Vogan’s method to construct \( \theta \)-stable data, actually, the main work is to determine the structure of the quasisplit subgroup \( L \).

#### 3.1. Basic facts.

First, we rewrite Proposition 5.3.3 [8], since we assume \( \text{rank} G = \text{rank} K \) and \( \mathfrak{h} = t^c \).

**Proposition 3.2 ([8]).** For each \( \Delta^+(t, t^c) \)-dominant weight \( \mu \in \hat{T} \), there is a unique element \( \lambda \in (t^c)^* \) having the following properties: Fix a \( \theta \)-invariant positive root system \( \Delta^+(\mathfrak{g}, t^c) \) for \( t^c \) in \( \mathfrak{g} \), making \( \mu + 2\rho_c \) dominant; and write \( \rho = \rho(\Delta^+(\mathfrak{g}, t^c)) \). Then \( \lambda \) is dominant for \( \Delta^+(\mathfrak{g}, t^c) \), and there is a set \( \Gamma = \{\beta_1, \ldots, \beta_r\} \subseteq \Delta^+(\mathfrak{g}, t^c) \) satisfying:

a) If we put
\[
\bar{\lambda} = \mu + 2\rho_c - \rho, \\
c_i = -\langle \bar{\lambda}, \beta_i^\vee \rangle,
\]
then
\[
0 \leq c_i \leq 1,
\]
and
\[
\lambda = \bar{\lambda} + \frac{1}{2} \sum c_i \beta_i.
\]

b) If \( (\lambda, \alpha) = 0 \) for \( \alpha \in \Delta(\mathfrak{g}, t^c) \), then \( (\alpha, \beta_i) \neq 0 \) for some \( i \).

c) The root \( \beta_1 \) is noncompact and simple.
d) Write
\[ \mathfrak{g}^1 = \mathfrak{g}^{\beta_1}, \quad \mathfrak{h}^1 = (t^c)^{\beta_1}. \]
Then the positive system \( \Delta^+(\mathfrak{g}, t^c) \cap \beta_1 \) and its subset \( \{\beta_2, \ldots, \beta_r\} \) for \( \Delta(\mathfrak{g}^1, \mathfrak{h}^1) \) satisfy these same conditions for \( \mathfrak{g}^1 \) and the weight \( \mu|_{\mathfrak{g}^1 \cap t^c} \).

e) If \( c_i \neq 0 \) and \( c_j = 0 \), then \( i < j \).

Under Assumption 3.1, we can get a stronger result.

**Lemma 3.3.** Let the notation be as above. Then
\[ c_i = 0 \text{ or } 1. \]

**Proof.** By Lemma 7.7.6 [1], we have
\[ \exp(2\pi\sqrt{-1} \alpha^\vee) = e, \]
where \( e \) is the unit of \( G \). Then \( (\mu, \alpha^\vee) \) is an integer. \( \square \)

For convenience, we denote
\[ \Gamma_1 = \{\beta_i \in \Gamma | c_i = 1\}, \]
\[ \Gamma_0 = \{\beta_i \in \Gamma | c_i = 0\}. \]

Let \( \Pi \) be the system of simple roots of \( \Delta^+(\mathfrak{g}, t^c) \). Set
\[ \Sigma_1 = \{\alpha \in \Pi | (\tilde{\lambda}, \alpha^\vee) = -1\}, \]
\[ \Sigma_0 = \{\alpha \in \Pi | (\tilde{\lambda}, \alpha^\vee) = 0\}. \]

Now we can define \( l \) by
\[ \Delta(l, t^c) = \{\alpha \in \Delta(\mathfrak{g}, t^c) | (\lambda, \alpha^\vee) = 0\}. \]

Obviously, the Dynkin diagram of \( l \) is a subdiagram of that of \( \mathfrak{g} \) if we choose compatible orderings, i.e.,
\[ \Delta^+(l, t^c) \subseteq \Delta^+(\mathfrak{g}, t^c). \]

Denote by \( \Pi_l \) the system of simple roots of \( l \). First we establish some lemmas.

**Lemma 3.4.** Let \( \alpha \) and \( \beta \) be adjacent simple roots of the same length. Then
\[ (\tilde{\lambda}, (\alpha + \beta)^\vee) \geq 0. \]

**Proof.** If both \( \alpha \) and \( \beta \) are compact or noncompact, then \( \alpha + \beta \) is compact, so
\[ (\tilde{\lambda}, (\alpha + \beta)^\vee) \geq (\mu, (\alpha + \beta)^\vee) \geq 0. \]
Thus we can assume \( \alpha \) is compact and \( \beta \) is noncompact. Then
\[ (\tilde{\lambda}, \beta^\vee) \geq -1 \]
and
\[ (\tilde{\lambda}, \alpha^\vee) = (\mu, \alpha^\vee) + 1. \]
So

\[(\lambda, (\alpha + \beta)\vee) \geq (\mu, \alpha\vee) \geq 0.\]

\[\Box\]

**Lemma 3.5.** Let \(\alpha, \beta\) and \(\alpha + \beta \in \Delta(g, t^c)\). If \((\lambda, \alpha\vee) \geq 0\) and \((\lambda, \beta\vee) \geq 0\), then

\[(\lambda, (\alpha + \beta)\vee) \geq 0.\]

**Proof.** \((\alpha + \beta)\vee = a\alpha\vee + b\beta\vee\), where \(a\) and \(b\) are positive. \[\Box\]

**Lemma 3.6.** Assume \(\mu\) is regular, i.e., \((\mu, \gamma\vee) \geq 1\), for all \(\gamma \in \Delta^+(t, t^c)\). Let \(\alpha\) and \(\beta\) be adjacent simple roots. If \((\alpha, \alpha) = 2(\beta, \beta)\), then

\[(\lambda, (\alpha + \beta)\vee) \geq 0.\]

If \(\alpha\) and \(\beta\) have the same length, then

\[(\lambda, (\alpha + \beta)\vee) \geq 1.\]

**Proof.** Only the first assertion needs to prove. We treat it case by case.

**Case I.** Both \(\alpha\) and \(\beta\) are noncompact.

\[(\lambda, (\alpha + \beta)\vee) = (\mu, (\alpha + \beta)\vee) + 2 - (\rho, 2\alpha\vee + \beta\vee) \geq 1 + 2 - 3 \geq 0.\]

**Case II.** \(\alpha\) is compact while \(\beta\) is noncompact.

\[(\lambda, (\alpha + \beta)\vee) = (\lambda, 2\alpha\vee + \beta\vee) \geq 4 - 1 \geq 3.\]

**Case III.** \(\alpha\) is noncompact while \(\beta\) is compact.

\[(\lambda, (\alpha + \beta)\vee) = (\lambda, 2\alpha\vee + \beta\vee) \geq -2 + 2 \geq 0.\]

\[\Box\]

**Corollary 3.7.** Let \(\alpha \in \Sigma_1, \beta \in \Sigma_1 \cup \Sigma_0\). Then \((\alpha, \beta) = 0\) for types \(AIII\) and \(D_4\). If \(\mu\) is regular for \(\Delta^+(t, t^c)\), then it is true for any type.

**Lemma 3.8.** Assume \(\mu\) is regular for \(\Delta(t, t^c)\) and \(\Gamma\) consists of simple roots of \(\Delta(g, t^c)\). Then the simple roots of \(l\) are noncompact.

**Proof.** If \(\alpha \in \Pi\) is compact, then

\[(\lambda, \alpha\vee) = (\lambda, \alpha\vee) + \frac{1}{2} \sum c_i(\beta_i, \alpha\vee) \geq 2 - \frac{3}{2} > 0.\]

The first inequality holds because \(\alpha\) is adjacent to at most three simple roots of the same length or two simple roots of different length. So \(\alpha \notin \Pi_l\). \[\Box\]
3.2. **Main theorem.** Now we can study the structure of $I$. Our purpose is to prove the following theorem:

**Theorem 3.9.** Let $\mu$ be a $K$-type of a $(g, K)$-module. Assume $\mu$ is regular for $\Delta(t, t^c)$. Then:

1) For types $AIII$, $CII$ and $D_l$, the Dynkin diagram of $I$ is discrete.
2) For types $B_l$ and $CI$, the Dynkin diagram of $I$ is either discrete or of the form
   \[ A_1 \times \cdots \times A_1 \times B_2. \]
3) For type $B_l$. If $\mu$ is regular for $\Delta(g, t^c)$, then the Dynkin diagram of $I$ is discrete.

Let’s deal with the problem case by case.

3.2.1. **Type $AIII$.** In this subsection, we assume that the Lie algebra $g_0$ is of type $AIII$.

**Proposition 3.10.**

1) Let $\alpha \in \Delta^+(g, t^c)$. If $(\alpha, \beta) = 0$ for any $\beta \in \Sigma_1$, then
   \[ (\widetilde{\lambda}, \alpha^\vee) \geq 0. \]
2) Those $\beta_i$ in Proposition 3.2 can be chosen to be simple.
3) If $\mu$ is regular for $\Delta(t, t^c)$, then
   \[ \Pi_1 = \Gamma. \]

**Proof.**

1) Let $\alpha = \alpha_i + \cdots + \alpha_k$. If $\alpha$ is not adjacent to any $\beta \in \Sigma_1$, then $\alpha_i, \alpha_k \notin \Sigma_1$, so $(\lambda, \alpha^\vee) \geq 0$ by Lemma 3.4.
2) Choose a maximal subset $\Sigma_0'$ of $\Sigma_0$ such that the elements of $\Sigma_0'$ are orthogonal to each other. Then we claim that the set $\Gamma = \Sigma_1 \cup \Sigma_0'$ satisfies the condition of Proposition 3.2.

   Firstly, we choose $\Gamma_1$ containing $\Sigma_1$. By 1) we have
   \[ \Gamma_1 = \Sigma_1. \]

   Secondly, we choose $\Gamma_0$ containing $\Sigma_0'$. If $\alpha = \alpha_i + \cdots + \alpha_k$ is orthogonal to $\Sigma_1 \cup \Sigma_0'$ and
   \[ (\widetilde{\lambda}, \alpha^\vee) = 0, \]
   then $\alpha_i, \alpha_k \notin \Sigma_1 \cup \Sigma_0'$. We claim that $\alpha_i, \ldots, \alpha_k \in \Sigma_0$. By Lemma 3.4, we have $(\widetilde{\lambda}, (\alpha_i + \cdots + \alpha_k)^\vee) \geq 0$. The equality holds and $\alpha_i \in \Sigma_0$ by Equation (1). Furthermore, for the same reason we have $(\widetilde{\lambda}, (\alpha_i + \alpha_{i+1})^\vee) = 0$, that is $\alpha_{i+1} \in \Sigma_0$. Then our claim follows. But one can easily see that the claim contradicts the fact that $\Sigma_0'$ is maximal.
3) Obviously, \( \Gamma \subseteq \Pi_l \). Let \( \alpha \in \Pi_l \setminus \Gamma \). By Lemma 3.8, \( \alpha \) is noncompact. Then \( \alpha \) must be adjacent to some \( \beta \in \Gamma \), so \( (\bar{\lambda}, (\alpha + \beta)^\vee) \geq 1 \). Hence \((\lambda, \alpha^\vee) > 0\). Contradiction. \( \square \)

**Corollary 3.11.** If \( \mu \) is regular for \( \Delta(g, t^c) \), then \( \bar{\lambda} \) is strictly dominant for \( \Delta(u) \), that is,
\[
(\bar{\lambda}, \alpha^\vee) > 0
\]
for any \( \alpha \in \Delta(u) \).

**Proof.** Just follow the proof of the above proposition. \( \square \)

### 3.2.2. Types \( B_l \) and \( C_l \).

**Proposition 3.12.** Assume \( \mu \) is regular for \( \Delta(t, t^c) \). If \( (\alpha, \beta) = 0 \) for any \( \beta \in \Sigma_1 \), then
\[
(\bar{\lambda}, \alpha^\vee) \geq 0
\]

**Proof.** First assume \( g \) is of type \( B_l \). Let \( \alpha \in \Delta(g, t^c) \). Assume \( (\alpha, \beta) = 0 \) for any \( \beta \in \Sigma_1 \). If \( \alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_k \), then \( \alpha_i \notin \Sigma_1 \). Similar to the proof of type \( A_{III} \), one can get \((\bar{\lambda}, \alpha^\vee) \geq 0\).

Now we assume \( \alpha = \alpha_i + \cdots + \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_l \), so \( \alpha_k \notin \Sigma_1 \). If \( \alpha_l \notin \Sigma_1 \), then we have
\[
(\bar{\lambda}, (\alpha_i + \cdots + \alpha_l)^\vee) \geq 0
\]
and
\[
(\bar{\lambda}, (\alpha_k + \cdots + \alpha_l)^\vee) \geq 0
\]
by Lemmas 3.5 and 3.6. Hence \((\bar{\lambda}, \alpha^\vee) \geq 0\).

If \( \alpha_i \notin \Sigma_1 \), the proof is similar. So we just need to check the case that \( \alpha_i, \alpha_l \in \Sigma_1 \). Obviously \( i + 1 = k < l \) and \( \alpha_{l-1} \notin \Sigma_1 \). We have
\[
(\bar{\lambda}, \alpha^\vee) = (\bar{\lambda}, (\alpha_i + \cdots + \alpha_{l-1})^\vee + \alpha_l^\vee) \geq 0.
\]
This completes the proof for type \( B_l \). And the proof for type \( C_l \) is similar. \( \square \)

This Proposition tells us that those \( \beta_i \), which satisfy \((\bar{\lambda}, \beta_i^\vee) < 0\), can be chosen to be simple, that is, \( \Gamma_1 = \Sigma_1 \). Then we get the element \( \lambda = \bar{\lambda} + \frac{1}{2} \sum_{\beta_i \in \Gamma_1} \beta_i \).

**Lemma 3.13.** The simple roots of \( l \) are noncompact simple roots of \( \Delta(g, t^c) \).

**Proof.** If \( \alpha \) is compact simple, then
\[
(\lambda, \alpha^\vee) = (\bar{\lambda}, \alpha^\vee) + \frac{1}{2} \sum c_i(\beta_i, \alpha^\vee) \geq 2 - \frac{3}{2} > 0.
\]
The first inequality follows from that \( \alpha \) is adjacent to at most three simple roots of the same length or two simple roots. So \( \alpha \notin \Pi_l \). \( \square \)
\(\alpha_1, \ldots, \alpha_{l - 1}\) generate a subsystem of type \(A_{l - 1}\). Let \(\Pi'_l = \Delta(I, t^c) \cap \{\alpha_1, \ldots, \alpha_{l - 1}\}\). Then we have:

**Lemma 3.14.** Let \(\beta_j \in \Pi'_l \cap \Sigma_1\) and \(\alpha \in \Pi'_l\). Then \((\beta_j, \alpha^\vee) = 0\).

**Proof.** If \(\alpha\) is adjacent to \(\beta\), then \(\alpha + \beta\) is compact and we have

\[
(\lambda, (\alpha + \beta)^\vee) = (\tilde{\lambda}, (\alpha + \beta)^\vee) + \frac{1}{2} \sum c_i(\beta_i, \beta_j^\vee) + \frac{1}{2} \sum c_i(\beta_i, \alpha^\vee) \\
\geq 1 + c_j - \frac{1}{2}c_j - 1 = \frac{1}{2}.
\]

This leads to a contradiction. \(\square\)

**Corollary 3.15.** The Dynkin diagram of \(\Pi'_l\) is discrete.

**Proof.** If it is not true, then there exist two adjacent noncompact simple roots \(\alpha, \beta \in \Pi'_l\). By the above Lemma, neither of them is adjacent to some \(\beta_i \in \Pi'_l \cap \Sigma_1\). Then \(\alpha + \beta\) is compact.

**Case I.** \(\alpha_l \notin \Sigma_1\).

(2) \((\lambda, (\alpha + \beta)^\vee) = (\tilde{\lambda}, (\alpha + \beta)^\vee) \geq 1 + 2 - 2 = 1\).

**Case II.** \(\alpha_l \in \Sigma_1\).

1) If \(g\) is of type \(C_l\), then \(\alpha_{l - 1} \notin \Pi_l\) by the following Lemma 3.16. The inequality (2) is also correct.

2) If \(g\) is of type \(B_l\), then

\[
(\lambda, (\alpha + \beta)^\vee) \geq (\tilde{\lambda}, (\alpha + \beta)^\vee) + \frac{1}{2}(\alpha_l, (\alpha + \beta)^\vee) \geq 1 + 2 - \frac{1}{2} = \frac{1}{2}.
\]

Thus for all the cases, we have \((\lambda, (\alpha + \beta)^\vee) > 0\). Contradiction. \(\square\)

\(\alpha_{l - 1}\) and \(\alpha_l\) generate a subsystem of type \(B_2 = \langle \alpha, \beta \rangle\), where \(\alpha\) is the long root.

**Lemma 3.16.** Let the notation be as above.

1) If \(\alpha \in \Sigma_1\), then \((\lambda, \beta^\vee) > 0\), i.e., \(\beta \notin \Pi_l\).

2) If \(\beta \in \Sigma_1\), then \((\lambda, \alpha^\vee) > 0\), i.e., \(\beta \notin \Pi_l\).

**Proof.** Thanks to Lemma 3.13, we can assume that \(\beta\) is noncompact. Then \(\alpha + \beta\) is compact and

\[
(\tilde{\lambda}, (\alpha + \beta)^\vee) \geq 1 + 2 - (\rho, 2\alpha^\vee + \beta^\vee) = 0.
\]

1) \(\alpha \in \Sigma_1\). Then \((\tilde{\lambda}, \beta^\vee) \geq 2c\). For type \(B_l\),

\[
(\lambda, \beta^\vee) \geq 2 + \frac{1}{2}(\alpha, \beta^\vee) = 1.
\]

For type \(C_l\),

\[
(\lambda, \beta^\vee) \geq 2 + \frac{1}{2}(\alpha, \beta^\vee) + \frac{1}{2}(\alpha_{l - 2}, \beta^\vee) = \frac{1}{2}.
\]
2) $\beta \in \Sigma_1$. Then $(\tilde{\lambda}, \beta^\vee) = -1$. Since
\[ (\tilde{\lambda}, (\alpha + \beta)^\vee) \geq 0, \]
then
\[ (\tilde{\lambda}, \alpha^\vee) \geq \frac{1}{2}. \]
Consequently
\[ (\tilde{\lambda}, \alpha^\vee) \geq 1 \]
since $(\tilde{\lambda}, \alpha^\vee)$ is an integer. Then
\[ (\lambda, \alpha^\vee) = (\tilde{\lambda}, \alpha^\vee) + \frac{1}{2}(\beta, \alpha^\vee) \geq \frac{1}{2}, \]
namely, $\alpha \notin \pi_1$. \hfill $\Box$

If $\alpha_{l-1}, \alpha_l \notin \Sigma_1$, that is,
\[ (\tilde{\lambda}, \alpha_{l-1}^\vee) \geq 0, \quad (\tilde{\lambda}, \alpha_l^\vee) \geq 0, \]
then
\[ (\tilde{\lambda}, (\alpha_{l-1} + \alpha_l)^\vee) = 2(\tilde{\lambda}, \alpha_{l-1}^\vee) + (\tilde{\lambda}, \alpha_l^\vee) \geq 0. \]
Here the equality holds if and only if
\[ (\tilde{\lambda}, \alpha_{l-1}^\vee) = (\tilde{\lambda}, \alpha_l^\vee) = 0. \]

Now we assume $g$ is of type $B_l$. First we prove a lemma.

**Lemma 3.17.** Assume $g$ is of type $B_l$. If $\alpha_l$ is noncompact, then $(\rho_c, \alpha_l) = 0$.

**Proof.** The compact root $\alpha$ which is adjacent to $\alpha_l$ must have one of the two forms: 1) $\alpha = \alpha_i + \cdots + \alpha_{l-1}$, 2) $\alpha = \alpha_i + \cdots + \alpha_{l-1} + 2\alpha_l$. Two such forms occur in a pair. A simple calculation leads to the lemma. \hfill $\Box$

If $\alpha_{l-1} \notin \Sigma_1$ and $\mu$ is regular for $\Delta(g, t^c)$, that is, $(\mu, \alpha) \neq 0$ for any $\alpha \in \Delta(g, t^c)$, then $(\mu + 2\rho_c, \alpha_l^\vee) \geq 1$ since $(\mu + 2\rho_c, \alpha_l^\vee)$ is an integer. Then $\alpha_l \notin \Sigma_1$. If $\alpha_{l-1}, \alpha_l \in \Pi_l$, then we have $(\tilde{\lambda}, \alpha_{l-1}^\vee) = (\tilde{\lambda}, \alpha_l^\vee) = 0$, that is, $(\mu + 2\rho_c, \alpha_{l-1}^\vee) = (\mu + 2\rho_c, \alpha_l^\vee) = 1$. Then $(\mu + 2\rho_c, (\alpha_{l-1} + \alpha_l)^\vee) = (\mu, (\alpha_{l-1} + \alpha_l)^\vee) + (2\rho_c, (\alpha_{l-1} + \alpha_l)^\vee) = (\mu, (\alpha_{l-1} + \alpha_l)^\vee) + 2 = 3$. So we get $(\mu, \alpha_{l-1}^\vee) = 0$, which contradicts the assumption that $\mu$ is regular. Actually, we have proved:

**Theorem 3.18.** Assume $g$ is of type $B_l$ and $\mu$ is regular for $\Delta(g, t^c)$. Then the Dynkin diagram of $l$ is discrete. Consequently,
\[ \Gamma = \Pi_l. \]
3.2.3. Type $D_l$. Since all roots of $D_l$ have the same length, some results on $AIII$ can be applied and we can get some similar results.

**Proposition 3.19.** Assume $\mu$ is regular for $\Delta(t, t^e)$.

1) Let $\alpha \in \Delta^+(g, t^e)$. If $(\alpha, \beta) = 0$ for any $\beta \in \Sigma_1$, then

$$<\tilde{\lambda}, \alpha^\vee> \geq 0.$$ 

2) $\Pi = \Gamma$. Consequently, $\Gamma$ consists of simple roots.

*Proof.* 1) Let $\{\alpha_1, \ldots, \alpha_l\}$ be the simple roots. If $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_k$, similar to the proof of Proposition 3.10, one can easily get $<\tilde{\lambda}, \alpha^\vee> \geq 0$. Now we assume $\alpha = \alpha_i + \cdots + \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l$.

If $k > i + 1$, then $\alpha_i, \alpha_k \notin \Sigma_1$. So we have

$$<\tilde{\lambda}, (\alpha_i + \cdots + \alpha_{l-1})^\vee> \geq 0$$

and

$$<\tilde{\lambda}, (\alpha_k + \cdots + \alpha_{l-2} + \alpha_l)^\vee> \geq 0.$$ 

Hence $<\tilde{\lambda}, \alpha^\vee> \geq 0$.

If $k = i + 1$, that is, $\alpha = \alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l$, then we have $\alpha_{i+1} \notin \Sigma_1$. In this case we may have $\alpha_i \in \Sigma_1$. If $\alpha_{l-1} \notin \Sigma_1$ or $\alpha_l \notin \Sigma_1$, the proof is similar to the above. Now we assume $\alpha_{l-1}, \alpha_l \in \Sigma_1$, then $\alpha_{l-2} \notin \Sigma_1$. If $\alpha_{l-3} \notin \Sigma_1$, then we write

$$\alpha = (\alpha_i + \cdots + \alpha_{l-3}) + (\alpha_{i+1} + \cdots + \alpha_{l-1}) + (\alpha_{l-2} + \alpha_l).$$

If $\alpha_{l-3} \in \Sigma_1$, then $\alpha_{l-4} \notin \Sigma_1$, then write

$$\alpha = (\alpha_i + \cdots + \alpha_{l-4}) + (\alpha_{i+1} + \cdots + \alpha_{l-3}) + (\alpha_{l-3} + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l).$$

So we only need to show

$$<\tilde{\lambda}, (\alpha_{l-3} + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l)^\vee> \geq 0,$$

where $\alpha_{l-3}, \alpha_{l-1}, \alpha_l \in \Sigma_1$. If $\alpha_{l-2}$ is compact, then $<\tilde{\lambda}, \alpha_{l-2}^\vee> \geq 2$. (3) holds. If $\alpha_{l-2}$ is noncompact, then $\alpha_{l-3} + \alpha_{l-2}$ and $\alpha_{l-1} + \alpha_{l-2}$ are compact, hence

$$<\tilde{\lambda}, \alpha^\vee> = <\tilde{\lambda}, (\alpha_{l-3} + \alpha_{l-2})^\vee> + <\alpha_{l-1} + \alpha_{l-2})^\vee + \alpha_l^\vee> \geq 1.$$

(3) holds.

2) Let $\alpha \in \Pi_l$. If $\alpha$ is adjacent to $\beta_j \in \Sigma_1$, then $\alpha + \beta_j$ is compact. Then

$$<\lambda, (\alpha + \beta)^\vee> = <\tilde{\lambda}, (\alpha + \beta)^\vee> + \frac{1}{2} \sum c_i(\beta_i, \beta^\vee) + \frac{1}{2} \sum c_i(\beta_i, \alpha^\vee) \geq 1 + c_j - \frac{1}{2} c_j - 1 = \frac{1}{2}.$$

For the first inequality, we use the assumption that $\mu$ is regular. But it contradicts the fact that $\alpha \in \Pi_l$. 

Now let $\alpha, \beta \in \Pi_l$ be adjacent. Then neither $\alpha$ nor $\beta$ is adjacent to elements in $\Sigma_1$. Again the fact that $\alpha + \beta$ is compact implies it is impossible. So the Dynkin diagram of $l$ is discrete. We must have

$$\Pi_l = \Gamma.$$ 

Combining the above results, Theorem 3.9 follows.

4. Dirac cohomology of unitary representations with regular lambda-lowest $K$-types.

In this section, we will consider the simple group $G$ of types $AIII$ ($SU(p,q)$), $BI$ ($BI$) ($SO_0(p,q)$, $p+q$ odd), $CII$ ($Sp(p,q)$), $DI$ ($SO_0(p,q)$, $p$ and $q$ even), $DIII$ ($SO^*(2n)$), that is all the classical groups except $CI$ ($Sp(n,\mathbb{R})$) with rank $G = \text{rank} K$. Also we will make the following assumption:

**Assumption 4.1.** $\mu$ is regular for $\Delta(g, t^c)$.

**4.1. The dominance of $\mu - \rho_n$.** Since we assume $\mu$ is regular for $\Delta(g, t^c)$, we can choose the following positive root system for $\Delta(g, t^c)$: $\alpha \in \Delta^+(g, t^c)$ if $(\mu + 2\rho_c, \alpha^\vee) > 0$ or $(\mu + 2\rho_c, \alpha^\vee) = 0$ and $(\mu, \alpha^\vee) > 0$. Let $\rho_n = \rho - \rho_c$.

Since $\mu \in K$ is a lambda-lowest $K$-type, then the associate fine $L \cap K$-type with respect to $L$ is $\mu^L = \mu - 2\rho(u \cap p)$. Since $\Delta^+(l, t^c) = \Pi_l$ consists of noncompact imaginary roots, we have $\Delta^+(l, t^c) \subset \Delta^+(p, t^c)$, hence

$$\Delta^+(p, t^c) = \Delta^+(l, t^c) \cup \Delta(u, t^c).$$

So $\rho(u \cap p) = \rho_n - \rho_l = \rho - \rho_c - \rho_l$. Consequently,

$$\mu^L = \mu - 2\rho(u \cap p)$$

$$= \mu + 2\rho_c - 2(\rho - \rho_l).$$

So

$$(\mu^L, \beta_i^\vee) = 1 - c_i$$

for $\beta_i \in \Pi_l$.

**Proposition 4.2.** $\mu - \rho_n$ is dominant for $\Delta^+(l, t^c)$.

We have to deal with it case by case.

**Proof of the case AIII.** We just need to prove $\bar{\lambda}$ is strictly dominant for $\Delta(t, t^c)$, that is,

$$\bar{\lambda}(\alpha^\vee) \geq 1$$

for any $\alpha$ compact. Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be the simple roots of $\Delta^+(g, t^c)$. Then the system of simple roots $\Pi_t$ of $\Delta^+(l, t^c)$ consists of two kinds of elements

$$\Pi_t = \Pi_c \cup \Pi_n.$$
\( \Pi_c \) consists of those compact simple roots of \( \Pi \). Elements of \( \Pi_n \) are of the form
\[
\alpha = \alpha_i + \cdots + \alpha_k,
\]
where \( \alpha_i \) and \( \alpha_k \) are noncompact and \( \alpha_{i+1}, \ldots, \alpha_{k-1} \) are compact. If \( \alpha \in \Pi_c \) then
\[
(\lambda, \alpha^\vee) \geq 2.
\]
If \( \alpha \in \Pi_n \), we treat it case by case.

**Case I.** \( \alpha = \alpha_i + \alpha_i+1 \), where \( \alpha_i \) and \( \alpha_i+1 \) are noncompact. Then
\[
(\tilde{\lambda}, \alpha^\vee) \geq 1 + 2 - 2 = 1.
\]

**Case II.** \( \alpha = \alpha_i + \cdots + \alpha_k \), where \( k > i \). In this case \( \alpha_{i+1} \) and \( \alpha_{i+2} \) are compact. Then
\[
(\tilde{\lambda}, \alpha^\vee) = (\tilde{\lambda}, (\alpha_i + \alpha_{i+1})^\vee) + (\tilde{\lambda}, (\alpha_i+1 + \cdots + \alpha_k)^\vee) \geq 1.
\]

**Case III.** \( \alpha = \alpha_i + \cdots + \alpha_k + 2(\alpha_k+1 + \cdots + \alpha_l) \), where \( \alpha_i \), \( \alpha_k \) and \( \alpha_{k+1} \) are noncompact and the others are compact.
Since $\alpha_i + \cdots + \alpha_k \in \Pi_\ell$, we need only to prove
\[ (\bar{\lambda}, (\alpha_{k+1} + \cdots + \alpha_l)^\vee) \geq 0, \]
which is obvious thanks to Lemma 3.17.

Proof of the case CII. Since $\mathfrak{g}$ is of type CII, $\mathfrak{k}$ has no center and $\alpha_l$ must be a compact root. Let $\alpha \in \Pi_\ell$ be simple. Then $\alpha$ must be one of the forms in the following cases:

Case I. Similar to type $B_l$.

Case II. $\alpha = \alpha_i + \cdots + \alpha_k$, where $k < l$. Similar to type $B_l$.

Case III. $\alpha = 2(\alpha_i + \cdots + \alpha_{l-1}) + \alpha_l$, where only $\alpha_i$ is noncompact.

Since $(\bar{\lambda}, \alpha_i^\vee) = (\bar{\lambda}, \alpha_i^\vee + \cdots + \alpha_l^\vee)$, the conclusion is clear.

Proof of the case $D_l$. Let $\alpha \in \Pi_\ell$ be simple. Then $\alpha$ must be one of the forms in the following cases:

Case I. Similar to type $B_l$.

Case II. $\alpha = \alpha_i + \cdots + \alpha_k$ ($k \leq l - 2$), $\alpha = \alpha_i + \cdots + \alpha_{l-1}$ or $\alpha = \alpha_i + \cdots + \alpha_{l-2} + \alpha_l$ or $\alpha_{l-1} + \alpha_{l-2} + \alpha_l$. Still similar to type $B_l$.

Case III. $\alpha = \alpha_i + \cdots + \alpha_l$, where $\alpha_i, \alpha_{l-1}, \alpha_{l-2}$ and $\alpha_l$ are noncompact and others are compact.

The only hard case is that $i = l - 3$. Since at least one of $\alpha_{l-3}, \alpha_{l-1}$ and $\alpha_l$ is not in $\Sigma_1$, all the simple root of $\mathfrak{k}$ is one of the three forms:

1) $\alpha_i \in \Pi_1$, $i < l - 3$.

2) $\alpha_i + \cdots + \alpha_k, k < l - 2$.

3) $\alpha_{l-3} + \alpha_{l-2}, \alpha_{l-2} + \alpha_{l-1}, \alpha_{l-1} + \alpha_l$ or $\alpha_{l-1} + \alpha_{l-2} + \alpha_{l-1} + \alpha_l$.

In this case $\mathfrak{k}$ is a sum of two simple Lie algebras of type $D_l$, say $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\Pi_\ell = \Pi_{\ell_1} \cup \Pi_{\ell_2}$. One can easily see $\alpha_{l-3} + \alpha_{l-2}$ and $\alpha_{l-3} + \alpha_{l-2} + \alpha_{l-1} + \alpha_l$ belong to the same subsystem, say $\Pi_{\ell_1}$, while $\alpha_{l-3} + \alpha_{l-1}, \alpha_{l-2} + \alpha_l \in \Pi_{\ell_2}$. And they play the role of $\alpha_{l-1}$ and $\alpha_l$. One can easily see that
\[ (\rho_c, \alpha_{l-1}^\vee) = (\rho_c, \alpha_l^\vee) = 0. \]

Then $(\mu, \alpha_{l-1}^\vee) > 0$ and $(\mu, \alpha_l^\vee) > 0$ and $(\rho_c, \alpha_{l-2}^\vee) = 1$. If $\mu$ is an integral weight, then $\alpha_{l-1}, \alpha_l \notin \Sigma_1$. The assertion follows.

Case IV. $\alpha = \alpha_i + \cdots + \alpha_k + 2(\alpha_{k+1} + \cdots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l$, where $\alpha_i + \cdots + \alpha_k (k \leq l - 2)$ is a compact root in case II and $\alpha_{k+1}, \alpha_{l-1}$ and $\alpha_l$ are noncompact. We can easily get the $(\mu - \rho_n, \alpha^\vee) \leq 0$. \qed
4.2. The dominance of $\Lambda$. Let $\Lambda = \lambda + \frac{1}{2} \sum c_i \beta_i = \tilde{\lambda} + \sum c_i \beta_i$. Then we have:

**Proposition 4.3.** $\Lambda$ is dominant for $\Delta^+(g, t^c)$.

**Proof.** Let $\sigma_i \in W(g, t^c)$ be the reflection with respect to $\beta_i \in \Gamma$. Set $\Delta' = \{ \alpha \in \Delta^+(g, t^c) | \alpha \notin \Gamma \}$. Then $\Delta'$ is stable under each $\sigma_i$ and their product $\sigma = \sigma_1 \ldots \sigma_r$.

$\Lambda = \sigma(\tilde{\lambda})$ is dominant for $\Delta'$ if and only if $\tilde{\lambda}$ is dominant for $\Delta'$. The assertion follows by the following lemma. □

**Lemma 4.4.** $\tilde{\lambda}$ is dominant for $\Delta'$.

**Proof.** In the above subsection, we have proved that $\tilde{\lambda}$ is strictly dominant for $\Delta^+(k, t^c)$, so the only left is to check our assertion for noncompact roots in $\Delta'$.

Let $\alpha \in \Delta'$ be noncompact. If $\alpha$ is not adjacent to any element in $\Sigma_1$, then

$$(\tilde{\lambda}, \alpha^\vee) = (\lambda, \alpha^\vee) > 0.$$

Now assume $\alpha$ is adjacent to $\beta \in \Sigma_1$. If $\alpha + \beta$ is a root, then it is compact. So we have

$$(\tilde{\lambda}, (\alpha + \beta)^\vee) \geq 1.$$

One can easily get $(\tilde{\lambda}, \alpha^\vee) > 0$. If $\gamma = \alpha - \beta$ is a root then

$$(\tilde{\lambda}, \alpha^\vee) = (\tilde{\lambda}, (\gamma + \beta)^\vee).$$

Also we have $(\tilde{\lambda}, \alpha^\vee) \geq 0$ if $(\gamma, \gamma) \geq (\beta, \beta)$ or $\gamma$ is not a simple compact root. So we just need to consider the case that $(\gamma, \gamma) < (\beta, \beta)$ and $\gamma$ is a simple compact root. Obviously, $g$ is of type $B_l$ and $\gamma = \alpha_i + \cdots + \alpha_l$. According to the proof in Chapter 3, the assertion follows. □

4.3. The representations of $L$. Let $L_1$ be the commutator subgroup of $L$. It is a connected semisimple Lie group by [8, Lemma 4.3.4]. Then $L = TL_1$ (see [8, Lemma 0.4.2]) and $T_1 = T \cap L_1$ is a finite product of $\mathbb{Z}_2$.

Let $(\delta, V) \in \tilde{T}$ and $\delta_1 = \delta|_{T_1}$. Then $(\delta_1, V) \in \tilde{T}$. Define $(\pi, \mathcal{H}) = \pi(P, \delta \otimes \nu)$ and $(\pi_1, \mathcal{H}_1) = \pi_1(P_1, \delta \otimes \nu)$, where $P = TAN$, $P_1 = T_1AN$ and $\nu \in \hat{A}$.

**Lemma 4.5.** $\pi|_{L_1} \cong \pi_1$ as representations of $L_1$. Consequently, $\pi$ is irreducible (resp. unitary) if and only if $\pi_1$ is irreducible (resp. unitary).

Locally, $L_1$ is a product of some copies of $SL(2, \mathbb{R})$, i.e., there exists a canonical covering map:

$$p : \tilde{L}_1 = SL(2, \mathbb{R}) \times \cdots \times SL(2, \mathbb{R}) \to L_1$$

with finite kernel $Z$. Then $\pi_1$ can be regarded as a representation of $\tilde{L}$ with $Z$ acting trivially. Let $\tilde{T} = p^{-1}(T_1)$. Then $\delta_1$ can be regarded as a
representation of $\widetilde{T}_1$. Let $\pi = \pi(P, \delta_1 \otimes \nu)$, which is equivalent with $\pi_1$ as representations of $L_1$. Obviously, $\pi_1$ is a tensor product of representations of $SL(2, \mathbb{R})$. Then $\pi_1$ is unitary (irreducible, resp.) if and only if every component of the tensor product is unitary (irreducible, resp.). We can easily get the unitaribility and irreducibility of representations of $L_1$ since the representations of $SL(2, \mathbb{R})$ is so clear. Let us recall the following:

**Theorem 4.6** ([4], Theorem 16.3). *The only irreducible unitary representations of $SL(2, \mathbb{R})$ up to unitary equivalence are:

a) The trivial representation;
b) the discrete series $D^\pm_n$, $n \geq 2$, and the limits of discrete series $D^\pm_1$;
c) the irreducible members of the unitary principal series, $P^{\pm,iy}$ with $y$ real and $P^{-,iy}$ with $y$ nonzero real;
d) the complementary series $\varphi^x$ with $0 < x < 1$.

Moreover the only equivalences among these representations are $P^{+,iy} \cong P^{+,iy}$ and $P^{-,iy} \cong P^{-,iy}$.

The fine representation $\mu^0$ (see [8], Corollary 5.4.7) corresponding to $\mu$ is just $\mu^0 = \mu - 2\rho(u \cap p) = \mu - 2\rho(p) + 2\rho(l) = (\mu + 2\rho_c) - 2(\rho - \rho(l))$. Then we have

$$(\mu^0, \beta_i^\nu) = 1 - c_i,$$

that is, $\mu^0$ is weight 0 of those $l(\beta_i)$ for $\beta_i \in \Gamma_1$ (Here $l(\beta_i)$ is the TDS generated by $\beta_i$) and weight 1 of those $l(\beta_i)$ for $\beta_i \in \Gamma_0$. Consequently $L(\beta_i) \cong SL(2, \mathbb{R})$ since $L(\beta_i)$ is either $SL(2, \mathbb{R})$ or $PSL(2, \mathbb{R})$, but the representations of the latter has no odd weight.

4.4. **Proof of Theorem 1.2.** Let $X$ be an irreducible $(g, K)$-module with lambda-lowest $K$-type $\mu$ satisfying $\mu$ is regular for $\Delta(g, K)$. By the discussion above, we have known the following facts:

1) $\lambda = \mu + 2\rho_c - \rho + \frac{1}{2} \sum_{\beta_i \in \Gamma_1} \beta_i$. Let $q = l + u$ be the parabolic associated to $\mu$. Then the Dynkin diagram of $l$ is discrete.

2) $\mu - \rho_n$ is dominant for $\Delta^+(t, K')$.

3) $\Lambda = (\lambda, \frac{1}{2} \sum_{\beta_i \in \Gamma_1} \beta_i)$ is dominant for $\Delta^+(g, K)$.

Now we assume the $\theta$-stable data corresponding to $X$ is $(q, H, \delta, \nu)$, where $\nu = \frac{1}{2} \sum_{\beta_i \in \Gamma_1} \beta_i$. Consider the standard $(g, K)$-module:

$$\mathcal{R}^S(X_L(P, \delta \otimes \nu)).$$

By Theorem 6.5.12 [8], $X \cong \mathcal{R}^S(X_L(P, \delta \otimes \nu))(\mu)$ and a canonical cohomology class is $Y = \widetilde{X}_L(P, \delta \otimes \nu)(\mu - 2\rho(u \cap p))$, which is unitary as one can easily see. $\mathcal{R}^S(Y)$ is a submodule of $\mathcal{R}^S(X_L(P, \delta \otimes \nu))$. We have $X \subseteq \mathcal{R}^S(Y)$ since they have the same lambda-lowest $K$-type $\mu$. Since $(\Lambda, \alpha^\nu) \geq 0$, then
\( R^S(Y) \) is unitary and irreducible (it is nonzero since it contains \( X \)), hence
\[ X = R^S(Y) \] is unitary. By Dirac inequality, we have
\[ (\omega(\mu' - \rho'_n) + \rho_c, \omega(\mu' - \rho'_n) + \rho_c) \geq (\Lambda, \Lambda), \]
for all \( K \)-type \( \mu' \) of \( X \), all \( \rho'_n \) and for some \( \omega \in W_K \). Note that \( \mu - \rho_n \) is
dominant for \( \Delta^+(t, t^c) \) and
\[ \mu - \rho_n + \rho_c = \Lambda, \]
we have the equality holds. Using \( t\nu \), \( 0 < t < 1 \), instead of \( \nu \), one can see
that \( X_L(\delta \otimes t\nu) \) is unitary since it is a tensor product of complementary
series and discrete series of \( SL(2, \mathbb{R}) \). Let \( \Lambda_t = (\lambda, t\nu) \). Then \( \Lambda_0 = \lambda \) and
\( \Lambda_t = (1 - t)\Lambda_0 + t\Lambda_1 \). Hence \( (\Lambda_t, \alpha^\vee) = (1 - t)(\lambda, \alpha^\vee) + t(\lambda, \alpha^\vee) > 0 \), for all \( \alpha \in \Delta(u) \). Then by Theorem 1.3 [9], we have
\[ R^S(X_L(\delta \otimes t\nu)) \]
is unitary. So we have
\[ (\omega(\mu' - \rho'_n) + \rho_c, \omega(\mu' - \rho'_n) + \rho_c) \geq (\Lambda_t, \Lambda_t), \]
for all \( t \in (0, 1) \) by Dirac inequality. Since all the \( K \) types of \( X_G(q, H, \delta, \nu) \)
are independent of the choice of \( \nu \), when \( t \) tends to 1, we get
\[ (\omega(\mu' - \rho'_n) + \rho_c, \omega(\mu' - \rho'_n) + \rho_c) \geq (\Lambda, \Lambda) \]
which implies \( (\mu - \rho_n + \rho_c, \mu - \rho_n + \rho_c) = (\Lambda, \Lambda) \), hence \( X \) has nonzero Dirac cohomology.

Conversely, if \( X \) has nonzero Dirac cohomology, then the infinitesimal
character of \( X \) is \( \mu - \rho_n + \rho_c = (\lambda, \nu) \) by the same argument. One can easily
show that \( \nu = \frac{1}{2} \sum_{\beta_i \in \Gamma_1} \beta_i \).

References


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HILBERT SPACES OF TENSOR-VALUED HOLOMORPHIC FUNCTIONS ON THE UNIT BALL OF $\mathbb{C}^n$

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We study expansion of reproducing kernels for Hilbert spaces of holomorphic functions on the unit ball in $\mathbb{C}^n$ with values in the antisymmetric tensor of the tangent and the cotangent spaces. As an application we find the composition series for the analytic continuation of certain families of holomorphic discrete series.

1. Introduction.

The expansion of reproducing kernels of Bergman spaces of holomorphic functions on a domain $D$ in $\mathbb{C}^n$ is of considerable interests. If the domain admits an action of a compact group $K$, then naturally one would like to decompose the space of polynomials into irreducible subspaces of $K$ and expand the reproducing kernels in terms of the reproducing kernels of the finite dimensional subspaces. In [5] Hua found the expansion of the Bergman reproducing kernels on classical bounded symmetric domains. In that case the domain is a Hermitian symmetric space $D = G/K$, and the Bergman space forms also a unitary representation of the group $G$. Faraut and Koranyi [3] have recently found the expansion of weighted Bergman reproducing kernels on a general bounded symmetric domain and gave several implications. Since the appearance of that paper there have been more interests in the problem of finding the expansions.

In the previous papers [6] and [7] the authors studied the expansions of reproducing kernels for spaces of vector-valued holomorphic functions on classical matrix tube domains $U(n,n)/(U(n) \times U(n))$, $Sp(n,\mathbb{R})/U(n)$ and $O^*(2n)/U(n)$ (for even integers $n$). The tangent spaces of the domains are $\mathbb{C}^n \otimes \mathbb{C}^n$, $\mathbb{C}^n \circ \mathbb{C}^n$ and $\mathbb{C}^n \wedge \mathbb{C}^n$ respectively. The holomorphic functions considered there take values in the space $\mathbb{C}^n$, which roughly speaking corresponds to the half of the tangent bundle, or the spin bundle. In this paper we take the simplest non-tube type domain, namely the unit ball $B^n$ in $\mathbb{C}^n$, and consider $G$-invariant Hilbert spaces of holomorphic functions with value in the antisymmetric $p$-tensor $\wedge^p V$ and $q$-tensor $\wedge^q V$ of the the cotangent space $V'$ and respectively tangent space $V$; they are viewed as holomorphic sections of the respective bundles tensored with a line bundle.
The spaces of cotangent vector-valued holomorphic functions appear naturally in the study of the harmonic analysis on those domains, in particular in Hodge theory. Also, if we consider the analytic continuation of a scalar discrete series on a non-tube domain, the unitary quotients in the composition series are equivalent to space of vector-valued holomorphic functions; see Remark 3.5. The other spaces that we consider in this paper are those of tangent vector-valued holomorphic functions. Some of those spaces can be realized as a relative discrete series of a weighted $L^2$-space on the unit ball $B^n$, cf. [13] and Remark 4.1. We find in this paper the expansion of the reproducing kernels of those spaces. As a direct consequence of our result we get the composition series for the analytic continuation of the holomorphic discrete series; for a general bounded symmetric domain, the composition series is still not known and quite a mystery. See also [4] for a detailed study of composition series of (scalar-valued) principal series representations of some classical groups.

We remark that the problem of finding the expansion of matrix-valued reproducing kernels has also been addressed in mathematical physics [10]. Furthermore, those spaces also provide some nontrivial and new examples of Hilbert modules over the function algebra $H^\infty(B^n)$, and might be of independent interests; see [1] and [2].

As it is pointed in [12] our question is also related to Capelli type and to the invariant differential operators on vector-valued functions on the symmetric domain; see further [11] Proposition 2.2. More precisely there corresponds to each $K$-invariant (matrix-valued) polynomial an invariant differential operators on vector-valued functions on the unit ball $B^n$. There are various open questions [11] concerning invariant differential operators on vector-valued functions on general bounded symmetric domains. We hope that our result will shed some light in understanding those problems.

2. Preliminaries.

Let $V = \mathbb{C}^n$ be the complex vector space equipped with the standard Hermitian inner product $(z, w)$. Let $B = B^n$ be the unit ball in $V$. The vectors in $V$ will be written as column vectors. Let $G = U(n, 1)$ be the Lie group of linear transformations of $\mathbb{C}^{n+1}$ that are isometric with respect to the sesquilinear form $|z_1|^2 + \cdots + |z_n|^2 - |z_{n+1}|^2$. We write any element of $G$ as a block matrix \[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \] under the decomposition of $\mathbb{C}^{n+1} = \mathbb{C}^n + \mathbb{C}$. It acts on the unit ball $B^n$ as biholomorphic mappings:

\[ g(z) = (az + b)(cz + d)^{-1}, \quad g \in G, \ z \in B^n. \]

$B^n$ is thus a symmetric space, $B^n = G/K$, where $K = U(n) \times U(1)$ is a maximal compact subgroup of $G$. 


Let \( g_0 \) and \( k_0 \) be the Lie algebra of \( G \) and \( K \) respectively. We denote by \( g \) and \( k \) their complexifications. Then \( g = \mathfrak{gl}(n+1, \mathbb{C}) \), the Lie algebra of the complex \((n+1) \times (n+1)\) complex matrices. Denote \( E_{j,k} \) the matrix with 1 in the \((j,k)\)-entry and 0 in the rest, and \( E_j = E_{j,j} \). We choose a Cartan algebra \( \mathfrak{h} \) to be the space of all diagonal matrices with basis \( \{E_{j,j} \mid j = 1, \ldots, n+1\} \), and let \( \{\varepsilon_j, j = 1, \ldots, n+1\} \) be the dual basis of \( \mathfrak{h}' \), \( \varepsilon_j(E_k) = \delta_{j,k} \). The roots of \((g, h)\) are then \( \varepsilon_j - \varepsilon_k, j \neq k \), and \( j, k = 1, 2, \ldots, n+1 \). We choose an ordering of the roots by requiring \( \varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_{n+1} \). Note that the compact roots are \( \varepsilon_j - \varepsilon_k \) for \( j, k = 1, \ldots, n \).

The group \( K \) acts on \( V = \mathbb{C}^n \) via the defining action. We let \( e_1, \ldots, e_n \) be the standard basis vectors of \( V \). Let \( V' \) be the dual space of \( V \) with the dual basis vectors \( e'_1, \ldots, e'_n \), with the coadjoint action of \( K \). \( V' \) will be realized as the space of row vectors. When viewed as holomorphic tangent vectors and forms we have \( e_j = \frac{\partial}{\partial z_j} \) and \( e'_j = dz_j, j = 1, \ldots, n \). The representations of \( K \) on \( V \) and \( V' \) are of highest weights \( \varepsilon_1 \) and \( -\varepsilon_n \), with highest weight vectors \( e_1 \) and \( e'_n \), respectively.

Denote \( P = P(B^n) \) the space of all holomorphic polynomials on \( B^n \). There is a natural \( K \)-invariant Hermitian inner product on \( P \) whose completion is the so-called Fock space, namely

\[
(p, q)_{\mathcal{F}} = p(\overline{\partial} q^*(z))|_{z=0},
\]

where the polynomial \( q^* \) is obtained from \( q \) by taking the complex conjugate of the coefficients of the monomials \( z_1^{m_1} \cdots z_n^{m_n} \). As a representation space of \( K \) it is decomposed into

\[
P = \sum_{m=0}^{\infty} P_m
\]

where \( P_m \) is the space of all homogeneous polynomials of degree \( m \), and it has highest weight \(-m\varepsilon_n \) and highest weight vector \( v_{-m\varepsilon_n} = \frac{z^m}{\sqrt{m!}} \). The vector \( v_{-m\varepsilon_n} \) then has norm 1 in the Fock space. With this inner product the whole space \( P \) has reproducing kernel \( e(z, w) \) and the spaces \( P_m \) have \( \frac{(z, w)^m}{m!} \).

If \( W \) is any finite-dimensional vector space with a \( K \)-invariant inner product we can equip as well the space \( P(W) = P \otimes W \) of all \( W \)-valued polynomials on \( B^n \) with the tensor of the Fock norm on \( P \) with the given norm in \( W \).

We introduce now the Bergman operator on \( V \)

\[
B(z, w) = (1 - (z, w))(1 - z \otimes w^*),
\]

where \( z \otimes w^* \) stands for the rank one operator given by \( (z \otimes w^*)(v) = (v, w)z \). Viewed as a matrix it is

\[
B(z, w) = (1 - (z, w))(1 - z w^*),
\]
where as before $w'$ is the transpose of $w$. The Bergman metric at $z \in \mathcal{B}_n$, when we identify the tangent space with $V$, is $(B(z, z)^{-1}u, v)$ for $u, v \in V$.

We note that

\[
B(z, w)^{-1} = (1 - (z, w))^{-1} - 2(1 - (z, w) + z \otimes w^*). \tag{2.1}
\]

We denote $B^t(z, w)$ the dual of $B(z, w)$ acting on the dual space $V'$ of $V$. When acting on vector in $v' \in V'$ it is

\[
B^t(z, w)v' = (1 - (z, w))v'(1 - zw').
\]


For $1 \leq p \leq n$ let $\wedge^p V'$ be the anti-symmetric tensor of $V'$, viewed as the constant holomorphic differential $p$-forms on $\mathcal{B}_n$. The space $\wedge^p V'$ is equipped with a natural Hermitian inner product induced from that of $V'$, so that

\[
(dz_1 \wedge \cdots \wedge dz_p, dz_1 \wedge \cdots \wedge dz_p) = \frac{1}{p!}
\]

Let $dm(z)$ be the Lebesgue measure on $\mathcal{B}_n$ and

\[
d\mu_\nu(z) = (1 - |z|^2)^{\nu - n - 1}dm(z)
\]

be the weighted measure. For $\nu > n - 1$ we consider the space $\mathcal{A}_{\nu, p} = L^2_\nu(\mathcal{B}_n, \wedge^p V', \mu_\nu)$ of $\wedge^p V'$-valued holomorphic functions $f$ on $\mathcal{B}_n$ so that

\[
\|f\|_\nu^2 = \frac{1}{d_\nu} \int_{\mathcal{B}_n} (\wedge^p B(z, z)^t f(z), f(z))d\mu_\nu(z) < \infty
\]

where

\[
d_\nu = \pi^n \frac{\nu}{\nu + p} \frac{\Gamma(\nu + p - n)}{\Gamma(\nu + p)}.
\]

Here and after $\wedge^p B^t(z, z)$ on $\wedge^p V'$ denotes the induced action of $B^t(z, z)$ on $V'$. The inner product $(\wedge^p B(z, z)^t f(z), f(z))$ is the one in the space $\wedge^p V'$. With the above normalization, the constant function $dz_1 \wedge \cdots \wedge dz_p$ has norm 1 in the weighted Bergman space. The computation of the normalization constant is done below.

When $p = n$, the space is the usual weighted Bergman space of scalar holomorphic functions $f(z) = f(z)dz_1 \wedge \cdots \wedge dz_n$ so that

\[
\|f\|_\nu^2 = \frac{1}{d_\nu} \int_{\mathcal{B}_n} |f(z)|^2(1 - |z|^2)^\nu dm(z) < \infty
\]

for $B(z, z)^t dz_1 \wedge \cdots \wedge dz_n = (1 - |z|^2)^{n+1}dz_1 \wedge \cdots \wedge dz_n$.

The group $G$ acts on $\mathcal{A}_{\nu, p}$ via

\[
\pi_\nu(g)f(z) = J_{g^{-1}}(z)^{\nu} (dg^{-1}(z))^t f(g^{-1}z), \tag{3.1}
\]
where \( J_{g^{-1}}(z) \) stands for the complex Jacobian of \( g^{-1} \) and \((dg^{-1}(z))^t : \wedge^p V' \rightarrow \wedge^p V' \) the pull-back of the differential \( dg^{-1}(z) : V \rightarrow V \). The reproducing kernel of the space \( \mathcal{A}_{\nu,p} \) is

\[
(1 - (z, w))^{-\nu} \otimes^p (B^t(z, w))^{-1};
\]

this can easily be obtained by the transformation rule of the kernel under \( G \). To find the expansion of the reproducing kernel we need to find the orthogonal decomposition of the space \( \mathcal{A}_{\nu,p} \) under the compact group \( K \).

The space \( P(\wedge^p V') \) of all \( \wedge^p V' \)-valued holomorphic polynomials is dense in the space \( \mathcal{A}_{\nu,p} \), and as representation space of \( K \) it is

\[
P \otimes \wedge^p V' = \sum_{m=0}^{\infty} P_m \otimes \wedge^p V'.
\]

We can further decompose it into irreducible representations of \( K \). Let \( q = n - p \). Recall that \( P_m \) has highest weight \(-m\varepsilon_n\) and \( \wedge^p V' \) has highest weight \(-\varepsilon_{q+1} - \cdots - \varepsilon_n\). Thus, if \( m \geq 1 \), \( P_m \otimes \wedge^p V' \) is decomposed into a sum of two modules with highest weights \(-m\varepsilon_n - \varepsilon_{q+1} - \cdots - \varepsilon_n\) and \(-m\varepsilon_n - \varepsilon_q - \cdots - \varepsilon_{n-1}\); we denote the two modules by \( V^{-m\varepsilon_n-\varepsilon_{q+1} \cdots -\varepsilon_n} \) and \( V^{-m\varepsilon_n-\varepsilon_q \cdots -\varepsilon_{n-1}} \) respectively. That is

\[
P \otimes \wedge^p V' = \sum_{m=0}^{\infty} V^{-m\varepsilon_n-\varepsilon_{q+1} \cdots -\varepsilon_n} \oplus V^{-m\varepsilon_n-\varepsilon_q \cdots -\varepsilon_{n-1}},
\]

where it is understood that \( V^{-m\varepsilon_n-\varepsilon_q \cdots -\varepsilon_{n-1}} \) for \( m = 0 \) will not appear. Note that for \( p = n \) the second summand does not make sense and only the first summand appears.

This next result is similar to Lemma 1 in [7].

**Lemma 3.1.** The highest weight vectors of the above spaces are given by

\[
v_{-m\varepsilon_n-\varepsilon_{q+1} \cdots -\varepsilon_n} = \sqrt{\frac{p!}{m!}} z_n^m dz_{q+1} \wedge \cdots \wedge dz_n
\]

and

\[
v_{-m\varepsilon_n-\varepsilon_q \cdots -\varepsilon_{n-1}} = \sqrt{\frac{p!}{(m-1)!(m+p)}} \sum_{j=q}^{n} (-1)^{n-j} z_n^{m-1} z_j dz_q \wedge \cdots \wedge dz_j \wedge \cdots \wedge dz_n,
\]

where \( \widehat{dz}_j \) indicates that the term is missing in the wedge product. The two vectors are unit vectors with respect to the Fock norm.

**Proof.** The first formula is clear. We prove that the second vector is annihilated by the positive root vectors. Observe that, for \( j < k \),

\[
E_{j,k} dz_j = -dz_k, \quad E_{j,k} dz_l = 0, \quad l \neq j
\]
and
\begin{equation}
E_{j,k}z_n^{m-1}z_j = -z_n^{m-1}z_k, \ E_{j,k}z_n^{m-1}z_l = 0, \ l \neq j.
\end{equation}
Consider a positive root vector \(E_{j_0,k_0}, \ j_0 < k_0\). If \(j_0 < q\) it annihilates every term in the sum. Let \(q \leq j_0 < k_0\). If \(k_0 \leq n - 1\) we have
\[E_{j_0,k_0}dz_q \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n = 0\]
for \(j \neq k_0\) and
\[E_{j_0,k_0}dz_q \wedge \cdots \wedge \widehat{dz_{k_0}} \wedge \cdots \wedge dz_n = (-1)dz_q \wedge \cdots \wedge dz_{k_0} \wedge \cdots \wedge dz_n\]
with \(dz_{k_0}\) appearing in the \(j_0\)th position. We rewrite it as
\[E_{j_0,k_0}dz_q \wedge \cdots \wedge \widehat{dz_{k_0}} \wedge \cdots \wedge dz_n = (-1)^{k_0-j_0}dz_q \wedge \cdots \wedge dz_{j_0} \wedge \cdots \wedge dz_{k_0} \wedge \cdots \wedge dz_n\]
by moving \(dz_{k_0}\) to the \(k_0\)th (empty) position. Using (3.2) and (3.3) we see that \(E_{j_0,k_0}\) acting on \(v^{m_\varepsilon_n}_{-\varepsilon_q} \cdots \varepsilon_n-1\) is, disregarding the normalization constant factor,
\[-(-1)^{n-j_0}z_n^{m-1}z_{k_0}dz_q \wedge \cdots \wedge \widehat{dz_{j_0}} \wedge \cdots \wedge dz_n + (-1)^{n-k_0}(-1)^{k_0-j_0}z_n^{m-1}dz_{j_0} \wedge \cdots \wedge dz_{k_0} \wedge \cdots \wedge dz_n,\]
which is clearly 0. The case when \(k_0 = n\) can be proved similarly. \(\square\)

For any highest weight \(m_1\varepsilon_1 + \cdots + m_n\varepsilon_n\) appearing in the space \(\mathcal{P}(V')\) we let \(K_{m_1\varepsilon_1 + \cdots + m_n\varepsilon_n}(z, w)\) be the reproducing kernel of the space with respect to the Fock norm. To find the expansion of the reproducing kernel in terms of \(K_{m_1\varepsilon_1 + \cdots + m_n\varepsilon_n}\) we need to calculate the norm of the highest weight vector in the space.

The computation of the norms of the highest weight vectors in the case \(p = 1\) differs somewhat from the case \(p > 1\). We start with \(p = 1\).

**Lemma 3.2.** The norm of \(v^{m_\varepsilon_n}_{-\varepsilon_n}\) and \(v^{m_\varepsilon_n}_{-\varepsilon_n-1}\) in the weighted Bergman space is given by the following:

\[
\|v^{m_\varepsilon_n}_{-\varepsilon_n}\|_{\nu}^2 = \frac{1}{(\nu + 2)_m}, \quad \|v^{m_\varepsilon_n}_{-\varepsilon_n-1}\|_{\nu}^2 = \frac{\nu + 1}{(\nu)_{m+1}}.
\]

**Proof.** Consider first the vector \(v = v^{m_\varepsilon_n}_{-\varepsilon_n}\). We have
\[(B(z, z)'v, v) = (vB(z, z), v) = \frac{1}{m!}(1 - |z|^2)(|z_n|^2 - |z_n+1|^2)\]
The norm then becomes, disregarding temporarily the normalizing constant and \(\frac{1}{m!}\),
\[
\int_B \frac{(|z_n|^2 - |z_n+1|^2)(1 - |z|^2)\nu + 1}{(1 - |z|^2)^{\nu+1}} \frac{dm(z)}{(1 - |z|^2)^{n+1}}.
\]
Recalling that (see e.g., [9], Prop. 1.4.9)
\[
\int_B |z_1| \cdots |z_n| (1 - |z|^2)^\nu \frac{dm(z)}{(1 - |z|^2)^{n+1}} = C_{\nu}^n k_1! \cdots k_n!\]

\[
\int_B |z_1| \cdots |z_n| (1 - |z|^2)^\nu \frac{dm(z)}{(1 - |z|^2)^{n+1}} = C_{\nu}^n k_1! \cdots k_n!\]
with
\[ C_\nu = \pi \frac{\Gamma(\nu - n)}{\Gamma(\nu)}, \]
we find the above integral is
\[ C_{\nu+1} \left( \frac{m!}{(\nu + 1)_m} - \frac{(m + 1)!}{(\nu + 1)_{m+1}} \right) = C_{\nu+1} \frac{m!}{(\nu + 1)_m} \frac{\nu}{\nu + m + 1}. \]

Putting these calculations together we find
\[ \|v\|_\nu^2 = \|v - m\varepsilon_n - \varepsilon_n\|_\nu^2 = \frac{1}{(\nu + 2)_m}. \]

Now let \( v = v_{-m\varepsilon_n - \varepsilon_n-1} = \frac{1}{\sqrt{(m-1)!(m+1)}} \sqrt{w}. \)

The rest is straightforward computation. □

Consequently, we have:

**Theorem 3.3.** The reproducing kernel \( (B^t(z,w))^{-\nu} (1 - (z,w))^{-\nu} \) has the following expansion:
\[
(B^t(z,w))^{-\nu} (1 - (z,w))^{-\nu} = I + \sum_{m=1}^{\infty} \left[ (\nu + 2)_m K_{-m\varepsilon_n - \varepsilon_n}(z,w) + \frac{(\nu)_{m+1}}{\nu + 1} K_{-m\varepsilon_n - \varepsilon_n-1}(z,w) \right].
\]

We can then determine the composition series by reading off the order of the poles of the \( g \)-invariant Hermitian structure on \( \mathcal{P} \otimes V' \) for all parameters \( \nu \).

**Theorem 3.4.** The representation \( \mathcal{P} \otimes V' \) of \( g \) is reducible if and only if \( \nu = 0 \) or \( \nu \leq -2 \) is an integer, in which case we have a composition series
\[ 0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 = \mathcal{P} \otimes V', \]
and the only unitarizable quotient is \( \mathcal{M}_1 / \mathcal{M}_0 = \mathcal{M}_1 \) when \( \nu = 0 \). The \( K \)-types in composition factors are described as follows:
\[ M_1 = V' + \sum_{m \geq 0} V^{-m\varepsilon_n - \varepsilon_n}. \]
if $\nu = 0$;

$$M_1 = V' + V^{-\varepsilon_n - \varepsilon_{n-1}}$$

if $\nu = -2$; and

$$M_1 = V' + \sum_{1 \leq m \leq -(\nu+1)} V^{-m\varepsilon_n - \varepsilon_n} + \sum_{1 \leq m \leq -\nu} V^{-m\varepsilon_n - \varepsilon_{n-1}},$$

if $\nu \leq -3$.

**Remark 3.5.** When $\nu = 0$, the expansion becomes

$$(B^t(z, w))^{-1} = \sum_{m=0}^{\infty} (\nu + 2)_m K_{-m\varepsilon_n - \varepsilon_n}(z, w)$$

the corresponding unitary representation $M_1$ consists of $K$-types $-m\varepsilon_n - \varepsilon_n$, $m = 0, 1, \ldots$, and it realizes the so-called Dirichlet space. Consider the analytic continuation of the scalar holomorphic discrete series with reproducing kernel $(1 - (z, w))^{-\kappa}$ with the corresponding action of $G$. When $\kappa = 0$, the space $\mathcal{P}$ of all polynomials is reducible and the space $\mathbb{C}$ of constant function is a submodule, considered as representations of $g^\mathbb{C}$. The quotient $\mathcal{P}/\mathbb{C}$ is also unitarizable and is equivalent to our module $M_1$ with $\nu = 0$; so the quotient is no longer a continuation of the scalar discrete series. The intertwining operator from the quotient $\mathcal{P}/\mathbb{C}$ onto $M_1$ is $f \mapsto \partial f$. We remark that Theorem 5.4 in [3] gives the equivalence of the top quotient of a composition series at the reducible point in analytic continuation of a scalar discrete series, with another highest weight module of the same series. However, the theorem, as it stands there, holds only for tube domains. For non-tube domains the quotients there is equivalent to an analytic continuation of the vector-valued discrete series, since the highest weight given in that theorem is not one-dimensional; the case of the unit ball is as explained here. One of the authors (GZ) pointed this out to Professor Jacques Faraut in Mittag-Leffler institute in May 1996, who immediately realized that the theorem should be modified for non-tube domains.

**Remark 3.6.** When $\nu = -2$, the expansion reads as follows:

$$\begin{equation}
(B^t(z, w))^{-1}(1 - |z|^2)^2 = I - 2K_{-\varepsilon_n - \varepsilon_{n-1}}(z, z).
\end{equation}$$

The representation $-\varepsilon_n - \varepsilon_{n-1}$ is the space of anti-symmetric tensor of $V'$ with $V'$, and as polynomials it has orthogonal basis $z_ie'_j - zje'_i$, $i < j$. One can easily compute the reproducing kernel and finds that

$$K_{-\varepsilon_n - \varepsilon_{n-1}} = \frac{1}{2}(|z|^2 - z \otimes z^*)^t.$$

On the other hand, by (2.1) we have

$$\begin{equation}
(B^t(z, w))^{-1}(1 - |z|^2)^2 = (1 - |z|^2 + z \otimes z^*)^t = 1 - (|z|^2 + z \otimes z^*)^t,
\end{equation}$$

which clearly coincides with (3.5) and (3.6).
Consider now \( p > 1 \). Let \( v = v_{-m\varepsilon_n-\varepsilon_q+1-\cdots-\varepsilon_n} \) and write \( v = \sqrt{\frac{p}{m}} w \). We first compute \( B(z, z)w, \)
\[ \otimes^p B(z, z)^i w = (1 - |z|^2)^p z^m \left( dz_{q+1} \wedge \cdots \wedge d z_n - \sum_{j=q+1}^n dz_{q+1} \wedge \cdots \wedge z_j z^* \wedge \cdots \wedge d z_n \right), \]
where in the summation the term \( z_j z^* \) appears in the \( j \)th position. Indeed,
\[ \otimes^p B(z, z)^i dz_{q+1} \wedge \cdots \wedge dz_n = (1 - |z|^2)^p (dz_{q+1} - z_{q+1} z^*) \wedge \cdots \wedge (dz_n - z_n z^*) \]
\[ = (1 - |z|^2)^p \left( dz_{q+1} \wedge \cdots \wedge dz_n - \sum_{j=q+1}^n dz_{q+1} \wedge \cdots \wedge z_j z^* \wedge \cdots \wedge dz_n \right) \]
since all other wedge products contain two factors of \( z^* \) and are vanishing. Consequently
\[ (\otimes^p B(z, z)^i w, w) = (1 - |z|^2)^p z^m \left( 1 - \sum_{j=q+1}^n |z_j|^2 \right) \]
We can compute its integration, noticing that the integral involving \( |z_n|^{2m} |z_j|^2 \) for \( j = n \) is different from that for \( j \neq n \),
\[ \|v\|^2 = \frac{1}{m!} \frac{1}{d\nu} \int_B (1 - |z|^2)^p \left( |z_n|^{2m} - \sum_{j=q+1}^{n-1} |z_n|^{2m} |z_j|^2 - |z_n|^{2m} |z_n|^2 \right) (1 - |z|^2)^{\nu-n-1} dm(z) \]
\[ = \frac{\pi^n \Gamma(\nu + p - n)}{m! \Gamma(\nu + p)} \left( \frac{m!}{(\nu + p)_m} - (p - 1) \frac{m!}{(\nu + p)_{m+1}} - (m + 1)! \frac{m!}{(\nu + p)_{m+1}} \right) \]
\[ = \frac{1}{d\nu} \Gamma(\nu + p) \left( \frac{1}{(\nu + p)_m} - \frac{1}{(\nu + p)_{m+1}} - \frac{m + 1}{\nu + p + m} \right) \]
\[ = \frac{1}{d\nu} \Gamma(\nu + p) \left( \frac{1}{(\nu + p)_{m+1}} - \frac{m + 1}{\nu + p + m} \right) \]
\[ = \frac{1}{(\nu + p + 1)_m}. \]
Next let $v = v - m\varepsilon_n - \varepsilon_{n-1} - \cdots - \varepsilon_1$ and write $v = \sqrt{\frac{p!}{(m-1)!(m+p)}} w$. We have

$$(B(z, z)^t w, w) = (1 - |z|^2)^p (\otimes^p (1 - z \otimes z^*)^t w, w).$$

However,

$$\otimes^p (1 - z \otimes z^*)^t w$$

$$= \sum_{j=q}^{n} (-1)^{n-j} z_n^{m-1} z_j (dz_q - z_q z^*) \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge (dz_n - z_n z^*)$$

$$= w - \sum_{j=q}^{n} \sum_{i \neq j, i=q}^{n} (-1)^{n-j} z_n^{m-1} z_j dz_q \wedge \cdots \wedge z_i z^* \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n,$$

so that

$$(\otimes^p (1 - z \otimes z^*)^t w, w)$$

$$= (w, w) - \sum_{j=q}^{n} \sum_{i \neq j, i=q}^{n} (-1)^{n-j} z_n^{m-1} z_j z_i$$

$$\cdot (dz_q \wedge \cdots \wedge z^* \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n, w).$$

We prove that the second double sum actually vanishes. Indeed writing $z^* = z_1 dz_1 + \cdots + z_n dz_n$, we see that only the term $z_i dz_i + z_j dz_j$ contributes to the wedge product and then the inner product, namely

$$(dz_q \wedge \cdots \wedge z^* \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n, w)$$

$$= (dz_q \wedge \cdots \wedge (z_i dz_i + z_j dz_j) \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n, w).$$

Furthermore

$$dz_q \wedge \cdots \wedge (z_i dz_i + z_j dz_j) \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n$$

$$= z_i dz_q \wedge \cdots \wedge dz_i \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n$$

$$+ z_j dz_q \wedge \cdots \wedge dz_i \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n$$

$$= z_i dz_q \wedge \cdots \wedge dz_i \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n$$

$$+ z_j (-1)^{j-1} dz_q \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_j \wedge \cdots \wedge dz_n,$$

where in the last term we move $dz_j$ in the $i$th position to the $j$th empty position (leaving the $i$th position empty). The double sum is then
\[
\sum_{j=q}^{n} \sum_{i \neq j, i=q}^{n} (-1)^{n-j} z_n^{m-1} z_j |z_i|^2 (dz_q \wedge \cdots \wedge dz_i \wedge \cdots \wedge dz_j \wedge \cdots \wedge dz_n, w) \\
+ \sum_{j=q}^{n} \sum_{i \neq j, i=q}^{n} z_i \bar{z}_j (-1)^{j-i-1} (dz_q \wedge \cdots \wedge \bar{dz}_i \wedge \cdots \wedge dz_j \wedge \cdots \wedge dz_n, w)
\]
\[
= \sum_{j=q}^{n} \sum_{i \neq j, i=q}^{n} |(-1)^{n-j} z_n^{m-1} z_j|^2 |z_i|^2 \frac{1}{p!} \\
+ \sum_{j=q}^{n} \sum_{i \neq j, i=q}^{n} (-1)^{n-j}(-1)^{j-i-1} z_n^{m-1} z_j z_i \bar{z}_j \bar{z}_n^{m-1} \bar{z}_i \frac{1}{p!}
\]
\[
= 0.
\]

So that only the first term contributes to the integral, and
\[
\|w\|_{\nu}^2 = \frac{1}{d_{\nu}} \int_B (w, w)(1 - |z|^2)^{p+\nu-n-1} dm(z)
\]
\[
= \frac{1}{d_{\nu}} \frac{1}{p!} \int_B \sum_{j=q}^{n} |z_n^{m-1} z_j|^2 (1 - |z|^2)^{p+\nu-n-1} dm(z)
\]
\[
= \frac{1}{d_{\nu}} \frac{1}{p!} \frac{\pi^n \Gamma(p+\nu-n)}{\pi^n \Gamma(p+\nu)} \left( (n-q)(m-1)! + \frac{m!}{(\nu+p)_m} \right)
\]
\[
= \frac{1}{d_{\nu}} \frac{1}{p!} \frac{\pi^n \Gamma(p+\nu-n)}{\pi^n \Gamma(p+\nu)} \frac{(m-1)!}{(\nu+p)_m} (n-q+m).
\]

We find eventually
\[
\|w\|_{\nu}^2 = \frac{1}{\nu(\nu+p+1)_{m-1}}.
\]

We have thus established the following expansion; the case \( p = n \) reduces to the known expansion for the weighted Bergman reproducing kernel \( (1 - (z, w))^{-\nu+n+1} \).

**Theorem 3.7.** Let \( 1 < p \leq n \). The reproducing kernel \( \otimes^p (B^t(z, w))^{-1} (1 - (z, w))^{-\nu} \) has the following expansion:
\[
\otimes^p (B^t(z, w))^{-1} (1 - (z, w))^{-\nu} = I + \sum_{m=1}^{\infty} \left[ (\nu+p+1)_m K_{-m\varepsilon_n - \varepsilon_{q+1} \cdots - \varepsilon_n}(z, w) \right.
\]
\[
+ \left. \nu(\nu+p+1)_{m-1} K_{-m\varepsilon_n - \varepsilon_{q} \cdots \varepsilon_{n-1}}(z, w) \right]
\]

where \( q = n - p \).

Similar to Theorem 3.4 we have:
Theorem 3.8. Let \( 1 < p \leq n \). The representation \( \mathcal{P} \otimes \wedge^p V' \) is reducible if and only if \( \nu = 0 \) or \( \nu \leq -(p+1) \) is an integer, in which case the composition factors \( 0 = M_0 \subset M_1 \subset M_2 = \mathcal{P} \otimes \wedge^p V' \) are described as follows:

\[
M_1 = \sum_{m=0}^{\infty} V^{-m\varepsilon_n - \varepsilon_{q+1} - \cdots - \varepsilon_n}
\]

if \( \nu = 0 \); \[
M_1 = V^{-\varepsilon_{q+1} - \cdots - \varepsilon_n} + V^{-\varepsilon_n - \varepsilon_{q} - \cdots - \varepsilon_n}
\]

if \( \nu = -(p+1) \); and \[
M_1 = \sum_{0 \leq m \leq -\nu - p - 1} V^{-m\varepsilon_n - \varepsilon_{q+1} - \cdots - \varepsilon_n} + \sum_{1 \leq m \leq -\nu - p} V^{-m\varepsilon_n - \varepsilon_{q} - \cdots - \varepsilon_n}
\]

if \( \nu < -(p+1) \).

It is interesting to observe that when \( \nu = 0 \) we have

\[
\otimes^p (B^q(z,w))^{-1} = \sum_{m=0}^{\infty} (\nu + p + 1)_m K_{-m\varepsilon_n - \varepsilon_{q+1} - \cdots - \varepsilon_n} (z,w)
\]

so that there exists a unitarizable submodule of \( \mathcal{P}(\wedge^p V') \) consisting of the \( K \)-types \(-m\varepsilon_n - \varepsilon_{q+1} - \cdots - \varepsilon_n\), for all \( p = 1, \ldots , n \); only when \( p = n \) is that a discrete series, namely the Bergman space.


Let \( 1 \leq q \leq n \). In this section we study the space \( \mathcal{B}_{\nu,q} = L^2_\mu (B^n, \wedge^q V, \mu_\nu) \), \( \nu > 2q + n \), of \( \wedge^q V \)-valued holomorphic functions \( f \) on \( B^n \) so that

\[
\|f\|_\nu^2 = \frac{1}{b_\nu} \int_{B^n} (\wedge^q B(z,z))^{-1} |f(z), f(z)| d\mu_\nu(z) < \infty
\]

and its analytic continuation in \( \nu \). Here

\[
b_\nu = \pi^n \frac{\Gamma(\nu - q - 1 - n)(\nu - 1 - n)}{\Gamma(\nu - q)}.
\]

Notice that if \( q = n \) the space \( \wedge^n V \) is one dimensional and the above space is the weighted Bergman space of holomorphic functions \( f(z) = f(z)e_1 \wedge \cdots \wedge e_n \) so that

\[
\|f\|_\nu^2 = \frac{1}{b_\nu} \int_{B^n} |f(z)|^2 (1 - |z|^2)^{\nu - 2n - 2} d\mu(z) < \infty.
\]

The group \( G \) acts on \( \mathcal{B}_{\nu,q} \) via

\[
\tau_\nu f(z) = J_g^{-1}(z) \frac{\pi^\nu}{\pi^{\nu+1}} (dg^{-1}(z))^{-1} f(g^{-1} z).
\]

The reproducing kernel of \( \mathcal{B}_{\nu,q} \) is then

\[
(1 - (z,w))^{-\nu \wedge B(z,w)}.
\]
Remark 4.1. Let \( q = 1 \). The space \( L^2_\alpha(B^n, V, \mu_\nu) \) appears also naturally in the study of discrete spectrum of Laplace-Beltrami operators on line bundles over the unit ball. Indeed consider the \( L^2 \)-space \( L^2_\alpha(B^n, \mu_\nu) \) on \( B^n \) with respect to the weighted measure \( \mu_\nu \). The group \( G \) acts unitarily on the space and there is an \( G \)-invariant (nonpositive) Laplace-Beltrami operator. It has some discrete spectrum, if \( \nu > n+1 \), the biggest discrete spectrum corresponds to the weighted Bergman space \( L^2_\alpha(B^n, \mu_\nu) \); if \( \nu > n+2 \), the next one, say \( A^\nu_1(B^n) \) gives an invariant subspace consisting of non-holomorphic functions, see [14]. The invariant Cauchy-Riemann operator \( \bar{D} \) is the unitary intertwining operator from \( A^\nu_1(B^n) \) onto \( B^\nu,q = L^2_\alpha(B^n, V, \mu_\nu) \); see [8] and [13].

The space of \( P(\wedge^q V) \) all \( \wedge^q V \)-valued holomorphic polynomials is

\[
P(\wedge^q V) = P \otimes \wedge^q V = \sum_{m=0}^{\infty} P_m \otimes \wedge^q V.
\]

As a representation of \( K \) the space \( \wedge^q V \) is an irreducible representation with highest weight \( \varepsilon_1 + \cdots + \varepsilon_q \). Thus the polynomial space is decomposed further as

\[
P \otimes \wedge^q V = V^{\varepsilon_1 + \cdots + \varepsilon_q} + \sum_{m=1}^{\infty} \left( V^{-m\varepsilon_n+\varepsilon_1+\cdots+\varepsilon_q} + V^{-m\varepsilon_n+\varepsilon_1+\cdots+\varepsilon_{q-1}+\varepsilon_n} \right),
\]

where it is understood that \( V^{-m\varepsilon_n+\varepsilon_1+\cdots+\varepsilon_{q-1}+\varepsilon_n} \) for \( m = 0 \) will not appear.

We give now formulas for the highest weight vectors of the spaces above. For \( q = 1 \) they are proved in Lemma 1, [7], the general case is similar and we omit the proof.

Lemma 4.2. We have the following formulas for the highest weight vectors in the respective spaces:

\[
v_{-m\varepsilon_n+\varepsilon_1+\cdots+\varepsilon_q} = \sqrt{\frac{q!}{m!}} e_1 \wedge \cdots \wedge e_q
\]

and respectively

\[
v_{-m\varepsilon_n+\varepsilon_1+\cdots+\varepsilon_{q-1}+\varepsilon_n} = \frac{\sqrt{q!}}{\sqrt{(m-1)!(m+n-q)}} \sum_{j=q}^{n} z_n^{m-1} z^j e_1 \wedge \cdots \wedge e_{q-1} \wedge e_j.
\]

The two vectors are unit vectors with respect to the Fock norm.

We now compute the norm of the highest weight vectors in the weighted Bergman space. Let \( q = 1 \). Notice that

\[
(B(z, z)^{-1} v_{-m\varepsilon_n+\varepsilon_1}, v_{-m\varepsilon_n+\varepsilon_1}) = \frac{1}{m!} \left( (1 - |z|^2)^{-1} |z_n|^2 + (1 - |z|^2)^{-2} |z_1|^2 |z_n|^2 \right),
\]
its integral against $d\mu_\nu$ is
\[
\frac{1}{m!} b_\nu \left( C_{\nu-1} \frac{m!}{(\nu-1)_m} + C_{\nu-2} \frac{(m+1)!}{(\nu-2)_{m+1}(m+1)!} \right) = \frac{1}{(\nu-1)_m}.
\]

Similarly,
\[
\frac{1}{(m-1)! (m+n+1)} (1-|z|^2)^{-1} |z|^2 |z_n^{m-1}|^2 + (1-|z|^2)^{-2} |z|^4 |z_n^{m-1}|^2.
\]

To calculate its integral against the measure $d\mu_\nu$ we write $|z|^2 = 1 - (1-|z|^2)$, the above is then
\[
\frac{1}{(m-1)! (m+n+1)} \left( |z_n^{m-1}|^2 (1-|z|^2)^{-2} - |z_n^{m-1}|^2 (1-|z|^2)^{-1} \right).
\]

We calculate the integral of (4.2) of the term in the parenthesis, and find
\[
b_\nu^{-1} C_{\nu-2} \frac{(m-1)!}{(\nu-2)_{m-1}} - b_\nu^{-1} C_{\nu-1} \frac{(m-1)!}{(\nu-1)_{m-1}} = \frac{\Gamma(\nu-1) (m-1)! (m+n+1)}{(\nu-1-n) \Gamma(\nu+m-2)}.
\]

Thus the norm of $v_{-m\epsilon_n+\epsilon_n}$ is
\[
\|v_{-m\epsilon_n+\epsilon_n}\|_\nu^2 = \frac{\Gamma(\nu-1)}{\nu-1-n} \frac{1}{\Gamma(\nu+m-2)} = \frac{1}{(\nu-1-n)(\nu-1)_m}.
\]

We have the following expansion for the reproducing kernel:

**Theorem 4.3.** The reproducing kernel $B(z, w)(1-(z, w))^{-\nu}$ has the following expansion:

\[
B(z, w)(1-(z, w))^{-\nu} = I + \sum_{m=1}^\infty [(\nu-1)_m K_{-m\epsilon_n+\epsilon_1} + (\nu-1-n)(\nu-1)_{m-1} K_{-m\epsilon_n+\epsilon_n}].
\]

**Theorem 4.4.** The representation $\mathcal{P} \otimes V$ of $\mathfrak{g}^\mathbb{C}$ is reducible if and only if when $\nu = n+1$ or $\nu \leq 1$ is an integer; it has a composition series

\[
0 = M_0 \subset M_1 \subset M_2 = \mathcal{P} \otimes V;
\]

the composition factor are given as follows:

\[
M_1 = \sum_{m=0}^\infty V_{-m\epsilon_n+\epsilon_1},
\]
if \( \nu = n + 1 \), and the composition factor are given as follows:

\[
M_1 = V^{\varepsilon_1} + \sum_{1 \leq m < 2 - \nu} V^{-m\varepsilon_n + \varepsilon_1} + \sum_{1 \leq m < 3 - \nu} V^{-m\varepsilon_n + \varepsilon_n},
\]

the submodule \( M_1 \) is unitarizable if and only if \( \nu = n + 1 \) and no quotient \( M_2 / M_1 \) of the composition series is unitarizable.

**Remark 4.5.** We mention here that in the analytic continuation for the scalar holomorphic discrete series, the composition series has two factors at the reducible points, and the quotient is always unitarizable. Thus we see here that the unitarizable property behaves differently in the vector case.

**Corollary 4.6.** The Bergman operator has the following expansion:

\[
B(z, z) = I - K_{-\varepsilon_n + \varepsilon_1} - (1 + n)K_0 + (1 + n)K_{-2\varepsilon_n + \varepsilon_n}
\]

where

\[
K_{-\varepsilon_n + \varepsilon_1} = |z|^2 - \frac{1}{n} z \otimes z^*, \quad K_0 = \frac{1}{n} z \otimes z^*, \quad K_{-2\varepsilon_n + \varepsilon_n} = \frac{1}{n+1} |z|^2 z \otimes z^*.
\]

**Remark 4.7.** The above formula has the following nice explanation: Consider the space of \( V \)-valued polynomial space

\[
M_1 = V^{\varepsilon_1} \oplus V^{-\varepsilon_n + \varepsilon_1} \oplus V^0 \oplus V^{-2\varepsilon_n + \varepsilon_n}.
\]

Under the \( G \)-action

\[
g f(z) = g'(z)^{-1} f(g \cdot z)
\]

where \( g \in G \) and \( f \) is in the polynomial space, the space has an invariant Hermitian (non-unitary) form for which \( B(z, z) \) is the reproducing kernel. Corollary 4.6 gives an expansion of \( B(z, z) \) in terms of the reproducing kernels in each irreducible \( K \)-spaces. Actually \( M_1 \) is \( G \)-equivalent to the Lie algebra \( \mathfrak{g}^C \) with the adjoint action of \( G \); and the above decomposition of \( M_1 \) is simply the decomposition of \( \mathfrak{g}^C \)

\[
\mathfrak{g}^C = \mathfrak{p}^+ \oplus \mathfrak{k}^C \oplus \mathfrak{z} \oplus \mathfrak{p}^-
\]

with respect to the action of \( K \). Here \( \mathfrak{z} \) is the center of \( \mathfrak{k}^C \), which is one-dimensional. Clearly this can be done for all the bounded symmetric domain, we have thus given a different property of the Bergman operator.

We consider now the case \( q > 1 \). Let \( v = v_{-m\varepsilon_n + \varepsilon_1 + \cdots + \varepsilon_q} \). Using (2.1) and by similar consideration as in §3 we have

\[
(\otimes^q B(z, z)^{-1} v, v) = \frac{1}{m!} (1 - |z|^2)^{-q-1} |z_n|^{2q} \left( 1 - |z|^2 + \sum_{j=1}^q |z_j|^2 \right).
\]
It is now easy to compute $\|v\|_\nu^2$. If $1 < q < n$, $\|v\|_\nu^2$ is, disregarding the normalization constant,

$$\frac{1}{m!} \int_B (1 - |z|^2)^{-q} |z_n|^2 \left( (1 - |z|^2)^q + (1 - |z|^2)^{q-1} \sum_{j=1}^q |z_j|^2 \right) (1 - |z|^2)^{\nu-n-1} dm(z)$$

$$= \frac{1}{m!} \left( \pi^n \Gamma(\nu - q - n) \frac{m!}{(\nu - q)_m} + q \pi^n \Gamma(\nu - q - 1 - n) \frac{m!}{(\nu - q - 1)_{m+1}} \right)$$

$$= \frac{\pi^n \Gamma(\nu - q - 1 - n)}{\Gamma(\nu - q + m)} (\nu - n),$$

and that

$$\|v\|_\nu^2 = \frac{1}{(\nu - q)_m}. $$

If $q = n$, Formula (4.4) becomes

$$(\otimes^q B(z, z)^{-1} v, v) = \frac{1}{m!} (1 - |z|^2)^{-q-1} |z_n|^2$$

and

$$\|v\|_\nu^2 = \frac{1}{(\nu - n - 1)_m}, $$

which is the known formula for the norm of $\frac{z_n}{\sqrt{m!}}$ in the scalar weighted Bergman space with weight $(1 - |z|^2)^{\nu-2n-2}$.

Finally consider $v = v_{-m \epsilon_n + \epsilon_1 + \cdots + \epsilon_{q-1} + \epsilon_n}$. Write $w = \sum_{j=q}^n z_j e_1 \wedge \cdots \wedge e_{q-1} \wedge e_j$.

$$B(z, z)^{-1} w = (1 - |z|^2)^{-q} \otimes^q (1 - |z|^2 + z \otimes z^*) w$$

$$= (1 - |z|^2)^{-q} z_n^{m-1} w$$

$$+ \sum_{j=q}^n z_j \left( \sum_{i=1}^{q-1} e_1 \wedge \cdots \wedge \overline{e}_i z \wedge \cdots \wedge e_j + e_1 \wedge \cdots \wedge e_{q-1} \wedge \overline{e}_j z \right)$$

where in the summation $\sum_{i=1}^{q-1}$ the factor $\overline{e}_i z$ appears in the $i$th position. Therefore $B(z, z)^{-1} v, v) = \frac{1}{(m-1)!(n+q)} |z_n^{m-1}|^2 (I + II)$, with

$$I = (1 - |z|^2)^{-q} \sum_{j=q}^n |z_j|^2;$$

$$II = q!(1 - |z|^2)^{-q-1} \sum_{j=q}^n \left( z_j \sum_{i=1}^{q-1} e_1 \wedge \cdots \wedge \overline{e}_i z \wedge \cdots \wedge e_j + |z_j|^2 e_1 \wedge \cdots \wedge e_{q-1} \wedge z, w \right).$$
moreover,
\[
\sum_{j=q}^{n} \left( \sum_{i=1}^{q-1} z_{j}e_{1} \wedge \cdots \wedge z_{i} \wedge \cdots \wedge e_{j}, w \right) \\
= \frac{1}{q!} \sum_{j=q}^{n} \sum_{i=1}^{q-1} |z_{j}|^2 |z_{i}|^2 \\
= \frac{1}{q!} \left( \sum_{j=q}^{n} |z_{j}|^2 \right) \left( \sum_{i=1}^{q-1} |z_{i}|^2 \right), \\
\sum_{j=q}^{n} |z_{j}|^2 (e_{1} \wedge \cdots \wedge e_{q-1} \wedge z, w) \\
= \sum_{j=q}^{n} \sum_{k=q}^{n} |z_{j}|^2 (e_{1} \wedge \cdots \wedge e_{q-1} \wedge z, e_{1} \wedge \cdots \wedge e_{q-1} \wedge e_{k}) \\
= \frac{1}{q!} \sum_{j=q}^{n} \sum_{k=q}^{n} |z_{j}|^2 |z_{k}|^2 = \frac{1}{q!} \left( \sum_{j=q}^{n} |z_{j}|^2 \right)^2,
\]
and the summation of the inner products in II is
\[
\frac{1}{q!} \left( \sum_{j=q}^{n} |z_{j}|^2 \right)^2 + \frac{1}{q!} \left( \sum_{j=q}^{n} |z_{j}|^2 \right) \left( \sum_{i=1}^{q-1} |z_{i}|^2 \right) = |z|^2 \sum_{j=q}^{n} |z_{j}|^2.
\]
Eventually we find
\[
(B(z, z)^{-1} v, v) = \frac{1}{(m-1)! (m+n-q)} |z_{n}^{m-1}|^2 \\
\cdot \left( \sum_{j=q}^{n} |z_{j}|^2 \right) \left( (1 - |z|^2)^{-q} + (1 - |z|^2)^{-q-1} |z|^2 \right) \\
= \frac{1}{(m-1)! (m+n-q)} |z_{n}^{m-1}|^2 \left( \sum_{j=q}^{n} |z_{j}|^2 \right) \left( 1 - |z|^2 \right)^{-q-1}
\]
which has a simpler form. The norm $\|v\|_\nu^2$ is given by

$$b_\nu \frac{\pi^n \Gamma(\nu - q - 1 - n)}{\Gamma(\nu - q - 1)} \frac{1}{(m - 1)! (m + n - q)} \frac{(n - q)(m - 1)! + m!}{(\nu - q - 1)_m}$$

$$= b_\nu \frac{\pi^n \Gamma(\nu - q - 1 - n)}{\Gamma(\nu - q - 1)} \frac{1}{(m - 1)! (m + n - q)} \frac{(n - q + m)(m - 1)!}{(\nu - q - 1)_m}$$

$$= \frac{1}{(\nu - 1 - n)(\nu - q)_{m-1}}.$$ 

Thus we have:

**Theorem 4.8.** Suppose $1 < q \leq n$. The following expansion holds:

$$\otimes^q B(z, w)(1 - (z, w))^{-\nu}$$

$$= I + \sum_{m=1}^{\infty} (\nu - q)_m K_{-m\varepsilon_n + \varepsilon_1 + \cdots + \varepsilon_q}$$

$$+ \sum_{m=1}^{\infty} (\nu - 1 - n)(\nu - q)_{m-1} K_{-m\varepsilon_n + \varepsilon_1 + \cdots + \varepsilon_q - 1 + \varepsilon_n}$$

if $q < n$. For $q = n$ we have the known expansion for the weighted Bergman kernel

$$\otimes^n B(z, w)(1 - (z, w))^{-\nu} = (1 - (z, w))^{-(\nu - n - 1)}$$

$$= \sum_{m=0}^{\infty} (\nu - n - 1)_m \frac{1}{m!} (z, w)^m.$$ 

**Theorem 4.9.** Let $1 < q < n$. The representation $\mathcal{P} \otimes \wedge^q V$ is reducible if and only if $\nu = n + 1$ or $\nu \leq q$ is an integer, in which case the composition factors $0 = M_0 \subset M_1 \subset M_2 = \mathcal{P} \otimes \wedge^p V'$ are described as follows:

$$M_1 = \sum_{m=0}^{\infty} V^{-m\varepsilon_n + \varepsilon_1 + \cdots + \varepsilon_q}$$

if $\nu = n + 1$;

$$M_1 = V^{\varepsilon_1 + \cdots + \varepsilon_q} + V^{\varepsilon_1 + \cdots + \varepsilon_q - 1}$$

if $\nu = q$; and

$$M_1 = \sum_{0 \leq m \leq q - \nu} V^{-m\varepsilon_n + \varepsilon_1 + \cdots + \varepsilon_q} + \sum_{1 \leq m \leq q - \nu + 1} V^{-m\varepsilon_n + \varepsilon_1 + \cdots + \varepsilon_q - 1 + \varepsilon_n}$$

if $\nu < q$. 

References


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A SINGULARLY PERTURBED LINEAR EIGENVALUE PROBLEM IN $C^1$ DOMAINS

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Some non-existence result is established for bounded solutions of a Neumann problem on the upper half space. Based on this non-existence result, precise asymptotic behavior is given for the principal eigenvalue of some linear eigenvalue problem in bounded $C^1$ domains, and this answers a question that appeared in Lacey et al, 1998.

1. Introduction.

For any $\gamma > 0$, set

\[ \Lambda(\gamma) = \sup_{u \in H^1(\Omega) \setminus \{0\}} \frac{\gamma \int_{\partial \Omega} u^2 - \int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2}, \]

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with boundary $\partial \Omega$. It is straightforward to show that the supremum of (1) is attained by some positive function $u_\gamma \in H^1(\Omega)$, which is a weak solution of

\[ \Delta u = \Lambda(\gamma) u \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \nu} = \gamma u \quad \text{on} \quad \partial \Omega, \]

where $\nu$ is the outward unit normal vector on $\partial \Omega$; $\nu$ exists a.e. for Lipschitz domains. The goal of this paper is to understand the asymptotic behavior of $\Lambda(\gamma)$ as $\gamma \to \infty$ when $\partial \Omega \in C^1$. Since $\Lambda(\gamma) \to \infty$ when $\gamma \to \infty$, (2) can be viewed as a singularly perturbed linear eigenvalue problem.

The asymptotic behavior of $\Lambda(\gamma)$ was first studied by Lacey, Ockendon and Sabina in [3], where they investigated some reaction-diffusion model in which distributed nonlinear absorption mechanisms compete with nonlinear boundary sources. In order to describe the long time behaviors of solutions to this reaction-diffusion model, it is important to understand the asymptotic behavior of $\Lambda(\gamma)$ as $\gamma \to \infty$ (see [3] and the references therein). Among other things, Lacey, Ockendon and Sabina showed in [3] that

\[ \lim_{\gamma \to \infty} \frac{\Lambda(\gamma)}{\gamma^2} = 1 \]
if ∂Ω is $C^2$ and is differentially equivalent to the unit sphere. On the other hand, when Ω is a planar domain and ∂Ω is piecewise $C^1$, they proved that

$$\liminf_{\gamma \to \infty} \frac{\Lambda(\gamma)}{\gamma^2} \geq \csc^2 \alpha \geq 1,$$

where $\alpha$ is the smallest interior semiangle on $\partial \Omega$. These considerations indicate that the asymptotic behavior of $\Lambda(\gamma)$ is strongly affected by the smoothness of the boundary. In this connection, we prove:

**Theorem 1.1.** (3) holds for any bounded $C^1$ domain.

**Remark 1.1.** Similar result can be established for the problem

$$\Delta u = \Lambda(\gamma)u \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \nu} = \gamma b(x)u \quad \text{on} \quad \partial \Omega.$$  

More precisely, if $b(x) \in C(\partial \Omega)$ is positive somewhere, then

$$\lim_{r \to \infty} \frac{\Lambda(\gamma)}{\gamma^2} = \max_{\partial \Omega} (b_+)^2.$$  

In the following we briefly sketch our approach: Since (2) is a singularly perturbed problem, it is natural to “blow up” $u_\gamma$, the solution of (2), near its maximum which must be attained on $\partial \Omega$ (via the Maximum Principle). That is, straightening out the boundary and rescaling $u_\gamma$ suitably, by passing to the limit we are led to the following Neumann problem on the upper half space:

$$\Delta u = au \quad \text{in} \quad \mathbb{R}^n_+, \quad \frac{\partial u}{\partial x_n} = -u \quad \text{on} \quad \partial \mathbb{R}^n_+,$$

where $a$ is the limit of $\Lambda(\gamma)/\gamma^2$ (subject to a subsequence) as $\gamma \to \infty$. By adequate choice of test function in (1), one can show that $a \geq 1$. On the other hand, using some similar ideas as the sliding method developed in [1], we are able to show:

**Theorem 1.2.** If $a > 1$, (4) has no bounded nontrivial solutions.

By some non-degeneracy result in Section 3, the solution of (4) obtained via the blowup process is indeed nontrivial. Hence, Theorem 1.2 ensures that $a = 1$, which in turn yields Theorem 1.1.

**Remark 1.2.** It turns out that Theorem 1.2 is sharp: For every $a \leq 1$, (4) has bounded nontrivial solutions of the form $w(x')e^{-x_n}$, where $x = (x', x_n)$ and $w$ is a solution of

$$\Delta w = (a - 1)w \quad \text{in} \quad \mathbb{R}^{n-1};$$

Theorem 1.2 also fails without the boundedness condition: (4) has positive (unbounded) solutions of the form $w(x')e^{-x_n}$, where $w$ is a positive solution of (5) for $a > 1$. We refer to [2] for the classification of positive solutions to (5).
The plan of this paper is as follows: Theorem 1.2 is established in Section 2. In Section 3, we first derive the relation between \( \lim_{\gamma \to \infty} \Lambda(\gamma)/\gamma^2 \) and (4), and then use it to complete the proof of Theorem 1.1.

2. The proof of Theorem 1.2.

We prove Theorem 1.2 in this section. Our idea is to construct some super-solution of (4), by employing some similar ideas as the sliding method of Berestycki, Caffarelli and Nirenberg (see, e.g., [1]).

Throughout this section, we assume that \( u(x) \) is a bounded solution to (4) and \( a > 0 \); without loss of generality, we may assume that \( \sup_{\mathbb{R}^n_+} u > 0 \).

Lemma 2.1. \( \sup_{\mathbb{R}^n_+} u = \sup_{\partial \mathbb{R}^n_+} u \).

Proof. Let \( \{x^j\}_{j=1}^\infty \) be a sequence of points with \( u(x^j) \to \sup_{\mathbb{R}^n_+} u \). Denote \( x^j_n \) the last component of \( x^j \). We first show that \( x^j_n \to 0 \) as \( j \to \infty \). If not, then there is a \( \delta > 0 \), such that \( x^j_n > \delta \) (after passing to some subsequence). We consider \( u^j(x) = u(x + x^j) \) for \( |x| \leq \delta \). By standard elliptic estimates, we know that after passing to some subsequence, \( u^j \to u_0 \) in \( C^2(B_{\delta/2}(0)) \), where \( u_0 \) satisfies

\[
\Delta u_0 = au_0 \quad \text{in} \quad B_{\delta/2}(0),
\]

and \( u_0 \) assumes its positive maximum at the origin. This is clearly impossible.

Again, standard elliptic estimates yield that \( |\nabla u| \leq C \) for some positive constant \( C \). Therefore, \( \sup_{\mathbb{R}^n_+} u = \sup_{\partial \mathbb{R}^n_+} u \).

We normalize \( u(x) \) so that \( \sup_{\mathbb{R}^n_+} u = 1 \). By Lemma 2.1,

\[
(6) \quad \sup_{\partial \mathbb{R}^n_+} u = 1.
\]

Define

\[
\Omega_h = \{(x', x_n) \mid x' \in \mathbb{R}^{n-1}, \ 0 < x_n < h\},
\]

\[
\Omega_{h,r} = \{(x', x_n) \mid |x'| < r, \ 0 < x_n < h\},
\]

\[
\Gamma_{1,h,r} = \partial \Omega_{h,r} \cap \{x_n > 0\},
\]

\[
\Gamma_{2,h,r} = \partial \Omega_{h,r} \cap \{x_n = 0\}.
\]

We first state a lemma concerning the sub- and super-solution method. Consider

\[
(7) \quad \begin{align*}
-\Delta u &= f(x, u) &\text{in} &\quad \Omega_{h,r}, \\
u &= g(x) &\text{on} &\quad \Gamma_{1,h,r}, \\
\frac{\partial u}{\partial \nu} &= h(x, u) &\text{on} &\quad \Gamma_{2,h,r},
\end{align*}
\]
where \( f(x, u), h(x, u) \) are Carathéodory functions, \( g(x) \) is continuous.

A function \( u \in H^1(\Omega_{h, r}) \) is called a sub-solution to (7) if \( u \leq g(x) \) on \( \Gamma_{1, h, r} \) and
\[
\int_{\Omega_{h, r}} \left[ \nabla u \nabla \eta - f(x, u) \eta \right] - \int_{\Gamma_{2, h, r}} h(x, u) \eta \leq 0,
\]
for all \( \eta \in C^\infty(\Omega_{h, r}), \eta \geq 0, \) and \( \eta = 0 \) on \( \Gamma_{1, h, r} \). Similarly \( u \in H^1(\Omega_{h, r}) \) is called a super-solution to (7) if the above inequalities are reversed. We refer the proof of the following result to [5]:

**Lemma 2.2.** Suppose that \( \bar{u}, u \in H^1(\Omega_{h, r}) \) are both bounded, \( \bar{u} \geq u \), and they are super-solution, sub-solution of (7), respectively. Then there is a solution \( u \in H^1(\Omega_{h, r}) \) to (7) such that \( \bar{u} \leq u \leq u \) holds a.e. in \( \Omega_{h, r} \).

We now apply Lemma 2.2 to construct a super-solution of (4).

**Lemma 2.3.** For any fixed \( h, r > 0 \), there is a unique solution to
\[
\begin{align*}
\Delta \psi &= a \psi & \text{in } \Omega_{h, r}, \\
\psi &= 1 & \text{on } \Gamma_{1, h, r}, \\
\frac{\partial \psi}{\partial \nu} &= 1 & \text{on } \Gamma_{2, h, r}.
\end{align*}
\]

(8)

**Proof.** It is easy to check that \( u = 0 \) is a sub-solution and \( \bar{u} = 1 + h - x_n \) is a super-solution to (8). The existence follows from Lemma 2.2, whereas the uniqueness follows from the Maximum Principle.

**Lemma 2.4.** Let \( u(x) \) be a solution to (4) and \( \psi_{h, r} \) be the unique solution to (8). Then \( \psi_{h, r} \geq u \) in \( \Omega_{h, r} \).

**Proof.** Set \( w = \psi_{h, r} - u \). Then \( w \) satisfies
\[
\begin{align*}
\Delta w &= aw & \text{in } \Omega_{h, r}, \\
w &\geq 0 & \text{on } \Gamma_{1, h, r}, \\
\frac{\partial w}{\partial \nu} &= 1 - u & \text{on } \Gamma_{2, h, r}.
\end{align*}
\]

If \( w(x_0) = \min_{\Omega_{h, r}} w(x) < 0 \), then \( x_0 \in \Gamma_{2, h, r} \setminus \{|x'| = r\} \). By the Hopf Boundary Lemma, we know that \( \partial w/\partial \nu(x_0) < 0 \), which contradicts
\[
\partial w/\partial \nu(x_0) = 1 - u(x_0) \geq 0.
\]

This proves Lemma 2.4.

Now consider
\[
\begin{align*}
\Delta \psi &= a \psi & \text{in } \Omega_{h, r}, \\
\psi &= 1 & \text{on } \Gamma_{1, h, r}, \\
\frac{\partial \psi}{\partial \nu} &= 0 & \text{on } \Gamma_{2, h, r}.
\end{align*}
\]

(9)
It is easy to see that $u = 0$ and $v = 1$ are sub- and super-solutions to (9), respectively. Thus there is a unique solution to (9). We denote it by $\psi_{1,h,r}$. Decompose $\psi_{h,r}$ as

$$
\psi_{h,r} = \psi_{1,h,r} + \psi_{2,h,r},
$$

where $\psi_{2,h,r}$ is the unique solution to

$$
\begin{align*}
\Delta \psi &= a\psi & \text{in} & \Omega_{h,r}, \\
\psi &= 0 & \text{on} & \Gamma_{1,h,r}, \\
\frac{\partial \psi}{\partial \nu} &= 1 & \text{on} & \Gamma_{2,h,r}.
\end{align*}
$$

It again follows from the Maximum Principle that

$$
0 < \psi_{1,h,r} < 1, \quad 0 < \psi_{2,h,r} < 1 + h, \quad \forall x \in \Omega_{h,r}.
$$

Furthermore, we can show:

**Lemma 2.5.**

a) $\psi_{1,h,r}$ is non-increasing in $r$. For any fixed $h > 0$, as $r \to \infty$, $\psi_{1,h,r}$ converges monotonically to $\psi_{1,h}$, where $\psi_{1,h}(x)$ is a function of $x_n$ alone and satisfies

$$
\begin{align*}
\Delta \psi &= a\psi & \text{in} & \Omega_h, \\
\psi &= 1 & \text{on} & \{x_n = h\}, \\
\frac{\partial \psi}{\partial \nu} &= 0 & \text{on} & \{x_n = 0\}.
\end{align*}
$$

(10)

b) $\psi_{2,h,r}$ is non-decreasing in $r$. For any fixed $h > 0$, as $r \to \infty$, $\psi_{2,h,r}$ converges monotonically to $\psi_{2,h}$, where $\psi_{2,h}(x)$ is a function of $x_n$ alone and satisfies

$$
\begin{align*}
\Delta \psi &= a\psi & \text{in} & \Omega_h, \\
\psi &= 0 & \text{on} & \{x_n = h\}, \\
\frac{\partial \psi}{\partial \nu} &= 1 & \text{on} & \{x_n = 0\}.
\end{align*}
$$

(11)

**Proof.** We only give the proof of Part (a) since Part (b) can be established in the same spirit. For any $r' > r$, set $w = \psi_{1,h,r'} - \psi_{1,h,r}$ in $\Omega_{h,r}$. Then $w$ satisfies

$$
\begin{align*}
\Delta w &= aw & \text{in} & \Omega_{h,r}, \\
w &\leq 0 & \text{on} & \Gamma_{1,h,r}, \\
\frac{\partial w}{\partial \nu} &= 0 & \text{on} & \Gamma_{2,h,r}.
\end{align*}
$$

(12)

It follows from the Maximum Principle and the Hopf Boundary Lemma that $w \leq 0$ in $\Omega_{h,r}$. This proves the monotonicity of $\psi_{1,h,r}$.

Therefore, for any fixed $h > 0$, as $r \to \infty$, $\psi_{1,h,r}$ monotonically converges to $\psi_{1,h}$ in $\mathbb{R}^n$ and $\psi_{1,h}$ satisfies (10).
We still need to show that $\psi_{1,h}$ is a function of $x_n$ only. For any $P \in \mathbb{R}^{n-1}$ and $r', r > 0$ with $r' - r > |P|$, consider the difference $w_1 = \psi_{1,h,r'}(x' + P, x_n) - \psi_{1,h,r}(x', x_n)$ in $\Omega_{h,r}$. It is easy to see that $w_1$ satisfies (12), thus $w_1 \leq 0$ in $\Omega_{h,r}$. That is,

$$
\psi_{1,h,r'}(x' + P, x_n) \leq \psi_{1,h,r}(x', x_n), \quad \forall x \in \Omega_{h,r}, \quad \forall P \in \mathbb{R}^{n-1}.
$$

Sending $r, r' \to \infty$, we have

$$
\psi_{1,h}(x' + P, x_n) \leq \psi_{1,h}(x), \quad \forall x \in \Omega_h.
$$

Hence $\psi_{1,h}(x) = \psi_{1,h}(x_n)$.

**Proof of Theorem 1.2.** Denote $\psi_h = \psi_{1,h} + \psi_{2,h}$. Then by Lemmas 2.4 and 2.5, $\psi_h(x) \geq u(x)$ in $\Omega_h$ and $\psi_h$ satisfies

$$
\begin{align*}
\psi'' - a\psi &= 0 \quad \text{in } (0, h), \\
\psi'(0) &= -1, \quad \psi(h) = 1.
\end{align*}
$$

Direct calculation shows that

$$
\psi_h(x_n) = c_1 e^{\sqrt{a} x_n} + c_2 e^{-\sqrt{a} x_n},
$$

where

$$
\begin{align*}
c_1 &= \frac{1 - \frac{1}{\sqrt{a}} e^{-\sqrt{a} h}}{e^{\sqrt{a} h} + e^{-\sqrt{a} h}}, \\
c_2 &= \frac{1 + \frac{1}{\sqrt{a}} e^{\sqrt{a} h}}{e^{\sqrt{a} h} + e^{-\sqrt{a} h}}.
\end{align*}
$$

Sending $h \to \infty$, we have $\psi_h(x_n) \to \frac{1}{\sqrt{a}} e^{-\sqrt{a} x_n}$. Thus by Lemma 2.4,

$$
u(x) \leq \frac{1}{\sqrt{a}} e^{-\sqrt{a} x_n}, \quad \forall x \in \mathbb{R}_+^n.
$$

It follows from (6) that

$$
1 = \sup_{\partial \mathbb{R}_+^n} u(x) \leq \frac{1}{\sqrt{a}},
$$

i.e., $a \leq 1$. This proves Theorem 1.2.

**3. Asymptotic behaviors of eigenvalues.**

We prove Theorem 1.1 in this section. For every piecewise smooth domain $\Omega$, it was proved in [3] that

$$
\lim_{\gamma \to \infty} \frac{\Lambda(\gamma)}{\gamma^2} \geq 1.
$$

To prove Theorem 1.1, we need to show that when $\partial \Omega$ is $C^1$,

$$
\lim_{\gamma \to \infty} \frac{\Lambda(\gamma)}{\gamma^2} \leq 1.
$$

(13)
Proof of Theorem 1.1. For any $\gamma > 1$, let $u_\gamma$ be a positive solution of (2) and $u_\gamma$ attains its maximum at $x_\gamma$. By the Maximum Principle, we know that $x_\gamma \in \partial \Omega$. After normalization we can assume that $\max_{\Omega} u_\gamma = 1$ and $x_\gamma \to 0 \in \partial \Omega$. Further, we can assume that there is a $C^1$ function $\phi$ such that $\partial \Omega \cap B_2(0)$ can be represented by $x_n = \phi(x')$ for $|x'| \leq 2$ with $\phi(0) = 0$ and $\partial \phi / \partial x_i(0) = 0$ for $i = 1, \ldots, n - 1$.

For any $\eta \in C^\infty_0(B_2(0))$, $u_\gamma$ satisfies

$$\int_\Omega \nabla u_\gamma \cdot \nabla \eta + \Lambda(\gamma) \int_\Omega u_\gamma \eta - \gamma \int_{\partial \Omega} u_\gamma \eta = 0. \tag{14}$$

Now we flatten $\partial \Omega$ near the origin. Let $y = \Phi(x) : \Omega \cap B_2(0) \to \Omega \equiv \Phi(\Omega \cap B_2(0))$, be such that

$$\phi_i(x) = x_i, \quad i = 1, 2, \ldots, n - 1,$$

$$\phi_n(x) = x_n - \phi(x').$$

Denote the inverse of $y = \Phi(x)$ by $x = \Psi(y)$. Then (14) can be rewritten as

$$\sum_{k,l=1}^n \int_{\Omega} \frac{\partial u_\gamma}{\partial y_k} \frac{\partial \eta}{\partial y_l} \frac{\partial \phi_k}{\partial x_i}(\psi(y)) \frac{\partial \phi_l}{\partial x_i}(\psi(y)) |D\psi| dy$$

$$+ \Lambda(\gamma) \int_{\Omega} u_\gamma \eta |D\psi| dy - \gamma \int_{\partial \Omega} u_\gamma \eta \sqrt{1 + |\nabla \phi(y')|^2} dy' = 0,$$

where $|D\psi|$ is the determinant of $D\psi$. Notice that $|\nabla \phi| = o(1)$ as $x' \to 0$. Thus $D\psi \to I$ as $|y| \to 0$, where $I$ is the $n \times n$ identity matrix. We now consider two different cases.

Case 1.

$$\lim_{\gamma \to \infty} \frac{\Lambda(\gamma)}{\gamma^2} = a < +\infty.$$

Without loss of generality, we may assume that

$$\frac{\gamma \int_{\partial \Omega} u_\gamma^2 - \int_\Omega |\nabla u_\gamma|^2}{\gamma^2 \int_\Omega u_\gamma^2} = \sup_{u \in H^1(\Omega) \setminus \{0\}} \frac{\gamma \int_{\partial \Omega} u^2 - \int_\Omega |\nabla u|^2}{\gamma^2 \int_\Omega u^2} \to a,$$

and $u_\gamma(x_\gamma) = \max_\Omega u_\gamma(x) = 1$, $x_\gamma \to 0$. We let $z = (y_\gamma, y_n)$, where $y_\gamma = (x'_\gamma, 0)$, and set $v_\gamma(z) = u_\gamma(y)$. Then for any $R > 0$ and $\eta$ with
compact support in $B_{2R}$, as $\gamma$ becomes sufficiently large, $v_\gamma$ satisfies

\[
(15) \quad \sum_{k,l=1}^{n} \int_{B_{2R}^+} \frac{\partial v_\gamma}{\partial z_k} \frac{\partial \eta}{\partial z_l} \partial \Phi_k \left( \Psi \left( y_\gamma + \frac{z}{\gamma} \right) \right) \partial \Phi_l \left( \Psi \left( y_\gamma + \frac{z}{\gamma} \right) \right) |D\Psi| \left( y_\gamma + \frac{z}{\gamma} \right) dz \\
+ \frac{\Lambda(\gamma)}{\gamma^2} \int_{B_{2R}^+} v_\gamma \eta |D\Psi| \left( y_\gamma + \frac{z}{\gamma} \right) dz \\
- \int_{z_n=0} v_\gamma \eta \sqrt{1 + |\nabla \phi'(y_\gamma' + z'/\gamma)|^2} dz' = 0.
\]

Since for $z \in B_{2R}^+$, $y_\gamma + z/\gamma \to 0$ as $\gamma \to \infty$, we know that for sufficiently large $\gamma$,

\[
\left( \sum_{i=1}^{n} \frac{\partial \Phi_k}{\partial x_i} \left( \Psi \left( y_\gamma + \frac{z}{\gamma} \right) \right) \frac{\partial \Phi_l}{\partial x_i} \left( \Psi \left( y_\gamma + \frac{z}{\gamma} \right) \right) \right)_{kl} > \frac{I}{2}.
\]

Let $\eta_R$ be a cutoff function satisfying $\eta_R = 1$ in $B_R$ with compact support in $B_{2R}$ and $|\nabla \eta_R| \leq C$. Choosing $\eta = u \cdot \eta_R$ in (15), we have,

\[
\| \nabla v_\gamma \|_{L^2(B_R)} \leq C \| v_\gamma \|_{L^\infty} = C.
\]

Therefore, after passing to a subsequence, $v_\gamma \to v_0$ weakly in $H^1_{\text{loc}}(\mathbb{R}^n_+)$ as $\gamma \to \infty$, where $v_0 \in H^1_{\text{loc}}(\mathbb{R}^n_+)$ satisfies

\[
\Delta v_0 = av_0 \quad \text{in } \mathbb{R}^n_+, \\
\frac{\partial v_0}{\partial \nu} = v_0 \quad \text{on } \partial \mathbb{R}^n_+, \\
0 \leq v_0 \leq 1 \quad \text{in } \mathbb{R}^n_+.
\]

To show that $v_0$ is nontrivial, we claim that there is a constant $C > 0$ such that

\[
(16) \quad 1 = \| v_\gamma \|_{L^\infty(B_1^+)} \leq C \left( \| v_\gamma \|_{L^2(\partial \mathbb{R}^n_+ \cap \{|x'| < 2\})} + \| v_\gamma \|_{L^{\frac{2n}{n-1}}(B_1^+(0))} \right).
\]

Since the embeddings from $W^{1,2}(B_2^+(0))$ to $L^{\frac{2n}{n-1}}(B_1^+(0))$ and $L^2(\partial \mathbb{R}^n_+ \cap \{|x'| < 2\})$ are both compact, we know that

\[
1 \leq C \left( \| v_0 \|_{L^2(\partial \mathbb{R}^n_+ \cap \{|x'| < 1\})} + \| v_0 \|_{L^{\frac{2n}{n-1}}(B_1^+(0))} \right),
\]

from which it follows that $v_0 \neq 0$. From Theorem 1.2 we see that $a \leq 1$. Hence it suffices to establish (16).

Inequality (16) can be obtained via Moser iteration. Though it seems to be a standard result (see, e.g., [4]), we include a proof here for the completeness.
Direct calculation shows that
\[
\int_{B_2^+} \nabla v_\gamma \cdot \nabla (v_\gamma^k \xi^2) = 4k \frac{1}{(k+1)^2} \int_{B_2^+} \Delta (v_\gamma^k \xi) + \int_{B_2^+} \nabla (v_\gamma^k \xi) + \frac{k-1}{(k+1)^2} \int_{B_2^+} \nabla \xi^2
\]

Choosing \(\eta = v_\gamma^k \xi^2\) in (15) with \(k > 1\) and \(\xi\) having compact support in \(B_2^+\), we have
\[
\int_{B_2^+} \nabla \left( \frac{k+1}{v_\gamma^2} \xi \right) \leq \frac{k-1}{4k} \int_{B_2^+} v_\gamma^{k+1} \Delta \xi + \int_{B_2^+} v_\gamma^{k+1} \nabla \xi^2 + \frac{k-1}{4k} \int_{\{x_n=0, \ |x'| \leq 2\}} v_\gamma^{k+1} \nabla \xi^2 \cdot \nu
\]

Let
\[
r_i = 1 + \frac{1}{2^{i-1}}, \quad i = 1, 2, \ldots,
\]
and choose \(\xi_i\) satisfying
\[
\xi_i = 1, \quad |x| \leq r_{i+1};
\]
\[
\xi_i = 0, \quad |x| > r_i;
\]
\[
|\nabla \xi_i| \leq 2 \cdot 2^i, \quad |\nabla^2 \xi_i| \leq 4 \cdot 4^i.
\]

Replacing \(\xi\) by \(\xi_i\) in (17) and using the Sobolev inequality and the trace inequality, we have, for \(n \geq 3\),
\[
\int_{B_i} v_\gamma^{k+1} \leq C \left( \int_{B_i} v_\gamma^{(k+1) \frac{n}{n-1}} \right)^{\frac{n-1}{n}},
\]
where \(C\) is some universal constant. By
\[
\int_{B_i} v_\gamma^{k+1} \leq C \left( \int_{B_i} v_\gamma^{(k+1) \frac{n}{n-1}} \right)^{\frac{n-1}{n}},
\]
we arrive at

\[
\left( \int_{B_{r_i}^+} \left( v_{\gamma}^{k+1} \xi_i \right)^{\frac{2n}{n-2}} \right) \left( \int_{\partial B_{r_i}^+ \cap \{ x_n = 0 \}} \left( \frac{v_{\gamma}^{k+1} \xi_i}{v_{\gamma}^{k+1}} \right)^{\frac{n-2}{n}} \right)^{\frac{2(n-1)}{n-2}} \\ \leq C \cdot \left( 4^i + \frac{(k+1)^2}{4k} \right)^{\frac{n-1}{n}} \int_{B_{r_i}^+} v_{\gamma}^{k+1}.
\]

Define \( \beta = \frac{(n-1)/(n-2)}{2q_i} \), \( q_0 = 2 \), \( q_i = \beta q_i \) and \( p_i = nq_i/(n-1) \) for \( i = 0, 1, \ldots \), choosing \( k = q_i - 1 \) in (18), we have

\[
\| v_{\gamma} \|_{L^{p_i+1}(B_{r_i}^+)} + \| v_{\gamma} \|_{L^{q_i+1}(\Gamma_{i+1})} \\ \leq C \cdot \left( 4^i + \frac{q_i^2}{q_i - 1} \right) \left( \| v_{\gamma} \|_{L^{p_i}(B_{r_i}^+)} + \| v_{\gamma} \|_{L^{q_i}(\Gamma_i)} \right),
\]

where \( \Gamma_i = \partial B_{r_i}^+ \cap \{ x_n = 0 \} \) for \( i = 0, 1, \ldots \); since \( \beta > 1 \), \( (a^\beta + b^\beta)^{1/\beta} \leq a + b \).

It follows that

\[
\left( \int_{B_{r_i}^+} \left( v_{\gamma}^{q_i+1} \xi_i \right)^{\frac{2n}{n-2}} \right) \left( \int_{\partial B_{r_i}^+ \cap \{ x_n = 0 \}} \left( \frac{v_{\gamma}^{q_i+1} \xi_i}{v_{\gamma}^{q_i+1}} \right)^{\frac{n-2}{n}} \right)^{\frac{2(n-1)}{n-2}} \\ \leq C \cdot \left( 4^i + \frac{q_i^2}{q_i - 1} \right) \left( \| v_{\gamma} \|_{L^{p_i}(B_{r_i}^+)} + \| v_{\gamma} \|_{L^{q_i}(\Gamma_i)} \right)^{1/q_i}.
\]

Since \( q_i = 2^\beta \), it is easy to see that

\[
\left( C4^i + \frac{Cq_i^2}{q_i - 1} \right)^{1/p_i} \leq \left[ C(4^i + 2^\beta) \right]^{1/(2^\beta)} \leq C^{1/(2^\beta)} (4 + \beta)^{i/(2^\beta)}.
\]

Thus

\[
\prod_{i=1}^{\infty} \left( 4^i C + \frac{q_i^2 C}{q_i - 1} \right)^{1/q_i} \leq C < \infty.
\]

It follows that

\[
\| v_{\gamma} \|_{L^{p_i+1}(B_{r_i+1}^+)} \leq C \left( \| v_{\gamma} \|_{L^2(\partial B_{r_i}^+ \cap \{ |x'| < 2 \})} + \| v_{\gamma} \|_{L^{2n/(n-1)}(\partial B_{r_i}^+ (0))} \right).
\]

Sending \( i \to \infty \), we obtain (16). For \( n = 2 \), we can obtain (16) in the same spirit. We thereby complete the proof of Theorem 1.1 in Case 1.

Case 2.

\[
\lim_{\gamma \to \infty} \frac{\Lambda(\gamma)}{\gamma^2} = \infty.
\]
We will also rule out this possibility. Let \( u_\gamma \) be the sequence of positive functions such that

\[
\frac{\gamma \int_{\partial \Omega} u_\gamma^2 - \int_{\Omega} |\nabla u_\gamma|^2}{\gamma^2 \int_{\Omega} u_\gamma^2} = \sup_{u \in H^1(\Omega) \setminus \{0\}} \frac{\gamma \int_{\partial \Omega} u^2 - \int_{\Omega} |\nabla u|^2}{\gamma^2 \int_{\Omega} u^2} = a(\gamma) \to \infty
\]

as \( \gamma \to \infty \), and \( u_\gamma(x_\gamma) = \max_{\Omega} u(x) \equiv 1 \). Define \( z = \sqrt{a(\gamma)} \gamma(y - y_\gamma) \) and \( v_\gamma(z) = u_\gamma(y_\gamma) \). Then for any \( R > 0 \) and \( \eta \) with compact support in \( B_{2R}^+ \), as \( \gamma \) becomes sufficiently large, \( v_\gamma \) satisfies

\[
\sum_{k,l=1}^{n} \int_{B_{2R}^+} \frac{\partial v_\gamma}{\partial \gamma} \frac{\partial \eta}{\partial \gamma} \frac{\partial \Phi_k}{\partial x_i} \left( \Psi \left( y_\gamma + \frac{z}{\gamma} \right) \right) \\
\cdot \frac{\partial \Phi_l}{\partial x_i} \left( \Psi \left( y_\gamma + \frac{z}{\gamma} \right) \right) |D\Psi| \left( y_\gamma + \frac{z}{\gamma} \right) dz \\
+ \frac{\Lambda(\gamma)}{a(\gamma) \gamma^2} \int_{B_{2R}^+} v_\gamma \eta |D\Psi| \left( y_\gamma + \frac{z}{\gamma} \right) dz \\
- \frac{1}{\sqrt{a(\gamma)}} \int_{z_\gamma = 0} v_\gamma \eta \sqrt{1 + |\nabla \phi(y'_{\gamma} + z'/\gamma)|^2} dz' = 0.
\]

Similarly as in Case 1, we can show that \( v_\gamma \to v_0 \) weakly in \( H^1_{\text{loc}}(B_R^+) \), where \( 0 \leq v_0 \leq 1, \ v_0 \not\equiv 0 \), and \( v_0 \in H^1_{\text{loc}}(\mathbb{R}^n_+) \) is a weak solution of

\[
\Delta v_0 = v_0 \quad \text{in} \quad \mathbb{R}^n_+, \\
\frac{\partial v_0}{\partial \nu} = 0 \quad \text{on} \quad \partial \mathbb{R}^n_+.
\]

Using even reflection, from (20) we know that there is a positive bounded function satisfying

\[
\Delta v_0 = v_0 \quad \text{in} \quad \mathbb{R}^n.
\]

On the other hand, it is well-known that there is no nontrivial positive solution to the above equation. This finishes the proof of Theorem 1.1.

References


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THE SIMULATION TECHNIQUE AND ITS
APPLICATIONS TO INFINITARY COMBINATORICS
UNDER THE AXIOM OF BLACKWELL DETERMINACY

Benedikt Löwe

The Axiom of Blackwell Determinacy is a set-theoretic axiom motivated by games used in statistics. It is known that the Axiom of Determinacy implies the Axiom of Blackwell Determinacy. Tony Martin has conjectured that the two axioms are equivalent.

We develop the “simulation technique” which allows us to simulate boundedness proofs under the assumption of Blackwell Determinacy and deduce strong combinatorial consequences that can be seen as an important step towards proving Tony Martin’s conjecture.

1. Introduction.

Set-theoretic game theory is an important part of Higher Set Theory. The research of the Cabal seminar and its successors unearthed deep connections between two-person perfect information games, inner models of set theory, and large cardinals.

The core concept of set-theoretic game theory is the notion of a strategy. Since set theorists usually worked with perfect information games, a strategy is a tree of moves in the set-theoretic setting. The strategy is a winning strategy for one of the two players if all its branches lie in the set designating a win for that player. Using the notion of a winning strategy, we can define when we call a set determined, viz. if one of the two players has a winning strategy.

For imperfect information games, this concept is too coarse. If one or both of the players are not completely informed about the current state of the game, we cannot expect winning strategies in the above sense to exist.

If we now look only at special classes of imperfect information (in this paper, this will be Blackwell games, or — as Blackwell calls them in [Bl97] — “Games with Slightly Imperfect Information”), we can define the notions of a mixed strategy and of strong optimality and use them to define the notion of Blackwell determinacy. Based on the notion of Blackwell determinacy, we can reformulate the questions of set-theoretic game theory for games of this type.
The notion of Blackwell determinacy was introduced for finite games by von Neumann, and generalized to infinite games by David Blackwell [Bl69] who also proved the first theorems about Blackwell determinacy. At the MSRI Workshop in 1994 on Combinatorial Games, it was still open whether the Blackwell determinacy of all $G_{\delta\sigma}$ sets is a theorem of $\text{ZFC}$ as Blackwell’s extended abstract [Bl97] witnesses. Soon thereafter, Marco Vervoort proved the Blackwell determinacy of all $G_{\delta\sigma}$ sets in his Master’s thesis [Ve95], and then Tony Martin developed a coding technique to derive Blackwell determinacy from perfect information determinacy in [Ma98]. In his Master’s thesis, Vervoort also introduced the “Axiom of Blackwell Determinacy” which will be the protagonist of this paper.

In this paper, we shall discuss consequences of axioms of Blackwell Determinacy in the field of Infinitary Combinatorics. We shall define several axioms of Blackwell Determinacy in Section 2 and discuss Martin’s conjecture on Blackwell Determinacy, Conjecture 2.6, which is the motivation for the rest of the paper. Section 2 is mostly expository and almost all results are either folklore or from the published literature.

In Section 3 we shall develop the simulation technique which will be used throughout this paper.

Section 4 is the main part of this paper, and its contents can be seen as an important step towards proving Conjecture 2.6. After introducing the basics of Infinitary Combinatorics under $\text{AD}$ (Section 4.1), we transfer these results to Blackwell Determinacy using the simulation technique in Sections 4.2 and 4.3. We prove that $\mathbb{N}_1$ has the strong partition property and that the odd projective ordinals are measurable cardinals. Some of the results of this paper have been announced together with additional results on a Blackwell Lipschitz hierarchy (without proof) in the survey paper [Lö02b].

The paper closes with a discussion of open problems in Section 5.

2. Definitions and notation.

2.1. Set-theoretic standard notation. The notation used in this paper is standard. The reader is assumed to have a firm grasp of descriptive set theory and large cardinal theory as contained in [Mo80] and [Ka94]. All theorems and definitions that are not found in this paper can be found in one or both of the mentioned textbooks. Of course, we shall say reals or real numbers when we talk about elements of Baire space $\omega^\omega$ as is customary in set theory.

Since Blackwell determinacy contradicts the full Axiom of Choice $\text{AC}$, we shall work throughout this paper in the theory

$$\text{ZF} + \text{DC}.$$
We shall need DC in the context of Infinitary Combinatorics, yet for most of this paper, ACω(ℝ) will be enough. Note that its an open problem whether ACω(ℝ) follows from Blackwell determinacy (as it does from AD).

We now fix some notation. In the following, let X ⊆ ω be the set of possible moves.

Let us write $X^{\text{even}} := \{ s \in X^{<\omega} : \text{lh}(s) \text{ is even} \}$, $X^{\text{odd}} := \{ s \in X^{<\omega} : \text{lh}(s) \text{ is odd} \}$, and $\text{Prob}(X)$ for the set of probability measures on $X$.

Fix a recursive bijection $\langle \cdot, \cdot \rangle : ω \times ω \rightarrow ω$. If $x \in X^ω$, we define $(x)_i(n) := x(\langle i, n \rangle)$. Using this notation we can easily code countable sequences of elements of $X^ω$ into one element of $X^ω$.

We shall be using the standard notation for infinite games: If $x \in X^ω$ is the sequence of moves for player I and $y \in X^ω$ is the sequence of moves for player II, we let $x * y$ be the sequence constructed by playing $x$ against $y$, i.e.,

$$(x * y)(n) := \begin{cases} x(k) & \text{if } n = 2k, \\ y(k) & \text{if } n = 2k + 1. \end{cases}$$

Conversely, if $x \in X^ω$ is a run of a game, then we let $x_1$ be the part played by player I and $x_2$ be the part played by player II, i.e., $x_1(n) = x(2n)$ and $x_2(n) = x(2n + 1)$. We shall extend this notation to sets $A \subseteq X^ω$ in the obvious way: $A_1 := \{ x_1 : x \in A \}$ and $A_2 := \{ x_2 : x \in A \}$. If $Y \subseteq X^ω$, we write $A_1^Y$ for the set $\{ x \in A : x_1 \in Y \}$ and $A_2^Y$ for the set $\{ x \in A : x_2 \in Y \}$.

(In most situations, $Y$ will be of the form $\{ y \}$.) We denote by WO the $\Pi^1_1$-complete set of all codes of wellorderings relative to the bijection $\langle \cdot, \cdot \rangle$.

A standard result that we shall be using a lot is the Boundedness Lemma in the following abstract form:

**Theorem 2.1.** Let $Γ \subseteq ϕ(X^ω)$ be a pointclass with the following properties:

- $Γ$ is boldface (i.e., closed under continuous preimages),
- $Γ$ has the prewellordering property, and
- $Γ$ is closed under $\forall^\mathcal{R}$ and finite unions.

Suppose that $X$ is a $Γ$-complete set, $ϕ$ is a $Γ$ prewellordering on $X$, and $A \subseteq X$ is $Γ$.

Then $A$ is bounded in the following sense: There is an $α < \text{lh}(ϕ)$ such that for all $a \in A$, $ϕ(a) < α$.

**Proof.** First of all, note that the assumptions on $Γ$ imply that $Γ$ is nonself-dual (no selfdual pointclass can have the prewellordering property, [Ka94, Exercise 29.2, Proposition 29.3 & Proposition 29.7]). The closure properties of $Γ$ give that $Γ$ is boldface, closed under $\exists^\mathcal{R}$ and finite intersections.

Now we assume that boundedness fails and derive a contradiction: Let $A \in Γ$ be such that for all $α < \text{lh}(ϕ)$ there is $a \in A$ with $ϕ(a) > α$. Then (as in the standard proof of boundedness), we can show that every set in $Γ$ is in $Γ$, thereby showing that $Γ$ is selfdual, which is a contradiction. □
2.2. Blackwell games. As mentioned in the Introduction, we shall not be dealing with imperfect information games in full generality, but with the very special subclass that Blackwell called “Games with Slightly Imperfect Information”, and we shall call Blackwell games. We imagine a perfect information game in which both players have to move simultaneously in each step. Consequently, player II has only partial information: He is aware of all the game information up to the current step, but doesn’t know what player I plays in the current round. This situation is modelled by the notion of a Blackwell strategy (see below).

We call a function $\sigma : X^{\text{Even}} \to \text{Prob}(X)$ a mixed strategy for player I and a function $\sigma : X^{\text{Odd}} \to \text{Prob}(X)$ a mixed strategy for player II.

Let us describe two particularly interesting types of mixed strategies:

- A mixed strategy is called Blackwell strategy if it doesn’t depend on the moves in the same turn, i.e., if $s$ and $t$ have the same longest even subsequence, then we have $\sigma(s) = \sigma(t)$.
- A mixed strategy $\sigma$ is called pure if for all $s \in \text{dom}(\sigma)$ the measure $\sigma(s)$ is a Dirac measure, i.e., there is a natural number $n$ such that $\sigma(s)(\{n\}) = 1$. (This is of course equivalent to being a strategy in the perfect information sense.)

Let

$$\nu(\sigma, \tau)(s) := \begin{cases} \sigma(s) & \text{if } \text{lh}(s) \text{ is even}, \\ \tau(s) & \text{if } \text{lh}(s) \text{ is odd}. \end{cases}$$

Then for any $s \in \omega^{< \omega}$, we can define

$$\mu_{\sigma, \tau}([s]) := \prod_{i=0}^{\text{lh}(s)-1} \nu(\sigma, \tau)(s|i)(\{s_i\}).$$

This generates a Borel probability measure on $\omega^\omega$ which can be seen as an indicator of how well the strategies $\sigma$ and $\tau$ perform against each other. If $B$ is a Borel set, $\mu_{\sigma, \tau}(B)$ is interpreted as the probability that the result of the game ends up in the set $B$ when player I randomizes according to $\sigma$ and player II according to $\tau$. Note that if $\sigma$ and $\tau$ are both pure, then $\mu_{\sigma, \tau}$ is a Dirac measure concentrated on the unique real that is the outcome of this game, denoted by $\sigma \ast \tau$.

If $\sigma$ and $\tau$ are Blackwell strategies, the game modelled by $\sigma$ and $\tau$ corresponds to the “Games with Slightly Imperfect Information” of [BL97]: Both players can be understood as moving simultaneously, so player II may not use any information about the move of player I in the same round of the game.

- We call a pure strategy $\sigma$ for player I ($\tau$ for player II) a winning strategy if for all pure counterstrategies $\tau(\sigma)$, we have that $\sigma \ast \tau \in A$ ($\sigma \ast \tau \notin A$).
We define a measure of quality (the **mixed value of the strategy**) for mixed strategies \( \sigma \) (for player I) or \( \tau \) (for player II) by

\[
mval^A_I(\sigma) := \inf \{ \mu^- - \sigma, \tau(A) ; \tau \text{ is a mixed strategy for player II} \},
\]

\[
mval^A_{II}(\tau) := \sup \{ \mu^+ + \sigma, \tau(A) ; \sigma \text{ is a mixed strategy for player I} \}.
\]

Also for Blackwell strategies \( \sigma \) (for player I) or \( \tau \) (for player II), we define a measure of quality (the **Blackwell value of the strategy**) by

\[
Bval^A_I(\sigma) := \inf \{ \mu^- - \sigma, \tau(A) ; \tau \text{ is a Blackwell strategy for player II} \},
\]

\[
Bval^A_{II}(\tau) := \sup \{ \mu^+ + \sigma, \tau(A) ; \sigma \text{ is a Blackwell strategy for player I} \}.
\]

We call a set \( A \) **determined** if either player I or player II has a winning strategy, and we call a pointclass \( \Gamma \) determined if all sets in \( \Gamma \) are determined (in symbols: \( \text{Det}(\Gamma) \)).

We define the **mixed and Blackwell value sets** for player I and player II by

\[
V^{\text{mix}}_I(A) := \{ mval^A_I(\sigma) ; \sigma \text{ is a mixed strategy for player I} \},
\]

\[
V^{\text{mix}}_{II}(A) := \{ mval^A_{II}(\tau) ; \tau \text{ is a mixed strategy for player II} \},
\]

\[
V^{\text{Bl}}_I(A) := \{ Bval^A_I(\sigma) ; \sigma \text{ is a Blackwell strategy for player I} \},
\]

\[
V^{\text{Bl}}_{II}(A) := \{ Bval^A_{II}(\tau) ; \tau \text{ is a Blackwell strategy for player II} \}.
\]

Then \( V^{\text{mix}}_{II}(A) \) lies entirely above \( V^{\text{mix}}_I(A) \) in the sense that for all \( v \in V^{\text{mix}}_{II}(A) \) and \( v^* \in V^{\text{mix}}_I(A) \) we have \( v \geq v^* \) (and the same for \( V^{\text{Bl}}_I(A) \) and \( V^{\text{Bl}}_{II}(A) \)).

If now \( \inf V^{\text{Bl}}_{II}(A) = \sup V^{\text{Bl}}_I(A) =: p \), then the outcome of the game is stochastically determined as follows: Both players can approximate the outcome that player I wins with probability \( p \). In this case, we call the payoff set **(imperfect information) Blackwell determined**. Similarly, if \( \inf V^{\text{mix}}_{II}(A) = \sup V^{\text{mix}}_I(A) \), we call the payoff set \( A \) **perfect information Blackwell determined**.

It is well-known that for infinite sets \( X \) it’s easy to construct clopen payoff sets \( A \subseteq X^\omega \) such that \( A \) is not imperfect information Blackwell determined. Thus, in order to talk about this property, we have to restrict ourselves to looking at \( A \subseteq X^\omega \) where \( X \) is finite. For a pointclass \( \Gamma \), we say it’s (imperfect information) Blackwell determined (in symbols: \( \text{Bl-Det}(\Gamma) \)) if all sets \( A \in \Gamma \) with \( A \subseteq X^\omega \) are (imperfect information) Blackwell determined (where \( X \) is a finite set), and we say that it is perfect information Blackwell determined (in symbols: \( \text{pBl-Det}(\Gamma) \)) if for all \( A \in \Gamma \), the set \( A \) is perfect information Blackwell determined.

---

1. Here, \( \mu^+ \) denotes outer measure and \( \mu^- \) denotes inner measure with respect to \( \mu \) in the usual sense of measure theory. If \( A \) is Borel, then \( \mu^+(A) = \mu^-(A) = \mu(A) \) for Borel measures \( \mu \).
In 2000, Martin and Vervoort proved a crucial result about games with mixed strategies [MaNeVe03, Lemma 3.7]:

**Theorem 2.2** (Martin-Vervoort Zero-One Law, 2000). If \( A \subseteq \omega^\omega \) is perfect information Blackwell determined. Then \( \inf V^\text{mix}_I(A) = \sup V^\text{mix}_I(A) \) is either 0 or 1.

A mixed strategy for player I is now called **strongly optimal for** \( A \) if \( \text{mval}_I^\preceq(A) = 1 \), and a mixed strategy \( \tau \) for player II is called **strongly optimal for** \( A \) if \( \text{mval}_I^\preceq(A) = 0 \).

Vervoort was able to use the Martin-Vervoort Zero-One-Law to prove the existence of strongly optimal strategies [MaNeVe03, Lemma 3.10] (cf. also [Lö02a] for a transfer of a finite branching version of Theorem 2.3 to an infinite branching version):

**Theorem 2.3.** Let \( \Gamma \) be a boldface pointclass. Suppose that \( \text{pBl-Det}(\Gamma) \) holds and \( A \in \Gamma \) where \( A \) is a subset of \( \omega^\omega \). Then there is either a strongly optimal strategy for player I or a strongly optimal strategy for player II in the game with payoff \( A \).

**Corollary 2.4.** Let \( \Gamma \) be a boldface pointclass. Then \( \text{pBl-Det}(\Gamma) \) is equivalent to “for each \( A \in \Gamma \), either player I or player II has a strongly optimal strategy.”

We now defined three notions of determinacy allowing us to look at three different axioms of determinacy \( \text{AD} \), \( \text{Bi-AD} \), and \( \text{pBl-AD} \) (and their restrictions to the projective sets \( \text{PD} \), \( \text{Bi-PD} \) and \( \text{pBl-PD} \)):

- The Axiom of Determinacy \( \text{AD} \): All subsets of \( \omega^\omega \) are determined.
- The Axiom of (imperfect information) Blackwell Determinacy \( \text{Bi-AD} \): All subsets of \( 2^\omega \) are (imperfect information) Blackwell determined.
- The Axiom of perfect information Blackwell Determinacy \( \text{pBl-AD} \): All subsets of \( \omega^\omega \) are perfect information Blackwell determined.

To these three axioms, we shall add another one: The **Axiom of blindfolded Blackwell Determinacy**. Instead of defining optimality by looking at all mixed counterstrategies, we can look at smaller classes of counterstrategies. One class that will happen to play a rôle is the class of all trivial strategies:

We call a mixed strategy \( \sigma \) **trivial**, if it is a pure strategy and its values don’t depend on the input, i.e., whenever \( s \) and \( t \) are sequences of the same length, then \( \sigma(s) = \sigma(t) \). A trivial strategy corresponds to a real that is fixed in advance before a single move of the game is played and which the player following the strategy is using as his predetermined moves. Playing according to a trivial strategy is like playing blindfoldedly: You have no idea what is going on in the game and just follow a previously fixed sequence of moves no matter what happens.
Now we call a mixed strategy $\sigma$ for player I (player II) **weakly optimal** if for all trivial counterstrategies $\tau$, we have $\mu_{\sigma,\tau}(A) = 1$ ($\mu_{\tau,\sigma}(A) = 0$). The existence of weakly optimal strategies is a very weak property. It is possible that both players have weakly optimal strategies, and it’s even possible that player I has a weakly optimal strategy, but player II has a winning strategy.

We say a set $A$ is **blindfoldedly Blackwell determined** if one of the two players has a weakly optimal strategy. A pointclass $\Gamma$ is called blindfoldedly Blackwell determined (in symbols: $\text{bBl-}\text{Det}(\Gamma)$). Finally, the Axiom of blindfolded Blackwell determinacy $\text{bBl-AD}$ says that all sets are blindfoldedly Blackwell determined. To make this axiom sound less bizarre, let us note that for a pure strategy $\sigma$, being winning and being winning against all trivial counterstrategies are equivalent, so that a notion of “blindfolded determinacy” would be equivalent to determinacy in the usual perfect information setting in a very strong sense.

We now have four determinacy axioms—what are the relations between them?

**Theorem 2.5.**

(a) $\text{AD}$ implies $\text{BI-AD}$,

(b) $\text{BI-AD}$ is equivalent to $\text{pBI-AD}$, and

(c) $\text{pBI-AD}$ implies $\text{bBl-AD}$.

**Proof.** (a) is an instance of the main theorem of [Ma98] and (c) is obvious.

For the “$\Rightarrow$”-direction of (b), we model a game with mixed strategies by a game with Blackwell strategies in which the players pass every second move. The “$\Leftarrow$”-direction follows from the remarks on [Ma98, p. 1579] and [MaNeVe03, p. 618sq.]: The main theorem of [Ma98] can be proved using mixed strategies. □

The main open question in this area which is also the motivation for the work in this paper is Tony Martin’s conjecture that the converse of Theorem 2.5 (a) also holds:

**Conjecture 2.6.** $\text{BI-AD}$ implies $\text{AD}$.

Tony Martin, Itay Neeman and Marco Vervoort proved (cf. [MaNeVe03, Theorems 5.1 & 5.7]) the following result, thereby determining the consistency strength of $\text{pBI-AD}$:

**Theorem 2.7** (Martin–Neeman–Vervoort 1999/2000). $\text{pBI-PD}$ implies $\text{PD}$ and $(\text{pBI-AD})^{L(\mathbb{R})}$ implies $(\text{AD})^{L(\mathbb{R})}$.

However, while their result yields the equiconsistency of $\text{pBI-AD}$ and $\text{AD}$, it doesn’t give an equivalence. So far, Conjecture 2.6 is open. In this paper we shall get a rich structure theory of the ordinals up to $\aleph_{\omega+1}$ that is usually
seen as being very characteristic of AD and thus can serve as an indication that Martin’s Conjecture 2.6 is true.

Let us close by mentioning that the usual proof of Lebesgue measurability from determinacy doesn’t work with Blackwell determinacy. Vervoort proved (using a new proof, cf. [Ve95, Theorem 6.11]) that Bl-Det(Γ) implies that every set in Γ is Lebesgue measurable. Thus we have (by standard arguments):

**Theorem 2.8.** Assume Bl-AD and let μ be a σ-finite measure (in particular, all of our measures μσ,τ derived from mixed strategies σ and τ will do). Let \( \langle B_\xi : \xi < \lambda \rangle \) be a wellordered sequence of sets with \( \mu(B_\xi) = 1 \). Then \( \mu(\bigcap_{\xi<\lambda} B_\xi) = 1 \).

Let us sum up the implications between our axioms of determinacy and some other set-theoretic statements as they were known before this paper in Figure 1. Dotted lines indicate results proved in this paper; \( M^{\text{strong}} \) abbreviates “there is an inner model with a strong cardinal”.

![Diagram of axioms of determinacy and consequences.](image)

**Figure 1.** Diagram of axioms of determinacy and consequences.

3. The simulation technique.

In this section we shall introduce a technique called the **simulation technique**. This technique will be our main tool in getting consequences from Blackwell determinacy.
Many proofs of some statement $\text{AD} \rightarrow \Psi$ involve some game $G$ and split up into two cases: Either player I has a winning strategy in the game $G$ (we denote this situation by $\Phi_I$) or player II has a winning strategy in the game $G$ (we denote this situation by $\Phi_{II}$). The proof then proceeds by showing $\Phi_I \rightarrow \Psi_I$ and $\Phi_{II} \rightarrow \Psi_{II}$, where $\Psi_I \lor \Psi_{II}$ implies $\Psi$ (very often, $\Psi_I \equiv 0=1$ and $\Psi_{II} \equiv \Psi$).

Now, if we want to simulate these proofs to get a proof of $\text{pBl} \text{-AD} \rightarrow \Psi$, we weaken the two cases to $\Phi_I^*$ (“Player I has a strongly optimal strategy”) and $\Phi_{II}^*$ (“Player II has a strongly optimal strategy”), and need to show that $\Phi_I^* \rightarrow \Psi_I$ and $\Phi_{II}^* \rightarrow \Psi_{II}$ are still provable.

The simulation technique allows to do this in special cases. In particular, the simulation technique does not generate essentially new proofs but proves that in some situations, the classical AD proofs can be simulated in the Blackwell context.


**Definition 3.1.** Let $\Gamma$ be a boldface pointclass. We shall call it a **Kechris-Tanaka pointclass** if:

1. $\Gamma$ is closed under existential real quantification,
2. for all $A \subseteq (\omega^\omega)^2$ with $A \in \Gamma$, and all reals $\varepsilon$ the sets
   \[
   \{ \langle x, \sigma, \tau \rangle ; \mu_{\sigma,\tau}^{-}(A_x) > \varepsilon \} \quad \text{and} \quad \{ \langle x, \sigma, \tau \rangle ; \mu_{\sigma,\tau}^{+}(A_x) > \varepsilon \}
   \]
   are in $\Gamma$, and
3. $\Gamma$ is closed under countable intersections.

We say that a boldface pointclass $\Gamma$ has the **weak scale property** if every set in $\Gamma$ admits a $\forall R \bar{R}$-scale (this should not be confused with the weak scale property of [MaNeVe03]). Under PD, every universal projective class has the weak scale property, and that the class of Borel sets has the weak scale property (without any assumptions).

**Theorem 3.2** (Kechris-Tanaka). Suppose $\Gamma$ is a boldface pointclass closed under countable intersections. Suppose that there is a pointclass $\Gamma^*$ with the following properties:

1) $\exists R \bar{R} \Gamma^* = \Gamma$,
2) $\Gamma^*$ has the weak scale property, and
3) every set in $\Gamma^*$ is Lebesgue measurable.

Then $\Gamma$ is a Kechris-Tanaka pointclass.

A proof can be found in [Ke73, Theorem 2.2.3 & Corollary 2.2.2]. Theorem 3.2 yields that (under the assumption of PD for $n > 1$) $\Sigma^1_n$ is a Kechris-Tanaka pointclass.
Let \( A \subseteq \omega^\omega \) be a set of reals and \( \leq \) any prewellordering of \( \omega^\omega \). Also fix a mixed strategy \( \sigma \) for player I and a mixed strategy \( \tau \) for player II.

Let \( U^I_\leq := \{ u \; ; \; x_1 \leq u_1 \} \) and \( U^{\leq}_{II} := \{ u \; ; \; x_{II} \leq u_{II} \} \). Using this notation, we define the \( \leq \)-pseudoimage of \( A \) under \( \sigma \) (under \( \tau \)):

\[
\Psi^{\sigma I}_{\leq}(A) := \{ x_1 \; ; \; \exists z \in A \left( \mu_{\sigma,z}(U^I_\leq) > 0 \right) \}, \quad \text{and} \\
\Psi^{\tau II}_{\leq}(A) := \{ x_{II} \; ; \; \exists z \in A \left( \mu_{z,\tau}(U^{\leq}_{II}) > 0 \right) \}.
\]

**Proposition 3.3.** If \( \Gamma \) is a Kechris-Tanaka pointclass, and \( \leq \) and \( A \) are in \( \Gamma \), then for all strategies \( \sigma \) for player I and \( \tau \) for player II, both \( \Psi^{\sigma I}_{\leq}(A) \) and \( \Psi^{\tau II}_{\leq}(A) \) are in \( \Gamma \).

**Proof.** The sets \( U^I_\leq \) and \( U^{\leq}_{II} \) are in \( \Gamma \), so (KT1) and (KT2) give us that the pseudoimage is in \( \exists^\# \Gamma = \Gamma \). \( \square \)

**Proposition 3.4.** Let \( \leq \) be a prewellordering of \( \omega^\omega \), and \( Y \) an arbitrary nonempty set of reals. If \( \sigma \) is a strongly optimal strategy for player I (\( \tau \) a strongly optimal strategy for player II) for the set \( A \), then

\[
\Phi^{\sigma I}_{\leq}(Y) \cap (A^I_\leq) \neq \emptyset \quad \left( \Psi^{\tau II}_{\leq}(Y) \cap (A^II_{\leq}) \neq \emptyset \right).
\]

**Proof.** Let \( y \in Y \). Since \( \sigma \) is strongly optimal, we know that \( \mu_{\sigma,y}(A) = \mu_{\sigma,y}(A^I_\leq) = 1 \). Let now \( x \in A^I_\leq \) such that the \( \leq \)-rank of \( x_1 \) is minimal. Clearly, \( \mu_{\sigma,y}(U^I_\leq) = 1 \), and so \( x_1 \in \Psi^{\sigma I}_{\leq}(Y) \). \( \square \)

With the notion of pseudoimage at hand, we can prove a rather abstract version of the boundedness lemma for the following type of games:

**Definition 3.5.** Let \( A \subseteq \omega^\omega \). Then \( A \) is called of boundedness type if there is a boldface Kechris-Tanaka pointclass \( \Gamma \) such that \( \bar{\Gamma} \) has the prewellordering property and a set \( X \in \bar{\Gamma} \setminus \Gamma \) with \( \bar{\Gamma} \)-norm \( \varphi : X \to \alpha \) such that:

\[
(B1) \quad x_1 \notin X \text{ implies } x \notin A, \quad \text{and} \\
(B2) \quad \text{there is a cofinal function } \rho : \omega^\omega \to \alpha \text{ with the property that } \rho(x_{II}) \geq \varphi(x_1) \text{ implies } x \notin A.
\]

**Theorem 3.6** (Boundedness Lemma). Assume \( \text{pBl-AD} \). Let \( A \) be of boundedness type. Then there is a strongly optimal strategy for player II.

**Proof.** Let \( \Gamma, X \subseteq \bar{\Gamma} \setminus \Gamma \), a \( \bar{\Gamma} \)-norm \( \varphi : X \to \alpha \), and a cofinal function \( \rho : \omega^\omega \to \alpha \) witness that \( A \) is of boundedness type. Let \( \leq \varphi \) be the prewellordering associated to \( \varphi \).

By \( \text{pBl-AD} \), we know that there is either a strongly optimal strategy for player I or for player II. Towards a contradiction, suppose that there is a strongly optimal strategy \( \sigma \) for player I.
Since \( \varphi \) is a \( \bar{\Gamma} \)-norm, the sets \( U_{x}^{\leq \varphi} \) are in \( \Gamma \) for all \( x \in \omega^{\omega} \). Thus \( B := \{ (x, z) \mid \mu_{\sigma, z}(U_{x}^{\leq \varphi}) > 0 \} \) is in \( \Gamma \) by (KT2). By (KT1), \( \Psi_{x}^{\sigma, 1} (\omega^{\omega}) = \exists B \in \Gamma \).

Now the Boundedness Lemma 2.1 gives us an ordinal \( \beta < \alpha \) such that for all \( y \in \Psi_{x}^{\sigma, 1} (\omega^{\omega}) \) we have \( \varphi(y) < \beta \). Since \( \rho \) was cofinal, let \( z \) be any real such that \( \rho(z) > \beta \).

If \( x \in \Psi_{x}^{\sigma, 1} (\{z\}) \subseteq B \), then \( \varphi(x_{1}) < \rho(x_{1}) = \rho(z) \) by choice of \( z \), so \( x \notin A \). But this contradicts Proposition 3.4, so \( \sigma \) can’t be strongly optimal for player I.

□

Note that the proof of the Boundedness Lemma 3.6 showed that player I can’t have a weakly optimal strategy, since we only needed information about the measures \( \mu_{\sigma, z} \) (as opposed to all measures \( \mu_{\sigma, \tau} \)). This yields the following corollary:

**Corollary 3.7.** Assume blBl-AD. Let \( A \) be of boundedness type. Then there is a weakly optimal strategy for player II.

### 3.2. Analytic Blackwell determinacy: An application

Tony Martin announced in [Ma98] a proof of sharps from Bl-Det(\( \Pi_{1}^{1} \)). The proof is unpublished, therefore we include it in this paper with Martin’s permission. The proof given here is essentially Martin’s original proof, but uses the language of the simulation technique developed by the present author for the applications in Section 4. Our analysis of Martin’s proof shows that instead of Bl-Det(\( \Pi_{1}^{1} \)) we only need blBl-Det(\( \Pi_{1}^{1} \)).

**Theorem 3.8 (Martin).** Suppose that blBl-Det(\( \Pi_{1}^{1} \)) holds. Then for all reals \( x \) the sharp \( x^{\#} \) exists.

**Proof.** As usual, we show the theorem for \( x = 0 \) since it relativizes easily. We shall prove the theorem with Harrington’s original proof (cf. [Ha78]) in mind.

We consider the following game: Player I must play a code for a countable ordinal \( \alpha \); if player I succeeds then player II’s play must code a model with domain \( \omega \) that is an end extension of \( L_{\alpha} \). Denote by \( A \) the set of winning plays for player I. This set is \( \Pi_{1}^{1} \) and of boundedness type, hence by assumption blindfoldedly Blackwell determined, and by Corollary 3.7, player II has a weakly optimal strategy \( \tau \). By Harrington’s proof, it is enough to show that \( \tau \) has the following property:

Let \( \gamma < \alpha < \omega_{1} \) and \( \beta < \omega_{1} \) be ordinals. If \( b \in L_{\beta} \) with \( b \subseteq \gamma \), and \( b \) codes a wellordering of \( \gamma \) of order type \( \alpha \), then for every \( z \in \text{WO} \) with \( \|z\| = \beta \), we have \( b \in L_{\gamma + \omega_{2}}[z, \tau] \).
Set $B := (2^\omega \setminus A)^{\Pi_1}(z)$. For every formula $\varphi$ and every natural number $m$, we define

$$B_{\varphi,m} := \{x : x \in B \& \forall \delta < \gamma (\delta \in b \leftrightarrow L_{\gamma+\omega}[x_{\Pi}] \models \varphi[x_{\Pi}, m, \delta])\}.$$ 

By the usual argument, we get $\bigcup_{\varphi,m} B_{\varphi,m} = B$, and thus know that at least one of the sets $B_{\varphi,m}$ must have positive measure. Fix this set $B^* := B_{\varphi^*,m^*}$.

By the continuity property of Borel measures we know that we find an $s \in 2^{<\omega}$ such that the basic open set $[s]$ has positive measure and we have

$$\mu_{z,\tau}(B^* \cap [s]) > \frac{1}{2} \cdot \mu_{z,\tau}([s]).$$

We now finish the proof by defining $b$ in $L_{\gamma+\omega}2[z, \tau]$ as follows:

$$\delta \in b \iff \mu_{z,\tau}\{x : L_{\gamma+\omega}[x_{\Pi}] \models \varphi^*[\delta, m^*, x_{\Pi}]) \cap [s]\} > \frac{1}{2} \cdot \mu_{z,\tau}([s]).$$

"⇒" If $\delta \in b$, then $B^* \subseteq \{x : L_{\gamma+\omega}[x_{\Pi}] \models \varphi^*[\delta, m^*, x_{\Pi}])\}$, so the claim follows from the choice of $s$.

"⇐" If $\delta \notin b$, then we again invoke the choice of $s$ to see that more than half of the measure of $[s]$ is taken by reals $x$ that are in $B^*$, thus they can’t satisfy $L_{\gamma+\omega}[x_{\Pi}] \models \varphi^*[\delta, m^*, x_{\Pi}]).$ \hfill \Box

Consequently, at the $\Pi_1^1$ level, all mentioned forms of determinacy are equivalent:

**Corollary 3.9.** The following are equivalent:

1) Det$(\Pi_1^1)$,
2) Bl-Det$(\Pi_1^1)$,
3) pBl-Det$(\Pi_1^1)$,
4) bBl-Det$(\Pi_1^1)$, and
5) for all $x \in \omega^\omega$, $x^\#$ exists.

This result suggests looking for other pointclasses with this property. It is unknown whether similar theorems can be proved for other pointclasses.\footnote{In [MaNeVe03, Theorem 5.1, Corollary 5.3, Theorem 5.4, Theorem 5.6], the authors show equivalence of perfect information Blackwell determinacy and standard determinacy for the pointclasses $\Delta^0_3$, $\Pi^0_2$, and $\mathcal{D}(\omega^2 \Pi^0_1)$; cf. Theorem 2.7.}

### 3.3. Another application.

We show that under Blackwell determinacy no wellordered sequence of pairwise different Borel sets can have length $\omega_2$.

This has been proved under the assumption of AD by Leo Harrington [Ha78, Theorem 4.5]. Our proof follows Harrington’s proof closely, so we shall only mention the changes necessary for the Blackwell situation.

\footnote{In [MaNeVe03, Theorem 5.1, Corollary 5.3, Theorem 5.4, Theorem 5.6], the authors show equivalence of perfect information Blackwell determinacy and standard determinacy for the pointclasses $\Delta^0_3$, $\Pi^0_2$, and $\mathcal{D}(\omega^2 \Pi^0_1)$; cf. Theorem 2.7.}
Lemma 3.10. Assume blBl-AD. Then for every sequence of sets of reals $\langle B_\xi ; \xi < \omega_1 \rangle$ with $B_\xi \in \Sigma^0_\beta$ for some fixed $\beta < \omega_1$ there are $\gamma_0$ and $\gamma_1$ such that $B_{\gamma_0} = B_{\gamma_1}$.

Corollary 3.11. blBl-AD implies $\neg$AC.

From Lemma 3.10 we can easily deduce the theorem modulo an assumption which will be proved later.

Theorem 3.12. Assume either blBl-AD+ “$\aleph_2$ is regular” or Bl-AD. (Corollary 4.17 will show that Bl-AD implies that $\aleph_2$ is a regular cardinal.) Then no wellordered sequence of pairwise different Borel sets has length $\omega_2$.

Proof of Lemma 3.10. The proof is essentially Harrington’s AD-proof with modifications where necessary. In this proof we shall use notation rather loosely and identify the reals played in the game with the objects coded by them. For example, when player I plays a code $v$ for a pair $\langle \eta,T \rangle$ against the strategy $\tau$, we denote the product measure derived from this by $\mu_{\langle \eta,T \rangle,\tau}$ instead of $\mu_{v,\tau}$.

Fix a universal $\Sigma^0_\beta$ set $C$ with Borel code $c$ and play the game with the following winning conditions:

Player I plays $\langle \eta,T \rangle$ where $T$ is a tree on $\omega$ and $\eta \in T$ with defined height $h_T(\eta) \in \omega_1$; player II plays $\langle b,w \rangle$ where $b = \langle b_i ; i \in \omega \rangle$ is a countable sequence of reals and $w \in WO$. Player II wins if

$$\|w\| > h_T(\eta) & \{ C_{b_i} ; i \in \omega \} = \{ B_\xi ; \xi < \|w\| \}.$$  

This game is clearly of boundedness type, so by Corollary 3.7, there is a weakly optimal strategy $\tau$ for player II.

We fix some $\gamma > \beta$ with $\omega \cdot \gamma = \gamma$. We define a prewellordering

$$x \leq y : \iff x \text{ codes } \langle b,w \rangle, y \text{ codes } \langle b^*, w^* \rangle, \text{ and } \|w\| \leq \|w^*\|.$$  

Consider the set $X_\gamma := \{ x ; x \text{ codes } \langle \eta,T \rangle \text{ with } h_T(\eta) < \gamma \}$ which is a Borel set. Then $\{ w ; \exists b \exists y \in \Psi_{\leq \gamma}^r(H(X_\gamma))(y = \langle b,w \rangle) \}$ is a $\Sigma^1_1$ subset of WO ($\Sigma^1_1$ is a Kechris-Tanaka pointclass), and thus it is bounded by some countable ordinal. Call that ordinal $\rho$ and assume without loss of generality that $\rho > \gamma$.

We shall now show that there is a $\xi < \rho$ such that $B_\xi = B_\rho$.

Suppose that the claim is false. Fix an enumeration of $\rho = \{ \xi_i ; i \in \omega \}$. Then for all natural numbers $i$, the symmetric difference $D_i := B_{\xi_i} \triangle B_\rho$ is nonempty, say $d_i \in D_i$. Define a function $f \in 2^\omega$ by

$$f(i) = 1 : \iff d_i \notin B_\rho.$$
and take a tree $T$ to be $L_{c+\omega}[\tau,d,f,c]$-generic in the sense of [Ha78, Definition 2.8]. There must be an $\eta \in T$ such that $h_T(\eta) = \varrho$ by definition of genericity. Then

$$(\psi_{\eta,T})_{\mu_{\eta,T},\tau}(\{ (\eta, T) \ast (b, w) ; \exists i \forall j (d_j \in C_{b_i} \iff f(j) = 0) \}) = 1,$$

since the condition is true for all winning plays $(b, w)$ for player II against $(\eta, T)$.

Using Steel forcing as in Harrington’s original proof, we get $T'$ with $\eta \in T'$, $h_{T'}(\eta) < \gamma$ and $\psi_{\eta,T'}$. This means that

$$\gamma := \{ (\eta, T') \ast (b, w) ; \exists i \forall j (d_j \in C_{b_i} \iff f(j) = 0) \}$$

is a $\mu_{(\eta,T')},\tau$-measure 1 set. Since $h_{T'}(\eta) < \gamma$, we know that $(\eta, T') \in X_\gamma$, so we can get some $(b, w) \in \Psi^H_\gamma(\{ (\eta, T') \}) \cap (A^H_\{ (\eta, T') \}) \cap Y_{II}$. In particular, $||w|| < \varrho$. This is enough to derive a contradiction. □

4. Infinitary combinatorics under the assumption of Blackwell determinacy.

4.1. Infinitary combinatorics under AD. We shall give a brief overview of the basic structure theory below $\Theta$ under AD. For a historical account of infinitary combinatorics under AD, we refer the reader to [Ka94, Chapter 28] and [Lö02b, §3.1]. The reader can also find proofs or pointers to proofs there.

**Definition 4.1.** Let $\kappa$ be a cardinal. We say that $\kappa$ has the strong partition property if $\kappa \rightarrow (\kappa)^\kappa$ holds.

The strong partition property of any infinite cardinal severely violates the Axiom of Choice. Of the rich structure theory of $\omega_1$ under AD, the following two results will be of most interest to us:

**Theorem 4.2** (Solovay’s Lemma). Assume AD. Then for every $A \subseteq \omega_1$ there is a real $x \in \omega^\omega$ such that $A \in L[x]$.

**Theorem 4.3** (Martin). Assume AD. Then $\aleph_1$ has the strong partition property.

We shall use the following convention: If $U$ is a normal $\sigma$-complete ultrafilter on an ordinal $\alpha$ and $\beta$ is another ordinal, then $\beta^\alpha / U$ is a well-ordered structure (using DC). We shall identify this structure with its ordertype.

**Definition 4.4.** Let $\kappa$ be a cardinal with the strong partition property and $\mu$ a normal measure on $\kappa$. We then define a sequence of well-ordered structures $\langle \kappa^\mu_n ; n \leq \omega \rangle$ as follows:

- $\kappa^\mu_1 := \kappa$,
- $\kappa^\mu_{n+1} := (\kappa^\mu_n)^\kappa / \mu$, and
• $\kappa_n^\mu := \sup\{\kappa_n^\mu \mid n \in \omega\}$.  

This sequence is called the **Kleinberg sequence derived from $\mu$**.

**Theorem 4.5** (Kleinberg). Let $\kappa$ be a cardinal with the strong partition property, $\mu$ be a normal ultrafilter on $\kappa$, and $\langle \kappa_n^\mu \mid i \leq \omega \rangle$ the derived Kleinberg sequence. Then:

1. For all natural numbers $n \in \omega$, $\kappa_n^\mu < \kappa_{n+1}^\mu$.
2. $\kappa_1^\mu$ and $\kappa_2^\mu$ are measurable,
3. for all $n \geq 2$, $\text{cf}(\kappa_n^\mu) = \kappa_2^\mu$,
4. for all $n \geq 3$, $\kappa_n^\mu$ is a Jónsson cardinal, and
5. $\sup\{\kappa_n^\mu \mid n \in \omega\}$ is a Rowbottom cardinal.

Moreover, if $\kappa^\mu / U = \kappa^+$, then $\kappa_{n+1}^\mu = (\kappa_n^\mu)^+$ for all $n \in \omega$.

**Corollary 4.6.** Assume AD. Then $\aleph_1$ and $\aleph_2$ are measurable,3 $\aleph_n$ for $3 \leq n < \omega$ is Jónsson, and $\aleph_\omega$ is Rowbottom.

Let $\lambda < \kappa$ be regular cardinals. Let us denote by $C_\kappa$ the closed unbounded filter on $\kappa$. Then we define

$$C_\kappa^\lambda := \{X \subseteq \kappa \mid \exists C \in C_\kappa (C \cap \{\xi < \kappa \mid \text{cf}(\xi) = \lambda\} \subseteq X)\}.$$ 

We furthermore define the projective ordinals by

$$\delta_n^\kappa := \sup\{\xi \mid \xi \text{ is the length of a prewellordering of } \omega^\omega \text{ in } \Delta_n^1\}.$$ 

**Fact 4.7.** Let $n$ be a natural number. Assume AD. Then:

1. (Kunen, Martin 1971) $\delta_{2n+2}^1 = (\delta_{2n+1}^1)^+$,
2. (Kechris 1974) $\delta_{2n+1}^1$ is the cardinal successor of a cardinal of cofinality $\omega$,
3. (Martin, Kunen 1971) all $\delta_n^1$ are measurable,
4. (Martin, Kunen 1971) $\delta_1^1 = \aleph_1$, $\delta_2^1 = \aleph_2$, $\delta_3^1 = \aleph_{\omega+1}$, and $\delta_4^1 = \aleph_{\omega+2}$,
5. (Martin, Paris 1971) $\delta_1^1 \rightarrow (\delta_1^1)^\delta_1$, and for all $\alpha < \delta_2^1$, the relation $\delta_2^1 \rightarrow (\delta_2^1)^\alpha$ holds,
6. (Martin 1971) for all $\alpha < \omega_1$ the partition relation $\delta_{2n+1}^1 \rightarrow (\delta_{2n+1}^1)^\alpha$ holds,
7. (Kunen 1971) the $\omega$-cofinal measure $C_{\delta_{2n+1}^1}^\omega$ is a normal measure on $\delta_{2n+1}^1$ with $\delta_{2n+1}^1 \delta_{2n+1}^1 / C_{\delta_{2n+1}^1}^\omega = \delta_{2n+2}^1$, and
8. (Martin 1980) $\delta_3^1 / C_{\delta_3^1}^\omega = \aleph_{\omega+2}$ and $\delta_3^1 / C_{\delta_3^1}^\omega = \aleph_{\omega+1}$, and these two cardinals are measurable.
9. (Jackson 1983) Let $E$ be the function recursively defined by $E(1) = 0$ and $E(n+1) = \omega^{E(n)}$. Then for every $n \in \omega$,

$$\delta_{2n+1}^1 = \aleph_{E(2n+1)+1}.$$ 

---

3The measurability of $\aleph_2$ had been proved by Solovay long before that using ideas of Tony Martin [Ka94, Theorem 28.6].
and all odd projective ordinals have the strong partition property.

4.2. The combinatorial theory of $\aleph_1$ under the assumption of Blackwell determinacy. In this section we shall prove that the combinatorial theory of $\aleph_1$ behaves under Blackwell determinacy axioms as it does under $\text{AD}$ in many respects. We start with the Blackwell analogue of Solovay’s Lemma 4.2.

**Theorem 4.8.** Assume $\text{blBl-AD}$. Then for every $Y \subseteq \omega_1$ there is an $a \in \omega_\omega$ and a formula $\vartheta$ such that

$$\xi \in Y \iff L[a] \models \vartheta[\xi,a].$$

**Proof.** The proof follows Solovay’s original proof idea closely; cf. [Ka94, Theorem 28.5].

We look at the following game: Given $Y \subseteq \omega_1$, player I plays an ordinal $\alpha$ and player II tries to play $Y \cap \beta$ for some $\beta > \alpha$. Obviously, $A$ is of boundedness type. Thus by Corollary 3.7, we have a weakly optimal strategy $\tau$ for player II.

For each $z \in \text{WO}$, we let $B_z := \{y : y_1 = z \& \text{there is an } i \in \omega \text{ such that } \langle \omega, E_{y_1} \rangle \text{ and } \langle \omega, E_{(y_1)i} \rangle \text{ are isomorphic}\}$.

Since the sets $B_z$ are $\Sigma^1_1$, and $\Sigma^1_1$ is a Kechris-Tanaka pointclass by Theorem 3.2, we have that

$$\varphi(v_0,v_1) : \iff v_0 \in \text{WO} \& v_1 \text{ is a strategy for player II, and } \mu_{v_0,v_1}(B_{v_0}) > \varepsilon$$

is $2\text{-}\Pi^1_1$ (in the Hausdorff difference hierarchy). By Shoenfield’s absoluteness lemma, $2\text{-}\Pi^1_1$ relations are absolute.

We let $\mathbb{P}_\xi$ denote the forcing partial order adding a bijection from $\omega$ to $\xi$, and $\check{\xi}$ be a name for this bijection (defined uniformly in $\xi$). Then the formula $\vartheta(v_0,v_1)$ saying “$v_0$ is an ordinal, $v_1$ is a strategy for player II, and $\mathbb{P}_{v_0} \models \varphi(z_{v_0},v_1)$” proves the theorem as in Solovay’s original proof. □

**Corollary 4.9.** Assume $\text{blBl-AD}$. Then:

1) For every $Y \subseteq \omega_1$ there is a real $a \in \omega_\omega$ such that $Y \in L[a]$, and

2) $\delta^1_2 = \aleph_2$, and

3) there is an inner model with a strong cardinal.

**Proof.** The first two claims are standard (note that the second claim uses the existence of sharps which we get from Corollary 3.9). As to the third, sharps plus “$\delta^1_2 = \aleph_2$” give an inner model with a strong cardinal by a theorem of Steel and Welch [StWe98, Theorem 3.9]. □

Actually, the methods of Steel and Welch allow to go a bit beyond a strong cardinal. Even beyond that, if we allow a measurable cardinal in our metatheoretical assumptions, Hjorth claims that the validity of Solovay’s Lemma alone implies the existence of an inner model with a Woodin cardinal (personal communication, 2001).
4.3. A general technique to prove infinite partition relations. We shall now develop the analogue of a general technique of proving infinitary partition relations. The technique is in essence due to Martin. Terminology and notation are due to Kechris [Ke78]:

**Definition 4.10.** Let $\lambda \leq \kappa$ be ordinals. We call a family of objects

$$C := \langle C_\xi, C_{\xi, \vartheta}, f^\xi; \xi < \omega \cdot \lambda, \vartheta < \kappa \rangle$$

a $\langle \lambda, \kappa \rangle$-**Martin system** if it satisfies the following conditions:

(M1) For every $\xi < \omega \cdot \lambda$ and every $\vartheta < \kappa$, $C_\xi$ and $C_{\xi, \vartheta}$ are sets of reals.

(M2) For every $\xi < \omega \cdot \lambda$, $f^\xi : C_\xi \to \kappa$ is a function.

(M3) For every $\xi < \omega \cdot \lambda$ and every $\vartheta < \kappa$, we have that $C_{\xi, \vartheta} \subseteq \bigcap_{\eta \leq \xi} C_\eta$.

For any given Martin system $C$, we shall use the following notation:

The set $\text{core}(C) := \bigcap_{\xi \in \omega \cdot \lambda} C_\xi$ will be called the **core of** $C$. For each element $x$ of the core, we can define the function $f_x \in [\kappa]^{\omega \cdot \lambda}$ by $f_x(\xi) := f^\xi(x)$.\(^4\) For any continuous function $\sigma : \omega^\omega \to \omega^\omega$, we define

$$i^\sigma_\xi(\xi, \vartheta) := \sup \{ f^\xi(x) + 1 \mid x \in \sigma^n C_{\xi, \vartheta} \}.$$

**Definition 4.11.** Let $C$ be any $\langle \kappa, \lambda \rangle$-Martin system. We call $C$ **perfectly good** if the following conditions hold:

(P1) For any $f \in [\kappa]^{\omega \cdot \lambda}$ there is an $x \in \text{core}(C) \cap \bigcap_{\xi < \omega \cdot \lambda} C_{\xi, f(\xi)}$ such that $f_x = f$.

(P2) For any continuous function $\sigma : \omega^\omega \to \omega^\omega$ with the property

$$\sigma^n \bigcap_{\eta \leq \xi} C_\eta \subseteq C_\xi,$$

we have $i^\sigma_\xi(\xi, \vartheta) < \kappa$.

The existence of a perfectly good Martin system gives rise to an abstract proof of Martin’s Theorem 4.3:

**Theorem 4.12** (Martin). Let $\lambda \leq \kappa$ be ordinals, $\kappa$ a regular cardinal, and suppose that there is a perfectly good $\langle \lambda, \kappa \rangle$-Martin system. Then AD implies that $\kappa \rightarrow (\kappa^\lambda)$ holds. (Cf. [Ke78, Lemma 11.1].)

Using the simulation technique, we define the Blackwell analogue of this notion:

\(^4\)We extend the functions $f^\xi$ to $\omega^\omega$ by

$$f^\xi(x) := \begin{cases} f^\xi(x) & \text{if } x \in C_\xi \\ \infty & \text{otherwise.} \end{cases}$$

In the following we shall not distinguish between the functions $f^\xi : C_\xi \to \kappa$ and $f^\xi : \omega^\omega \to \kappa \cup \{\infty\}$.
Fixing any Martin system $C$ we define the prewellordering
\[ x \leq_{\xi} y :\iff f^\xi(x) \leq f^\xi(y) \lor y \notin C_\xi , \] and the functions
\[ \psi^\sigma_\xi(C, \vartheta) := \sup\{ f^\xi(x) + 1; x \in \Psi^\sigma_\leq \xi(C, \vartheta) \} , \] and
\[ \psi^\tau_\xi(C, \vartheta) := \sup\{ f^\xi(x) + 1; x \in \Psi^\tau_\leq \xi(C, \vartheta) \} . \]

We shall call a mixed strategy $\sigma$ for player I $C$-adequate if for all $\xi < \omega \cdot \lambda$, the following holds:
\[ \Psi^\sigma_\leq \xi \left( \bigcap_{\eta \leq \xi} C_\eta \right) \subseteq C_\xi . \]

Similarly, we call a mixed strategy $\tau$ for player II $C$-adequate if for all $\xi < \omega \cdot \lambda$, we have:
\[ \Psi^\tau_\leq \xi \left( \bigcap_{\eta \leq \xi} C_\eta \right) \subseteq C_\xi . \]

Now we can extend Definition 4.11 naturally to the Blackwell context:

**Definition 4.13.** Let $C$ be any $\langle \kappa, \lambda \rangle$-Martin system. We call $C$ imperfectly good if it satisfies (P1) and:

(I2) For any adequate strategy $\sigma$ and any $\xi < \omega \cdot \lambda$ and $\vartheta < \kappa$, we have $\psi^\sigma_\xi(\vartheta, \vartheta) < \kappa$.

Note that by (P1), if $C$ is an imperfectly good Martin system, then for all $\xi$, $f^\xi \upharpoonright C_\xi \supseteq \{ \nu; \xi \leq \nu \}$ holds.

**Theorem 4.14** (Abstract Martin Theorem). Let $\lambda \leq \kappa$ be ordinals, $\kappa$ a regular cardinal in $\text{L}(\mathbb{R})$ and suppose that there is an imperfectly good $\langle \lambda, \kappa \rangle$-Martin system in $\text{L}(\mathbb{R})$. Then $\text{Bl-AD}$ implies that $\kappa \rightarrow (\kappa)^\lambda$ holds. (If $\lambda < \omega_1$, we only need $\text{bBl-AD}$.)

**Proof.** In this proof we shall mostly be using only $\text{bBl-AD}$. There is only a single point in the proof where we have to use Theorem 2.8 if $\lambda$ is uncountable. The proof follows Martin’s proof closely: We fix an imperfectly good Martin system $C = \langle C_\xi, C_\xi, \vartheta, f^\xi; \xi < \omega \cdot \lambda, \vartheta < \kappa \rangle$ and a partition $a : [\kappa]^\lambda \rightarrow 2$. We want to show that there is an $a$-homogeneous set. As in the usual proof, we look at the following game:

Given a real $x$ such that $x_1, x_\Pi \in \text{core}(C)$, define $p_x : \lambda \rightarrow \kappa$ by
\[ p_x(\vartheta) := \max\{ \sup\{ f_{x_1}(\vartheta \cdot n + n); n \in \omega \}, \sup\{ f_{x_\Pi}(\vartheta \cdot n + n); n \in \omega \} \} . \]

Now set
\[ x \in A :\iff \exists \xi < \omega \cdot \lambda (x_\Pi \notin C_\xi \land \forall \zeta \leq \xi (x_1 \in C_\zeta) \lor (\forall \xi < \omega \cdot \lambda (x_1 \in C_\xi \land x_\Pi \in C_\xi) \land \neg p_x) \in [\kappa]^\lambda \land a(p_x) = 0) . \]
We shall show that a weakly optimal strategy $\sigma$ for player I in the game with payoff $A$ gives an $\alpha$-homogeneous set for the value 0 (and similarly a weakly optimal strategy for player II gives an $\alpha$-homogeneous set for the value 1).

**Claim 1.** $\sigma$ is $C$-adequate.

**Proof.** Take any $x \in \Psi_{\leq \xi}^{\sigma_1}(\bigcap_{\eta \leq \xi} C_{\eta})$ and $z$ witnessing this, i.e., $\mu_{\sigma,z}^{-}(U_{\leq \xi}^{\leq 1}) > 0$. By definition, $z \in \bigcap_{\eta \leq \xi} C_{\eta}$. Towards a contradiction, suppose that $x_{\xi} \notin C_{\xi}$. Then $U := (U_{\leq \xi}^{\leq 1})_{\{z\}}$ is disjoint from $A$: Every element of $U$ represents a run of the game where player II plays into every $C_{\eta}$ for $\eta \leq \xi$, but player I doesn’t play into $C_{\xi}$, hence player I loses. But $\mu_{\sigma,z}^{-}(U) = \mu_{\sigma,z}^{-}(U_{\leq \xi}^{\leq 1}) > 0$, contradicting the weak optimality of $\sigma$. □

With Claim 1 we know by (I2) that $\psi_{C}^{\sigma}(\xi, \vartheta) < \kappa$ for all $\xi < \omega \cdot \lambda$ and $\vartheta < \kappa$.

**Claim 2.** Let $\xi < \omega \cdot \lambda$ and $\vartheta < \kappa$. Then for every $x \in C_{\xi, \vartheta}$ the set \[ \{ u : f_{x}(u_{1}) < \psi_{C}^{\sigma}(\xi, \vartheta) \} \] has $\mu_{\sigma,x}-$measure 1.

**Proof.** Suppose not. Then there is an $x$ such that \[ \{ u : f_{x}(u_{1}) \geq \psi_{C}^{\sigma}(\xi, \vartheta) \} \] has positive $\mu_{\sigma,x}-$measure. Let $r$ be such that $f_{x}(r) = \psi_{C}^{\sigma}(\xi, \vartheta)$. Then $r \in \Psi_{\leq \xi}^{\sigma_1}(C_{\xi, \vartheta})$. But then $f_{x}(r) < \psi_{C}^{\sigma}(\xi, \vartheta)$ by the definition of $\psi_{C}^{\sigma}$. □

The set \[ D := \{ g : \forall \xi < \omega \cdot \lambda \forall \vartheta < \kappa (\xi < \rho \& \vartheta < \rho \rightarrow \psi_{C}^{\sigma}(\xi, \vartheta) < \vartheta) \} \] is a closed unbounded set in $\kappa$ (using the assumptions that $\kappa$ is regular in $L(\mathbb{R})$ and that the Martin system $C$ was in $L(\mathbb{R})$). As in the classical proof, let \[ [D]_{\alpha}^{\lambda} := \left\{ g \in [D]^{\lambda}; \exists f \in [\kappa]^{\omega \cdot \lambda} \forall \vartheta < \lambda (g(\vartheta) = \sup_{n \in \omega} f(\omega \cdot \vartheta + n)) \right\}. \]

It is enough to show $\alpha$” $[D]_{\alpha}^{\lambda} = \{0\}$ to complete the proof.

Pick any $g \in [D]_{\alpha}^{\lambda}$. This fact is witnessed by some $f \in [\kappa]^{\omega \cdot \lambda}$ such that $g(\vartheta) = \sup_{n \in \omega} f(\omega \cdot \vartheta + n)$. By Property (P1), we find an element $z \in \bigcap_{\xi \leq \omega \cdot \lambda} C_{\xi, f(\xi)}$ with $f_{z} = f$. We let player II play this $z$ and set $A^{\ast} := A_{\{z\}}^{1}$. If $U_{\xi} := \{ u : f_{x}(u_{1}) < \psi_{C}^{\sigma}(\xi, f(\xi)) \}$, then Claim 2 tells us that $\mu_{\sigma,z}^{-}(U_{\xi}) = 1$. By Theorem 2.8,\(^5\) we get

$$
\mu_{\sigma,z}^{-}\left( A^{\ast} \cap \bigcap_{\xi \leq \omega \cdot \lambda} U_{\xi} \right) = 1,
$$

\(^5\)If $\lambda < \omega_{1}$, $\sigma$-completeness of the measure suffices, and we don’t need to invoke $\text{BI-AD}$ here.
in particular, this set is nonempty. We pick any $w \in A^* \cap \bigcap_{\xi < \omega} U_\xi$; set $x := w \ast z$ and finish the proof as in the classical proof by proving $g = p_x$, and thus $a(g) = 0$ because $x \in A$. \hfill \Box

With the Abstract Martin Theorem 4.14 at hand, we can reproduce part of the structure theory under AD under the assumption of Bl-AD.

**Theorem 4.15.** Bl-AD implies that $\omega_1 \rightarrow (\omega_1)^\omega$ holds.

**Proof.** By Theorem 4.14, we just have to show that there is an imperfectly good $\langle \omega_1, \omega_1 \rangle$-Martin system in $L(\mathbb{R})$. The system is defined exactly as in the perfect information case (cf. [Ke78]).

For $\xi, \vartheta < \omega_1$, we define:

$$C_\xi := \left\{ x ; \text{ EMB}(x^+) \& \xi, t_{x(0)}^{M(x^+, \xi+\omega)}[\xi] \in \text{wfp}(M(x^+, \xi + \omega)) \right\},$$

$$f_\xi(x) := t_{x(0)}^{M(x^+, \xi+\omega)}[\xi],$$

$$C_{\xi, \vartheta} := \left\{ x ; \forall \xi^* \leq \xi \exists \vartheta^* \leq \vartheta \left( x \in C_{\xi^*} \& t_{x(0)}^{M(x^+, \xi^*+\omega)}[\xi^*] \leq \vartheta^* \right) \right\}.$$  

That $\mathcal{C}$ is a Martin system is immediate from the definitions. It is also clear that it is in $L(\mathbb{R})$. (P1) is the same as in the classical case, so we only have to show (I2) (we only show it for strategies for player I):

First of all, notice that $C_{\xi, \vartheta}$ is Borel. Since $\leq_\xi$ is a $\Pi^1_1$ norm, $U_{\xi}^{\leq_\xi}_1$ is $\Sigma^1_1$. We can combine Theorem 3.2 and Proposition 3.3 to get that $\Psi^{\leq_\xi}_1(C_{\xi, \vartheta})$ is $\Sigma^1_1$.

We define a function $e_\xi : C_\xi \rightarrow \text{WO}$ as follows:

For $x \in C_\xi$, let $E_{x, \xi}$ be the binary relation of the model $M(x^+, \xi + \omega)$ and

$$S_{x, \xi} := \{ n \in \omega ; \exists N (M(x^+, \xi + \omega) \models N = t_{x(0)}[\xi] \& n E_{x, \xi} N) \}.$$  

Let $s_{x, \xi} : \omega \rightarrow S_{x, \xi}$ be the increasing enumeration of $S_{x, \xi}$. Then we set

$$e_\xi(x)(\gamma(k, \ell)) = 1 : \iff s_{x, \xi}(k) E_{x, \xi} s_{x, \xi}(\ell).$$

Obviously, $e_\xi(x)$ is a code for the ordinal representing $t_{x(0)}^{M(x^+, \xi+\omega)}[\xi]$. Thus $\|e_\xi(x)\| = f_\xi(x)$.

---

6 We shall be using Kanamori’s notation for Ehrenfeucht-Mostowski blueprints and related objects from [Ka94, §9 & p. 393]: We have formulae $\text{EMB}(v_0)$ and $\text{WF}(v_0)$ such that the following are equivalent:

1) $x$ is a remarkable, wellfounded Ehrenfeucht-Mostowski blueprint, and

2) $\text{EMB}(x)$ and $\text{WF}(x)$.

The formula $\text{WF}(v_0)$ says that if the models $M(\alpha, v_0)$ (defined in [Ka94, Lemma 9.4]) exist, then they are wellfounded for all $\alpha$. Most importantly, the formula $\text{EMB}$ is an arithmetic formula. $\langle t_n : n \in \omega \rangle$ is a list of all terms with one free variable denoting ordinals, so if $M$ is a model and $\xi \in M$ is an ordinal, then $t_\xi^M[\xi]$ is an $M$-ordinal. For a (not necessarily wellfounded) model $M$, $\text{wfp}(M)$ denotes the wellfounded part of $M$. We shall also be using the notation $x^+(n) := x(n + 1)$. 
If $\sigma$ is adequate, we can apply $e_\xi$ to the inequality

$$\Psi_{\leq \xi}^{\sigma_1}(C_{\xi, \vartheta}) \subseteq C_{\xi},$$

and receive

$$e_\xi^n \left( \Psi_{\leq \xi}^{\sigma_1}(C_{\xi, \vartheta}) \right) \subseteq WO.$$

The left-hand side is a $\Sigma_1$ set, thus by boundedness, it is bounded in WO, say, by $\zeta < \omega_1$. But this means that $f_\xi^n \left( \Psi_{\leq \xi}^{\sigma_1}(C_{\xi, \vartheta}) \right)$ is bounded by $\zeta$ which was to be shown. $\square$

Kleinberg’s Theorem 4.5 now gives us a Kleinberg sequence starting from $\aleph_1$. We can determine their exact values with the “moreover” part of Theorem 4.5 holds which follows from the following theorem (for a proof, cf. [Kl77]):

**Theorem 4.16** (Solovay). Suppose that:

1) For every real there is a sharp,

2) for every subset $X \subseteq \omega_1$ there is a formula $\vartheta$ and a real $a$ such that $\xi \in X \iff L[a] = \vartheta[\xi, a]$, and

3) the closed unbounded filter on $\aleph_1$ is a normal ultrafilter.

Then $\omega_1^1/\mathcal{C}_{\omega_1}^\omega = \omega_2$. Moreover, $\mathcal{C}_{\omega_1}^\omega$ is a canonical measure.

**Corollary 4.17.** Assume $\text{Bl} - \text{AD}$. Then $\aleph_1$ and $\aleph_2$ are measurable, $\aleph_n$ is Jónsson (for $3 \leq n < \omega$) and $\aleph_\omega$ is Rowbottom.

We now move to look at the odd partition ordinals $\delta_{2n+1}^1$:

**Proposition 4.18.** Assume $\text{pBl-AD}$. Then $\delta_{2n+1}^1 \rightarrow (\delta_{2n+1}^1)^\lambda$ for all $\lambda < \omega_1$.

**Proof.** Since $\lambda < \omega_1$, we can fix a coding of sequences of reals of length $\omega \cdot \lambda$. Let $(\cdot)_\xi : \omega^\omega \rightarrow \omega^\omega$ be the $\xi$th component map of this coding. Let $W$ be a complete $\Pi_{2n+1}^1$ set with $\Pi_{2n+1}^1$ norm $\varphi$ onto $\delta_{2n+1}^1$.

Following [Ke78, Theorem 11.2], we define for $\xi < \omega \cdot \lambda$ and $\vartheta < \delta_{2n+1}^1$

$$C_\xi := \{ x \mid \varphi(x) \in W \},$$

$$f_\xi(x) := \varphi((x)_\xi),$$

$$C_{\xi, \vartheta} := \{ x; \forall \xi^* \leq \xi \exists \vartheta^* \leq \vartheta ((x)_{\vartheta^*} \in W \& \varphi((x)_{\xi^*}) \leq \vartheta^*) \}.$$

With Theorem 4.14, we only have to show that

$$\mathcal{C} := \langle C_\xi, C_{\xi, \vartheta}, f_\xi; \xi < \omega \cdot \lambda, \vartheta < \delta_{2n+1}^1 \rangle$$

is an imperfectly good Martin system. Note that since $\lambda < \omega_1$, we can apply Theorem 4.14 without Bl-AD.

Again, it is obvious that $\mathcal{C}$ is a Martin system in $L(\mathbb{R})$. We use pBl-AD to employ the Martin-Neeman-Vervoort Equivalence Theorem 2.7 and get
The projective ordinals are absolute between $V$ and $L(\mathbb{R})$, so we get that $\delta_n^1 = (\delta_n^1)^{L(\mathbb{R})}$ is regular in $L(\mathbb{R})$ from $(\text{AD})^{L(\mathbb{R})}$ and Fact 4.7 (3). Again, (P1) doesn’t have to be shown, leaving only (I2) to be checked:

Without loss of generality we show (I2) just for strategies of player I. Because $\varphi$ was a $\Pi^1_{2n+1}$-norm, the sets $C_{\xi,\vartheta}$ and $U_{x}^{\leq \xi}$ are $\Delta^1_{2n+1}$. Since $\Sigma^1_{2n+1}$ is a Kechris-Tanaka pointclass, the sets $\Psi_{\leq \xi}(C_{\xi,\vartheta})$ are $\Sigma^1_{2n+1}$. If $\sigma$ is adequate, we have $\{ (x)_{\xi} : x \in \Psi^\sigma_{\leq \xi}(C_{\xi,\vartheta}) \}$ is a $\Sigma^1_{2n+1}$ subset of $W$, hence bounded by Theorem 2.1, and thus $\psi^\sigma(\xi,\vartheta) < \delta^1_{2n+1}$.

**Corollary 4.19.** Assume $\text{pBl-AD}$. Then all odd projective ordinals $\delta^1_{2n+1}$ are measurable.

**Corollary 4.20.** Assume $\text{Bl-AD}$. Then $\delta^1_3 = \aleph_{\omega+1}$.

**Proof.** By the Martin-Neeman-Vervoort Equivalence Theorem 2.7 we know that PD holds whence (by a result of Kechris and Moschovakis [Ke78, Theorem 9.1 (5)]) the projective ordinals are strictly increasing and in particular $\delta^1_3 > \delta^1_2 = \aleph_2$ (using Corollary 4.9).

By Corollary 4.19, $\delta^1_3$ is regular. Using Theorem 4.5, we know the cofinalities below $\aleph_{\omega+1}$, since $\langle \aleph_n : n \geq 1 \rangle$ is a Kleinberg sequence: $\text{cf}(\aleph_n) = \aleph_2$ (for $n \geq 3$) and $\text{cf}(\aleph_\omega) = \aleph_0$. So, $\delta^1_3 \geq \aleph_{\omega+1}$. Since by Theorem 3.8 sharps for reals exist, we can employ the Martin-Solovay analysis of $\Sigma^1_3$ sets under the existence of sharps (cf. [Ke78, Theorem 6.3]) and get $\delta^1_3 \leq \aleph_{\omega+1}$. This proves the claim.

**5. Open problems.**

At the moment, we cannot say much on combinatorics on projective ordinals beyond $\delta^1_3$. Even of the following consequences of AD for $\delta^1_3$ it is unknown whether they hold under any sort of Axiom of Blackwell determinacy [listed from harder to easier]:

1) $\delta^1_3 \rightarrow (\delta^1_3)^{\delta^1_3}$,
2) $\delta^1_3 \rightarrow (\delta^1_3)^{\lambda}$ for all $\lambda < \delta^1_3$,
3) $C^\delta_{\delta^1_3}$ is a normal ultrafilter on $\delta^1_3$,
4) $\mathcal{C}^\delta_{\delta^1_3}$ is a normal ultrafilter on $\delta^1_3$.

Even less is know about the $\delta^1_n$ for $n > 3$. It is not even known that $\delta^1_4 \geq \aleph_{\omega+2}$. For all we know, it could be an ordinal between $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$.

The reason for that is the lack of an analogue of the Moschovakis Coding Lemma.

In the following, we shall abbreviate with $\text{CL}$ the statement of the Moschovakis Coding Lemma (cf. [Mo80, 7D.5]).
Given the Martin-Neeman-Vervoort Equivalence Theorem 2.7, a proof of $\text{CL}$ from Blackwell determinacy would immediately settle many of the questions on projective ordinals under Blackwell determinacy. For example, $\text{PD} + \text{CL}$ is enough to show that $\delta_1^n$ is a cardinal, that $\delta_{2n+2}^1 = (\delta_{2n+1}^1)^+$, and that $\delta_{2n+1}^1$ is the successor of a cardinal of cofinality $\omega$.

But so far, we do not know how to prove $\text{CL}$ from Blackwell Determinacy.

References


[Ve00] Benedikt Löwe, Games, Walks and Grammars, Problems I’ve worked on, Academisch Proefschrift ter verkrijging van de graad van doctor aan de Universiteit van Amsterdam, 2000.

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KOSZUL EQUIVALENCES AND DUALITIES

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For every positively graded algebra $A$, we show that its categories of linear complexes of projectives and almost injectives (see definition below) are both naturally equivalent to the category of graded modules over the quadratic dual algebra $A!$. In case $A = \Lambda$ is a graded factor of a path algebra with Yoneda algebra $\Gamma$, we show that the category $\mathcal{LC}_\Gamma$ of linear complexes of finitely generated right projectives over $\Gamma$ is dual to the category of locally finite graded left modules over the quadratic algebra $\tilde{\Lambda}$ associated to $\Lambda$. When $\Lambda$ is Koszul and $\Gamma$ is graded right coherent, we also prove that the suspended category $\mathcal{G}_\Lambda$ has a (triangulated) stabilization $S(\mathcal{G}_\Lambda)$ which is triangle-equivalent to the bounded derived category of the ‘category of tails’ $fpg\mathcal{G}_\Gamma/L\Gamma$.

1. Introduction and terminology.

The interest on Koszul equivalences and dualities arises mainly in the context of derived categories and, specially, dealing with Koszul algebras (see definitions below). In case $\Lambda$ is a graded Koszul algebra with Yoneda algebra $\Gamma$, Beilinson, Ginzburg and Soergel ([1]) showed the existence of an equivalence between certain full triangulated subcategories of the derived categories $\mathcal{D}(\Lambda Gr)$ and $\mathcal{D}(\Gamma Gr)$. When composing with the canonical duality defined by $\text{Hom}_{\Lambda_0}(-, \Lambda_0)$, one gets a duality between suitable subcategories of $\mathcal{D}(\Lambda Gr)$ and $\mathcal{D}(\Gamma Gr)$. The aim of this paper is to show that Koszul equivalences and dualities also appear naturally between some nice abelian categories associated to positively graded algebras. In this context, no restriction is needed a priori on the graded algebras, although quadratic algebras will play a predominant role as in the context of derived categories. On one side, our results generalize those of Yoshino ([18]) for symmetric and exterior algebras, although quadratic algebras will play a predominant role as in the context of derived categories. On the other, they show that the above mentioned triangulated equivalences of [1] already live in an abelian context.

Throughout the paper, $K$ will be a field and, for every $K$-algebra $R$, we shall denote by $\text{Mod}_R$ (resp. $\text{mod}_R$) the category of all right (resp. finitely generated right) $R$-modules and by $R\text{Mod}$ (resp. $R\text{mod}$) its left-right symmetric version. The term positively graded algebra will stand for a graded $K$-algebra $A = \oplus_{n \geq 0} A_n$ such that $A_0$ is a $K$-algebra isomorphic to a
finite direct product of copies of $K$ and $\dim_K A_1 < \infty$. We shall say that such an algebra is **locally finite** when $\dim_K A_n < \infty$, for all $n \in \mathbb{Z}$. A particular case comes as follows: Let $Q$ be a finite oriented graph or quiver and give $KQ$ a grading by assigning positive degrees to the arrows. Then, for every ideal $I$ of $KQ$, homogeneous with respect to that grading and contained in the ideal generated by the paths of length 2, the algebra $A = KQ/I$ is positively graded and locally finite. Every graded algebra isomorphic to one of this form will be called a **generalized graded factor of a path algebra**, reserving the term **graded factor of a path algebra**, for the case when the grading on $KQ$ is the classical one, i.e., obtained by assigning degree 1 to all arrows. We shall distinguish this latter case by putting $\Lambda \cong KQ/I$, reserving letters $A$, $B$ for general positively graded algebras.

When $Q$ is a finite quiver, we shall identify $Q_0 = \{1, \ldots, n\}$ with the set of vertices and will denote by $Q_n$ the set of paths in $Q$ of length $n$ while $KQ_n$ will be the vector subspace of $KQ$ generated by $Q_n$. When $p$ is a path $i \to \cdots \to j$ in $Q$, we shall put $i = o(p)$ and $j = t(p)$ for the **origin** and **terminus** of $p$. We write paths $\alpha_1 \cdots \alpha_n$ conveying that $t(\alpha_i) = o(\alpha_{i+1})$, for all $i = 1, \ldots, n-1$. The idempotent of $KQ$ given by $i \in Q_0$ will be denoted by $e_i$. The opposite quiver $Q^\text{op}$ of $Q$ has $Q_0^\text{op} = Q_0$ and is obtained from $Q$ by reversing the orientation of the arrows. Whenever $p = \alpha_1 \cdots \alpha_n \in Q_n$, we shall put $p^a = \alpha_n^a \cdots \alpha_1^a$ and then, clearly, $Q_n^a = \{p^a : p \in Q_n\}$.

Notice that if $A$ is a positively graded algebra, then the subalgebra $\overline{A}$ of $A$ generated by the subspace $A_0 \oplus A_1$ is a graded factor of a path algebra. Indeed, there is a uniquely determined (up to isomorphism) finite quiver $Q$ such that $KQ \cong A_0$, as $K$-algebras, and $KQ_1 \cong A_1$, as $KQ_0$-bimodules. Then $Q$ will be called the **quiver of $A$**, although $A$ may not be a graded factor algebra of $KQ$. The isomorphism $KQ_0 \oplus KQ_1 \cong A_0 \oplus A_1$ extends to a homomorphism of graded algebras $\pi_A : KQ \to A$ with image $\overline{A}$, where the grading on $KQ$ is the classical one. If $I = \text{Ker}(\pi_A)$ and $I_2 = \{x \in I : x$ is homogeneous of degree 2\}, then we denote by $\langle I_2 \rangle$ the homogeneous ideal of $KQ$ generated by $I_2$ and $\overline{A} = KQ/\langle I_2 \rangle$ will be called the **quadratic algebra associated to $A$**. We identify $KQ_0 = A_0$ all through the paper and unspecified tensors are tensors over $A_0$. The **canonical duality** $D = \text{Hom}_{A_0}(-, A_0) = \text{Hom}_{KQ_0}(-, KQ_0) : A_0 \text{mod} \to A_0 \text{mod} = \text{mod}_{A_0}$ is ‘inverse to itself’. If $Q^\text{op}$ denotes the opposite quiver of $Q$, then we have canonical isomorphisms of $KQ_0 - KQ_0$-bimodules $KQ_n^\text{op} \cong D(KQ_n)$, for all $n \geq 0$. When $W$ is a $KQ_0 - KQ_0$-subbimodule of either $KQ_n$ or $KQ_n^\text{op}$, we shall denote by $W^\perp$ its orthogonal with respect to the usual duality $KQ_n^\text{op} \otimes KQ_n \cong D(KQ_n) \otimes KQ_n \to KQ_0$. Notice that there are actually two dualities, namely, one for the case when $KQ_n$ is considered as a left $KQ_0$-module and one for the case when it is considered as a right $KQ_0$-modules. They map $p^a \otimes q$ onto $\delta_{pq} \epsilon_{t(q)}$ and $\delta_{pq} \epsilon_{o(q)}$, respectively, where
\( \delta_{pq} \) is the Kronecker symbol. Nonetheless, \( W^{-1} \) is the same for both dualities.

In the above situation, the algebra \( A' = KQ^{\text{op}} / \langle I_2 \rangle \) is called the \textbf{quadratic dual algebra} of \( A \). We shall put \( \tilde{A} = (A')^{\text{op}} \) for the opposite algebra, which is then a graded factor of \( KQ \). Up to graded isomorphism, \( \tilde{A} \) and \( A' \) do not depend on the presentation of \( A \), i.e., do not depend on the choice of the graded homomorphism \( \pi_A : KQ \to A \). If \( A \) and \( B \) are positively graded algebras, we shall say that they are \textbf{orthogonal} when \( A' \cong \tilde{B} \) and that they are \textbf{quadratically equivalent} when \( A \cong \tilde{B} \) (isomorphisms as graded algebras in both cases).

We will be concerned with the category \( Gr_A \) of \( \mathbb{Z} \)-graded right \( A \)-module and its full subcategories \( \text{lfgr}_A \), \( \text{gr}_A \) and \( \text{fpgr}_A \) consisting of locally finite (i.e., \( \dim_K M_i < \infty \), for all \( i \in \mathbb{Z} \)), finitely generated and finitely presented graded right \( A \)-modules, respectively. Of course, \( _A Gr \), \( _A \text{lfgr} \), \( _A \text{gr} \) and \( _A \text{fpgr} \) will stand for the left-right symmetric versions. To some of these categories, and also to some categories of cochain complexes that will eventually appear in the paper, we will add a superindex + or − meaning that we consider the corresponding full subcategory of lower or upper bounded objects (e.g., \( \text{lfgr}_A \) will be the full subcategory of \( \text{lfgr}_A \) with objects \( M = \oplus_{n \in \mathbb{Z}} M_n \) such that \( M_n = 0 \), for all \( n < 0 \)).

An object \( M = \oplus_{n \in \mathbb{Z}} M_n \) of \( Gr_A \) will be called \textbf{generated in degree} \( j \) when \( M_j \) generates \( M \) as a graded \( A \)-module. Dually, \( M \) will be called \textbf{cogenerated in degree} \( j \) when \( M_j \) cogenerated \( M \) as a graded module, i.e., when \( M = M < j = \oplus_{n \leq j} M_n \) and \( N \cap M_j \neq 0 \), for every nonzero graded submodule \( N \) of \( M \). For every \( k \in \mathbb{Z} \), the \( k \)-shifting \( M[k] \) of \( M \) coincides with \( M \) as an ungraded \( A \)-module, but its grading is given by \( M[k]_n = M_{k+n} \), for all \( n \in \mathbb{Z} \). In general, given any cocomplete abelian category \( \mathcal{A} \) and \( X \in \text{Ob}(\mathcal{A}) \), we shall denote by \( \text{Add} (X) \) the full subcategory of \( \mathcal{A} \) with objects the direct summands of direct sums of copies of \( X \). For instance, when \( \mathcal{A} = Gr_A \) and \( X = \oplus_{k \in \mathbb{Z}} A[k] \), \( \text{Add} (X) \) is just the class of projective objects in \( Gr_A \).

The canonical duality \( D : _A Gr \to Gr_A \) (resp. \( _A Gr \to _A Gr \)), for if \( M = \oplus_{n \in \mathbb{Z}} M_n \) is an object of \( _A Gr \) then \( D(M) = : \oplus_{n \in \mathbb{Z}} D(M)_n \), where \( D(M)_n = D(M_{-n}) \) for all \( n \in \mathbb{Z} \), is a graded right \( A \)-module with multiplication \( f \cdot a : x \to f(ax) \), for all \( a \in A_m \) and \( f \in D(M)_m \). Clearly, \( D \) restricts to a duality \( D : _A \text{lfgr} \to _A \text{lfgr} \) ‘inverse to itself’. The objects of \( \text{Add} (\oplus_{k \in \mathbb{Z}} D(A)[k]) \) (full subcategory of \( Gr_A \)) will be called \textbf{almost injective} graded \( A \)-modules. They need not be injective objects of \( Gr_A \), but they are so when \( A \) is right Noetherian. We shall denote by \( \text{Proj}^A_k = \text{Add} (A[k]) \) and \( \text{Inj}^A_k = \text{Add} (D(A)[k]) \) the full subcategories of \( Gr_A \) consisting of projective graded \( A \)-modules generated in degree \( -k \) and almost injective graded \( A \)-modules cogenerated in degree \( -k \), respectively.
In our situation, every \( M \in Gr_A \) has a projective cover in \( Gr_A \) (cf. [5, Prop. 2.6]), \( \varepsilon_M : P(M) \rightarrow M \). We define inductively \( \Omega^0 M = M, \Omega^1 M = \Omega M = \text{Ker} \varepsilon_M \) and, then, \( \Omega^n M = \Omega(\Omega^{n-1} M) \), for all \( n > 0 \). The category \( \text{Mod}_{A_0} \) (resp. \( \text{mod}_{A_0} \)) can be identified with the category of (finitely generated) semisimple graded right \( A \)-modules and homomorphisms as ungraded right \( A \)-modules. When \( X \in \text{Mod}_{A_0} \) and \( M \in Gr_A \), we have an isomorphism \( \text{Ext}_A^n(\Omega M, X) \cong \text{Hom}_A(\Omega^n M, X) \) (extensions and homomorphisms as ungraded right \( A \)-modules!), for all \( n \geq 0 \). In the particular case when \( M = X = \Lambda_0 \), we can consider the Yoneda algebra of \( A, \Gamma = \oplus_{n \geq 0} \text{Ext}_A^n(A_0, A_0) \). It is a graded algebra with the Yoneda product as multiplication. It is positively graded in our sense only in case \( \text{Ext}_A^1(A_0, A_0) \) is finite dimensional, something which always happens when \( A \) is a generalized graded factor of a path algebra. More restrictively, when \( A = A = \Lambda \) is a graded factor of a path algebra, the quiver of \( \Gamma \) is \( Q^\text{op} \).

Indeed, \( \Gamma_0 = \text{End}_{A_0}(\Lambda_0) \cong \Lambda_0 \cong KQ_0 \) and, from the projective presentation of \( \Lambda_0 \) as a left \( \Lambda \)-module, \( 0 \rightarrow \Lambda_{\geq 1} \rightarrow \Lambda_0 \rightarrow 0 \), one immediately gets that \( \Gamma_1 = \text{Ext}_A^1(\Lambda_0, \Lambda_0) \cong \text{Hom}_A(\Lambda_{\geq 1}, \Lambda_0) \cong \text{Hom}_{A_0}(\Lambda_1, \Lambda_0) \cong D(KQ_1) \)

Then \( \Gamma_0 \oplus \Gamma_1 \) can be identified with \( KQ_0^{\text{op}} \oplus KQ_1^{\text{op}} \).

A positively graded algebra \( A \) is a \textbf{Koszul algebra} in case \( \Omega^n(A_0) \) is locally finite and generated in degree \( n \), for all \( n \geq 0 \). In that case, \( A = \Lambda \) is a graded factor of a path algebra and \( \Gamma \cong A^! \).

The organization of the paper goes as follows: Let \( A = \oplus_{n \geq 0} A_n \) be a positively graded algebra with quiver \( Q \). In Section 2 we show that the graded versions of \( - \otimes A \) and \( \text{Hom}_{A_0}(A, -) \) embed \( KQGr \) in two different ways as a full subcategory of the category \( Gr_{A[X]} \) of \( \mathbb{Z} \times \mathbb{Z} \)-graded modules over \( A[X] \). That induces by restriction equivalences of categories between \( AGr = Gr_A \) and the categories \( LC_A \) and \( LC^*_A \) of linear complexes of projective and almost injective graded \( A \)-modules, respectively (Theorems 2.4 and 2.10). In Section 3 we show that in the case when \( A = \Lambda \) is a graded factor of a path algebra, \( \Lambda \) is orthogonal to its Yoneda algebra \( \Gamma \) and then there is an induced duality between \( \lambda^!fg \) and the category \( LC^! \) of linear complexes of finitely generated projective graded modules over \( \Gamma \) (Theorem 3.3). Among the consequences of these results, we characterize quadratic algebras in categorical terms (Corollary 3.4) and show that the categories of linear complexes of projective (resp. almost injective) graded modules are equivalent for quadratically equivalent algebras. In case the algebras are quadratically equivalent graded factors of path algebras, the categories of linear complexes of finitely generated projective graded modules over their Yoneda algebras are also equivalent (Corollary 3.5). In the final Section 4, somewhat independent from the rest, we extend some equivalences of derived categories obtained by Bernstein, Gelfand and Gelfand (cf. [4] and [9]) in the classification of algebraic vector bundles over the projective space.
2. Koszul equivalences.

All throughout this section $A = \bigoplus_{n \geq 0} A_n$ will be a positively graded algebra with quiver $Q$. We fix a homomorphism $\pi_A : KQ \to A$ of graded $K$-algebras and put $\overline{p} = \pi_A(p)$, for every path $p$ in $Q$. One-sided modules over $A_0 = KQ_0$ will be considered indistinctly as left or right modules, with the same action of $A_0$ on both sides. It is convenient now to make some comments concerning the canonical duality $D$. Suppose $A_0 S A_0$ is a bimodule and $B$ is a $K$-basis of $S$ satisfying the following property: For every $b \in B$ there exist (necessarily unique) $i, j \in Q_0$ such that $u = e_i u e_j$. We put $i = o(b)$ and $j = t(b)$. For each $b \in B$, we denote by $b^*$ the homomorphism of right $A_0$-modules defined by the rule $b^*(c) = \delta_{bc} e_i(c)$, where $\delta_{bc}$ is the Kronecker symbol. It is clear that $B^* = \{b^* : b \in B\}$ is a $K$-linearly independent subset of $D(S)$, which is a basis when $S$ is finite dimensional. A symmetric argument works when we consider homomorphisms of left $A_0$-module, but then $b^*(c) = \delta_{bc} e_o(c)$. We shall call $B^*$ the dual basis of $B$, the side of the $A_0$-homomorphisms being clear from the context. The following remark and the next two lemmas will be very useful in the sequel.

**Remark 2.1.** Let $S$ be finite dimensional in the above situation and let $X_{A_0}$ (resp. $A_0 X$) be an $A_0$-module. For each $b \in B$ and each $x \in X e_i(b) \mathbb{P}$ (resp $x \in e_o(b) X$), we consider the $A_0$-homomorphisms $x b^*(-) : S \to X$ (resp. $b^*(-) x : S \to X$), mapping $s \mapsto x b^*(s)$ (resp. $s \mapsto b^*(s) x$). Then the set $\{x b^*(-) : b \in B, x \in X e_i(b)\}$ (resp. $\{b^*(-) x : b \in B, x \in e_o(b) X\}$) generates $\text{Hom}_{A_0}(S, X)$ as a $K$-vector space.

**Proof.** Straightforward consequence of the isomorphism $X \otimes D(S) \cong \text{Hom}_{A_0}(S, X)$ (resp. $D(S) \otimes X \cong \text{Hom}_{A_0}(S, X)$), which maps $x \otimes b^*$ onto $x b^*(-)$ (resp. $b^* x$ onto $b^*(-) x$), for all $x \in X, b \in B$. □

**Lemma 2.2.** The assignment $X \mapsto X \otimes A$ extends to a fully faithful covariant exact functor $T : \text{Mod}_{A_0} \to \text{Gr}_A$ with essential image $\text{Proj}^0_A$. In particular, it induces an equivalence of categories $\text{Mod}_{A_0} \cong \text{Proj}^0_A$.

**Proof.** It is clear that the assignment extends to a covariant functor $T = - \otimes A : \text{Mod}_{A_0} \to \text{Gr}_A$ with essential image contained in $\text{Proj}^0_A$. Moreover, since $A_0 A$ is projective, the functor is clearly exact. We also have that $\text{Mod}_{A_0} = \text{Add}((A_0)_{A_0}), \text{Proj}^0_A = \text{Add}(A A)$, $T$ preserves direct sums and $T(A_0) \cong A$. From that it follows that $\text{Proj}^0_A \subseteq \text{Im}(T)$, and hence equality. It also follows that the fully faithful condition reduces to check that the functorial map $\text{Hom}_{A_0}(A_0, A_0) \to \text{Hom}_{\text{Gr}_A}(A, A), \lambda \mapsto T(\lambda)$ is bijective. That is straightforward. □

The isomorphisms of next lemma and Lemma 2.9 can be derived from appropriate adjunction settings, but we give their explicit definition for they are used in the proofs or our theorems.
Lemma 2.3. Let \( X, Y \) be \( A_0 \)-modules. The map \( \varphi : \text{Hom}_{KQ_0}(KQ_1 \otimes X, Y) \to \text{Hom}_{A_0}(X, Y \otimes A_1) \) taking \( \mu \) onto \( \varphi(\mu) : x \mapsto \sum_{\alpha \in Q_1} \mu(\alpha \otimes x) \otimes \alpha \) is an isomorphism of \( K \)-vector spaces. Moreover, if \( \mu \in \text{Hom}_{KQ_0}(KQ_1 \otimes X, Y), \mu' \in \text{Hom}_{KQ_0}(KQ_1 \otimes X', Y') \) and \( f : X \to X', g : Y \to Y' \) are \( A_0 \)-homomorphisms, then one of the following diagrams commutes iff the other does:

\[
\begin{array}{ccc}
KQ_1 \otimes X & \xrightarrow{\mu} & Y \\
1 \otimes f \downarrow & & \downarrow g \\
KQ_1 \otimes X' & \xrightarrow{\mu'} & Y'
\end{array}
\quad \begin{array}{ccc}
X & \xrightarrow{\varphi(\mu)} & Y \otimes A_1 \\
f \downarrow & & \downarrow g \otimes 1 \\
X' & \xrightarrow{\varphi(\mu')} & Y' \otimes A_1.
\end{array}
\]

Proof. Since \( \{ \sigma : \alpha \in Q_1 \} \) is a basis of \( A_1 \), every element of \( X \otimes A_1 \) can be written as a sum \( \sum_{\alpha \in Q_1} y_\alpha \otimes \sigma \), where \( y_\alpha \in Ye_\alpha \), for all \( \alpha \in Q_1 \). In particular, if \( f \in \text{Hom}_{A_0}(X, Y \otimes A_1) \) then it maps \( x \) onto a sum \( \sum_{\alpha \in Q_1} f_\alpha(x) \otimes \sigma \), with \( f_\alpha(x) \in Ye_\alpha \) for all \( \alpha \in Q_1 \). Moreover, if \( x \in X \sigma \), then the fact that \( f \) is a morphism in \( \text{Mod}_{A_0} \) implies that we can take \( f_\alpha(x) = 0 \) whenever \( i \neq t(\alpha) \). Hence, we get a uniquely determined family of \( K \)-linear maps \( \{ f_\alpha : Xe_\alpha \to Ye_\alpha : \alpha \in Q_1 \} \) such that \( f(x) = \sum_{\alpha \in Q_1} f_\alpha(x) \otimes \sigma \).

We now define \( \xi : \text{Hom}_{A_0}(X, Y \otimes A_1) \to \text{Hom}_{KQ_0}(KQ_1 \otimes X, Y) \) by the rule \( \xi(f)(\alpha \otimes x) = f_\alpha(x) \). The choice of the \( f_\alpha \) guarantees that \( \xi(f) \) is a morphism in \( A_0 \text{Mod} \). We leave as an easy exercise to check that \( \varphi \) and \( \xi \) are mutually inverse. The rest of the proof is then routine.

Let \( (A_k)_{k \in \mathbb{Z}} \) be a family of categories. We shall denote by \( \prod_{k \in \mathbb{Z}} A_k \) the corresponding product category. Its objects are the families \((U_k)_{k \in \mathbb{Z}}\) such that \( U_k \in A_k \), for every \( k \in \mathbb{Z} \). Its morphisms are families of morphisms \((f_k : U_k \to V_k)_{k \in \mathbb{Z}}\), with \( f_k \) a morphism in \( A_k \), for all \( k \in \mathbb{Z} \). The composition of morphisms is defined pointwise. In particular, we shall denote by \( A^\mathbb{Z} \) the category \( \prod_{k \in \mathbb{Z}} A_k \), where \( A_k = A \), for all \( k \in \mathbb{Z} \). If \( U \in A^\mathbb{Z} \) and \( n \in \mathbb{Z} \) then the object \( U\{n\} \) of \( A^\mathbb{Z} \) is defined by the rule \( U\{n\}_k = U_{n+k} \), for all \( k \in \mathbb{Z} \).

If \( f : U \to V\{n\} \) is a morphism in \( A^\mathbb{Z} \), we shall write \( f : U \to^{n} V \) and shall say that \( f \) is a morphism of degree \( n \) from \( U \) to \( V \).

We are mainly interested in the cases when \( A = KQ_0 \text{ Mod} = \text{Mod}_{A_0} \) and \( A = Gr_A \) in the above situation. For technical reasons, we shall still keep subindices for the first case, while we shall use superindices for the second case (e.g., an object of \( Gr_A^{\mathbb{Z}} \) will be denoted by \( P = (P_k)_{k \in \mathbb{Z}} \), where \( P^k \in Gr_A \) for all \( k \)). We introduce now a new (Grothendieck) category \( Gr_{A[X]} \) as follows: Its objects are pairs \((P, d')\), where \( P \in Gr_A^{\mathbb{Z}} \) and \( d' : P \xrightarrow{+1} P \) is a morphism in \( Gr_A^{\mathbb{Z}} \) of degree \(+1\). A morphism \( f : (P, d') \to (Q, \delta') \) in \( Gr_{A[X]} \) is just a morphism \( f : P \to Q \) in \( Gr_A^{\mathbb{Z}} \) such that \( f \circ d' = \delta' \circ f \).

The notation \( Gr_{A[X]} \) makes sense. Indeed, we can provide the polynomial
algebra \(A[X]\) with a \(\mathbb{Z} \times \mathbb{Z}\)-grading by putting \(A[X]_{(m,n)} = A_m X^n\), whenever \(m, n \geq 0\), and \(A[X]_{(m,n)} = 0\) otherwise. If \(M = \oplus M_{(m,n)}\) is a \(\mathbb{Z} \times \mathbb{Z}\)-graded right \(A[X]\)-module, then \(M^n = \oplus_{m \in \mathbb{Z}} M_{(m,n)}\) is an object of \(Gr_A\) and multiplication by \(X\) yields morphisms in \(Gr_A\), \(d^n : M^n \rightarrow M^{n+1}\), for all \(n \in \mathbb{Z}\).

In that way, we get an object of \(Gr_A[X]\) and the category of \(\mathbb{Z} \times \mathbb{Z}\)-graded right \(A[X]\)-modules is identified with \(Gr_A[X]\). We shall indistinctly use this and the classical interpretation of the category \(\Lambda KQ\) consisting of graded left \(\mathbb{Z}\)-modules is identified with \(\Lambda Gr\), i.e., such that \(P_k\) is a projective object of \(\Lambda Gr\) generated in degree \(-k\), for all \(k \in \mathbb{Z}\). Inside \(\Lambda Gr\) we consider the full subcategory \(\Lambda C\) consisting of those pairs \((P, d)\) such that \(P \in \prod_{k \in \mathbb{Z}} \text{Proj}_{A_k}^h\), i.e., such that \(P^k\) is a projective object of \(\Lambda Gr\) generated in degree \(-k\), for all \(k \in \mathbb{Z}\). Inside \(\Lambda C\) we consider the full subcategory \(\Lambda C\) consisting of those \((P, d)\) which are cochain complexes, i.e., such that \(d \circ d = 0\). The objects of \(\Lambda C\) are called linear complexes of projectives. The full subcategory of \(\Lambda C\) with objects \((P, d)\) such that \(P^k\) is finitely generated, for all \(k \in \mathbb{Z}\), will be denoted \(\Lambda C\).

Our main results in the section concern the category \(KQGr\). We point out that an object of that category can be identified with a pair \((M, \mu)\), where \(M = (M_k)\) is an object of \(KQ_0 \text{Mod}^\mathbb{Z}\) and \(\mu = (\mu_k : KQ_1 \otimes M_k \rightarrow M_{k+1})\) is a family of morphisms in \(KQ_0 \text{Mod}\). In that vein, a morphism \(f : (M, \mu) \rightarrow (N, \mu')\) in \(KQGr\) is identified with a morphism \(f = (f_k)_{k \in \mathbb{Z}}\) in \(KQ_0 \text{Mod}^\mathbb{Z} = \text{Mod}_{A_0}^\mathbb{Z}\) such that \(f_{k+1} \circ \mu_k = \mu'_k \circ (1_{KQ_1} \otimes f_k)\), for all \(k \in \mathbb{Z}\). We shall indistinctly use this and the classical interpretation of the category \(KQGr\).

When \(\Lambda \simeq KQ/I\) is a graded factor of a path algebra (e.g., \(\Lambda = \Lambda A\) in our case), the category \(\Lambda Gr\) can be identified with the full subcategory of \(KQGr\) consisting of graded left \(KQ\)-modules annihilated by \(I\). That is the sense of the word ‘restriction’ in our next theorem.

**Theorem 2.4.** Let \(A = \oplus_{n \geq 0} A_n\) be a positively graded algebra with quiver \(Q\). There is a fully faithful exact functor \(\psi : KQGr \rightarrow Gr_{A[X]}\) which induces by restriction equivalences of categories \(\Lambda Gr = Gr_A \simeq \Lambda C\) and \(\Lambda! \text{gr} = \text{fgr} \Lambda^! \simeq \Lambda C\).

**Proof.** By Lemma 2.2, the composition \(\text{Mod}_{A_0}^\mathbb{Z} \xrightarrow{T} Gr_A \xrightarrow{-[k]} Gr_A\), \(X \sim X \otimes A[k]\), is a fully faithful covariant exact functor, which we denote by \(T_k\) and induces an equivalence of categories \(\text{Mod}_{A_0}^\mathbb{Z} \cong \text{Proj}^k\), for every \(k \in \mathbb{Z}\). As a consequence the product \(\hat{T} = \prod_{k \in \mathbb{Z}} T_k : \text{Mod}_{A_0}^\mathbb{Z} \rightarrow Gr_A^\mathbb{Z}\) is a fully faithful exact functor inducing an equivalence of categories

\[
\hat{T} : \text{Mod}_{A_0}^\mathbb{Z} \cong \prod_{k \in \mathbb{Z}} \text{Proj}^k.
\]

With the above interpretation of the objects in \(KQGr\) and \(Gr_{A[X]}\), we are ready to define a functor \(\hat{\psi} : KQGr \rightarrow Gr_{A[X]}\) verifying the requirements. Using Lemma 2.3, to every \((M, \mu) \in KQ Gr\) we can assign a family \((\varphi(\mu_k) : M_k \rightarrow M_{k+1} \otimes A_1)_{k \in \mathbb{Z}}\) of morphisms in \(\text{Mod}_{A_0}\). But we
have a $K$-linear isomorphism $\text{Hom}_{A_0}(M_k, M_{k+1} \otimes A_1) \cong \text{Hom}_{Gr_A}(M_k \otimes A[k], M_{k+1} \otimes A[k+1])$, for every $k \in \mathbb{Z}$. Hence, the family $\mu = (\mu_k)$ induces a uniquely determined family $\varphi(\mu) = (\varphi(\mu_k) : \hat{T}(M)^k = M_k \otimes A[k] \rightarrow M_{k+1} \otimes A[k+1] = \hat{T}(M)^{k+1})_{k \in \mathbb{Z}}$ of morphisms in $Gr_A$. That is, we get a morphism $\varphi(\mu) : \hat{T}(M) \rightarrow \hat{T}(M)$ in $Gr_Z$ of degree +1. We then define $\psi : KQGr \rightarrow Gr_{A[X]}$ on objects by taking the pair $(M, \mu)$ onto $(\hat{T}(M), d')$, with $d' = \varphi(\mu)$. Suppose now that $(M, \mu)$ and $(N, \mu')$ are objects of $KQGr$ and let $g : \hat{T}(M) \rightarrow \hat{T}(N)$ be a morphism in $Gr_Z$. Since $\hat{T}(M)$ and $\hat{T}(N)$ belong to $\prod_{k \in \mathbb{Z}} \text{Proj}^k A$, the above equivalence $(\ast)$ gives a uniquely determined morphism $f : M \rightarrow N$ in $\text{Mod} \oplus \text{Mod}$ such that $T(f) = g$. Then $g^k = f_k \otimes 1_{A[k]}$ for all $k \in \mathbb{Z}$. We claim that $f$ is a morphism $(M, \mu) \rightarrow (N, \mu')$ in $KQGr$ if $g$ is a morphism $(\psi(M), \varphi(\mu)) \rightarrow (\psi(N), \varphi(\mu'))$ in $Gr_{A[X]}$. If that is proved it will follow that, defining $\psi(f) = \hat{T}(f)$ for every morphism $f$ in $KQGr$, one obtains a fully faithful exact functor $\psi : KQGr \rightarrow Gr_{A[X]}$ with essential image $\mathcal{L}_A$. Let us prove our claim. We know that $f$ is a morphism in $KQGr$ if $f_{k+1} \circ \mu_k = \mu'_k \circ (1 \otimes f_k)$, for all $k \in \mathbb{Z}$. By Lemma 2.3, that is equivalent to say that $(f_{k+1} \otimes 1_{A_1}) \otimes \varphi(\mu_k) = \varphi(\mu'_k) \circ f_k$, for all $k \in \mathbb{Z}$. This is in turn equivalent to say that $g^{k+1} \circ d^k = d^k \circ g^k$, for all $k \in \mathbb{Z}$, where $d' = \varphi(\mu)$. That occurs iff $g$ is a morphism in $Gr_{A[X]}$, thus proving our claim.

In the final part of the proof, we come back to the classical interpretation of objects in $KQGr = Gr_{KQ^{op}}$, which will be looked at as graded right $KQ^{op}$-modules. With the equality $\mathcal{L}_A = \text{Im} \psi$ at hand, the rest of the proof reduces to check that $\psi(M)$ is a cochain complex iff $M_k \cdot I^+_2 = 0$ for every $k \in \mathbb{Z}$, where $I = \text{Ker}(\pi_A)$. From that the equivalences of the last part of the theorem will follow. On one hand, $\psi(M)$ is a cochain complex iff the composition $M_k \otimes A[k] \xrightarrow{d^k} M_{k+1} \otimes A[k+1] \xrightarrow{d^{k+1}} M_{k+2} \otimes A[k+2]$ is zero, for each $k \in \mathbb{Z}$. But, since $M_k \otimes A[k]$ is generated by $M_k \otimes A_0 \cong M_k$, that is equivalent to say that $d^{k+1} \circ d^k$ vanish on $M_k$. Direct calculation shows that $(d^{k+1} \circ d^k)(x) = \sum_{p \in Q_2} px \otimes \bar{p} = \sum_{p \in Q_2} xp^\rho \otimes \bar{p}$, for all $x \in M_k$. In particular, $(d^{k+1} \circ d^k)(M_k) \subseteq M_{k+2} \otimes A^2_2$, where $A_2^2 = A_1 \cdot A_1$. On the other hand, the obvious sequences $0 \rightarrow I_2 \rightarrow KQ_2 \rightarrow A^1_2 \rightarrow 0$ and $0 \rightarrow I^+_2 \rightarrow KQ^{op}_2 \rightarrow D(I_2) \rightarrow 0$ (in $\text{Mod}$ and $\text{Mod} \oplus \text{Mod}$, respectively) are dual to each other. Hence, we have an isomorphism $A_2^2 \cong D(I^+_2)$ which maps $\bar{p}$ onto the restriction of $p^\rho$ to $I^+_2$, where $\{p^\rho : p \in Q_2\}$ is the dual basis in $D(KQ^{op}_2)$ of $Q^{op}_2$. Taking now the composition of $d^{k+1} \circ d^k : M_k \rightarrow M_{k+2} \otimes A_1^2$ followed by the canonical isomorphism $M_{k+2} \otimes A_1^2 \cong M_{k+2} \otimes D(I^+_2) \cong \text{Hom}_{A_0}(I^+_2, M_{k+2})$, we get a map $\delta^k : M_k \rightarrow \text{Hom}_{A_0}(I^+_2, M_{k+2})$. In a routine way, one checks that $\delta^k(x) : I^+_2 \rightarrow M_{k+2}$ is the restriction to $I^+_2$ of the map $f_x = \sum_{p \in Q_2} (xp^\rho) p^\rho(-) : KQ^{op}_2 \rightarrow M_{k+2}$ (with notation...
as in Remark 2.1). The latter maps \( q^o \) onto \( xq^o \), for every \( q \in Q_2 \), so that \( \delta^k(x)(a) = xa \) for all \( a \in I^+_2 \). Therefore \( d^{k+1} \circ d^k \) vanish on \( M_k \) iff \( M_k \cdot I^+_2 = 0 \) as desired.

**Remark 2.5.** The equivalences of the above theorem restrict to the corresponding full subcategories of upper or lower bounded objects. For instance, the equivalence \( {_A^\Lambda}lfgr = lfgr_{A^1} \xrightarrow{\cong} \mathcal{L}c_A \) restricts to an equivalence \( {_A^\Lambda}lfgr = lfgr_{A^1} \xrightarrow{\cong} \mathcal{L}c_A \).

In case \( \Lambda = KQ/I \) is a graded factor of a path algebra, the category \( Gr\Lambda \) can be seen in a canonical way as a subcategory of \( Gr\Lambda \). In particular, for every positively graded algebra \( A \) which is orthogonal to \( \Lambda \), swapping the roles of \( Q \) and \( Q^{op} \), Theorem 2.4 yields a fully faithful exact embedding \( \psi_A : Gr\Lambda \rightarrow Gr_{A[X]} \) such that \( \psi_A(M) \) is a cochain complex, for all \( M \in Gr\Lambda \).

We then have the following consequence:

**Corollary 2.6.** Let \( \Lambda = KQ/I \) be a graded factor of a path algebra. For every \( M \in Gr\Lambda \) and every \( j \in \mathbb{Z} \), the following assertions are equivalent:

1. \( M \) is cogenerated in degree 1.
2. For every positively graded algebra \( A = \oplus_{n \geq 0} A_n \) orthogonal to \( \Lambda \), the cohomology graded \( A \)-module \( H^k(\psi_A(M)) \) is generated in degrees \( > -k \), for all \( k \neq j \).
3. There is a positively graded algebra \( A \) orthogonal to \( \Lambda \) satisfying 2).

**Proof.** Suppose \( M = \oplus_{k \leq k_0} M_k \), with \( M_{k_0} \neq 0 \). Then \( M \) is cogenerated in degree 1 implies \( k_0 = j \). On the other hand, \( H^k(\psi_A(M)) = 0 \), for all \( k > k_0 \) and \( H^{k_0}(\psi_A(M))_{-k_0} = M_{k_0} \otimes A_0 \cong M_{k_0} \neq 0 \). Hence, \( j = k_0 \) in 1), 2) or 3). Now \( M \) is cogenerated in degree 1 iff for every \( k < j \) and for every \( 0 \neq x \in M_k, xA_{j-k} = 0 \). But, in our case, \( \Lambda_n = \Lambda^1 = \Lambda_1 \cdots \Lambda_n \) for all \( n > 0 \). We then get that \( M \) is cogenerated in degree 1 iff for every \( k < j \) and for every \( 0 \neq x \in M_k, xA_{j-k} = 0 \). On the other hand, given any positively graded algebra \( A \) orthogonal to \( \Lambda \), the graded \( A \)-module \( H^k(\psi_A(M)) \) has support contained in \( \{ n \in \mathbb{Z} : n \geq -k \} \), for all \( k \in \mathbb{Z} \). Moreover, the homogeneous component of degree \( -k \) of \( H^k(\psi_A(M)) \), denoted \( H^k(\psi_A(M))_{-k} \), is the kernel of the map \( d_k : M_k \cong M_k \otimes A_0 \rightarrow M_{k+1} \otimes A_1 \). This map, after the suitable adaptation from Theorem 2.2 due to the swapping of roles of \( Q \) and \( Q^{op} \), takes the form \( x \rightarrow \sum_{\alpha \in Q_1} x\alpha \circ \alpha^o \). We now compose this latter map with the canonical isomorphism \( M_{k+1} \otimes A_1 \cong M_{k+1} \otimes D(KQ_1) \cong \text{Hom}_{A_0}(KQ_1, M_{k+1}) \cong \text{Hom}_{A_0}(\Lambda_1, M_{k+1}) \). The resulting map \( M_k \rightarrow \text{Hom}_{A_0}(\Lambda_1, M_{k+1}) \) takes \( x \) onto \( \sum_{\alpha \in Q_1} (x\alpha) \circ \alpha^o (-) : \beta \rightarrow x\beta \) (with the notation of Remark 2.1). Then \( H^k(\psi_A(M))_{-k} \cong \{ x \in M_k : xA_1 = 0 \} \) and the desired equivalence of 1), 2) and 3) follows.

We leave as an exercise the proof of the following lemma, which will be useful in the proof of our next theorem:
Lemma 2.7. Let \( f, g : M \rightarrow N \) be two morphisms in \( \text{Gr}_A \) and assume that \( N \) is cogenerated in degree \( j \). Then \( f = g \) if, and only if, \( f_j = g_j \).

Given a \( A_0 \)-module \( X \) and viewing each \( A_i \) as a right \( A_0 \)-module, the vector space \( \text{Hom}_{A_0}(A, X) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{A_0}(A_{-k}, X) \) gets a canonical structure of graded right \( A \)-module by defining \( \text{Hom}_{A_0}(A, X)_k = \text{Hom}_{A_0}(A_{-k}, X) \) and \((fa)(x) = f(az)\) whenever \( a \in A_j \), \( f \in \text{Hom}_{A_0}(A, X) \) and \( x \in A_{-(j+k)} \). In the particular case when \( X = A_0 \), we get \( \text{Hom}_{A_0}(A, A_0) = D(A) \). We have now:

Lemma 2.8. Assume that \( A \) is locally finite. The assignment \( X \rightarrow \text{Hom}_{A_0}(A, X) \) extends to a fully faithful covariant exact functor \( H : \text{Mod}_{A_0} \rightarrow \text{Gr}_A \) with essential image \( \text{Inj}^0_A \). In particular, it induces an equivalence of categories \( \text{Mod}_{A_0} \cong \text{Inj}^0_A \).

Proof. We define \( H(f) = f_\ast : u \mapsto f \circ u \), for every composable morphisms \( f \) and \( u \) in \( \text{Mod}_{A_0} \). Then we clearly get a covariant functor \( H : \text{Mod}_{A_0} \rightarrow \text{Gr}_A \) such that \( H(X) = \text{Hom}_{A_0}(A, X) \), for every \( X \in \text{Mod}_{A_0} \). The functor is exact because \( A_{A_0} \) is projective. Since each \( A_k \) is a finitely generated \( A_0 \)-module, \( H \) preserves direct sums. Notice that \( \text{Mod}_{A_0} = \text{Add}(A_0, A_0) \) and \( \text{Add}(H(A_0)) = \text{Add}(D(A), A) = \text{Inj}^0_A \). Then, using the preservation of direct sums, we conclude that our task reduces to check that \( H \) induces a bijection

\[(*) \quad \text{Hom}_{A_0}(A_0, A_0) \rightarrow \text{Hom}_{\text{Gr}_A}(H(A_0), H(A_0)) = \text{Hom}_{\text{Gr}_A}(D(A), D(A)).\]

Let us prove this. By the canonical duality \( D : A_{lfg} \rightarrow A_{lfgr} \), every morphism \( g : D(A) \rightarrow D(A) \) in \( \text{Gr}_A \) is of the form \( D(\rho_a) \), where \( \rho_a : A \rightarrow A \) is right multiplication by \( a \), for a uniquely determined \( a \in A_0 \). We take the left multiplication by \( a \), \( \lambda_a : A_0 \rightarrow A_0 \), which is a morphism in \( \text{Mod}_{A_0} \). Then, for every \( x \in A_k \) and \( u \in \text{Hom}_{A_0}(A_k, A_0) = H(A_0)_k \), we have \( [H(\lambda_a)(u)](x) = (\lambda_a \circ u)(x) = u(\lambda_a)(x) = (\rho_a)(u)(x) = (g(u))(x) \). Then commutativity of \( A_0 \) yields \( au(x) = u(x)a = u(xa) = |D(\rho_a)(u)](x) = (g(u))(x) \), from which we get \( g = H(\lambda_a) \), for a uniquely determined \( a \in A_0 \). This proves that the map \((*)\) is bijective, thus ending the proof.

Lemma 2.9. Let \( X, Y \) be \( A_0 \)-modules. The map \( \eta : \text{Hom}_{KQ_0}(KQ_1 \otimes X, Y) \rightarrow \text{Hom}_{A_0}(\text{Hom}_{A_0}(A_1, X), Y) \) defined by \( \eta(\mu)(u) = \sum_{\alpha \in Q_1} \mu(\alpha) \otimes u(\bar{\alpha}) \), for all \( u \in \text{Hom}_{A_0}(A_1, X) \) and all \( \mu \in \text{Hom}_{KQ_0}(KQ_1 \otimes X, Y) \), is an isomorphism of \( K \)-vector spaces. Moreover, if \( \mu \in \text{Hom}_{KQ_0}(KQ_1 \otimes X, Y) \), \( \mu' \in \text{Hom}_{KQ_0}(KQ_1 \otimes X', Y') \) and \( f : X \rightarrow X' \), \( g : Y \rightarrow Y' \) are \( A_0 \)-homomorphisms, then one of the following diagrams commutes iff the other does:
The same terminology of Remark 2.1), for every morphism $u \in \text{Hom}_{A_0}(A_1, X)$, using the rule $\rho(h)(\alpha \otimes x) = h(x\overline{\alpha}(-))$, where $x\overline{\alpha}(-) : A_1 \rightarrow X$ is as in Remark 2.1, using $\{\overline{\alpha} = \pi_A(\alpha) : \alpha \in Q_1\}$ as basis of $A_1$. The fact that $\eta$ and $\rho$ are mutually inverse follows easily using that $\eta \circ \rho = \rho \circ \eta$.

**Proof.** We define $\rho : \text{Hom}_{A_0}(A_1, X) \rightarrow \text{Hom}_{KQ_0}(KQ_1 \otimes X, Y)$ by the rule $\rho(h)(\alpha \otimes x) = h(x\overline{\alpha}(-))$, where $x\overline{\alpha}(-) : A_1 \rightarrow X$ is as in Remark 2.1, using $\{\overline{\alpha} = \pi_A(\alpha) : \alpha \in Q_1\}$ as basis of $A_1$. The same terminology of Remark 2.1), for every morphism $u : A_1 \rightarrow X$ in $\text{Mod}_{A_0}$. The rest is routine.

In the sequel we shall denote by $\mathcal{L}_G^A$ the full subcategory of $Gr_{A[X]}$ consisting of those $(I, d')$ such that $I' \in \prod_{k \in \mathbb{Z}} \text{Inj}^k_A$, i.e., such that $I^k$ is an almost injective graded right $A$-module cogenerated in degree $-k$, for every $k \in \mathbb{Z}$. Also, we denote by $\mathcal{L}_C^A$ the full subcategory of $\mathcal{L}_G^A$ consisting of those $(I, d')$ which are cochain complexes. The objects of $\mathcal{L}_C^A$ are called **linear complexes of almost injectives**. Within $\mathcal{L}_C^A$ we shall also consider the full subcategory $\mathcal{Lc}^A$ consisting of those $(I, d') \in \mathcal{L}_C^A$ such that $I^k$ is finitely cogenerated, for every $k \in \mathbb{Z}$.

**Theorem 2.10.** Let $A = \oplus_{n \geq 0} A_n$ a locally finite positively graded algebra with quiver $Q$. There is a fully faithful covariant exact functor $\nu = \nu_A : KQGr \rightarrow Gr_{A[X]}$ which induces, by restriction, equivalences of categories $\nu_! : Gr_A \rightarrow \mathcal{L}_C^A$ and $\nu_! : Gr_A \rightarrow \mathcal{Lc}^A$.

**Proof.** The first part of the proof is parallel to the corresponding one in the proof of Theorem 2.4, using the functor $H$ of Lemma 2.8 instead of the functor $T$ and Lemma 2.9 instead of Lemma 2.3. Indeed, the functor $H$ induces a fully faithful exact functor $\hat{H} : KQ_0 \rightarrow \text{Mod}^{\mathbb{Z}} = \text{Mod}_{A_0} \rightarrow Gr_A^{\mathbb{Z}}$ with essential image $\prod_{k \in \mathbb{Z}} \text{Inj}^k_A$. Notice that, due to Lemma 2.7, we have $K$-linear isomorphisms

\[
\text{Hom}_{Gr_A}(\hat{H}(N)^k, \hat{H}(N)^{k+1}) = \text{Hom}_{Gr_A}(\text{Hom}(A, N_k)[k], \text{Hom}(A, N_{k+1})[k + 1]) \cong \text{Hom}_{A_0}(\text{Hom}_{A_0}(A_1, N_k), N_{k+1}),
\]

for all $k \in \mathbb{Z}$. We now define the desired functor $\nu$. On objects, it will take an object $(N, \mu) \in KQ_0$ $Gr$ onto the object $(I, d')$, where $I' = \hat{H}(N)$ and $d' : \hat{H}(N) \rightarrow \hat{H}(N)$ is the morphism of degree $+1$ in $Gr_A^{\mathbb{Z}}$ induced by the family of maps $(\eta(\mu_k) : \text{Hom}_{A_0}(A_1, N_k) \rightarrow N_{k+1})_{k \in \mathbb{Z}}$ (see Lemma 2.9) and the above isomorphisms $(\ast)$. On the other hand, if $(N, \mu)$ and $(N', \mu')$ are objects of $KQGr$ and $g : \hat{H}(N) \rightarrow \hat{H}(N')$ is a morphism in $Gr_A^{\mathbb{Z}}$, then there
is a uniquely determined morphism $f : N \rightarrow N'$ in $KQ_{0}\text{Mod}^{Z} = \text{Mod}^{Z}_{A_{0}}$ such that $H(f) = g$. Using the last part of Lemma 2.9, one sees that $f$ is a morphism $(N, \mu) \rightarrow (N', \mu')$ in $KQ\text{Gr}$ iff $g$ is a morphism $(H(N), \eta(\mu)) \rightarrow (H(N'), \eta(\mu'))$ in $Gr_{A[X]}$. That proves that the assignment $f \rightarrow v(f) =: H(f)$ defines a faithfully exact functor $v : KQ\text{Gr} \rightarrow Gr_{A[X]}$ with essential image $L\Gamma_{A}$.

For the final part, as in the proof of Theorem 2.4, we view the objects of $KQ\text{Gr}$ as graded right $KQ^{\text{op}}$-modules. We want to prove that $v(N)$ is a cochain complex iff $N_{k} \cdot I_{2}^{\perp} = 0$ for all $k \in \mathbb{Z}$, where $I = \text{Ker}(\pi_{A})$. From that it will follow that $v$ induces an equivalence of categories $\mathcal{L}_{A} \rightarrow \mathcal{L}^{\ast}_{A}$. We have that $\nu(N)$ is a cochain complex iff the composition $\mathcal{H}\text{Hom}_{A_{0}}(A, N_{k})[k] \xrightarrow{d^{k}} \mathcal{H}\text{Hom}_{A_{0}}(A, N_{k+1})[k+1] \xrightarrow{d^{k+1}} \mathcal{H}\text{Hom}_{A_{0}}(A, N_{k+2})[k+2]$ is zero, for all $k \in \mathbb{Z}$. According to Lemma 2.7, that is equivalent to say that its $-(k+2)$-component $\mathcal{H}\text{Hom}_{A_{0}}(A_{2}, N_{k}) \xrightarrow{d^{k}} \mathcal{H}\text{Hom}_{A_{0}}(A_{1}, N_{k+1}) \xrightarrow{d^{k+1}} \mathcal{H}\text{Hom}_{A_{0}}(A_{0}, N_{k+2}) \equiv N_{k+2}$ is zero, for all $k \in \mathbb{Z}$. Using the explicit definition of $d'$, one sees that $(d^{k+1} \circ d^{k})(f) = \sum_{p \in Q_{2}} f(\bar{p})p^{n}$, for all $f \in \text{Hom}_{A_{0}}(A_{2}, N_{k})$. Since $A_{2}^{1} = A_{1} \cdot A_{1}$ is a direct summand of $A_{2}$ in $\text{Mod}_{A_{0}}$ and $\bar{p} \in A_{2}^{1}$, for all $p \in Q_{2}$, we get that $\nu(N)$ is a cochain complex iff $(d^{k+1} \circ d^{k})(f) = 0$, for all $f \in \text{Hom}_{A_{0}}(A_{2}, N_{k})$. Now we argue as in the corresponding part of the proof of Theorem 2.4. We have $A_{1}^{1} \cong D(I_{2}^{\perp})$ and the composition of the canonical isomorphism $N_{k} \otimes I_{2}^{\perp} \cong N_{k} \otimes DD(I_{2}^{\perp}) \cong \text{Hom}_{A_{0}}(D(I_{2}^{\perp}), N_{k}) \cong \text{Hom}_{A_{0}}(A_{1}^{2}, N_{k})$ followed by $d^{k+1} \circ d^{k} : \text{Hom}_{A_{0}}(A_{1}^{2}, N_{k}) \rightarrow N_{k+2}$ is just the canonical multiplication map $N_{k} \otimes I_{2}^{\perp} \rightarrow N_{k+2}$ coming from the right $KQ$-module structure of $N$. Therefore $d^{k+1} \circ d^{k}$ vanishes iff $N_{k} \cdot I_{2}^{\perp} = 0$ as desired.

It only remains to see that $v$ also induces an equivalence $\mathcal{L}_{A} \text{lfgr} = \text{lfgr}_{A} \cong \mathcal{L}^{\ast}_{A}$. That reduces to prove that if $X \in \text{Mod}_{A_{0}}$ then $X$ is finite dimensional iff $H(X) = \text{Hom}_{A_{0}}(A, X)$ is a finitely cogenerated graded right $A$-module. By Lemma 2.8, that follows immediately from the fact that an object of $\text{Inj}_{A}^{0}$ is finitely cogenerated in $Gr_{A}$ iff it is a direct summand of a finite direct sum of copies of $D(A)$.

The following is dual to Corollary 2.6 and we leave the proof as an exercise:

**Corollary 2.11.** Let $\Lambda = KQ/I$ be a graded factor of a path algebra. For $M \in Gr_{\Lambda}^{+}$ and $j \in \mathbb{Z}$, the following assertions are equivalent:

1. $M$ is generated in degree $j$.
2. For every locally finite positively graded algebra $A = \oplus_{n \geq 0} A_{n}$ orthogonal to $\Lambda$, the cohomology graded $A$-module $H^{k}(v_{A}(M))$ is cogenerated in degrees $< -k$, for all $k \neq j$.
3. There is a positively graded algebra $A$ orthogonal to $\Lambda$ satisfying 2).

All throughout this section, $\Lambda = KQ/I$ will be a graded factor of a path algebra and $\Gamma$ will be its Yoneda algebra. The main goal of this section is to see that $\Lambda$ and $\Gamma$ are orthogonal. A first ingredient for that is the next straightforward consequence of Theorem 2.4, which is valid for an arbitrary positively graded algebra.

**Corollary 3.1.** Let $A = \oplus_{n \geq 0} A_n$ be a positively graded algebra with quiver $Q$, let $D : lfgr_{KQ} \longrightarrow KQlfgr$ be the canonical duality and $\iota : KQlfgr \hookrightarrow KQGr$ be the inclusion functor. The composition $\phi = \phi_A : lfgr_{KQ} \xrightarrow{D} KQlfgr \xrightarrow{\iota} KQGr \xrightarrow{\psi_A} Gr_{A[X]}$ is a fully faithful contravariant exact functor which induces by restriction a duality of categories $A!lfgr = lfgr_A \cong \mathcal{L}c_A$.

**Remark 3.2.** When restricted to the full subcategories of lower or upper bounded graded modules, the above duality ‘changes signs’. For instance, it induces a duality of categories $A!lfgr^+ \cong \mathcal{L}c_A^{-1}$.

The following is the main result of this section:

**Theorem 3.3.** Let $\Lambda = KQ/I$ be a graded factor of a path algebra and let $\Gamma = \oplus_{n \geq 0} \Gamma_n$ be its Yoneda algebra. Then $\Lambda$ and $\Gamma$ are orthogonal graded algebras. In particular, $\phi = \phi_\Gamma : KQlfgr = lfgr_{KQ^{op}} \longrightarrow Gr_{\Gamma[X]}$ induces a duality of categories $\Lambda!lfgr \cong \mathcal{L}c_\Gamma$.

**Proof.** Let $\pi_\Lambda : KQ \longrightarrow \Lambda$ and $\pi_\Gamma : KQ^{op} \longrightarrow \Gamma$ the canonical homomorphisms, with kernels $I$ and $J$. We want to prove that $I^2 = J_2$. Notice that $\pi_\Gamma(\alpha') = \tilde{\alpha}$ can be identified, via the isomorphism $\Gamma_1 = \text{Ext}_\Lambda^1(A_0, A_0) \cong \text{Hom}_\Lambda(A_{\geq 1}, A_0)$, with the unique $\Lambda$-homomorphism $\tilde{\alpha} : A_{\geq 1} \longrightarrow A_0$ mapping an arrow $\gamma$ onto $\delta_{\alpha\gamma} e_{\alpha(\gamma)}$, where $\delta_{\alpha\gamma}$ is the Kronecker symbol. Our goal is to interpret the Yoneda product $\tilde{\alpha} \cdot \tilde{\beta}$ as a $\Lambda$-homomorphism $\Omega^2(A_0) \longrightarrow A_0$, bearing in mind that $\Gamma_2 = \text{Ext}_\Lambda^2(A_0, A_0) \cong \text{Hom}_\Lambda(\Omega^2(A_0), A_0)$. First, for the convenience of the reader, we shall adapt to our terminology a known explicit description of $\Omega^2(A_0)$. Recall that $\Omega^2(A_0) = \Omega^1(A_{\geq 1})$ is the kernel of the canonical multiplication map $\mu : \Lambda \otimes \Lambda_1 \longrightarrow \Lambda_2$. Suppose $\rho$ is a homogeneous generating set of the ideal $I$ of relations. We write every $r \in \rho$ as a linear combination $\sum_{\gamma \in Q_1} r_{\gamma} \gamma$, where $r_{\gamma} \in KQe_{\alpha(\gamma)}$ is uniquely determined for every $\gamma \in Q_1$. We claim that $\Omega^2(A_0)$ is the $\Lambda$-submodule of $\Lambda \otimes \Lambda_1$ generated by the set $\{ \sum_{\gamma \in Q_1} r_{\gamma} \otimes \gamma : r \in \rho \}$, where the bar on top of an element of $KQ$ always means its image by $\pi_\Lambda$. Indeed, let $\{ a_\gamma : \gamma \in Q_1 \}$ be a family of elements of $KQ$ such that $a_\gamma \in KQe_{\alpha(\gamma)}$ for all $\gamma \in Q_1$ and $\mu(\sum_{\gamma \in Q_1} \bar{a}_\gamma \otimes \gamma) = \sum_{\gamma \in Q_1} a_\gamma \gamma = 0$ in $\Lambda$. Then $\sum_{\gamma \in Q_1} a_\gamma \gamma \in I$. This means...
that we have an equality

\[(*) \quad \sum_{\gamma \in Q_1} a_{\gamma} \gamma = \sum_{r \in \rho} f_r \gamma + \sum_{r \in \rho} h_r \gamma \]

in \(KQ\), where \(f_r, h_r \in KQ\) are all zero but finitely many and where \(g_r \in KQ_{\geq 1}\), for all \(r \in \rho\). We write \(g_r = \sum_{\gamma \in Q_1} g_r \gamma \gamma\) and \(r = \sum_{\gamma \in Q_1} r \gamma \gamma\), with \(g_r, \gamma \in KQ e_{\rho(\gamma)}\) for every \(\gamma \in Q_1\). By substituting in the equality \((*)\), we get that \(\sum_{\gamma \in Q_1} a_{\gamma} \gamma = \sum_{r \in \rho} (\sum_{\gamma \in Q_1} f_r \gamma + \sum_{r \in \rho} h_r \gamma) \gamma\) in \(KQ\). From that it follows that \(a_{\gamma} = \sum_{r \in \rho} f_r \gamma + \sum_{r \in \rho} h_r \gamma\) in \(KQ\), which implies that \(\bar{a}_{\gamma} = \sum_{r \in \rho} \bar{f}_r \gamma \) in \(\Lambda\). We then get an equality \(\sum_{\gamma \in Q_1} \bar{a}_{\gamma} \gamma = \sum_{r \in \rho} \bar{h}_r (\sum_{\gamma \in Q_1} \bar{\gamma} \gamma) \) in \(\Lambda \otimes \Lambda_1\) which proves the claim.

Once we have an explicit description of \(\Omega^2(\Lambda_0)\), we are ready for an identification of \(\tilde{\alpha} \cdot \tilde{\beta}\). We consider the morphism \(v : \Lambda \otimes \Lambda_1 \to \Lambda\) in \(\Lambda\text{Mod}\) mapping \(\sum_{\gamma \in Q_1} \tilde{a}_{\gamma} \otimes \gamma\) onto \(\tilde{a}_{\rho(e(\alpha))}\). Clearly, \(v(\Omega^2(\Lambda_0)) \subseteq \Lambda_{\geq 1}\), so that we get by restriction a \(\Lambda\)-homomorphism \(u : \Omega^2(\Lambda_0) \to \Lambda_{\geq 1}\) making commute the following diagram:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \Omega^2(\Lambda_0) & \longrightarrow & \Lambda \otimes \Lambda_1 & \longrightarrow & \Lambda_{\geq 1} & \longrightarrow & 0 \\
\downarrow{u} & & \downarrow{v} & & \downarrow{\tilde{\alpha}} & & \\
0 & \longrightarrow & \Lambda_{\geq 1} & \longrightarrow & \Lambda & \longrightarrow & \Lambda_0 & \longrightarrow & 0
\end{array}
\]

where the rows are the obvious exact sequences. Then the Yoneda product \(\tilde{\alpha} \cdot \tilde{\beta}\) is represented by the composition \(\Omega^2(\Lambda_0) \xrightarrow{u} \Lambda_{\geq 1} \xrightarrow{\tilde{\beta}} \Lambda_0\). Take a generator \(x_r = \sum_{\gamma \in Q_1} \bar{\gamma} \gamma\) of \(\Omega^2(\Lambda_0)\). When the path \(\beta \alpha\) does not appear in \(r\), that element is mapped onto zero by \(\tilde{\alpha} \cdot \tilde{\beta} = \tilde{\beta} \circ u\). When \(\beta \alpha\) does appear in \(r\), it is mapped onto \(e_{\rho(\beta)}\). That implies that \(\tilde{\alpha} \cdot \tilde{\beta}\) vanishes on \(\{x_r : r \in \rho, \text{length}(r) > 2\}\). That is, the action of \(\tilde{\alpha} \cdot \tilde{\beta}\) on \(\Omega^2(\Lambda_0)\) is completely identified by its action on the \(K\)-subspace generated by \(\{x_r : r \in \rho, \text{length}(r) = 2\}\), which is a \(\Lambda_0\)-submodule of \(\Lambda_0 \Omega^2(\Lambda_0)\) isomorphic to \(I_2\). We then have a restriction map \(\varphi : \Gamma_2 \cong \text{Hom}_\Lambda(\Omega^2(\Lambda_0), \Lambda_0) \to \text{Hom}_{\Lambda_0}(I_2, \Lambda_0)\) and our argument says that the kernel \(J_2\) of the canonical map \(\pi : KQ^\text{op}_2 \to \Gamma_2\) coincides with the kernel of \(\varphi \circ \pi\). But the action of \((\varphi \circ \pi)(\alpha^\rho) = \tilde{\alpha} \cdot \tilde{\beta}\) on \(I_2\) is the restriction to \(I_2\) of the action of the morphism \((\beta \alpha)^* : KQ_2 \to KQ_0\) in \(\Lambda_0\text{Mod}\), which maps \(\beta \alpha\) onto \(e_{\rho(\beta)}\), and the remaining paths of length 2 to zero. Hence, via the isomorphism \(KQ^\text{op}_2 \cong D(KQ_2)\), we have that \(J_2\) is identified with the kernel of the restriction map \(D(KQ_2) \to \text{Hom}_{\Lambda_0}(I_2, \Lambda_0)\), \(f \mapsto f|_{I_2}\). Therefore \(J_2 = I_2^\perp\) as desired.

The last assertion of the theorem follows now directly from Corollary 3.1.
We now get the following categorical characterization of quadratic algebras:

**Corollary 3.4.** Let \( \Lambda = KQ/I \) be a graded factor of a path algebra and \( \Gamma \) be its Yoneda algebra. The following assertions are equivalent:

1) \( \Lambda \) is quadratic;
2) \( \psi_A : \text{Gr}_KQ = KQ^{op}\text{Gr} \to \text{Gr}_A[X] \) induces by restriction an equivalence of categories \( \text{Gr}_\Lambda \cong \mathcal{L}_A \) (resp. \( \text{lfgr}_\Lambda \cong \mathcal{L}_A \)), for every (or some) positively graded algebra \( A \) which is orthogonal to \( \Lambda \);
3) \( \psi_B : \text{Gr}_KQ = KQ^{op}\text{Gr} \to \text{Gr}_\Gamma[X] \) induces by restriction an equivalence of categories \( \text{Gr}_\Lambda \cong \mathcal{L}_\Gamma \) (resp. \( \text{lfgr}_\Lambda \cong \mathcal{L}_\Gamma \)), for every (or some) locally finite positively graded algebra \( A \) which is orthogonal to \( \Lambda \);
4) \( \varphi_B : \text{Gr}_{\text{lfgr}KQ} = \text{lfgr}_KQ^{op}\text{Gr} \to \text{Gr}_\Gamma[X] \) induces by restriction a duality of categories \( \text{lfgr}_\Lambda \cong \mathcal{L}_\Gamma \) (resp. \( \text{lfgr}_\Lambda \cong \mathcal{L}_\Gamma \)), for every (or some) locally finite positively graded algebra \( A \) which is orthogonal to \( \Lambda \).

**Proof.** \( \Lambda \) is quadratic if \( \Lambda = \tilde{\Lambda} \) and this is equivalent to say that one (or all) of the categories \( \text{Gr}_\Lambda \), \( \text{lfgr}_\Lambda \) or \( \text{A lfgr} \cong \mathcal{L}_\Gamma \) coincides with the corresponding category of graded modules over \( \tilde{\Lambda} \), viewed as full subcategories of \( \text{Gr}_KQ \) or \( KQ\text{Gr} \) according to the case. Since \( \tilde{\Lambda} \cong A^l \), for every positively graded algebra \( A \) which is orthogonal to \( \Lambda \), the result follows directly from Theorems 2.4, 2.10 and 3.3.

We end this section with an interesting consequence of our theorems.

**Corollary 3.5.** Let \( A \) and \( B \) be two quadratically equivalent positively graded algebras. The following assertions hold:

1) There is an equivalence of categories \( \mathcal{L}_A \cong \mathcal{L}_B \) (resp. \( \mathcal{L}_A^* \cong \mathcal{L}_B^* \)).
2) When \( A \) and \( B \) are locally finite, there is an equivalence of categories \( \mathcal{L}_A^* \cong \mathcal{L}_B^* \) (resp. \( \mathcal{L}_A^* \cong \mathcal{L}_B^* \)).
3) When \( A \), \( B \) are graded factors of path algebras, there is an equivalence of categories \( \mathcal{L}_\Gamma \cong \mathcal{L}_{\Gamma'} \), where \( \Gamma \) and \( \Gamma' \) are the Yoneda algebras of \( A \) and \( B \), respectively.

**Proof.** Since \( \tilde{\Lambda} \cong \tilde{B} \) and \( A^l \cong B^l \), the result is a direct consequence of Theorems 2.4, 2.10 and 3.3.

### 4. Some equivalences of derived categories.

Throughout this section, for every abelian category \( A \), \( D^b(A) \) will be the full subcategory of its derived category \( D(A) \) with objects those isomorphic (in \( D(A) \)) to bounded complexes of objects in \( A \). We follow [17] for the terminology concerning derived categories. In the sequel, \( \Lambda \) will be a Koszul algebra with Yoneda algebra \( \Gamma \cong \Lambda^l \). Recall from [1] that we have mutually
inverse equivalences of triangulated categories $F : \mathcal{D}^1(\Lambda) \cong \mathcal{D}^1(\Gamma) : G$, where we adopt the same terminology of [1], but working with right instead of left graded modules. With that in mind, if $M'$ is an object in $\mathcal{D}^1(\Lambda)$ then $F(M')$ is a complex of graded $\Gamma$-modules with $F(M')^p = \oplus_{i+j=p} M_i \otimes \Gamma[j] = \oplus_{i+j=p} \psi(M)^i_j$, where $\psi = \psi_\Gamma$. Conversely if $N$ is an object in $\mathcal{D}^1(\Gamma)$, then one has $G(N)^p = \oplus_{i+j=p} v(N)^i_j$, with $v = v_\Lambda$. Moreover, in the particular case when $\Lambda$ is finite dimensional and $\Gamma$ is Noetherian, those equivalences induce mutually inverse equivalences $\mathcal{D}^b(gr^-\Lambda) \cong \mathcal{D}^b(gr^-\Gamma)$ ([1, Theor. 2.12.6]). Our next result is a slight extension of this. Recall that a positively graded algebra $A = \oplus_{n \geq 0} A_n$ is graded right coherent in case every finitely generated graded right ideal is finitely presented. In that case, $fpgr A$ is an abelian category with exact inclusion functor $fpgr A \rightarrow Gr A$.

**Proposition 4.1.** Let $\Lambda$ be a Koszul finite dimensional algebra with graded right coherent Yoneda algebra. The equivalences of categories $F : \mathcal{D}^1(\Lambda) \cong \mathcal{D}^1(\Gamma) : G$ induce by restriction mutually inverse equivalences of triangulated categories $F : \mathcal{D}^b(gr\Lambda) \cong \mathcal{D}^b(fpgr\Gamma) : G$.

**Proof.** The restriction of $F$ to $Gr^-\Lambda$, viewed as the full subcategory of $\mathcal{D}^1(\Lambda)$ consisting of stalk complexes at the 0-position, is just $\psi_\Gamma$. Then $F$ takes the simple objects of $Gr^-\Lambda$ onto indecomposable projective objects of $Gr\Gamma$. But then $F$ and $G$ induce by restriction mutually inverse equivalences between the triangulated subcategories of $\mathcal{D}^1(\Lambda)$ and $\mathcal{D}^1(\Gamma)$ generated by the simple objects of $Gr\Lambda$ and the indecomposable projective objects of $Gr\Gamma$, respectively. Those subcategories are $\mathcal{D}^b(gr\Lambda)$ and $\mathcal{D}^b(fpgr\Gamma)$. The latter is due to the fact that $\Gamma$ has finite graded global dimension, because its Yoneda algebra $\Lambda$, is finite-dimensional (cf. [15, Cor., p. 424]).

We shall consider the full subcategory $I_\Lambda$ of $\mathcal{D}^b(gr\Lambda)$ whose objects are the complexes isomorphic (in $\mathcal{D}^1(\Lambda)$) to bounded complexes of injective graded $\Lambda$-modules. On the other hand, when $\Gamma$ is right coherent, every finite-dimensional graded $\Gamma$-module is finitely presented, because so are the simples. We denote by $\mathcal{F}_\Gamma$ the full subcategory of $\mathcal{D}^b(fpgr\Gamma)$ consisting (up to isomorphism in $\mathcal{D}^b(fpgr\Gamma)$) of bounded complexes of finite dimensional graded $\Gamma$-modules.

**Corollary 4.2.** Let $\Lambda$ be a finite dimensional Koszul algebra such that its Yoneda algebra $\Gamma$ is graded right coherent. The equivalences $F : \mathcal{D}^b(gr\Lambda) \cong \mathcal{D}^b(fpgr\Gamma) : G$ induce mutually inverse equivalences of triangulated categories $I_\Lambda \cong \mathcal{F}_\Gamma$ and $\mathcal{D}^b(gr\Lambda)/I_\Lambda \cong \mathcal{D}^b(fpgr\Gamma)/\mathcal{F}_\Gamma$.

**Proof.** The restriction of $G$ to $Gr^+\Lambda$ is $v_\Lambda$, so that $G(\Gamma_0)$ is the stalk complex $D(\Lambda)$ at the 0-position. Then $F$ and $G$ induce mutually inverse equivalences...
between the triangulated subcategories of $\mathcal{D}^b(gr_\mathcal{A})$ and $\mathcal{D}^b(fpgr_\mathcal{A})$ generated by $D(\Lambda)$ and $\Gamma_0$, respectively, and their corresponding Verdier quotients. Those subcategories are precisely $\mathcal{I}_\mathcal{A}$ and $\mathcal{F}_\mathcal{T}$.

Let $\mathcal{A}$ be an abelian category. Recall that a Serre subcategory $T$ of $\mathcal{A}$ is a full subcategory satisfying the property that in every short exact sequence of $\mathcal{A}$, say $0 \to A \to B \to C \to 0$, the central object $B$ belongs to $T$ if and only if so do $A, C$. By [7], we have a quotient abelian category $\mathcal{A}/T$ and an exact quotient functor $\pi : \mathcal{A} \to \mathcal{A}/T$. In case of existence of the respective derived categories, we have an induced exact functor $D\pi : D(\mathcal{A}) \to D(\mathcal{A}/T)$. The following is well-known (cf. [13, Theorem 3.2]):

**Lemma 4.3.** The kernel $K$ of $D\pi$ is the full subcategory with objects the complexes having cohomology in $T$ and $D\pi$ induces an equivalence of triangulated categories $D^*(\mathcal{A})/K \cong D^*(\mathcal{A}/T)$, for $* = +, -, b$.

For every positively graded algebra $A = \oplus_{n \geq 0} A_n$, we shall denote by $L_A$ the full subcategory of $Gr_A$ consisting of finite dimensional graded $A$-modules. Within the derived category $D(Gr_A)$ we shall consider the full subcategory $\mathcal{F}_A$ consisting of bounded complexes of objects of $L_A$. We close $\mathcal{F}_A$ under isomorphism in any full subcategory of $D(Gr_A)$ containing it that we use in the sequel.

**Theorem 4.4.** Let $A = \oplus_{n \geq 0} A_n$ be a positively graded right coherent algebra such that each ideal $A_{\geq s} = \oplus_{n \geq s} A_n$ is finitely generated on the right, for every $s \geq 0$ (e.g., when $A = KQ/I$ is a right coherent generalized graded factor of a path algebra). Then all graded right $A$-modules of finite length are finitely presented and the canonical quotient functor $\pi : fpgr_A \to fpgr_A/L_A$ induces an exact functor of triangulated categories $D\pi : D^b(fpgr_A) \to D^b(fpgr_A/L_A)$ with kernel $\mathcal{F}_A$. In particular, it also induces an equivalence of triangulated categories

$$D^b(fpgr_A)/\mathcal{F}_A \cong D^b(fpgr_A/L_A).$$

**Proof.** If $S = \oplus_{n \in \mathbb{Z}} S_n$ is a simple object of $Gr_A$, then it is generated by a homogeneous element, so that $S = S_{\geq m}$, with $S_m \neq 0$, for some $m \in \mathbb{Z}$. But the chain $S : A_{\geq 1} \subseteq S_{\geq m+1} \subseteq S_{\geq m} = S$ of inclusions in $Gr_A$ and the simplicity of $S$ imply that $S \cdot A_{\geq 1} = S_{\geq m+1} = 0$. Then $S = S_m$ is a necessarily simple $A_0$-module. This argument also proves that $A_{\geq 1}$ is the graded Jacobson radical of $A$. In particular, $S$ is a direct summand of $A/A_{\geq 1} \cong A_0$ in $Gr_A$. Since $A/A_{\geq 1}$ is a finitely presented graded right $A$-module by hypothesis, we conclude that every simple object of $Gr_A$, and hence every object of finite length, belongs to $fpgr_A$. Clearly, $L_A$ is then a Serre subcategory of $fpgr_A$ and the quotient functor $\pi : fpgr_A \to fpgr_A/L_A$ makes sense. Let $K$ be the full subcategory of $D^b(fpgr_A)$ with objects the complexes $X\cdot$ having cohomology in $L_A$. Bearing in mind the previous lemma, in order to prove the assertions concerning
derived categories, we only need to prove the equality $K = \mathcal{F}_A$. Since we clearly have $\mathcal{F}_A \subseteq K$, we just need to check the converse inclusion. Let $X^r \in K$, then $H^s(X^r) = \oplus_{i \in \mathbb{Z}} H^i(X^r)$ is a (finitely presented) graded $A$-module of finite length. Then it is finitely graded, i.e., $H^s(X^r)_n = 0$ for all but finitely many $n \in \mathbb{Z}$. We pick up an interval of integers $[k, m]$ such that $H^s(X^r)_n = 0$, for all $n \notin [k, m]$. Then the complexes $X^r_{>m}$ and $X^r/X^r_{>k}$ having in their $i$-th position $X^r_{>m} = \oplus_{n>m} X^i_n$ and $X^i/X^i_{>k}$, respectively, are acyclic. Notice that our hypothesis on each $A_{\geq s}$ guarantees that all these complexes are complexes of finitely presented graded modules. Now, from the exact sequences of complexes $0 \to X^r_{>m} \to X^r \to X^r/X^r_{>m} \to 0$ and $0 \to X^r_{>k}/X^r_{>m} \to X^r/X^r_{>m} \to X^r/X^r_{>k} \to 0$, we immediately deduce that $X^r$ is isomorphic in $D^b(\text{fpgr}_R)$ to $X^r_{>k}/X^r_{>m}$. But the fact that $A_{\geq s}$ is finitely generated as a right ideal, for every $s \geq 0$, easily implies that each $A_k$ has finite length as an $A_0$-module. From that it follows that, for every finitely presented graded right $A$-module $M$, its homogeneous components $M_s$ are all $A_0$-modules of finite length. In particular, $X^r_{>k}/X^r_{>m}$ is a complex in $\mathcal{F}_A$. Hence $K \subseteq \mathcal{F}_A$, and the converse inclusion is clear.

Finally, we clarify the assertion between brackets. If $A = KQ/I$ is a right coherent generalized graded factor of a path algebra, take $d = \max\{\deg(\alpha) : \alpha \in Q_1\}$. If $p$ is a path in $Q$ such that $\bar{p} = p + I \in A_n$, with $n > s + d$, then the decomposition $p = q\alpha$, with $\alpha \in Q_1$, yields that $\deg(q) \geq s$. By iteration of this argument, we conclude that $A_{\geq s}$ is generated by $A_s \oplus \cdots \oplus A_{s+d}$, which is finite dimensional over $K$. Therefore $A_{\geq s}$ is a finitely generated graded right ideal of $A$, for every $s \geq 0$.

If $A$ is a positively graded finite dimensional algebra, we shall denote by $\overline{\text{gr}}_A$ the stable category (module injectives) of $\text{gr}_A$. Its objects are those of $\text{gr}_A$ and $\text{Hom}_{\overline{\text{gr}}_A}(M, N) = \text{Hom}_{\text{gr}_A}(M, N)/I(M, N)$, for all $M, N \in \text{gr}_A$, where $I(M, N)$ is the vector subspace of $\text{Hom}_{\text{gr}_A}(M, N)$ given by the morphisms which factor through an injective object of $\text{gr}_A$. The stable category (modulo projectives) of $\text{gr}_A$, denoted $\overline{\text{gr}}_A$, is defined dually. It is well-known that $\overline{\text{gr}}_A$ has a structure of suspended category (in the terminology of [12]) or right triangulated category (in the terminology of [3]) whose stabilization $S(\overline{\text{gr}}_A)$ is the triangulated category $D^b(\text{gr}_A)/\mathcal{I}_A$ (cf. [2][dual of Theorem 3.8]). We now have:

**Corollary 4.5.** Let $\Lambda = KQ/I$ be a finite dimensional Koszul algebra with graded right coherent Yoneda algebra $\Gamma$. There is an equivalence of triangulated categories $S(\overline{\text{gr}}_A) \cong D^b(\text{fpgr}_\Gamma)/\mathcal{I}_\Gamma$. In particular, when $\Lambda$ is self-injective, there is an equivalence of triangulated categories $\overline{\text{gr}}_A = \overline{\text{gr}}_\Lambda \cong D^b(\text{fpgr}_\Gamma)/\mathcal{I}_\Gamma$. 
Proof. Just apply Corollary 4.2 and Theorem 4.4 and the fact that, when $\Lambda$ is selfinjective, $gr_{\Lambda} = gr_{\Lambda}$ is already a triangulated category coincident with $S(\mathcal{F}_{\Lambda})$. □

Remark 4.6. When $V \subseteq P^n$ is a projective irreducible variety with coordinate algebra $K[V]$, Serre's theorem (see [16], Chap. III) says that the category coh$(V)$ of coherent sheaves on $V$ is equivalent to the category $gr_{K[V]}/L_{K[V]}$. Hence, last corollary extends and reproves in a different way the well-known result of Bernstein, Gelfand and Gelfand (see, e.g., [4] or [9, Ch. IV, Section 3]) stating that $\mathcal{D}^b(coh(P^n))$ is equivalent to $gr_{\Lambda}$, where $\Lambda$ is the exterior algebra of a $(n+1)$-dimensional vector space.

References


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GENERALIZED FOCK SPACES AND WEYL COMMUTATION RELATIONS FOR THE DUNKL KERNEL

Fethi Soltani

In this paper we study a class of generalized Fock spaces associated with the Dunkl operators. Next we introduce the commutator relations between the Dunkl operators and multiplication operators which lead to a generalized class of Weyl commutation relations for the Dunkl kernel.

1. Introduction.

Fock space (called also Segal-Bargmann space \([5]\)) is the Hilbert space of entire functions of \(\mathbb{C}^d\) with inner product given by

\[
(f, g) := \frac{1}{\pi^d} \int_{\mathbb{C}^d} f(z) \overline{g(z)} e^{-|z|^2} \, dx \, dy, \quad z = x + iy,
\]

where

\[
|z|^2 = \sum_{i=1}^{d} x_i^2 + y_i^2, \quad dx \, dy = \prod_{i=1}^{d} dx_i \, dy_i.
\]

This space which associated with Fock’s \([12]\) realization of the creation and annihilation operators of Bose particles is studied by Bargmann \([4]\). Next, the ordinary Fock space \(\mathcal{A}\) is the subject of many works \((\text{\cite{5, 7}} \text{ and \cite{8}})\).

In 2001, M. Sifi and F. Soltani \([21]\) introduced a Hilbert space \(\mathcal{A}_\gamma\) of entire functions where the inner product is weighted by a generalized Gaussian distribution. On \(\mathcal{A}_\gamma\) the Dunkl operator on the real line:

\[
T_\gamma(f)(z) := \frac{d}{dz} f(z) + 2\gamma \frac{f(z) - f(-z)}{2}, \quad \gamma > 0,
\]

and the multiplication by \(z\) are adjoints and satisfy the commutation rule

\[
[T_\gamma, z] = I + 2\gamma B, \quad \text{where} \quad Bf(x) = f(-x).
\]

This commutator rule leads to a generalized class of Weyl commutation relations for the Dunkl kernel in the one dimensional.

In this paper we consider the differential-difference operators \(T_j, \ j = 1, \ldots, d, \) on \(\mathbb{R}^d\) introduced by C.F. Dunkl in \([9]\) and called Dunkl operators in the literature. These operators are very important in pure Mathematics and in Physics. They provide a useful tool in the study of special functions associated with root systems \([10]\). They are closely related to certain
representations of degenerated affine Hecke algebras ([6] and [16]). Moreover the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum Mechanics, namely the Calogero-Sutherland-Moser models, which deal with systems of identical particles in the one dimensional space ([2, 3] and [14]).

The Dunkl kernel $E_k(x, y)$ is the unique solution of the initial problem

$$T_j^x u(x, y) = y_j u(x, y); \quad u(0, y) = 1; \quad j = 1, \ldots, d,$$

see [10, 17] and [18]. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$. Furthermore, the Dunkl kernel $E_k(z, w); \quad z, w \in \mathbb{C}^d$ can be expanded in a power series in the form

$$E_k(z, w) = \sum_{\nu \in \mathbb{N}^d} \varphi_\nu(z) \varphi_\nu(w),$$

with some homogeneous orthonormal basis $\{\varphi_\nu\}_{\nu \in \mathbb{N}^d}$ of polynomials ([17] and [19]).

We introduce in this paper the generalized Fock space $A_k$ associated with the Dunkl operators. This is a Hilbert space of functions $f$ on $\mathbb{C}^d$ which can be written $f(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu \varphi_\nu(z)$ with

$$\|f\|_k^2 = (f, f)_k := \sum_{\nu \in \mathbb{N}^d} |a_\nu|^2 < \infty.$$

If $f, g \in A_k$, having series expansions $f(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu \varphi_\nu(z)$ and $g(z) = \sum_{\nu \in \mathbb{N}^d} b_\nu \varphi_\nu(z)$. Then the inner product is given by the generalized spherical harmonics

$$(f, g)_k = \left( f(T) \tilde{g} \right)(0),$$

where $f(T) = f(T_1, \ldots, T_d)$ and $\tilde{g}(z) = \sum_{\nu \in \mathbb{N}^d} \overline{b_\nu} \varphi_\nu(z)$.

The generalized Fock space $A_k$, has also a reproducing kernel $K$ given by

$$(K(z, w) = E_k(z, \overline{w}); \quad z, w \in \mathbb{C}^d).$$

If $f \in A_k$, then we have

$$f(w) = (f, E_k(\cdot, w))_k, \quad w \in \mathbb{C}^d.$$

Thus the Dunkl kernel serves as the generalized Dirac delta function in $A_k$.

The associated operators for the generalized Fock space $A_k$ are $T_j$ and the multiplication operator by $z_j$. They are adjoints in $A_k$ and satisfy a commutation rule:

$$[T_i, z_j] = \delta_{i,j} I + \sum_{\alpha \in R_+} k(\alpha) \alpha_i \alpha_j B_\alpha; \quad i, j = 1, \ldots, d,$$

where $B_\alpha$ a reflection operator, $k(\alpha)$ a multiplicity function and $R_+$ is a positive root system.

These commutators rule lead to a generalized class of Weyl commutation relations for the Dunkl kernel.

These relations are studied in the classical case ($k = 0$) in [13].
Throughout this paper we shall use the standard multi-index notations. For multi-indices $\nu, s \in \mathbb{N}^d$, we write $|\nu| = \sum_{i=1}^{d} \nu_i$, $\nu! = \prod_{i=1}^{d} \nu_i!$, $(\nu / s) = \prod_{i=1}^{d} (\nu_i / s_i)$, as well as $z^{\nu} = \prod_{i=1}^{d} z_i^{\nu_i}$, $D^{\nu} = \prod_{i=1}^{d} D_i^{\nu_i}$, for $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ and any family $D = (D_1, \ldots, D_d)$ of commuting operators.

Finally, we will need the partial ordering $\leq$ on $\mathbb{N}^d$, which is defined by $s \leq \nu \iff s_i \leq \nu_i$, $i = 1, \ldots, d$.

2. Preliminaries.

In this section we collect some notations and results on Dunkl operators and Dunkl kernel that will be important later on. General references here are [9, 17, 18, 19] and [20].

We consider $\mathbb{R}^d$ with the Euclidean scalar $\langle ., . \rangle$ and $|x| = \sqrt{\langle x, x \rangle}$. On $\mathbb{C}^d$, $|.|$ denotes also the standard Hermitian norm, while $\langle z, w \rangle = \sum_{j=1}^{d} z_j w_j$ and $\ell(z) = \langle z, z \rangle$.

For $\alpha \in \mathbb{R}^d \setminus \{0\}$, let $\sigma_\alpha$ be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to $\alpha$,

$$\sigma_\alpha x := x - \frac{2 \langle \alpha, x \rangle}{|\alpha|^2} \alpha.$$

A finite set $R \subset \mathbb{R}^d \setminus \{0\}$ is called a root system if $R \cap \mathbb{R} \alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. We assume that it is normalized by $|\alpha|^2 = 2$ for all $\alpha \in R$. For a given root system $R$ the reflections $\sigma_\alpha$, $\alpha \in R$ generated a finite group $G \subset O(d)$, the reflection group associated with $R$. All reflections in $G$ correspond to suitable pairs of roots. For a given root system $R$ with positive subsystem $R_+ = \{\alpha \in R / \langle \alpha, \alpha \rangle > 0\}$, then for each $\alpha \in R$ either $\alpha \in R_+$ or $-\alpha \in R_+$. The connected components of $H$ are called the Weyl chambers of $G$.

A function $k : R \to \mathbb{C}$ on a root system $R$ is called a multiplicity function if it is invariant under the action of the associated reflection group $G$. If one regards $k$ as a function on the corresponding reflections, this means that $k$ is constant on the conjugacy classes of reflections in $G$. For abbreviation, we introduce the index

$$\gamma = \gamma(k) := \sum_{\alpha \in R_+} k(\alpha).$$

Moreover, let $w_k$ denotes the weight function:

$$w_k(x) := \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}, \quad x \in \mathbb{R}^d,$$

which is $G$-invariant and homogeneous of degree $2\gamma$. 
For $d = 1$ and $G = \mathbb{Z}_2$, the multiplicity function $k$ is a simple parameter denoted $\gamma > 0$ and

$$w_k(x) = |x|^{2\gamma}, \quad x \in \mathbb{R}.$$  

The Dunkl operators $T_j; \ j = 1, \ldots, d$, on $\mathbb{R}^d$ associated with the finite reflection group $G$ and multiplicity function $k$ are given for a function $f$ of class $C^1$ on $\mathbb{R}^d$, by

$$T_j f(x) := \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathbb{R}_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}.$$  

In the case $k = 0$, the $T_j; \ j = 1, \ldots, d$, reduce to the corresponding partial derivatives. In this paper we will assume throughout that $k \geq 0$.

For $y \in \mathbb{R}^d$, the initial problem

$$\begin{cases}
T_j^x u(x,y) = y_j u(x,y); \ j = 1, \ldots, d, \\
u(0,y) = 1,
\end{cases}$$

admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted $E_k(x,y)$ and called the Dunkl kernel ([17, 18, 19] and [20]). This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

Examples.

1) If $k = 0$, then $E_k(z,w) = e^{\langle z,w \rangle}$ for $z, w \in \mathbb{C}^d$. (Recall that $\langle \cdot, \cdot \rangle$ was defined to be bilinear on $\mathbb{C}^d \times \mathbb{C}^d$.)

2) If $d = 1$ and $G = \mathbb{Z}_2$, the Dunkl kernel is given by

$$E_{\gamma}(z,w) = \Im_{\gamma - \frac{1}{2}}(zw) + \frac{zw}{2\gamma + 1} \Im_{\gamma + \frac{1}{2}}(zw),$$

where

$$\Im_{\gamma - \frac{1}{2}}(zw) = \Gamma \left( \gamma + \frac{1}{2} \right) \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n + \gamma + \frac{1}{2})} \left( \frac{zw}{2} \right)^{2n},$$

is the modified Bessel function of order $\gamma - \frac{1}{2}$ [21].

Let $\mathcal{P} = \mathbb{C}[\mathbb{R}^d]$ denotes the $\mathbb{C}$- Algebra of polynomial functions on $\mathbb{R}^d$ and $\mathcal{P}_n, \ n \in \mathbb{N}$, the subspace of homogeneous polynomials of degree $n$. In the context of generalized spherical harmonics, C.F. Dunkl in [9] introduced on $\mathcal{P}$ the following bilinear form:

$$(p, q)_k := \left( p(T)q \right)(0); \quad p, q \in \mathcal{P}.$$  

Here $p(T)$ is the operator derived from $p(x)$ by replacing $x_i$ by $T_i$. A useful collection of its properties can be found in [9] and [17]. We recall that $(\cdot, \cdot)_k$ is symmetric, positive-definite and $(p, q)_k = 0$, for $p \in \mathcal{P}_n, \ q \in \mathcal{P}_m$ with $n \neq m$. Moreover, for all $j = 1, \ldots, d$ and $p, q \in \mathcal{P}$,

$$(x_j p, q)_k = (p, T_j q)_k.$$
Let \( \{ \varphi_\nu \}_{\nu \in \mathbb{N}^d} \) be an orthonormal basis of \( \mathcal{P} \) with respect to the scalar product \( (.,.)_k \) such that \( \varphi_\nu \in \mathcal{P}_{|\nu|} \) and the coefficients of the \( \varphi_\nu \) are real. As \( \mathcal{P} = \bigoplus_{n \in \mathbb{N}} \mathcal{P}_n \) and \( \mathcal{P}_n \perp \mathcal{P}_m \) for \( n \neq m \), the \( \varphi_\nu \) with \( |\nu| = n \) can for example be constructed by Gram-Schmidt orthogonalization within \( \mathcal{P}_n \) from an arbitrary ordered real-coefficients basis of \( \mathcal{P}_n \). If \( k = 0 \) the canonical choice of the homogeneous orthonormal basis \( \varphi_\nu \) is just \( \varphi_\nu(x) = x^{\nu} / \sqrt{\nu!} \).

As in the classical case, M. Rösler obtained in [17, p. 524] the following connection of the basis \( \varphi_\nu \) with the Dunkl kernel:

\[
E_k(z, w) = \sum_{\nu \in \mathbb{N}^d} \varphi_\nu(z) \varphi_\nu(w); \quad z, w \in \mathbb{C}^d,
\]  

where the convergence is normal on \( \mathbb{C}^d \times \mathbb{C}^d \).

**Example.** If \( d = 1 \) and \( G = \mathbb{Z}_2 \) every homogeneous orthonormal basis is of the form

\[
\varphi_n(z) = \frac{z^n}{\sqrt{b_n(\gamma)}}, \quad b_n(\gamma) = \frac{2^n ([n/2])!}{\Gamma(\gamma + \frac{1}{2})} \Gamma \left( \left\lfloor \frac{n + 1}{2} \right\rfloor + \gamma + \frac{1}{2} \right).
\]  

Here \( [n/2] \) is the integer part of \( n/2 \).

From (2), the Dunkl kernel \( E_k \) possesses the following properties ([17, 19] and [20]):

\[
E_k(z, w) = E_k(w, z), \quad E_k(\lambda z, w) = E_k(z, \lambda w),
\]

\[
E_k(z, w) = E_k(z, \overline{w}), \quad E_k(z, \overline{z}) = \sum_{\nu \in \mathbb{N}^d} |\varphi_\nu(z)|^2,
\]

\[
|E_k(z, w)| \leq e^{||z|| ||w||}.
\]

In [18], M. Rösler establish the Bochner-type representation of the Dunkl kernel

\[
E_k(x, z) = \int_{\mathbb{R}^d} e^{(\xi, z)} d\mu_x(\xi); \quad x \in \mathbb{R}^d, \quad z \in \mathbb{C}^d,
\]

where \( \mu_x \) is a probability measure on \( \mathbb{R}^d \) with support in \( \{ \xi \in \mathbb{R}^d / |\xi| \leq |x| \} \).

The Dunkl kernel \( E_k \) is analytic on \( \mathbb{C}^d \times \mathbb{C}^d \). Therefore, there exist unique analytic functions \( m_\nu, \nu \in \mathbb{N}^d \), on \( \mathbb{C}^d \) with

\[
E_k(z, w) = \sum_{\nu \in \mathbb{N}^d} \frac{m_\nu(z)}{\nu!} w^\nu; \quad z, w \in \mathbb{C}^d.
\]

The restriction of \( m_\nu \) to \( \mathbb{R}^d \) are called the \( \nu \)-th moment functions ([18, 19] and [20]). It is given explicitly by

\[
m_\nu(x) = \int_{\mathbb{R}^d} \xi^\nu d\mu_x(\xi), \quad x \in \mathbb{R}^d,
\]

where \( \mu_x \) is the measure given by (7).
The functions $m_{\nu}$ are homogeneous polynomials of degree $|\nu|$. Among the applications of these moments, we mention the construction of martingales from Dunkl-type Markov processes \cite{19}.


In this section we define and study the generalized Fock space for the Dunkl kernel in $d$-dimensions.

Definition 1. The generalized Fock space $A_k$ associated with the Dunkl operators is the space of holomorphic functions $f$ on $\mathbb{C}^d$ which can be written $f(z) = \sum_{\nu \in \mathbb{N}^d} a_{\nu} \varphi_{\nu}(z)$ with

$$\|f\|_k^2 := \sum_{\nu \in \mathbb{N}^d} |a_{\nu}|^2 < \infty.$$  

Hence the inner product in $A_k$ is given for $f(z) = \sum_{\nu \in \mathbb{N}^d} a_{\nu} \varphi_{\nu}(z) \in A_k$ and $g(z) = \sum_{\nu \in \mathbb{N}^d} b_{\nu} \varphi_{\nu}(z) \in A_k$, by

$$(f,g)_k := \sum_{\nu \in \mathbb{N}^d} a_{\nu} \overline{b_{\nu}}.$$  

Remark. If $k = 0$, $A_0$ is the ordinary Fock space $A$ \cite{4}.

Proposition 1.  

i) If $f,g \in A_k$ with $f(z) = \sum_{\nu \in \mathbb{N}^d} a_{\nu} \varphi_{\nu}(z)$ and $g(z) = \sum_{\nu \in \mathbb{N}^d} b_{\nu} \varphi_{\nu}(z)$, we have

$$(f,g)_k = \left( f(T) \overline{g} \right)(0),$$

where $\overline{g}(z) = \sum_{\nu \in \mathbb{N}^d} \overline{b_{\nu}} \varphi_{\nu}(z)$.

ii) If $f \in A_k$ with $f(z) = \sum_{\nu \in \mathbb{N}^d} a_{\nu} \varphi_{\nu}(z)$, we have

$$|f(z)| \leq e^{\|z\|^2/2} \|f\|_k.$$

Proof.  

i) From \cite[p. 529]{17}, we have

$$\left( \varphi_{\nu}(T) \varphi_{s} \right)(0) = \delta_{\nu,s},$$

where $\delta_{\nu,s}$ is the Kronecker symbol.

Thus

$$(f,g)_k = \sum_{\nu,s \in \mathbb{N}^d} a_{\nu} \overline{b_{s}} \left( \varphi_{\nu}(T) \varphi_{s} \right)(0).$$

Using the continuously of the inner product, we obtain the result.

ii) Using Cauchy-Schwarz’s inequality, then

$$|f(z)|^2 \leq \left[ \sum_{\nu \in \mathbb{N}^d} |a_{\nu}|^2 \right] \left[ \sum_{\nu \in \mathbb{N}^d} |\varphi_{\nu}(z)|^2 \right] = \|f\|_k^2 E_k(z,\overline{z}).$$
Thus
\[ |f(z)| \leq [E_k(z, z)]^{1/2} \|f\|_k. \]
The result follows from the inequality (6).

From Proposition 1 ii), the map \( f \rightarrow f(z), z \in \mathbb{C}^d \), is a continuous linear functional on \( A_k \). Thus from Riesz theorem [1], \( A_k \) has a reproducing kernel.

**Proposition 2.** The function \( K \) given for \( w, z \in \mathbb{C}^d \), by
\[ K(z, w) = E_k(z, w), \]
is a reproducing kernel for the generalized Fock spaces \( A_k \), that is:

i) For every \( w \in \mathbb{C}^d \), the function \( z \rightarrow K(z, w) \) belongs to \( A_k \).

ii) The reproducing property: For every \( w \in \mathbb{C}^d \) and \( f \in A_k \), we have
\[ (f, K(\cdot, w))_k = f(w). \]

**Proof.**

i) Using (5) and (6), we deduce for \( w \in \mathbb{C}^d \),
\[ \|E_k(\cdot, w)\|_k^2 = \sum_{\nu \in \mathbb{N}^d} |\varphi_\nu(w)|^2 = E_k(w, w) \leq e^{\|w\|^2}, \]
which proves i).

ii) If \( f(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu \varphi_\nu(z) \in A_k \), it follows from (9) that
\[ (f, E_k(\cdot, w))_k = \sum_{\nu \in \mathbb{N}^d} a_\nu \varphi_\nu(w) = f(w). \]

\[ \square \]

**Corollary 1.**

i) The set \( \{ E_k(\cdot, w), w \in \mathbb{C}^d \} \) is complete in \( A_k \).

ii) For all \( z, w \in \mathbb{C}^d \), we have
\[ E_k(z, w) = (E_k(\cdot, z), E_k(\cdot, w))_k. \]

iii) Let \( m \in \mathbb{N} \setminus \{0\} \) and \( z_1, z_2, \ldots, z_m \in \mathbb{C}^d \), with \( z_i \neq z_j \), then
\[ \det \left[ E_k(z_i, z_j) \right]_{i,j=1}^m > 0. \]

**Notation.** We denote by \( L^2(\mu_k) \) the Hilbert space of measurable functions on \( \mathbb{R}^d \), for which
\[ \|f\|_{2,k} := \left[ \int_{\mathbb{R}^d} |f(x)|^2 d\mu_k(x) \right]^{1/2} < \infty. \]
Here \( \mu_k \) is the measure defined on \( \mathbb{R}^d \), by
\[ d\mu_k(x) := c_k w_k(x) dx, \quad \text{with} \quad c_k = \left( \int_{\mathbb{R}^d} e^{-|x|^2} d\mu_k(x) \right)^{-1}, \]
is the Mehta-type constant.
In the next part of this section we establish the unitary equivalence of $L^2(\mu_k)$ and $A_k$. First we recall some properties of the generalized Hermite functions ([17] and [19]):

**Definition 2.** The generalized Hermite polynomials $\{H_\nu\}_{\nu \in \mathbb{N}^d}$ associated with the basis $\{\varphi_\nu\}_{\nu \in \mathbb{N}^d}$ on $\mathbb{C}^d$, are given by

$$H_\nu(z) := 2^{\nu} e^{-\frac{\Delta_k}{4} z^2} \varphi_\nu(z) = 2^{\nu} \sum_{n=0}^{\lfloor |\nu|/2 \rfloor} \frac{(-1)^n}{n!} \triangle_k^n \varphi_\nu(z),$$

where $\Delta_k = \sum_{i=1}^d T_i^2$ is the Dunkl Laplacian [17].

Moreover, we define the generalized Hermite functions on $\mathbb{C}^d$, by

$$h_\nu(z) := 2^{-\nu/2} e^{-\ell(z)/2} H_\nu(z).$$

**Examples.**
1) If $k = 0$, we obtain

$$H_\nu(x) = \frac{2^{\nu}}{\sqrt{\nu!}} \prod_{i=1}^d e^{-\frac{1}{4} x_i^2} \left( x_i^{\nu_i} \right) = \frac{1}{\sqrt{\nu!}} \prod_{i=1}^d \tilde{H}_{\nu_i}(x_i), \quad x \in \mathbb{R}^d,$

where

$$\tilde{H}_{x_i} = (-1)^{\nu_i} e^{x_i^2} \frac{\partial^{\nu_i}}{\partial x_i^{\nu_i}} (e^{-x_i^2}).$$

2) If $d = 1$ and $G = \mathbb{Z}_2$, we obtain

$$H_n(z) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{(-1)^i}{i! b_{n-2i}(\gamma)} (2z)^{n-2i}, \quad x \in \mathbb{R},$$

where $b_n(\gamma)$ are the constants given by (3).

The following lemma is shown in [17, p. 525-529]:

**Lemma 1.**

i) The set $\{h_\nu\}_{\nu \in \mathbb{N}^d}$ is an orthonormal basis of $L^2(\mu_k)$.

ii) For all $z,w \in \mathbb{C}^d$, there is a generating function for the generalized Hermite polynomials,

$$e^{-\ell(w)} E_k(2z,w) = \sum_{\nu \in \mathbb{N}^d} h_\nu(z) \varphi_\nu(w).$$

**Notation.** We denote by $U_k$ the kernel given for $z,w \in \mathbb{C}^d$, by

(10) $$U_k(z,w) := e^{-(\ell(z)+\ell(w))/2} E_k(\sqrt{2}z,w).$$

**Lemma 2.** For $w,z \in \mathbb{C}^d$, we have

$$U_k(z,w) = \sum_{\nu \in \mathbb{N}^d} h_\nu(z) \varphi_\nu(w).$$
Proof. From Definition 2, we have
\[ \sum_{\nu \in \mathbb{N}^d} h_\nu(z)\varphi_\nu(w) = e^{-\ell(z)/2} \sum_{\nu \in \mathbb{N}^d} 2^{-|\nu|/2} H_\nu(z)\varphi_\nu(w). \]
As \( \varphi_\nu \) is homogeneous of degree \( |\nu| \), then
\[ \varphi_\nu\left(\frac{w}{\sqrt{2}}\right) = 2^{-|\nu|/2} \varphi_\nu(w). \]
Thus
\[ \sum_{\nu \in \mathbb{N}^d} h_\nu(z)\varphi_\nu(w) = e^{-\frac{\ell(z)}{2}} \sum_{\nu \in \mathbb{N}^d} H_\nu(z)\varphi_\nu\left(\frac{w}{\sqrt{2}}\right). \]
Applying Lemma 1 ii) and (4), we obtain
\[ \sum_{\nu \in \mathbb{N}^d} h_\nu(z)\varphi_\nu(w) = e^{-\frac{(\ell(z) + \ell(w))}{2}} E_k(2z, \frac{w}{\sqrt{2}}) = U_k(z, w). \]
\[ \square \]

Lemma 3.

i) For all \( z, w \in \mathbb{C}^d \), we have
\[ E_k(z, w) = \int_{\mathbb{R}^d} U_k(z, x)U_k(w, x)d\mu_k(x). \]

ii) For all \( z \in \mathbb{C}^d \), the function \( x \to U_k(z, x) \) belongs to \( L^2(\mu_k) \), and we have
\[ \|U_k(z, \cdot)\|_{2, k}^2 = E_k(z, \overline{z}). \]

iii) For all \( x \in \mathbb{R}^d \), the function \( z \to U_k(z, x) \) belongs to \( \mathcal{A}_k \), and we have
\[ \|U_k(\cdot, x)\|_{2, k}^2 = e^{-3|x|^2} E_k(2x, x). \]

Proof. i) We put
\[ I = \int_{\mathbb{R}^d} U_k(z, x)U_k(w, x)d\mu_k(x). \]
From (10), we have
\[ I = e^{-\frac{(\ell(z) + \ell(w))}{2}} \int_{\mathbb{R}^d} e^{-|x|^2} E_k(\sqrt{2}z, x)E_k(\sqrt{2}w, x)d\mu_k(x). \]
So from [17, p. 523] and (4), we get
\[ \int_{\mathbb{R}^d} e^{-|x|^2} E_k(\sqrt{2}z, x)E_k(\sqrt{2}w, x)d\mu_k(x) = e^{\frac{(\ell(z) + \ell(w))}{2}} E_k(z, w), \]
which proves i).

ii) This assertion follows from i) and (5).

iii) For \( z \in \mathbb{C}^d \), we put
\[ \phi(z) := e^{-\frac{(\ell(z) + \ell(\overline{z}))}{2}}. \]
Let \( x \in \mathbb{R}^d \), then from Proposition 2 ii), (10) and (4), we have
\[
\|U_k(\cdot, x)\|^2_k = e^{-|x|^2} \phi(\cdot) E_k(\cdot, \sqrt{2}x), E_k(\cdot, \sqrt{2}x) = e^{-3|x|^2} E_k(2x, x).
\]

\[\square\]

**Definition 3.** The chaotic transform \( C_k \) (also called S-transform in the stochastic calculus [15]) is the transformation defined on \( L^2(\mu_k) \), by
\[
C_k(f)(z) := \int_{\mathbb{R}^d} U_k(z, x) f(x) d\mu_k(x), \quad z \in \mathbb{C}^d.
\]

**Remark.** The basis elements of \( L^2(\mu_k) \) and \( A_k \) are called chaos. In the following theorem we shall prove that the transformation \( C_k \) maps the chaos of \( L^2(\mu_k) \) to these of \( A_k \).

**Theorem 1.** The chaotic transform \( C_k \) is a unitary mapping of \( L^2(\mu_k) \) on \( A_k \). Moreover, the basis elements are related by
\[
C_k(h_\nu) = \varphi_\nu.
\]

**Proof.** It follows directly from Lemma 1 i) and Lemma 2, that for \( \nu \in \mathbb{N}^d \),
\[
C(h_\nu)(z) = \int_{\mathbb{R}^d} U_k(z, x) h_\nu(x) d\mu_k(x) = \varphi_\nu(z), \quad z \in \mathbb{C}^d.
\]

Consequently \( C_k \) maps the subspace generated by the family \( \{h_\nu\}_{\nu \in \mathbb{N}^d} \) into the polynomials in \( A_k \). Thus \( C_k \) maps a dense set in \( L^2(\mu_k) \) into a dense set in \( A_k \). Further, if \( f \in L^2(\mu_k) \), then \( f(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu h_\nu(x) \). For \( \nu \in \mathbb{N}^d \), let \( f_N(x) = \sum_{j=0}^N \sum_{|\nu|=j} a_\nu h_\nu(x), \quad x \in \mathbb{R}^d \). Then
\[
C_k(f_N)(z) = \sum_{j=0}^N \sum_{|\nu|=j} a_\nu \varphi_\nu(z); \quad \lim_{N \to \infty} \|f - f_N\|_{2,k} = 0.
\]

On the other hand, from H"older’s inequality and Lemma 3 ii), we have
\[
|C_k(f - f_N)(z)| \leq \|E_k(z, \overline{z})\|^{1/2} \|f - f_N\|_{2,k}.
\]
Thus we obtain
\[
C_k(f)(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu \varphi_\nu(z).
\]

Hence
\[
\|C_k(f)\|^2_k = \sum_{\nu \in \mathbb{N}^d} |a_\nu|^2 = \|f\|^2_{2,k}.
\]

It follows that \( C_k \) is a unitary transformation from \( L^2(\mu_k) \) into \( A_k \).

Clearly, if \( g(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu \varphi_\nu(z) \in A_k \), we have
\[
C_k^{-1}(g)(x) = \sum_{\nu \in \mathbb{N}^d} a_\nu h_\nu(x), \quad x \in \mathbb{R}^d.
\]

Which completes the proof. \[\square\]
Proposition 3. If \( g \in \mathcal{A}_k \), we have
\[
C_k^{-1}(g)(x) = (g, U_k(\cdot, x))_k, \quad x \in \mathbb{R}^d.
\]

Proof. Let \( g \in \mathcal{A}_k \). We put for \( x \in \mathbb{R}^d \),
\[
\Psi_k(g)(x) = (g, U_k(\cdot, x))_k.
\]
Using Lemma 2, Lemma 3 iii) and the same method as in the proof of Theorem 1 we obtain
\[
\Psi_k(g)(x) = \sum_{\nu \in \mathbb{N}^d} a_{\nu} h_\nu(x) = C_k^{-1}(g)(x), \quad x \in \mathbb{R}^d.
\]
\( \Box \)


We define the multiplication operators \( Q_i; \ i = 1, \ldots, d \) on \( \mathcal{A}_k \) by
\[
Q_i f(z) := z_i f(z), \quad z \in \mathbb{C}^d.
\]
We denote also by \( T_i; \ i = 1, \ldots, d \) the operators defined on \( \mathcal{A}_k \).
Let
\[
\mathcal{D}(Q_i) = \{ f \in \mathcal{A}_k / Q_i f \in \mathcal{A}_k \},
\]
\[
\mathcal{D}(T_i) = \{ f \in \mathcal{A}_k / T_i f \in \mathcal{A}_k \}
\]
denote the domains of \( Q_i \) and \( T_i \) respectively.
We denote by \([\cdot, \cdot]\) the commutator product \([A, B] = AB - BA\). As in [11], we have the following relations:

Lemma 4. The operators \( Q_i \) and \( T_i, \ i = 1, \ldots, d \) satisfy on \( \mathcal{A}_k \) the commutation relations:
\[
[T_i, T_j] = [Q_i, Q_j] = 0; \quad i, j = 1, \ldots, d,
\]
\[
[T_i, Q_j] = \delta_{i,j} I + \sum_{\alpha \in \mathbb{R}_+} k(\alpha) \alpha_i \alpha_j B_\alpha; \quad i, j = 1, \ldots, d,
\]
where \( I \) the identity operator and \( B_\alpha \) is the reflection operator \((B_\alpha^2 = I)\) given by
\[
B_\alpha f(z) = f(\sigma_\alpha z).
\]

Proof. Using the fact that \( \sigma_\alpha^2 = id \) and \( \langle \alpha, \sigma_\alpha z \rangle = -\langle \alpha, z \rangle \), we obtain
\[
T_i T_j f(z) = T_i \left( \frac{\partial}{\partial z_j} f \right)(z) + \sum_{\alpha \in \mathbb{R}_+} k(\alpha) \alpha_i \frac{\partial}{\partial z_i} \left( \frac{f(z) - f(\sigma_\alpha z)}{\langle \alpha, z \rangle} \right).
\]
Since
\[
\frac{\partial}{\partial z_i} (f(\sigma_\alpha z)) = \frac{\partial}{\partial z_i} f(\sigma_\alpha z) - \sum_{\ell=1}^d \alpha_i \alpha_\ell \frac{\partial}{\partial z_\ell} f(\sigma_\alpha z),
\]
we have
\[ T_i T_j f(z) = -\frac{\partial^2}{\partial z_j \partial z_i} f(z) + T_i \left( \frac{\partial}{\partial z_j} f \right)(z) + T_j \left( \frac{\partial}{\partial z_i} f \right)(z) - \sum_{\alpha \in R_+} k(\alpha) \alpha_i \alpha_j \left[ f(z) - f(\sigma_\alpha z)^R_{\alpha, z}^2 - \sum_{\ell=1}^d \alpha_\ell \frac{\partial}{\partial z_\ell} f(\sigma_\alpha z) \right]. \]

Thus
\[ [T_i, T_j] f(z) = 0. \]
The other relations are evident. \qed

**Proposition 4.** Let
\[ f(z) = \sum_{\nu \in \mathbb{N}^d} a_{\nu} \varphi_{\nu}(z) \in D(Q_i) \quad \text{and} \]
\[ g(z) = \sum_{\nu \in \mathbb{N}^d} a_{\nu} \varphi_{\nu}(z) \in D(T_i), \quad \text{then} \]
\[ (Q_i f, g)_k = (f, T_i g)_k. \]

**Proof.** Applying Proposition 1 i), we get
\[ (Q_i f, g)_k = (Q_i f(T) \tilde{g})(0) = \sum_{\nu, s \in \mathbb{N}^d} a_{\nu} \overline{b_{\nu}} T_i \varphi_{\nu}(T) \varphi_s(0). \]
Then from (12) we obtain
\[ (Q_i f, g)_k = \sum_{\nu, s \in \mathbb{N}^d} a_{\nu} \overline{b_{\nu}} \varphi_{\nu}(T) T_i \varphi_s(0) = (f, T_i g)_k. \]
\qed

**Lemma 5.** If \( f \in A_k \), then \( B_\alpha f \in A_k \), and we have
\[ \|Q_i f\|_k^2 = \|T_i f\|_k^2 + \|f\|_k^2 + \sum_{\alpha \in R_+} k(\alpha) \alpha_i^2 (f, B_\alpha f)_k, \]
where \( B_\alpha \) is the operator given by (14).

**Proof.** Let \( f \in A_k \). Applying the chaotic transform, in view of Theorem 1, it suffices to show that \( C_k^{-1}(B_\alpha f) \in L^2(\mu_k) \). From (11), we have
\[ C_k^{-1}(B_\alpha f)(x) = C_k^{-1}(f)(\sigma_\alpha x), \quad x \in \mathbb{R}^d. \]
Putting \( u = \sigma_\alpha x \), we get
\[ d\mu_k(x) = |J_\alpha| d\mu_k(u) \quad \text{where} \quad J_\alpha = \det \left[ \delta_{i,j} - \alpha_i \alpha_j \right]_{i,j=1}^d. \]
Since \( J_\alpha = -1 \), we obtain
\[ \|C_k^{-1}(B_\alpha f)\|_{2,k}^2 = \int_{\mathbb{R}^d} |C_k^{-1}(f)(u)|^2 d\mu_k(u). \]
Which proves that $B_\alpha f \in A_k$.

On the other hand, from Proposition 4, we deduce
\[ \|Q_if\|^2_k = (f, T_iQ_if)_k. \]

But from (13), we have
\[ T_iQ_if = Q_iT_if + f + \sum_{\alpha \in R_+} k(\alpha)\alpha_i^2 B_\alpha f. \]

Thus
\[ \|Q_if\|^2_k = (f, Q_iT_if)_k + \|f\|^2_k + \sum_{\alpha \in R_+} k(\alpha)\alpha_i^2(f, B_\alpha f)_k. \]

Using another time Proposition 4, we obtain the result. \[ \square \]

**Proposition 5.** The operators $Q_i$ and $T_i$ are closed densely defined operators on $A_k$, and we have
\[ \mathcal{D}(Q_i) = \mathcal{D}(T_i); \quad Q_i^* = T_i; \quad T_i^* = Q_i, \]
where $Q_i^*$ and $T_i^*$ are the adjoints operators of $Q_i$ and $T_i$, respectively.

**Proof.** These results follow from [4, Theorem 1.2], Lemma 5 and Proposition 4 by using the same method as [21, Proposition 6]. \[ \square \]

**Lemma 6.** For $\nu \in \mathbb{N}^d \setminus \{0\}$, we have the following relations:

i) \[ [T_\nu, Q_j] = \nu_j T_1^\nu_1 \cdots T_{i-1}^\nu_{i-1} T_i^\nu_i - 1 T_{i+1}^\nu_{i+1} \cdots T_d^\nu_d \]
\[ + B_\alpha \sum_{i=1}^d \sum_{\ell=0}^{\nu_i-1} \sum_{\alpha \in R_+} k(\alpha)\alpha_i\alpha_j H_1^\nu_1 \cdots H_{i-1}^\nu_{i-1} H_i^\ell T_1^\nu_i - \ell - 1 T_{i+1}^\nu_{i+1} \cdots T_d^\nu_d, \]

where $H_i; \ i = 1, \ldots, d$, are given by the differential-difference operators
\[ H_i = -T_i + 2 \frac{\partial}{\partial x_i} - \sum_{\ell=1}^d \alpha_i\alpha_\ell \frac{\partial}{\partial x_\ell}. \]

ii) \[ [T_j, Q_\nu] = \nu_j Q_1^\nu_1 \cdots Q_{i-1}^\nu_{i-1} Q_i^\nu_i - 1 Q_{i+1}^\nu_{i+1} \cdots Q_d^\nu_d \]
\[ + B_\alpha \sum_{i=1}^d \sum_{\ell=0}^{\nu_i-1} \sum_{\alpha \in R_+} k(\alpha)\alpha_i\alpha_j Z_1^\nu_1 \cdots Z_{i-1}^\nu_{i-1} Z_i^\ell Q_1^\nu_i - \ell - 1 Q_{i+1}^\nu_{i+1} \cdots Q_d^\nu_d, \]

where $Z_i; \ i = 1, \ldots, d$, are the multiplication operators given by
\[ Z_i = Q_i - \sum_{\ell=1}^d \alpha_i\alpha_\ell Q_\ell. \]
Proof. From (13), we have
\[
[T^\nu_i, Q_j] = \sum_{\ell=0}^{\nu_i-1} T^\ell_i [T_i, Q_j] T^{\nu_i-\ell-1}_i
\]
\[
= \nu_i \delta_{ij} T^{\nu_i-1}_i + \sum_{\ell=0}^{\nu_i-1} k(\alpha) \alpha_i \alpha_j T^\ell_i B_\alpha T^{\nu_i-\ell-1}_i.
\]
From this equality, we get
\[
[T^\nu, Q_j] = \sum_{i=1}^d T^\nu_i \ldots T^{\nu_{i-1}}_i [T^\nu_i, Q_j] T^{\nu_{i+1}}_i \ldots T^\nu_d
\]
\[
= \nu_j T^\nu_1 \ldots T^{\nu_{i-1}}_i T^{\nu_{i+1}}_i \ldots T^\nu_d
\]
\[
+ \sum_{i=1}^d \sum_{\ell=0}^{\nu_i-1} \sum_{\alpha \in R_+} k(\alpha) \alpha_i \alpha_j T^\nu_i \ldots T^{\nu_{i-1}}_i T^\ell_i B_\alpha T^{\nu_{i+1}}_i \ldots T^{\nu_d}_i.
\]
But
\[
T^\nu_i B_\alpha = B_\alpha H^\nu_i,
\]
where
\[
H_i = -T_i + 2 \frac{\partial}{\partial x_i} - \sum_{\ell=1}^d \alpha_i \alpha_\ell \frac{\partial}{\partial x_\ell}.
\]
Thus we obtain Assertion i). And similarly we get ii). \(\square\)

**Notation.** For \(x \in \mathbb{R}^d\) and \(z \in \mathbb{C}^d\), we denote by
\[
I_k(z, x) := \frac{E_k(z, x) - E_k(z, \sigma_\alpha x)}{\langle \alpha, x \rangle}.
\]
From [17, p. 533], we can write the function \(I_k(z, x)\) in the form
\[
I_k(z, x) = \langle \nabla_x E_k(z, x), \alpha \rangle + \frac{1}{2} \langle \alpha, x \rangle \alpha^t D^2_x E_k(z, x) \alpha,
\]
with some \(\xi\) on the line segment between \(x\) and \(\sigma_\alpha x\).
(Here \(\nabla\) and \(D^2 f(x)\) denote the usual gradient and Hessian of \(f\) in \(x\).)

**Lemma 7.** For \(a, b \in \mathbb{C}^d\), we have the following commutation relations:

i) \([E_k(a, T), Q_j] = a_j E_k(a, T) - R_{k,j}(a, T)\), where
\[
R_{k,j}(a, T) = \sum_{\alpha \in R_+} k(\alpha) \alpha_j I_k(a, T)
\]
\[
- B_\alpha \sum_{\nu \in \mathbb{N}^d} \sum_{i=1}^d \sum_{\ell=0}^{\nu_i-1} k(\alpha) \alpha_i \alpha_j \frac{m_\nu(a)}{\nu!} H^\nu_1 \ldots H^{\nu_{i-1}}_i H^\ell_i T^{\nu_i-\ell-1}_i T^{\nu_{i+1}}_i \ldots T^{\nu_d}_i.
\]
ii) \([T_j, E_k(b, Q)] = b_j E_k(b, Q) - S_{k,j}(b, Q)\), where

\[
S_{k,j}(b, Q) = \sum_{\alpha \in R_+} k(\alpha) \alpha j I_k(b, Q) - B_{\alpha} \sum_{\nu \in \mathbb{N}_d} \sum_{i=1}^{d} \sum_{\ell=0}^{\nu_i-1} k(\alpha) \alpha_i \alpha_j \frac{m_\nu(b)}{\nu!} Z_1^{\nu_1} \cdots Z_{i-1}^{\nu_{i-1}} Z_i^{\nu_i - \ell} Q_i^{\nu_i - \ell - 1} Q_{i+1}^{\nu_{i+1}} \cdots Q_d^{\nu_d}.
\]

**Proof.** Using (8) and Lemma 6 i), we obtain

\[
[E_k(a, T), Q_j] = \sum_{\nu \in \mathbb{N}_d} \frac{m_\nu(a)}{\nu!} [T^\nu, Q_j]
\]

\[
= \sum_{\nu \in \mathbb{N}_d} \frac{m_\nu(a)}{\nu!} v_j T_1^{\nu_1} \cdots T_{i-1}^{\nu_{i-1}} T_{i+1}^{\nu_{i+1}} \cdots T_d^{\nu_d}
+ B_{\alpha} \sum_{\nu \in \mathbb{N}_d} \sum_{i=1}^{d} \sum_{\ell=0}^{\nu_i-1} k(\alpha) \alpha_i \alpha_j \frac{m_\nu(a)}{\nu!} H_1^{\nu_1} \cdots H_{i-1}^{\nu_{i-1}} H_i^{\nu_i - \ell} H_{i+1}^{\nu_{i+1}} \cdots H_d^{\nu_d}.
\]

Applying the relation

\[
\frac{\partial}{\partial w} E_k(z, w) = z E_k(z, w) - \sum_{\alpha \in R_+} k(\alpha) \alpha_j I_k(z, w); \quad z, w \in \mathbb{C}^d,
\]

we obtain

\[
[E_k(a, T), Q_j] = a_j E_k(a, T) - R_{k,j}(a, T).
\]

This proves i). Similarly, we can prove ii). \(\square\)

**Remark.** If \(d = 1\) and \(G = \mathbb{Z}_2 [21]\), we have

\[
R_{\gamma}(a, T_\gamma) = \frac{2\gamma}{2\gamma + 1} a(I - B) 3_\gamma + \frac{1}{2}(aT_\gamma),
\]

\[
S_{\gamma}(b, Q) = \frac{2\gamma}{2\gamma + 1} b(I - B) 3_\gamma + \frac{1}{2}(bQ),
\]

where \(Bf(x) = f(-x)\).

Since \(E_k(a, 0) = 1\), the Dunkl kernel \(E_k(a, z); a, z \in \mathbb{C}^d\), is a unit in the integral domain formal power series over \(\mathbb{C}^d\). We define

\[
E_k^{-1}(a, z) := \sum_{\nu \in \mathbb{N}_d} \frac{t_\nu(a)}{\nu!} z^\nu.
\]

Writing

\[
E_k(a, z) E_k^{-1}(a, z) = E_k^{-1}(a, z) E_k(a, z) = 1,
\]
we obtain
\[
 t_0(a) = 1, \quad \sum_{\nu \in \mathbb{N}^d} \left\{ \sum_{s \leq \nu} \binom{\nu}{s} m_{\nu-s}(a)t_s(a) \right\} \frac{z^\nu}{\nu!} = 1.
\]
Thus \( \{t_\nu(a)\}_{\nu \in \mathbb{N}^d} \) is a sequence of moment functions in \( a \) determined by
\[
 t_0(a) = 1, \quad t_\nu(a) = -\sum_{s \leq \nu-1} \binom{\nu}{s} m_{\nu-s}(a)t_s(a).
\]
The function \( E_{-1}^{'k}(a, z) \) occurs in the generalized Weyl commutation relations for the Dunkl kernel.

**Theorem 2.** Let \( a, b \in \mathbb{C}^d \), then:

i) \( E_k(b, Q)E_k(a, T) = E_k(a, T)E_k(b, P_a), \quad P_a = (P_{a,1}, \ldots, P_{a,d}), \) where
\[
 P_{a,j} = Q_j - a_j I + E_k^{-1}(a, T)R_{k,j}(a, T).
\]

ii) \( E_k(a, T)E_k(b, Q) = E_k(b, Q)E_k(a, L_b), \quad L_b = (L_{b,1}, \ldots, L_{b,d}), \) where
\[
 L_{b,j} = T_j + b_j I - E_k^{-1}(b, Q)S_{k,j}(b, Q).
\]

iii) \( E_k(a, Q)E_k(b, Q) = E_k(a \# b, Q) \), \( E_k(a, T)E_k(b, T) = E_k(a \# b, T), \) where \( a \# b \) is the convolution of \( a \) and \( b \) given by
\[
 m_\nu(a \# b) = \sum_{s \leq \nu} \binom{\nu}{s} m_s(a)m_{\nu-s}(b).
\]

**Proof.** We shall prove i), ii) follows in the same way. For \( j = 1, 2, \ldots, d \), we have
\[
 E_{-1}^{'k}(a, T)Q_j E_k(a, T) = E_{-1}^{'k}(a, T)\left\{ E_k(a, T)Q_j - [E_k(a, T), Q_j] \right\}.
\]
Using Lemma 7 i), we obtain
\[
 E_{-1}^{'k}(a, T)Q_j E_k(a, T) = Q_j - a_j I + E_{-1}^{'k}(a, T)R_{k,j}(a, T).
\]
Thus implies that for \( \nu \in \mathbb{N}^d: \)
\[
 E_{-1}^{'k}(a, T)Q^\nu E_k(a, T) = P_\nu^a, \quad P_a = (P_{a,1}, \ldots, P_{a,d}),
\]
where
\[
 P_{a,j} = Q_j - a_j I + E_{-1}^{'k}(a, T)R_{k,j}(a, T).
\]
Multiplying by \( \frac{m_\nu(b)}{\nu!} \) and summing, we get
\[
 E_{-1}^{'k}(a, T)E_k(b, Q)E_k(a, T) = E_k(b, P_a).
\]
Then i) follows upon multiplication by \( E_k(a, T) \).

iii) It suffices to prove the first relation.
Using (8) and (12), we can write

\[ E_k(a, Q)E_k(b, Q) = \sum_{\nu, s \in \mathbb{N}^d} \frac{m_\nu(a)m_\nu(b)}{\nu! s!} Q^{\nu+s} \]

\[ = \sum_{\nu \in \mathbb{N}^d} \left\{ \sum_{s \leq \nu} \binom{\nu}{s} m_s(a)m_{\nu-s}(b) \right\} \frac{Q^\nu}{\nu!} \]

\[ = \sum_{\nu \in \mathbb{N}^d} \frac{m_\nu(a\#b)}{\nu!} Q^\nu. \]

Thus we obtain

\[ E_k(a, Q)E_k(b, Q) = E_k(a\#b, Q). \]

□

Remarks. 1) In the classical case \((k = 0)\) [13, p. 223], the Weyl commutation relations are given by

\[ e^{(a, P)}e^{(b, Q)} = e^{(a, b)}e^{(b, Q)}e^{(a, P)}, \]

\[ e^{(a, P)}e^{(b, P)} = e^{(a+b, P)}, \]

\[ e^{(a, Q)}e^{(b, Q)} = e^{(a+b, Q)}, \]

where \(P = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})\) and \(Q = (Q_1, \ldots, Q_d)\).

2) If \(d = 1\) and \(G = \mathbb{Z}_2\) [21], the Weyl commutation relations are given by

\[ E_\gamma(bQ)E_\gamma(aT_\gamma) = E_\gamma(aT_\gamma)E_\gamma(bP_a); \]

\[ E_\gamma(aT_\gamma)E_\gamma(bQ) = E_\gamma(bQ)E_\gamma(aL_b), \]

where

\[ P_a = Q - aI + \frac{2\gamma}{2\gamma + 1} aE^{-1}_\gamma(aT_\gamma)(I - B)\Sigma_{\alpha+1}(aT_\gamma), \]

and

\[ L_b = T_\gamma + bI - \frac{2\gamma}{2\gamma + 1} bE^{-1}_\gamma(bQ)(I - B)\Sigma_{\alpha+1}(bQ). \]

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References


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