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UNITARY REPRESENTATIONS OF CLASSICAL LIE GROUPS OF EQUAL RANK WITH NONZERO DIRAC COHOMOLOGY

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In this paper, we consider unitary representations of classical groups of equal rank (rank$G$ = rank$K$) except type CI with regular lambda-lowest K-type and get the necessary and sufficient condition such that those unitary representations considered have nonzero Dirac cohomology.

1. Introduction.

In the past twenty years, people are interested in unitary representations with nonzero cohomologies, that is, $(\mathfrak{g}, K)$-cohomology and Dirac cohomology. The former was studied by Vogan and Zuckerman in [**10**]. Since every representation with nonzero $(\mathfrak{g}, K)$-cohomology has nonzero Dirac cohomology, maybe it is this fact that motivates people to pay more attention to Dirac cohomology.

In 1997, Vogan explained a conjecture on Dirac cohomology at MIT Lie groups seminar. The conjecture can be stated as follows: Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}_0$ and let K be the maximal compact subgroup of G corresponding to the Cartan involution θ. Suppose X is an irreducible unitarizable $(\mathfrak{g}, K)$-module and (γ, S) is a space of spinors for $\mathfrak{p}_0$. Here $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ is the Cartan decomposition of $\mathfrak{g}_0$. Let $x_1, \ldots, x_n$ be an orthonormal basis of $\mathfrak{p}_0$, then the Dirac operator $D = \sum \pi(x_i) \otimes \gamma(x_i)$ acts on $X \otimes S$. Vogan's conjecture says that if D has nonzero Dirac cohomology, which by definition is just $\operatorname{Ker} D$, then the infinitesimal character of X can be expressed in terms of the highest weight of a K-type of X.

The conjecture was proved by Huang and Pandžić [**2**]. Furthermore, they get that an irreducible unitarizable $(\mathfrak{g}, K)$-module X has nonzero Dirac cohomology, say $\gamma \subseteq \operatorname{Ker} D$, if and only if the infinitesimal character Λ of X is given by $\gamma + \rho_c$. To be precise, γ has highest weight $\omega(\mu - \rho_n)$, where μ is a K-type of X, $\omega \in W(K)$ such that $\omega(\mu - \rho_n)$ is dominant and $\Lambda = \omega(\mu - \rho_n) + \rho_c$. One could ask: For what kinds of K-types does the expression $\|\omega(\mu - \rho_n) + \rho_c\|$ reach the minimum? For what cases is μ a lambda-lowest K-type of X when $\omega(\mu - \rho_n) \subset \operatorname{Ker} D$?

In this paper, we will answer the above problems partially. We study the representations of classical group G of equal rank except type CI, with regular lambda-lowest K-type. First we recall the definition of θ-stable data.

Definition 1.1 (Vogan [**8**], Definition 6.5.1)**.** A set of **θ-stable data for** G is a quadruple $(\mathfrak{q}, H, \delta, \nu)$, such that:

a) $\mathfrak{q} = \mathfrak{l}+\mathfrak{u}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$. Let L be the normalizer of $\mathfrak{q}$ in G.
b) L is quasisplit, and $H = TA \subseteq L$ is a maximally split θ-stable Cartan subgroup of L.
c) $\delta \in \hat{T}$ is fine with respect to L.
d) $\nu \in \hat{A}$.
e) Write $\lambda^L \in \mathfrak{t}^*$ for the differential of δ, and $\lambda = \lambda^L + \rho(\Delta(\mathfrak{u}, \mathfrak{t})) \in \mathfrak{t}^* \subseteq \mathfrak{h}^*$. Then λ is strictly dominant for $\Delta(\mathfrak{u}, \mathfrak{h})$.

There is a surjective map from the set of equivalence classes of irreducible $(\mathfrak{g}, K)$-module to K conjugacy classes of set of θ-stable data for G ([**8**], Corollary 6.5.13). And following Vogan's method ([**8**], Chapter 5), one can construct θ-stable data from any given irreducible $(\mathfrak{g}, K)$-module X.

Now we can state our main theorem.

Theorem 1.2. *Let X be an irreducible $(\mathfrak{g}, K)$-module with regular lambda-lowest K-type μ. Then X is unitary and has nonzero Dirac cohomology if and only if the parameter ν in the θ-stable data $(\mathfrak{q}, H, \delta, \nu)$ corresponding to X is just $\frac{1}{2}\sum_{\beta_i \in \Gamma_1} \beta_i$ under G-conjugation. Here, Γ_1 is a set of roots defined by the lambda-lowest K-type μ during the construction of θ-stable data (see Section* 3.1 *for details).*

The paper is organized as follows: We first collected some notations and results on Dirac operator and Dirac cohomology in Section 2. Then we followed Vogan's method to construct θ-stable data $(\mathfrak{q}, H, \delta, \nu)$ for corresponding $(\mathfrak{g}, K)$-module X. Actually, we found that the quasisplit subgroup L is simple enough under our assumption. Locally L is a product of copies of $SL(2, \mathbb{R})$ and Euclidean space. In Section 4, we find out that if a lambda-lowest K-type μ of X is regular, then $\mu - \rho_n$ is dominant (Proposition 4.2) and $\|\mu - \rho_n + \rho_c\| \leq \|\omega(\mu' - \rho_n') + \rho_c\|$. Then X has nonzero Dirac cohomology only if $\|\Lambda\| = \|\mu - \rho_n + \rho_c\|$. Fortunately, in this case, Λ is dominant. Then Vogan's result, Theorem 1.3 [**9**], implies that X is unitary, hence X has nonzero Dirac cohomology by Huang and Pandžić's result (Proposition 2.4) since $\|\Lambda\| = \|\mu - \rho_n + \rho_c\|$. Thus we get the main theorem.

2. Preliminary.

Let G be a real semisimple group with Lie algebra $\mathfrak{g}_0$ and let K be the maximal compact subgroup of G corresponding to Cartan involution θ. Let

$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition of $\mathfrak{g}_0$. Fix a maximally compact Cartan subalgebra $\mathfrak{h}_0^c$ of $\mathfrak{g}_0$ with decomposition $\mathfrak{h}_0^c = \mathfrak{t}_0^c + \mathfrak{a}_0^c$. Denote by be $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{h}^c$, $\mathfrak{t}^c$ and $\mathfrak{a}^c$ the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{h}_0^c$, $\mathfrak{t}_0^c$ and $\mathfrak{a}_0^c$, respectively. Let $\Delta(\mathfrak{g}, \mathfrak{h}^c)$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}^c$. Fix a system of positive roots, $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$, for $\Delta(\mathfrak{k}, \mathfrak{t}^c)$ and choose a compatible system of positive roots, $\Delta^+(\mathfrak{g}, \mathfrak{h}^c)$, for $\Delta(\mathfrak{g}, \mathfrak{h}^c)$ with the set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_l\}$. Let G_0 be the identity component of G.

Definition 2.1 ([**11**])**.** Let (π, X) be a $(\mathfrak{g}, K)$-module, set $S = S(\mathfrak{p}_0)$, a space of spinors of $\mathfrak{p}_0$. Let $x_1, \ldots, x_n$ be an orthonormal basis of $\mathfrak{p}_0$, then the **Dirac operator**

$$D : X \bigotimes S \to X \bigotimes S$$

is defined by

$$D = \sum \pi(x_i) \otimes \gamma(x_i),$$

which is a K-module homomorphism (sometime $\widetilde{K}$-module homomorphism, where $\widetilde{K}$ is a two-fold spin cover of K).

The **Dirac cohomology** of X is defined by

$$\operatorname{Ker} D / (\operatorname{Ker} D \cap \operatorname{Im} D).$$

When X is unitary, then Dirac operator is self-dual, then we can see that the Dirac cohomology of X is just $\operatorname{Ker} D$.

The following result of Pathasarathy is well-known. It can be found in many papers.

Proposition 2.2 (Pathasarathy's Dirac Inequality)**.** *Let X be an irreducible unitary $(\mathfrak{g}, K)$-module with infinitesimal character Λ. Fix a representation of K occurring in X of highest weight $\mu \in (\mathfrak{t}^c)^*$, and a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{t}^c)$ for $\mathfrak{t}^c$ in $\mathfrak{g}$. Here $\mathfrak{t}^c$ is Cartan subalgebra of $\mathfrak{k}$. Write*

$$\rho_c = \rho(\Delta^+(\mathfrak{k}, \mathfrak{t})), \quad \rho_n = \rho(\Delta^+(\mathfrak{p}, \mathfrak{t})).$$

Fix an element $\omega \in W_K$ such that $\omega(\mu - \rho_n)$ is dominant for $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Then

$$(\omega(\mu - \rho_n) + \rho_c, \omega(\mu - \rho_n) + \rho_c) \geq (\Lambda, \Lambda).$$

The equality holds if and only if

$$\Lambda = \omega(\mu - \rho_n) + \rho_c.$$

The last assertion was obtained by Huang and Pandžić [**2**].

We also have another similar inequality.

Proposition 2.3. *Let V be an irreducible unitary $(\mathfrak{g}, K)$-module with Hermitian form $\langle , \rangle$ and infinitesimal character Λ. Assume $\mu \in \hat{K}$ occurs in V. Then*

$$\|\Lambda\|^2 \leq \|\mu + \rho_c\|^2 - \|\rho_c\|^2 + \|\rho\|^2.$$

Proof. Let $\{x_i\}$ be a orthonormal basis of $\mathfrak{p}$ with respect to the Killing form. For $v \in V_\mu$ we have

$$\langle x_i v, x_i v\rangle \geq 0 \Rightarrow \langle x_i^2 v, v\rangle \leq 0 \Rightarrow \langle (c - c_{\mathfrak{k}})v, v\rangle \leq 0.$$

Then the assertion follows easily. □

In 1997, Vogan explained a conjecture on Dirac cohomology, which was proved by Huang and Pandžić [**2**]. We summarize their results as follows:

Proposition 2.4 ([**2**]). *Let X be an irreducible unitarizable $(\mathfrak{g}, K)$-module with infinitesimal character Λ. Assume $X \otimes S$ contains a $\widetilde{K}$-type γ, i.e., $(X \otimes S)(\gamma) \neq 0$. Then the Dirac cohomology of X,* $\operatorname{Ker} D$*, contains $(X \otimes S)(\gamma)$ if and only if $\Lambda = \gamma + \rho_c$. Here γ must be of the form $\omega(\mu - \rho_n)$ for some ρ_n and K-type μ contained in X.*

3. Construction of θ-stable data.

In this section, we will make the following assumption:

Assumption 3.1. G is a classical group with $\operatorname{rank} G = \operatorname{rank} K$, i.e., θ is an inner automorphism of $\mathfrak{g}_0$. Consequently $\mathfrak{h}_0^c = \mathfrak{t}_0^c$.

We will follow Vogan's method to construct θ-stable data, actually, the main work is to determine the structure of the quasisplit subgroup L.

3.1. Basic facts. First, we rewrite Proposition 5.3.3 [**8**], since we assume $\operatorname{rank} G = \operatorname{rank} K$ and $\mathfrak{h} = \mathfrak{t}^c$.

Proposition 3.2 ([**8**]). *For each $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$-dominant weight $\mu \in \hat{T}$, there is a unique element $\lambda \in (\mathfrak{t}^c)^*$ having the following properties: Fix a θ-invariant positive root system $\Delta^+(\mathfrak{g}, \mathfrak{t}^c)$ for $\mathfrak{t}^c$ in $\mathfrak{g}$, making $\mu + 2\rho_c$ dominant; and write $\rho = \rho(\Delta^+(\mathfrak{g}, \mathfrak{t}^c))$. Then λ is dominant for $\Delta^+(\mathfrak{g}, \mathfrak{t}^c)$, and there is a set*

$$\Gamma = \{\beta_1, \ldots, \beta_r\} \subseteq \Delta^+(\mathfrak{g}, \mathfrak{t}^c)$$

satisfying:

a) *If we put*

$$\widetilde{\lambda} = \mu + 2\rho_c - \rho,$$
$$c_i = -(\widetilde{\lambda}, \beta_i^\vee),$$

then

$$0 \leq c_i \leq 1,$$

and

$$\lambda = \widetilde{\lambda} + \frac{1}{2}\sum c_i \beta_i.$$

b) *If $(\lambda, \alpha) = 0$ for $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}^c)$, then $(\alpha, \beta_i) \neq 0$ for some i.*

c) *The root β_1 is noncompact and simple.*

d) *Write*

$$\mathfrak{g}^1 = \mathfrak{g}^{\beta_1}, \quad \mathfrak{h}^1 = (\mathfrak{t}^c)^{\beta_1}.$$

Then the positive system $\Delta^+(\mathfrak{g}, \mathfrak{t}^c) \cap \beta_1^{\perp}$ *and its subset* $\{\beta_2, \ldots, \beta_r\}$ *for* $\Delta(\mathfrak{g}^1, \mathfrak{h}^1)$ *satisfy these same conditions for* $\mathfrak{g}^1$ *and the weight* $\mu|_{\mathfrak{g}^1 \cap \mathfrak{t}^c}$.

e) *If* $c_i \neq 0$ *and* $c_j = 0$, *then* $i < j$.

Under Assumption 3.1, we can get a stronger result.

Lemma 3.3. *Let the notation be as above. Then*

$$c_i = 0 \text{ or } 1.$$

Proof. By Lemma 7.7.6 [**1**], we have

$$\exp(2\pi\sqrt{-1}\ \alpha^{\vee}) = e,$$

where e is the unit of G. Then $(\mu, \alpha^{\vee})$ is an integer. □

For convenience, we denote

$$\Gamma_1 = \{\beta_i \in \Gamma | c_i = 1\},$$
$$\Gamma_0 = \{\beta_i \in \Gamma | c_i = 0\}.$$

Let Π be the system of simple roots of $\Delta^+(\mathfrak{g}, \mathfrak{t}^c)$. Set

$$\Sigma_1 = \{\alpha \in \Pi | (\widetilde{\lambda}, \alpha^{\vee}) = -1\},$$
$$\Sigma_0 = \{\alpha \in \Pi | (\widetilde{\lambda}, \alpha^{\vee}) = 0\}.$$

Now we can define $\mathfrak{l}$ by

$$\Delta(\mathfrak{l}, \mathfrak{t}^c) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}^c) | (\lambda, \alpha^{\vee}) = 0\}.$$

Obviously, the Dynkin diagram of $\mathfrak{l}$ is a subdiagram of that of $\mathfrak{g}$ if we choose compatible orderings, i.e.,

$$\Delta^+(\mathfrak{l}, \mathfrak{t}^c) \subseteq \Delta^+(\mathfrak{g}, \mathfrak{t}^c).$$

Denote by $\Pi_{\mathfrak{l}}$ the system of simple roots of $\mathfrak{l}$.

First we establish some lemmas.

Lemma 3.4. *Let* α *and* β *be adjacent simple roots of the same length. Then*

$$(\widetilde{\lambda}, (\alpha + \beta)^{\vee}) \geq 0.$$

Proof. If both α and β are compact or noncompact, then $\alpha + \beta$ is compact, so

$$(\widetilde{\lambda}, (\alpha + \beta)^{\vee}) \geq (\mu, (\alpha + \beta)^{\vee}) \geq 0.$$

Thus we can assume α is compact and β is noncompact. Then

$$(\widetilde{\lambda}, \beta^{\vee}) \geq -1$$

and

$$(\widetilde{\lambda}, \alpha^{\vee}) = (\mu, \alpha^{\vee}) + 1.$$

So

$$(\widetilde{\lambda}, (\alpha+\beta)^\vee) \geq (\mu, \alpha^\vee) \geq 0.$$

□

Lemma 3.5. *Let α, β and $\alpha+\beta \in \Delta(\mathfrak{g}, \mathfrak{t}^c)$. If $(\widetilde{\lambda}, \alpha^\vee) \geq 0$ and $(\widetilde{\lambda}, \beta^\vee) \geq 0$, then*

$$(\widetilde{\lambda}, (\alpha+\beta)^\vee) \geq 0.$$

Proof. $(\alpha+\beta)^\vee = a\alpha^\vee + b\beta^\vee$, where a and b are positive. □

Lemma 3.6. *Assume μ is regular, i.e., $(\mu, \gamma^\vee) \geq 1$, for all $\gamma \in \Delta^+(\mathfrak{k}, \mathfrak{t}^c)$. Let α and β be adjacent simple roots. If $(\alpha, \alpha) = 2(\beta, \beta)$, then*

$$(\widetilde{\lambda}, (\alpha+\beta)^\vee) \geq 0.$$

If α and β have the same length, then

$$(\widetilde{\lambda}, (\alpha+\beta)^\vee) \geq 1.$$

Proof. Only the first assertion needs to prove. We treat it case by case.

Case I. Both α and β are noncompact.

$$(\widetilde{\lambda}, (\alpha+\beta)^\vee) = (\mu, (\alpha+\beta)^\vee) + 2 - (\rho, 2\alpha^\vee + \beta^\vee) \geq 1 + 2 - 3 \geq 0.$$

Case II. α is compact while β is noncompact.

$$(\widetilde{\lambda}, (\alpha+\beta)^\vee) = (\widetilde{\lambda}, 2\alpha^\vee + \beta^\vee) \geq 4 - 1 \geq 3.$$

Case III. α is noncompact while β is compact.

$$(\widetilde{\lambda}, (\alpha+\beta)^\vee) = (\widetilde{\lambda}, 2\alpha^\vee + \beta^\vee) \geq -2 + 2 \geq 0.$$

□

Corollary 3.7. *Let $\alpha \in \Sigma_1$, $\beta \in \Sigma_1 \cup \Sigma_0$. Then $(\alpha, \beta) = 0$ for types AIII and D_l. If μ is regular for $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$, then it is true for any type.*

Lemma 3.8. *Assume μ is regular for $\Delta(\mathfrak{k}, \mathfrak{t}^c)$ and Γ consists of simple roots of $\Delta(\mathfrak{g}, \mathfrak{t}^c)$. Then the simple roots of $\mathfrak{l}$ are noncompact.*

Proof. If $\alpha \in \Pi$ is compact, then

$$(\lambda, \alpha^\vee) = (\widetilde{\lambda}, \alpha^\vee) + \frac{1}{2}\sum c_i(\beta_i, \alpha^\vee) \geq 2 - \frac{3}{2} > 0.$$

The first inequality holds because α is adjacent to at most three simple roots of the same length or two simple roots of different length. So $\alpha \notin \Pi_{\mathfrak{l}}$. □

3.2. Main theorem. Now we can study the structure of $\mathfrak{l}$. Our purpose is to prove the following theorem:

Theorem 3.9. *Let μ be a K-type of a $(\mathfrak{g}, K)$-module. Assume μ is regular for $\Delta(\mathfrak{k}, \mathfrak{t}^c)$. Then:*

1) *For types AIII , CII and D_l, the Dynkin diagram of $\mathfrak{l}$ is discrete.*
2) *For types B_l and CI, the Dynkin diagram of $\mathfrak{l}$ is either discrete or of the form*
$$\underbrace{A_1 \times \cdots \times A_1}_{r-2} \times B_2.$$
3) *For type B_l. If μ is regular for $\Delta(\mathfrak{g}, \mathfrak{t}^c)$, then the Dynkin diagram of $\mathfrak{l}$ is discrete.*

Let's deal with the problem case by case.

3.2.1. Type *AIII* **.** In this subsection, we assume that the Lie algebra $\mathfrak{g}_0$ is of type *AIII* .

Proposition 3.10.

1) *Let $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{t}^c)$. If $(\alpha, \beta) = 0$ for any $\beta \in \Sigma_1$, then*
$$(\widetilde{\lambda}, \alpha^\vee) \geq 0.$$
2) *Those β_i in Proposition 3.2 can be chosen to be simple.*
3) *If μ is regular for $\Delta(\mathfrak{k}, \mathfrak{t}^c)$, then*
$$\Pi_{\mathfrak{l}} = \Gamma.$$

Proof.
1) Let $\alpha = \alpha_i + \cdots + \alpha_k$. If α is not adjacent to any $\beta \in \Sigma_1$, then α_i, $\alpha_k \notin \Sigma_1$, so $(\widetilde{\lambda}, \alpha^\vee) \geq 0$ by Lemma 3.4.
2) Choose a maximal subset Σ_0' of Σ_0 such that the elements of Σ_0' are orthogonal to each other. Then we claim that the set $\Gamma = \Sigma_1 \cup \Sigma_0'$ satisfies the condition of Proposition 3.2.

Firstly, we choose Γ_1 containing Σ_1. By 1) we have
$$\Gamma_1 = \Sigma_1.$$

Secondly, we choose Γ_0 containing Σ_0'. If $\alpha = \alpha_i + \cdots + \alpha_k$ is orthogonal to $\Sigma_1 \cup \Sigma_0'$ and
$$(\widetilde{\lambda}, \alpha^\vee) = 0, \tag{1}$$
then α_i, $\alpha_k \notin \Sigma_1 \cup \Sigma_0'$. We claim that $\alpha_i, \ldots, \alpha_k \in \Sigma_0$. By Lemma 3.4, we have $(\widetilde{\lambda}, (\alpha_{i+1} + \cdots + \alpha_k)^\vee) \geq 0$. The equality holds and $\alpha_i \in \Sigma_0$ by Equation (1). Furthermore, for the same reason we have $(\widetilde{\lambda}, (\alpha_i + \alpha_{i+1})^\vee) = 0$, that is $\alpha_{i+1} \in \Sigma_0$. Then our claim follows. But one can easily see that the claim contradicts the fact that Σ_0' is maximal.

3) Obviously, $\Gamma \subseteq \Pi_{\mathfrak{l}}$. Let $\alpha \in \Pi_{\mathfrak{l}} \backslash \Gamma$. By Lemma 3.8, α is noncompact. Then α must be adjacent to some $\beta \in \Gamma$, so $(\widetilde{\lambda}, (\alpha+\beta)^\vee) \geq 1$. Hence $(\lambda, \alpha^\vee) > 0$. Contradiction. □

Corollary 3.11. *If μ is regular for $\Delta(\mathfrak{g}, \mathfrak{t}^c)$, then $\widetilde{\lambda}$ is strictly dominant for $\Delta(\mathfrak{u})$, that is,*

$$(\widetilde{\lambda}, \alpha^\vee) > 0$$

for any $\alpha \in \Delta(\mathfrak{u})$.

Proof. Just follow the proof of the above proposition. □

3.2.2. Types B_l and C_l.

Proposition 3.12. *Assume μ is regular for $\Delta(\mathfrak{k}, \mathfrak{t}^c)$. If $(\alpha, \beta) = 0$ for any $\beta \in \Sigma_1$, then*

$$(\widetilde{\lambda}, \alpha^\vee) \geq 0.$$

Proof. First assume $\mathfrak{g}$ is of type B_l. Let $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}^c)$. Assume $(\alpha, \beta) = 0$ for any $\beta \in \Sigma_1$. If $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_k$, then $\alpha_i \notin \Sigma_1$. Similar to the proof of type *AIII* , one can get $(\widetilde{\lambda}, \alpha^\vee) \geq 0$.

Now we assume $\alpha = \alpha_i + \cdots + \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_l$, so $\alpha_k \notin \Sigma_1$. If $\alpha_l \notin \Sigma_1$, then we have

$$(\widetilde{\lambda}, (\alpha_i + \cdots + \alpha_l)^\vee) \geq 0$$

and

$$(\widetilde{\lambda}, (\alpha_k + \cdots + \alpha_l)^\vee) \geq 0$$

by Lemmas 3.5 and 3.6. Hence $(\widetilde{\lambda}, \alpha^\vee) \geq 0$.

If $\alpha_i \notin \Sigma_1$, the proof is similar. So we just need to check the case that α_i, $\alpha_l \in \Sigma_1$. Obviously $i + 1 = k < l$ and $\alpha_{l-1} \notin \Sigma_1$. We have

$$(\widetilde{\lambda}, \alpha^\vee) = (\widetilde{\lambda}, (\alpha_i + \cdots + \alpha_{l-1})^\vee + \alpha_l^\vee) \geq 0.$$

This completes the proof for type B_l. And the proof for type C_l is similar. □

This Proposition tells us that those β_i, which satisfy $(\widetilde{\lambda}, \beta_i^\vee) < 0$, can be chosen to be simple, that is, $\Gamma_1 = \Sigma_1$. Then we get the element $\lambda = \widetilde{\lambda} + \frac{1}{2}\sum_{\beta_i \in \Gamma_1} \beta_i$.

Lemma 3.13. *The simple roots of $\mathfrak{l}$ are noncompact simple roots of $\Delta(\mathfrak{g}, \mathfrak{t}^c)$.*

Proof. If α is compact simple, then

$$(\lambda, \alpha^\vee) = (\widetilde{\lambda}, \alpha^\vee) + \frac{1}{2}\sum c_i(\beta_i, \alpha^\vee) \geq 2 - \frac{3}{2} > 0.$$

The first inequality follows from that α is adjacent to at most three simple roots of the same length or two simple roots. So $\alpha \notin \Pi_{\mathfrak{l}}$. □

$\alpha_1, \dots, \alpha_{l-1}$ generate a subsystem of type A_{l-1}. Let $\Pi'_{\mathfrak{l}} = \Delta(\mathfrak{l}, \mathfrak{t}^c) \cap \{\alpha_1, \dots, \alpha_{l-1}\}$. Then we have:

Lemma 3.14. *Let $\beta_j \in \Pi'_{\mathfrak{l}} \cap \Sigma_1$ and $\alpha \in \Pi'_{\mathfrak{l}}$. Then $(\beta_j, \alpha^\vee) = 0$.*

Proof. If α is adjacent to β, then $\alpha + \beta$ is compact and we have

$$\begin{aligned}(\lambda, (\alpha+\beta_j)^\vee) &= (\widetilde{\lambda}, (\alpha+\beta_j)^\vee) + \frac{1}{2}\sum c_i(\beta_i, \beta_j^\vee) + \frac{1}{2}\sum c_i(\beta_i, \alpha^\vee)\\ &\geq 1 + c_j - \frac{1}{2}c_j - 1 = \frac{1}{2}.\end{aligned}$$

This leads to a contradiction. □

Corollary 3.15. *The Dynkin diagram of $\Pi'_{\mathfrak{l}}$ is discrete.*

Proof. If it is not true, then there exist two adjacent noncompact simple roots α, $\beta \in \Pi'_{\mathfrak{l}}$. By the above Lemma, neither of them is adjacent to some $\beta_i \in \Pi'_{\mathfrak{l}} \cap \Sigma_1$. Then $\alpha + \beta$ is compact.

Case I. $\alpha_l \notin \Sigma_1$.

$$(\lambda, (\alpha+\beta)^\vee) = (\widetilde{\lambda}, (\alpha+\beta)^\vee) \geq 1 + 2 - 2 = 1. \tag{2}$$

Case II. $\alpha_l \in \Sigma_1$.

1) If $\mathfrak{g}$ is of type C_l, then $\alpha_{l-1} \notin \Pi_{\mathfrak{l}}$ by the following Lemma 3.16. The inequality (2) is also correct.
2) If $\mathfrak{g}$ is of type B_l, then

$$(\lambda, (\alpha+\beta)^\vee) \geq (\widetilde{\lambda}, (\alpha+\beta)^\vee) + \frac{1}{2}(\alpha_l, (\alpha+\beta)^\vee) \geq 1 + 2 - 2 - \frac{1}{2} = \frac{1}{2}.$$

Thus for all the cases, we have $(\lambda, (\alpha+\beta)^\vee) > 0$. Contradiction. □

α_{l-1} and α_l generate a subsystem of type $B_2 = \langle \alpha, \beta \rangle$, where α is the long root.

Lemma 3.16. *Let the notation be as above.*

1) *If $\alpha \in \Sigma_1$, then $(\lambda, \beta^\vee) > 0$, i.e., $\beta \notin \Pi_{\mathfrak{l}}$.*
2) *If $\beta \in \Sigma_1$, then $(\lambda, \alpha^\vee) > 0$, i.e., $\beta \notin \Pi_{\mathfrak{l}}$.*

Proof. Thanks to Lemma 3.13, we can assume that β is noncompact. Then $\alpha + \beta$ is compact and

$$(\widetilde{\lambda}, (\alpha+\beta)^\vee) \geq 1 + 2 - (\rho, 2\alpha^\vee + \beta^\vee) = 0.$$

1) $\alpha \in \Sigma_1$. Then $(\widetilde{\lambda}, \beta^\vee) \geq 2c$. For type B_l,

$$(\lambda, \beta^\vee) \geq 2 + \frac{1}{2}(\alpha, \beta^\vee) = 1.$$

For type C_l,

$$(\lambda, \beta^\vee) \geq 2 + \frac{1}{2}(\alpha, \beta^\vee) + \frac{1}{2}(\alpha_{l-2}, \beta^\vee) = \frac{1}{2}.$$

2) $\beta \in \Sigma_1$. Then $(\widetilde{\lambda}, \beta^\vee) = -1$. Since

$$(\widetilde{\lambda}, (\alpha+\beta)^\vee) \geq 0,$$

then

$$(\widetilde{\lambda}, \alpha^\vee) \geq \frac{1}{2}.$$

Consequently

$$(\widetilde{\lambda}, \alpha^\vee) \geq 1$$

since $(\widetilde{\lambda}, \alpha^\vee)$ is an integer. Then

$$(\lambda, \alpha^\vee) = (\widetilde{\lambda}, \alpha^\vee) + \frac{1}{2}(\beta, \alpha^\vee) \geq \frac{1}{2},$$

namely, $\alpha \notin \pi_{\mathfrak{l}}$. □

If α_{l-1}, $\alpha_l \notin \Sigma_1$, that is,

$$(\widetilde{\lambda}, \alpha_{l-1}^\vee) \geq 0, \quad (\widetilde{\lambda}, \alpha_l^\vee) \geq 0,$$

then

$$(\widetilde{\lambda}, (\alpha_{l-1}+\alpha_l)^\vee) = 2(\widetilde{\lambda}, \alpha_{l-1}^\vee) + (\widetilde{\lambda}, \alpha_l^\vee) \geq 0.$$

Here the equality holds if and only if

$$(\widetilde{\lambda}, \alpha_{l-1}^\vee) = (\widetilde{\lambda}, \alpha_l^\vee) = 0.$$

Now we assume $\mathfrak{g}$ is of type B_l. First we prove a lemma.

Lemma 3.17. *Assume $\mathfrak{g}$ is of type B_l. If α_l is noncompact, then $(\rho_c, \alpha_l) = 0$.*

Proof. The compact root α which is adjacent to α_l must have one of the two forms: 1) $\alpha = \alpha_i + \cdots + \alpha_{l-1}$, 2) $\alpha = \alpha_i + \cdots + \alpha_{l-1} + 2\alpha_l$. Two such forms occur in a pair. A simple calculation leads to the lemma. □

If $\alpha_{l-1} \notin \Sigma_1$ and μ is regular for $\Delta(\mathfrak{g}, \mathfrak{t}^c)$, that is, $(\mu, \alpha) \neq 0$ for any $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}^c)$, then $(\mu + 2\rho_c, \alpha_l^\vee) \geq 1$ since $(\mu + 2\rho_c, \alpha_l^\vee)$ is an integer. Then $\alpha_l \notin \Sigma_1$. If α_{l-1}, $\alpha_l \in \Pi_{\mathfrak{l}}$, then we have $(\widetilde{\lambda}, \alpha_{l-1}^\vee) = (\widetilde{\lambda}, \alpha_l^\vee) = 0$, that is, $(\mu + 2\rho_c, \alpha_{l-1}^\vee) = (\mu + 2\rho_c, \alpha_l^\vee) = 1$. Then $(\mu + 2\rho_c, (\alpha_{l-1} + \alpha_l)^\vee) = (\mu, (\alpha_{l-1}+\alpha_l)^\vee) + (2\rho_c, (\alpha_{l-1}+\alpha_l)^\vee) = (\mu, (\alpha_{l-1}+\alpha_l)^\vee) + 2 = 3$. So we get $(\mu, \alpha_{l-1}^\vee) = 0$, which contradicts the assumption that μ is regular. Actually, we have proved:

Theorem 3.18. *Assume $\mathfrak{g}$ is of type B_l and μ is regular for $\Delta(\mathfrak{g}, \mathfrak{t}^c)$. Then the Dynkin diagram of $\mathfrak{l}$ is discrete. Consequently,*

$$\Gamma = \Pi_{\mathfrak{l}}.$$

3.2.3. Type D_l. Since all roots of D_l have the same length, some results on *AIII* can be applied and we can get some similar results.

Proposition 3.19. *Assume μ is regular for $\Delta(\mathfrak{k}, \mathfrak{t}^c)$.*

1) *Let $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{t}^c)$. If $(\alpha, \beta) = 0$ for any $\beta \in \Sigma_1$, then*

$$(\widetilde{\lambda}, \alpha^\vee) \geq 0.$$

2) $\Pi_{\mathfrak{l}} = \Gamma$. *Consequently, Γ consists of simple roots.*

Proof. 1) Let $\{\alpha_1, \ldots, \alpha_l\}$ be the simple roots. If $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_k$, similar to the proof of Proposition 3.10, one can easily get $(\widetilde{\lambda}, \alpha^\vee) \geq 0$. Now we assume $\alpha = \alpha_i + \cdots + \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l$.

If $k > i+1$, then α_i, $\alpha_k \notin \Sigma_1$. So we have

$$(\widetilde{\lambda}, (\alpha_i + \cdots + \alpha_{l-1})^\vee) \geq 0$$

and

$$(\widetilde{\lambda}, (\alpha_k + \cdots + \alpha_{l-2} + \alpha_l)^\vee) \geq 0.$$

Hence $(\widetilde{\lambda}, \alpha^\vee) \geq 0$.

If $k = i+1$, that is, $\alpha = \alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l$, then we have $\alpha_{i+1} \notin \Sigma_1$. In this case we may have $\alpha_i \in \Sigma_1$. If $\alpha_{l-1} \notin \Sigma_1$ or $\alpha_l \notin \Sigma_1$, the proof is similar to the above. Now we assume α_{l-1}, $\alpha_l \in \Sigma_1$, then $\alpha_{l-2} \notin \Sigma_1$. If $\alpha_{l-3} \notin \Sigma_1$, then we write

$$\alpha = (\alpha_i + \cdots + \alpha_{l-3}) + (\alpha_{i+1} + \cdots + \alpha_{l-1}) + (\alpha_{l-2} + \alpha_l).$$

If $\alpha_{l-3} \in \Sigma_1$, then $\alpha_{l-4} \notin \Sigma_1$, then write

$$\alpha = (\alpha_i + \cdots + \alpha_{l-4}) + (\alpha_{i+1} + \cdots + \alpha_{l-3}) + (\alpha_{l-3} + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l).$$

So we need only to show

$$(\widetilde{\lambda}, (\alpha_{l-3} + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l)^\vee) \geq 0, \tag{3}$$

where α_{l-3}, α_{l-1}, $\alpha_l \in \Sigma_1$. If α_{l-2} is compact, then $(\widetilde{\lambda}, \alpha_{l-2}^\vee) \geq 2$. (3) holds. If α_{l-2} is noncompact, then $\alpha_{l-3} + \alpha_{l-2}$ and $\alpha_{l-1} + \alpha_{l-2}$ are compact, hence

$$(\widetilde{\lambda}, \alpha^\vee) = (\widetilde{\lambda}, (\alpha_{l-3} + \alpha_{l-2})^\vee + (\alpha_{l-1} + \alpha_{l-2})^\vee + \alpha_l^\vee) \geq 1.$$

(3) holds.

2) Let $\alpha \in \Pi_{\mathfrak{l}}$. If α is adjacent to $\beta_j \in \Sigma_1$, then $\alpha + \beta_j$ is compact. Then

$$\begin{aligned}(\lambda, (\alpha+\beta)^\vee) &= (\widetilde{\lambda}, (\alpha+\beta)^\vee) + \frac{1}{2}\sum c_i(\beta_i, \beta^\vee) + \frac{1}{2}\sum c_i(\beta_i, \alpha^\vee) \\ &\geq 1 + c_j - \frac{1}{2}c_j - 1 = \frac{1}{2}.\end{aligned}$$

For the first inequality, we use the assumption that μ is regular. But it contradicts the fact that $\alpha \in \Pi_{\mathfrak{l}}$.

Now let α, $\beta \in \Pi_{\mathfrak{l}}$ be adjacent. Then neither α nor β is adjacent to elements in Σ_1. Again the fact that $\alpha+\beta$ is compact implies it is impossible. So the Dynkin diagram of $\mathfrak{l}$ is discrete. We must have

$$\Pi_{\mathfrak{l}} = \Gamma.$$

□

Combining the above results, Theorem 3.9 follows.

4. Dirac cohomology of unitary representations with regular lambda-lowest K-types.

In this section, we will consider the simple group G of types *AIII* ($SU(p,q)$), *BI*(*BII*) ($SO_0(p,q)$, $p+q$ odd), *CII*($Sp(p,q)$), *DI* ($SO_0(p,q)$, p and q even), *DIII*($SO^*(2n)$), that is all the classical groups except *CI*($Sp(n,\mathbb{R})$) with $\operatorname{rank} G = \operatorname{rank} K$. Also we will make the following assumption:

Assumption 4.1. μ is regular for $\Delta(\mathfrak{g},\mathfrak{t}^c)$.

4.1. The dominance of $\mu - \rho_n$. Since we assume μ is regular for $\Delta(\mathfrak{g},\mathfrak{t}^c)$, we can choose the following positive root system for $\Delta(\mathfrak{g},\mathfrak{t}^c)$: $\alpha \in \Delta^+(\mathfrak{g},\mathfrak{t}^c)$ if $(\mu+2\rho_c,\alpha^\vee) > 0$ or $(\mu+2\rho_c,\alpha^\vee) = 0$ and $(\mu,\alpha^\vee) > 0$. Set $\rho_n = \rho - \rho_c$.

Since $\mu \in \hat{K}$ is a lambda-lowest K-type, then the associate fine $L\cap K$-type with respect to L is $\mu^L = \mu - 2\rho(\mathfrak{u}\cap\mathfrak{p})$. Since $\Delta^+(\mathfrak{l},\mathfrak{t}^c) = \Pi_{\mathfrak{l}}$ consists of noncompact imaginary roots, we have $\Delta^+(\mathfrak{l},\mathfrak{t}^c) \subset \Delta^+(\mathfrak{p},\mathfrak{t}^c)$, hence

$$\Delta^+(\mathfrak{p},\mathfrak{t}^c) = \Delta^+(\mathfrak{l},\mathfrak{t}^c) \cup \Delta(\mathfrak{u},\mathfrak{t}^c).$$

So $\rho(\mathfrak{u}\cap\mathfrak{p}) = \rho_n - \rho_{\mathfrak{l}} = \rho - \rho_c - \rho_{\mathfrak{l}}$. Consequently,

$$\begin{aligned}\mu^L &= \mu - 2\rho(\mathfrak{u}\cap\mathfrak{p})\\ &= \mu + 2\rho_c - 2(\rho - \rho_{\mathfrak{l}}).\end{aligned}$$

So

$$(\mu^L, \beta_i^\vee) = 1 - c_i$$

for $\beta_i \in \Pi_{\mathfrak{l}}$.

Proposition 4.2. *$\mu - \rho_n$ is dominant for $\Delta^+(\mathfrak{k},\mathfrak{t}^c)$.*

We have to deal with it case by case.

Proof of the case AIII . We just need to prove $\widetilde{\lambda}$ is strictly dominant for $\Delta(\mathfrak{k},\mathfrak{t}^c)$, that is,

$$(\widetilde{\lambda},\alpha^\vee) \geq 1$$

for any α compact. Let $\Pi = \{\alpha_1,\ldots,\alpha_l\}$ be the simple roots of $\Delta^+(\mathfrak{g},\mathfrak{t}^c)$. Then the system of simple roots $\Pi_{\mathfrak{k}}$ of $\Delta^+(\mathfrak{k},\mathfrak{t}^c)$ consists of two kinds of elements

$$\Pi_{\mathfrak{k}} = \Pi_c \cup \Pi_n.$$

Π_c consists of those compact simple roots of Π. Elements of Π_n are of the form

$$\alpha = \alpha_i + \cdots + \alpha_k,$$

where α_i and α_k are noncompact and $\alpha_{i+1}, \ldots, \alpha_{k-1}$ are compact. If $\alpha \in \Pi_c$ then

$$(\widetilde{\lambda}, \alpha^\vee) \geq 2.$$

If $\alpha \in \Pi_n$, we treat it case by case.

Case I. $\alpha = \alpha_i + \alpha_{i+1}$, where α_i, α_{i+1} are noncompact. Then

$$(\widetilde{\lambda}, \alpha^\vee) \geq 1 + 2 - 2 = 1.$$

Case II. $\alpha = \alpha_i + \cdots + \alpha_k$, where $k - i > 2$. In this case α_{i+1} and α_{i+2} are compact. Then

$$(\widetilde{\lambda}, \alpha^\vee) = (\widetilde{\lambda}, (\alpha_i + \alpha_k)^\vee) + (\widetilde{\lambda}, (\alpha_{i+1} + \cdots + \alpha_{k-1})^\vee) \geq 1.$$

Case III. $\alpha = \alpha_i + \alpha_{i+1} + \alpha_{i+2}$. Here α_{i+1} is compact. Then $(\widetilde{\lambda}, \alpha^\vee) \geq 0$. If $(\widetilde{\lambda}, \alpha^\vee) = 0$, then

$$(\widetilde{\lambda}, \alpha_i) = (\widetilde{\lambda}, \alpha_{i+2}) = -1, \quad (\widetilde{\lambda}, \alpha_{i+1}) = 2.$$

Thus both (μ, α_i) and (μ, α_{i+2}) are integers. $(\rho_c, \alpha_i^\vee + \alpha_{i+2}^\vee) = (\rho_c, \alpha^\vee) - (\rho_c, \alpha_{i+1}^\vee) = 0$ implies that $(\mu, \alpha_i^\vee + \alpha_{i+2}^\vee) = 0$, which contradicts the choice of positive roots. □

Proof of the case B_l. Let $\alpha \in \Pi_{\mathfrak{k}}$ be simple. Then α must be one of the forms in the following cases:

Case I. $\alpha = \alpha_i \in \Pi$. Then $(\mu - \rho_n, \alpha^\vee) \geq 1$.

Case II. $\alpha = \alpha_i + \cdots + \alpha_k$, where α_i and α_k are noncompact and others are compact.

If $k < l$, the proof is similar to that of *AIII* .

If $k = l$, that is, α_l is noncompact, then $(\widetilde{\lambda}, \alpha^\vee) = (\widetilde{\lambda}, 2(\alpha_i^\vee + \cdots + \alpha_{l-1}^\vee) + \alpha_l^\vee)$. If $i < l - 1$, we can get the result easily. If $i = l - 1$, then $(\rho_c, \alpha_l^\vee) = 0$ implies $2(\rho_c, \alpha_{l-1}^\vee) = 1$ and $(\mu, \alpha_l^\vee) \geq 1$. Since μ is an integral weight and the choice of positive root system depends on μ, we have

$$(\mu, \alpha_{l-1}^\vee) \geq 0. \tag{4}$$

Since

$$(\mu, \alpha_l^\vee) \geq 0, \tag{5}$$

and the equalities (4) and (5) can't hold at the same time, we have

$$(\widetilde{\lambda}, (\alpha_{l-1} + \alpha_l)^\vee) \geq 1.$$

Case III. $\alpha = \alpha_i + \cdots + \alpha_k + 2(\alpha_{k+1} + \cdots + \alpha_l)$, where α_i, α_k and α_{k+1} are noncompact and the others are compact.

Since $\alpha_i + \cdots + \alpha_k \in \Pi_{\mathfrak{k}}$, we need only to prove

$$(\widetilde{\lambda}, (\alpha_{k+1} + \cdots + \alpha_l)^\vee) \geq 0,$$

which is obvious thanks to Lemma 3.17. □

Proof of the case CII. Since $\mathfrak{g}$ is of type *CII*, $\mathfrak{k}$ has no center and α_l must be a compact root. Let $\alpha \in \Pi_{\mathfrak{k}}$ be simple. Then α must be one of the forms in the following cases:

Case I. Similar to type B_l.

Case II. $\alpha = \alpha_i + \cdots + \alpha_k$, where $k < l$. Similar to type B_l.

Case III. $\alpha = 2(\alpha_i + \cdots + \alpha_{l-1}) + \alpha_l$, where only α_i is noncompact.
Since $(\widetilde{\lambda}, \alpha^\vee) = (\widetilde{\lambda}, \alpha_i^\vee + \cdots + \alpha_l^\vee)$, the conclusion is clear. □

Proof of the case D_l. Let $\alpha \in \Pi_{\mathfrak{k}}$ be simple. Then α must be one of the forms in the following cases:

Case I. Similar to type B_l.

Case II. $\alpha = \alpha_i + \cdots + \alpha_k$ $(k \leq l-2)$, $\alpha = \alpha_i + \cdots + \alpha_{l-1}$ or $\alpha = \alpha_i + \cdots + \alpha_{l-2} + \alpha_l$ or $\alpha_{l-1} + \alpha_{l-2} + \alpha_l$. Still similar to type B_l.

Case III. $\alpha = \alpha_i + \cdots + \alpha_l$, where α_i, α_{l-1}, α_{i-1} and α_l are noncompact and others are compact.

The only hard case is that $i = l-3$. Since at least one of α_{l-3}, α_{l-1} and α_l is not in Σ_1, all the simple root of $\mathfrak{k}$ is one of the three forms:

1) $\alpha_i \in \Pi$, $i < l-3$.

2) $\alpha_i + \cdots + \alpha_k$, $k < l-2$.

3) $\alpha_{l-3} + \alpha_{l-2}$, $\alpha_{l-2} + \alpha_{l-1}$, $\alpha_{l-2} + \alpha_l$ or $\alpha_{l-3} + \alpha_{l-2} + \alpha_{l-1} + \alpha_l$.

In this case $\mathfrak{k}$ is a sum of two simple Lie algebras of type D_l, say $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\Pi_{\mathfrak{k}} = \Pi_{\mathfrak{k}_1} \cup \Pi_{\mathfrak{k}_2}$. One can easily see $\alpha_{l-3}+\alpha_{l-2}$ and $\alpha_{l-3}+\alpha_{l-2}+\alpha_{l-1}+\alpha_l$ belong to the same subsystem, say $\Pi_{\mathfrak{k}_1}$, while $\alpha_{l-2} + \alpha_{l-1}$, $\alpha_{l-2} + \alpha_l \in \Pi_{\mathfrak{k}_2}$. And they play the role of α_{l-1} and α_l. One can easily see that

$$(\rho_c, \alpha_{l-1}^\vee) = (\rho_c, \alpha_l^\vee) = 0.$$

Then $(\mu, \alpha_{l-1}^\vee) > 0$ and $(\mu, \alpha_l^\vee) > 0$ and $(\rho_c, \alpha_{l-2}^\vee) = 1$. If μ is an integral weight, then α_{l-1}, $\alpha_l \notin \Sigma_1$. The assertion follows.

Case IV. $\alpha = \alpha_i + \cdots + \alpha_k + 2(\alpha_{k+1} + \cdots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l$, where $\alpha_i + \cdots + \alpha_k$ $(k \leq l-2)$ is a compact root in case II and α_{k+1}, α_{l-1} and α_l are noncompact. We can easily get the $(\mu - \rho_n, \alpha^\vee) \leq 0$. □

4.2. The dominance of Λ. Let $\Lambda = \lambda + \frac{1}{2}\sum c_i\beta_i = \widetilde{\lambda} + \sum c_i\beta_i$. Then we have:

Proposition 4.3. *Λ is dominant for $\Delta^+(\mathfrak{g}, \mathfrak{t}^c)$.*

Proof. Let $\sigma_i \in W(\mathfrak{g}, \mathfrak{t}^c)$ be the reflection with respect to $\beta_i \in \Gamma$. Set $\Delta' = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{t}^c) | \alpha \notin \Gamma\}$. Then Δ' is stable under each σ_i and their product $\sigma = \sigma_1 \dots \sigma_r$.

$\Lambda = \sigma(\widetilde{\lambda})$ is dominant for Δ' if and only if $\widetilde{\lambda}$ is dominant for Δ'. The assertion follows by the following lemma. □

Lemma 4.4. *$\widetilde{\lambda}$ is dominant for Δ'.*

Proof. In the above subsection, we have proved that $\widetilde{\lambda}$ is strictly dominant for $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$, so the only left is to check our assertion for noncompact roots in Δ'.

Let $\alpha \in \Delta'$ be noncompact. If α is not adjacent to any element in Σ_1, then

$$(\widetilde{\lambda}, \alpha^\vee) = (\lambda, \alpha^\vee) > 0.$$

Now assume α is adjacent to $\beta \in \Sigma_1$. If $\alpha+\beta$ is a root, then it is compact. So we have

$$(\widetilde{\lambda}, (\alpha+\beta)^\vee) \geq 1.$$

One can easily get $(\widetilde{\lambda}, \alpha^\vee) > 0$. If $\gamma = \alpha - \beta$ is a root then

$$(\widetilde{\lambda}, \alpha^\vee) = (\widetilde{\lambda}, (\gamma+\beta)^\vee).$$

Also we have $(\widetilde{\lambda}, \alpha^\vee) \geq 0$ if $(\gamma, \gamma) \geq (\beta, \beta)$ or γ is not a simple compact root. So we just need to consider the case that $(\gamma, \gamma) < (\beta, \beta)$ and γ is a simple compact root. Obviously, $\mathfrak{g}$ is of type B_l and $\gamma = \alpha_i + \cdots + \alpha_l$. According to the proof in Chapter 3, the assertion follows. □

4.3. The representations of L. Let L_1 be the commutator subgroup of L. It is a connected semisimple Lie group by [**8**, Lemma 4.3.4]. Then $L = TL_1$ (see [**8**, Lemma 0.4.2]) and $T_1 = T \cap L_1$ is a finite product of $\mathbb{Z}_2$. Let $(\delta, V) \in \hat{T}$ and $\delta_1 = \delta|_{T_1}$. Then $(\delta_1, V) \in \hat{T}$. Define $(\pi, \mathcal{H}) = \pi(P, \delta \otimes \nu)$ and $(\pi_1, \mathcal{H}_1) = \pi_1(P_1, \delta \otimes \nu)$, where $P = TAN$, $P_1 = T_1AN$ and $\nu \in \hat{A}$.

Lemma 4.5. *$\pi|_{L_1} \cong \pi_1$ as representations of L_1. Consequently, π is irreducible* (*resp. unitary*) *if and only if π_1 is irreducible* (*resp. unitary*).

Locally, L_1 is a product of some copies of $SL(2, \mathbb{R})$, i.e., there exists a canonical covering map:

$$p : \widetilde{L_1} = SL(2, \mathbb{R}) \times \cdots \times SL(2, \mathbb{R}) \to L_1$$

with finite kernel Z. Then π_1 can be regarded as a representation of $\widetilde{L}$ with Z acting trivially. Let $\widetilde{T} = p^{-1}(T_1)$. Then δ_1 can be regarded as a representation of $\widetilde{T_1}$. Let $\widetilde{\pi} = \pi(\widetilde{P}, \delta_1 \otimes \nu)$, which is equivalent with π_1 as representations of $\widetilde{L_1}$. Obviously, π_1 is a tensor product of representations of $SL(2,\mathbb{R})$. Then π_1 is unitary (irreducible, resp.) if and only if every component of the tensor product is unitary (irreducible, resp.). We can easily get the unitaribility and irreducibiliby of representations of L_1 since the representations of $SL(2,\mathbb{R})$ is so clear. Let us recall the following:

Theorem 4.6 ([**4**], Theorem 16.3). *The only irreducible unitary representations of $SL(2,\mathbb{R})$ up to unitary equivalence are:*

a) *The trivial representation;*
b) *the discrete series $\mathcal{D}_n^{\pm}$, $n \geq 2$, and the limits of discrete series $\mathcal{D}_1^{\pm}$,*
c) *the irreducible members of the unitary principal series, $\mathcal{P}^{+,iy}$ with y real and $\mathcal{P}^{-,iy}$ with y nonzero real,*
d) *the complementary series $\wp^x$ with $0 < x < 1$.*

Moreover the only equivalences among these representations are $\mathcal{P}^{+,iy} \cong \mathcal{P}^{+,-iy}$ and $\mathcal{P}^{-,iy} \cong \mathcal{P}^{-,-iy}$.

The fine representation μ^0 (see [**8**], Corollary 5.4.7) corresponding to μ is just $\mu^0 = \mu - 2\rho(\mathfrak{u} \cap \mathfrak{p}) = \mu - 2\rho(\mathfrak{p}) + 2\rho(\mathfrak{l}) = (\mu + 2\rho_c) - 2(\rho - \rho(\mathfrak{l}))$. Then we have

$$(\mu^0, \beta_i^{\vee}) = 1 - c_i,$$

that is, μ^0 is weight 0 of those $\mathfrak{l}(\beta_i)$ for $\beta_i \in \Gamma_1$ (Here $\mathfrak{l}(\beta_i)$ is the TDS generated by β_i) and weight 1 of those $\mathfrak{l}(\beta_i)$ for $\beta_i \in \Gamma_0$. Consequently $L(\beta_i) \cong SL(2,\mathbb{R})$ since $L(\beta_i)$ is either $SL(2,\mathbb{R})$ or $PSL(2,\mathbb{R})$, but the representations of the latter has no odd weight.

4.4. Proof of Theorem 1.2. Let X be an irreducible $(\mathfrak{g}, K)$-module with lambda-lowest K-type μ satisfying μ is regular for $\Delta(\mathfrak{g}, \mathfrak{t}^c)$. By the discussion above, we have known the following facts:

1) $\lambda = \mu + 2\rho_c - \rho + \frac{1}{2}\sum_{\beta_i \in \Gamma_1} \beta_i$. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be the parabolic associated to μ. Then the Dynkin diagram of $\mathfrak{l}$ is discrete.

2) $\mu - \rho_n$ is dominant for $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$.

3) $\Lambda = (\lambda, \frac{1}{2}\sum_{\beta_i \in \Gamma_1} \beta_i)$ is dominant for $\Delta^+(\mathfrak{g}, \mathfrak{t}^c)$.

Now we assume the θ-stable data corresponding to X is $(\mathfrak{q}, H, \delta, \nu)$, where $\nu = \frac{1}{2}\sum_{\beta_i \in \Gamma_1} \beta_i$. Consider the standard $(\mathfrak{g}, K)$-module:

$$\mathcal{R}^S(X_L(P, \delta \otimes \nu)).$$

By Theorem 6.5.12 [**8**], $X \cong \mathcal{R}^S(X_L(P, \delta \otimes \nu))(\mu)$ and a canonical cohomology class is $Y = \overline{X}_L(P, \delta \otimes \nu)(\mu - 2\rho(\mathfrak{u} \cap \mathfrak{p}))$, which is unitary as one can easily see. $\mathcal{R}^S(Y)$ is a submodule of $\mathcal{R}^S(X_L(P, \delta \otimes \nu))$. We have $X \subseteq \mathcal{R}^S(Y)$ since they have the same lambda-lowest K-type μ. Since $(\Lambda, \alpha^{\vee}) \geq 0$, then

$\mathcal{R}^S(Y)$ is unitary and irreducible (it is nonzero since it contains X), hence $X = \mathcal{R}^S(Y)$ is unitary. By Dirac inequality, we have

$$(\omega(\mu' - \rho_n') + \rho_c, \omega(\mu' - \rho_n') + \rho_c) \geq (\Lambda, \Lambda),$$

for all K-type μ' of X, all ρ_n' and for some $\omega \in W_K$. Note that $\mu - \rho_n$ is dominant for $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$ and

$$\mu - \rho_n + \rho_c = \Lambda,$$

we have the equality holds. Using $t\nu$, $0 < t < 1$, instead of ν, one can see that $X_L(\delta \otimes t\nu)$ is unitary since it is a tensor product of complementary series and discrete series of $SL(2, \mathbb{R})$. Let $\Lambda_t = (\lambda, t\nu)$. Then $\Lambda_0 = \lambda$ and $\Lambda_t = (1-t)\Lambda_0 + t\Lambda_1$. Hence $(\Lambda_t, \alpha^\vee) = (1-t)(\lambda, \alpha^\vee) + t(\widetilde{\lambda}, \alpha^\vee) > 0$, for all $\alpha \in \Delta(\mathfrak{u})$. Then by Theorem 1.3 [**9**], we have

$$\mathcal{R}^S(X_L(\delta \otimes t\nu))$$

is unitary. So we have

$$(\omega(\mu' - \rho_n') + \rho_c, \omega(\mu' - \rho_n') + \rho_c) \geq (\Lambda_t, \Lambda_t),$$

for all $t \in (0, 1)$ by Dirac inequality. Since all the K types of $X_G(\mathfrak{q}, H, \delta, \nu)$ are independent of the choice of ν, when t tends to 1, we get

$$(\omega(\mu' - \rho_n') + \rho_c, \omega(\mu' - \rho_n') + \rho_c) \geq (\Lambda, \Lambda)$$

which implies $(\mu - \rho_n + \rho_c, \mu - \rho_n + \rho_c) = (\Lambda, \Lambda)$, hence X has nonzero Dirac cohomology.

Conversely, if X has nonzero Dirac cohomology, then the infinitesimal character of X is $\mu - \rho_n + \rho_c = (\lambda, \nu)$ by the same argument. One can easily show that $\nu = \frac{1}{2}\sum_{\beta_i \in \Gamma_1} \beta_i$.

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