INTERSECTION OF CONJUGACY CLASSES WITH BRUHAT CELLS IN CHEVALLEY GROUPS

ERICH W. ELLERS AND NIKOLAI GORDEEV
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Let $G = \tilde{G}(K)$ where $\tilde{G}$ is a simple and simply-connected algebraic group that is defined and quasi-split over a field $K$. We investigate properties of intersections of Bruhat cells $B\dot{w}B$ of $G$ with conjugacy classes $C$ of $G$, in particular, we consider the question, when is $B\dot{w}B \cap C \neq \emptyset$.

1. Introduction.

Let $(G, B, N, S)$ be a Tits system. Some aspects of intersections of conjugacy classes of $G$ with Bruhat cells $B\dot{w}B$ have been investigated by several authors (see e.g., [St1], [K], [V] and [VS]). Here $w \in W = N/(B \cap N)$ and $\dot{w} \in N$ is a preimage of $w$ with respect to the natural surjection $N \to W$. In particular, it is desirable to learn how a conjugacy class $C$ of $G$ is related to those conjugacy classes $C_w$ of $W$ for which $B\dot{w}B \cap C \neq \emptyset$, where $w \in C_w$.

Here we deal with the case where $G$ is a Chevalley group, i.e., $G$ is the group of points $\tilde{G}(K)$ of a simple algebraic group $\tilde{G}$ that is defined and quasi-split over a field $K$ and let $\tilde{G}(K)$ be a Chevalley group (see [St2]). Therefore, one can define a Tits system $(G, B, N, S)$, where $S = \{w_{\alpha_i} | \alpha_i \in \Pi\}$ for a simple root system $\Pi$ corresponding to $G$ ([St2] and [C1]).

A crucial step to investigate intersections $B\dot{w}B \cap C$ was done by R. Steinberg [St1] who constructed the cross-section of regular conjugacy classes in $B\dot{w}S\dot{w}^{-1}B$, where $\dot{w}S$ is a Coxeter element of $W$ with respect to the fixed set of generators $S$ of $W$, i.e., $\dot{w}S$ is a product of elements in $S$ in any order, where each $s \in S$ occurs exactly once. The next natural step is to consider intersections of regular classes with cells of the form $B\dot{w}S\dot{w}^{-1}B$. Here we prove the following:

**Theorem 1.1.** Let $\tilde{G}$ be a simple and simply-connected algebraic group that is defined and quasi-split over a field $K$ and let $G = \tilde{G}(K)$. Further, let $C \subset G$ be a conjugacy class of $G$ such that

\[ B\dot{w}S\dot{w}^{-1}B \cap C \neq \emptyset, \]

where $w_S$ is a Coxeter element of $W$ with respect to $S$. Then $C$ intersects all cells of the form $B\dot{w}S\dot{w}^{-1}B$, where $w \in W$. 

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Note that the condition $B \dot{w} S B \cap C \neq \emptyset$ implies that every element of $C$ is regular in $\tilde{G}$, except the case when $\tilde{G}$ is not split and has the type $A_{2l}$ ([St1, Remark 8.8]). Condition (∗) holds, for instance, for regular conjugacy classes of $G$ in the following cases (as shown in Section 4):

(a) $G = SL_n(K)$;
(b) $K = \overline{K}$ (where $\overline{K}$ is the algebraic closure of $K$).

In the cases (c) to (f) below, the field $K$ is supposed to be perfect:

(c) $\tilde{G}$ is split over $K$ and $C = \overline{C} \cap G$ for a conjugacy class $\overline{C}$ of $\tilde{G}$;
(d) $\dim K \leq 1$ and $C$ is a semisimple class (here $\dim K$ is the homological dimension of $K$);
(e) $\tilde{G}$ is split over $K$, $C \cap B \neq \emptyset$, and $C$ is a semisimple class;
(f) $C$ is a unipotent class, char $K$ is not a bad prime for $\tilde{G}$, and if $\tilde{G}$ is not split, then $\tilde{G}$ is not of type $A_{2l}$.

Theorem 1.1 implies:

**Corollary 1.2.** Let $\tilde{G}$ be a simple and simply-connected algebraic group that is defined and quasi-split over a field $K$ and let $G = G(K)$. Further, let $C \subset G$ be a regular conjugacy class of $G$. If one of Conditions (a) to (f) holds, then $C$ intersects all Bruhat cells of the form $B \dot{w} \dot{w}^X \dot{w}^{-1} B$.

**Remark.** The statement of the Corollary in Case (a) follows from the existence of a normal rational form. Case (b) follows from a much more general fact: Every regular conjugacy class of a simple algebraic group (i.e., $G = \tilde{G}(K)$) intersects all Bruhat cells (see Appendix). Also, in Case (f), if $K$ is a finite field, then a theorem of Kawanaka [K] shows that any regular unipotent conjugacy class intersects all Bruhat cells.

Now let $X \subset S$, $W_X = \langle X \rangle$. By $w_X$ we denote a product (in any order) of elements of $X$, where each $x \in X$ occurs exactly once, i.e., $w_X$ is a Coxeter element of $W_X$ with respect to $X$. It is natural to consider intersections $B \dot{w} \dot{w}^X \dot{w}^{-1} B \cap C$ next. In [GS] it has been shown that $B \dot{w} \dot{w}^X B \cap C \neq \emptyset$ for some $X \subset S$ if $C$ is a semisimple class and $K$ is a finite field. Here we prove:

**Theorem 1.3.** Let $\tilde{G}$ be a simple and simply-connected algebraic group that is defined and quasi-split over a perfect field $K$ such that $\dim K \leq 1$, and let $G = \tilde{G}(K)$. Further, let $C \subset G$ be a noncentral semisimple conjugacy class of $G$. Then $C$ intersects all Bruhat cells of the form $B \dot{w} \dot{w}^X \dot{w}^{-1} B$ for some $X \subset S$, $X \neq \emptyset$.

**Remark.** This theorem generalizes Proposition 6 from [GS].

We thank the referee for drawing our attention to a result of Geck and Pfeiffer (see Proposition 3.3) which allows us to extend our results to all Chevalley groups.
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2. S-Coxeter elements in Coxeter groups.

Let $W$ be a finite group of orthogonal transformations of a Euclidean space $V$ generated by reflections. Then $W$ is a Coxeter group. Let $S = \{s_1, \ldots, s_r\}$ be a Coxeter system of generators of $W$, i.e., $s_i^2 = 1$ for every $i = 1, \ldots, r$ and $(s_is_j)^{m_{ij}} = 1$ is the system of basic relations for the group $W$ (see [Bou, IV, 1]). Then every element of the form $s_{\pi(1)}s_{\pi(2)} \ldots s_{\pi(r)}$, where $\pi \in S_r$, is called a Coxeter element of $W$. All Coxeter elements of $W$ constructed for all possible Coxeter systems of generators are conjugate in $W$ (see [Bou, V, 6, Proposition 1]), and if $V^W = \{0\}$, each Coxeter element acts on $V \setminus \{0\}$ without fixed points ([Bou, V, 6, 2]).

Definition 2.1. Let $X \subset S$ and let $W_X$ be the subgroup of $W$ generated by $X$. Every element of $W$ that is conjugate to a Coxeter element in $W_X$ will be called a generalized Coxeter element of $W$.

Definition 2.2. For a fixed system $S$ of generators the elements of the form $s_{\pi(1)}s_{\pi(2)} \ldots s_{\pi(r)}$, where $|S| = r$, will be called $S$-Coxeter elements. If $X \subset S$, then $X$-Coxeter elements in $W_X$ will be called generalized $S$-Coxeter elements of $W$.

Let $l_S(w)$ be the $S$-length of $w$, i.e., the length of $w$ with respect to $S$. Obviously, a Coxeter element $w \in W$ is $S$-Coxeter if and only if $l_S(w) = r$. Below, we shall work with a fixed system $S$ and we shall write $l(w)$ instead of $l_S(w)$. We shall use the well-known fact that $l_X(w) = l_S(w)$ for any $w \in W_X$.

Example 2.3. Let $W = S_4$ and $S = \{(12), (23), (34)\}$. Then we have six Coxeter elements (4-cycles) in $W$. Among them there are four $S$-Coxeter elements:

\[(12)(23)(34), \ (34)(23)(12), \ (23)(12)(34), \ (12)(34)(23),\]

and two elements that are not $S$-Coxeter elements:


Lemma 2.4. Let $w_1, w_2$ be two $S$-Coxeter elements of $W$. Then there exists a sequence $\sigma_1, \sigma_2, \ldots, \sigma_n \in S$ (possibly $\sigma_i = \sigma_j$ for $i \neq j$) such that

\[w_2 = \sigma_n\sigma_{n-1} \ldots \sigma_1w_1\sigma_1\sigma_2 \ldots \sigma_n\]

and $l(\sigma_1\sigma_{i-1} \ldots \sigma_1w_1\sigma_1\sigma_2 \ldots \sigma_{i-1}\sigma_i) = r$ for every $i = 1, \ldots, n$.

Proof. See [C2, Section 10.3].

3. A condition for the intersection of a conjugacy class with Bruhat cells and Gauss cells.

We are going to use the concepts of $S$-ascent and $S$-descent and derive some of their properties. The notion of descent was introduced and considered in
Definition 3.1. Let $w_1, w_2 \in W$. We say that there exists an $S$-ascent (resp. $S$-descent) from $w_1$ to $w_2$ if there is a sequence $\sigma_1, \ldots, \sigma_n \in S$ such that
\[ w_2 = \sigma_n \sigma_{n-1} \cdots \sigma_1 w_1 \sigma_1 \sigma_2 \cdots \sigma_n \]
and
\[ l(\sigma_i \sigma_{i-1} \cdots \sigma_1 w_1 \sigma_1 \sigma_2 \cdots \sigma_i) \geq (\text{resp. } \leq) l(\sigma_{i-1} \cdots \sigma_1 w_1 \sigma_1 \sigma_2 \cdots \sigma_{i-1}) \]
for every $i = 1, \ldots, n$.

Remark. As before, we fix a set $S$ of generators for $W$. In [GP] an $S$-descent from an element $w \in W$ to an element $w' \in W$ is denoted by $w \rightarrow w'$. It is logical to denote an $S$-ascent from $w' \in W$ to $w \in W$ by $w' \leftarrow w$.

Definition 3.2. Let $C \subset W$ be a conjugacy class. We define
\[ l(C) = \min \{ l(w) \mid w \in C \}. \]

The following proposition is due to M. Geck and G. Pfeifer ([GP, Theorem 3.2.9.(a)]):

Proposition 3.3. Let $C \subset W = W(R)$ be a conjugacy class. Then for every $w \in C$ there exists an $S$-descent to an element $w' \in C$ such that $l(w') = l(C)$.

Let $G$ be a Chevalley group (proper or twisted) corresponding to a root system $R$ in the sense of [St2]. We fix a simple root system $\Pi = \{ \alpha_1, \ldots, \alpha_r \}$ and a corresponding Borel subgroup $B = HU$. Let $W = W(R)$ be the Weyl group of $G$ and $S = \{ w_{\alpha_1}, \ldots, w_{\alpha_r} \}$ the corresponding Coxeter system of generators. By $X_\alpha$ we denote below a root subgroup of $G$ (see [St2]).

The meaning of Definition 3.1 becomes clear from the following:

Proposition 3.4. Let $g \in B\dot{w}B$ (resp. $g \in B^-\dot{w}B$) and let $w' \in W$ be an element that is conjugate to $w$. If there exists an $S$-ascent (resp. $S$-descent) from $w$ to $w'$, then there exists an element $g' \in B\dot{w}'B$ (resp. $B^-\dot{w}'B$) that is conjugate to $g$.

Proof. We shall use the following lemma:

Lemma 3.5. Let $w \in W$. Suppose
\[ w(\alpha_i) < 0, \text{ and } w^{-1}(\alpha_i) < 0 \]
for some $\alpha_i \in \Pi$. Then either $w = w_{\alpha_i} w' w_{\alpha_i}$, where $l(w') = l(w) - 2$, or $w = w_{\alpha_i} w' = w' w_{\alpha_i}$, where $l(w') = l(w) - 1$. 

Proof. The assumption \( w^{-1}(\alpha_i) < 0 \) implies
\[
w = w_{\alpha_i}w_1,
\]
where \( l(w_1) = l(w) - 1 \) ([C2, Section 2.2]). Suppose \( w_1(\alpha_i) = \beta > 0 \). Since \( w(\alpha_i) = w_{\alpha_i}(\beta) < 0 \), we have \( \beta = \alpha_i \) and we have the second possibility. Now let \( w_1(\alpha_i) < 0 \). Then \( w_1 = w'w_{\alpha_i} \), where \( l(w') = l(w_1) - 1 \) and we have the first possibility.

First, let \( g \in B\dot{w}B \), then \( g = b_1\dot{w}b_2 \). We may assume \( b_1 = 1 \) and \( b_2 = u \in U \). Also, it is sufficient to prove the assertion for an \( S \)-ascent of one step, i.e., \( w' = w_{\alpha_i}ww_{\alpha_i} \) for some \( \alpha_i \in \Pi \). We can write \( u = u_\alpha v \), where \( u_\alpha \) is a root subgroup element corresponding to \( \alpha \) and where \( v \in U \) is an element that has no \( \alpha \)-factors in any decomposition into positive root subgroup elements.

If \( u_\alpha = 1 \), then \( w' = w_{\alpha_i}w_{\alpha_i}^{-1} \in U \) and
\[
g' = w_{\alpha_i}gw_{\alpha_i}^{-1} = (w_{\alpha_i}\dot{w}w_{\alpha_i}^{-1})(w_{\alpha_i}\dot{w}w_{\alpha_i}^{-1}) = \dot{w}'u' \in B\dot{w}B.
\]

Let \( u_\alpha \neq 1 \). Suppose \( \beta = w(\alpha) > 0 \). We may assume \( \beta \neq \alpha \) (otherwise \( w' = w_{\alpha_i}ww_{\alpha_i}^{-1} = w \)). We have \( g = \dot{w}u_{\alpha_i}w_{\alpha_i}^{-1} \in U \). Now we can consider the element \( u_{\beta}^{-1}g_{\alpha_\beta} \) instead of \( g \) which satisfies the previous condition \( u_\alpha = 1 \).

Suppose \( \beta = w(\alpha) < 0 \) and \( \gamma = w^{-1}(\alpha) > 0 \). We have \( g = \dot{w}u_{\alpha_i}v = \dot{w}u_{\alpha_i}v_{\alpha_i}^{-1}u_{\alpha_i} \). Note that \( v' = u_{\alpha_i}v_{\alpha_i}^{-1} \) has no factors corresponding to \( \alpha \). Consider now the element \( \tilde{g} = u_{\alpha_i}g_{u_{\alpha_i}}^{-1} \) instead of \( g \). We have \( \tilde{g} = u_{\alpha_i}w_{\alpha_i} = \dot{w}^{-1}u_{\alpha_i}w_{\alpha_i} = \dot{w}_{\alpha_i}v' \), an element which also satisfies the condition \( u_\alpha = 1 \).

Now let \( \beta = w(\alpha) < 0 \), \( \gamma = w^{-1}(\alpha) < 0 \). Then, by Lemma 3.5, either \( u_{\alpha_i}ww_{\alpha_i} = w \) and, therefore, there is nothing to prove, or \( l(w_{\alpha_i}ww_{\alpha_i}) < l(w) \) which contradicts our assumption.

Second, let \( g \in B^{-}\dot{u}B \). We may assume \( g = vv_{\alpha_i}v_{\alpha_i}u \), where \( v \in U^- \), \( v_{\alpha_i} \in X_{-\alpha} \), \( u_{\alpha_i} \in X_{\alpha} \), \( u \in U \) and the elements \( v, u \) have no factors from the group \( X_{\pm \alpha} \). Note, \( \dot{w}_{\alpha_i}v_{\alpha_i}^{-1} \in U^- \), \( \dot{w}_{\alpha_i}w_{\alpha_i}^{-1} \in U \) (because \( \alpha \) is a simple root). Thus, if \( v_{\alpha_i} = u_{\alpha_i} = 1 \), then \( \dot{w}_{\alpha_i}v_{\alpha_i}^{-1} \in B^{-}\dot{u}B \). Now put \( \beta = w(\alpha), \gamma = w^{-1}(\alpha) \). If \( \beta < 0, \gamma < 0 \), we have \( g = vv_{\alpha_i}u_{\alpha_i}\dot{w}^{-1}u_{\alpha_i} = vv_{\alpha_i}v_{\beta}v_{\alpha_i}^{-1}v_{\alpha_i}u_{\alpha_i} = v(v_{\alpha_i}v_{\beta}v_{\alpha_i}^{-1})u_{\alpha_i}u_{\alpha_i} \in X_{\beta} \), \( u_{\alpha_i} = \dot{w}_{\alpha_i}^{-1}u_{\alpha_i} \). We may assume \( \beta, \gamma \neq -\alpha \) (otherwise we have \( v_{\alpha_i}ww_{\alpha_i} = w \)). Thus the elements \( v(v_{\alpha_i}v_{\beta}v_{\alpha_i}^{-1}) \), \( u_{\gamma}u \) have no factors from \( X_{\pm \alpha} \) and we are in the preceding case.

Let \( \beta > 0, \gamma < 0 \). Then \( g = vv_{\alpha_i}u_{\beta}v_{\gamma} = v_{\alpha_i}v_{\beta}v_{\alpha_i}^{-1}v_{\alpha_i}u_{\gamma} \), where the element \( v' \in U^- \) has no factor from \( X_{-\alpha} \). Put \( u_{\alpha_i} = \dot{w}_{\alpha_i}v_{\alpha_i}^{-1} \). Then \( \dot{w}_{\alpha_i}v_{\alpha_i}^{-1} = u_{\alpha_i}v_{\alpha_i}^{-1}u_{\alpha_i} \) for some \( v'' \in U^- \), \( u' \in U \). Thus \( u_{\alpha_i}^{-1}\dot{w}_{\alpha_i}v_{\alpha_i}^{-1}u_{\alpha_i} \in B^{-}\dot{w}B \).

The case \( \beta < 0, \gamma > 0 \) is similar to the preceding one.

Let \( \beta > 0, \gamma > 0 \). Again, as above, we may assume \( \beta, \gamma \neq \alpha \). Thus by Lemma 3.5, we have \( l(w') = l(w_{\alpha_i}ww_{\alpha_i}) = l(w) + 2 \) which contradicts our assumption.

\[\square\]
Example 3.6. Let \( G = SL_3(K) \) and let
\[
\hat{w} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \hat{w}' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]
Let \( g \) be a semisimple element of \( G \) that has no eigenvalues in \( K \). Then \( g \) is a regular element and therefore its conjugacy class \( C_g \) intersects the big Bruhat cell \( B\hat{w}B \) (see [EGH, Lemma 4]). But \( C_g \cap B\hat{w}'B = \emptyset \) because every element of the form \( b_1\hat{w}'b_2 \) is conjugate to an element of the form
\[
\hat{w}'b = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & a_{22} & a_{23} \\ -a_{11} & -a_{12} & -a_{13} \\ 0 & 0 & a_{33} \end{pmatrix}
\]
which has an eigenvalue \( a_{33} \in K \). Note that here \( S = \{ w_{12}, w_{23} \} \) (where \( w_{ij} \) is the matrix in which the \( i \)th and \( j \)th elements of the standard basis are interchanged) and \( l(w) = 3, l(w') = l(w_{12}) = 1 \).

Example 3.7. Let \( G = SL_4(K) \) and let
\[
\hat{w} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \hat{w}' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
\]
Here \( S = \{ w_{12}, w_{23}, w_{34} \} \) and \( \hat{w} = \hat{w}_{12}\hat{w}_{34} \) is a generalized \( S \)-Coxeter element. Thus every noncentral conjugacy class of \( G \) intersects \( B^-\hat{w}B \) ([EG]). In particular, one can find a transvection \( g \in B^-\hat{w}B \). But there are no transvections in \( B^-\hat{w}'B \) because \( B^-\hat{w}'B = \hat{w}'B\hat{w}'^{-1} \hat{w}'B = \hat{w}'B \), and every matrix \( x \in \hat{w}'B \) satisfies the condition \( \text{rank}(x) \geq 2 \).

The examples above show that if there is no \( S \)-ascent (resp. \( S \)-descent) from \( w \in W \) to its conjugate \( w' \in W \), the condition \( C \cap B\hat{w}B \neq \emptyset \) (resp. \( C \cap B^-\hat{w}B \neq \emptyset \)) for a conjugacy class \( C \subseteq G \) does not necessarily imply \( C \cap B\hat{w}B \neq \emptyset \) (resp. \( C \cap B^-\hat{w}B \neq \emptyset \)).

Proposition 3.8. Let \( g \in B\hat{w}B \) (resp. \( g \in B^-\hat{w}B \)). Suppose \( l(w_\alpha w_\alpha) = l(w) - 2 \) (resp. \( l(w_\alpha w_\alpha) = l(w) + 2 \)). Then the conjugacy class \( C_g \) of \( g \) intersects either \( B\hat{w}_\alpha \hat{w}_\alpha^{-1}B \) (resp. \( B^-\hat{w}_\alpha \hat{w}_\alpha^{-1}B \)) or \( B\hat{w}_\alpha \hat{w}_\alpha B \) (resp. \( B^-\hat{w}_\alpha \hat{w}_\alpha B \)).

Proof. Let \( g \in B\hat{w}B \). We may assume, as in the proof of Proposition 3.4, that \( g = \hat{w}_\alpha u \) and \( w = w_\alpha w_1 w_\alpha \), where \( l(w_1) = l(w) - 2 \). Moreover, \( \beta = w_1(\alpha) > 0 \), \( \gamma = w_1^{-1}(\alpha) > 0 \), and \( w_1(\alpha), w_1(\alpha) \neq \alpha \). If \( u_\alpha = 1 \), then \( \hat{w}_\alpha \hat{w}_\alpha^{-1} \in B\hat{w}B \). Suppose \( u_\alpha \neq 1 \). Put \( u_{-\alpha} = \hat{w}_\alpha u_\alpha \hat{w}_\alpha^{-1} \). There exists \( u_{-\alpha} \in X_\alpha \) (here \( X_\alpha \) is the corresponding root subgroup) such that \( u_{-\alpha} u_{-\alpha} = \hat{w}_\alpha u_\alpha \) for some \( u'' \in X_\alpha \). Further, \( g_1 = \hat{w}_\alpha \hat{w}_\alpha^{-1} = \hat{w}_1 u_{-\alpha} u' \) for some \( u' \in U \). Put \( u_\beta = \hat{w}_1 u''_\alpha \hat{w}_1^{-1} \) (recall \( \beta = w_1(\alpha) > 0 \)). Then
\[ g_2 = u_\beta g_1 u_\beta^{-1} = \tilde{w}_1 u_\alpha' \tilde{w}_1^{-1} \tilde{w}_1 u_\alpha' u_\beta' \tilde{w}_1 u_\beta' = \tilde{w}_1 \tilde{u}_\alpha u_\alpha'' u_\beta' \tilde{w}_1 \tilde{u}_\beta' B. \]
Since \( l(w_\alpha) = l(w_\alpha w_1) \), we also can find an element in \( C_g \cap B \tilde{w}_\alpha \tilde{w}_1 B \) (by Proposition 3.4).

Now let \( g \in B^{-} \tilde{w} B \). As in the proof of Proposition 3.4 we may assume \( g = v v_\alpha \tilde{w}_\alpha u , \alpha \neq (\omega(\alpha) > 0 , \alpha \neq \omega^{-1}(\alpha) > 0 \). If \( u_\alpha = u_\alpha = 1 \), then \( \tilde{w}_\alpha g \tilde{w}_\alpha^{-1} B = B^{-} \tilde{w}_\alpha \tilde{w}_\alpha^{-1} B \). Let \( v_\alpha = 1 \) and \( u_\alpha \neq 1 \). Then
\[
\begin{align*}
g_1 &= \tilde{w}_\alpha g \tilde{w}_\alpha^{-1} = (\tilde{w}_\alpha v v_\alpha) (\tilde{w}_\alpha \tilde{w}_\alpha^{-1}) (\tilde{w}_\alpha u_\alpha \tilde{w}_\alpha^{-1}) (\tilde{w}_\alpha u_\alpha \tilde{w}_\alpha^{-1}) \\
&= v' \tilde{w}_\alpha \tilde{w}_\alpha^{-1} u' \tilde{w}_\alpha \tilde{w}_\alpha^{-1} \end{align*}
\]
where \( v' \in U^{-} \), \( u' \in U \), \( u_\alpha \in X_\alpha \). Moreover, the element \( u' \) has no factors in \( X_\alpha \). Further, \( u_\alpha' g_1 \tilde{w}_\alpha^{-1} = u_\alpha' \tilde{w}_\alpha \tilde{w}_\alpha^{-1} u_\alpha' \tilde{w}_\alpha^{-1} \). Since \( u_\alpha' \tilde{w}_\alpha^{-1} \in U \), we have \( u_\alpha' g_1 \tilde{w}_\alpha^{-1} \in B^{-} \tilde{w}_\alpha \tilde{w}_\alpha^{-1} B \). Similar considerations work in the case \( v_\alpha \neq 1 \), \( u_\alpha = 1 \).

Let \( v_\alpha \neq 1 \), \( u_\alpha \neq 1 \). Put \( u_\alpha' = \tilde{w}_\alpha v_\alpha \tilde{w}_\alpha^{-1} \), \( v_\alpha' = \tilde{w}_\alpha u_\alpha \tilde{w}_\alpha^{-1} \), \( v' = \tilde{w}_\alpha v v_\alpha \tilde{w}_\alpha^{-1} \), \( u' = \tilde{w}_\alpha \tilde{w}_\alpha^{-1} \). Then
\[
\begin{align*}
g_1 &= \tilde{w}_\alpha g \tilde{w}_\alpha^{-1} = v' u_\alpha' \tilde{w}_\alpha \tilde{w}_\alpha^{-1} v_\alpha' u' = v' u_\alpha' \tilde{w}_\alpha \tilde{w}_\alpha^{-1} \tilde{u}_\alpha' (v_\alpha' u_\alpha' v_\alpha') v_\alpha' \end{align*}
\]
Put \( u'' = v_\alpha' u_\alpha' \), \( v'' = v_\alpha' v_\alpha' \). Then \( g_2 = v_\alpha' g_1 \tilde{w}_\alpha^{-1} = v'' u_\alpha' \tilde{w}_\alpha \tilde{w}_\alpha^{-1} u'' \).
Further, \( u_\alpha' \tilde{w}_\alpha = x_\alpha x_\alpha \) for some \( x_\alpha \in X_\alpha \), \( x_\alpha \in X_\alpha \). Hence
\[
\begin{align*}
g_2 &= v'' x_\alpha x_\alpha \tilde{w}_\alpha^{-1} u'' = v'' x_\alpha \tilde{w}_\alpha^{-1} (x_\alpha x_\alpha \tilde{w}_\alpha^{-1}) u'' \\
\end{align*}
\]
Since \( w^{-1}(\alpha) > 0 \) and \( w^{-1}(\alpha) \neq \alpha \), we get \( \tilde{w}_\alpha \tilde{w}_\alpha^{-1} x_\alpha \tilde{w}_\alpha^{-1} \in U \) and therefore \( g_2 \in B^{-} \tilde{w} \tilde{w}_\alpha B \). From Proposition 3.4 we get \( C_g \cap B^{-} \tilde{w}_\alpha \tilde{w} B \neq \emptyset \).

4. Proofs of the Theorems.

Here \( \tilde{G} \) is a simple algebraic group defined and quasi-split over a field \( K \), \( \tilde{B} = \tilde{T} \tilde{U} \) is a Borel subgroup defined over \( K \), \( \tilde{N} = N_{\tilde{G}}(\tilde{T}), \tilde{W} = \tilde{N}/\tilde{T} \) and \( G = \tilde{G}(K), B = \tilde{B}(K), T = \tilde{T}(K), U = \tilde{U}(K), N = \tilde{N}(K), W = N/T \). Further, let \( \Pi = \{ \gamma_1, \ldots, \gamma_s \} \) be a simple root system of \( \tilde{G} \) and \( \Pi = \{ \alpha_1, \ldots, \alpha_r \} \) be a simple root system (in the sense of \( [C2] \)) for \( G \) (which is obtained from \( \Pi \) by gluing of some roots).

**Proof of Theorem 1.1.** Assume that Condition (*) of Theorem 1.1 holds. Further let \( C_c \subset W \) be the conjugacy class of Coxeter elements and let \( \omega \in C_c \). By Proposition 3.3 there exists an \( S \)-descent from \( \omega \) to an element \( \omega' \in C_c \) such that \( l(\omega') = l(C_c) = r \). Since among the factors of \( \omega' \) there are all reflections \( w_{\alpha_i}, \alpha_i \in \Pi \), the element \( \omega' \) is an \( S \)-Coxeter element. By Lemma 2.4 we have an \( S \)-ascent from \( w_S \) (recall that \( w_S \) is the Coxeter element from Condition (*)) to \( \omega' \) and, therefore, we have an \( S \)-ascent from \( w_S \) to \( \omega \). Now our statement follows from Proposition 3.4.

**Condition (*).** Below, \( K \) is a perfect field.

We need the following simple (and known) facts:
Lemma 4.1. Let $\tilde{C}$ be a conjugacy class of $\tilde{G}$ such that $C = \tilde{C} \cap G \neq \emptyset$. Further, let $g \in C$. If $H^1(K, C_{\tilde{G}}(g)) = 1$ then $C$ is a conjugacy class of $G$ (here $C_{\tilde{G}}(g)$ is the centralizer of $g$ in $\tilde{G}$).

Proof. The argument here is the same as in ([C2, Proposition 3.7.3]). Indeed, if $g' \in C$, then there exists an element $\gamma \in \tilde{G}$ such that $g' = \gamma g \gamma^{-1}$. Thus, for every element $\sigma \in \text{Gal}(K/K)$ of the Galois group we have

$$\sigma(\gamma) g \sigma(\gamma^{-1}) = \gamma g \gamma^{-1}$$

and therefore $x_\sigma = \gamma^{-1} \sigma(\gamma) \in C_{\tilde{G}}(g)$. Since $x_\sigma$ is a 1-cocycle, we have $x_\sigma = y \sigma(y^{-1})$ for some $y \in C_{\tilde{G}}$ and therefore $\sigma(\gamma y) = \gamma y$ for every $\sigma \in \text{Gal}(K/K)$. Thus, $\gamma y \in G$ and $g' = \gamma yy^{-1} \gamma^{-1}$.

Lemma 4.2. Let $\tilde{C}$ be a semisimple conjugacy class of $\tilde{G}$ and let $C = \tilde{C} \cap G \neq \emptyset$. If $\dim K \leq 1$, then $C$ is a conjugacy class of $G$.

Proof. Since $\tilde{G}$ is simply-connected, $C_{\tilde{G}}(s)$ is a connected reductive group for $s \in \tilde{C} \cap \tilde{T}$ ([C2, Theorem 3.5.6]) and therefore $H^1(K, C_{\tilde{G}}(s)) = 1$ ([St1, 11.2]). Now the assertion follows from Lemma 4.1.

Lemma 4.3. Let $C$ be the same as in the preceding lemma. Suppose that $G$ is split and $C$ is a regular semisimple class such that $\tilde{C} \cap \tilde{T} \neq \emptyset$. Then $C$ is a conjugacy class of $G$.

Proof. If $s \in \tilde{C} \cap \tilde{T}$, then $C_{\tilde{G}}(s) = \tilde{T}$ is a $K$-split torus and therefore $H^1(K, C_{\tilde{G}}(s)) = 1$ ([Sp, 12.3.5.(3)]). Now the assertion follows from Lemma 4.1.

Lemma 4.4. Let $u_1, u_2 \in G$ be two regular unipotent elements of $\tilde{G}$. Assume that $\text{char } K$ is not a bad prime for $\tilde{G}$. Then there exists elements $t \in \tilde{T}$ and $\gamma \in G$ such that $u_1 = t \gamma u_2 \gamma^{-1} t^{-1}$.

Proof. Let $\tilde{G} = \tilde{G}/Z(\tilde{G}), \tilde{T} = \tilde{T}/Z(\tilde{G})$. Then $\tilde{G}$ is defined and quasi-split over $K$ and $Z(\tilde{G}) = 1$. Further, let $u \in G$ be a regular unipotent element and let $\bar{u}$ be its image in $\tilde{G}$. The char $K$ is not a bad prime for $\tilde{G}$, thus $V = C_{\tilde{G}}(u)$ is a connected unipotent subgroup of $\tilde{G}$ ([C2, Proposition 5.1.6]) which is defined and split over $K$ ([Sp, 14.3.8]) and therefore $H^1(K, V) = 1$ ([Sp, 12.3.5.(3)]). Hence any two regular unipotent elements of $\tilde{G}(K)$ are conjugate (Lemma 4.1). If $\tilde{G}_1(K) \leq \tilde{G}(K)$ is a subgroup generated by unipotent elements of $\tilde{G}$, then it is a normal subgroup and $\tilde{G}_1(K) = \tilde{G}_1(K)\tilde{T}(K)$ (this follows from the Bruhat decomposition). Now let $\bar{u}_1, \bar{u}_2 \in \tilde{G}(K)$ be images of regular unipotent elements $u_1, u_2 \in G$. Then there exist elements $\gamma \in \tilde{G}_1(K), t \in \tilde{T}(K)$ such that $\bar{u}_1 = t \gamma \bar{u}_2 \gamma^{-1} t^{-1}$. If $\gamma \in \tilde{G}(K) = G, t \in \tilde{T}$ are preimages of $\gamma, t$, then $u_1 \equiv t \gamma u_2 \gamma^{-1} t^{-1} (\text{mod } Z(\tilde{G}))$. Since $u_1, u_2$ are both unipotent elements, we have $u_1 = t \gamma u_2 \gamma^{-1} t^{-1}$.
Now we check Condition (*) for (a) to (f):

(a) If $G = SL_n(K)$, Condition (*) is an immediate consequence of the representation of elements of $GL_n(K)$ in rational canonical form.

(b) Consider the case where $K$ is an algebraically closed field. According to Steinberg’s theorem ([St1, 1.4]), the set

$$\mathfrak{N} = \bar{w}_{\gamma_1}X_{\gamma_1}\bar{w}_{\gamma_2}X_{\gamma_2}\ldots\bar{w}_{\gamma_s}X_{\gamma_s}$$

is a cross-section of all regular conjugacy classes of the group $\bar{G}$, where $\bar{w}_{\gamma_1}, \ldots, \bar{w}_{\gamma_s}$ is any fixed system of preimages of the basic reflections $w_{\gamma_1}, \ldots, w_{\gamma_s}$ in any fixed order (here $X_{\gamma_i}$ is the corresponding root subgroup). Moreover, we can rewrite $\mathfrak{N}$ in the form

$$\mathfrak{N} = \bar{w}_{\gamma_1}\bar{w}_{\gamma_2}\ldots\bar{w}_{\gamma_s}X_{\theta_1}X_{\theta_2}\ldots X_{\theta_s},$$

where $\theta_i = w_{\gamma_i}\ldots w_{\gamma_{i+1}}w_{\gamma_i-1}(\gamma_i) > 0$. Since $K$ is an algebraically closed field, $\{\alpha_1, \ldots, \alpha_s\} = \{\gamma_1, \ldots, \gamma_s\}$ and any element in the intersection $\mathfrak{N} \cap N(K)$ lies in the $S$-Coxeter cell $BW_{\alpha_1}W_{\alpha_2}\ldots W_{\alpha_s}B$. This proves (*).

(c) If $\bar{G}$ is split over $K$, the closed subset $\mathfrak{N}$ (defined above) of $\bar{G}$ is defined over $K$ and $\mathfrak{N} \cap \bar{C} \in G$ ([St1, Section 9]).

(d) There exists a closed subset $\mathfrak{N}'$ of $\mathfrak{N}$ which is defined over $K$ and such that every regular semisimple conjugacy class $\bar{C}$ of $\bar{G}$ intersects $\mathfrak{N}'$ in just one point (and this point belongs to $G$ if $\bar{C} \cap G \neq \emptyset$) ([St1, 9.11]). Since $\mathfrak{N} \subset BW_{\gamma}B$ for some $S$-Coxeter element $w_S$, the assertion follows from Lemma 4.2.

(e) We may use the same argument as in (d), and Lemma 4.3.

(f) If $\bar{G}$ is split or $\bar{G}$ is not of type $A_2$, the cross-section of regular classes $\mathfrak{N}$ is defined over $K$ and for the conjugacy class of regular unipotent elements $\bar{C}$ we have $u = \bar{C} \cap \mathfrak{N} \in BW_{\gamma}B$, where $w_{\gamma} \in N$ for some $S$-Coxeter element $w_{\gamma}$ in $W$ ([St1, Section 9]). Now let $u' \in \bar{C} \cap G$. By Lemma 4.4 we have

$$t\gamma u' \gamma^{-1} t^{-1} = u = u_1 \bar{w}_{\gamma} b_1$$

for some $t \in T, \gamma \in G$ and $u_1 \in U, b_1 \in B$. Hence

$$u'' = \gamma u' \gamma^{-1} = (t^{-1} u t)(t^{-1} \bar{w}_{\gamma} t)(t^{-1} b_1 t).$$

Thus $u'' \in BW_{\gamma}B$. But $u'' \in G$ and, therefore, $u'' \in BW_B$ for some $\bar{w} \in N$. Since $BW_B \subset BW_B$, we have $w = w_{\gamma}$. This implies that the conjugacy class $C$ of $u'$ in $G$ has a nontrivial intersection with $BW_{\gamma}B$, where $w_{\gamma} \in N$.

Proof of Theorem 1.3.

Below, $\bar{\Gamma}$ is a connected reductive algebraic group defined over a perfect field $K$ such that $\dim K \leq 1$.

Lemma 4.5. Let $\bar{P} = \bar{L}R_u(\bar{P})$ be a parabolic subgroup of $\bar{G}$ defined over $K$. Let $\bar{L}$ be a fixed Levi factor (defined over $K$) and let $R_u(\bar{P})$ be the unipotent radical of $\bar{P}$. Further, let $s \in \bar{P}(K)$, $s = lu$, where $l \in \bar{L}$ and $u \in R_u(\bar{P})$. 
If \( s \in \tilde{\Gamma}(K) \), then \( l \in \tilde{L}(K) \) and \( u \in R_u(\tilde{P})(K) \). If, in addition, \( s \) is a semisimple element, then \( s \) is conjugate to \( l \) in \( \tilde{P} \).

**Proof.** The first assertion follows from the uniqueness of the decomposition \( ku \).

Further, if \( s \) is semisimple, it is contained in a maximal torus in \( \tilde{P} \) which is contained in a Levi subgroup \( L' \) (\([\text{Sp}, 8.4.4]\)). Since all Levi subgroups are conjugate in \( \tilde{P} \) (\([\text{Sp}, 16.1.1]\)) by elements of \( \tilde{P} \), one can find an element \( p = l_1u_1 \in \tilde{P} \) where \( l_1 \in \tilde{L} \), \( u_1 \in R_u(\tilde{P}) \) such that \( psp^{-1} \in \tilde{L} \). Then \( l_1^{-1}psp^{-1}l_1 = u_1su_1^{-1} = l(l^{-1}u_1lu_1^{-1}) \in \tilde{L} \). Hence \( (l^{-1}u_1lu_1^{-1}) = 1 \) (because \( (l^{-1}u_1lu_1^{-1}) \in R_u(\tilde{P}) \)) and therefore \( l_1^{-1}psp^{-1}l_1 = l \). \( \Box \)

**Lemma 4.6.** Let \( s \in \tilde{\Gamma}(K) \) be a semisimple element of \( \tilde{\Gamma} \) such that \( C_{\tilde{\Gamma}}(s)^0 \) is not a torus. Then there exists a parabolic subgroup \( \tilde{P} \) of \( \tilde{\Gamma} \) defined over \( K \) such that \( s \in \tilde{P} \).

**Proof.** The group \( C_{\tilde{\Gamma}}(s)^0 \) is defined over \( K \) (\([\text{Sp}, 12.1.4]\)). Further, the condition \( \dim K \leq 1 \) implies that there exists a Borel subgroup \( \tilde{B}_s \) of \( C_{\tilde{\Gamma}}(s)^0 \) which is also defined over \( K \) (\([\text{St1}, 10.2]\)). Since \( C_{\tilde{\Gamma}}(s)^0 \) is not a torus, the unipotent radical \( R_u(\tilde{B}_s) \) is not trivial. The group \( \tilde{U}_1 = R_u(\tilde{B}_s) \) is also defined over \( K \) (\([\text{Sp}, 14.4.5(v)]\)). Further, let

\[
\tilde{N}_1 = N_{\tilde{G}}(\tilde{U}_1), \quad \tilde{U}_2 = \tilde{U}_1 \cdot R_u(\tilde{N}_1), \quad \tilde{N}_2 = N_{\tilde{G}}(\tilde{U}_2), \ldots,
\]

\[
\tilde{U}_i = \tilde{U}_{i-1} \cdot R_u(\tilde{N}_{i-1}), \quad \tilde{N}_i = N_{\tilde{G}}(\tilde{U}_i), \ldots.
\]

Then all members of \( (1) \) are closed subgroups of \( \tilde{\Gamma} \) and \( \tilde{U}_k = \tilde{N}_{k+1} \), \( \tilde{N}_k = \tilde{N}_{k+1} \) for some positive integer \( k \) (\([\text{Hu}, 30.3]\)). Further, all groups in \( (1) \) are defined over \( K \); indeed, the field \( K \) is perfect and all groups are defined as normalizers of \( K \)-defined groups, their unipotent radicals, and the images of \( K \)-defined groups with respect to maps \( \tilde{U}_{i-1} \times R_u(\tilde{N}_{i-1}) \rightarrow \tilde{U}_{i-1} \cdot R_u(\tilde{N}_{i-1}) \), induced by multiplication in \( \tilde{G} \). Since \( \tilde{U}_1 \) is connected, the last member \( \tilde{N}_k \) of this sequence is a parabolic subgroup of \( \tilde{\Gamma} \) (\([\text{Hu}, 30.3]\)). From the definitions we have \( s \in \tilde{N}_1 \leq \tilde{N}_k \). \( \Box \)

Now we can prove Theorem 1.3. Let \( s \in G \) be a noncentral semisimple element. We may assume that \( s \) is not a regular element of \( \tilde{G} \) (otherwise the statement follows from Theorem 1.1 and Property (d)). By Lemma 4.6 we have \( s \in \tilde{P} \) for some parabolic subgroup defined over \( K \). Since \( g\tilde{P}g^{-1} = \tilde{P}_l \) for some standard parabolic subgroup \( \tilde{P}_l \) and \( g \in G \) (\([\text{Sp}, 15.4.6]\)), we may assume \( s \in \tilde{P}_l \), where \( I \subset \tilde{\Pi} \) is a Gal (\( \overline{K}/K \))-invariant subset (note that the group \( \text{Gal}(\overline{K}/K) \) acts on \( \tilde{\Pi} \) by permutation and the orbits of this action
correspond to \( \Pi \); see [St1, Section 9]). Let \( \widetilde{L}_I = \widetilde{T}\widetilde{G}_I \), where \( \widetilde{G}_I = \langle X_\alpha \mid \alpha \in \langle I \rangle \rangle \). Then \( \widetilde{L}_I \) is a \( K \)-defined Levi factor of \( \widetilde{P}_I \).

By Lemmas 4.5 and 4.2 we may assume \( s \in \widetilde{L}_I \). (Indeed, by Lemma 4.5 we have an element \( l \in \widetilde{L}_I(K) \) which is conjugate to \( s \) in \( \widetilde{P}_I \). By Lemma 4.2 the elements \( s, l \) are conjugate by an element of the group \( G \). Hence we may take the element \( l \in C \) instead of \( s \).

Again by Lemma 4.6 we may assume that \( C_{\widetilde{L}_I}(s)^0 = \widetilde{T}' \), where \( \widetilde{T}' \) is a maximal torus of \( \widetilde{L}_I \) defined over \( K \) (otherwise, we can take a smaller set \( I \) using the same procedure as above). Note that the derived subgroup \( \widetilde{L}_I \) is equal to \( \widetilde{G}_I \) and therefore is simply-connected semisimple group (because \( \widetilde{G} \) is simply-connected). Hence \( C_{\widetilde{L}_I}(s)^0 = C_{\widetilde{L}_I}(s) \) ([C2, Theorem 3.5.6]) and thus

\[
(2) \quad C_{\widetilde{L}_I}(s) = \widetilde{T}'.
\]

Further, if \( I = \emptyset \) we have \( \widetilde{P}_I = \widetilde{B} \) and \( \widetilde{T}' = \widetilde{T} \). Hence \( s \in \widetilde{T}(K) = T \).

Since \( s \) is a noncentral element of \( G \), there exists a root \( \alpha \in \Pi \) such that \( s \) is not in the center of the group \( T\widetilde{G}_\alpha(K) \) (here, \( \widetilde{G}_\alpha = \langle X_\beta \mid \beta \in \langle I_\alpha \rangle \rangle \) where \( I_\alpha \subset \Pi \) is the Gal \((K/K)-orbit of \( \alpha \)). Since the Borel subgroup \( B_\alpha \) of \( T\widetilde{G}_\alpha(K) \) (with respect to \( T \)) is not a normal subgroup, one can find an element \( \gamma \in T\widetilde{G}_\alpha(K) \) such that \( \gamma s \gamma^{-1} = \check{w}_\alpha b \), where \( w_\alpha \in W \) is the corresponding reflection and \( b \in B_\alpha \). Hence \( C \cap B\check{w}_\alpha B \neq \emptyset \). Further, let \( \omega \in W \). Then \( \omega w_\alpha \omega^{-1} = \check{w}_\beta \), where \( \beta = \omega(\alpha) \). Let \( \check{\omega}, \check{\beta} \), be preimages of \( \omega, w_\beta \), in the group \( N \). Then \( \check{\omega}T\widetilde{G}_\alpha(K)\check{\omega}^{-1} = T\widetilde{G}_\beta(K) \). The element \( s' = \check{\omega}s\check{\omega}^{-1} \) is not a central element in \( T\widetilde{G}_\beta(K) \). Now, as above, we have \( \gamma' s' \gamma'^{-1} \in B\check{w}_\beta B \) for some \( \gamma' \in T\widetilde{G}_\beta(K) \). Thus, if \( I = \emptyset \), the assertion of the theorem holds for \( X = \{\alpha\} \).

Now we may assume that \( I \neq \emptyset \) and Condition \( (2) \) holds.

We have \( s = tg, t \in \check{T} \cap C_{\widetilde{L}_I}(\widetilde{G}_I) \), and \( g \in \widetilde{G}_I \) ([Hu, 27.5]). Note that the elements \( t \) and \( g \) do not necessarily belong to \( G \) but \( t, g \in \widetilde{L}_I(K') \) for some extension \( K'/K \). The element \( s \in G \) is Gal \((K/K)-invariant and \( t \in Z(\widetilde{L}_I) \). Hence \( g = h_1 g_1 \), where \( h_1 \in \check{T}(K') \), \( g_1 \in \widetilde{G}_I(K) \) (this follows from the Bruhat decomposition of \( g \)). Further, \( (2) \) implies that \( g \) is a regular element of \( \widetilde{G}_I \). If \( \mathfrak{N}' \) is a cross-section (defined over \( K \)) of regular semisimple conjugacy classes of \( \widetilde{G}_I \) ([St1, Section 9]) then \( h_1 \mathfrak{N}' \) is also a cross-section (defined over \( K' \)) of regular semisimple conjugacy classes of \( \widetilde{G}_I \). Hence the conjugacy class \( C_g \) of \( g \) in \( \widetilde{G}_I \) intersects \( h_1 \mathfrak{N}' \) in just one point. Thus the conjugacy class \( C_s = tC_g \) of \( s \) in \( \widetilde{L}_I \) intersects \( th_1 \mathfrak{N}' \) also in one point \( x \) (recall, \( t \in Z(\widetilde{L}_I) \)). Since the conjugacy class \( C_s \) is defined over \( K \) and the closed subset \( th_1 \mathfrak{N}' \) is also defined over \( K \) (because \( th_1 = sg_1^{-1} \in \widetilde{L}_I(K) \)),
the point $x$ is Gal($\overline{K}/K$)-invariant and therefore it belongs to $L_I(K)$. Since $s, x \in L_I(K) \leq G$ are conjugate in $L_I$ (and therefore in $\overline{G}$), we have $x = \sigma s \sigma^{-1}$ for some $\sigma \in G$ (Lemma 4.2). Further,

$$th_1 \mathfrak{g}' \subset \left( \prod_{\alpha \in X} \hat{w}_\alpha \right) \tilde{U},$$

where $X \subset \Pi$ is the set of Gal($\overline{K}/K$)-orbits of $I \subset \tilde{\Pi}$ and $w_\alpha$ in (3) is the product of basic reflections $w_\gamma$, where $\gamma$ runs through the orbit corresponding to $\alpha$ or $w_\alpha = w_{\gamma_1 + \gamma_2}$ if such orbit consists of two roots $\gamma_1, \gamma_2$ such that $\gamma_1 + \gamma_2$ is a root (see [St1, Section 9]). From (3) we obtain

$$x = \sigma s \sigma^{-1} \in \tilde{B} \prod_{\alpha \in X} \hat{w}_\alpha \tilde{B}. \quad (4)$$

Since $x \in G$, we have

$$x = \sigma s \sigma^{-1} \in B \hat{w} B \quad (5)$$

for some $w \in W$. But

$$B \hat{w} B \subset \tilde{B} \hat{w} \tilde{B}. \quad (6)$$

From (4), (5), (6) we get

$$w = \prod_{\alpha \in X} w_\alpha, \quad (7)$$

i.e., $w$ is a generalized $S$-Coxeter element of $W$. Now (5) and (7) imply that the conjugacy class of $s$ in $G$ intersects $B \hat{w} B$ for some generalized $S$-Coxeter element $w$ of $W$.

Suppose that $w' = \omega w \omega^{-1}$ is also an $S$-Coxeter element of $W$ for some $\omega \in W$. Then $w' = \prod_{\alpha \in Y} w_\alpha$ for some $Y \subset \Pi$, $|Y| = |X|$. Let $X' = \{\omega(\alpha) \mid \alpha \in X\}$. Then

$$w' = \prod_{\alpha \in Y} w_\alpha = \prod_{\beta \in X'} w_\beta. \quad (8)$$

The element $w'$ is a Coxeter element of the root systems generated by $Y$ and $X'$. It acts without fixed points on the vector space (over $\mathbb{R}$) generated by $Y$ and on the vector space generated by $X'$. Moreover, $l(w') = |Y| = |X'|$. Hence the vector spaces (over $\mathbb{R}$) generated by $Y$ and $X'$ coincide (it is the $\langle w' \rangle$-complement to the vector space of $w'$-invariant vectors). Since $X$ is a simple root system for the root system $\langle X \rangle$, the set $X'$ is a simple root system for $\langle X' \rangle$. On the other hand, the set $Y$ is a simple root system for the root system $\langle Y \rangle$. Now $X' \subset \omega(\Pi)$, $Y \subset \Pi$ and the linear spaces generated by $X'$ and $Y$ coincide. Moreover, the root subsystems $\langle X' \rangle$, $\langle Y \rangle$ have the same Coxeter element $w'$. Hence $\langle X' \rangle = \langle Y \rangle$. Now let $I'$ be a subset of $\Pi$
that is $\text{Gal} \left( \overline{K}/K \right)$-invariant and such that the set of $\text{Gal} \left( \overline{K}/K \right)$-orbits of $I'$ coincides with $Y$. Since $\omega(\langle X \rangle) = \langle Y' \rangle = \langle Y \rangle$, we have

\begin{align}
\tilde{G}' &= \langle X_\beta \mid \beta \in \langle I' \rangle \rangle = \omega \tilde{G} \omega^{-1}.
\end{align}

From (8) we get

\begin{align}
\tilde{L}' &= \tilde{T} \tilde{G}' = \omega \tilde{L} \omega^{-1}.
\end{align}

Since $\omega \in W$, we can choose the preimage $\hat{\omega} \in G$. From (9)

\begin{align}
s' = \hat{\omega}s\hat{\omega}^{-1} \in \tilde{L}' \cap G.
\end{align}

Now we have a semisimple regular element $s' \in \tilde{L}'(K)$. The same arguments as above show that there exists an element $\tau \in G$ such that $s'' = \tau s' \tau^{-1} \in B \hat{w}' B$, where

\begin{align}
\hat{w}' = \prod_{\beta \in Y} w_\beta
\end{align}

(the order of the roots $\beta$ in this product can be different from the order of the roots $\alpha$ in the product corresponding to $w'$). By Lemma 2.4 there exists an $S$-ascent from $\hat{w}'$ to $\tilde{w}' \in W$. Proposition 3.4 implies

\begin{align}
\delta s'' \delta^{-1} \in B \tilde{w}' B
\end{align}

for some $\delta \in G$.

The inclusions (5) and (10) show that the conjugacy class $C$ of $s$ in $G$ intersects all Bruhat cells $B \hat{w}'' B$, where $w''$ runs through all generalized $S$-Coxeter elements that are conjugate to $w$. Now let $\tilde{w} \in W$ be an element from the conjugacy class of $w$. Proposition 3.3 implies that there exists an $S$-ascent from some generalized $S$-Coxeter element $w''$ to $\tilde{w}$. Now the assertion of the theorem follows from Proposition 3.4. \(\Box\)

Theorem 1.3 has been proved.

Remarks to Theorem 1.3.

1. Intersection with a parabolic subgroup. In the proof of Theorem 1.3 we showed that

\begin{align}
C \cap P_X \neq \emptyset
\end{align}

for every noncentral semisimple conjugacy class $C$ that is not regular, where $X \not\subseteq \Pi$ and $P_X = BW_X B$ is the corresponding parabolic subgroup (if $K$ is a perfect field and $\dim K \leq 1$). More generally, Equation (**) holds for every noncentral conjugacy class $C$ that is not a regular semisimple class (if $K$ is a perfect field and $\dim K \leq 1$). Indeed, we consider the Jordan decomposition $g = su$ of an element $g \in C$. Applying the same construction as in Lemma 4.6, we get a parabolic subgroup $P$ which is defined over $K$ and contains $s, u$. Then by an appropriate conjugation we can embed $g$ in
2. The condition: \( \dim K \leq 1 \). The example below shows that if this condition does not hold, the conclusion of Theorem 1.3 may be false.

Let \( n = 4k \) and let \( V \) be a linear space over the real number field \( \mathbb{R} \) such that \( \dim V = 4k \). Further, let \( \{e_1, \ldots, e_{4k}\} \) be a fixed basis of \( V \) and let \( V^+ = \langle e_1, \ldots, e_{2k} \rangle \), \( V^- = \langle e_{2k+1}, \ldots, e_{4k} \rangle \). Further, let \( (x_1, \ldots, x_{4k}) \) be the coordinates of an element in \( V \) with respect to the basis \( \{e_i\} \) and let \( \Phi = x_1^2 + \cdots + x_{2k}^2 - x_{2k+1}^2 - \cdots - x_{4k}^2 \). Let \( \Omega = \Omega(V, \Phi) = [SO(V, \Phi), SO(V, \Phi)] \).

Then \( \Omega \) is a Chevalley group in the sense of [St2], corresponding to the root system \( D_{2k} \).

Let \( g \in GL(V) \) be the linear operator such that \( g|_{V^+} = -1, g|_{V^-} = 1 \). One can easily check that \( g \in \Omega \) and \( gug^{-1} \neq u^{-1} \) for every nontrivial unipotent element \( u \in \Omega \) (the latter follows from the fact that \( v \pm g(v) \) is not an isotropic vector if \( v \neq 0 \) is isotropic).

Hence the element \( g \) cannot normalize any nontrivial unipotent subgroup of \( \Omega \) and therefore \( g \) cannot belong to any proper parabolic subgroup of \( \Omega \). This implies that a preimage \( \hat{g} \) of \( g \in G = \text{Spin}_{4k}(\mathbb{R}) \) (with respect to the natural homomorphism \( G \longrightarrow \Omega \)) also cannot belong to a proper parabolic subgroup of \( G \). Hence \( C \cap Bw_XB = \emptyset \) for every \( X \subset \Pi \), where \( C \) is the conjugacy class of \( \hat{g} \) in \( G \), \( B \) is a Borel subgroup of \( G \), and \( \Pi \) is a simple root system corresponding to \( \hat{G} = \text{Spin}_{4k} \) (note, \( BW_XB = P_X \) is a standard parabolic subgroup).

3. The ordered set of \( \mathcal{X}_C \). Recall, for any set \( X \subset \Pi \) we define \( w_X = \prod_{\alpha \in X} w_\alpha \), where the product can be taken in any fixed order. For the set

\[
\mathcal{X}_C = \{X \subset \Pi \mid C \cap Bw_XB \neq \emptyset\}
\]

one can consider the natural order with respect to inclusion.

Let \( G = SL_n(\mathbb{C}) \) and \( C \) a noncentral semisimple conjugacy class. Let \( \lambda(C) = (\lambda_1, \ldots, \lambda_r) \) be the partition of \( n \), i.e., \( \lambda_1 \geq \cdots \geq \lambda_r \), where \( \lambda_1 + \cdots + \lambda_r = n \), which corresponds to the multiplicities of eigenvalues of elements of \( C \) (i.e., \( \lambda_1 \) is the biggest multiplicity, then \( \lambda_2 \), etc.) and let \( \lambda^*(C) \) be the dual partition (i.e., the rows and columns of \( \lambda \) are interchanged). Further, to every partition \( \mu = (\mu_1, \ldots, \mu_s) \) of \( n \) we assign a subset \( X(\mu) \subset \Pi = \{\alpha_1, \ldots, \alpha_{n-1}\} \), namely,

\[
X(\mu) \overset{\text{def}}{=} \Pi \setminus \{\alpha_{\mu_1}, \alpha_{\mu_1+\mu_2}, \ldots, \alpha_{\mu_1+\cdots+\mu_{s-1}}\}.
\]

It is easy to see that \( X(\lambda^*(C)) \) is a maximal element of \( \mathcal{X}_C \). Moreover, every maximal element \( Y \in \mathcal{X}_C \) is \( W \)-conjugate to \( X(\lambda^*(C)) \). Thus we have just one conjugacy class \( \{ww_Xw^{-1}\} \) in \( W \) for each maximal \( X \in \mathcal{X}_C \).

For other types of groups we can have several conjugacy classes in \( W \) of elements of the form \( w_X \), where \( X \in \mathcal{X}_C \) is a maximal element. Say, consider
the root system $R = B_2 = \langle \alpha_1, \alpha_2 \rangle$, where $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2$ (in the notation of [Bou]), and let $G$ be the corresponding simple and simply connected group over $\mathbb{C}$. Let $g = h_{\varepsilon_1}(t)h_{\varepsilon_2}(t^{-1}) \in G$ be a semisimple element, where $h_{\varepsilon_1}(t), h_{\varepsilon_2}(t^{-1})$ are the corresponding root semisimple elements (in the notation of [St2]) and $t \neq \pm 1$. Let $C$ be the conjugacy class of $g$. Then $\{\alpha_1\}$ and $\{\alpha_2\}$ both are maximal elements of $X_C$. Thus here we have two different conjugacy classes in $W$ of elements $w_X$ for maximal $X$ in $X_C$.

5. Appendix.

The following result as well as the line of proof was pointed out to the second author by T.A. Springer in the discussion of relevant questions:

**Proposition 5.1.** Let $\tilde{G}$ be a simple algebraic group defined over an algebraically closed field $K$ and let $G = \tilde{G}(K)$. Further, let $C$ be the conjugacy class of a regular element of $G$. Then $C \cap B \tilde{w}B \neq \emptyset$ for every $w \in W$.

**Proof.** For $b \in B$ we put $O_B(b) = \{xbx^{-1} \mid x \in B\}$.

**Lemma 5.2.** There exists a nonempty finite set $\{b_1, \ldots, b_n\} \subset C \cap B$ such that $C \cap B = \bigcup_{1 \leq i \leq n} O_B(b_i)$.

**Proof.** Let $x = s_1u_1, y = s_2u_2 \in B$ be two regular elements, where $s_1, s_2 \in T$ and $u_1, u_2 \in U$. We show

(11) $O_B(x) = O_B(y)$ if and only if $s_1 = s_2$.

Indeed, “only if” is obvious. Now let

(12) $b_1 = su_1, b_2 = su_2, s \in T, u_1, u_2 \in U$.

Since we can consider the Jordan decompositions of $x, y$ as elements of $B$, we may assume that (12) gives the Jordan decompositions of $b_1$ and $b_2$. Put $\Gamma = [C_G(s), C_G(s)], B_\Gamma = B \cap \Gamma$. Then (12) implies $u_1, u_2 \in B_\Gamma$. Moreover, the elements $u_1, u_2$ are regular unipotent elements of $\Gamma$ ([St1, 3.7]) and therefore the elements $u_1, u_2$ are conjugate in $B_\Gamma$ (see [C2, the proof of Proposition 5.1.3]). Hence we have (11).

Now let $b = su \in B, s \in T, u \in U, g \in G, gb^{-1} \in B$. Further, let $g \in B \tilde{w}B$. Then $gb^{-1} = w(s)u'$ for some $u' \in U$. Together with (11), this implies our assertion. □

**Lemma 5.3.** Let $b \in C \cap B$ be a fixed element and let $w \in W$. Then every irreducible component $C_w$ of $\overline{C} \cap \overline{B \tilde{w}B}$ such that $O_B(b) \subset C_w$ satisfies the following condition:

$\dim C_w = \dim \overline{C} + \dim \overline{B \tilde{w}B} - \dim G$. 

Proof. Since $b$ is a regular element, $\dim C_B(b) = \operatorname{rank} G$ ([St1, 3.11]). If $C_1$ is an irreducible component of $C \cap B$ containing $O_B(b)$, then Lemma 5.2 implies $C_1 = O_B(b)$ and, therefore,

(13) $\dim C_1 = \dim B - \operatorname{rank} G = \dim C + \dim B - \dim G.$

Let $O_B(b) \subset C_w$ for some irreducible component $C_w$ of $C \cap B\w' B$. Suppose $\dim C_w > \dim C + \dim B\w' B - \dim G.$

(14)

Since $B$ is a closed subset of $B\w' B$ ([Sp, 8.15]) and $C_1$ is an irreducible component of $C_w \cap B$, we have

(15) $\dim C_1 \geq \dim C_w + \dim B - \dim B\w' B.$

Now (14) and (15) contradict (13). Thus we have our statement. □

Now we return to the proof of Proposition 5.1.

Take $C_w$ as in Lemma 5.3. Assume $C \cap B\w' B = \emptyset$. Then

(16) $C_w \subset \bigcup_{w' < w} Bw'B = \bigcup_{w' < w} Bw'B$ ([Sp, 8.15]). From (16) we have $C_w \subset Bw' B$ for some $w' < w$ and we may consider $C_w$ as an irreducible component of $C \cap Bw' B$ that contains $O_B(b)$. Then, by Lemma 5.3, we have

(17) $\dim C_w = \dim C + \dim Bw' B - \dim G.$

But (17) contradicts Lemma 5.3 because $\dim Bw' B < \dim B\w' B$. This proves Proposition 5.1. □

References


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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TORONTO
100 ST. GEORGE STREET, TORONTO
ONTARIO M5S 3G3
CANADA
E-mail address: ellers@math.toronto.edu

RUSSIAN STATE PEDAGOGICAL UNIVERSITY
MOLKA 48
ST. PETERSBURG 191-186
RUSSIA
E-mail address: nickgordeev@mail.ru