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Let $G = \widetilde{G}(K)$ where $\widetilde{G}$ is a simple and simply-connected algebraic group that is defined and quasi-split over a field K. We investigate properties of intersections of Bruhat cells $B\dot{w}B$ of G with conjugacy classes C of G, in particular, we consider the question, when is $B\dot{w}B \cap C \neq \emptyset$.

1. Introduction.

Let (G, B, N, S) be a Tits system. Some aspects of intersections of conjugacy classes of G with Bruhat cells $B\dot{w}B$ have been investigated by several authors (see e.g., [**St1**], [**K**], [**V**] and [**VS**]). Here $w \in W = N/(B \cap N)$ and $\dot{w} \in N$ is a preimage of w with respect to the natural surjection $N \to W$. In particular, it is desirable to learn how a conjugacy class C of G is related to those conjugacy classes C_w of W for which $B\dot{w}B \cap C \neq \emptyset$, where $w \in C_w$.

Here we deal with the case where G is a Chevalley group, i.e., G is the group of points $\widetilde{G}(K)$ of a simple algebraic group $\widetilde{G}$ that is defined and quasi-split over a field K, thus G is a proper or a twisted Chevalley group (see [**St2**]). Therefore, one can define a Tits system (G, B, N, S), where $S = \{w_{\alpha_i} \mid \alpha_i \in \Pi\}$ for a simple root system Π corresponding to G ([**St2**] and [**C1**]).

A crucial step to investigate intersections $B\dot{w}B \cap C$ was done by R. Steinberg [**St1**] who constructed the cross-section of regular conjugacy classes in $B\dot{w}_S B$, where w_S is a Coxeter element of W with respect to the fixed set of generators S of W, i.e., w_S is a product of elements in S in any order, where each $s \in S$ occurs exactly once. The next natural step is to consider intersections of regular classes with cells of the form $B\dot{w}\dot{w}_S\dot{w}^{-1}B$. Here we prove the following:

Theorem 1.1. *Let $\widetilde{G}$ be a simple and simply-connected algebraic group that is defined and quasi-split over a field K and let $G = \widetilde{G}(K)$. Further, let $C \subset G$ be a conjugacy class of G such that*

$$(*) \qquad B\dot{w}_S B \cap C \neq \emptyset,$$

where w_S is a Coxeter element of W with respect to S. Then C intersects all cells of the form $B\dot{w}\dot{w}_S\dot{w}^{-1}B$, where $w \in W$.

Note that the condition $B\dot{w}_S B \cap C \neq \emptyset$ implies that every element of C is regular in $\widetilde{G}$, except the case when $\widetilde{G}$ is not split and has the type A_{2l} ([**St1**, Remark 8.8]). Condition $(*)$ holds, for instance, for regular conjugacy classes of G in the following cases (as shown in Section 4):

(a) $G = SL_n(K)$;
(b) $K = \overline{K}$ (where $\overline{K}$ is the algebraic closure of K).

In the cases (c) to (f) below, the field K is supposed to be perfect:

(c) $\widetilde{G}$ is split over K and $C = \widetilde{C} \cap G$ for a conjugacy class $\widetilde{C}$ of $\widetilde{G}$;
(d) $\dim K \leq 1$ and C is a semisimple class (here $\dim K$ is the homological dimension of K);
(e) $\widetilde{G}$ is split over K, $C \cap B \neq \emptyset$, and C is a semisimple class;
(f) C is a unipotent class, $\operatorname{char} K$ is not a bad prime for $\widetilde{G}$, and if $\widetilde{G}$ is not split, then $\widetilde{G}$ is not of type A_{2l}.

Theorem 1.1 implies:

Corollary 1.2. *Let $\widetilde{G}$ be a simple and simply-connected algebraic group that is defined and quasi-split over a field K and let $G = \widetilde{G}(K)$. Further, let $C \subset G$ be a regular conjugacy class of G. If one of Conditions (a) to (f) holds, then C intersects all Bruhat cells of the form $B\dot{w}\dot{w}_S\dot{w}^{-1}B$.*

Remark. The statement of the Corollary in Case (a) follows from the existence of a normal rational form. Case (b) follows from a much more general fact: Every regular conjugacy class of a simple algebraic group (i.e., $G = \widetilde{G}(\overline{K})$) intersects all Bruhat cells (see Appendix). Also, in Case (f), if K is a finite field, then a theorem of Kawanaka [**K**] shows that any regular unipotent conjugacy class intersects all Bruhat cells.

Now let $X \subset S$, $W_X = \langle X \rangle$. By w_X we denote a product (in any order) of elements of X, where each $x \in X$ occurs exactly once, i.e., w_X is a Coxeter element of W_X with respect to X. It is natural to consider intersections $B\dot{w}\dot{w}_X\dot{w}^{-1}B \cap C$ next. In [**GS**] it has been shown that $B\dot{w}_X B \cap C \neq \emptyset$ for some $X \subset S$ if C is a semisimple class and K is a finite field. Here we prove:

Theorem 1.3. *Let $\widetilde{G}$ be a simple and simply-connected algebraic group that is defined and quasi-split over a perfect field K such that $\dim K \leq 1$, and let $G = \widetilde{G}(K)$. Further, let $C \subset G$ be a noncentral semisimple conjugacy class of G. Then C intersects all Bruhat cells of the form $B\dot{w}\dot{w}_X\dot{w}^{-1}B$ for some $X \subset S$, $X \neq \emptyset$.*

Remark. This theorem generalizes Proposition 6 from [**GS**].

We thank the referee for drawing our attention to a result of Geck and Pfeiffer (see Proposition 3.3) which allows us to extend our results to all Chevalley groups.

2. S-Coxeter elements in Coxeter groups.

Let W be a finite group of orthogonal transformations of a Euclidean space V generated by reflections. Then W is a Coxeter group. Let $S = \{s_1, \dots, s_r\}$ be a Coxeter system of generators of W, i.e., $s_i^2 = 1$ for every $i = 1, \dots, r$ and $(s_i s_j)^{m_{ij}} = 1$ is the system of basic relations for the group W (see [**Bou**, IV, 1]). Then every element of the form $s_{\pi(1)} s_{\pi(2)} \dots s_{\pi(r)}$, where $\pi \in S_r$, is called a Coxeter element of W. All Coxeter elements of W constructed for all possible Coxeter systems of generators are conjugate in W (see [**Bou**, V, 6, Proposition 1]), and if $V^W = \{0\}$, each Coxeter element acts on $V \setminus \{0\}$ without fixed points ([**Bou**, V, 6, 2]).

Definition 2.1. Let $X \subset S$ and let W_X be the subgroup of W generated by X. Every element of W that is conjugate to a Coxeter element in W_X will be called a generalized Coxeter element of W.

Definition 2.2. For a fixed system S of generators the elements of the form $s_{\pi(1)} s_{\pi(2)} \dots s_{\pi(r)}$, where $|S| = r$, will be called S-Coxeter elements. If $X \subset S$, then X-Coxeter elements in W_X will be called generalized S-Coxeter elements of W.

Let $l_S(w)$ be the S-length of w, i.e., the length of w with respect to S. Obviously, a Coxeter element $w \in W$ is S-Coxeter if and only if $l_S(w) = r$. Below, we shall work with a fixed system S and we shall write $l(w)$ instead of $l_S(w)$. We shall use the well-known fact that $l_X(w) = l_S(w)$ for any $w \in W_X$.

Example 2.3. Let $W = S_4$ and $S = \{(12), (23), (34)\}$. Then we have six Coxeter elements (4-cycles) in W. Among them there are four S-Coxeter elements:

$$(12)(23)(34),\ (34)(23)(12),\ (23)(12)(34),\ (12)(34)(23),$$

and two elements that are not S-Coxeter elements:

$$(23)(12)(23)(34)(23),\ (23)(34)(23)(12)(23).$$

Lemma 2.4. *Let w_1, w_2 be two S-Coxeter elements of W. Then there exists a sequence $\sigma_1, \sigma_2, \dots, \sigma_n \in S$ (possibly $\sigma_i = \sigma_j$ for $i \neq j$) such that*

$$w_2 = \sigma_n \sigma_{n-1} \dots \sigma_1 w_1 \sigma_1 \sigma_2 \dots \sigma_n$$

and $l(\sigma_i \sigma_{i-1} \dots \sigma_1 w_1 \sigma_1 \sigma_2 \dots \sigma_{i-1} \sigma_i) = r$ for every $i = 1, \dots, n$.

Proof. See [**C2**, Section 10.3]. □

3. A condition for the intersection of a conjugacy class with Bruhat cells and Gauss cells.

We are going to use the concepts of S-ascent and S-descent and derive some of their properties. The notion of descent was introduced and considered in

[**GP**] (without the name "descent") as a binary relation between elements of conjugacy classes of Coxeter groups. The notion of ascent is dual to that of descent.

Definition 3.1. Let $w_1, w_2 \in W$. We say that there exists an S-ascent (resp. S-descent) from w_1 to w_2 if there is a sequence $\sigma_1, \ldots, \sigma_n \in S$ such that

$$w_2 = \sigma_n \sigma_{n-1} \ldots \sigma_1 w_1 \sigma_1 \sigma_2 \ldots \sigma_n$$

and

$$\begin{aligned} & l(\sigma_i \sigma_{i-1} \ldots \sigma_1 w_1 \sigma_1 \sigma_2 \ldots \sigma_i) \\ & \geq \text{ (resp. } \leq \text{) } l(\sigma_{i-1} \ldots \sigma_1 w_1 \sigma_1 \sigma_2 \ldots \sigma_{i-1}) \end{aligned}$$

for every $i = 1, \ldots, n$.

Remark. As before, we fix a set S of generators for W. In [**GP**] an S-descent from an element $w \in W$ to an element $w' \in W$ is denoted by $w \longrightarrow w'$. It is logical to denote an S-ascent from $w' \in W$ to $w \in W$ by $w \longleftarrow w'$.

Definition 3.2. Let $C \subset W$ be a conjugacy class. We define

$$l(C) = \min \{ l(w) \ \mid \ w \in C \}.$$

The following proposition is due to M. Geck and G. Pfeifer ([**GP**, Theorem 3.2.9.(a)]):

Proposition 3.3. *Let $C \subset W = W(R)$ be a conjugacy class. Then for every $w \in C$ there exists an S-descent to an element $w' \in C$ such that $l(w') = l(C)$.*

Let G be a Chevalley group (proper or twisted) corresponding to a root system R in the sense of [**St2**]. We fix a simple root system $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ and a corresponding Borel subgroup $B = HU$. Let $W = W(R)$ be the Weyl group of G and $S = \{w_{\alpha_1}, \ldots, w_{\alpha_r}\}$ the corresponding Coxeter system of generators. By X_α we denote below a root subgroup of G (see [**St2**]).

The meaning of Definition 3.1 becomes clear from the following:

Proposition 3.4. *Let $g \in B\dot{w}B$ (resp. $g \in B^- \dot{w} B$) and let $w' \in W$ be an element that is conjugate to w. If there exists an S-ascent (resp. S-descent) from w to w', then there exists an element $g' \in B\dot{w}'B$ (resp. $B^- \dot{w}' B$) that is conjugate to g.*

Proof. We shall use the following lemma:

Lemma 3.5. *Let $w \in W$. Suppose*

$$w(\alpha_i) \ < \ 0, \quad \text{and} \quad w^{-1}(\alpha_i) \ < \ 0$$

for some $\alpha_i \in \Pi$. Then either $w = w_{\alpha_i} w' w_{\alpha_i}$, where $l(w') = l(w) - 2$, or $w = w_{\alpha_i} w' = w' w_{\alpha_i}$, where $l(w') = l(w) - 1$.

Proof. The assumption $w^{-1}(\alpha_i) < 0$ implies

$$w = w_{\alpha_i} w_1,$$

where $l(w_1) = l(w) - 1$ ([**C2**, Section 2.2]). Suppose $w_1(\alpha_i) = \beta > 0$. Since $w(\alpha_i) = w_{\alpha_i}(\beta) < 0$, we have $\beta = \alpha_i$ and we have the second possibility. Now let $w_1(\alpha_i) < 0$. Then $w_1 = w' w_{\alpha_i}$ where $l(w') = l(w_1) - 1$ and we have the first possibility. □

First, let $g \in B\dot{w}B$, then $g = b_1 \dot{w} b_2$. We may assume $b_1 = 1$ and $b_2 = u \in U$. Also, it is sufficient to prove the assertion for an S-ascent of one step, i.e., $w' = w_\alpha w w_\alpha$ for some $\alpha \in \Pi$. We can write $u = u_\alpha v$, where u_α is a root subgroup element corresponding to α and where $v \in U$ is an element that has no α-factors in any decomposition into positive root subgroup elements.

If $u_\alpha = 1$, then $u' = \dot{w}_\alpha u \dot{w}_\alpha^{-1} \in U$ and

$$g' = \dot{w}_\alpha g \dot{w}_\alpha^{-1} = (\dot{w}_\alpha \dot{w} \dot{w}_\alpha^{-1})(\dot{w}_\alpha u \dot{w}_\alpha^{-1}) = \dot{w}' u' \in B\dot{w}'B.$$

Let $u_\alpha \neq 1$. Suppose $\beta = w(\alpha) > 0$. We may assume $\beta \neq \alpha$ (otherwise $w' = w_\alpha w w_\alpha^{-1} = w$). We have $g = \dot{w} u_\alpha \dot{w}^{-1} \dot{w} v = u_\beta \dot{w} v$. Now we can consider the element $u_\beta^{-1} g u_\beta$ instead of g which satisfies the previous condition $u_\alpha = 1$.

Suppose $\beta = w(\alpha) < 0$ and $\gamma = w^{-1}(\alpha) > 0$. We have $g = \dot{w} u_\alpha v = \dot{w} u_\alpha v u_\alpha^{-1} u_\alpha$. Note that $v' = u_\alpha v u_\alpha^{-1}$ has no factors corresponding to α. Consider now the element $\widetilde{g} = u_\alpha g u_\alpha^{-1}$ instead of g. We have $\widetilde{g} = u_\alpha \dot{w} v' = \dot{w} \dot{w}^{-1} u_\alpha \dot{w} v' = \dot{w} u_\gamma v'$, an element which also satisfies the condition $u_\alpha = 1$.

Now let $\beta = w(\alpha) < 0$, $\gamma = w^{-1}(\alpha) < 0$. Then, by Lemma 3.5, either $w_\alpha w w_\alpha = w$ and, therefore, there is nothing to prove, or $l(w_\alpha w w_\alpha) < l(w)$ which contradicts our assumption.

Second, let $g \in B^- \dot{w} B$. We may assume $g = v v_\alpha \dot{w} u_\alpha u$, where $v \in U^-$, $v_\alpha \in X_{-\alpha}$, $u_\alpha \in X_\alpha$, $u \in U$ and the elements v, u have no factors from the group $X_{\pm\alpha}$. Note, $\dot{w}_\alpha v \dot{w}_\alpha^{-1} \in U^-$, $\dot{w}_\alpha u \dot{w}_\alpha^{-1} \in U$ (because α is a simple root). Thus, if $v_\alpha = u_\alpha = 1$, then $\dot{w}_\alpha g \dot{w}_\alpha^{-1} \in B^- \dot{w}' B$. Now put $\beta = w(\alpha)$, $\gamma = w^{-1}(\alpha)$. If $\beta < 0$, $\gamma < 0$, we have $g = v v_\alpha \dot{w} u_\alpha \dot{w}^{-1} \dot{w} u = v v_\alpha v_\beta \dot{w} u = v v_\alpha v_\beta v_\alpha^{-1} v_\alpha \dot{w} u = v(v_\alpha v_\beta v_\alpha^{-1}) \dot{w} u_\gamma u$, where $v_\beta = \dot{w} u_\alpha \dot{w}^{-1} \in X_\beta$, $u_\gamma = \dot{w}^{-1} v_\alpha \dot{w} \in X_{-\gamma}$. We may assume $\beta, \gamma \neq -\alpha$ (otherwise we have $w_\alpha w w_\alpha = w$). Thus the elements $v(v_\alpha v_\beta v_\alpha^{-1})$, $u_\gamma u$ have no factors from $X_{\pm\alpha}$ and we are in the preceding case.

Let $\beta > 0$, $\gamma < 0$. Then $g = v v_\alpha u_\beta \dot{w} u = v_\alpha v' \dot{w} u$, where the element $v' \in U^-$ has no factor from $X_{-\alpha}$. Put $u_\alpha = \dot{w}_\alpha v_\alpha \dot{w}_\alpha^{-1}$. Then $\dot{w}_\alpha g \dot{w}_\alpha^{-1} = u_\alpha v'' \dot{w}' u'$ for some $v'' \in U^-$, $u' \in U$. Thus $u_\alpha^{-1} \dot{w}_\alpha g \dot{w}_\alpha^{-1} u_\alpha \in B^- \dot{w}' B$.

The case $\beta < 0$, $\gamma > 0$ is similar to the preceding one.

Let $\beta > 0$, $\gamma > 0$. Again, as above, we may assume $\beta, \gamma \neq \alpha$. Thus by Lemma 3.5, we have $l(w') = l(w_\alpha w w_\alpha) = l(w) + 2$ which contradicts our assumption. □

Example 3.6. Let $G = SL_3(K)$ and let

$$\dot{w} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \dot{w}' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let g be a semisimple element of G that has no eigenvalues in K. Then g is a regular element and therefore its conjugacy class C_g intersects the big Bruhat cell $B\dot{w}B$ (see [**EGH**, Lemma 4]). But $C_g \cap B\dot{w}'B = \emptyset$ because every element of the form $b_1\dot{w}'b_2$ is conjugate to an element of the form

$$\dot{w}'b = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & a_{22} & a_{23} \\ -a_{11} & -a_{12} & -a_{13} \\ 0 & 0 & a_{33} \end{pmatrix}$$

which has an eigenvalue $a_{33} \in K$. Note that here $S = \{w_{12}, w_{23}\}$ (where w_{ij} is the matrix in which the ith and jth elements of the standard basis are interchanged) and $l(w) = 3$, $l(w') = l(w_{12}) = 1$.

Example 3.7. Let $G = SL_4(K)$ and let

$$\dot{w} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \dot{w}' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Here $S = \{w_{12}, w_{23}, w_{34}\}$ and $\dot{w} = \dot{w}_{12}\dot{w}_{34}$ is a generalized S-Coxeter element. Thus every noncentral conjugacy class of G intersects $B^-\dot{w}B$ ([**EG**]). In particular, one can find a transvection $g \in B^-\dot{w}B$. But there are no transvections in $B^-\dot{w}'B$ because $B^-\dot{w}'B = \dot{w}'B\dot{w}'^{-1}\dot{w}'B = \dot{w}'B$, and every matrix $x \in \dot{w}'B$ satisfies the condition $\operatorname{rank}(x-1) \geq 2$.

The examples above show that if there is no S-ascent (resp. S-descent) from $w \in W$ to its conjugate $w' \in W$, the condition $C \cap B\dot{w}B \neq \emptyset$ (resp. $C \cap B^-\dot{w}B \neq \emptyset$) for a conjugacy class $C \in G$ does not necessarily imply $C \cap B\dot{w}'B \neq \emptyset$ (resp. $C \cap B^-\dot{w}'B \neq \emptyset$).

Proposition 3.8. *Let $g \in B\dot{w}B$ (resp. $g \in B^-\dot{w}B$). Suppose $l(w_\alpha w w_\alpha) = l(w) - 2$ (resp. $l(w_\alpha w w_\alpha) = l(w) + 2$). Then the conjugacy class C_g of g intersects either $B\dot{w}_\alpha\dot{w}\dot{w}_\alpha^{-1}B$ (resp. $B^-\dot{w}_\alpha\dot{w}\dot{w}_\alpha^{-1}B$) or $B\dot{w}_\alpha\dot{w}B$ and $B\dot{w}\dot{w}_\alpha B$ (resp. $B^-\dot{w}_\alpha\dot{w}B$ and $B^-\dot{w}\dot{w}_\alpha B$).*

Proof. Let $g \in B\dot{w}B$. We may assume, as in the proof of Proposition 3.4, that $g = \dot{w}u_\alpha u$ and $w = w_\alpha w_1 w_\alpha$, where $l(w_1) = l(w) - 2$. Moreover, $\beta = w_1(\alpha) > 0$, $\gamma = w_1^{-1}(\alpha) > 0$, and $w_1(\alpha), w_1^{-1}(\alpha) \neq \alpha$. If $u_\alpha = 1$, then $\dot{w}_\alpha g\dot{w}_\alpha^{-1} \in B\dot{w}_1B$. Suppose $u_\alpha \neq 1$. Put $u_{-\alpha} = \dot{w}_\alpha u_\alpha \dot{w}_\alpha^{-1}$. There exists $u'_\alpha \in X_\alpha$ (here X_α is the corresponding root subgroup) such that $u'_\alpha u_{-\alpha} = \dot{w}_\alpha u''_\alpha$ for some $u''_\alpha \in X_\alpha$. Further, $g_1 = \dot{w}_\alpha g\dot{w}_\alpha^{-1} = \dot{w}_1 u_{-\alpha} u'$ for some $u' \in U$. Put $u_\beta = \dot{w}_1 u'_\alpha \dot{w}_1^{-1}$ (recall $\beta = w_1(\alpha) > 0$). Then

$g_2 = u_\beta g_1 u_\beta^{-1} = \dot w_1 u'_\alpha \dot w_1^{-1} \dot w_1 u_{-\alpha} u' u_\beta^{-1} = \dot w_1 \dot w_\alpha u''_\alpha u' u_\beta^{-1} \in B\dot w_1 \dot w_\alpha B$. Since $l(w_1 w_\alpha) = l(w_\alpha w_1)$, we also can find an element in $C_g \cap B\dot w_\alpha \dot w_1 B$ (by Proposition 3.4).

Now let $g \in B^- \dot w B$. As in the proof of Proposition 3.4 we may assume $g = v v_\alpha \dot w u_\alpha u$, $\alpha \neq w(\alpha) > 0$, $\alpha \neq w^{-1}(\alpha) > 0$. If $v_\alpha = u_\alpha = 1$, then $\dot w_\alpha g \dot w_\alpha^{-1} \in B^- \dot w_\alpha \dot w \dot w_\alpha^{-1} B$. Let $v_\alpha = 1$, $u_\alpha \neq 1$. Then

$$\begin{aligned} g_1 = \dot w_\alpha g \dot w_\alpha^{-1} &= (\dot w_\alpha v \dot w_\alpha^{-1})(\dot w_\alpha \dot w \dot w_\alpha^{-1})(\dot w_\alpha u_\alpha \dot w_\alpha^{-1})(\dot w_\alpha u \dot w_\alpha^{-1}) \\ &= v' \dot w_\alpha \dot w \dot w_\alpha^{-1} u_{-\alpha} u', \end{aligned}$$

where $v' \in U^-$, $u' \in U$, $u_{-\alpha} \in X_{-\alpha}$. Moreover, the element u' has no factors in X_α. Further, $u_{-\alpha} g_1 u_{-\alpha}^{-1} = u_{-\alpha} v' \dot w_\alpha \dot w \dot w_\alpha^{-1} u_{-\alpha} u' u_{-\alpha}^{-1}$. Since $u_{-\alpha} u' u_{-\alpha}^{-1} \in U$, we have $u_{-\alpha} g_1 u_{-\alpha}^{-1} \in B^- \dot w_\alpha \dot w \dot w_\alpha^{-1} B$. Similar considerations work in the case $v_\alpha \neq 1$, $u_\alpha = 1$.

Let $v_\alpha \neq 1$, $u_\alpha \neq 1$. Put $u'_\alpha = \dot w_\alpha v_\alpha \dot w_\alpha^{-1}$, $v'_\alpha = \dot w_\alpha u_\alpha \dot w_\alpha^{-1}$, $v' = \dot w_\alpha v \dot w_\alpha^{-1}$, $u' = \dot w_\alpha u \dot w_\alpha^{-1}$. Then

$$g_1 = \dot w_\alpha g \dot w_\alpha^{-1} = v' u'_\alpha \dot w_\alpha \dot w \dot w_\alpha^{-1} v'_\alpha u' = v' u'_\alpha \dot w_\alpha \dot w \dot w_\alpha^{-1} (v'_\alpha u' v_{\alpha'}^{-1}) v'_\alpha.$$

Put $u'' = v'_\alpha u' v_{\alpha'}^{-1}$, $v'' = v'_\alpha v'$. Then $g_2 = v'_\alpha g_1 v_{\alpha'}^{-1} = v'' u'_\alpha \dot w_\alpha \dot w \dot w_\alpha^{-1} u''$. Further, $u'_\alpha \dot w_\alpha = x_{-\alpha} x_\alpha$ for some $x_{-\alpha} \in X_{-\alpha}$, $x_\alpha \in X_\alpha$. Hence

$$g_2 = v'' x_{-\alpha} x_\alpha \dot w \dot w_\alpha^{-1} u'' = v'' x_{-\alpha} \dot w \dot w_\alpha^{-1} (\dot w_\alpha \dot w^{-1} x_\alpha \dot w \dot w_\alpha^{-1}) u''.$$

Since $w^{-1}(\alpha) > 0$ and $w^{-1}(\alpha) \neq \alpha$, we get $\dot w_\alpha \dot w^{-1} x_\alpha \dot w \dot w_\alpha^{-1} \in U$ and therefore $g_2 \in B^- \dot w \dot w_\alpha B$. From Proposition 3.4 we get $C_g \cap B^- \dot w_\alpha \dot w B \neq \emptyset$. □

4. Proofs of the Theorems.

Here $\widetilde G$ is a simple algebraic group defined and quasi-split over a field K, $\widetilde B = \widetilde T \widetilde U$ is a Borel subgroup defined over K, $\widetilde N = N_{\widetilde G}(\widetilde T), \widetilde W = \widetilde N/\widetilde T$ and $G = \widetilde G(K), B = \widetilde B(K), T = \widetilde T(K), U = \widetilde U(K), N = \widetilde N(K), W = N/T$. Further, let $\widetilde \Pi = \{\gamma_1, \dots, \gamma_s\}$ be a simple root system of $\widetilde G$ and $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be a simple root system (in the sense of [**C2**]) for G (which is obtained from $\widetilde \Pi$ by gluing of some roots).

Proof of Theorem 1.1. Assume that Condition $(*)$ of Theorem 1.1 holds. Further let $C_c \subset W$ be the conjugacy class of Coxeter elements and let $\omega \in C_c$. By Proposition 3.3 there exists an S-descent from ω to an element $\omega' \in C_c$ such that $l(\omega') = l(C_c) = r$. Since among the factors of ω' there are all reflections $w_{\alpha_i}, \alpha_i \in \Pi$, the element ω' is an S-Coxeter element. By Lemma 2.4 we have an S-ascent from w_S (recall that w_S is the Coxeter element from Condition $(*)$) to ω' and, therefore, we have an S-ascent from w_S to ω. Now our statement follows from Proposition 3.4. □

Condition $(*)$. Below, K is a perfect field.

We need the following simple (and known) facts:

Lemma 4.1. *Let $\widetilde{C}$ be a conjugacy class of $\widetilde{G}$ such that $C = \widetilde{C} \cap G \neq \emptyset$. Further, let $g \in C$. If $H^1(K, C_{\widetilde{G}}(g)) = 1$ then C is a conjugacy class of G (here $C_{\widetilde{G}}(g)$ is the centralizer of g in $\widetilde{G}$).*

Proof. The argument here is the same as in ([**C2**, Proposition 3.7.3]). Indeed, if $g' \in C$, then there exists an element $\gamma \in \widetilde{G}$ such that $g' = \gamma g \gamma^{-1}$. Thus, for every element $\sigma \in \mathrm{Gal}(\overline{K}/K)$ of the Galois group we have

$$\sigma(\gamma) g \sigma(\gamma^{-1}) = \gamma g \gamma^{-1}$$

and therefore $x_\sigma = \gamma^{-1}\sigma(\gamma) \in C_{\widetilde{G}}(g)$. Since x_σ is a 1-cocycle, we have $x_\sigma = y\sigma(y^{-1})$ for some $y \in C_{\widetilde{G}}$ and therefore $\sigma(\gamma y) = \gamma y$ for every $\sigma \in \mathrm{Gal}(\overline{K}/K)$. Thus, $\gamma y \in G$ and $g' = \gamma y g y^{-1} \gamma^{-1}$. □

Lemma 4.2. *Let $\widetilde{C}$ be a semisimple conjugacy class of $\widetilde{G}$ and let $C = \widetilde{C} \cap G \neq \emptyset$. If $\dim K \leq 1$, then C is a conjugacy class of G.*

Proof. Since $\widetilde{G}$ is simply-connected, $C_{\widetilde{G}}(s)$ is a connected reductive group for $s \in \widetilde{C} \cap G$ ([**C2**, Theorem 3.5.6]) and therefore $H^1(K, C_{\widetilde{G}}(s)) = 1$ ([**St1**, 11.2]). Now the assertion follows from Lemma 4.1. □

Lemma 4.3. *Let C be the same as in the preceding lemma. Suppose that $\widetilde{G}$ is split and $\widetilde{C}$ is a regular semisimple class such that $\widetilde{C} \cap T \neq \emptyset$. Then C is a conjugacy class of G.*

Proof. If $s \in \widetilde{C} \cap T$, then $C_{\widetilde{G}}(s) = \widetilde{T}$ is a K-split torus and therefore $H^1(K, C_{\widetilde{G}}(s)) = 1$ ([**Sp**, 12.3.5.(3)]). Now the assertion follows from Lemma 4.1. □

Lemma 4.4. *Let $u_1, u_2 \in G$ be two regular unipotent elements of $\widetilde{G}$. Assume that* $\mathrm{char}\, K$ *is not a bad prime for $\widetilde{G}$. Then there exist elements $t \in \widetilde{T}$ and $\gamma \in G$ such that $u_1 = t\gamma u_2 \gamma^{-1} t^{-1}$.*

Proof. Let $\overline{G} = \widetilde{G}/Z(\widetilde{G})$, $\overline{T} = \widetilde{T}/Z(\widetilde{G})$. Then $\overline{G}$ is defined and quasi-split over K and $Z(\overline{G}) = 1$. Further, let $u \in \widetilde{G}$ be a regular unipotent element and let $\overline{u}$ be its image in $\overline{G}$. The $\mathrm{char}\, K$ is not a bad prime for $\overline{G}$, thus $V = C_{\overline{G}}(\overline{u})$ is a connected unipotent subgroup of $\overline{G}$ ([**C2**, Proposition 5.1.6]) which is defined and split over K ([**Sp**, 14.3.8]) and therefore $H^1(K, V) = 1$ ([**Sp**, 12.3.5.(3)]). Hence any two regular unipotent elements of $\overline{G}(K)$ are conjugate (Lemma 4.1). If $\overline{G}_1(K) \leq \overline{G}(K)$ is a subgroup generated by unipotent elements of $\overline{G}$, then it is a normal subgroup and $\overline{G}(K) = \overline{G}_1(K)\overline{T}(K)$ (this follows from the Bruhat decomposition). Now let $\bar{u}_1, \bar{u}_2 \in \overline{G}(K)$ be images of regular unipotent elements $u_1, u_2 \in G$. Then there exist elements $\bar{\gamma} \in \overline{G}_1(K)$, $\bar{t} \in \overline{T}(K)$ such that $\bar{u}_1 = \bar{t}\bar{\gamma}\bar{u}_2\bar{\gamma}^{-1}\bar{t}^{-1}$. If $\gamma \in \widetilde{G}(K) = G$, $t \in \widetilde{T}$ are preimages of $\bar{\gamma}, \bar{t}$, then $u_1 \equiv t\gamma u_2 \gamma^{-1} t^{-1} (\mathrm{mod}\ Z(\widetilde{G}))$. Since u_1, u_2 are both unipotent elements, we have $u_1 = t\gamma u_2 \gamma^{-1} t^{-1}$. □

Now we check Condition $(*)$ for (a) to (f):

(a) If $G = SL_n(K)$, Condition $(*)$ is an immediate consequence of the representation of elements of $GL_n(K)$ in rational canonical form.

(b) Consider the case where K is an algebraically closed field. According to Steinberg's theorem ([**St1**, 1.4]), the set

$$\mathfrak{N} = \dot{w}_{\gamma_1} X_{\gamma_1} \dot{w}_{\gamma_2} X_{\gamma_2} \dots \dot{w}_{\gamma_s} X_{\gamma_s}$$

is a cross-section of all regular conjugacy classes of the group $\widetilde{G}$, where $\dot{w}_{\gamma_1}, \dots, \dot{w}_{\gamma_s}$ is any fixed system of preimages of the basic reflections $w_{\gamma_1}, \dots,$ w_{γ_s} in any fixed order (here X_{γ_i} is the corresponding root subgroup). Moreover, we can rewrite $\mathfrak{N}$ in the form

$$\mathfrak{N} = \dot{w}_{\gamma_1} \dot{w}_{\gamma_2} \dots \dot{w}_{\gamma_s} X_{\theta_1} X_{\theta_2} \dots X_{\theta_s},$$

where $\theta_i = w_{\gamma_s} \dots w_{\gamma_{i+2}} w_{\gamma_{i+1}}(\gamma_i) > 0$. Since K is an algebraically closed field, $\{\alpha_1, \dots, \alpha_r\} = \{\gamma_1, \dots, \gamma_s\}$ and any element in the intersection $C \cap \widetilde{N}(K)$ lies in the S-Coxeter cell $B\dot{w}_{\alpha_1}\dot{w}_{\alpha_2} \dots \dot{w}_{\alpha_r}B$. This proves $(*)$.

(c) If $\widetilde{G}$ is split over K, the closed subset $\mathfrak{N}$ (defined above) of $\widetilde{G}$ is defined over K and $\mathfrak{N} \cap \widetilde{C} \in G$ ([**St1**, Section 9]).

(d) There exists a closed subset $\mathfrak{N}'$ of $\mathfrak{N}$ which is defined over K and such that every regular semisimple conjugacy class $\widetilde{C}$ of $\widetilde{G}$ intersects $\mathfrak{N}'$ in just one point (and this point belongs to G if $\widetilde{C} \cap G \neq \emptyset$) ([**St1**, 9.11]). Since $\mathfrak{N} \subset B\dot{w}_S B$ for some S-Coxeter element w_S, the assertion follows from Lemma 4.2.

(e) We may use the same argument as in (d), and Lemma 4.3.

(f) If $\widetilde{G}$ is split or $\widetilde{G}$ is not of type A_{2l}, the cross-section of regular classes $\mathfrak{N}$ is defined over K and for the conjugacy class of regular unipotent elements $\widetilde{C}$ we have $u = \widetilde{C} \cap \mathfrak{N} \in B\dot{w}_S B$, where $\dot{w}_S \in N$ for some S-Coxeter element w_S in W ([**St1**, Section 9]). Now let $u' \in \widetilde{C} \cap G$. By Lemma 4.4 we have $t\gamma u' \gamma^{-1} t^{-1} = u = u_1 \dot{w}_S b_1$ for some $t \in \widetilde{T}, \gamma \in G$ and $u_1 \in U, b_1 \in B$. Hence $u'' = \gamma u' \gamma^{-1} = (t^{-1}u_1 t)(t^{-1}\dot{w}_S t)(t^{-1} b_1 t)$. Thus $u'' \in \widetilde{B}\dot{w}_S\widetilde{B}$. But $u'' \in G$ and, therefore, $u'' \in B\dot{w}B$ for some $\dot{w} \in N$. Since $B\dot{w}B \subset \widetilde{B}\dot{w}\widetilde{B}$, we have $w = w_S$. This implies that the conjugacy class C of u' in G has a nontrivial intersection with $B\dot{w}_S B$, where $\dot{w}_S \in N$.

Proof of Theorem 1.3.

Below, $\widetilde{\Gamma}$ is a connected reductive algebraic group defined over a perfect field K such that $\dim K \leq 1$.

Lemma 4.5. *Let $\widetilde{P} = \widetilde{L}R_u(\widetilde{P})$ be a parabolic subgroup of $\widetilde{\Gamma}$ defined over K. Let $\widetilde{L}$ be a fixed Levi factor* (*defined over K*) *and let $R_u(\widetilde{P})$ be the unipotent radical of $\widetilde{P}$. Further, let $s \in \widetilde{P}(K)$, $s = lu$, where $l \in \widetilde{L}$ and $u \in R_u(\widetilde{P})$.*

If $s \in \widetilde{\Gamma}(K)$, then $l \in \widetilde{L}(K)$ and $u \in R_u(\widetilde{P})(K)$. If, in addition, s is a semisimple element, then s is conjugate to l in $\widetilde{P}$.

Proof. The first assertion follows from the uniqueness of the decomposition lu.

Further, if s is semisimple, it is contained in a maximal torus in $\widetilde{P}$ which is contained in a Levi subgroup L'. ([**Sp**, 8.4.4]). Since all Levi subgroups are conjugate in $\widetilde{P}$ ([**Sp**, 16.1.1]) by elements of $\widetilde{P}$, one can find an element $p = l_1u_1 \in \widetilde{P}$ where $l_1 \in \widetilde{L}$, $u_1 \in R_u(\widetilde{P})$ such that $psp^{-1} \in \widetilde{L}$. Then $l_1^{-1}psp^{-1}l_1 = u_1su_1^{-1} = l(l^{-1}u_1l)uu_1^{-1} \in \widetilde{L}$. Hence $(l^{-1}u_1l)uu_1^{-1} = 1$ (because $(l^{-1}u_1l)uu_1^{-1} \in R_u(\widetilde{P})$) and therefore $l_1^{-1}psp^{-1}l_1 = l$. □

Lemma 4.6. *Let $s \in \widetilde{\Gamma}(K)$ be a semisimple element of $\widetilde{\Gamma}$ such that $C_{\widetilde{\Gamma}}(s)^0$ is not a torus. Then there exists a parabolic subgroup $\widetilde{P}$ of $\widetilde{\Gamma}$ defined over K such that $s \in \widetilde{P}$.*

Proof. The group $C_{\widetilde{\Gamma}}(s)^0$ is defined over K ([**Sp**, 12.1.4]). Further, the condition $\dim K \leq 1$ implies that there exists a Borel subgroup $\widetilde{B}_s$ of $C_{\widetilde{G}}(s)^0$ which is also defined over K ([**St1**, 10.2]). Since $C_{\widetilde{\Gamma}}(s)^0$ is not a torus, the unipotent radical $R_u(\widetilde{B}_s)$ is not trivial. The group $\widetilde{U}_1 = R_u(\widetilde{B}_s)$ is also defined over K ([**Sp**, 14.4.5(v)]). Further, let

$$\begin{aligned}(1)\qquad &\widetilde{N}_1 = N_{\widetilde{G}}(\widetilde{U}_1),\ \widetilde{U}_2 = \widetilde{U}_1 \cdot R_u(\widetilde{N}_1),\ \widetilde{N}_2 = N_{\widetilde{G}}(\widetilde{U}_2), \dots,\\ &\widetilde{U}_i = \widetilde{U}_{i-1} \cdot R_u(\widetilde{N}_{i-1}),\ \widetilde{N}_i = N_{\widetilde{G}}(\widetilde{U}_i), \dots .\end{aligned}$$

Then all members of (1) are closed subgroups of $\widetilde{\Gamma}$ and $\widetilde{U}_k = \widetilde{U}_{k+1}$, $\widetilde{N}_k = \widetilde{N}_{k+1}$ for some positive integer k ([**Hu**, 30.3]). Further, all groups in (1) are defined over K; indeed, the field K is perfect and all groups are defined as normalizers of K-defined groups, their unipotent radicals, and the images of K-defined groups with respect to maps $\widetilde{U}_{i-1} \times R_u(\widetilde{N}_{i-1}) \longrightarrow \widetilde{U}_{i-1} \cdot R_u(\widetilde{N}_{i-1})$, induced by multiplication in $\widetilde{G}$. Since $\widetilde{U}_1$ is connected, the last member $\widetilde{N}_k$ of this sequence is a parabolic subgroup of $\widetilde{\Gamma}$ ([**Hu**, 30.3]). From the definitions we have $s \in \widetilde{N}_1 \leq \widetilde{N}_k$. □

Now we can prove Theorem 1.3. Let $s \in G$ be a noncentral semisimple element. We may assume that s is not a regular element of $\widetilde{G}$ (otherwise the statement follows from Theorem 1.1 and Property (d)). By Lemma 4.6 we have $s \in \widetilde{P}$ for some parabolic subgroup defined over K. Since $g\widetilde{P}g^{-1} = \widetilde{P}_I$ for some standard parabolic subgroup $\widetilde{P}_I$ and $g \in G$ ([**Sp**, 15.4.6]), we may assume $s \in \widetilde{P}_I$, where $I \subset \widetilde{\Pi}$ is a $\operatorname{Gal}(\overline{K}/K)$-invariant subset (note that the group $\operatorname{Gal}(\overline{K}/K)$ acts on $\widetilde{\Pi}$ by permutation and the orbits of this action correspond to Π; see [**St1**, Section 9]). Let $\widetilde{L}_I = \widetilde{T}\widetilde{G}_I$, where $\widetilde{G_I} = \langle X_\alpha \mid \alpha \in \langle I\rangle\rangle$. Then $\widetilde{L}_I$ is a K-defined Levi factor of $\widetilde{P}_I$.

By Lemmas 4.5 and 4.2 we may assume $s \in \widetilde{L}_I$. (Indeed, by Lemma 4.5 we have en element $l \in \widetilde{L}_I(K)$ which is conjugate to s in $\widetilde{P}_I$. By Lemma 4.2 the elements s, l are conjugate by an element of the group G. Hence we may take the element $l \in C$ instead of s.)

Again by Lemma 4.6 we may assume that $C_{\widetilde{L}_I}(s)^0 = \widetilde{T}'$, where $\widetilde{T}'$ is a maximal torus of $\widetilde{L}_I$ defined over K (otherwise, we can take a smaller set I using the same procedure as above). Note that the derived subgroup $\widetilde{L}_I$ is equal to $\widetilde{G}_I$ and therefore is a simply-connected semisimple group (because $\widetilde{G}$ is simply-connected). Hence $C_{\widetilde{L}_I}(s)^0 = C_{\widetilde{L}_I}(s)$ ([**C2**, Theorem 3.5.6]) and thus

$$C_{\widetilde{L}_I}(s) = \widetilde{T}'. \tag{2}$$

Further, if $I = \emptyset$ we have $\widetilde{P}_I = \widetilde{B}$ and $\widetilde{T}' = \widetilde{T}$. Hence $s \in \widetilde{T}(K) = T$. Since s is a noncentral element of G, there exists a root $\alpha \in \Pi$ such that s is not in the center of the group $T\widetilde{G}_\alpha(K)$ (here, $\widetilde{G}_\alpha = \langle X_\beta \mid \beta \in \langle I_\alpha\rangle\rangle$ where $I_\alpha \subset \widetilde{\Pi}$ is the $\mathrm{Gal}(\overline{K}/K)$-orbit of α). Since the Borel subgroup B_α of $T\widetilde{G}_\alpha(K)$ (with respect to T) is not a normal subgroup, one can find an element $\gamma \in T\widetilde{G}_\alpha(K)$ such that $\gamma s\gamma^{-1} = \dot{w}_\alpha b$, where $w_\alpha \in W$ is the corresponding reflection and $b \in B_\alpha$. Hence $C \cap B\dot{w}_\alpha B \neq \emptyset$. Further, let $\omega \in W$. Then $\omega w_\alpha \omega^{-1} = w_\beta$, where $\beta = \omega(\alpha)$. Let $\dot{\omega}, \dot{w}_\beta$ be preimages of ω, w_β in the group N. Then $\dot{\omega}T\widetilde{G}_\alpha(K)\dot{\omega}^{-1} = T\widetilde{G}_\beta(K)$. The element $s' = \dot{\omega}s\dot{\omega}^{-1}$ is not a central element in $T\widetilde{G}_\beta(K)$. Now, as above, we have $\gamma' s'\gamma'^{-1} \in B\dot{w}_\beta B$ for some $\gamma' \in T\widetilde{G}_\beta(K)$. Thus, if $I = \emptyset$, the assertion of the theorem holds for $X = \{\alpha\}$.

Now we may assume that $I \neq \emptyset$ and Condition (2) holds.

We have $s = tg$, $t \in \widetilde{T} \cap C_{\widetilde{L}_I}(\widetilde{G}_I)$, and $g \in \widetilde{G}_I$ ([**Hu**, 27.5]). Note that the elements t and g do not necessarily belong to G but $t, g \in \widetilde{L}_I(K')$ for some extension K'/K. The element $s \in G$ is $\mathrm{Gal}(\overline{K}/K)$-invariant and $t \in Z(\widetilde{L}_I)$. Hence $g = h_1 g_1$, where $h_1 \in \widetilde{T}(K')$, $g_1 \in \widetilde{G}_I(K)$ (this follows from the Bruhat decomposition of g). Further, (2) implies that g is a regular element of $\widetilde{G}_I$. If $\mathfrak{N}'$ is a cross-section (defined over K) of regular semisimple conjugacy classes of $\widetilde{G}_I$ ([**St1**, Section 9]) then $h_1\mathfrak{N}'$ is also a cross-section (defined over K') of regular semisimple conjugacy classes of $\widetilde{G}_I$. Hence the conjugacy class C_g of g in $\widetilde{G}_I$ intersects $h_1\mathfrak{N}'$ in just one point. Thus the conjugacy class $C_s = tC_g$ of s in $\widetilde{L}_I$ intersects $th_1\mathfrak{N}'$ also in one point x (recall, $t \in Z(\widetilde{L}_I)$). Since the conjugacy class C_s is defined over K and the closed subset $th_1\mathfrak{N}'$ is also defined over K (because $th_1 = sg_1^{-1} \in \widetilde{L}_I(K)$), the point x is $\mathrm{Gal}\,(\overline{K}/K)$-invariant and therefore it belongs to $L_I(K)$. Since $s, x \in L_I(K) \leq G$ are conjugate in L_I (and therefore in $\widetilde{G}$), we have $x = \sigma s \sigma^{-1}$ for some $\sigma \in G$ (Lemma 4.2). Further,

$$th_1\mathfrak{N}' \subset \left(\prod_{\alpha\in X} \dot{w}_\alpha\right)\widetilde{U}, \tag{3}$$

where $X \subset \Pi$ is the set of $\mathrm{Gal}\,(\overline{K}/K)$-orbits of $I \subset \widetilde{\Pi}$ and w_α in (3) is the product of basic reflections w_γ, where γ runs through the orbit corresponding to α or $w_\alpha = w_{\gamma_1+\gamma_2}$ if such orbit consists of two roots γ_1, γ_2 such that $\gamma_1+\gamma_2$ is a root (see [**St1**, Section 9]). From (3) we obtain

$$x = \sigma s \sigma^{-1} \in \widetilde{B} \prod_{\alpha\in X} \dot{w}_\alpha \widetilde{B}. \tag{4}$$

Since $x \in G$, we have

$$x = \sigma s \sigma^{-1} \in B\dot{w}B \tag{5}$$

for some $w \in W$. But

$$B\dot{w}B \subset \widetilde{B}\dot{w}\widetilde{B}. \tag{6}$$

From (4), (5), (6) we get

$$w = \prod_{\alpha\in X} w_\alpha, \tag{7}$$

i.e., w is a generalized S-Coxeter element of W. Now (5) and (7) imply that the conjugacy class of s in G intersects $B\dot{w}B$ for some generalized S-Coxeter element w of W.

Suppose that $w' = \omega w \omega^{-1}$ is also an S-Coxeter element of W for some $\omega \in W$. Then $w' = \prod_{\alpha\in Y} w_\alpha$ for some $Y \subset \Pi$, $|Y| = |X|$. Let $X' = \{\omega(\alpha) \mid \alpha \in X\}$. Then

$$w' = \prod_{\alpha\in Y} w_\alpha = \prod_{\beta\in X'} w_\beta.$$

The element w' is a Coxeter element of the root systems generated by Y and X'. It acts without fixed points on the vector space (over $\mathbb{R}$) generated by Y and on the vector space generated by X'. Moreover, $l(w') = |Y| = |X'|$. Hence the vector spaces (over $\mathbb{R}$) generated by Y and X' coincide (it is the $\langle w'\rangle$-complement to the vector space of w'-invariant vectors). Since X is a simple root system for the root system $\langle X\rangle$, the set X' is a simple root system for $\langle X'\rangle$. On the other hand, the set Y is a simple root system for the root system $\langle Y\rangle$. Now $X' \subset \omega(\Pi)$, $Y \subset \Pi$ and the linear spaces generated by X' and Y coincide. Moreover, the root subsystems $\langle X'\rangle, \langle Y\rangle$ have the same Coxeter element w'. Hence $\langle X'\rangle = \langle Y\rangle$. Now let I' be a subset of $\widetilde{\Pi}$

that is $\mathrm{Gal}\,(\overline{K}/K)$-invariant and such that the set of $\mathrm{Gal}\,(\overline{K}/K)$-orbits of I' coincides with Y. Since $\omega(\langle X\rangle) = \langle X'\rangle = \langle Y\rangle$, we have

$$\widetilde{G}_{I'} = \langle X_\beta \mid \beta \in \langle I'\rangle\rangle = \dot{\omega}\widetilde{G}_I\dot{\omega}^{-1}. \tag{8}$$

From (8) we get

$$\widetilde{L}_{I'} = \widetilde{T}\widetilde{G}_{I'} = \dot{\omega}\widetilde{L}_I\dot{\omega}^{-1}. \tag{9}$$

Since $\omega \in W$, we can choose the preimage $\dot{\omega} \in G$. From (9)

$$s' = \dot{\omega}s\dot{\omega}^{-1} \in \widetilde{L}_{I'} \cap G.$$

Now we have a semisimple regular element $s' \in \widetilde{L}_{I'}(K)$. The same arguments as above show that there exists an element $\tau \in G$ such that $s'' = \tau s'\tau^{-1} \in B\dot{w}''B$, where

$$w'' = \prod_{\beta\in Y} w_\beta$$

(the order of the roots β in this product can be different from the order of the roots α in the product corresponding to w'). By Lemma 2.4 there exists an S-ascent from w'' to w' (both elements are Y-Coxeter elements for the Weyl group of the system $\langle Y\rangle$). Proposition 3.4 implies

$$\delta s''\delta^{-1} \in B\dot{w}'B \tag{10}$$

for some $\delta \in G$.

The inclusions (5) and (10) show that the conjugacy class C of s in G intersects all Bruhat cells $B\dot{w}'''B$, where w''' runs through all generalized S-Coxeter elements that are conjugate to w. Now let $\widetilde{w} \in W$ be an element from the conjugacy class of w. Proposition 3.3 implies that there exists an S-ascent from some generalized S-Coxeter element w''' to $\widetilde{w}$. Now the assertion of the theorem follows from Proposition 3.4. □

Theorem 1.3 has been proved.

Remarks to Theorem 1.3.

1. Intersection with a parabolic subgroup. In the proof of Theorem 1.3 we showed that

$$C \cap P_X \neq \emptyset \tag{$**$}$$

for every noncentral semisimple conjugacy class C that is not regular, where $X \subsetneqq \Pi$ and $P_X = BW_XB$ is the corresponding parabolic subgroup (if K is a perfect field and $\dim K \le 1$). More generally, Equation $(**)$ holds for every noncentral conjugacy class C that is not a regular semisimple class (if K is a perfect field and $\dim K \le 1$). Indeed, we consider the Jordan decomposition $g = su$ of an element $g \in C$. Applying the same construction as in Lemma 4.6, we get a parabolic subgroup P which is defined over K and contains s, u. Then by an appropriate conjugation we can embed g in a standard parabolic subgroup. (Note, if K is a finite field, then Condition $(**)$ is a consequence of the properties of the Steinberg representation [**C2**, Proposition 6.4.5].)

2. The condition: $\dim \mathbf{K} \leq 1$. The example below shows that if this condition does not hold, the conclusion of Theorem 1.3 may be false.

Let $n = 4k$ and let V be a linear space over the real number field $\mathbb{R}$ such that $\dim V = 4k$. Further, let $\{e_1, \ldots, e_{4k}\}$ be a fixed basis of V and let $V^+ = \langle e_1, \ldots, e_{2k} \rangle$, $V^- = \langle e_{2k+1}, \ldots, e_{4k} \rangle$. Further, let $(x_1, \ldots, x_{4k})$ be the coordinates of an element in V with respect to the basis $\{e_i\}$ and let $\Phi = x_1^2 + \cdots + x_{2k}^2 - x_{2k+1}^2 - \cdots - x_{4k}^2$. Let $\Omega = \Omega(V, \Phi) = [SO(V, \Phi), SO(V, \Phi)]$. Then Ω is a Chevalley group in the sense of [**St2**], corresponding to the root system D_{2k}. Let $g \in GL(V)$ be the linear operator such that $g|_{V^+} = -1, g|_{V^-} = 1$. One can easily check that $g \in \Omega$ and $gug^{-1} \neq u^{\pm 1}$ for every nontrivial unipotent element $u \in \Omega$ (the latter follows from the fact that $v \pm g(v)$ is not an isotropic vector if $v \neq 0$ is isotropic). Hence the element g cannot normalize any nontrivial unipotent subgroup of Ω and therefore g cannot belong to any proper parabolic subgroup of Ω. This implies that a preimage $\hat{g}$ of g in $G = \mathrm{Spin}_{4k}(\mathbb{R})$ (with respect to the natural homomorphism $G \longrightarrow \Omega$) also cannot belong to a proper parabolic subgroup of G. Hence $C \cap Bw_X B = \emptyset$ for every $X \subset \Pi$, where C is the conjugacy class of $\hat{g}$ in G, B is a Borel subgroup of G, and Π is a simple root system corresponding to $\widetilde{G} = \mathbf{Spin}_{4k}$ (note, $BW_X B = P_X$ is a standard parabolic subgroup).

3. The ordered set of $\mathfrak{X}_\mathbf{C}$. Recall, for any set $X \subset \Pi$ we define $w_X = \prod_{\alpha \in X} w_\alpha$, where the product can be taken in any fixed order. For the set

$$\mathfrak{X}_C = \{X \subset \Pi \mid C \cap B\dot{w}_X B \neq \emptyset\}$$

one can consider the natural order with respect to inclusion.

Let $G = SL_n(\mathbb{C})$ and C a noncentral semisimple conjugacy class. Let $\lambda(C) = (\lambda_1, \ldots, \lambda_r)$ be the partition of n, i.e., $\lambda_1 \geq \cdots \geq \lambda_r$, where $\lambda_1 + \cdots + \lambda_r = n$, which corresponds to the multiplicities of eigenvalues of elements of C (i.e., λ_1 is the biggest multiplicity, then λ_2, etc.) and let $\lambda^*(C)$ be the dual partition (i.e., the rows and columns of λ are interchanged). Further, to every partition $\mu = (\mu_1, \ldots, \mu_s)$ of n we assign a subset $X(\mu) \subset \Pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$, namely,

$$X(\mu) \stackrel{\text{def}}{=} \Pi \setminus \{\alpha_{\mu_1}, \alpha_{\mu_1+\mu_2}, \ldots, \alpha_{\mu_1+\cdots+\mu_{s-1}}\}.$$

It is easy to see that $X(\lambda^*(C))$ is a maximal element of $\mathfrak{X}_C$. Moreover, every maximal element $Y \in \mathfrak{X}_C$ is W-conjugate to $X(\lambda^*(C))$. Thus we have just one conjugacy class $\{ww_Xw^{-1}\}$ in W for each maximal $X \in \mathfrak{X}_C$.

For other types of groups we can have several conjugacy classes in W of elements of the form w_X, where $X \in \mathfrak{X}_C$ is a maximal element. Say, consider the root system $R = B_2 = \langle \alpha_1, \alpha_2 \rangle$, where $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2$ (in the notation of [**Bou**]), and let G be the corresponding simple and simply connected group over $\mathbb{C}$. Let $g = h_{\varepsilon_1}(t)h_{\varepsilon_2}(t^{-1}) \in G$ be a semisimple element, where $h_{\varepsilon_1}(t), h_{\varepsilon_2}(t^{-1})$ are the corresponding root semisimple elements (in the notation of [**St2**]) and $t \neq \pm 1$. Let C be the conjugacy class of g. Then $\{\alpha_1\}$ and $\{\alpha_2\}$ both are maximal elements of $\mathfrak{X}_C$. Thus here we have two different conjugacy classes in W of elements w_X for maximal X in $\mathfrak{X}_C$.

5. Appendix.

The following result as well as the line of proof was pointed out to the second author by T.A. Springer in the discussion of relevant questions:

Proposition 5.1. *Let $\widetilde{G}$ be a simple algebraic group defined over an algebraically closed field $\overline{K}$ and let $G = \widetilde{G}(\overline{K})$. Further, let C be the conjugacy class of a regular element of G. Then $C \cap B\dot{w}B \neq \emptyset$ for every $w \in W$.*

Proof. For $b \in B$ we put

$$\mathfrak{O}_B(b) = \{xbx^{-1} \mid x \in B\}.$$

Lemma 5.2. *There exists a nonempty finite set $\{b_1, \dots, b_n\} \subset C \cap B$ such that*

$$C \cap B = \bigcup_{1 \le i \le n} \mathfrak{O}_B(b_i).$$

Proof. Let $x = s_1u_1, y = s_2u_2 \in B$ be two regular elements, where $s_1, s_2 \in T$ and $u_1, u_2 \in U$. We show

$$\mathfrak{O}_B(x) = \mathfrak{O}_B(y) \text{ if and only if } s_1 = s_2. \tag{11}$$

Indeed, "only if" is obvious. Now let

$$b_1 = su_1,\ b_2 = su_2,\ s \in T,\ u_1, u_2 \in U. \tag{12}$$

Since we can consider the Jordan decompositions of x, y as elements of B, we may assume that (12) gives the Jordan decompositions of b_1 and b_2. Put $\Gamma = [C_G(s), C_G(s)], B_\Gamma = B \cap \Gamma$. Then (12) implies $u_1, u_2 \in B_\Gamma$. Moreover, the elements u_1, u_2 are regular unipotent elements of Γ ([**St1**, 3.7]) and therefore the elements u_1, u_2 are conjugate in B_Γ (see [**C2**, the proof of Proposition 5.1.3]). Hence we have (11).

Now let $b = su \in B$, $s \in T$, $u \in U$, $g \in G$, $gbg^{-1} \in B$. Further, let $g \in B\dot{w}B$. Then $gbg^{-1} = w(s)u'$ for some $u' \in U$. Together with (11), this implies our assertion. □

Lemma 5.3. *Let $b \in C \cap B$ be a fixed element and let $w \in W$. Then every irreducible component $\mathfrak{C}_w$ of $\overline{C} \cap \overline{B\dot{w}B}$ such that $\mathfrak{O}_B(b) \subset \mathfrak{C}_w$ satisfies the following condition:*

$$\dim\, \mathfrak{C}_w = \dim\, \overline{C} + \dim\, \overline{B\dot{w}B} - \dim\, G.$$

Proof. Since b is a regular element, $\dim C_B(b) = \operatorname{rank} G$ ([**St1**, 3.11]). If $\mathfrak{C}_1$ is an irreducible component of $\overline{C} \cap B$ containing $\mathfrak{O}_B(b)$, then Lemma 5.2 implies $\mathfrak{C}_1 = \overline{\mathfrak{O}_B(b)}$ and, therefore,

$$\dim \mathfrak{C}_1 = \dim B - \operatorname{rank} G = \dim \overline{C} + \dim B - \dim G. \tag{13}$$

Let $\mathfrak{O}_B(b) \subset \mathfrak{C}_w$ for some irreducible component $\mathfrak{C}_w$ of $\overline{C} \cap \overline{B\dot{w}B}$. Suppose

$$\dim \mathfrak{C}_w > \dim \overline{C} + \dim \overline{B\dot{w}B} - \dim G. \tag{14}$$

Since B is a closed subset of $\overline{B\dot{w}B}$ ([**Sp**, 8.15]) and $\mathfrak{C}_1$ is an irreducible component of $\mathfrak{C}_w \cap B$, we have

$$\dim \mathfrak{C}_1 \geq \dim \mathfrak{C}_w + \dim B - \dim \overline{B\dot{w}B}. \tag{15}$$

Now (14) and (15) contradict (13). Thus we have our statement. □

Now we return to the proof of Proposition 5.1.

Take $\mathfrak{C}_w$ as in Lemma 5.3. Assume $C \cap B\dot{w}B = \emptyset$. Then

$$\mathfrak{C}_w \subset \bigcup_{w' < w} B\dot{w}'B = \bigcup_{w' < w} \overline{B\dot{w}'B} \tag{16}$$

([**Sp**, 8.15]). From (16) we have $\mathfrak{C}_w \subset \overline{B\dot{w}'B}$ for some $w' < w$ and we may consider $\mathfrak{C}_w$ as an irreducible component of $\overline{C} \cap \overline{B\dot{w}'B}$ that contains $\mathfrak{O}_B(b)$. Then, by Lemma 5.3, we have

$$\dim \mathfrak{C}_w = \dim \overline{C} + \dim \overline{B\dot{w}'B} - \dim G. \tag{17}$$

But (17) contradicts Lemma 5.3 because $\dim \overline{B\dot{w}'B} < \dim \overline{B\dot{w}B}$. This proves Proposition 5.1. □

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